ECE 340: PROBABILISTIC METHODS IN ENGINEERING

SOLUTIONS TO HOMEWORK #5

3.10.
Solution

a) Note that the size of the alphabet of each bit in the password is 2. Then there is a total 2^m number of possible passwords. The sample space S can be expressed as $S = \{B, FB, FFB,\ FFFB, \ldots\ \}$, where we denote an 'F' as fail and an 'B' as 'BINGO'. Note that the samples space contains a total 2^m number of outcomes.

b) The mapping from S to S_X is simply:

S	B	FB	FFB	FFFB	…	FFF…FB
	↓	↓	↓	↓	↓	↓
S_X	1	2	3	4	…	2^m

c) P({X=1}) = $\frac{1}{2^m}$

P({X=2})= P(FB) =P(B at 2nd time| 1st time fail) X P(1st time fail) =$\frac{2^m-1}{2^m}\frac{1}{2^m-1} = \frac{1}{2^m}$

P({X=3}) = P(FFB)

=P(B at 3rd time| 1st time fail and 2nd time fail) X P(1st time fail and 2nd time fail)

=P(B at 3rd time| 1st time fail and 2nd time fail) X P(2nd time fail | 1st time fail) X P(1st time fail) =$\frac{1}{2^m-2}\frac{2^m-2}{2^m-1}\frac{2^m-1}{2^m} = \frac{1}{2^m}$

So, for $1 \le n \le 2^m$

P({X=n}) = $\frac{1}{2^m}$

3.13. Solution

a) By the properties of pmfs, we know that,

$$\sum_{k \in S_X} p_K = 1$$

So,

$$1 = \sum_{k \in S_X} \frac{c}{k^2}$$

$$1 = c \sum_{k \in S_X} \frac{1}{k^2}$$

Recalling the following result from the sum of infinite series: $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, we have:

$$1 = \frac{c\pi^2}{6}$$

So, $$c = \frac{6}{\pi^2}$$

b) P({X>4})= $\sum_{k=5}^{\infty} \frac{6}{\pi^2 k^2}$

or,

P({X>4})=1-P{(X≤ 4)} $= 1 - \sum_{k=1}^{4} \frac{6}{\pi^2 k^2} = 1 - 0.8655 = 0.1345$

c) P({6 ≤ X ≤ 8})=P[X ≤ 8] - P[X ≤ 5] = $\sum_{k=1}^{8} \frac{6}{\pi^2 k^2} - \sum_{k=1}^{5} \frac{6}{\pi^2 k^2}$

P({6 ≤ X ≤ 8})= 0.9286 – 0.8898

P({6 ≤ X ≤ 8})= 0.0388

3.16a i) We want to calculate P({X>2}). However, because {X=2} U {X=51} = Ω, we know that the event of {X>2} is the same as the event of {X=51}. Thus, P({X>2})=0.2. (see the solution to problem 3.7).

The event {X>50} is also the same as {X=51}, so P({X>50})=0.2.

3.17.

a) First, note that S_Y={-1, 0, 1, 2}. Given the probabilities, we obtain the pmf:

p_{-1} = P{(Y = -1)} = $\frac{1}{10}$

p_0 = P{(Y = 0)} = $\frac{2}{10}$

p_1 = P{(Y = 1)} = $\frac{3}{10}$

p_2 = P{(Y = 2)} = $\frac{4}{10}$

b) The output is equal to the input if and only if the noise is zero. Hence, the probability that the output is equal to the input is : p_2 = P{(*Y*=2)} = $\frac{4}{10}$

c) The probability that the output of the channel is positive is:

P{(Y > 0)} = P{(Y=1)} + P{(Y=2)} = $\frac{3}{10}$+ $\frac{4}{10}$ = $\frac{7}{10}$

3.21.

a) Compare E[*Y*] to E[*X*].

First, remember that *X* is the maximum of the number of heads that Michael gets *and* the number of heads that Carlos gets in pair of flips. The pmf for *X* is given below:

P({*X*=0}) = P({(T,T,T,T)}) = 1/16

P({*X*=1}) = P({(T,H,T,H), (H,T,H,T), (T,H,H,T), (H,T,T,H), (T,H,T,T), (H,T,T,T), (T,T,T,H), (T,T,H,T)}) = 8/16

P({*X*=2}) = P({(H,H,T,T), (H,H,T,H), (H,H,H,T), (H,H,H,H), (T,T,H,H), (T,H,H,H), (H,T,H,H)}) = 7/16.

So,

$$E\{X\} = \sum_{k=0}^{2} k * p_X(k) = 0 * \frac{1}{16} + 1 * \frac{8}{16} + 2 * \frac{7}{16} = \frac{22}{16} = \frac{11}{8}$$

Y is the number of heads in two tosses of a fair coin. Its pmf is given by:

P({Y=0}) = P({(T,T)}) = 1/4

P({Y=1}) = P({(H,T), (T,H)}) = 2/4 = 1/2

P({Y=2}) = P({(H,H)}) = 1/4

$$E\{Y\} = \sum_{k=0}^{2} k * p_Y(k) = 0 * \frac{1}{4} + 1 * \frac{1}{2} + 2 * \frac{1}{4} = 1$$

3.28.

$$E\{X\} = \sum_{k=-1}^{2} k * p_X(k) = -1 * \frac{1}{10} + 0 * \frac{2}{10} + 1 * \frac{3}{10} + 2 * \frac{4}{10} = 1$$

$$VAR\{X\} = E\{X^2\} - E\{X\}^2 = \left((-1)^2 * \frac{1}{10} + 0^2 * \frac{2}{10} + 1^2 * \frac{3}{10} + 2^2 * \frac{4}{10}\right) - 1^2 = 1$$

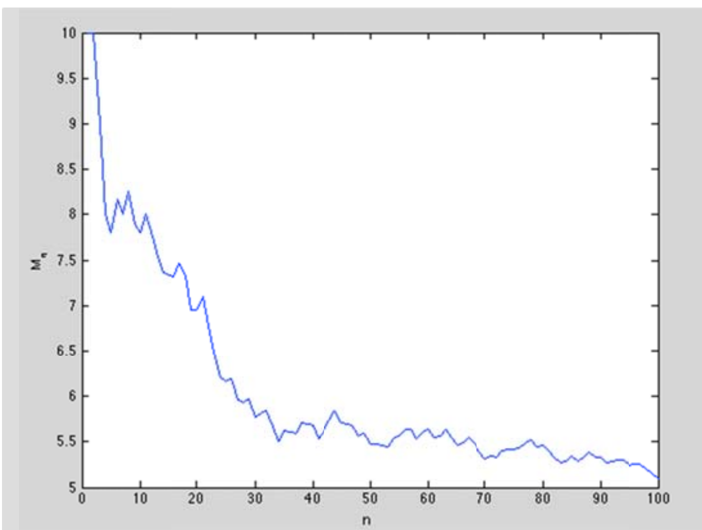
Special Problem

The code to generate X n times and obtain an estimate of the mean of X is provided as follows:

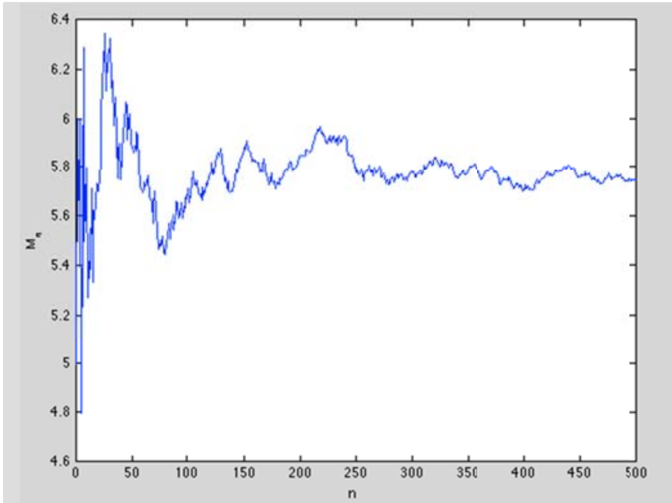
```
clc
clear all
close all

n=input('Give a value for n: ');

for i=1:n
    outcome=ceil(4.*rand(1));                      %generate a random integer
                                                   %between 1 and 4 with 1/4 of
                                                   %probability each

    switch(outcome)                                %assign the value to X according
                                                   %to the outcome of the experiment
        case(1)                                    %(H,H)
            X(i)=10;
        case(2)                                    %(H,T)
            X(i)=5;
        case(3)                                    %(T,H)
            X(i)=7;
        case(4)                                    %(T,T)
            X(i)=0;
    end
    meanX(i) = sum(X)/i;
end

plot(1:n,meanX);
xlabel('n')
ylabel('M_n')
```

==

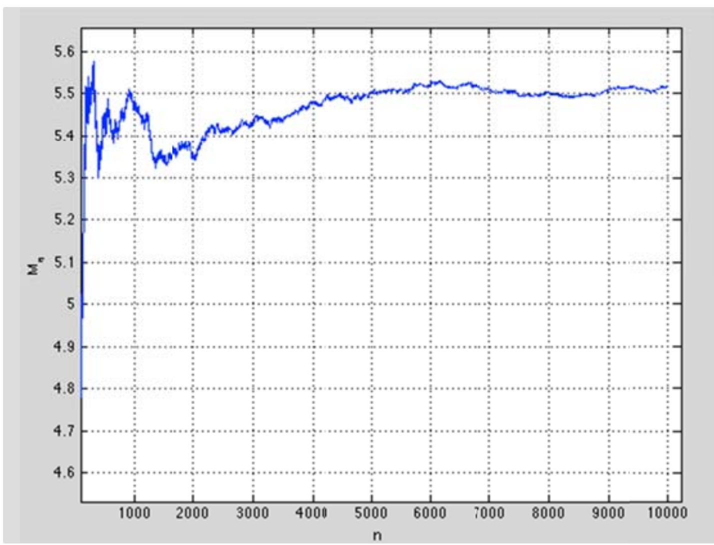
For n = 100,



For n = 5000,



For n = 10000,



It seems that E{*X*} is converging to a value around 5.5.

Calculating the actual mean, we can see that this estimate is correct:

$$E\{X\} = 10 * \frac{1}{4} + 5 * \frac{1}{4} + 7 * \frac{1}{4} + 0 * \frac{1}{4} = 5.5$$

The first observation is that if we use small values of *n*, we cannot obtain a reliable estimate of the expected value of *X*, E[*X*]; however, as we increase the number of repetitions of the experiment, the estimate of the expected value of *X* obtained from the measurements (also called the *sample mean*) starts approaching the true value of E[*X*].

In summary, if *n* is sufficiently large, we can use the sample mean as an estimate of E[*X*]. The larger *n*, the better the estimate of E[*X*] is. Later in the course we shall see that this is a result of the so-called Law of Large Numbers.