ECE 340: PROBABILISTIC METHODS IN ENGINEERING

SOLUTIONS TO HOMEWORK #4

2.54

a) The number of ways of choosing (without replacement nor caring about order) *M* out of 100 is $\binom{100}{M}$. This is the total number of equiprobable outcomes in the sample space.

We are interested in the outcomes in which *m* of the chosen items are defective and *M*-*m* are non defective.

The number of ways of choosing *m* defectives out of a total of *k* defective items is $\binom{k}{m}$.

The number of ways of choosing *M*-*m* non-defectives out of a total of 100-*k* non-defective items is is $\binom{100-k}{M-m}$.

Hence, the number of ways of choosing m defectives out of *k* and *M*-*m* non-defectives out of 100-*k* is

$$\binom{k}{m}\binom{100-k}{M-m}$$

P{*m* defectives in *M* samples} = (# outcomes with *k* defective) / (total # of outcomes)

$$P\{m \text{ defectives in } M \text{ samples}\} = \frac{\binom{k}{m}\binom{100-k}{M-m}}{\binom{100}{M}}$$

b) Apply the result from part (a) to *m*=0 and *m*=1, sum up, and obtain:

$$P\{\text{lot accepted}\} = P\{m=0 \text{ or } m=1\} = \frac{\binom{100-k}{M}}{\binom{100}{M}} + \frac{k\binom{100-k}{M-1}}{\binom{100}{M}}$$

2.62 A={$N_1 \geq N_2$} , B={$N_1 = 6$}

First note that B $\subset$ A. Next,

$$P(A\,|\,B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

and

$$P(B\,|\,A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)}{P(A)} = (6/36)/(21/36) = 2/7.$$

2.69 First, notice that the Ω=[-1,2], and the event B can be rewritten as the open interval B=(0 < x < 1). It is clear that A and B are disjoint and P(B)=length(B)/3>0, P(C)=length(C)/3 >0 and P(C^c)=length([-1,0.75]) >0; therefore,

$$P(A\,|\,B) = \frac{P(A \cap B)}{P(B)} = 0$$

$$P(B\,|\,C) = \frac{P(B \cap C)}{P(C)} = \frac{length([0.75 < x < 1))/3}{length((x > 0.75))/3} = \frac{0.25/3}{1.25/3} = 1/5$$

$$P(A\,|\,C^C) = \frac{P(A \cap C^C)}{P(C^C)} = \frac{length([-1,0))/3}{length([-1,0.75])/3} = \frac{1/3}{1.75/3} = 4/7$$

$$P(B\,|\,C^C) = \frac{P(B \cap C^C)}{P(C^C)} = \frac{length(0,0.75])/3}{P([-1,0.75])/3} = \frac{0.75/3}{1.75/3} = 3/7$$

2.76

a) We use conditional probability to solve this problem. For i=1,2, let A_i be the event that a non-defective chip is found in ith test. A lot is accepted if the items in tests 1 and 2 are both non-defective, that is, if the event $A_1 \cap A_2$ occurs. Therefore,

P(event lot accepted) = P($A_2 \cap A_1$)
= P($A_2|A_1$) P(A_1).

This equation simply states that we must have A_1 occurring, and then A_2 occurring given that A_1 has already occurred. If the lot of 100 items contains k defective items then

$P(A_1) = \frac{100-k}{100}$ and

$P(A_2\,|\,A_1) = \frac{99-k}{99}$ since 99-k of the 99 items are non-defective.

Thus,

P(lot accepted) $= \frac{99-k}{99} \times \frac{100-k}{100}$

b) P(1 or more items in *m* tested chips are defective)
=1- P(no items in *m* are defective)
= 1- P($A_m \cap A_{m-1} \cap \ldots \cap A_1$) $=1-\frac{50}{100}\times\frac{49}{99}\times\ldots\ldots\times\frac{50-m+1}{100-m+1}$ (why?).

Hence, P(1 or more items in *m* tested chips are defective) > 0.99 is equivalent to P($A_m \cap A_{m-1} \cap \ldots \cap A_1$) < 0.01.

Now our problem is to find the smallest *m* for which P($A_m \cap A_{m-1} \cap \ldots \cap A_1$) < 0.01. Therefore, we set

P($A_m \cap A_{m-1} \cap \ldots \cap A_1$) = $\frac{50}{100}\times\frac{49}{99}\times\ldots\ldots\times\frac{50-m+1}{100-m+1}=0.01$; the smallest *m* for which the left hand side is less than 0.01 is the desired *m*. It turns out that

when *m*=6, P($A_m \cap A_{m-1} \cap \ldots \cap A_1$) =0.133, and

when *m*=7, P($A_m \cap A_{m-1} \cap \ldots \cap A_1$) =0.00624.

Hence, the desired *m* is 7.

2.97

a) P(0 or 1 errors) = $(1-p)^{100} + 100(1-p)^{99}p$
= 0.3660 + 0.3697
= 0.7357

b) p_R=P(retransmission required) = 1 – P(0 or 1 errors) = 0.2642

P(M retransmissions in total before block is accepted) = $(1-p_R)p_R^M$.

2.98

a) Let *k* be the number of defective items in a batch of *n* tested items, then *k* is a binomial random variable with parameters *n* and *p.* Using Corollary 1, we have

P(k>1) = 1 – P(k ≤ 1) = 1- [P(k=0) + P(k=1)]
= 1- $(1-p)^n - n(1-p)^{n-1}p$

b) Let us define the following events:
G = {production line functions well}
G^C = {production line malfunctions}
D_n = {at least one item in the batch with size n is defective}

We want to find the smallest *n* for which $P(G^C \mid D_n) \geq .99$.

If we assume that $P(G^C) = 0.05$ and $P(G) = 0.95$, then we have

$$P(G^C \mid D_n) = \frac{P(D_n \mid G^C)P(G^C)}{P(D_n \mid G^C)P(G^C) + P(D_n \mid G)P(G)}$$

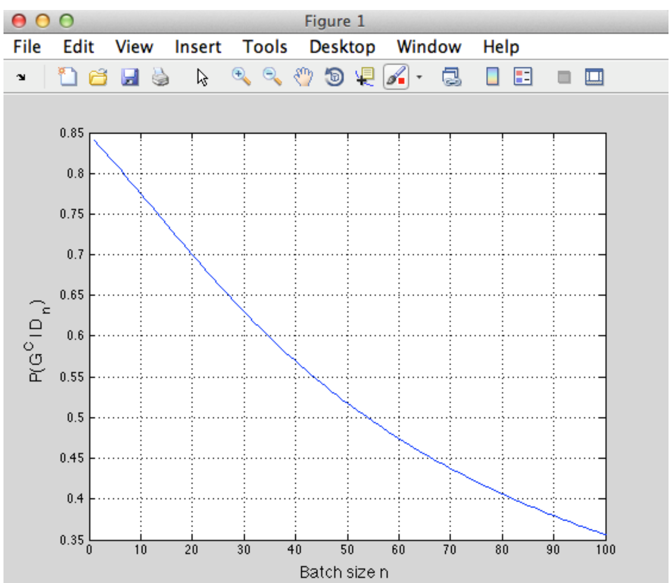
$$= \frac{[1-(1-10^{-1})^n]0.05}{[1-(1-10^{-1})^n]0.05 + [1-(1-10^{-3})^n]0.95}$$

To plot the probability of $P(G^C \mid D_n)$ as a function of n in Matlab, we can use the following code:

```
clc
clear all
close all

p1 = 0.001;
p2 = 0.1;
for n = 1:100
p_sure(n) = (1-(1-p2)^n)*0.05/((1-(1-p2)^n)*0.05+(1-(1-p1)^n)*0.95);
end
plot(1:100,p_sure)
grid on
xlabel('Batch size n','fontsize',14)
ylabel('P(G^C | D_n)','fontsize',14)
```

To get this plot:



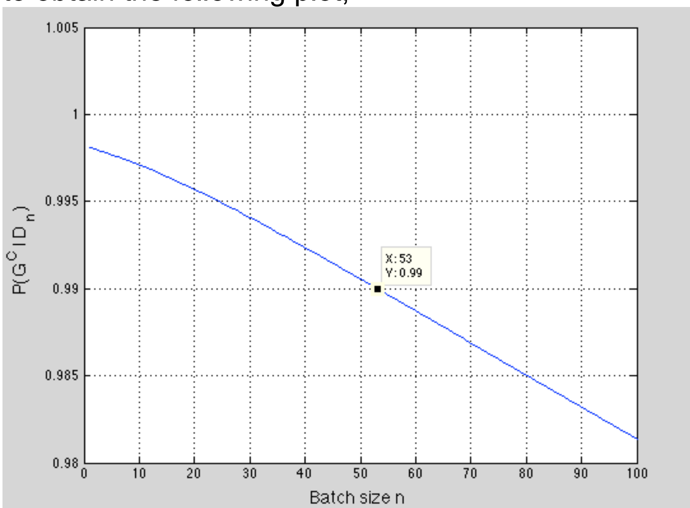
Note that this probability does not exceed 0.85. Then we conclude that there is no way that we can be 99% sure about the production line being malfunctional. However, if we change the values: p1 = 0.00001 and p2 = 0.1, and we may use the following code

```
clc
clear all
close all

p1 = 0.00001;
p2 = 0.1;
```

```
for n = 1:100
p_sure(n) = (1-(1-p2)^n)*0.05/((1-(1-p2)^n)*0.05+(1-(1-p1)^n)*0.95);
end
plot(1:100,p_sure)
grid on
xlabel('Batch size n','fontsize',14)
ylabel('P(G^C | D_n)','fontsize',14)
```

to obtain the following plot,



In this case, we conclude that when n=53, and if we find any defective item, then we are 99% sure that there is a production malfunction. And yes, when n=10, and if we find any defective item, we are about 99.7% sure that there is a production malfunction.

2.99

p = P(success) = 95/100.
Pick *n* so that P(k≥ 8) ≥ 0.9

P(k≥ 8) = $\sum_{k=8}^{n}\binom{n}{k}p^k(1-p)^{n-k}$

For n=8 P(k≥ 8) = 0.66
For n=9 P(k ≥8) = 0.93,
so the student needs to buy 9 chips.

3.6.

a) The mapping from *S* to S_X is given by:

S	000	111	010	101	001	110	100	011
	↓	↓	↓	↓	↓	↓	↓	↓
***Range*(*X*)**	2	2	3	3	4	4	4	4

b) The probabilities of the various values of X are:

P({X=2})=P({(000), (111)})= 1/4+ 1/4 = 1/2

P({X=3})=P({(010), (101)})= 1/8 + 1/8 = 1/4

P({X=4})=P({(001), (110), (100), (011)})= 1/16+1/16+1/16+1/16=1/4

3.7

a) Let us define the sample space Ω as the collection of all possibilities of pairs of dollar bills drawn with ordering. Namely, Ω = {(1,1), (1,50), (50,1)}. Note that we cannot have (50,50) as an outcome since there is only one fifty-dollar bill without replacement.

P({(1,1)}) = $\dfrac{\binom{9}{2}}{\binom{10}{2}} = 0.8$. The term on the denominator is the number of ways we can pick two objects from 10 objects without replacement without ordering. The term on the numerator is the number of ways we can pick two objects from 9 objects (picking two 1-dollar bills from a total of nine 1-dollar bills). Another way to see this is to think of the probability of drawing a 1-dollar bill in the first round, which is 9/10. The probability of drawing another 1-dollar bill from the remaining in the second round is 8/9, since there is no replacement, i.e. only 8 1-dollar bills left. So the resulting probability is then (9/10)X(8/9) = 0.8.

P({(1,50)} = P({(50,1)}) = (9/10)(1/9) =(1/10)(9/9)= 0.1. Note that we have used an approach similar to the second approach used to obtain P({(1,1)}).

Alternatively, P({(1,50)}= P({(50,1)} = $\dfrac{\binom{9}{1}\binom{1}{1}}{\binom{10}{2}\times 2!} = 1/10$.

Note that in the denominator, we have to have the term 2!, since we do care about the order when counting the number of possible outcomes.

b) The random variable X is defined as follows X: $\Omega \rightarrow \mathrm{R}$ according to the rule $X(\omega) = \omega_1 + \omega_2$. Clearly, the range of X is $R(X) = \{2,51\}$.

c) P({X=2}) = P({(1,1)}) =0.8.

P({X=51}) = P({(1,50),(50,1)}) = P({(1,50)} + P{(50,1)}) = 0.1 + 0.1 = 0.2.
Note that for $x\neq 2$ or 51, P({X=x})=0.

3.8. Solution

a) Let us define the sample space Ω as the collection of all possibilities of pairs of dollar bill drawn, in the order in which they were drawn. Namely, Ω = {(1,1), (1,50), (50,1), (50,50)}. Note that in this case we can have (50,50) as an outcome because we are drawing bills from the urn with replacement. The probabilities of the elementary events can be easily calculated by noting that the probabilities of drawing a $1 bill is always $\frac{9}{10}$ and a $50 bill is $\frac{1}{10}$, (because we have replacement). So,

$P(\{(1,1)\}) = \frac{9}{10} \times \frac{9}{10} = \frac{81}{100} = 0.81$

$P(\{(1,50)\}) = P(\{(50,1)\}) = \frac{9}{10} \times \frac{1}{10} = \frac{9}{100} = 0.09$

$P(\{(50,50)\}) = \frac{1}{10} \times \frac{1}{10} = \frac{1}{100} = 0.01$

b) The mapping from S to S_X is given by:

S	(1,1)	(1,50)	(50,1)	(50,50)
	↓	↓	↓	↓
***R*(*X*)**	2	51	51	100

And the range of X is $R(X) = \{2,51,100\}$.

c) The probabilities of X assuming the various values are:

$P(\{X=2\})=P(\{(1,1)\})= 0.81$

$P(\{X=51\})=P(\{(1,50), (50,1)\})= 0.09 + 0.09 = 0.18$

$P(\{X=100\})=P(\{(50,50)\})= 0.01$

Note that $P(\{X=2\}) + P(\{X=51\}) + P(\{X=100\}) = 1$, which is expected. (Why?)