ECE 314 – Signals and Systems Fall/2012

Solutions to Homework 3

Problem 1.61 For clarity, let us rename the signal described in the textbook from x to x_Δ. Now differentiate x and observe that the derivative is zero outside the interval $(-\Delta/2, -\Delta/2)$, and it is Δ^{-1} over this interval. Clearly, $\lim_{\Delta\to 0} x_\Delta(t) = 0$ for any $t \neq 0$. At the same time, the integral of $x_\Delta(t)$ over the interval $(-\infty, \infty)$ is always unity. These two properties are those that define a delta function.

Problem 1.64 The systems that follow have input $x(t)$ or $x[n]$ and output $y(t)$ or $y[n]$. For each system, determine whether it is (i) memoryless, (ii) stable, (iii) causal, and (v) time-invariant.

(a) $y(t) = \cos(x(t))$

Solution:

(i) Is the system memoryless? Yes, since $y(t)$ only depends on the present value of $x(t)$.

(ii) Is the system stable? Yes, since $|y(t)| \leq 1$ (property of the cosine function).

(iii) Is the system causal? Yes, since it is memoryless, it only depends on the present input (For a system to be causal, its present output must not depend on future values of the input).

(v) Is the system time-invariant? Yes, since $y(t+\tau) = \cos(x(t+\tau))$, for any t and τ.

(b) $y[n] = 2x[n]u[n]$

Solution:

(i) Is the system memoryless? Yes, since $y[n]$ only depends on the present value of $x[n]$.

(ii) Is the system stable? Yes, since $|y(t)| = 2|x[n]|u[n] \leq 2|x[n]||$. Hence, if $x[n]$ is bounded by M ($|x[n]| \leq M$), then $y[n]$ is bounded by $2M$.

(iii) Is the system causal? Yes, since it is memoryless, it only depends on the present input.

(v) Is the system time-invariant? No. The time-invariance condition does not hold, because the signal that is being multiplied by $x[n]$ varies with time.

(d) $y(t) = \int_{-\infty}^{t/2} x(\tau)d\tau$

Solution:

(i) Is the system memoryless? No, since the integral is evaluated on the input over all the time from $-\infty$ to $t/2$.

(ii) Is the system stable? No. A simple counter example is when the input signal is $x(t) \equiv 1$, which is obviously bounded, while the output $y(t)$ is not finite, since the integral of 1 from $-\infty$ to $t/2$ is not finite.

(iii) Is the system causal? No. For negative t, the output depends on all values of $x(t)$, from $-\infty$ to $t/2$, which is greater than t. Hence, the output depends on future values of $x(t)$.

(v) Is the system time-invariant? No. The output $y_d(t)$ for a time-shifted version of the input $x(t-d)$ is

$$\begin{aligned} y_d(t) &= \int_{-\infty}^{t/2} x(\tau - d)d\tau \\ &= \int_{-\infty}^{t/2-d} x(s)ds \\ &= \int_{-\infty}^{(t-2d)/2} x(s)ds \\ &= y(t-2d). \end{aligned}$$

Therefore, it does not obey the time-invariance condition.

(f) $y(t) = \frac{d}{dt}x(t)$

Solution:

(i) Is the system memoryless? No, since the derivative of a function at a specific point t_o cannot be determined just from the knowledge of the value of the function on t_o. (ex. you cannot determine the derivative of $x(t)$ at $t = 2$, if you only know that $x(2) = 10$).

(ii) Is the system stable? No. A counter example is when $x(t) = \sqrt{1-t^2}, -1 < t < 1$, and it is zero otherwise. $x(t)$ is bounded, but its derivative, which is given by

$$\frac{dx(t)}{dt} = -\frac{t}{\sqrt{1-t^2}},$$

goes to $-\infty$, when t approaches 1.

(iii) Is the system causal? Yes, since the derivative can be determined from the expression

$$\frac{dx(t)}{dt} = \lim_{h \to 0^+} \frac{x(t) - x(t-h)}{h},$$

which only depends on past values of $x(t)$.
(v) Is the system time-invariant? Yes. The derivative of a time-shifted signal is

$$y_d(t) = \frac{d}{dt}[x(t-d)] = \frac{dx}{dt}(t-d)\frac{d}{dt}(t-d) = \frac{dx}{dt}(t-d) = y(t-d).$$

(i) $y(t) = x(2-t)$
(i) Is the system memoryless? No, since the output depends on the value of the input at a time-instant other then t.
(ii) Is the system stable? Yes, since $|y(t)| \leq M$, if $|x(t)| \leq M$.
(iii) Is the system causal? No. For negative t, $2-t$ is positive, therefore the output depends on the future.
(v) Is the system time-invariant? No.

$$y_d(t) = x(2-t-d)) = x(2-(t+d)) = y(t+d).$$

Problem 1.71
(a) Yes. Consider the system defined by the rule $\mathcal{O}(i)(t) = \int_{-\infty}^{t} \frac{1}{M(s)} f(s)\, ds$, where $M(t) = 50e^{-0.01}tu(t)$. Show that this system is linear but it is time variant. Can you give an example of a physical system that can be modeled by the above system?
(b) The equation for this circuit is: $i(t)R(t) + v_2(t) = v_i(t)$. Assume that $v_2(\infty) = 0$. Since $i(t) = cv_2'(t)$, we can rewrite the circuit equation as $v_2'(t) + \frac{1}{R(t)C}v_2(t) = \frac{1}{R(t)C}v_i(t)$. Following class notes on linearity of ODE's, show that a system represent by a differential equation with time varying coefficients is still linear.

Problem 1.76 A linear system H has the input-ouput pairs depicted in Fig. 1.76(a) (in the book). Answer the following questions, and explain your answers:

(a) Could this system be causal?

Solution: No. The system is linear, therefore, for an input $x(t) \equiv 0$, the output should be $y(t) \equiv 0$. This is true because, by the homogeneity property, when $x(t) = 0 \cdot f(t)$, the output must be $y(t) = 0 \cdot H(f)(t) \equiv 0$.

If the system is causal, it doesn't know anything about the future. Therefore, if for some input $x(\tau) = 0$, for $\tau < t$, then the output must be $y(\tau) = 0$, for $\tau < t$. Because, as for what the system knows, $x(t)$ could be 0 for every t.

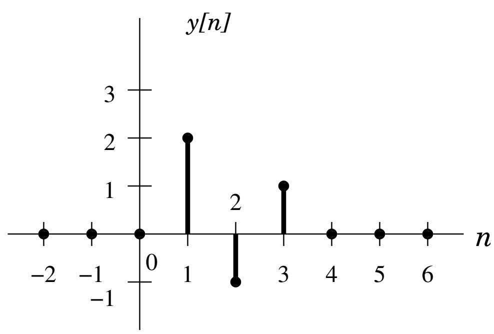
On the figure, we notice that $y_2(t) = 1$ for $t \in (0, 1)$, while $x_2(t) = 0$ for $t \in (-\infty, 1)$. This contradicts the conclusions discussed before. Therefore, the system cannot be causal.

Problem 1.77 A discrete-time system is both linear and time-invariant. Suppose the output due to an input $x[n] = \delta[n]$ is given in Fig. 1.77(a) (in the book).

(a) Find the output due to an input $x[n] = \delta[n-1]$

Solution: Let's call the signal in Fig. 1.77(a) $h[n]$. Since the system is time-invariant, for $x[n] = \delta[n-1]$,

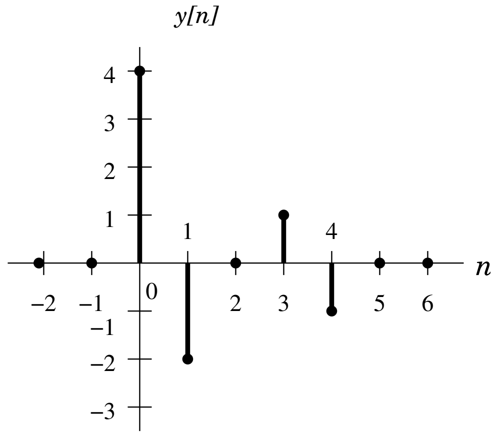
$$y[n] = h[n-1].$$



(b) Find the output due to an input $x[n] = 2\delta[n] - \delta[n-2]$.

Solution: Now we use the linearity property, as well as time-invariance:

$$\begin{aligned} x[n] &= 2\delta[n] - \delta[n-2] \Rightarrow \\ y[n] &= 2h[n] - h[n-2]. \end{aligned}$$

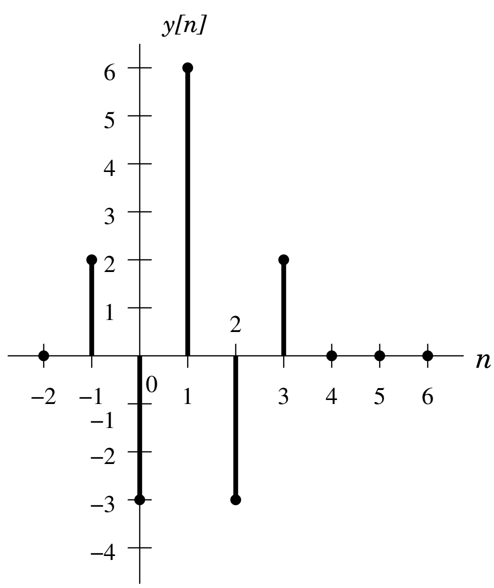


(c) Find the output due to the input depicted in Fig. 1.77(b).
Solution: Now, we can see that

$$x[n] = \delta[n+1] - \delta[n] + 2\delta[n-1].$$

Therefore,

$$y[n] = h[n+1] - h[n] + 2h[n-1].$$



Problem 1.93 **(a)** The solution of a linear differential equation is given by

$$x(t) = 10e^{-t} - 5e^{-0.5t}.$$

Using MATLAB, plot $x(t)$ versus t for `t = 0:0.01:5`.

Solution: A possible MATLAB code to do that is:

```
t = 0 : 0.01 : 5 ;
x = 10 * exp(-t) - 5 * exp(-0.5 * t);
plot(t,x);
```