ECE 314 – Signals and Systems – Fall 2012

Solutions to Homework 6

Problem 3.50

(a) $x(t) = \sin(3\pi t) + \cos(4\pi t)$.

Solution: The fundamental frequency of the signal above is $\omega_o = \pi$. We may write

$$x(t) = \frac{1}{2j}\left(e^{j3\pi t} - e^{-j3\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right).$$

Hence, we have

$$X[k] = \begin{cases} j/2, & k = -3, \\ -j/2, & k = 3 \\ 1/2, & k = -4, 4 \\ 0, & \text{otherwise.} \end{cases}$$

(d) $x(t)$ as depicted in Figure P3.50(a).

Solution: From the graph, we can see that $T = 1$. We can calculate $X[k]$ using the formula over the period $[0, 1]$:

$$\begin{aligned} X[k] &= \int_0^1 \sin(\pi t) e^{-j2\pi k t} dt \\ &= \frac{1}{2j}\int_0^1 \left(e^{j(\pi - 2\pi k)t} - e^{-j(\pi + 2\pi k)t}\right) dt \\ &= \frac{1}{\pi - 2\pi k} + \frac{1}{\pi + 2\pi k} \end{aligned}$$

Problem 3.51

(d) $X[k]$ as depicted in Figure P3.51(a), $\omega_o = \pi$.

Solution: From the definition,

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t} \\
&= X[-4]e^{-j4\omega_o t} + X[-3]e^{-j3\omega_o t} + X[3]e^{j3\omega_o t} + X[4]e^{j4\omega_o t} \\
&= 2e^{-j\pi/4}e^{-j4\omega_o t} + e^{j\pi/4}e^{-j3\omega_o t} + e^{-j\pi/4}e^{j3\omega_o t} + 2e^{j\pi/4}e^{j4\omega_o t} \\
&= 2\left[e^{-j(4\omega_o t+\pi/4)} + e^{j(4\omega_o t+\pi/4)}\right] + \left[e^{-j(3\omega_o t-\pi/4)} + e^{j(3\omega_o t-\pi/4)}\right] \\
&= 4\cos\left(4\omega_o t + \pi/4\right) + 2\cos\left(3\omega_o t - \pi/4\right).
\end{aligned}$$

Special Problem 1 Let $x(n) = 2[u(n) - u(n-11)]$ and $y(n) = (n+2)[u(n+2) - u(n-17)]$.

(a) Determine $z(n) = x(n) * y(n)$ analytically.

Solution: By definition

$$\begin{aligned}
z[n] &= \sum_{k=-\infty}^{\infty} x[k]y[n-k], \\
&= \sum_{k=-\infty}^{\infty} 2[u(k) - u(k-11)](n-k+2)[u(n-k+2) - u(n-k-17)] \\
&= 2\sum_{k=0}^{10}(n-k+2)[u(n-k+2) - u(n-k-17)].
\end{aligned}$$

As we can see, $[u(n-k+2) - u(n-k-17)] = 1$, when $k \in [n-16, n+2]$, and it is zero, otherwise. So, we have to consider five situations:

- When $n+2 < 0 \Rightarrow n < -2$, $z(n) = 0$;

- when $n+2 \in [0,10] \Rightarrow n \in [-2,8]$,

$$\begin{aligned} z(n) &= 2\sum_{k=0}^{n+2}(n-k+2)) \\ &= 2\sum_{j=0}^{n+2} j \\ &= 2(n+3)(n+2)/2 \\ &= n^2+5n+6; \end{aligned}$$

- when $n-16<0$ and $n+2>10 \Rightarrow n \in (8,16)$,

$$\begin{aligned} z(n) &= 2\sum_{k=0}^{10}(n-k+2)) \\ &= 2\sum_{j=n-8}^{n+2} j \\ &= 2\cdot 11(2n-6)/2 \\ &= 22n-66; \end{aligned}$$

- when $n-16 \in [0,10] \Rightarrow n \in [16,26]$,

$$\begin{aligned} z(n) &= 2\sum_{k=n-16}^{10}(n-k+2)) \\ &= 2\sum_{j=n-8}^{18} j \\ &= 2(n+10)(-n+27)/2 \\ &= -n^2+17n+270; \end{aligned}$$

- and when $n-16>10 \Rightarrow n>26$, $z(n)=0$.

(b) What are the values of n for which $z(n) \neq 0$?

Solution: As we have seen from the previous item, $z(n) \neq 0$ when $n \in [-2,26]$.

(c) Compute $z(n)$ using the *conv* command in MATLAB. Plot z and compare it to your results from part (a). Does your answer agree with part a?

Solution: Run the file `hw6_prob1.m`.

Special Problem 2 Write a Matlab code to compute the following discrete convolutions $z(n) = x(n) * y(n)$ for the cases below. Plot $z(n)$ in the range $-5 \leq n \leq 15$. In addition, turn in your code and a table of values for z(n) in each case. The plots and their axes must be appropriately labeled.

1. $x(n) = 0.8^n u(n)$; $y(n) = 0.5^n u(n)$.
2. $x(n) = 0.8^n u(n-1)$; $y(n) = 0.5^n u(n-2)$.
3. $x(n) = \cos(0.4\pi n)u(n)$; $y(n) = 0.85^n u(n)$.

Solution: The MATLAB code can be found in the files: `hw6_prob2_part1.m`, `hw6_prob2_part2.m`, and `hw6_prob2_part3.m`.

	$z(n)$		
n	Part 1	Part 2	Part 3
-5	0.0	0.0	0.0
-4	0.0	0.0	0.0
-3	0.0	0.0	0.0
-2	0.0	0.0	0.0
-1	0.0	0.0	0.0
0	1.0	0.0	1.0
1	1.3	0.0	1.159
2	1.29	0.0	0.176
3	1.157	0.2	-0.659
4	0.988	0.26	-0.251
5	0.822	0.258	0.786
6	0.673	0.231	0.977
7	0.546	0.198	0.0218
8	0.441	0.164	-0.791
9	0.355	0.135	-0.363
10	0.285	0.109	0.692
11	0.228	0.088	0.897
12	0.183	0.071	-0.0467
13	0.146	0.057	-0.849
14	0.117	0.046	-0.412
15	0.094	0.037	0.649

Special Problem 3 Compute $z(n)$ for part (1) above analytically and verify that your analytical answer agrees with the Matlab result.

Solution: Below, we determine the convolution of $\alpha^n u(n)$ and $\beta^n u(n)$, when $\alpha \neq \beta$.

$$\begin{aligned}
z(n) &= [\alpha^n u(n)] * [\beta^n u(n)] \\
&= \sum_{k=-\infty}^{\infty} \alpha^k u(k) \beta^{n-k} u(n-k) \\
&= u(n) \sum_{k=0}^{n} \alpha^k \beta^{n-k} \\
&= u(n) \beta^n \sum_{k=0}^{n} (\alpha\beta^{-1})^k \\
&= u(n) \beta^n \frac{(\alpha\beta^{-1})^{n+1} - 1}{\alpha\beta^{-1} - 1} \\
&= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n)
\end{aligned}$$

In the case of this exercise, $\alpha = 0.5$ and $\beta = 0.8$. So,

$$z(n) = [0.8^{n+1} - 0.5^{n+1}] \frac{1}{0.3} u(n).$$

n	$z(n)$
0	1.0
1	1.3
2	1.29
3	1.157
4	0.988
5	0.822
6	0.673
7	0.546
8	0.441
9	0.355
10	0.285
11	0.228
12	0.183
13	0.146
14	0.117
15	0.094

Special Problem 4 Consider the difference equation

$$y(n) = \frac{4}{3}y(n-1) - \frac{1}{3}y(n-2) + u(n) + \left(\frac{1}{3}\right)^n u(n)$$

with initial conditions $y(-2) = 1$ and $y(-1) = 3$.

(a) Find the zero-input response $y_{0x}(n)$.

Solution: If we consider that the input is zero, the difference equation above becomes the homogeneous equation:

$$y(n) = \frac{4}{3}y(n-1) - \frac{1}{3}y(n-2).$$

Now, let us suppose that the response for the homogeneous equation is $y_{0x}(n) = r^n$, where r is a constant. Then, we have

$$r^n = \frac{4}{3}r^{n-1} - \frac{1}{3}r^{n-2},$$

and from that expression, we can derive the characteristic equation

$$r^2 - \frac{4}{3}r + \frac{1}{3} = 0.$$

The roots of this characteristic equation are: $r_1 = 1$ and $r_2 = 1/3$.

Since, any linear combination of the responses $y_1(n) = r_1^n$ and $y_2(n) = r_2^n$ is also a response to the homogeneous equation, we may write the zero-input response as

$$y_{0x}(n) = c_1 + c_2 \left(\frac{1}{3}\right)^n,$$

where c_1 and c_2 are constants.

To determine the zero-input response corresponding to the initial conditions described in the problem statement, we have to determine the constants c_1 and c_2. This can be done by evaluating $y_{0x}(n)$ for $n = -2, -1$:

$$\left\{ \begin{array}{rcccl} h(-1) & = & c_1 + 3c_2 & = & 3 \\ h(-2) & = & c_1 + 9c_2 & = & 1 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{rcl} c_1 & = & 4 \\ c_2 & = & -\frac{1}{3} \end{array} \right.$$

Hence,

$$y_{0x}(n) = 4 - \left(\frac{1}{3}\right)^{n+1}$$

(b) Find the impulse response $h(n)$.

Solution: Considering that the input $x(n) = \delta(n)$, we will have the following equation

$$h(n) = \frac{4}{3}h(n-1) - \frac{1}{3}h(n-2) + \delta(n),$$

where $y(n)$ has been replaced by $h(n)$ (the impulse response).

Assuming that the system is causal, and, therefore, $h(n) = 0$ for $n < 0$, we can determine the first two samples of $h(n)$ from the equation above:

$$\begin{aligned} h(0) &= \frac{4}{3}h(-1) - \frac{1}{3}h(-2) + \delta(0) = 1 \\ h(1) &= \frac{4}{3}h(0) - \frac{1}{3}h(-1) + \delta(1) = \frac{4}{3} \end{aligned}$$

To find out what are the other values of $h(n)$, we take into account the fact that $h(n)$ satisfies the homogeneous equation for $n > 0$. Therefore, $h(n) = d_1 + d_2(1/3)^n$ is good candidate. We just have to determine the constants d_1 and d_2, based on the values of $h(n)$ for $n = 0, 1$. Hence, we have

$$\left\{ \begin{array}{rcccl} h(0) &=& d_1 + d_2 &=& 1 \\ h(1) &=& d_1 + \frac{1}{3}d_2 &=& \frac{4}{3} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{rcl} d_1 &=& \frac{3}{2} \\ d_2 &=& -\frac{1}{2} \end{array} \right.$$

So,

$$h(n) = \left[\frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^n\right] u(n).$$

(c) Find the zero-initial-condition response $y_{0ic}(n)$.

Solution: The zero-initial-condition response is given by

$$y_{0ic}(n) = h(n) * x(n),$$

where $x(n)$ is the input signal, given by

$$x(n) = u(n) + \left(\frac{1}{3}\right)^n u(n).$$

Hence,

$$y_{0ic}(n) = \left[\frac{3}{2}u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)\right] * \left[u(n) + \left(\frac{1}{3}\right)^n u(n)\right].$$

We can, then, apply the distributive property of the convolution with respect to addition, and use the following identities:

1. $u(n) * u(n) = (n+1)u(n)$;
2. $u(n) * \alpha^n u(n) = \frac{1-\alpha^{n+1}}{1-\alpha} u(n)$;
3. $\alpha^n u(n) * \alpha^n u(n) = (n+1)\alpha^n u(n)$.

Thus,

$$\begin{aligned}
y_{0ic}(n) &= \frac{3}{2}u(n) * u(n) + \frac{3}{2}u(n) * \left(\frac{1}{3}\right)^n u(n) \\
&- \frac{1}{2}\left(\frac{1}{3}\right)^n u(n) * u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n) \\
&= \frac{3}{2}(n+1)u(n) + \frac{3}{2}\left[1 - (1/3)^{n+1}\right] u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n (n+1)u(n)
\end{aligned}$$

(d) Find and plot the total response $y(n)$ using the results obtained in (a)-(c). Evaluate $y(n)$ at $n = 0, \ldots, 5$. Compute $y(n)$ interactively (directly from the difference equation) for $n = 0, \ldots, 5$, and compare to the values obtained from the total solution.

Solution: The total solution is

$$\begin{aligned}
y(n) &= y_{0x}(n) + y_{0ic}(n) \\
&= 4 - \left(\frac{1}{3}\right)^{n+1} + \frac{3}{2}(n+1)u(n) + \frac{3}{2}\left[1 - (1/3)^{n+1}\right] u(n) \\
&- \frac{1}{2}\left(\frac{1}{3}\right)^n (n+1)u(n) \qquad (1)
\end{aligned}$$

The results have been calculated using MATLAB (see file `hw6_prob4_d.m`). The graph below shows the results.

(e) Is the LTI system described by the above difference equation stable? Justify your answer. What is $\lim_{n\to\infty} h(n)$? Is this limit necessarily zero if the system is stable? Justify your answer in light of the condition for stability for LTI systems. Is $\lim_{n\to\infty} y_{0x}(n)$ necessarily zero if the system is stable?

Solution: The system is **unstable**, because

$$\sum_{n=-\infty}^{\infty} |h(n)| = \infty.$$

We can see that $\lim_{n\to\infty} h(n) = 3/2$. This limit has to be necessarily zero in order to the infinite sum of the absolute values of $h(n)$ have a limit. Since this condition does not hold, the system cannot be stable.

As we have also seen, $y_{0x}(n)$ is directly related to $h(n)$. Therefore, $\lim_{n\to\infty} y_{0x}(n)$ has to be zero, in order to $\lim_{n\to\infty} h(n)$ also be zero. A final comment, $\lim_{n\to\infty} h(n) = 0$ is a necessary, but insufficient, condition for stability.