ECE 340: PROBABILISTIC METHODS IN ENGINEERING

SOLUTIONS TO HOMEWORK #2

2.16 Solution

a) Write an expression for the event "overall system is up."

$A= (A_{11} \cap A_{12}) \cup (A_{21} \cap A_{22}) \cup (A_{31} \cap A_{32})$

b) We begin by stating that signal transmission through two switches *in series* amounts to the transmission of the signal through the first switch *and* its transmission through the second switch. Similarly, transmission through two switches in parallel amounts to transmission through one switch *or* transmission through the other. From Fig. P2.2 in the text, we see that if we want to transmit the signal from input (left side) to output (right side), the following condition must be satisfied: at least one of the pairs of switches that are next to each other horizontally (e.g., A_{11} with A_{12}) must be up.

2.24 Solution

a) P(A occurs and B does not occur) = $P(A\backslash B)=P(A) – P(A \cap B)$.
Proof) Note that we can always write A as the *disjoint* union $A\backslash B \cup (A\cap B)$ (can you prove this?). Hence, by taking the probability of both sides we obtain $P(A)= P(A\backslash B) + P(A\cap B)$; hence, by rearranging we obtain $P(A\backslash B)= P(A)- P(A\cap B)$
P(B occurs and A does not occur) = $P(B) – P(A \cap B)$
Proof) Similar to above.

b) P(exactly one of A or B occurs) = $P(A) + P(B) – 2P(A \cap B)$
Proof) We want $P((A^c\cap B) \cup (B^c\cap A))$, which is equal to $P((B\backslash A) \cup (A\backslash B))$. Now since $A\backslash B$ and $B\backslash A$ are disjoint, $P((A\backslash B) \cup (B\backslash A)) = P((A\backslash B)) + P((B\backslash A))$. If we use the results in the previous part, we obtain the desired result.

c) P(neither A nor B occur) = $1 – (P(A) + P(B) – P(A \cap B))$
Proof) We want $P(A^c \cap B^c)$. But $A^c \cap B^c =(A \cup B)^c$ (what is this formula called? DeMorgan's Law), so $P(A^c \cap B^c)= P((A \cup B)^c)= 1- P(A \cup B)= 1 – (P(A) + P(B) – P(A \cap B))$.

2.26 Identities of this type are shown by application of the axioms. We begin by treating (AUB) as a single event. Namely,

$$
\begin{aligned}
P(A \cup B \cup C) &= P((A \cup B) \cup C) \\
&= P(A \cup B) + P(C) – P((A \cup B) \cap C) && \textit{by cor. 5} \\
&= P(A) + P(B) – P(A \cap B) + P(C) – P((A \cap C) \cup (B \cap C))
\end{aligned}
$$

by cor. 5 and distributive property

= P(A) + P(B) + P(C) – P(A ∩ B) – P(A ∩ C) – P(B ∩ C) + P((A∩B) ∩ (B∩C))

by cor. 5 on (A ∩ C) U (B ∩ C)

= P(A) + P(B) + P(C) – P(A ∩ B) – P(A ∩ C) – P(B ∩ C) + P(A ∩ B ∩ C)

since (A ∩ B) ∩ (B ∩ C) =A ∩ B ∩ C

2.30

(a) Note that P(AUB) ≤P(A)+P(B); this is because we can write AUB as the disjoint union A U (B\A), so P(AUB)=P(A)+P(B\A), but P(B\A) ≤ P(B) since B\A is a subset of B. Hence, we now replace P(B\A) in P(AUB)=P(A)+P(B\A) by P(B) and make the right hand side bigger, and obtain P(AUB) ≤P(A)+P(B).

Now P(A U B U C) = P(A U (B U C)) ≤ P(A)+P(B U C) by the earlier result above (think of the earlier B as BUC). Also by the earlier result we know that P(BUC) ≤P(B)+P(C). Putting these together we conclude that P(A U (B U C)) ≤ P(A)+P(B U C)

(b).

Note that $P(\bigcup_{k=1}^{n} A_k) = P(\bigcup_{k=2}^{n} A_k \cup A_1) \le P(\bigcup_{k=2}^{n} A_k) + P(A_1)$

$= P(\bigcup_{k=3}^{n} A_k \cup A_2) + P(A_1)$

$\le P(\bigcup_{k=3}^{n} A_k) + P(A_2) + P(A_1)$

Continuing this process, we arrive at

$P(\bigcup_{k=1}^{n} A_k) \le P(A_{n-1} \cup A_n)) + P(A_{n-2})) + \ldots + P(A_1)$, from which we obtain the desired result

$P(\bigcup_{k=1}^{n} A_k) \le P(A_n) + P(A_{n-1}) + P(A_{n-2})) + \ldots + P(A_1)$,

(c) Replace each A_k in part b by B_k^c; hence, we can write $P(\bigcup_{k=1}^{n} A_k) = P(\bigcup_{k=1}^{n} B_k^c)$, but $\bigcup_{k=1}^{n} B_k^c$ is by De Morgan's law equal to $\left(\bigcap_{k=1}^{n} B_k\right)^c$. Hence, $P(\bigcup_{k=1}^{n} B_k^c) = P(\left(\bigcap_{k=1}^{n} B_k\right)^c) = 1 - P\left(\bigcap_{k=1}^{n} B_k\right)$. Now we use the fact that $P(\bigcup_{k=1}^{n} B_k^c) \le P(B_n^c) + P(B_{n-1}^c) + \ldots + P(B_1^c)$ from part (b), and obtain

$1 - P\left(\bigcap_{k=1}^{n} B_k\right) \le P(B_n^c) + P(B_{n-1}^c) + \ldots + P(B_1^c)$, or

$P\left(\bigcap_{k=1}^{n} B_k\right) \geq$ 1- P(B_n^c) - P(B_{n-1}^c) - …- P(B_1^c), as desired.

2.37 Solution:

a) Since $(-\infty,r] \subset (-\infty,s]$, when r < s, then necessarily $P((-\infty,r]) \leq P((-\infty,s])$.

b) $P((-\infty,s]) = P((-\infty,r] \cup (r,s])$
$= P((-\infty,r)) + P((r,s])$
Thus, $P((r,s]) = P((-\infty,s]) - P((-\infty,r])$

c) $P((s,\infty)) = 1-P((s,\infty)^c)$ $\quad 1- P((-\infty,s])$.

Special Problem:

a) We can think of a Geiger-count occurrence in the *n*th interval, $(ndt, (n+1)dt]$, as the occurrence of a head in the *n*th coin flip in the successive coin-flipping example considered in class, and the non-occurrence of a count in the same interval as the occurrence of a tail. Note that here we are taking dt as t/n.

b) The event that no count occurs in the big interval $[0,t]$ is the same as the event that no count occurs in any of the little intervals $[0,dt)$, $[dt, 2dt)$, …, $((n-1)dt, ndt]$.

Now consider the probability of this event, $(1-a\,dt)^n$. We now need to take the limit of this quantity as n tends to infinity. We will need the fact from sequences that $\lim_{n\to\infty} (1-n^{-1})^n=1/e$. More precisely, we write

$\lim_{n\to\infty} (1-a\,dt)^n = \lim_{n\to\infty} [1-(a\,t/n)]^n = \lim_{n\to\infty} \{[1-(a\,t/n)]^{n/(a\,t)}\}^{a\,t}$
$= \{\lim_{n\to\infty} [1-(a\,t/n)]^{n/(a\,t)}\}^{a\,t} = (1/e)^{at} = e^{-at}$.