#41 Normal Modes Pre-Class

Due: 11:00am on Friday, November 30, 2012

Note: You will receive no credit for late submissions. To learn more, read your instructor's Grading Policy

± Harmonics of a Piano Wire

A piano tuner stretches a steel piano wire with a tension of 765 N. The steel wire has a length of 0.700m and a mass of 5.25g .

Part A

What is the frequency f_1 of the string's fundamental mode of vibration?

Express your answer numerically in hertz using three significant figures.

Hint 1. How to approach the problem

Find the mass per unit length for the wire. Then apply the equation for the fundamental frequency of the wire.

Hint 2. Find the mass per unit length

What is the mass per unit length μ for the wire?

Express your answer numerically in kilograms per meter using three significant figures.

ANSWER:

μ = 7.50×10^{-3} $\mathrm{kg/m}$

Hint 3. Equation for the fundamental frequency of a string under tension

The fundamental frequency of a string under tension is given by

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}},$$

where L is the length of the string, F is the tension in the string, and μ is its mass per unit length.

ANSWER:

f_1 = 228 Hz

Correct

Part B

What is the number n of the highest harmonic that could be heard by a person who is capable of hearing frequencies up to f = 16 kHz?

Express your answer exactly.

Hint 1. Harmonics of a string

The harmonics of a string are given by $f_n = nf_1$, where f_n is the n th harmonic of a string with fundamental frequency f_1. Be careful if you get a noninteger answer for n, as harmonics can only be integer multiples of the fundamental frequency.

ANSWER:

n = 70

Correct

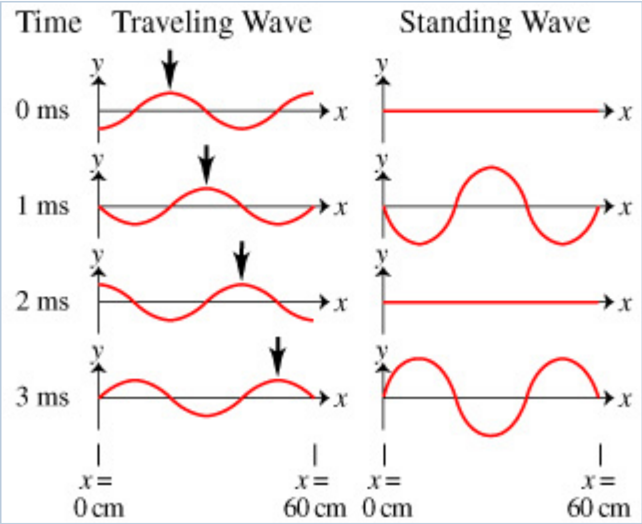
When solving this problem, you may have found a noninteger value for n, but harmonics can only be integer multiples of the fundamental frequency.

± Standing Waves on a Guitar String

Learning Goal:

To understand standing waves, including calculation of λ and f, and to learn the physical meaning behind some musical terms.

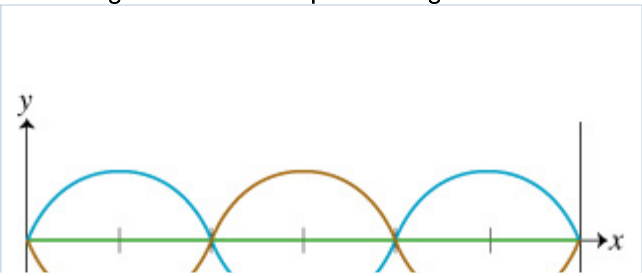
The columns in the figure show the instantaneous shape of a vibrating guitar string drawn every 1 ms. The guitar string is 60 cm long.

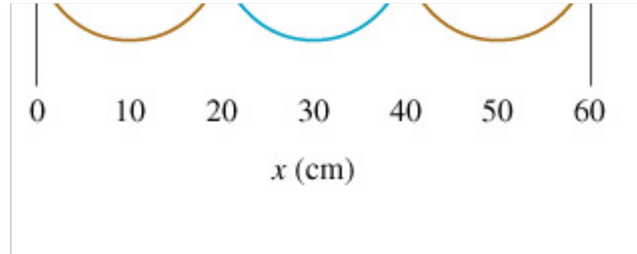


The left column shows the guitar string shape as a sinusoidal *traveling* wave passes through it. Notice that the shape is sinusoidal at all times and specific features, such as the crest indicated with the arrow, travel along the string to the right at a constant speed.

The right column shows snapshots of the sinusoidal *standing* wave formed when this sinusoidal traveling wave passes through an identically shaped wave moving in the opposite direction on the same guitar string. The string is momentarily flat when the underlying traveling waves are exactly out of phase. The shape is sinusoidal with twice the original amplitude when the underlying waves are momentarily in phase. This pattern is called a *standing* wave because no wave features travel down the length of the string.

Standing waves on a guitar string form when waves traveling down the string reflect off a point where the string is tied down or pressed against the fingerboard. The entire series of distortions may be superimposed on a single figure, like this , indicating different moments in time using traces of different colors or line styles.





Part A

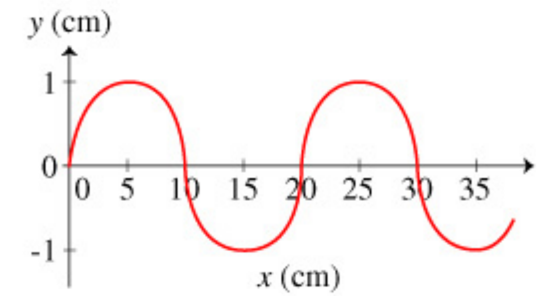
What is the wavelength λ of the standing wave shown on the guitar string?

Express your answer in centimeters.

Hint 1. Identify the wavelength of a sinusoidal shape

The wavelength of a sinusoidal shape is the distance from a given feature to the next instance of that same feature. Wavelengths are usually measured from one peak to the next peak. What is the wavelength λ of this sinusoidal pattern?

Express your answer in centimeters.



ANSWER:

λ = 20 cm

ANSWER:

λ = 40 cm

Correct

Nodes are locations in the standing wave pattern where the string doesn't move at all, and hence the traces on the figure intersect. In between nodes are *antinodes*, where the string moves up and down the most.

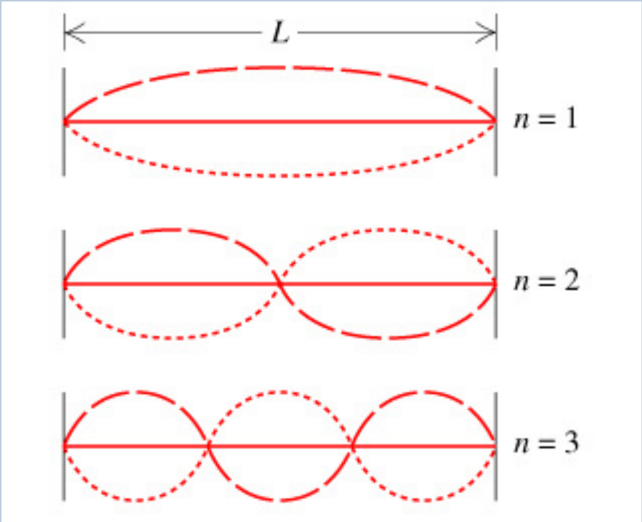
This standing wave pattern has three antinodes, at $x = 10\ \mathrm{cm}$, $30\ \mathrm{cm}$, and $50\ \mathrm{cm}$. The pattern also has four nodes, at $x = 0\ \mathrm{cm}$, $20\ \mathrm{cm}$, $40\ \mathrm{cm}$, and $60\ \mathrm{cm}$. Notice that the spacing between adjacent antinodes is only half of one wavelength, not one full wavelength. The same is true for the spacing between adjacent nodes.

This figure shows the first three standing wave patterns that fit on any string with length L tied down at both ends. A pattern's number n is the number of antinodes it contains. The wavelength of the nth pattern is denoted λ_n. The nth pattern has n half-wavelengths along the length of the string, so

$$n\frac{\lambda_n}{2} = L.$$

Thus the wavelength of the nth pattern is

$$\lambda_n = \frac{2L}{n}.$$



Part B

What is the wavelength of the longest wavelength standing wave pattern that can fit on this guitar string?

Express your answer in centimeters.

Hint 1. How to approach the problem

Look at the figure to determine the pattern number n for the longest wavelength pattern, then calculate its wavelength λ_n. Recall that the guitar string is 60 cm long.

Hint 2. Determine n for the longest wavelength pattern

What is the pattern number n for the longest wavelength standing wave pattern?

ANSWER:

n =
- (selected) 1
- 2
- 3
- some other integer

ANSWER:

λ_1 = 120 cm

Correct

This longest wavelength pattern is so important it is given a special name—the *fundamental*. The wavelength of the fundamental is always given by $\lambda_1 = 2L$ for a string that is held fixed at both ends.

Waves of all wavelengths travel at the same speed v on a given string. Traveling wave velocity and wavelength are related by

$$v = \lambda f,$$

where v is the wave speed (in meters per second), λ is the wavelength (in meters), and f is the frequency [in inverse seconds, also known as hertz (Hz)].

Since only certain wavelengths fit properly to form standing waves on a specific string, only certain frequencies will be represented in that string's standing wave series. The frequency of the nth pattern is

$$f_n = \frac{v}{\lambda_n} = \frac{v}{(2L/n)} = n\frac{v}{2L}.$$

Note that the frequency of the fundamental is $f_1 = v/(2L)$, so f_n can also be thought of as an integer multiple of f_1: $f_n = nf_1$.

Part C

The frequency of the fundamental of the guitar string is 320 Hz. At what speed v do waves move along that string?

Express your answer in meters per second.

Hint 1. How to approach the problem

The velocity of waves on the string equals the product of the wavelength and the corresponding frequency for any standing wave pattern:

$$v = \lambda_1 f_1 = \lambda_2 f_2 = \ldots.$$

You know enough about the fundamental to calculate v.

ANSWER:

v = 384 $\mathrm{m/s}$

Correct

Notice that these transverse waves travel slightly faster than the speed of sound waves in air, which is about 340 $\mathrm{m/s}$.

We are now in a position to understand certain musical terms from a physics perspective.

The standing wave frequencies for this string are $f_1 = 320\ \mathrm{Hz}$, $f_2 = 2f_1 = 640\ \mathrm{Hz}$, $f_3 = 3f_1 = 960\ \mathrm{Hz}$, etc. This set of frequencies is called a *harmonic series* and it contains common musical intervals such as the *octave* (in which the ratio of frequencies of the two notes is 2:1) and the *perfect fifth* (in which the ratio of frequencies of the two notes is 3:2). Here f_2 is one octave above f_1, f_3 is a perfect fifth above f_2, and so on. Standing wave patterns with frequencies higher than the fundamental frequency are called *overtones*. The $n = 2$ pattern is called the first overtone, the $n = 3$ pattern is called the second overtone, and so on.

Part D

How does the overtone number relate to the standing wave pattern number, previously denoted with the variable n?

ANSWER:

- ○ $\text{overtone number} = \text{pattern number}$
- ○ $\text{overtone number} = \text{pattern number} + 1$
- ◉ $\text{overtone number} = \text{pattern number} - 1$
- ○ There is no strict relationship between overtone number and pattern number.

Correct

The overtone number and the pattern number are easy to confuse but they differ by one. When referring to a standing wave pattern using a number, be explicit about which numbering scheme you are using.

When you pluck a guitar string, you actually excite many of its possible standing waves simultaneously. Typically, the fundamental is the loudest, so that is the pitch you hear. However, the unique mix of the fundamental plus overtones is what makes a guitar sound different from a violin or a flute, even if they are playing the same note (i.e., producing the same fundamental). This characteristic of a sound is called its *timbre* (rhymes with *amber*).

A sound containing just a single frequency is called a *pure tone*. A *complex tone*, in contrast, contains multiple frequencies such as a fundamental plus some of its overtones. Interestingly enough, it is possible to fool someone into identifying a frequency that is not present by playing just its overtones. For example, consider a sound containing pure tones at 450 Hz, 600 Hz, and 750 Hz. Here 600 Hz and 750 Hz are not integer multiples of 450 Hz, so 450 Hz would not be considered the fundamental with the other two as overtones. However, because all three frequencies are consecutive overtones of 150 Hz a listener might claim to hear 150 Hz, over an octave below any of the frequencies present. This 150 Hz is called a *virtual pitch* or a *missing fundamental*.

Part E

A certain sound contains the following frequencies: 400 Hz, 1600 Hz, and 2400 Hz. Select the best description of this sound.

Hint 1. How to identify a fundamental within a series of frequencies

All frequencies in a harmonic series are integer multiples of the fundamental frequency. The lowest frequency listed will be the fundamental, but only if all the other frequencies are integer multiples of it.

ANSWER:

- This is a pure tone.
- This is a complex tone with a fundamental of 400 Hz, plus some of its overtones.
- This is a complex tone with a virtual pitch of 800 Hz.
- These frequencies are unrelated, so they are probably pure tones from three different sound sources.

Correct

These concepts of fundamentals and overtones can be applied to other types of musical instruments besides string instruments. Hollow-tube instruments, such as brass instruments and reed instruments, have standing wave patterns in the air within them. Percussion instruments, such as bells and cymbals, often exhibit standing wave vibrations in the solid material of their bodies. Even the human voice can be analyzed this way, with the fundamental setting the pitch of the voice and the presence or absence of overtones setting the unique vowel or consonant being sounded.

Normal Modes and Resonance Frequencies

Learning Goal:

To understand the concept of normal modes of oscillation and to derive some properties of normal modes of waves on a string.

A *normal mode* of a closed system is an oscillation of the system in which all parts oscillate at a single frequency. In general there are an infinite number of such modes, each one with a distinctive frequency f_i and associated pattern of oscillation.

Consider an example of a system with normal modes: a string of length L held fixed at both ends, located at $x = 0$ and $x = L$. Assume that waves on this string propagate with speed v. The string extends in the *x* direction, and the waves are transverse with displacement along the *y* direction.

In this problem, you will investigate the shape of the normal modes and then their frequency.

The normal modes of this system are products of trigonometric functions. (For linear systems, the time dependance of a normal mode is always sinusoidal, but the spatial dependence need not be.) Specifically, for this system a normal mode is described by

$$y_i(x,t) = A_i \sin\left(2\pi \frac{x}{\lambda_i}\right) \sin\left(2\pi f_i t\right).$$

Part A

The string described in the problem introduction is oscillating in one of its normal modes. Which of the following statements about the wave in the string is correct?

Hint 1. Normal mode constraints

The key constraint with normal modes is that there are two spatial boundary conditions, $y_i(0,t) = 0$ and $y_i(L,t) = 0$, which correspond to the string being fixed at its two ends.

ANSWER:

- ○ The wave is traveling in the +*x* direction.
- ○ The wave is traveling in the -*x* direction.
- ○ The wave will satisfy the given boundary conditions for any arbitrary wavelength λ_i.
- ◉ The wavelength λ_i can have only certain specific values if the boundary conditions are to be satisfied.
- ○ The wave does not satisfy the boundary condition $y_i(0;t) = 0$.

Correct

Part B

Which of the following statements are true?

ANSWER:

- ◉ The system can resonate at only certain resonance frequencies f_i and the wavelength λ_i must be such that $y_i(0;t) = y_i(L;t) = 0$.
- ○ A_i must be chosen so that the wave fits exactly on the string.
- ○ Any one of A_i or λ_i or f_i can be chosen to make the solution a normal mode.

Correct

The key factor producing the normal modes is that there are two spatial boundary conditions, $y_i(0,t) = 0$ and $y_i(L,t) = 0$, that are satisfied only for particular values of λ_i.

Part C

Find the three longest wavelengths (call them λ_1, λ_2, and λ_3) that "fit" on the string, that is, those that satisfy the boundary conditions at $x = 0$ and $x = L$. These longest wavelengths have the lowest frequencies.

Express the three wavelengths in terms of L. List them in *decreasing* order of length, separated by commas.

Hint 1. How to approach the problem

The nodes of the wave occur where

$$\sin\left(2\pi\frac{x}{\lambda_i}\right) = 0.$$

This equation is trivially satisfied at one end of the string (with $x = 0$), since $\sin(0) = 0$.

The three largest wavelengths that satisfy this equation at the other end of the string (with $x = L$) are given by 2\pi L/\lambda_i=z_i, where the z_i are the three *smallest, nonzero* values of z that satisfy the equation \sin(z)=0.

Hint 2. Values of z that satisfy \sin(z)=0

The spatial part of the normal mode solution is a sine wave. Find the three smallest (nonzero) values of z (call them z_1, z_2, and z_3) that satisfy \sin(z) = 0.

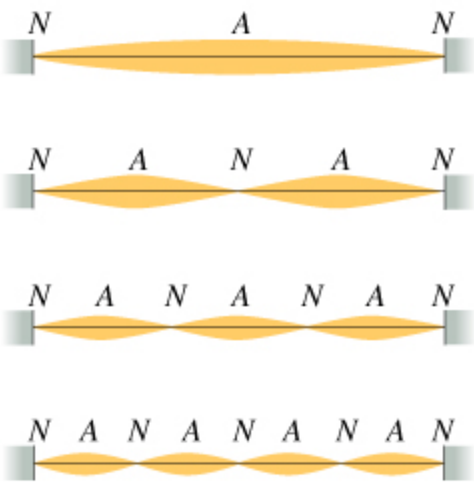
Express the three nonzero values of z as multiples of π. List them in increasing order, separated by commas.

ANSWER:

z_1, z_2, z_3 = 1,2,3

Hint 3. Picture of the normal modes

Consider the lowest four modes of the string as shown in the figure.



The letter N is written over each of the *nodes* defined as places where the string does not move. (Note that there are nodes in addition to those at the end of the string.) The letter A is written over the *antinodes*, which are where the oscillation amplitude is maximum.

ANSWER:

λ_1, λ_2, λ_3 = $2L, L, \frac{2}{3}L$

Correct

The procedure described here contains the same mathematics that leads to *quantization* in quantum mechanics.

Part D

The frequency of each normal mode depends on the spatial part of the wave function, which is characterized by its wavelength λ_i.

Find the frequency f_i of the *i*th normal mode.

Express f_i in terms of its particular wavelength λ_i and the speed of propagation of the wave v.

Hint 1. Propagation speed for standing waves

Your expression will involve v, the speed of propagation of a wave on the string. Of course, the normal modes are standing waves and do not travel along the string the way that traveling waves do. Nevertheless, the speed of wave propagation is a physical property that has a well-defined value that happens to appear in the relationship between frequency and wavelength of normal modes.

Hint 2. Use what you know about traveling waves

The relationship between the wavelength and the frequency for standing waves is the same as that for traveling waves and involves the speed of propagation v.

ANSWER:

f_i = $\frac{v}{\lambda_i}$

Correct

The frequencies f_i are the *only* frequencies at which the system can oscillate. If the string is excited at one of these *resonance frequencies* it will respond by oscillating in the pattern given by $y_i(x,t)$, that is, with wavelength λ_i associated with the f_i at which it is excited. In quantum mechanics these frequencies are called the *eigenfrequencies*, which are equal to the energy of that mode divided by Planck's constant h. In SI units, Planck's constant has the value $h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s}$.

Part E

Find the three lowest normal mode frequencies f_1, f_2, and f_3.

Express the frequencies in terms of L, v, and any constants. List them in *increasing* order, separated by commas.

ANSWER:

f_1, f_2, f_3 = $\frac{v}{2L}, 2\frac{v}{2L}, 3\frac{v}{2L}$

Correct

Note that, for the string, these frequencies are multiples of the lowest frequency. For this reason the lowest frequency is called the *fundamental* and the higher frequencies are called *harmonics* of the fundamental. When other physical approximations (for example, the stiffness of the string) are not valid, the normal mode frequencies are not exactly harmonic, and they are called *partials*. In an acoustic piano, the highest audible normal frequencies for a given string can be a significant fraction of a semitone sharper than a simple integer multiple of the fundamental. Consequently, the fundamental frequencies of the lower notes are deliberately tuned a bit flat so that their higher partials are closer in frequency to the higher notes.

Score Summary:

Your score on this assignment is 101.3%.
You received 15.19 out of a possible total of 15 points.