Lecture 26
(R-C Circuits)

Physics 161-01 Spring 2012

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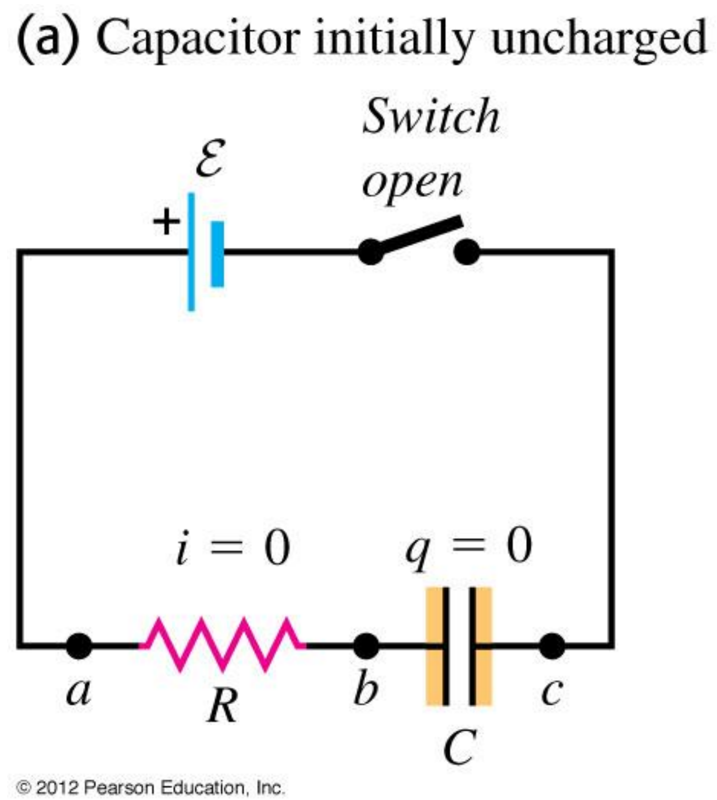
An R-C Circuit

- We are now going to explore the time dependence of the charge on a capacitor.
- If we start with a circuit with a resistor and a capacitor in series with a EMF, but with the circuit broken by an open switch, the capacitor is uncharged, since there is no potential difference across it.



An R-C Circuit (Charging)

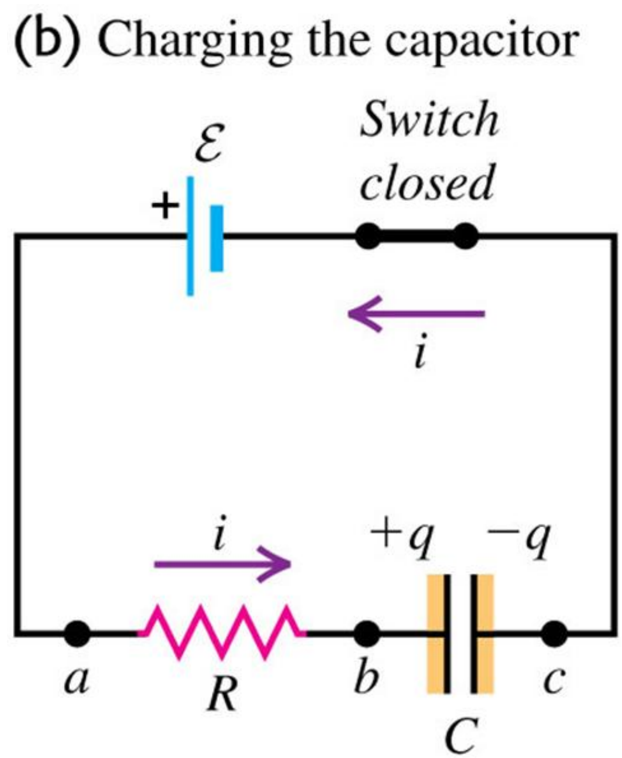
- At some time, t_0, we close the switch, let us now examine, using Kirchhoff's loop rule:

$$\sum \Delta V = \mathcal{E} - i(t)R - \frac{q(t)}{C} = 0$$

- But this equation relates two quantities that vary over time: *i* and *q*.
- But, since the charge on the capacitor, *q*, comes from the current, *i*, we can relate these two as well:

$$\mathcal{E} - \frac{dq}{dt}R - \frac{q}{C} = 0 \Rightarrow$$

$$\frac{dq}{dt} + \frac{1}{RC}q - \frac{\mathcal{E}}{R} = 0$$



An R-C Circuit (Charging)

- Let's set this up so that we can integrate it:

$$\frac{dq}{dt} = \frac{\mathcal{E}}{R} - \frac{1}{RC}q = \frac{-1}{RC}\left(q - C\mathcal{E}\right) \Rightarrow$$

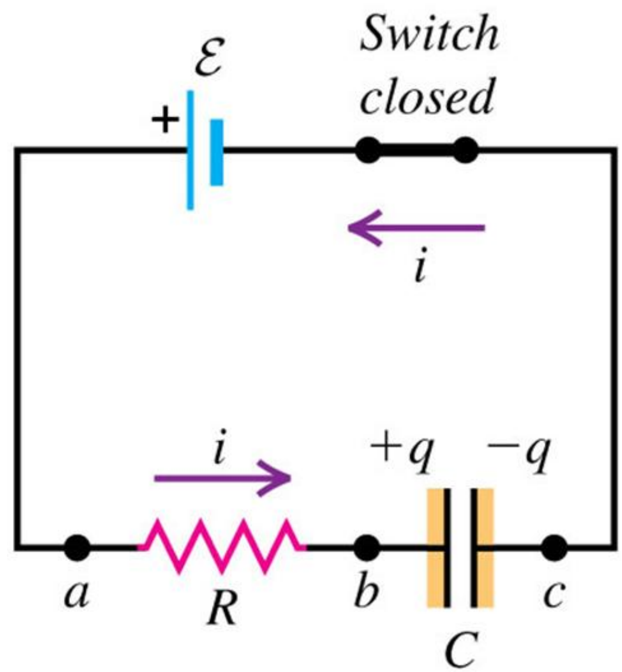
$$\frac{dq}{\left(q - C\mathcal{E}\right)} = \frac{-dt}{RC}$$

- And now, change the variable names so we don't get confused about variables of integration and limits of the integration:

$$\int_0^q \frac{dq'}{\left(q' - C\mathcal{E}\right)} = \frac{-1}{RC}\int_0^t dt' \Rightarrow$$

$$\ln\left(\frac{q - C\mathcal{E}}{-C\mathcal{E}}\right) = \frac{-t}{RC}$$

(b) Charging the capacitor



An R-C Circuit (Charging)

- And now we can take the inverse natural log of each side and solve for q(t):

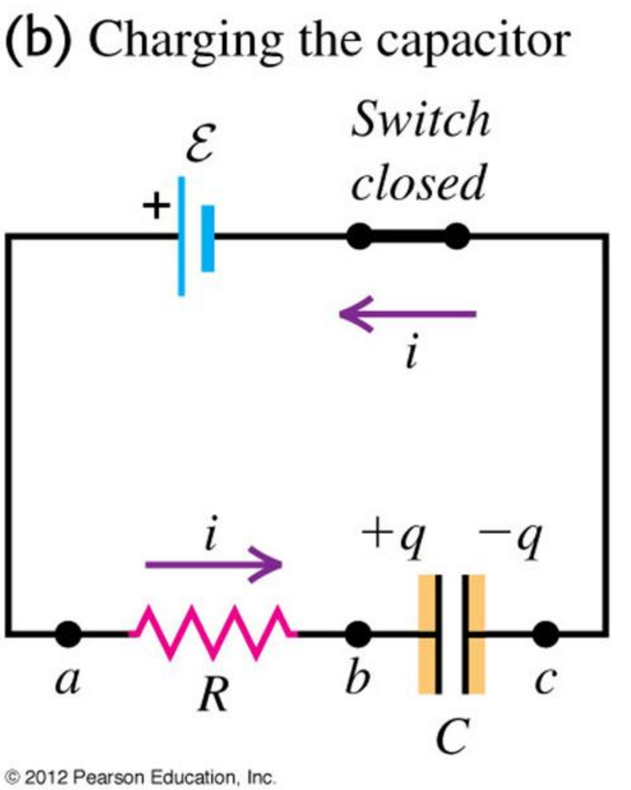
$$\ln\left(\frac{q-C\mathcal{E}}{-C\mathcal{E}}\right)=\frac{-t}{RC}$$

$$\left(\frac{q-C\mathcal{E}}{-C\mathcal{E}}\right)=e^{\frac{-t}{RC}}\Rightarrow$$

$$q(t)=C\mathcal{E}-C\mathcal{E}e^{\frac{-t}{RC}}=C\mathcal{E}\left(1-e^{\frac{-t}{RC}}\right)$$

- Then we can take the derivative to solve for the current:

$$\Rightarrow i(t)=\frac{dq}{dt}=\frac{\mathcal{E}}{R}e^{\frac{-t}{RC}}$$



An R-C Circuit (Charging)

- Let's check our boundary values of q(t) at t=0 and as t goes to infinity:

$$q(t) = C\mathcal{E}\left(1 - e^{\frac{-t}{RC}}\right)$$

$$q(0) = C\mathcal{E}\left(1 - e^{\frac{-0}{RC}}\right) = C\mathcal{E}(1-1) = 0$$

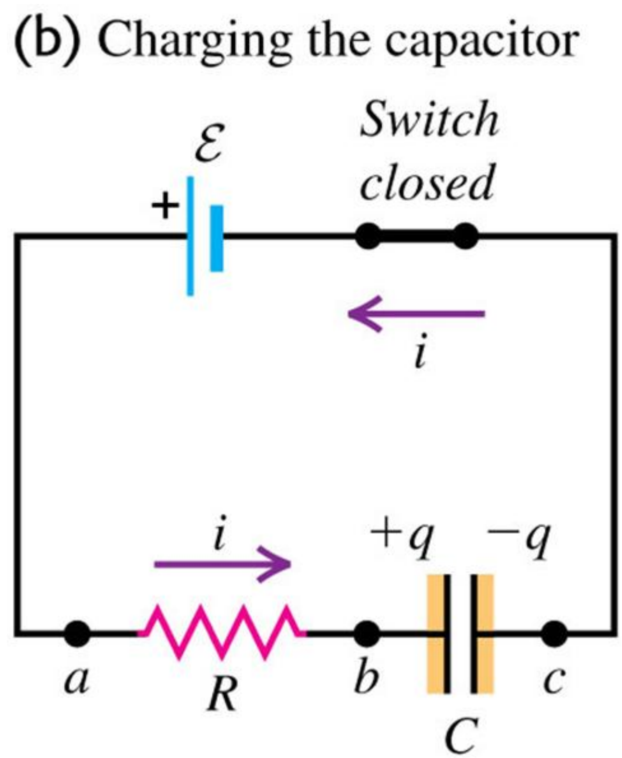
$$q(\infty) = C\mathcal{E}\left(1 - e^{\frac{-\infty}{RC}}\right) = C\mathcal{E}(1-0) = C\mathcal{E}$$

- And do the same for the current:

$$i(t) = \frac{\mathcal{E}}{R} e^{\frac{-t}{RC}} \Rightarrow$$

$$i(0) = \frac{\mathcal{E}}{R}\left(e^{\frac{-0}{RC}}\right) = \frac{\mathcal{E}}{R}(1) = \frac{\mathcal{E}}{R}$$

$$i(\infty) = \frac{\mathcal{E}}{R}\left(e^{\frac{-\infty}{RC}}\right) = \frac{\mathcal{E}}{R}(0) = 0$$



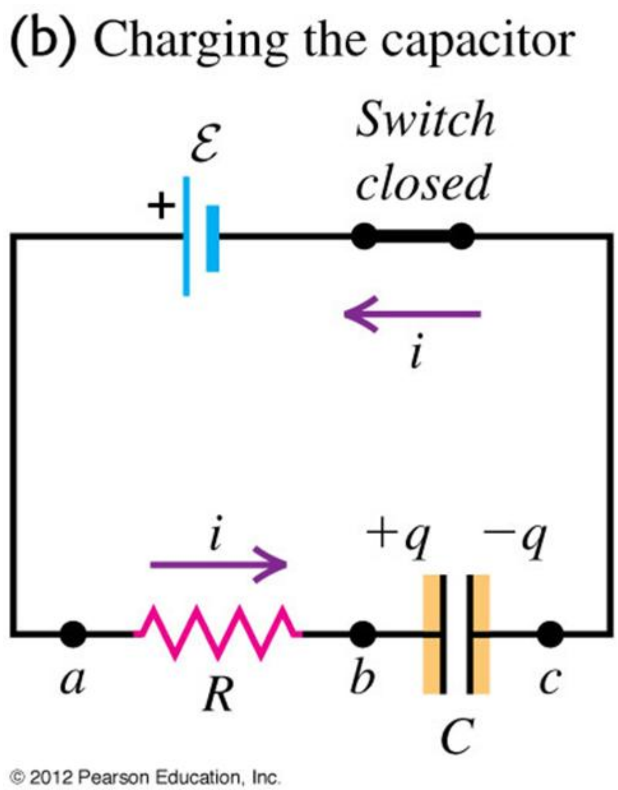
An R-C Circuit (Charging)

- Note how the capacitor acts at these extremes of times.
- At t=0, immediately after the switch is closed, current flows like the capacitor is not there.
- At very long times after the switch is closed, the capacitor acts like a broken circuit, so no current passes through that part of the circuit.

$$i(t) = \frac{\mathcal{E}}{R} e^{\frac{-t}{RC}} \Rightarrow$$

$$i(0) = \frac{\mathcal{E}}{R}\left(e^{\frac{-0}{RC}}\right) = \frac{\mathcal{E}}{R}(1) = \frac{\mathcal{E}}{R}$$

$$i(\infty) = \frac{\mathcal{E}}{R}\left(e^{\frac{-\infty}{RC}}\right) = \frac{\mathcal{E}}{R}(0) = 0$$



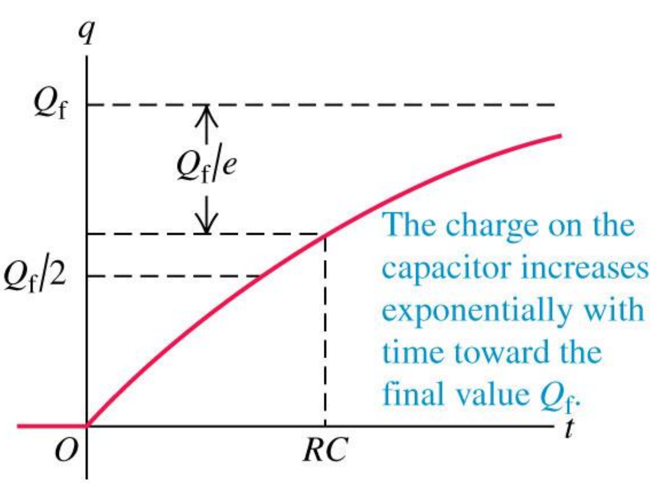
An R-C Circuit (Charging)

- Just to makes sure that we understand the behavior of the charge and the current for all the times in between, let's plot them:

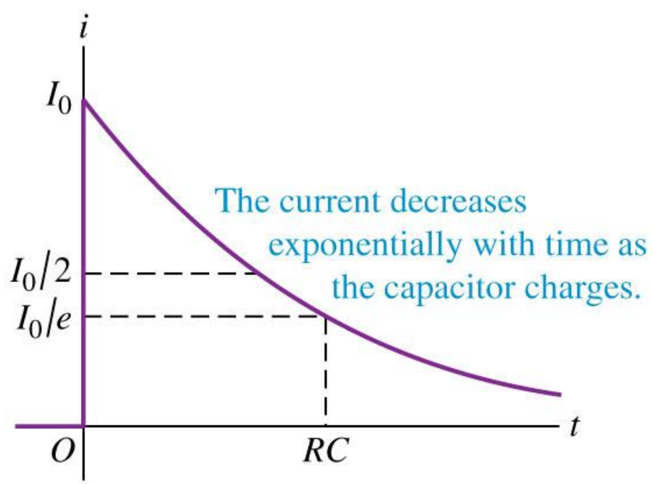
$$q(t) = C\mathcal{E}\left(1 - e^{\frac{-t}{RC}}\right)$$

$$i(t) = \frac{\mathcal{E}}{R} e^{\frac{-t}{RC}}$$

(b) Graph of capacitor charge versus time for a charging capacitor



(a) Graph of current versus time for a charging capacitor



Time Constant

- The factor RC in the denominator of the exponential is commonly called the RC time constant, and determines the rate at which the capacitor charges (and discharges, as we will soon see).

$$\tau = RC$$

$$q(t) = C\mathcal{E}\left(1 - e^{\frac{-t}{RC}}\right) = C\mathcal{E}\left(1 - e^{\frac{-t}{\tau}}\right)$$

$$i(t) = \frac{\mathcal{E}}{R} e^{\frac{-t}{RC}} = \frac{\mathcal{E}}{R} e^{\frac{-t}{\tau}}$$

CPS 26-1

A battery, a capacitor, and a resistor are connected in series. Which of the following affect(s) the maximum charge stored on the capacitor?

A. the emf ε of the battery

B. the capacitance C of the capacitor

C. the resistance R of the resistor

D. both ε and C

E. all three of ε, C, and R

CPS 26-1

A battery, a capacitor, and a resistor are connected in series. Which of the following affect(s) the maximum charge stored on the capacitor?

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CPS 26-2

A battery, a capacitor, and a resistor are connected in series. Which of the following affect(s) the rate at which the capacitor charges?

A. the emf ε of the battery

B. the capacitance C of the capacitor

C. the resistance R of the resistor

D. both C and R

E. all three of ε, C, and R

CPS 26-2

A battery, a capacitor, and a resistor are connected in series. Which of the following affect(s) the rate at which the capacitor charges?

A. the emf ε of the battery

B. the capacitance C of the capacitor

C. the resistance R of the resistor

D. both C and R

E. all three of ε, C, and R

Example 26.12 Charging a capacitor

A 10-MΩ resistor is connected in series with a 1.0-μF capacitor and a battery with emf 12.0 V. Before the switch is closed at time $t = 0$, the capacitor is uncharged. (a) What is the time constant? (b) What fraction of the final charge Q_f is on the capacitor at $t = 46$ s? (c) What fraction of the initial current I_0 is still flowing at $t = 46$ s?

SOLUTION

IDENTIFY and SET UP: This is the same situation as shown in Fig. 26.20, with $R = 10$ MΩ, $C = 1.0$ μF, and $\mathcal{E} = 12.0$ V. The charge q and current i vary with time as shown in Fig. 26.21. Our target variables are (a) the time constant τ, (b) the ratio q/Q_f at $t = 46$ s, and (c) the ratio i/I_0 at $t = 46$ s. Equation (26.14) gives τ. For a capacitor being charged, Eq. (26.12) gives q and Eq. (26.13) gives i.

EXECUTE: (a) From Eq. (26.14),

$$\tau = RC = (10 \times 10^6\ \Omega)(1.0 \times 10^{-6}\ \text{F}) = 10\ \text{s}$$

(b) From Eq. (26.12),

$$\frac{q}{Q_f} = 1 - e^{-t/RC} = 1 - e^{-(46\ \text{s})/(10\ \text{s})} = 0.99$$

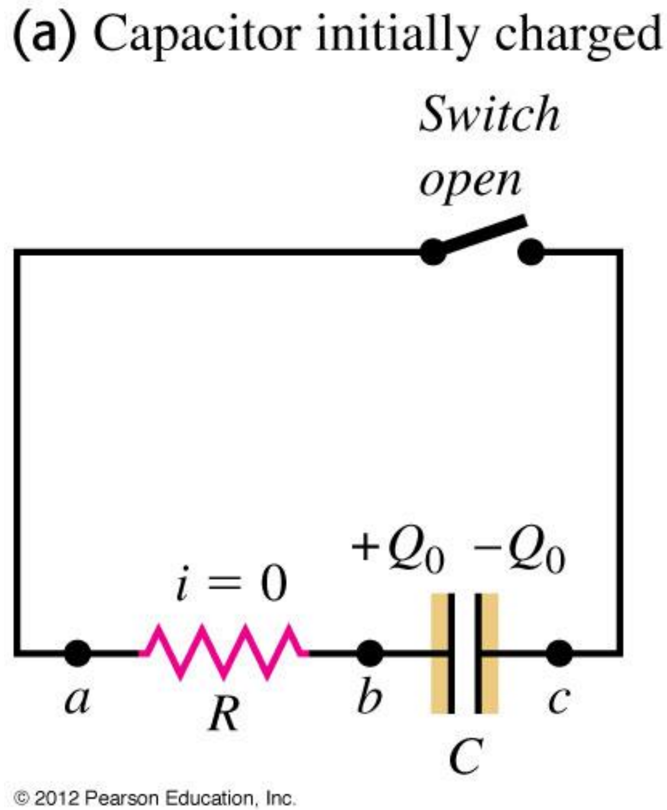
(c) From Eq. (26.13),

$$\frac{i}{I_0} = e^{-t/RC} = e^{-(46\ \text{s})/(10\ \text{s})} = 0.010$$

EVALUATE: After 4.6 time constants the capacitor is 99% charged and the charging current has decreased to 1.0% of its initial value. The circuit would charge more rapidly if we reduced the time constant by using a smaller resistance.

An R-C Circuit (Discharging)

- Now, let's start with a fully charged capacitor in an open circuit with no battery.
- Initially, there is charge Q_0 on the capacitor.
- At some time, t=0, we will close the switch, and charge will begin to flow around the circuit, through the resistor and back to the other side of the capacitor.



An R-C Circuit (Discharging)

- Let's again apply the loop rule to the situation:

$$\sum \Delta V = i(t)R - \frac{q(t)}{C} = 0$$

- But, we have to be careful about the signs here, since with a positive current, the charge decreases, so:

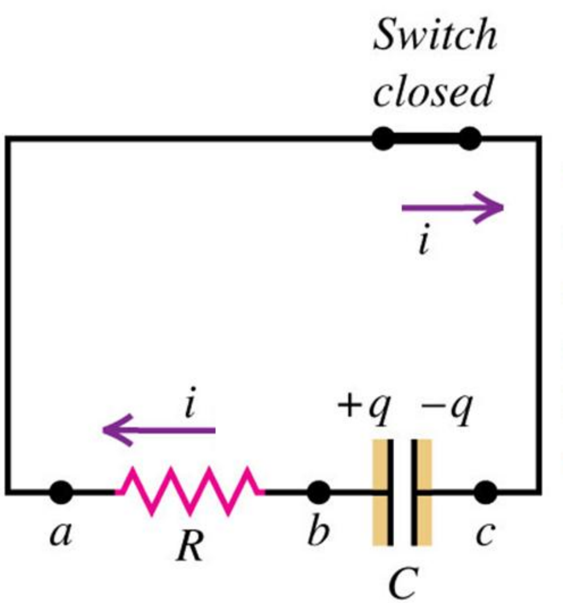
$$i(t)R = -\frac{q(t)}{C}$$

- Now, we can do similar things as we did with the charging equation:

$$\frac{dq}{dt} = \frac{-1}{RC} q(t) \Rightarrow$$

$$\frac{dq}{q(t)} = \frac{-1}{RC} dt$$

(b) Discharging the capacitor



An R-C Circuit (Discharging)

- And, rename the variables…

$$\frac{dq}{q(t)} = \frac{-1}{RC}dt \Rightarrow$$

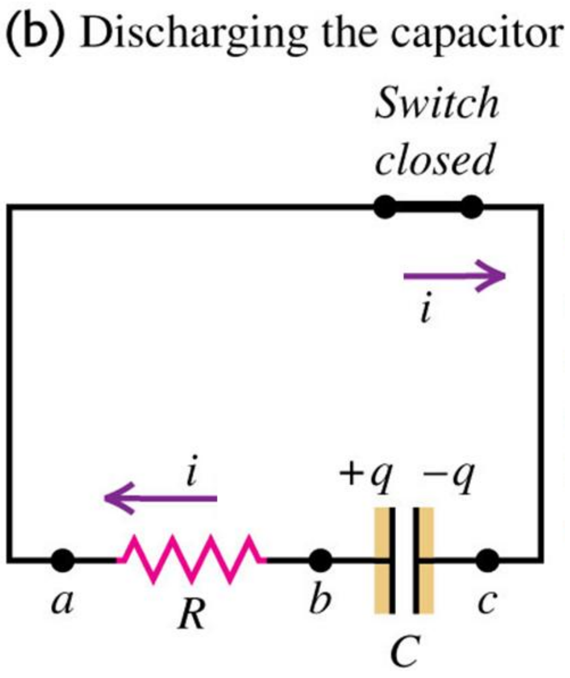
$$\frac{dq'}{q'(t)} = \frac{-1}{RC}dt'$$

- And then integrate, paying attention to the limits of integration (we start the charge at Q_0, and take it to some value q:

$$\int_{Q_0}^{q} \frac{dq'}{q'(t)} = \frac{-1}{RC}\int_{o}^{t} dt' \Rightarrow$$

$$\ln\left(\frac{q(t)}{Q_0}\right) = \frac{-t}{RC} \Rightarrow$$

$$q(t) = Q_0 e^{\frac{-t}{RC}}$$

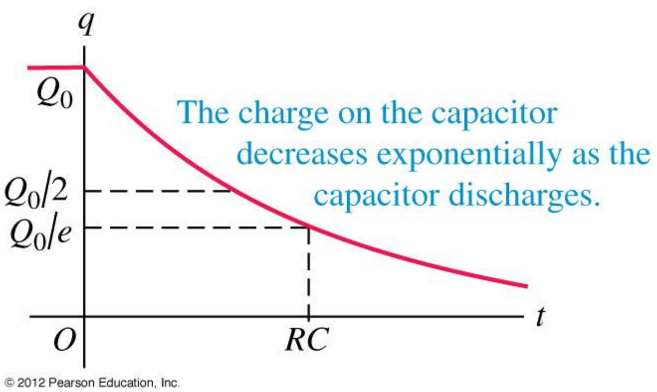


An R-C Circuit (Discharging)

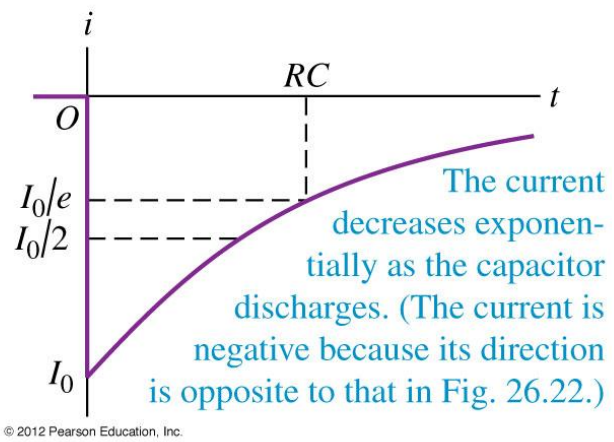
- To find the time dependence of the current, we just have to take the time derivative of this solution:

$$q(t) = Q_0 e^{\frac{-t}{RC}} \Rightarrow \qquad i(t) = \frac{dq}{dt} = -\frac{Q_0}{RC} e^{\frac{-t}{RC}}$$

(b) Graph of capacitor charge versus time for a discharging capacitor



(a) Graph of current versus time for a discharging capacitor



Example 26.13 Discharging a capacitor

The resistor and capacitor of Example 26.12 are reconnected as shown in Fig. 26.22. The capacitor has an initial charge of 5.0 μC and is discharged by closing the switch at $t = 0$. (a) At what time will the charge be equal to 0.50 μC? (b) What is the current at this time?

SOLUTION

IDENTIFY and SET UP: Now the capacitor is being discharged, so q and i vary with time as in Fig. 26.23, with $Q_0 = 5.0 \times 10^{-6}$ C. Again we have $RC = \tau = 10$ s. Our target variables are (a) the value of t at which $q = 0.50\ \mu$C and (b) the value of i at this time. We first solve Eq. (26.16) for t, and then solve Eq. (26.17) for i.

EXECUTE: (a) Solving Eq. (26.16) for the time t gives

$$t = -RC \ln \frac{q}{Q_0} = -(10\ \text{s}) \ln \frac{0.50\ \mu\text{C}}{5.0\ \mu\text{C}} = 23\ \text{s} = 2.3\tau$$

(b) From Eq. (26.17), with $Q_0 = 5.0\ \mu\text{C} = 5.0 \times 10^{-6}$ C,

$$i = -\frac{Q_0}{RC} e^{-t/RC} = -\frac{5.0 \times 10^{-6}\ \text{C}}{10\ \text{s}} e^{-2.3} = -5.0 \times 10^{-8}\ \text{A}$$

EVALUATE: The current in part (b) is negative because i has the opposite sign when the capacitor is discharging than when it is charging. Note that we could have avoided evaluating $e^{-t/RC}$ by noticing that at the time in question, $q = 0.10Q_0$; from Eq. (26.16) this means that $e^{-t/RC} = 0.10$.