# F O U R

# Time Response

## SOLUTIONS TO CASE STUDIES CHALLENGES

### Antenna Control: Open-Loop Response

The forward transfer function for angular velocity is,

$$G(s) = \frac{\omega_0(s)}{V_P(s)} = \frac{24}{(s+150)(s+1.32)}$$

**a.** $\omega_0(t) = A + Be^{-150t} + Ce^{-1.32t}$

**b.** $G(s) = \dfrac{24}{s^2+151.32s+198}$. Therefore, $2\zeta\omega_n = 151.32$, $\omega_n = 14.07$, and $\zeta = 5.38$.

**c.** $\omega_0(s) = \dfrac{24}{s(s^2+151.32s+198)} =$

$$\frac{24}{s(s+150)(s+1.32)} = 0.12121\frac{1}{s} + 0.0010761\frac{1}{s+150} - 0.12229\frac{1}{s+1.32}$$

Therefore, $\omega_0(t) = 0.12121 + .0010761\, e^{-150t} - 0.12229e^{-1.32t}$.

**d.** Using G(s),

$$\ddot{\omega}_0 + 151.32\dot{\omega}_0 + 198\omega_0 = 24v_p(t)$$

Defining,

$$x_1 = \omega_0$$

$$x_2 = \dot{\omega}_0$$

Thus, the state equations are,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -198x_1 - 151.32x_2 + 24v_p(t)$$

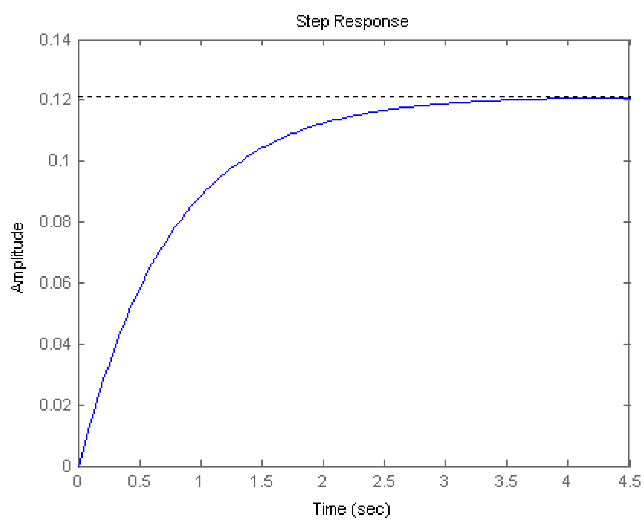
$$y = x_1$$

In vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -198 & -151.32 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 24 \end{bmatrix}v_p(t);\; y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}$$

**e.**
**Program:**
```
'Case Study 1 Challenge (e)'
num=24;
den=poly([-150  -1.32]);
G=tf(num,den)
step(G)
```

**Computer response:**
```
ans =

Case Study 1 Challenge (e)


Transfer function:
        24
-------------------
s^2 + 151.3 s + 198
```




## Ship at Sea: Open-Loop Response

**a.** Assuming a second-order approximation: $\omega_n^2 = 2.25$, $2\zeta\omega_n = 0.5$. Therefore $\zeta = 0.167$, $\omega_n = 1.5$.

$$T_s = \frac{4}{\zeta\omega_n} = 16;\ T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 2.12\ ;$$

$$\%OS = e^{-\zeta\pi\ /\sqrt{1-\zeta^2}} \times 100 = 58.8\%;\ \omega_n T_r = 1.169 \text{ therefore, } T_r = 0.77.$$

**b.** $$\theta(s) = \frac{2.25}{s\left(s^2+0.5s+2.25\right)} = \frac{1}{s} - \frac{s+0.5}{s^2+0.5s+2.25}$$

$$= \frac{1}{s} - \frac{(s+0.25)+\frac{0.25}{\sqrt{2.1875}}\sqrt{2.1875}}{(s+0.25)^2+2.1875}$$

$$= \frac{1}{s} - \frac{(s+0.25)+0.16903 \cdot 1.479}{(s+0.25)^2+2.1875}$$

Taking the inverse Laplace transform,

$\theta(t) = 1 - e^{-0.25t}$ ( cos1.479t +0.16903 sin1.479t)

**c.**

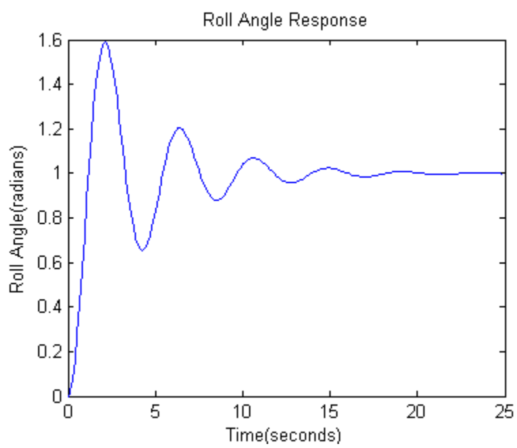
**Program:**

```
'Case Study 2 Challenge (C)'
'(a)'
numg=2.25;
deng=[1 0.5 2.25];
G=tf(numg,deng)
omegan=sqrt(deng(3))
zeta=deng(2)/(2*omegan)
Ts=4/(zeta*omegan)
Tp=pi/(omegan*sqrt(1-zeta^2))
pos=exp(-zeta*pi/sqrt(1-zeta^2))*100
t=0:.1:2;
[y,t]=step(G,t);
Tlow=interp1(y,t,.1);
Thi=interp1(y,t,.9);
Tr=Thi-Tlow
'(b)'
numc=2.25*[1 2];
denc=conv(poly([0 -3.57]),[1 2 2.25]);
[K,p,k]=residue(numc,denc)
'(c)'
[y,t]=step(G);
plot(t,y)
title('Roll Angle Response')
xlabel('Time(seconds)')
ylabel('Roll Angle(radians)')
```

**Computer response:**

```
ans =

Case Study 2 Challenge (C)


ans =

(a)


Transfer function:
       2.25
------------------
s^2 + 0.5 s + 2.25


omegan =

    1.5000


zeta =

    0.1667


Ts =

    16
```

```
Tp =

    2.1241


pos =

   58.8001


Tr =

    0.7801


ans =

(b)


K =

   0.1260
  -0.3431 + 0.1058i
  -0.3431 - 0.1058i
   0.5602


p =

  -3.5700
  -1.0000 + 1.1180i
  -1.0000 - 1.1180i
        0


k =

     []


ans =

(c)
```



# ANSWERS TO REVIEW QUESTIONS

**1.**Time constant

**2.** The time for the step response to reach 63% of its final value

**3.** The input pole

**4.** The system poles

**5.** The radian frequency of a sinusoidal response

**6.** The time constant of an exponential response

**7.** Natural frequency is the frequency of the system with all damping removed; the damped frequency of oscillation is the frequency of oscillation with damping in the system.

**8.** Their damped frequency of oscillation will be the same.

**9.** They will all exist under the same exponential decay envelop.

**10.** They will all have the same percent overshoot and the same shape although differently scaled in time.

**11.** $\zeta$, $\omega_n$, $T_p$, %OS, $T_s$

**12.** Only two since a second-order system is completely defined by two component parameters

**13.** (1) Complex, (2) Real, (3) Multiple real

**14.** Pole's real part is large compared to the dominant poles, (2) Pole is near a zero

**15.** If the residue at that pole is much smaller than the residues at other poles

**16.** No; one must then use the output equation

**17.** The Laplace transform of the state transition matrix is $(sI - \mathbf{A})^{-1}$

**18.** Computer simulation

**19.** Pole-zero concepts give one an intuitive feel for the problem.

**20.** State equations, output equations, and initial value for the state-vector

**21**. $\mathrm{Det}(s\mathbf{I}-\mathbf{A}) = 0$

# SOLUTIONS TO PROBLEMS

**1.**

**a.** Overdamped Case:

$$C(s) = \frac{9}{s(s^2 + 9s + 9)}$$

Expanding into partial fractions,

$$C(s) = \frac{9}{s(s+7.854)(s+1.146)} = \frac{1}{s} + \frac{0.171}{(s+7.854)} - \frac{1.171}{(s+1.146)}$$

Taking the inverse Laplace transform,

$$c(t) = 1 + 0.171\, e^{-7.854t} - 1.171\, e^{-1.146t}$$

**b.** Underdamped Case:

$$C(s) = \frac{9}{s(s^2 + 3s + 9)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{(s^2 + 3s + 9)}$$

$$K_1 = \left.\frac{9}{(s^2 + 3s + 9)}\right|_{s \to 0} = 1$$

$K_2$ and $K_3$ can be found by clearing fractions with $K_1$ replaced by its value. Thus,

$$9 = (s^2 + 3s + 9) + (K_2 s + K_3)s$$

or

$$9 = s^2 + 3s + 9 + K_2 s^2 + K_3 s$$

Hence $K_2 = -1$ and $K_3 = -3$. Thus,

$$C(s) = \frac{1}{s} - \frac{(s+3)}{(s^2 + 3s + 9)}$$

$$C(s) = \frac{1}{s} - \frac{(s + \frac{3}{2})}{(s + \frac{3}{2})^2 + \frac{27}{4}} - \frac{\frac{3}{2}}{(s + \frac{3}{2})^2 + \frac{27}{4}}$$

$$C(s) = \frac{1}{s} - \frac{(s + \frac{3}{2})}{(s + \frac{3}{2})^2 + \frac{27}{4}} - \frac{\frac{3}{\sqrt{27}}\sqrt{\frac{27}{4}}}{(s + \frac{3}{2})^2 + \frac{27}{4}}$$

$$c(t) = 1 - e^{\frac{-3}{2}t} \cos\sqrt{\frac{27}{4}}\,t - \frac{3}{\sqrt{27}} e^{\frac{-3}{2}t} \sin\sqrt{\frac{27}{4}}\,t$$

$$c(t) = 1 - \frac{2}{\sqrt{3}} e^{-3t/2} \cos(\sqrt{\frac{27}{4}}\ t - \phi)$$

$$= 1 - 1.155\, e^{-1.5t} \cos(2.598t - \phi)$$

where

$$\phi = \arctan\left(\frac{3}{\sqrt{27}}\right) = 30^o$$

**c.** Oscillatory Case:

$$C(s) = \frac{9}{s(s^2+9)}$$

$$C(s) = \frac{9}{s(s^2+9)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{(s^2+9)}$$

The evaluation of the constants in the numerator are found the same way as they were for the underdamped case. The results are $K_2$ = -1 and $K_3$ = 0. Hence,

$$C(s) = \frac{1}{s} - \frac{s}{(s^2+9)}$$

Therefore,

$$c(t) = 1 - \cos 3t$$

**d.** Critically Damped

$$C(s) = \frac{9}{s(s^2+6s+9)}$$

$$C(s) = \frac{9}{s(s^2+6s+9)} = \frac{K_1}{s} + \frac{K_2}{(s+3)^2} + \frac{K_3}{(s+3)}$$

The constants are then evaluated as

$$K_1 = \left.\frac{9}{(s^2+6s+9)}\right|_{s\to 0} = 1 \; ; \; K_2 = \left.\frac{9}{s}\right|_{s\to -3} = -3 \; ; \; K_3 = \left.\frac{d}{ds}\left(\frac{9}{s}\right)\right|_{s\to -3} = -1$$

Now, the transform of the response is

$$C(s) = \frac{9}{s(s^2+6s+9)} = \frac{1}{s} - \frac{3}{(s+3)^2} - \frac{1}{(s+3)}$$

$$c(t) = 1 - 3t\,e^{-3t} - e^{-3t}$$

**2.**

**a.** $C(s) = \frac{5}{s(s+5)} = \frac{1}{s} - \frac{1}{s+5}$ . Therefore, $c(t) = 1 - e^{-5t}$.

Also, $T = \frac{1}{5}$ , $T_r = \frac{2.2}{a} = \frac{2.2}{5} = 0.44$, $T_s = \frac{4}{a} = \frac{4}{5} = 0.8$.

**b.** $C(s) = \frac{20}{s(s+20)} = \frac{1}{s} - \frac{1}{s+20}$ . Therefore, $c(t) = 1 - e^{-20t}$. Also, $T = \frac{1}{20}$ ,

$T_r = \frac{2.2}{a} = \frac{2.2}{20} = 0.11$, $T_s = \frac{4}{a} = \frac{4}{20} = 0.2$.

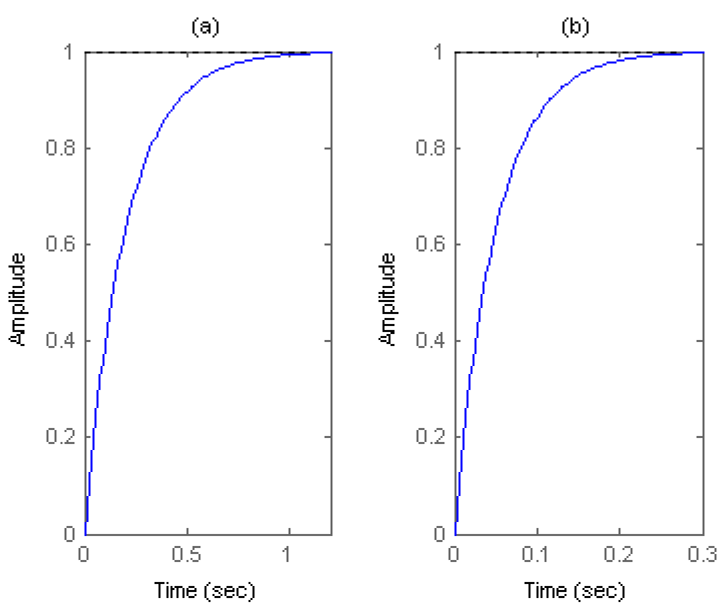
**3.**

**Program:**

```
'(a)'
num=5;
den=[1 5];
Ga=tf(num,den)
subplot(1,2,1)
step(Ga)
title('(a)')
'(b)'
num=20;
den=[1 20];
Gb=tf(num,den)
subplot(1,2,2)
step(Gb)
title('(b)')
```

**Computer response:**

```
ans =

(a)

Transfer function:
  5
-----
s + 5

ans =

(b)

Transfer function:
  20
------
s + 20
```



**4.**

Using voltage division, $\frac{V_C(s)}{V_i(s)} = \frac{1/RC}{S+\frac{1}{RC}} = \frac{0.703}{s+0.703}$. Since $V_i(s) = \frac{5}{s}$

$$V_c(s) = \frac{5}{s}\left(\frac{0.703}{s+0.703}\right) = \frac{5}{s} - \frac{5}{s+0.703}.$$

Therefore $v_c(t) = 5 - 5e^{-0.703t}$. Also,

$$T = \frac{1}{0.703} = 1.422;\ T_r = \frac{2.2}{0.703} = 3.129;\ \mathrm{T_s} = \frac{4}{0.703} = 5.69.$$

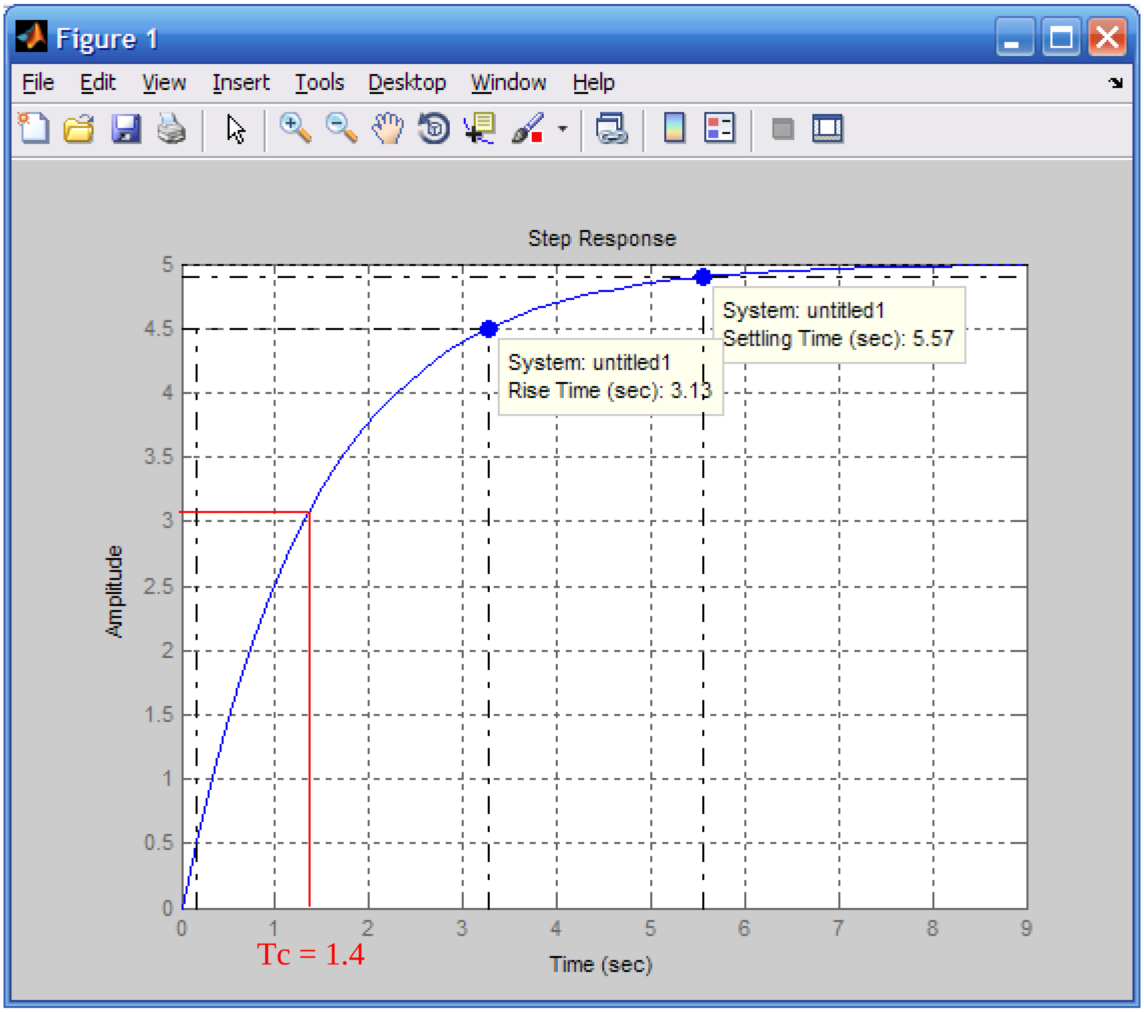
**5.**

**Program:**

```
clf
num=0.703;
den=[1 0.703];
G=tf(num,den)
step(5*G)
```

**Computer response:**

```
Transfer function:
  0.703
---------
s + 0.703
```




**6.**

Writing the equation of motion,

$$(Ms^2 + 6s)X(s) = F(s)$$

Thus, the transfer function is,

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + 6s}$$

Differentiating to yield the transfer function in terms of velocity,

$$\frac{sX(s)}{F(s)} = \frac{1}{Ms + 6} = \frac{1/M}{s + \dfrac{6}{M}}$$

Thus, the settling time, $T_s$, and the rise time, $T_r$, are given by

$$T_s = \frac{4}{6/M} = \frac{2}{3}M = 0.667M; \quad T_r = \frac{2.2}{6/M} = \frac{1.1}{3}M = 0.367M$$

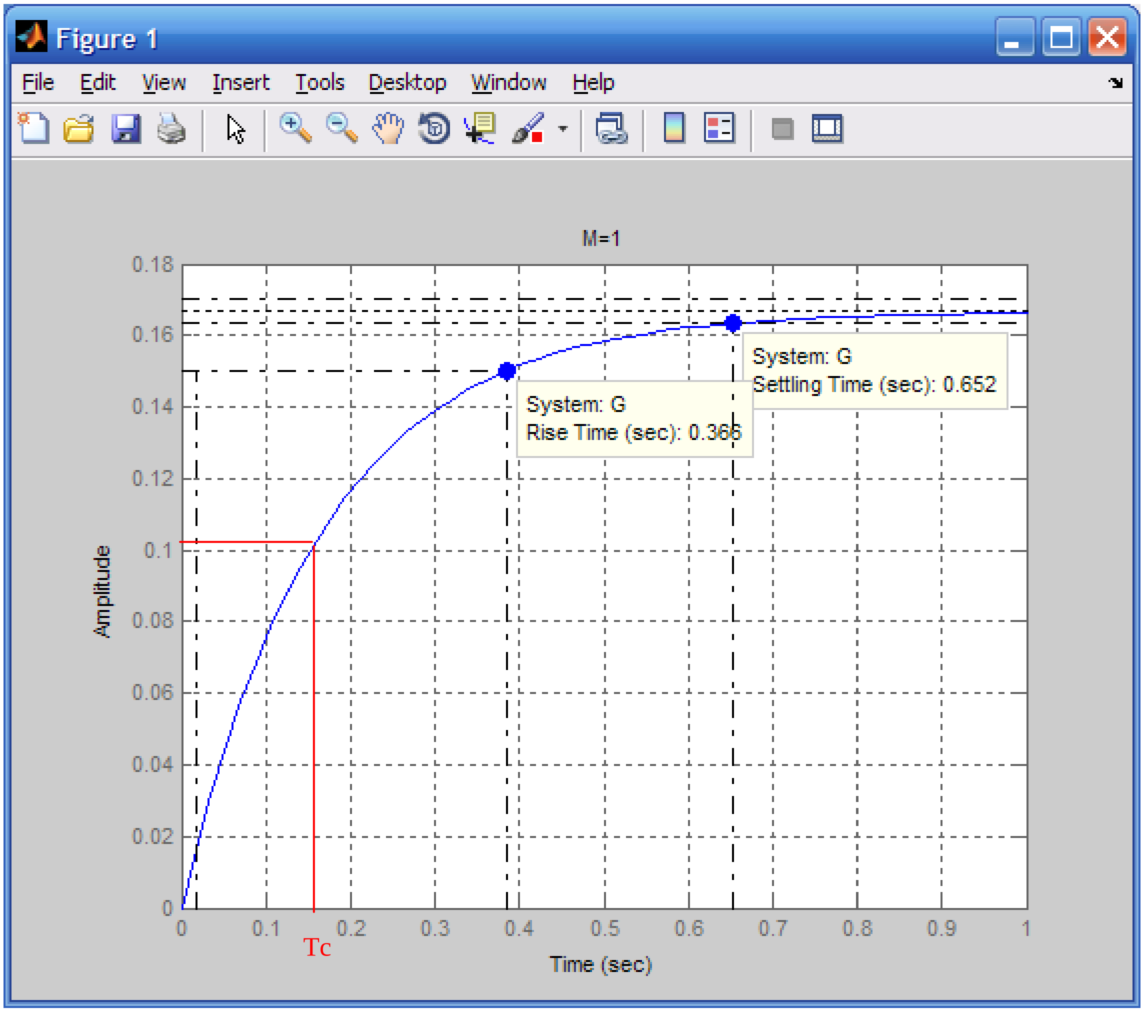
**7.**

**Program:**

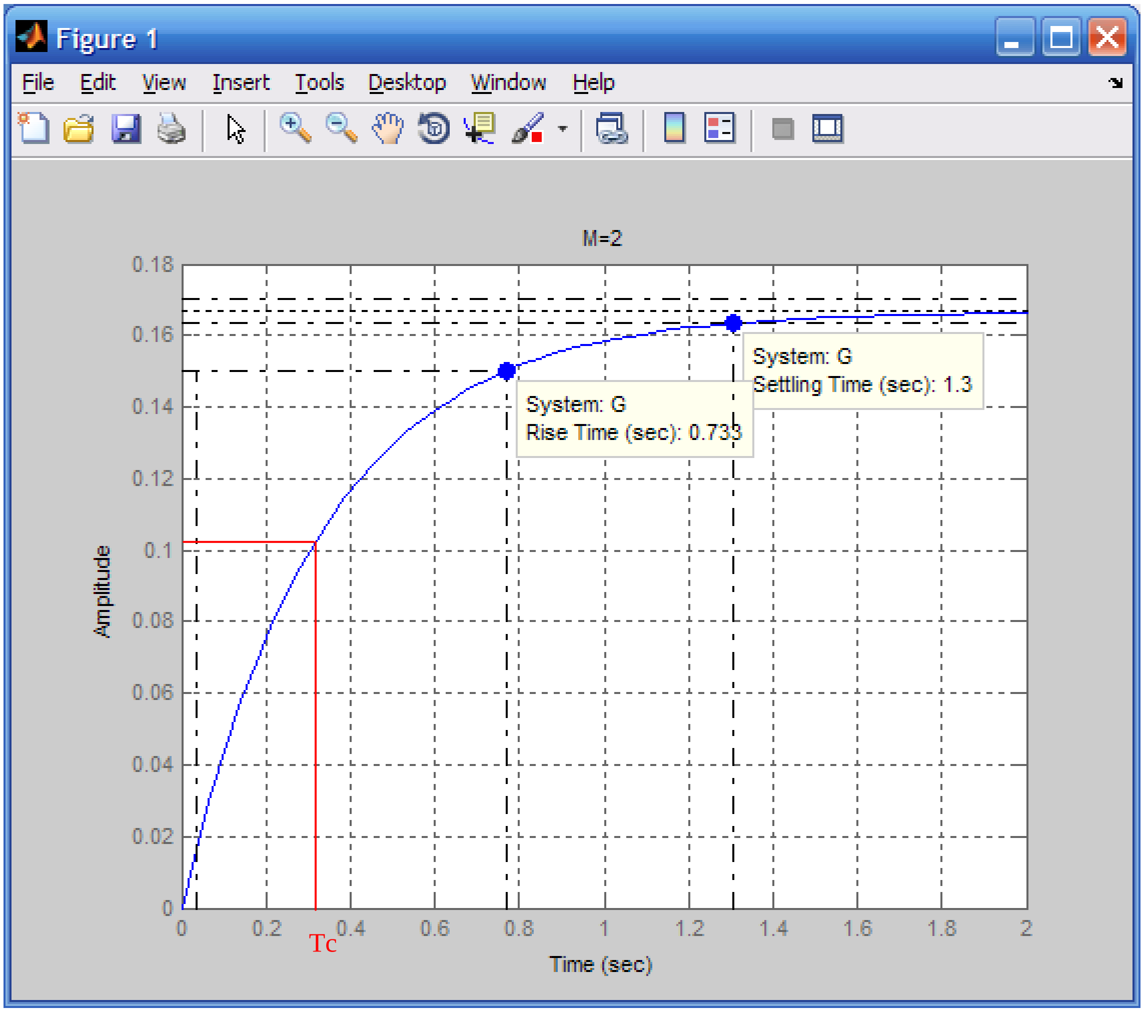
```
Clf
M=1
num=1/M;
den=[1 6/M];
G=tf(num,den)
step(G)
pause
M=2
num=1/M;
den=[1 6/M];
G=tf(num,den)
step(G)
```

**Computer response:**

```
M =

     1


Transfer function:
  1
-----
s + 6


M =

     2


Transfer function:
 0.5
-----
s + 3
```




From plot, time constant =.0.16 s.




From plot, time constant = 0.33 s.

**8.**

**a.** Pole: -2; $c(t) = A + Be^{-2t}$ ; first-order response.

**b.** Poles: -3, -6; $c(t) = A + Be^{-3t} + Ce^{-6t}$; overdamped response.

**c.** Poles: -10, -20; Zero: -7; $c(t) = A + Be^{-10t} + Ce^{-20t}$; overdamped response.

**d.** Poles: $(-3+j3\sqrt{15})$, $(-3-j3\sqrt{15})$ ; $c(t) = A + Be^{-3t} \cos(3\sqrt{15}\ t + \phi)$; underdamped.

**e.** Poles: j3, -j3; Zero: -2; $c(t) = A + B \cos(3t + \phi)$; undamped.

**f.** Poles: -10, -10; Zero: -5; $c(t) = A + Be^{-10t} + Cte^{-10t}$; critically damped.

**9.**

**Program:**
```
p=roots([1 6 4 7 2])
```

**Computer response:**
```
p =
```

```
-5.4917
-0.0955 + 1.0671i
-0.0955 - 1.0671i
-0.3173
```

**10.**

G(s) = **C** (s**I**-**A**)$^{-1}$ **B**

$$\mathbf{A}=\begin{bmatrix}8 & -4 & 1\\ -3 & 2 & 0\\ 5 & 7 & -9\end{bmatrix};\ \mathbf{B}=\begin{bmatrix}-4\\ -3\\ 4\end{bmatrix};\ \mathbf{C}=\begin{bmatrix}2 & 8 & -3\end{bmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s^3-s^2-91s+67}\begin{bmatrix}(s-2)(s+9) & -(4s+29) & (s-2)\\ -(3s+27) & (s^2+s-77) & -3\\ 5s-31 & 7s-76 & (s^2-10s+4)\end{bmatrix}$$

Therefore, G(s ) = $\dfrac{-44s^2+291s+1814}{s^3-s^2-91s+67}$.

Factoring the denominator, or using det(s**I**-**A**), we find the poles to be 9.683, 0.7347, -9.4179.

**11.**

**Program:**

```
A=[8 -4 1;-3 2 0;5 7 -9]
B=[-4;-3;4]
C=[2 8 -3]
D=0
[numg,deng]=ss2tf(A,B,C,D,1);
G=tf(numg,deng)
poles=roots(deng)
```

**Computer response:**

```
A =

     8    -4     1

    -3     2     0

     5     7    -9

B =

    -4

    -3

     4
```

```
C =

     2     8    -3

D =

     0

Transfer function:

-44 s^2 + 291 s + 1814

----------------------

s^3 - s^2 - 91 s + 67

poles =

   -9.4179

    9.6832

    0.7347
```

**12.**

Writing the node equation at the capacitor, $V_C(s)\left(\frac{1}{R_2} + \frac{1}{Ls} + Cs\right) + \frac{V_C(s) - V(s)}{R_1} = 0$.

Hence, $\frac{V_C(s)}{V(s)} = \frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{Ls} + Cs} = \frac{10s}{s^2+20s+500}$. The step response is $\frac{10}{s^2+20s+500}$. The poles

are at

$-10 \pm j20$. Therefore, $v_C(t) = Ae^{-10t}\cos(20t + \phi)$.

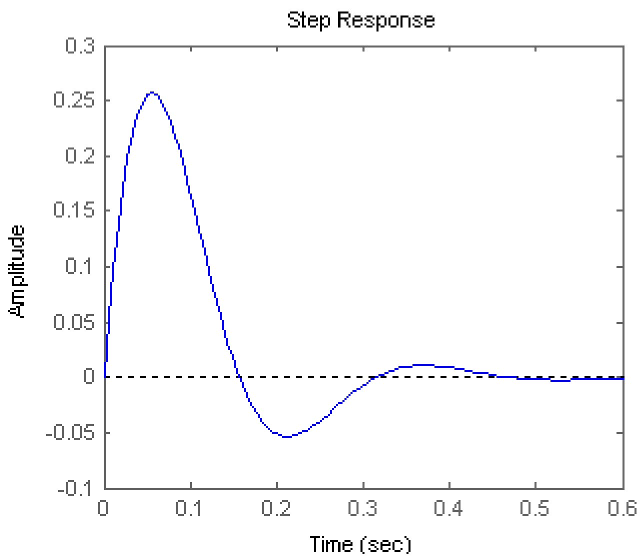
**13.**

**Program:**

```
num=[10 0];
den=[1 20 500];
G=tf(num,den)
step(G)
```

**Computer response:**

```
Transfer function:
      10 s
----------------
s^2 + 20 s + 500
```




**14.**

The equation of motion is: $(Ms^2+f_vs+K_s)X(s) = F(s)$. Hence, $\frac{X(s)}{F(s)} = \frac{1}{Ms^2+f_vs+K_s} = \frac{1}{s^2+s+5}$.

The step response is now evaluated: $X(s) = \frac{1}{s(s^2+s+5)} = \frac{1/5}{s} - \frac{\frac{1}{5}s+\frac{1}{5}}{(s+\frac{1}{2})^2+\frac{19}{4}} =$

$$\frac{\frac{1}{5}(s+\frac{1}{2}) + \frac{1}{5\sqrt{19}}\frac{\sqrt{19}}{2}}{(s+\frac{1}{2})^2+\frac{19}{4}}.$$

Taking the inverse Laplace transform, $x(t) = \frac{1}{5} - \frac{1}{5}e^{-0.5t}\left(\cos\frac{\sqrt{19}}{2}t + \frac{1}{\sqrt{19}}\sin\frac{\sqrt{19}}{2}t\right)$

$$= \frac{1}{5}\left[1 - 2\sqrt{\frac{5}{19}}\,e^{-0.5t}\cos\left(\frac{\sqrt{19}}{2}t - 12.92^o\right)\right].$$

**15.**

$$C(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_ns+\omega_n^2)} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_ns+\omega_n^2} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2-\zeta^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{(s+\zeta\omega_n)+\zeta\omega_n}{(s+\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2} = \frac{1}{s} - \frac{(s+\zeta\omega_n)+\frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2}$$

Hence, $c(t) = 1 - e^{-\zeta\omega_nt}\left(\cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right)$

$$= 1 - e^{-\zeta\omega_n t}\sqrt{1+\frac{\zeta^2}{1-\zeta^2}}\cos(\omega_n\sqrt{1-\zeta^2}\,t-\phi) = 1 - e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}}\cos(\omega_n\sqrt{1-\zeta^2}\,t-\phi),$$

where $\phi = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$

**16.**

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$. Dividing by 100 and taking the natural log of both sides, $\ln\left(\frac{\%OS}{100}\right) = -\frac{\zeta\pi}{\sqrt{1-\zeta^2}}$. Squaring both sides and solving for $\zeta^2$, $\zeta^2 = \frac{\ln^2\left(\frac{\%OS}{100}\right)}{\pi^2+\ln^2\left(\frac{\%OS}{100}\right)}$. Taking the negative square root, $\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2+\ln^2\left(\frac{\%OS}{100}\right)}}$.

**17.**

**a.**

$$C(s) = \frac{2}{s(s+2)}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+2}$$

$$c(t) = 1 - e^{-2t}$$

**b.**

$$C(s) = \frac{5}{s(s+3)(s+6)}$$

$$C(s) = \frac{5}{18}\frac{1}{s} - \frac{5}{9}\frac{1}{s+3} + \frac{5}{18}\frac{1}{s+6}$$

$$c(t) = \frac{5}{18} - \frac{5}{9}e^{-3t} + \frac{5}{18}e^{-6t}$$

**c.**

$$C(s) = \frac{10(s+7)}{s(s+10)(s+20)}$$

$$C(s) = \frac{7}{20}\frac{1}{s} + \frac{3}{10}\frac{1}{s+10} - \frac{13}{20}\frac{1}{s+20}$$

$$c(t) = \frac{7}{20} + \frac{3}{10}e^{-10t} - \frac{13}{20}e^{-20t}$$

**d.**

$$C(s) = \frac{20}{s(s^2+6s+144)}$$

$$C(s) = \frac{5}{36}\frac{1}{s} - \frac{5}{36}\frac{s+6}{s^2+6s+144}$$

$$C(s) = \frac{5}{36}\frac{1}{s} - \frac{5}{36}\frac{(s+3)+\frac{3}{\sqrt{135}}\sqrt{135}}{(s+3)^2+135}$$

$$c(t) = \frac{5}{36} - \frac{5}{36}e^{-3t}\left(\cos\left[\sqrt{135}\right]t + \frac{3}{\sqrt{135}}\sin\left[\sqrt{135}\right]t\right)$$

**e.**

$$C(s) = \frac{s+2}{s(s^2+9)}$$

$$C(s) = \frac{2}{9}\frac{1}{s} + \frac{1}{9}\frac{-2s+9}{s^2+9}$$

$$C(s) = \frac{2}{9}\frac{1}{s} + \frac{1}{9}\frac{-2s+3.3}{s^2+9}$$

$$c(t) = \frac{2}{9} - \left(\frac{2}{9}\cos 3t - \frac{1}{3}\sin 3t\right)$$

**f.**

$$C(s) = \frac{s+5}{s(s+10)^2}$$

$$C(s) = \frac{1}{20}\frac{1}{s} - \frac{1}{20}\frac{1}{s+10} + \frac{1}{2}\frac{1}{(s+10)^2}$$

$$c(t) = \frac{1}{20} - \frac{1}{20}e^{-10t} + \frac{1}{2}te^{-10t}$$

**18.**

**a.** N/A

**b.** $s^2+9s+18$, $\omega_n^2 = 18$, $2\zeta\omega_n = 9$, Therefore $\zeta = 1.06$, $\omega_n = 4.24$, overdamped.

**c.** $s^2+30s+200$, $\omega_n^2 = 200$, $2\zeta\omega_n = 30$, Therefore $\zeta = 1.06$, $\omega_n = 14.14$, overdamped.

**d.** $s^2+6s+144$, $\omega_n^2 = 144$, $2\zeta\omega_n = 6$, Therefore $\zeta = 0.25$, $\omega_n = 12$, underdamped.

**e.** $s^2+9$, $\omega_n^2 = 9$, $2\zeta\omega_n = 0$, Therefore $\zeta = 0$, $\omega_n = 3$, undamped.

**f.** $s^2+20s+100$, $\omega_n^2 = 100$, $2\zeta\omega_n = 20$, Therefore $\zeta = 1$, $\omega_n = 10$, critically damped.

**19.**

$$X(s) = \frac{100^2}{s(s^2+100s+100^2)} = \frac{1}{s} - \frac{s+100}{(s+50)^2+7500} = \frac{1}{s} - \frac{(s+50)+50}{(s+50)^2+7500} = \frac{1}{s} - \frac{(s+50) + \frac{50}{\sqrt{7500}}\sqrt{7500}}{(s+50)^2+7500}$$

Therefore, $x(t) = 1 - e^{-50t}\left(\cos\sqrt{7500}\ t + \frac{50}{\sqrt{7500}}\sin\sqrt{7500}\ t\right)$

$= 1 - \frac{2}{\sqrt{3}}e^{-50t}\cos\left(50\sqrt{3}\ t - \tan^{-1}\frac{1}{\sqrt{3}}\right)$

**20.**

**a.** $\omega_n^2 = 16$ r/s, $2\zeta\omega_n = 3$. Therefore $\zeta = 0.375$, $\omega_n = 4$. $T_s = \frac{4}{\zeta\omega_n} = 2.667$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} =$ 0.8472 s; $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 28.06\ \%$; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1) = 1.4238$; therefore, $T_r = 0.356$ s.

**b.** $\omega_n^2 = 0.04$ r/s, $2\zeta\omega_n = 0.02$. Therefore $\zeta = 0.05$, $\omega_n = 0.2$. $T_s = \frac{4}{\zeta\omega_n} = 400$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} =$ 15.73 s; $\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 85.45$ %; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$; therefore, $T_r = 5.26$ s.

**c.** $\omega_n^2 = 1.05 \times 10^7$ r/s, $2\zeta\omega_n = 1.6 \times 10^3$. Therefore $\zeta = 0.247$, $\omega_n = 3240$. $T_s = \frac{4}{\zeta\omega_n} = 0.005$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.001$ s; $\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 44.92$ %; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$; therefore, $T_r = 3.88 \times 10^{-4}$ s.

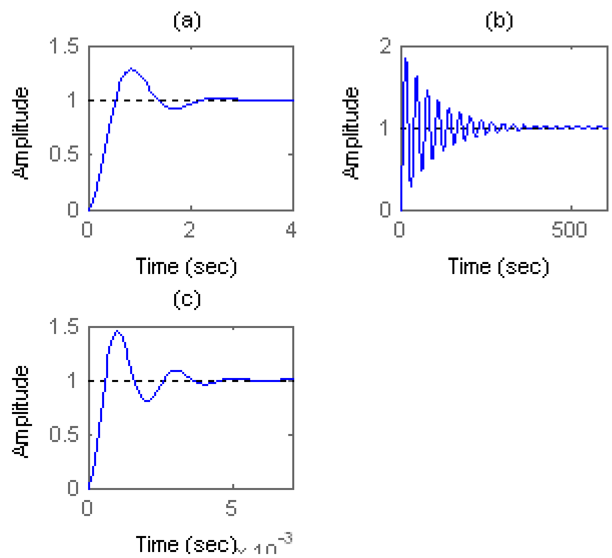
**21.**

**Program:**
```
'(a)'
clf
numa=16;
dena=[1 3 16];
Ta=tf(numa,dena)
omegana=sqrt(dena(3))
zetaa=dena(2)/(2*omegana)
Tsa=4/(zetaa*omegana)
Tpa=pi/(omegana*sqrt(1-zetaa^2))
Tra=(1.76*zetaa^3 - 0.417*zetaa^2 + 1.039*zetaa + 1)/omegana
percenta=exp(-zetaa*pi/sqrt(1-zetaa^2))*100
subplot(221)
step(Ta)
title('(a)')
'(b)'
numb=0.04;
denb=[1 0.02 0.04];
Tb=tf(numb,denb)
omeganb=sqrt(denb(3))
zetab=denb(2)/(2*omeganb)
Tsb=4/(zetab*omeganb)
Tpb=pi/(omeganb*sqrt(1-zetab^2))
Trb=(1.76*zetab^3 - 0.417*zetab^2 + 1.039*zetab + 1)/omeganb
percentb=exp(-zetab*pi/sqrt(1-zetab^2))*100
subplot(222)
step(Tb)
title('(b)')
'(c)'
numc=1.05E7;
denc=[1 1.6E3 1.05E7];
Tc=tf(numc,denc)
omeganc=sqrt(denc(3))
zetac=denc(2)/(2*omeganc)
Tsc=4/(zetac*omeganc)
Tpc=pi/(omeganc*sqrt(1-zetac^2))
Trc=(1.76*zetac^3 - 0.417*zetac^2 + 1.039*zetac + 1)/omeganc
percentc=exp(-zetac*pi/sqrt(1-zetac^2))*100
subplot(223)
step(Tc)
title('(c)')
```

**Computer response:**

```
ans =

(a)

Transfer function:
      16
--------------
s^2 + 3 s + 16


omegana =

     4


zetaa =

    0.3750


Tsa =

    2.6667


Tpa =

    0.8472


Tra =

    0.3559


percenta =

   28.0597


ans =

(b)


Transfer function:
       0.04
-------------------
s^2 + 0.02 s + 0.04


omeganb =

    0.2000

zetab =

    0.0500

Tsb =
   400
```

```
Tpb =

   15.7276


Trb =

    5.2556


percentb =

   85.4468


ans =

(c)


Transfer function:
       1.05e007
-----------------------
s^2 + 1600 s + 1.05e007


omeganc =

  3.2404e+003


zetac =

    0.2469


Tsc =

    0.0050


Tpc =

    0.0010


Trc =

  3.8810e-004


percentc =

   44.9154
```



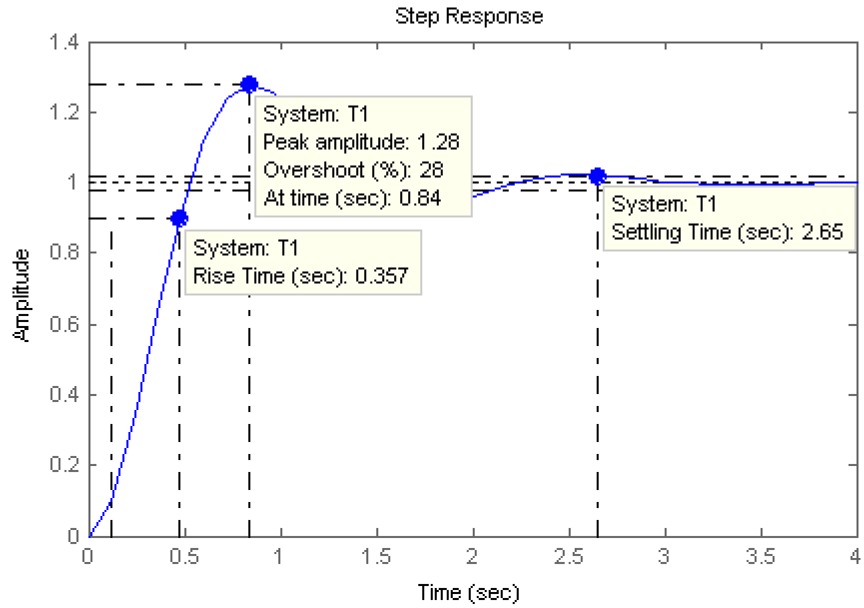

**22.**

**Program:**

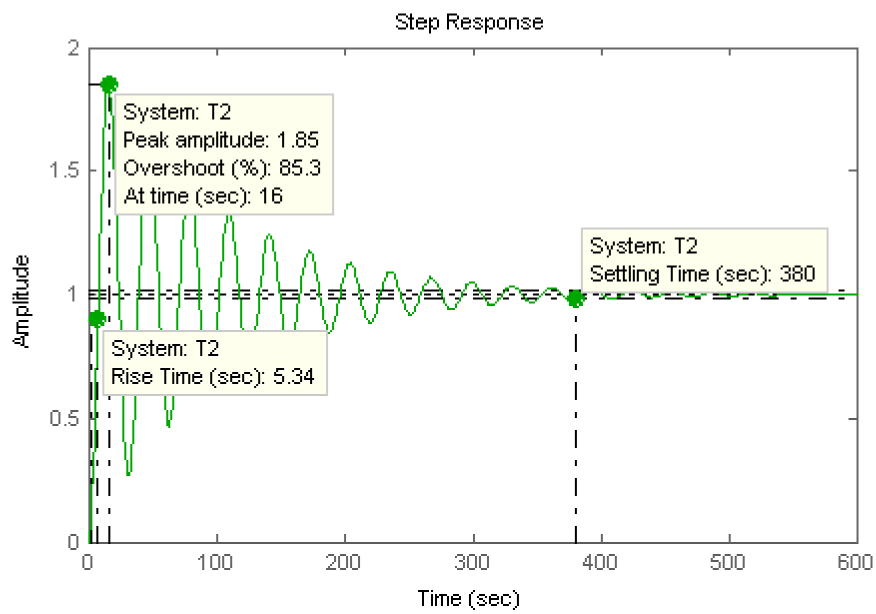
```
T1=tf(16,[1 3 16])
T2=tf(0.04,[1 0.02 0.04])
T3=tf(1.05e7,[1 1.6e3 1.05e7])
ltiview
```

**Computer response:**

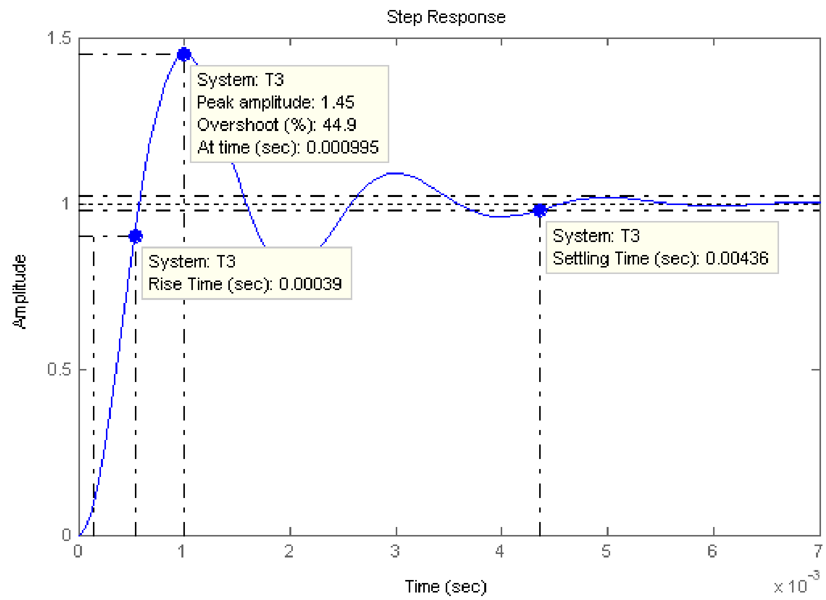
```
Transfer function:
      16
--------------
s^2 + 3 s + 16

Transfer function:
       0.04
-------------------
s^2 + 0.02 s + 0.04


Transfer function:
       1.05e007
-----------------------
s^2 + 1600 s + 1.05e007
```



Step Response
System: T1
Peak amplitude: 1.28
Overshoot (%): 28
At time (sec): 0.84
System: T1
Settling Time (sec): 2.65
System: T1
Rise Time (sec): 0.357
Amplitude
Time (sec)

Step Response
System: T2
Peak amplitude: 1.85
Overshoot (%): 85.3
At time (sec): 16
System: T2
Settling Time (sec): 380
System: T2
Rise Time (sec): 5.34
Amplitude
Time (sec)




**23.**

**a.** $\zeta = \dfrac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}} = 0.56$, $\omega_n = \dfrac{4}{\zeta T_s} = 11.92$. Therefore, poles $= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$= -6.67 \pm j9.88$.

**b.** $\zeta = \dfrac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}} = 0.591$, $\omega_n = \dfrac{\pi}{T_p\sqrt{1-\zeta^2}} = 0.779$.

Therefore, poles $= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -0.4605 \pm j0.6283$.

**c.** $\zeta\omega_n = \dfrac{4}{T_s} = 0.571$, $\omega_n\sqrt{1-\zeta^2} = \dfrac{\pi}{T_p} = 1.047$. Therefore, poles $= -0.571 \pm j1.047$.

**24.**

$$\text{Re} = \frac{4}{T_s} = 4;\quad \zeta = \frac{-\ln(12.3/100)}{\sqrt{\pi^2 + \ln^2(12.3/100)}} = 0.5549$$

$$\text{Re} = \zeta\omega_n = 0.5549\omega_n = 4;\quad \therefore \omega_n = 7.21$$

$$\text{Im} = \omega_n\sqrt{1-\zeta^2} = 6$$

$$\therefore G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{51.96}{s^2 + 8s + 51.96}$$

**25.**

**a.** Writing the equation of motion yields, $(5s^2 + 5s + 28)X(s) = F(s)$

Solving for the transfer function,

$$\frac{X(s)}{F(s)} = \frac{1/5}{s^2 + s + \frac{28}{5}}$$

**b.** $\omega_n^2 = 28/5$ r/s, $2\zeta\omega_n = 1$. Therefore $\zeta = 0.211$, $\omega_n = 2.37$. $T_s = \frac{4}{\zeta\omega_n} = 8.01$ s; $T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} =$ 1.36 s; $\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100 = 50.7$ %; $\omega_n T_r = (1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)$; therefore, $T_r = 0.514$ s.

**26.**

Writing the loop equations,

$$(1.07s^2 + 1.53s)\theta_1(s) - 1.53\theta_2(s) = T(s)$$
$$-1.53s\theta_1(s) + (1.53s + 1.92)\theta_2(s) = 0$$

Solving for $\theta_2(s)$,

$$\theta_2(s) = \frac{\begin{vmatrix} (1.07s^2 + 1.53s) & T(s) \\ -1.53s & 0 \end{vmatrix}}{\begin{vmatrix} (1.07s^2 + 1.53s) & -1.53s \\ -1.53s & (1.53s + 1.92) \end{vmatrix}} = \frac{0.935T(s)}{s^2 + 1.25s + 1.79}$$

Forming the transfer function,

$$\frac{\theta_2(s)}{T(s)} = \frac{0.935}{s^2 + 1.25s + 1.79}$$

Thus $\omega_n = 1.34$, $2\zeta\omega_n = 1.25$. Thus, $\zeta = 0.467$. From Eq. (4.38), %OS = 19.0%. From Eq. (4.42), $T_s$ = 6.4 seconds. From Eq. (4.34), $T_p = 2.66$ seconds.

**27.**

**a.** $$\frac{24.542}{s(s^2 + 4s + 24.542)} = \frac{1}{s} - \frac{s+4}{(s+2)^2 + 20.542} = \frac{1}{s} - \frac{(s+2) + \frac{2}{4.532}4.532}{(s+2)^2 + 20.542}.$$

Thus $c(t) = 1 - e^{-2t}(\cos 4.532t + 0.441 \sin 4.532t) = 1 - 1.09e^{-2t}\cos(4.532t - 23.8^0)$.

**b.**

$$\frac{245.42}{s(s+10)(s^2 + 4s + 24.542)} = \frac{1}{s} - 0.29029\frac{1}{s+10} - \frac{0.70971s + 5.7418}{s^2 + 4s + 24.542}$$
$$\frac{245.42}{s(s+10)(s^2 + 4s + 24.542)} = \frac{1}{s} - 0.29029\frac{1}{s+10} - \frac{0.70971s + 5.7418}{(s+2)^2 + 20.542}$$

$$\frac{245.42}{s(s+10)(s^2+4s+24.542)} = \frac{1}{s} - 0.29029\frac{1}{s+10} - \frac{0.70971(s+2) + \frac{4.3223}{\sqrt{20.542}}\sqrt{20.542}}{(s+2)^2+20.542}$$

$$\frac{245.42}{s(s+10)(s^2+4s+24.542)} = \frac{1}{s} - 0.29029\frac{1}{s+10} - \frac{0.70971(s+2) + 0.95367\sqrt{20.542}}{(s+2)^2+20.542}$$

Therefore, $c(t) = 1 - 0.29e^{-10t} - e^{-2t}(0.71 \cos 4.532t + 0.954 \sin 4.532t)$

$= 1 - 0.29e^{-10t} - 1.189 \cos(4.532t - 53.34^{o})$.

**c.**

$$\frac{73.626}{s(s+3)(s^2+4s+24.542)} = \frac{1}{s} - 1.1393\frac{1}{s+3} + \frac{0.13926s - 2.8607}{s^2+4s+24.542}$$

$$\frac{73.626}{s(s+3)(s^2+4s+24.542)} = \frac{1}{s} - 1.1393\frac{1}{s+3} + \frac{0.13926(s+2) - \frac{3.1393}{\sqrt{20.542}}\sqrt{20.542}}{(s+2)^2+20.542}$$

$$\frac{73.626}{s(s+3)(s^2+4s+24.542)} = \frac{1}{s} - 1.1393\frac{1}{s+3} + \frac{0.13926(s+2) - 0.69264\sqrt{20.542}}{(s+2)^2+20.542}$$

Therefore, $c(t) = 1 - 1.14e^{-3t} + e^{-2t}(0.14 \cos 4.532t - 0.69 \sin 4.532t)$

$= 1 - 1.14e^{-3t} + 0.704 \cos(4.532t + 78.53^{o})$.

**28.**

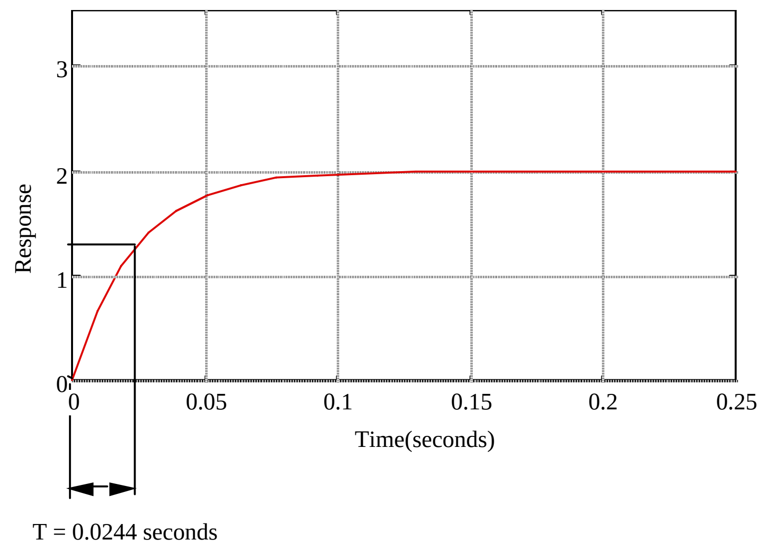
Since the third pole is more than five times the real part of the dominant pole, $s^2+0.842s+2.829$ determines the transient response. Since $2\zeta\omega_n = 0.842$, and $\omega_n = \sqrt{2.829} = \omega_n = 1.682$, $\zeta = 0.25$,

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 44.4\%$, $T_s = \frac{4}{\zeta\omega_n} = 9.50$ sec, $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.93$ sec; $\omega_n T_r =$

$(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1) = 1.26$, therefore, $T_r = 0.75$.

**29.**

**a.** Measuring the time constant from the graph, T = 0.0244 seconds.

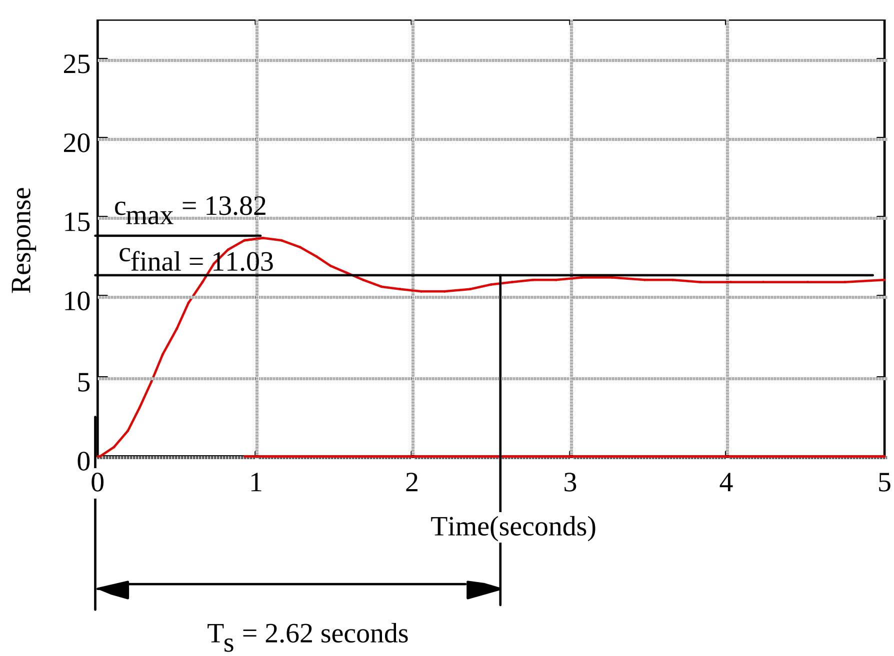



Estimating a first-order system, $G(s) = \frac{K}{s+a}$. But, $a = 1/T = 40.984$, and $\frac{K}{a} = 2$. Hence, K = 81.967.

Thus,

$$G(s) = \frac{81.967}{s+40.984}$$

**b.** Measuring the percent overshoot and settling time from the graph: %OS = (13.82-11.03)/11.03 = 25.3%,



and $T_s$ = 2.62 seconds. Estimating a second-order system, we use Eq. (4.39) to find $\zeta = 0.4$ , and Eq. (4.42) to find $\omega_n = 3.82$. Thus, $G(s) = \frac{K}{s^2+2\zeta\omega_n s+\omega_n^2}$ . Since $C_{final}$ = 11.03, $\frac{K}{\omega_n^2}$ = 11.03. Hence, K = 160.95. Substituting all values,

$$G(s) = \frac{160.95}{s^2+3.056s+14.59}$$

**c.** From the graph, %OS = 40%. Using Eq. (4.39), $\zeta = 0.28$. Also from the graph, $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 4.$ Substituting $\zeta = 0.28$, we find $\omega_n = 0.818$.

Thus,

$$G(s) = \frac{K}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{0.669}{s^2+0.458s+0.669}.$$

**30.**

**a.**

$$\frac{s+3}{s(s+2)(s^2+3s+10)} = \frac{3}{20}\frac{1}{s} - \frac{1}{16}\frac{1}{s+2} - \frac{1}{80}\frac{7s+31}{\left(s+\frac{3}{2}\right)^2+\frac{31}{4}}$$

$$\frac{s+3}{s(s+2)(s^2+3s+10)} = \frac{3}{20}\frac{1}{s} - \frac{1}{16}\frac{1}{s+2} - \frac{1}{80}\frac{7\left(s+\frac{3}{2}\right)+7.3638\sqrt{\frac{31}{4}}}{\left(s+\frac{3}{2}\right)^2+\frac{31}{4}}$$

Since the amplitude of the sinusoids are of the same order of magnitude as the residue of the pole at -2, pole-zero cancellation cannot be assumed.

**b.**

$$\frac{s+2.5}{s(s+2)(s^2+4s+20)} = \frac{1}{16}\frac{1}{s} - \frac{1}{64}\frac{1}{s+2} - \frac{1}{64}\frac{3s+14}{s^2+4s+20}$$

$$\frac{s+2.5}{s(s+2)(s^2+4s+20)} = \frac{1}{16}\frac{1}{s} - \frac{1}{64}\frac{1}{s+2} - \frac{1}{64}\frac{3(s+2)+2\sqrt{16}}{(s+2)^2+16}$$

Since the amplitude of the sinusoids are of the same order of magnitude as the residue of the pole at -2, pole-zero cancellation cannot be assumed.

**c.**

$$\frac{s+2.1}{s(s+2)(s^2+s+5)} = 0.21\frac{1}{s} - 0.0071429\frac{1}{s+2} - \frac{0.20286s+0.21714}{s^2+1s+5}$$

$$\frac{s+2.1}{s(s+2)(s^2+s+5)} = 0.21\frac{1}{s} - 0.0071429\frac{1}{s+2} - \frac{0.20286\left(s+\frac{1}{2}\right)+0.053093\sqrt{\frac{19}{4}}}{\left(s+\frac{1}{2}\right)^2+\frac{19}{4}}$$

Since the amplitude of the sinusoids are of two orders of magnitude larger than the residue of the pole at -2, pole-zero cancellation can be assumed. Since $2\zeta\omega_n = 1$, and $\omega_n = \sqrt{5} = 2.236$, $\zeta = 0.224$, $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 48.64\%$, $T_s = \dfrac{4}{\zeta\omega_n} = 8$ sec, $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.44$ sec; $\omega_n T_r = 1.23$, therefore, $T_r = 0.55$.

**d.**

$$\frac{s+2.01}{s(s+2)(s^2+5s+20)} = 0.05025\frac{1}{s} - 0.00035714\frac{1}{s+2} - \frac{0.049893s+0.25018}{\left(s+\frac{5}{2}\right)^2+\frac{55}{4}}$$

$$\frac{s+2.01}{s(s+2)(s^2+5s+20)} = 0.05025\frac{1}{s} - 0.00035714\frac{1}{s+2} - \frac{0.049893\left(s+\frac{5}{2}\right)+0.03383\sqrt{\frac{55}{4}}}{\left(s+\frac{5}{2}\right)^2+\frac{55}{4}}$$

Since the amplitude of the sinusoids are of two orders of magnitude larger than the residue of the pole at -2, pole-zero cancellation can be assumed. Since $2\zeta\omega_n = 5$, and $\omega_n = \sqrt{20} = 4.472$, $\zeta = 0.559$, $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} x100 = 12.03\%$, $T_s = \dfrac{4}{\zeta\omega_n} = 1.6$ sec, $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.847$ sec; $\omega_n T_r = 1.852$, therefore, $T_r = 0.414$.

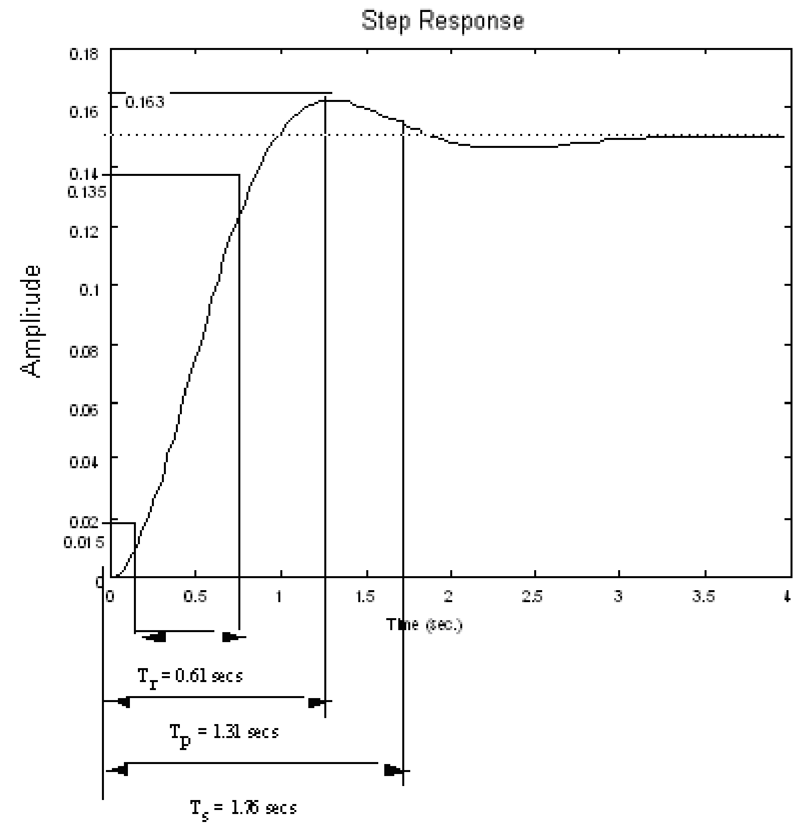
**31.**

**Program:**

```
%Form sC(s) to get transfer function
clf
num=[1 3];
den=conv([1 3 10],[1 2]);
T=tf(num,den)
step(T)
```

**Computer response:**

```
Transfer function:
         s + 3
-----------------------
s^3 + 5 s^2 + 16 s + 20
```




$$\%OS = \frac{(0.163 - 0.15)}{0.15} = 8.67\%$$

**32.**

Part **c** can be approximated as a second-order system. From the exponentially decaying cosine, the poles are located at $s_{1,2} = -2 \pm j9.796$. Thus,

$$T_s = \frac{4}{|\text{Re}|} = \frac{4}{2} = 2 \text{ s}; \; T_p = \frac{\pi}{|\text{Im}|} = \frac{\pi}{9.796} = 0.3207 \text{ s}$$

Also, $\omega_n = \sqrt{2^2 + 9.796^2} = 10$ and $\zeta\omega_n = |\text{Re}| = 2$. Hence, $\zeta = 0.2$, yielding 52.66 percent overshoot.

Part d can be approximated as a second-order system. From the exponentially decaying cosine, the poles are located at $S_{1,2} = -2 \pm j9.951$. Thus,

$$T_s = \frac{4}{|\text{Re}|} = \frac{4}{2} = 2 \text{ s}; T_p = \frac{\pi}{|\text{Im}|} = \frac{\pi}{9.951} = 0.3157 \text{ s}$$

Also, $\omega_n = \sqrt{2^2 + 9.951^2} = 10.15$ and $\zeta\omega_n = |\text{Re}| = 2$. Hence, $\zeta = 0.197$, yielding 53.19 percent overshoot.

**33.**

**a.**

(1) $$C_{a1}(s) = \frac{1}{s^2+3s+36} = \frac{\frac{1}{\sqrt{33.75}}\sqrt{33.75}}{(s+1.5)^2+33.75} = \frac{0.17213\sqrt{33.75}}{(s+1.5)^2+33.75} = \frac{0.17213 \cdot 5.8095}{(s+1.5)^2+33.75}$$

Taking the inverse Laplace transform

$$C_{a1}(t) = 0.17213\, e^{-1.5t} \sin 5.8095t$$

(2)

$$C_{a2}(s)s = \frac{2}{s\left(s^2+3s+36\right)} = \frac{1}{18}\,\frac{1}{s} - \frac{\frac{1}{18}s + \frac{1}{6}}{s^2+3s+36} =$$

$$\frac{1}{18}\,\frac{1}{s} - \frac{\frac{1}{18}\left(s+\frac{3}{2}\right) + \frac{0.083333}{\sqrt{33.75}}\sqrt{33.75}}{\left(s+\frac{3}{2}\right)^2+33.75}$$

$$= 0.055556\,\frac{1}{s} - \frac{0.055556\left(s+\frac{3}{2}\right)+0.014344\sqrt{33.75}}{\left(s+\frac{3}{2}\right)^2+33.75}$$

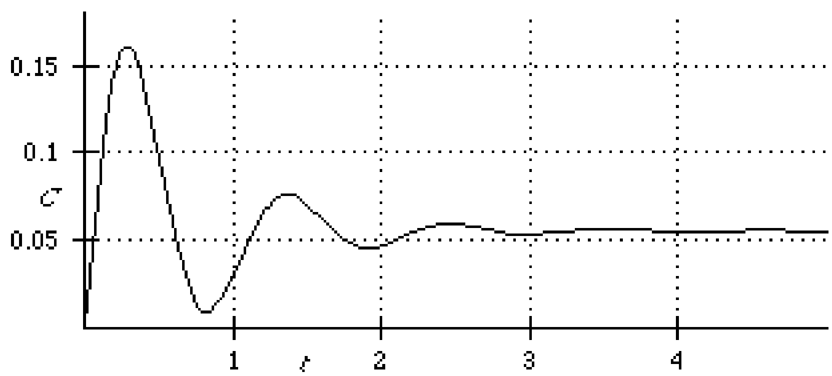
Taking the inverse Laplace transform

$$C_{a2}(t) = 0.055556 - e^{-1.5t}\,(0.055556 \cos 5.809t + 0.014344 \sin 5.809t)$$

The total response is found as follows:

$C_{at}(t) = C_{a1}(t) + C_{a2}(t) = 0.055556 - e^{-1.5t}\,(0.055556 \cos 5.809t - 0.157786 \sin 5.809t)$

Plotting the total response:




**b.**

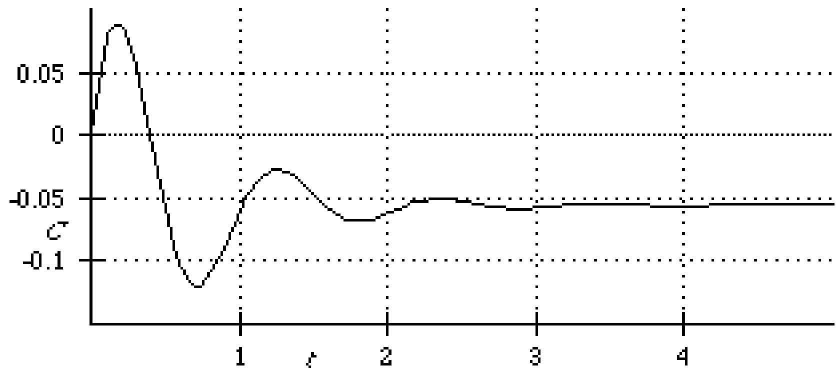
(1) Same as (1) from part (a), or $C_{b1}(t) = C_{a1}(t)$

(2) Same as the negative of (2) of part (a), or $C_{b2}(t) = -C_{a2}(t)$

The total response is

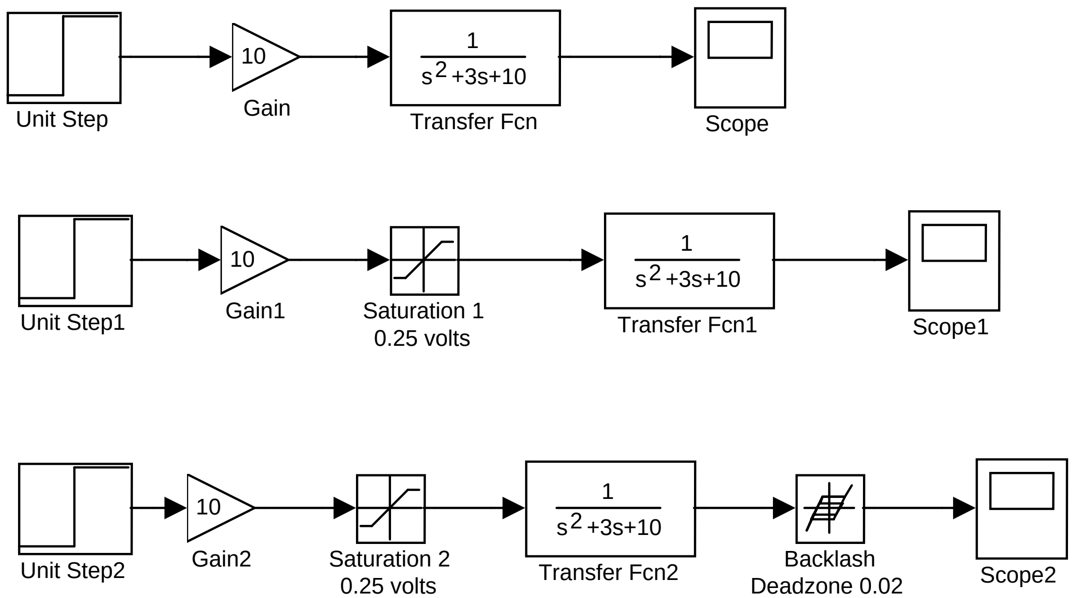
$C_{bt}(t) = C_{b1}(t) + C_{b2}(t) = C_{a1}(t) - C_{a2}(t) = -0.055556 + e^{-1.5t}\,(0.055556 \cos 5.809t + 0.186474 \sin$

$5.809t)$

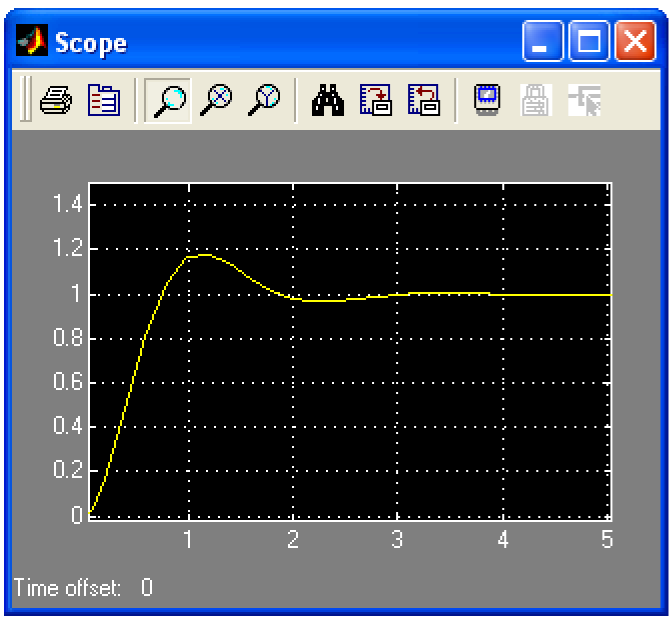



Notice the nonminimum phase behavior for $C_{bt}(t)$.

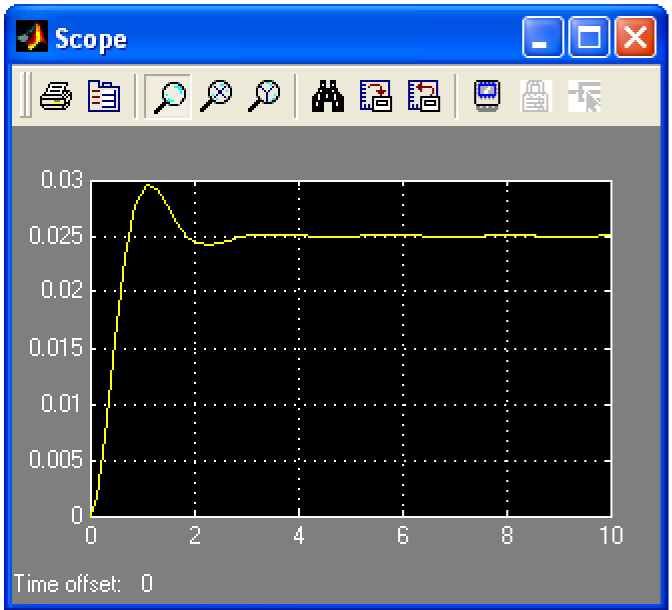
**34.**

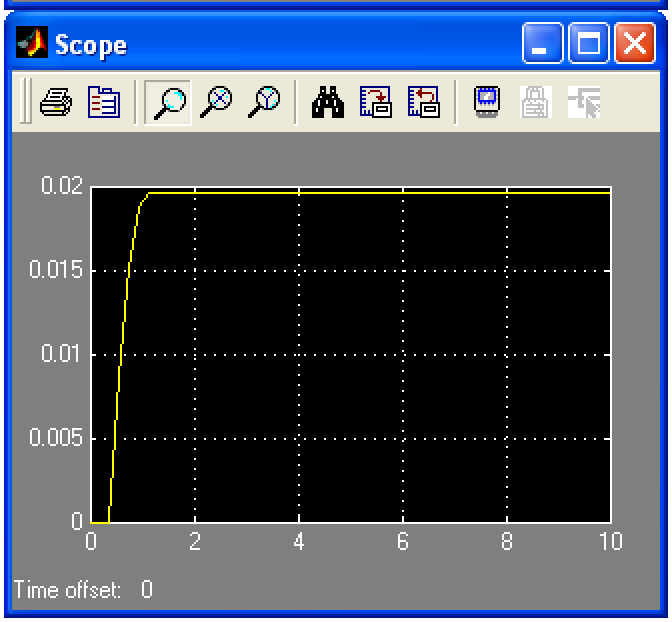


Unit Step
10
Gain
1
s2 +3s+10
Transfer Fcn
Scope
Unit Step1
10
Gain1
Saturation 1
0.25 volts
1
s2 +3s+10
Transfer Fcn1
Scope1
Unit Step2
10
Gain2
Saturation 2
0.25 volts
1
s2 +3s+10
Transfer Fcn2
Backlash
Deadzone 0.02
Scope2

Scope
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
Time offset: 0




**35.**

$$s\mathbf{I}-\mathbf{A}=s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}-2 & -1\\ -3 & -5\end{bmatrix}=\begin{bmatrix}(s+2) & 1\\ 3 & (s+5)\end{bmatrix}$$

$$|s\mathbf{I}-\mathbf{A}|=s^2+7s+7$$

Factoring yields poles at –5.7913 and –1.2087.

**36.**

**a.**

$$s\mathbf{I}-\mathbf{A}=s\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0&2&3\\0&6&5\\1&4&2\end{bmatrix}=\begin{bmatrix}s&-2&-3\\0&(s-6)&-5\\-1&-4&(s-2)\end{bmatrix}$$

$$|s\mathbf{I}-\mathbf{A}|=s^3-8s^2-11s+8$$

**b.** Factoring yields poles at 9.111, 0.5338, and –1.6448.

**37.**

$$\boldsymbol{x}=(s\boldsymbol{I}-\boldsymbol{A})^{-1}(\boldsymbol{x_0}+\boldsymbol{B}u)$$

$$\boldsymbol{X}=\left(s\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}1&2\\-3&-1\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}3\\1\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\cdot 3\frac{1}{s^2+9}\right)$$

$$\boldsymbol{X}=\begin{pmatrix}\dfrac{3s^3+5s^2+30s+54}{[s^2+5][s^2+9]}\\[2ex]\dfrac{s^3-10s^2+12s-102}{[s^2+5][s^2+9]}\end{pmatrix}$$

$$Y(s)=[1\quad 2]\boldsymbol{X}$$

$$Y(s)=\left(\frac{5s^3-15s^2+54s-150}{[s^2+9][s^2+5]}\right)$$

**38.**

$$\boldsymbol{x}=(s\boldsymbol{I}-\boldsymbol{A})^{-1}(\boldsymbol{x_0}+\boldsymbol{B}u)$$

$$\boldsymbol{x}=\left(s\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0&1&0\\-2&-4&1\\0&0&-6\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}\frac{1}{s+1}\right)$$

$$\boldsymbol{x}=\begin{pmatrix}\dfrac{1}{[s+6][s+1][s+0.58579][s+3.4142]}\\[2ex]\dfrac{s}{[s+6][s+1][s+0.58579][s+3.4142]}\\[2ex]\dfrac{s^2+4s+2}{[s+6][s+1][s+0.58579][s+3.4142]}\end{pmatrix}$$

$$Y(s)=[0\quad 0\quad 1]\mathbf{X}$$

$$Y(s)=\frac{s^2+4s+2}{[s+6][s+1][s+0.58579][s+3.4142]}$$

**39.**

$$\boldsymbol{x} = (s\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{x_0} + \boldsymbol{B}u)$$

$$\boldsymbol{X} = \left(s\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}-2 & 0\\-1 & -1\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}3\\0\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}\frac{1}{s}\right)$$

$$\boldsymbol{X} = \begin{pmatrix}\dfrac{3s+1}{s[s+2]}\\ \dfrac{1-2s}{s[s+1][s+2]}\end{pmatrix}$$

$$Y(s) = [0 \quad 1]\mathbf{X}$$

$$Y(s) = \left(\frac{1-2s}{s[s+1][s+2]}\right)$$

Applying partial fraction decomposition,

$$Y(s) = \left(\frac{1}{2}\frac{1}{s} - \frac{3}{s+1} + \frac{5}{2}\frac{1}{s+2}\right)$$

$$y(t) = \frac{1}{2}u(t) - 3e^{-t} + \frac{5}{2}e^{-2t}$$

**40.**

$$\mathbf{x} = (s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{x_0} + \mathbf{B}u)$$

$$\mathbf{x} = \left(s\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}-3 & 1 & 0\\0 & -6 & 1\\0 & 0 & -5\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix}\frac{1}{s}\right)$$

$$\mathbf{x} = \begin{bmatrix}\dfrac{1}{s(s+3)(s+5)}\\ \dfrac{1}{s(s+5)}\\ \dfrac{1}{s(s+5)}\end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix}\dfrac{1}{15} - \dfrac{1}{6}e^{-3t} + \dfrac{1}{10}e^{-5t}\\ \dfrac{1}{5} - \dfrac{1}{5}e^{-5t}\\ \dfrac{1}{5} - \dfrac{1}{5}e^{-5t}\end{bmatrix}$$

$$y(t) = \begin{bmatrix}0 & 1 & 1\end{bmatrix}\mathbf{x} = \frac{2}{5} - \frac{2}{5}e^{-5t}$$

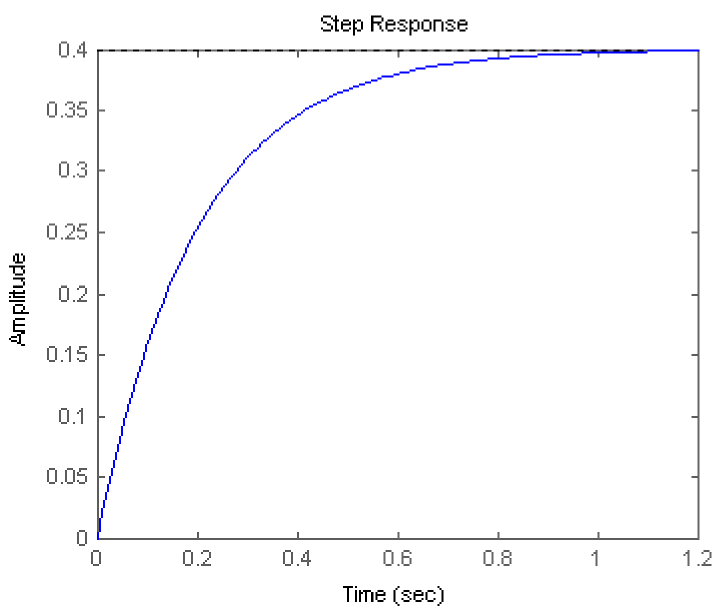
**41.**

**Program:**

```
A=[-3 1 0;0 -6 1;0 0 -5];
B=[0;1;1];
C=[0 1 1];
D=0;
S=ss(A,B,C,D)
step(S)
```

**Computer response:**

```
a =
        x1   x2   x3
   x1   -3    1    0
   x2    0   -6    1
   x3    0    0   -5




b =
        u1
   x1    0
   x2    1
   x3    1


c =
        x1   x2   x3
   y1    0    1    1


d =
        u1
   y1    0

Continuous-time model.
```



**42.**

**Program:**

```
syms s                          %Construct symbolic object for
                                %frequency variable 's'.
'a'                             %Display label
A=[-3 1 0;0 -6 1;0 0 -5]        %Create matrix A.
B=[0;1;1];                      %Create vector B.
C=[0 1 1];                      %Create C vector
X0=[1;1;0]                      %Create initial condition vector,X(0).
U=1/s;                          %Create U(s).
I=[1 0 0;0 1 0;0 0 1];          %Create identity matrix.
X=((s*I-A)^-1)*(X0+B*U);        %Find Laplace transform of state vector.
x1=ilaplace(X(1))               %Solve for X1(t).
x2=ilaplace(X(2))               %Solve for X2(t).
x3=ilaplace(X(3))               %Solve for X3(t).
y=C*[x1;x2;x3]                  %Solve for output, y(t).
y=simplify(y)                   %Simplify y(t).
'y(t)'                          %Display label.
pretty(y)                       %Pretty print y(t).
```

**Computer response:**

```
ans =

a

A =
    -3     1     0
     0    -6     1
     0     0    -5

X0 =
     1
     1
     0
x1 =

7/6*exp(-3*t)-1/3*exp(-6*t)+1/15+1/10*exp(-5*t)

x2 =

exp(-6*t)+1/5-1/5*exp(-5*t)

x3 =

1/5-1/5*exp(-5*t)

y =

2/5+exp(-6*t)-2/5*exp(-5*t)

y =

2/5+exp(-6*t)-2/5*exp(-5*t)

ans =

y(t)
                        2/5 + exp(-6 t) - 2/5 exp(-5 t)
```

**43.**

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^2 + 5\lambda + 1$$

$$|\lambda \mathbf{I} - \mathbf{A}| = (\lambda + 0.20871)\,(\lambda + 4.7913)$$

$$P = -0.20871$$

$$G = -4.7913$$
$$\Phi = \begin{pmatrix} A_1 e^{-0.20871t} + A_2 e^{-4.7913t} & A_5 e^{-0.20871t} + A_6 e^{-4.7913t} \\ A_3 e^{-0.20871t} + A_4 e^{-4.7913t} & A_7 e^{-0.20871t} + A_8 e^{-4.7913t} \end{pmatrix}$$
$$\Phi_0 = \begin{pmatrix} A_2 + A_1 & A_6 + A_5 \\ A_4 + A_3 & A_8 + A_7 \end{pmatrix}$$

$$\frac{\partial}{\partial t}\Phi = \begin{pmatrix} -0.20871 A_1 e^{-0.20871t} - 4.7913 A_2 e^{-4.7913t} & -0.20871 A_5 e^{-0.20871t} - 4.7913 A_6 e^{-4.7913t} \\ -0.20871 A_3 e^{-0.20871t} - 4.7913 A_4 e^{-4.7913t} & -0.20871 A_7 e^{-0.20871t} - 4.7913 A_8 e^{-4.7913t} \end{pmatrix}$$
$$\frac{d}{\partial t}\Phi_0 = \begin{pmatrix} -4.7913 A_2 - 0.20871 A_1 & -4.7913 A_6 - 0.20871 A_5 \\ -4.7913 A_4 - 0.20871 A_3 & -4.7913 A_8 - 0.20871 A_7 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} A_2 + A_1 & A_6 + A_5 \\ A_4 + A_3 & A_8 + A_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} -4.7913 A_2 - 0.20871 A_1 & -4.7913 A_6 - 0.20871 A_5 \\ -4.7913 A_4 - 0.20871 A_3 & -4.7913 A_8 - 0.20871 A_7 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -5 \end{pmatrix}$$

Solving for $A_i$'s two at a time, and substituting into the state-transition matrix

$$\Phi = \begin{pmatrix} 1.0455 e^{-0.20871t} - 0.045545 e^{-4.7913t} & 0.21822 e^{-0.20871t} - 0.21822 e^{-4.7913t} \\ -0.21822 e^{-0.20871t} + 0.21822 e^{-4.7913t} & -0.045545 e^{-0.20871t} + 1.0455 e^{-4.7913t} \end{pmatrix}$$

To find **x**(t),

$$x = \Phi x_0$$
$$x = \begin{pmatrix} 1.0455 e^{-0.20871t} - 0.045545 e^{-4.7913t} & 0.21822 e^{-0.20871t} - 0.21822 e^{-4.7913t} \\ -0.21822 e^{-0.20871t} + 0.21822 e^{-4.7913t} & -0.045545 e^{-0.20871t} + 1.0455 e^{-4.7913t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$x = \begin{pmatrix} 1.0455 e^{-0.20871t} - 0.045545 e^{-4.7913t} \\ -0.21822 e^{-0.20871t} + 0.21822 e^{-4.7913t} \end{pmatrix}$$

To find the output,

$$y = (1, 2)\,x$$
$$y = (1, 2)\begin{pmatrix} 1.0455 e^{-0.20871t} - 0.045545 e^{-4.7913t} \\ -0.21822 e^{-0.20871t} + 0.21822 e^{-4.7913t} \end{pmatrix}$$
$$y = \left(0.60911 e^{-0.20871t} + 0.39089 e^{-4.7913t}\right)$$

**44.**

$$\lambda I - A = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$$

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^2 + 1$$
$$\Phi = \begin{pmatrix} A_1 \cos[t] + A_2 \sin[t] & A_5 \cos[t] + A_6 \sin[t] \\ A_3 \cos[t] + A_4 \sin[t] & A_7 \cos[t] + A_8 \sin[t] \end{pmatrix}$$

$$\frac{d}{\partial t}\Phi = \begin{pmatrix} A_2\cos[t] - A_1\sin[t] & A_6\cos[t] - A_5\sin[t] \\ A_4\cos[t] - A_3\sin[t] & A_8\cos[t] - A_7\sin[t] \end{pmatrix}$$

$$\Phi_0 = \begin{pmatrix} A_1 & A_5 \\ A_3 & A_7 \end{pmatrix}$$

$$\frac{d}{\partial t}\Phi_0 = \begin{pmatrix} A_2 & A_6 \\ A_4 & A_8 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & A_5 \\ A_3 & A_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A_2 & A_6 \\ A_4 & A_8 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Solving for the $A_i$'s and substituting into the state-transition matrix,

$$\Phi = \begin{pmatrix} \cos[t] & \sin[t] \\ -\sin[t] & \cos[t] \end{pmatrix}$$

To find the state vector,

$$x = \int_0^t (\Phi[t-\tau]\,B\,u[\tau])\,d\tau$$

$$x = \int_0^t \left(\begin{bmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) d\tau$$

$$x = \int_0^t \begin{pmatrix} \sin[t-\tau] \\ \cos[t-\tau] \end{pmatrix} d\tau$$

$$t - \tau = \theta$$

$$x = \int_t^0 \begin{pmatrix} -\sin[\theta] \\ -\cos[\theta] \end{pmatrix} d\theta$$

$$x = \begin{pmatrix} 1-\cos[t] \\ \sin[t] \end{pmatrix}$$

$$y = (3,4)x$$

$$\Delta y = (3,4)\begin{pmatrix} 1-\cos[t] \\ \sin[t] \end{pmatrix}$$

$$y = (-3\cos[t] + 4\sin[t] + 3)$$

**45.**

$$|\lambda \mathbf{I} - \mathbf{A}| = (\lambda + 2)(\lambda + 0.5 - 2.3979i)(\lambda + 0.5 + 2.3979i)$$

Let the state-transition matrix be

$$\phi = \begin{bmatrix} A_1 e^{-.5t}\cos(2.3979t) + A_2 e^{-.5t}\sin(2.3979t) + A_3 e^{-2t} & A_{10} e^{-.5t}\cos(2.3979t) + A_{11} e^{-.5t}\sin(2.3979t) + A_{12} & \bullet \\ A_4 e^{-.5t}\cos(2.3979t) + A_5 e^{-.5t}\sin(2.3979t) + A_6 e^{-2t} & A_{13} e^{-.5t}\cos(2.3979t) + A_{14} e^{-.5t}\sin(2.3979t) + A_{15} & \bullet \\ A_7 e^{-.5t}\cos(2.3979t) + A_8 e^{-.5t}\sin(2.3979t) + A_9 e^{-2t} & A_{16} e^{-.5t}\cos(2.3979t) + A_{17} e^{-.5t}\sin(2.3979t) + A_{18} & \bullet \end{bmatrix}$$

Since $\phi(0) = \mathbf{I}$, $\dot{\Phi}(0) = \mathbf{A}$, and $\ddot{\phi}(0) = \mathbf{A}^2$, we can evaluate the coefficients, $A_i$'s. Thus,

$$\begin{pmatrix} A_3 + A_1 & A_{12} + A_{10} & A_{21} + A_{19} \\ A_6 + A_4 & A_{15} + A_{13} & A_{24} + A_{22} \\ A_9 + A_7 & A_{18} + A_{16} & A_{27} + A_{25} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -2A_3 + 2.3979A_2 - 0.5A_1 & -2A_{12} + 2.3979A_{11} - 0.5A_{10} & -2A_{21} + 2.3979A_{20} - 0.5A_{19} \\ -2A_6 + 2.3979A_5 - 0.5A_4 & -2A_{15} + 2.3979A_{14} - 0.5A_{13} & -2A_{24} + 2.3979A_{23} - 0.5A_{22} \\ -2A_9 + 2.3979A_8 - 0.5A_7 & -2A_{18} + 2.3979A_{17} - 0.5A_{16} & -2A_{27} + 2.3979A_{26} - 0.5A_{25} \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 4A_3 - 2.3979A_2 - 5.4999A_1 & 4A_{12} - 2.3979A_{11} - 5.4999A_{10} & 4A_{21} - 2.3979A_{20} - 5.4999A_{19} \\ 4A_6 - 2.3979A_5 - 5.4999A_4 & 4A_{15} - 2.3979A_{14} - 5.4999A_{13} & 4A_{24} - 2.3979A_{23} - 5.4999A_{22} \\ 4A_9 - 2.3979A_8 - 5.4999A_7 & 4A_{18} - 2.3979A_{17} - 5.4999A_{16} & 4A_{27} - 2.3979A_{26} - 5.4999A_{25} \end{pmatrix} = \begin{pmatrix} 4 & -2 & 1 \\ 0 & -6 & -1 \\ 0 & 6 & -5 \end{pmatrix}$$

Solving for the $A_i$'s taking three equations at a time,

$$\Phi = \begin{pmatrix} e^{-2t} & 0.125e^{-0.5t}\cos[2.3979t] + 0.33884e^{-0.5t}\sin[2.3979t] - 0.125e^{-2t} & -0.125e^{-0.5t}\cos[2.3979t] + 0.078194e^{-0.5t}\sin[2.3979t] + 0.125e^{-2t} \\ 0 & e^{-0.5t}\cos[2.3979t] + 0.20851e^{-0.5t}\sin[2.3979t] & 0.41703e^{-0.5t}\sin[2.3979t] \\ 0 & -2.502e^{-0.5t}\sin[2.3979t] & e^{-0.5t}\cos[2.3979t] - 0.20852e^{-0.5t}\sin[2.3979t] \end{pmatrix}$$

Using $\mathbf{x}(t) = \phi(t)\mathbf{x}(0) + \int_0^t \phi(t-\tau)\mathbf{B}\,\mathbf{u}(\tau)d\tau$, and $y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}(t)$,

$$y = \int_0^t e^{-2(t-\tau)}\,d\tau$$

$$= \frac{1}{2} - \frac{1}{2}e^{-2t}$$

**46.**

**Program:**

```
syms s t tau                    %Construct symbolic object for
                                %frequency variable 's', 't', and 'tau.
'a'                             %Display label.
A=[-2 1 0;0 0 1;0 -6 -1]        %Create matrix A.
B=[1;0;0]                       %Create vector B.
C=[1 0 0]                       %Create vector C.
X0=[1;1;0]                      %Create initial condition vector,X(0).
I=[1 0 0;0 1 0;0 0 1];          %Create identity matrix.
'E=(s*I-A)^-1'                  %Display label.
E=((s*I-A)^-1)                  %Find Laplace transform of state
                                %transition matrix, (sI-A)^-1.
Fi11=ilaplace(E(1,1));          %Take inverse Laplace transform
Fi12=ilaplace(E(1,2));          %of each element
Fi13=ilaplace(E(1,3));
Fi21=ilaplace(E(2,1));
Fi22=ilaplace(E(2,2));
Fi23=ilaplace(E(2,3));
```

```
Fi31=ilaplace(E(3,1));
Fi32=ilaplace(E(3,2));       %to find state transition matrix.
Fi33=ilaplace(E(3,3));       %of (sI-A)^-1.
'Fi(t)'                      %Display label.
Fi=[Fi11 Fi12 Fi13           %Form Fi(t).
Fi21 Fi22 Fi23
Fi31 Fi32 Fi33];
pretty(Fi)                   %Pretty print state transition matrix, Fi.
Fitmtau=subs(Fi,t,t-tau);    %Form Fi(t-tau).
'Fi(t-tau)'                  %Display label.
pretty(Fitmtau)              %Pretty print Fi(t-tau).
x=Fi*X0+int(Fitmtau*B*1,tau,0,t);
                             %Solve for x(t).
x=simple(x);                 %Collect terms.
x=simplify(x);               %Simplify x(t).
x=vpa(x,3);
'x(t)'                       %Display label.
pretty(x)                    %Pretty print x(t).
y=C*x;                       %Find y(t)
y=simplify(y);
y=vpa(simple(y),3);
y=collect(y);
'y(t)'
pretty(y)                    %Pretty print y(t).
```

**Computer response:**

```
ans =

a



A =

    -2     1     0
     0     0     1
     0    -6    -1

B =

     1
     0
     0

C =

     1     0     0

X0 =

     1
     1
     0

ans =

E=(s*I-A)^-1

E =

[               1/(s+2), (s+1)/(s+2)/(s^2+s+6),     1/(s+2)/(s^2+s+6)]
[                     0,       (s+1)/(s^2+s+6),           1/(s^2+s+6)]
[                     0,          -6/(s^2+s+6),           s/(s^2+s+6)]
```

```
ans =

Fi(t)

       [                                       13
       [exp(-2 t) , - 1/8 exp(-2 t) + 1/8 %1 + --- %2 ,
       [                                       184

                                        ]
       1/8 exp(-2 t) - 1/8 %1 + 3/184 %2]
                                        ]

       [
       [0 , 1/23 %2 + %1 , - 1/23

            1/2                       1/2                            1/2
       (-23)    (exp((-1/2 + 1/2 (-23)   ) t) - exp((-1/2 - 1/2 (-23)   ) t))

       ]
       ]

       [
       [0 , 6/23

            1/2                       1/2                            1/2
       (-23)    (exp((-1/2 + 1/2 (-23)   ) t) - exp((-1/2 - 1/2 (-23)   ) t))

                         ]
        , - 1/23 %2 + %1]

                                     1/2
  %1 := exp(- 1/2 t) cos(1/2 23    t)

                              1/2          1/2
  %2 := exp(- 1/2 t) 23    sin(1/2 23    t)

ans =

Fi(t-tau)


        [
        [exp(-2 t + 2 tau) ,
        [

                                                      13       1/2
        - 1/8 exp(-2 t + 2 tau) + 1/8 %2 cos(%1) + --- %2 23    sin(%1) ,
                                                      184

                                                             1/2        ]
        1/8 exp(-2 t + 2 tau) - 1/8 %2 cos(%1) + 3/184 %2 23    sin(%1)]
                                                                        ]

        [              1/2                                     1/2
        [0 , 1/23 %2 23    sin(%1) + %2 cos(%1) , - 1/23 (-23)    (

                           1/2
        exp((-1/2 + 1/2 (-23)   ) (t - tau))

                              1/2              ]
         - exp((-1/2 - 1/2 (-23)   ) (t - tau)))]

        [              1/2                        1/2
        [0 , 6/23 (-23)    (exp((-1/2 + 1/2 (-23)   ) (t - tau))

                              1/2
         - exp((-1/2 - 1/2 (-23)   ) (t - tau))) ,
```

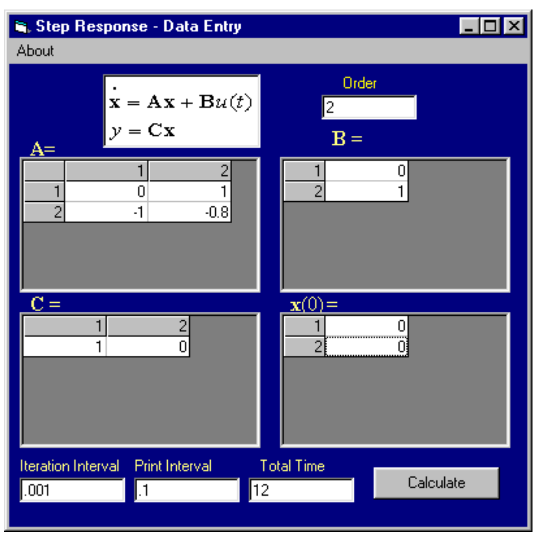
```
                  1/2                          ]
        - 1/23 %2 23     sin(%1) + %2 cos(%1)]

                   1/2
  %1 := 1/2 23     (t - tau)

  %2 := exp(- 1/2 t + 1/2 tau)

ans =

x(t)

        [.375 exp(-2. t) + .125 exp(-.500 t) cos(2.40 t)

         + .339 exp(-.500 t) sin(2.40 t) + .500]

        [.209 exp(-.500 t) sin(2.40 t) + exp(-.500 t) cos(2.40 t)]

        [1.25 i (exp((-.500 + 2.40 i) t) - 1. exp((-.500 - 2.40 i) t))]

ans =

y(t)

  .375 exp(-2. t) + .125 exp(-.500 t) cos(2.40 t)

         + .339 exp(-.500 t) sin(2.40 t) + .500
```

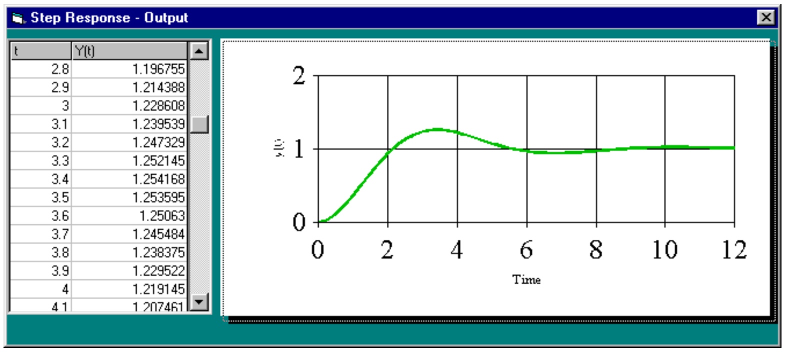
**47.**

The state-space representation used to obtain the plot is,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -0.8 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t); \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

Using the Step Response software,



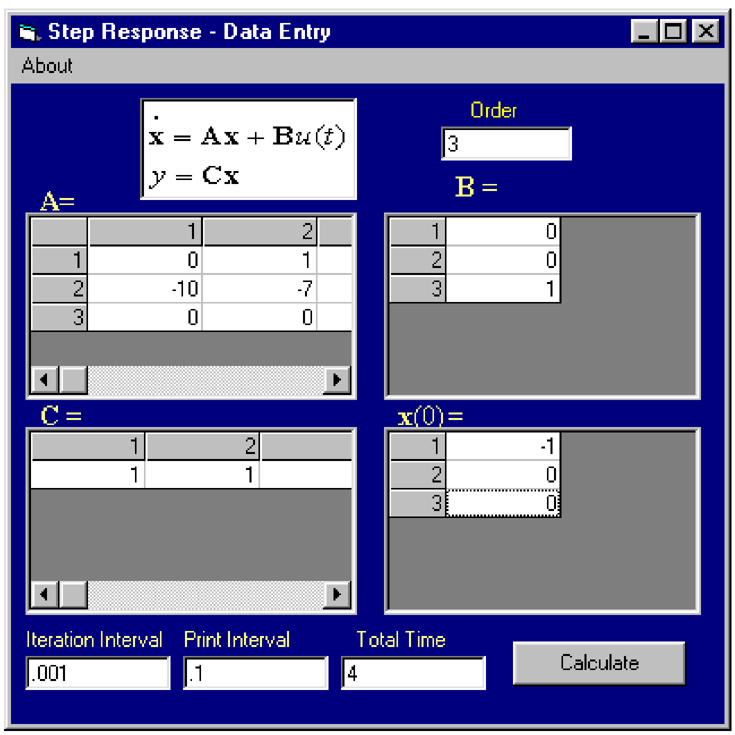


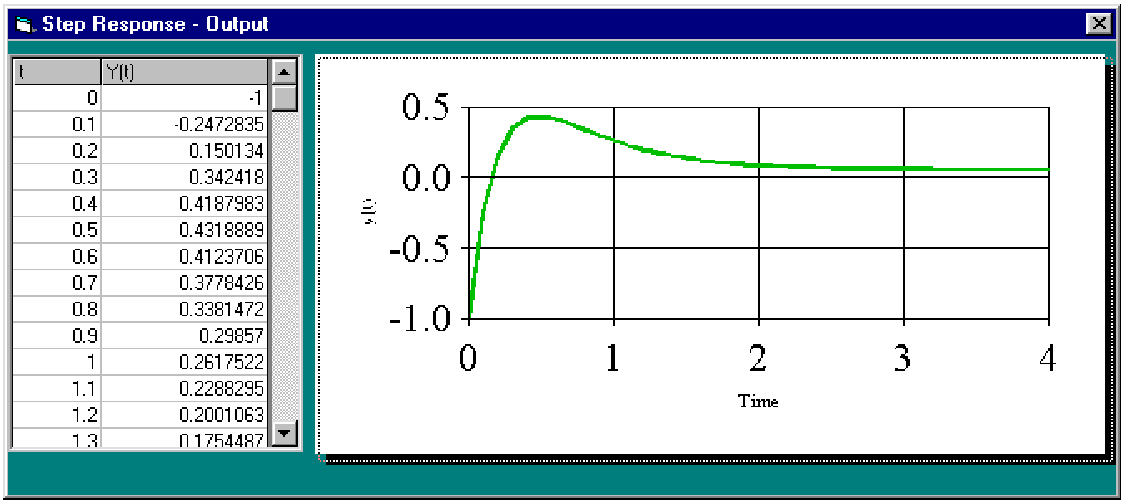

Calculating % overshoot, settling time, and peak time,

$2\zeta\omega_n = 0.8$, $\omega_n = 1$, $\zeta = 0.4$. Therefore, $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}}\text{x}100 = 25.38\%$, $T_s = \dfrac{4}{\zeta\omega_n} = 10$ sec,

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 3.43 \text{ sec.}$$

**48.**

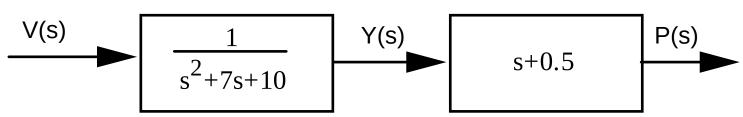





**49.**

**a.** $P(s) = \dfrac{s+0.5}{s(s+2)(s+5)} = \dfrac{1/20}{s} + \dfrac{1/4}{s+2} - \dfrac{3/10}{s+5}$. Therefore, $p(t) = \dfrac{1}{20} + \dfrac{1}{4}e^{-2t} - \dfrac{3}{10}e^{-5t}$.

**b.** To represent the system in state space, draw the following block diagram.




For the first block,

$$\ddot{y} + 7\dot{y} + 10y = v(t)$$

Let $x_1 = y$, and $x_2 = \dot{y}$. Therefore,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -10x_1 - 7x_2 + v(t)$$

Also,

$$p(t) = 0.5y + \dot{y} = 0.5x_1 + x_2$$

Thus,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1; \quad p(t) = \begin{bmatrix} 0.5 & 1 \end{bmatrix}\mathbf{x}$$

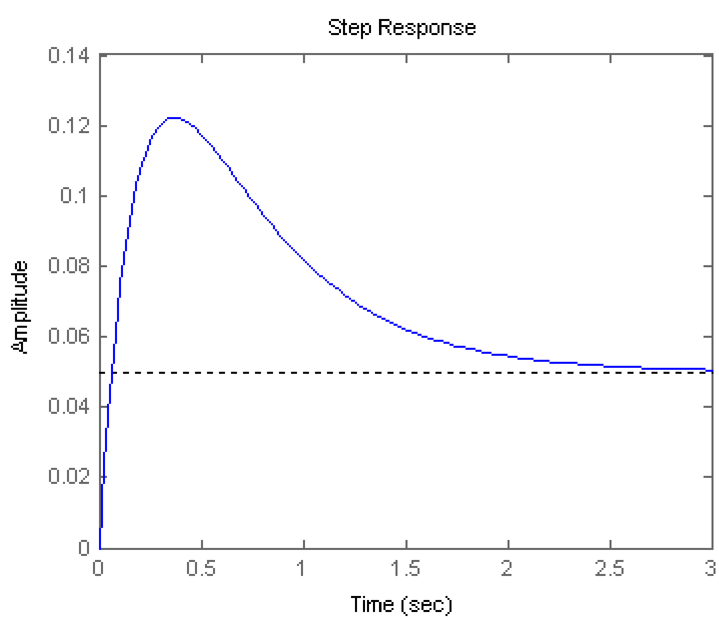
**c.**

**Program:**

```
A=[0 1;-10 -7];
B=[0;1];
C=[.5 1];
D=0;
S=ss(A,B,C,D)
step(S)
```

**Computer response:**

```
a =
          x1    x2
    x1     0     1
    x2   -10    -7


b =
        u1
    x1   0
    x2   1


c =
          x1    x2
    y1   0.5     1


d =
        u1
    y1   0

Continuous-time model.
```




**50.**

**a.** $\omega_n = \sqrt{10} = 3.16$; $2\zeta\omega_n = 4$. Therefore $\zeta = 0.632$. $\%OS = e^{-\xi\pi/\sqrt{1-\xi^2}} * 100 = 7.69\%$.

$T_s = \dfrac{4}{\xi\omega_n} = 2$ seconds. $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}} = 1.28$ seconds. From Figure 4.16, $T_r\omega_n = 1.93$.

Thus, $T_r = 0.611$ second. To justify second-order assumption, we see that the dominant poles are at –2 ± j2.449. The third pole is at -10, or 5 times further. The second-order approximation is valid.

**b.** $G_e(s) = \frac{K}{(s+10)(s^2+4s+10)} = \frac{K}{s^3+14s^2+50s+100}$ . Representing the system in phase-variable form:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -100 & -50 & -14 \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix};\ \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

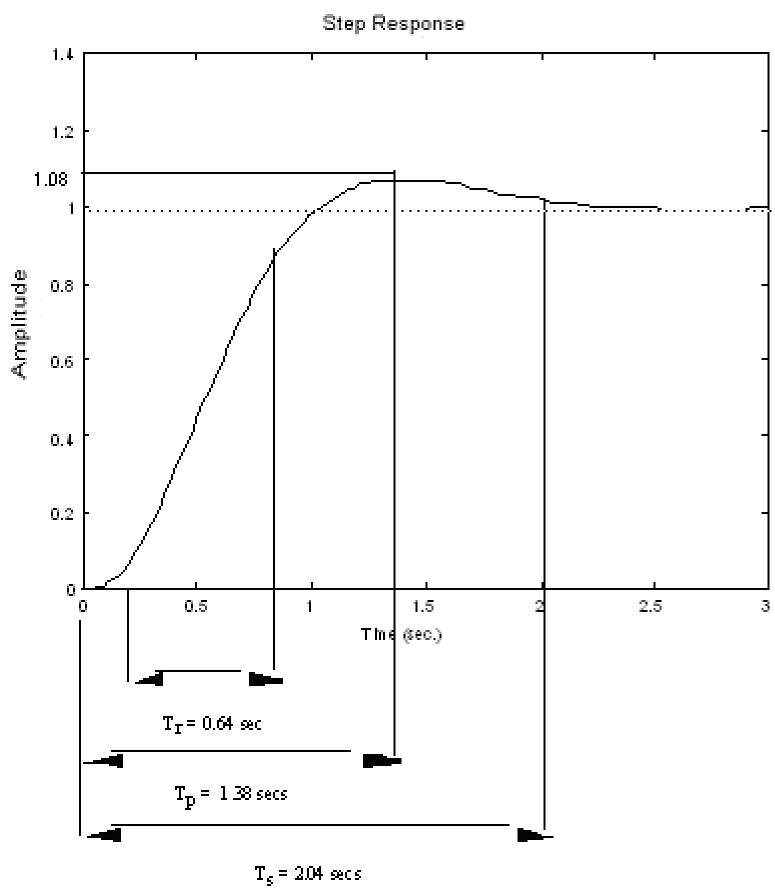
**c.**

**Program:**

```
numg=100;
deng=conv([1 10],[1 4 10]);
G=tf(numg,deng)
step(G)
```

**Computer response:**

```
Transfer function:
          100
-------------------------
s^3 + 14 s^2 + 50 s + 100
```




$$\%OS = \frac{(1.08-1)}{1} * 100 = 8\%$$

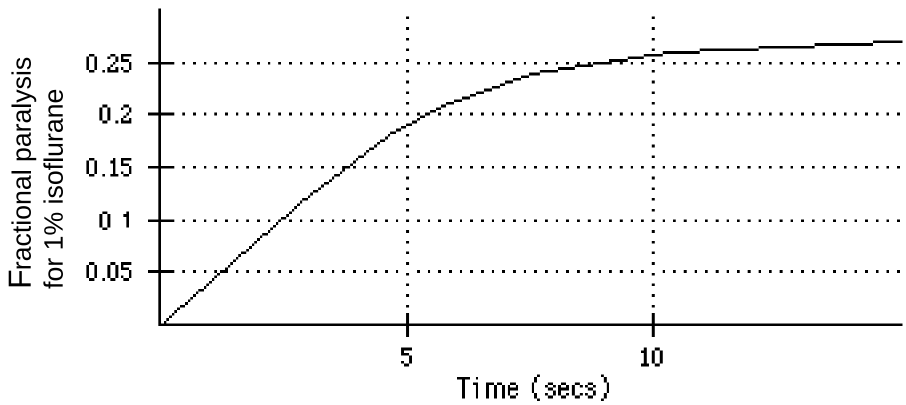
**51.**

**a.** $\omega_n = \sqrt{0.28} = 0.529$; $2\zeta\omega_n = 1.15$. Therefore $\zeta = 1.087$.

**b.** $P(s) = U(s)\frac{7.63\text{x}10^{-2}}{s^2+1.15s+0.28}$, where $U(s) = \frac{2}{s}$. Expanding by partial fractions, $P(s) = \frac{0.545}{s}$ + natural response terms. Thus percent paralysis = 54.5%.

**c.** $P(s) = \frac{7.63\text{x}10^{-2}}{s(s^2+1.15s+0.28)} = \frac{0.2725}{s} - \frac{0.48444}{s+0.35} + \frac{0.21194}{s+0.8}$.

Hence, $p(t) = 0.2725 - 0.48444e^{-0.35t} + 0.21194e^{-0.8t}$. Plotting,




**d.** $P(s) = \frac{K}{s} * \frac{7.63\text{x}10^{-2}}{s^2+1.15s+0.28} = \frac{1}{s}$ + natural response terms. Therefore, $\frac{7.63\text{x}10^{-2}\,K}{0.28} = 1$. Solving for K, K = 3.67%.

**52.**

**a.** Writing the differential equation,

$$\frac{dc(t)}{dt} = -k_{10}c(t) + \frac{i(t)}{V_d}$$

Taking the Laplace transform and rearranging,

$$(s+k_{10})C(s) = \frac{I(s)}{V_d}$$

from which the transfer function is found to be

$$\frac{C(s)}{I(s)} = \frac{\frac{1}{V_d}}{s+k_{10}}$$

For a step input, $I(s) = \frac{I_0}{s}$. Thus the response is

$$C(s) = \frac{\frac{I_0}{V_d}}{s(s+k_{10})} = \frac{I_0}{k_{10}V_d}\left(\frac{1}{s} - \frac{1}{s+k_{10}}\right)$$

Taking the inverse Laplace transform,

$$c(t) = \frac{I_0}{k_{10}V_d}(1 - e^{-k_{10}t})$$

where the steady-state value, $C_D$, is

$$C_D = \frac{I_0}{k_{10}V_d}$$

Solving for $I_R = I_0$,

$$I_R = C_D k_{10} V_d$$

**b.** $T_r = \frac{2.2}{k_{10}}$ ; $T_s = \frac{4}{k_{10}}$

**c.** $I_R = C_D k_{10} V_d = 12 \frac{\mu g}{ml}$ x 0.07 $hr^{-1}$ x 0.6 liters = 0.504 $\frac{mg}{h}$

**d.** Using the equations of part b, where $k_{10} = 0.07$, $T_r = 31.43$ hrs, and $T_s = 57.14$ hrs.

**53.** Consider the un-shifted Laplace transform of the output

$$Y(s) = \frac{2.5(1+0.172s)(1+0.008s)}{s(1+0.07s)^2(1+0.05s)^2} = \frac{280.82(s+5.814)(s+125)}{s(s+14.286)^2(s+20)^2}$$

$$= \frac{A}{s} + \frac{B}{(s+14.286)^2} + \frac{C}{(s+14.286)} + \frac{D}{(s+20)^2} + \frac{E}{(s+20)}$$

$$A = \frac{280.82(s+5.814)(s+125)}{(s+14.286)^2(s+20)^2}\Big|_{s=0} = 2.5$$

$$B = \frac{280.82(s+5.814)(s+125)}{s(s+20)^2}\Big|_{s=-14.286} = 564.7$$

$$C = \frac{d}{ds}\frac{280.82(s+5.814)(s+125)}{s(s+20)^2}\Big|_{s=-14.286} = \frac{d}{ds}\frac{280.82s^2+36735.2s+204085.94}{s^3+40s^2+400s}\Big|_{s=-14.286}$$

$$= \frac{(s^3+40s^2+400s)(561.64s+36735.2)-(280.82s^2+36735.2s+204085.94)(3s^2+80s+400)}{(s^3+40s^2+400s)^2}\Big|_{s=-}$$

$$= -219.7$$

$$D = \frac{280.82(s+5.814)(s+125)}{s(s+14.286)^2}\Big|_{s=-20} = 640.57$$

$$E = \frac{d}{ds}\frac{280.82(s+5.814)(s+125)}{s(s+14.286)^2}\Big|_{s=-20} = \frac{d}{ds}\frac{280.82s^2+36735.2s+204085.94}{s^3+28.572s^2+204.09s}\Big|_{s=-20}$$

$$= \frac{(s^3+28.572s^2+204.09s)(561.64s+36735.2)-(280.82s^2+36735.2s+204085.94)(3s^2+57.144s+204.09)}{(s^3+28.572s^2+204.09s)^2}\Big|_{s=-20}$$

$$= 217.18$$

thus

$$Y(s) = \frac{2.5}{s} + \frac{564.7}{(s+14.286)^2} - \frac{219.7}{(s+14.286)} + \frac{640.57}{(s+20)^2} + \frac{217.18}{(s+20)}$$

Obtaining the inverse Laplace transform of the latter and delaying the equation in time domain we get

$$y(t) = [2.5 + 564.7(t-0.008)e^{-14.286(t-0.008)} - 219.7e^{-14.286(t-0.008)} + 640.57(t-0.008)e^{-20(t-0.008)}$$
$$+ 217.18e^{-20(t-0.008)}]u(t-0.008)$$

**54.**

**a.** The transfer function can be written as

$$\frac{\theta}{I}(s) = \frac{2.5056(s+3.33)e^{-0.1s}}{(s+1)(s^2+0.72s+1.44)}$$

It has poles at s=-0.36±j1.145 and s=-1. A zero at s=-3.33

The ‘far away’ pole at -1 is relatively close to the complex conjugate poles as 0.36*5>1 so a dominant pole approximation can’t be applied.

b) In time domain the input can be expressed as:

$$i(t) = 250\mu A(u(t) - u(t-0.15))$$

Obtaining Laplace transforms this can be expressed as

$$I(s) = 250\mu \frac{1-e^{-0.15s}}{s}$$

We first obtain the response to an unshifted unit step:

$$\theta(s) = \frac{2.5056(s+3.33)}{s(s+1)(s^2+0.72s+1.44)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+0.72s+1.44}$$

$$A = \frac{2.5056(s+3.33)}{(s+1)(s^2+0.72s+1.44)}\bigg|_{s=0} = 5.8$$

$$B = \frac{2.5056(s+3.33)}{s(s^2+0.72s+1.44)}\bigg|_{s=-1} = \frac{2.5056(2.33)}{(-1)(1.72)} = -3.4$$

We will get C and D by equating coefficients. Substituting these two values and multiplying both sides by the denominator we get.

$$2.5056(s+3.33) = 5.8(s+1)(s^2+0.72s+1.44) - 3.4s(s^2+0.72s+1.44) + (Cs+D)s(s+1)$$

$$\begin{aligned}2.5056(s+3.33) = {} & 5.8(s^3+1.72s^2+2.16s+1.44) - 3.4(s^3+0.72s^2+1.44s) \\ & + (Cs^3+(C+D)s^2+Ds)\end{aligned}$$

$$2.5056(s+3.33) = (2.4+C)s^3 + (7.528+C+D)s^2 + (10.632+D)s + 8.352$$

We immediately get C=-2.4 and D=-5.128

So

$$\theta(s) = \frac{5.8}{s} - \frac{3.4}{s+1} - \frac{2.4s+5.128}{s^2+0.72s+1.44} = \frac{5.8}{s} - \frac{3.4}{s+1} - \frac{2.4s+5.128}{(s+0.36)^2+1.3104}$$

$$= \frac{5.8}{s} - \frac{3.4}{s+1} - \frac{2.4(s+0.15)+4.768}{(s+0.36)^2+1.3104} = \frac{5.8}{s} - \frac{3.4}{s+1} - \frac{2.4(s+0.15)}{(s+0.36)^2+1.3104}$$
$$-4.164\frac{1.145}{(s+0.36)^2+1.3104}$$

Obtaining inverse Laplace transform we get

$$\theta(t) = 5.8 - 3.4e^{-t} - 2.4e^{-0.36t}\cos(1.145t) - 4.164e^{-0.36t}\sin(1.145t)$$
$$= 5.8 - 3.4e^{-t} - 2.4e^{-0.36t}\sin(1.145t + 30^\circ)$$

So the actual (shifted) unit step response is given by

$$\theta(t) = [5.8 - 3.4e^{-(t-0.1)} - 2.4e^{-0.36(t-0.1)}\sin(1.145(t-0.1)+30^\circ)]u(t-0.1)$$

The response to the pulse is given by:

$$\theta(t) = [1.45m - 0.85me^{-(t-0.1)} - 0.6me^{-0.36(t-0.1)}\sin(1.145(t-0.1)+30^\circ)]u(t-0.1) -$$
$$[1.45m - 0.85me^{-(t-0.25)} - 0.6me^{-0.36(t-0.25)}\sin(1.145(t-0.25)+30^\circ)]u(t-0.25)$$

**55.**

At steady state the input is ≈ 9V and the output is ≈ 6V Thus G(0)=6/9=0.667

The maximum peak is achieved at ≈ 285μ with a %OS = (7.5/6-1)*100 = 25%

This corresponds to a damping factor of

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = \frac{1.3863}{\sqrt{\pi^2 + 1.9218}} \approx 0.4$$
$$\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}} = \frac{\pi}{(285\mu)(0.9165)} = 12027.2$$

So the approximated transfer function is

$$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{0.667*12027.2^2}{s^2+2*0.4*12027.2s+12027.2^2}=\frac{96.5*10^6}{s^2+9622s+14.5*10^7}$$

**56.**

The oscillation period is

$\frac{2\pi}{T}=\omega_n\sqrt{1-\zeta^2}$ and from the figure $\frac{T}{2}=0.0675s-0.0506s=0.0169s$

Thus T=0.0338sec from which we get $\omega_n\sqrt{1-\zeta^2}=185.8931$

The peaks of the response occur when the 'cos' term of the step response is ±1 thus from the figure we have:

$1+\frac{e^{-\zeta\omega_n(0.0506)}}{\sqrt{1-\zeta^2}}=1.1492$ and $1-\frac{e^{-\zeta\omega_n(0.0675)}}{\sqrt{1-\zeta^2}}=0.9215$

From which we get

$\frac{e^{-\zeta\omega_n(0.0506)}}{e^{-\zeta\omega_n(0.0675)}}=\frac{0.1492}{0.0785}=1.9006$ or $e^{-\zeta\omega_n(0.0169)}=1.9006$ or $\zeta\omega_n=38$

Substituting this result we get $\omega_n\sqrt{1-\zeta^2}=\frac{38}{\zeta}\sqrt{1-\zeta^2}=185.8931$

or $\frac{1444}{\zeta^2}(1-\zeta^2)=34556.2284$ or $\zeta^2=0.0436$ or $\zeta=0.21$

Finally $\omega_n=\frac{38}{\zeta}=180.9$

**57.**

The step input amplitude is the same for both responses so it will just be assumed to be unitary.

For the 'control' response we have:

$c_{final}=0.018$, $M_{pt}=0.024$ from which we get

$$\%OS=\frac{c_{\max}-c_{final}}{c_{final}}\times 100\%=\frac{0.024-0.018}{0.018}\times 100\%=33.33\%$$

$$\zeta=\frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}=\frac{-\ln(0.333)}{\sqrt{\pi^2+\ln^2(0.333)}}=0.33$$

$$\omega_n=\frac{\pi}{T_p\sqrt{1-\zeta^2}}=\frac{\pi}{0.1\sqrt{1-0.333^2}}=33.3$$

Leading a transfer function

$$G_c(s) = \frac{1108.9}{s^2 + 22s + 1108.9}$$

Similarly for the 'hot tail':

$$c_{final} = 0.023,\ M_{pt} = 0.029$$

$$\%OS = \frac{0.029 - 0.023}{0.023} \times 100\% = 26.1\%$$

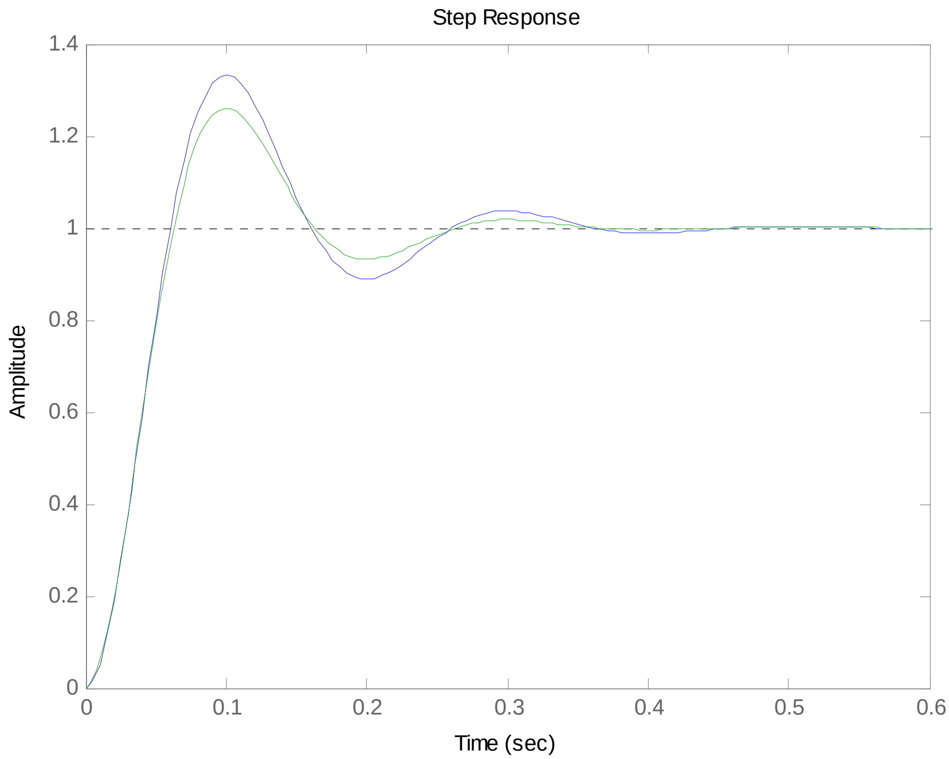
$$\zeta = \frac{-\ln(0.261)}{\sqrt{\pi^2 + \ln^2(0.261)}} = 0.393$$

$$\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}} = \frac{\pi}{0.1\sqrt{1-0.261^2}} = 34.17$$

$$G_h(s) = \frac{1167.6}{s^2 + 26.9s + 1167.6}$$

Using MATLAB:

```
>> syms s

>> s=tf('s')


Transfer function:

s

>> Gc = 1108.89/(s^2+22*s+1108.89);

>> Gh = 1167.6/(s^2+26.9*s+1167.6);

>> step(Gc,Gh)
```




Both responses are equivalent if error tolerances are considered.

**58.**

The original transfer function has zeros at $s = -7200 \pm j7400$

And poles at $s = -1900 \pm j4500$; $s = -120 \pm j1520$

With $G(0) = 0.1864$

The dominant poles are those with real parts at -120, so a real pole is added at

-1200 giving the following approximation:

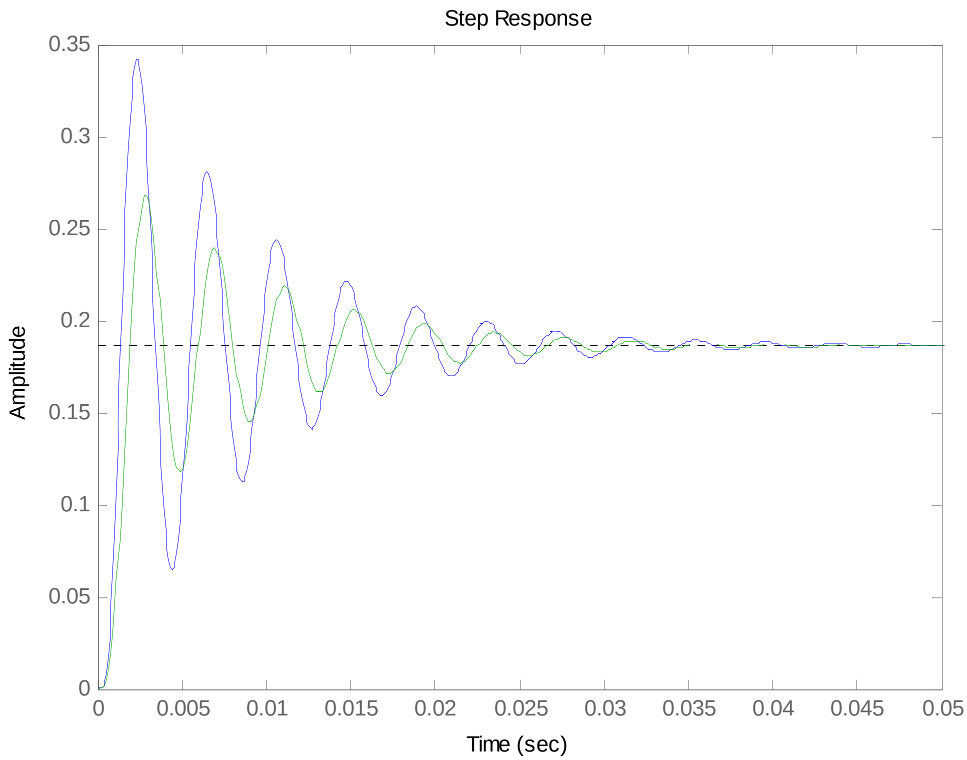
$$G(s) \approx 0.1864 \frac{(1200)(2324.8 \times 10^3)}{106.6 \times 10^6} \frac{(s^2 - 14400s + 106.6 \times 10^6)}{(s^2 + 240s + 2324.8 \times 10^3)(s + 1200)}$$

$$= \frac{4.8782(s^2 - 14400s + 106.6 \times 10^6)}{(s^2 + 240s + 2324.8 \times 10^3)(s + 1200)}$$

Using MATLAB:

```
>> syms s

>> s=tf('s');
```

```
>>G=9.7e4*(s^2-14400*s+106.6e6)…
        /(s^2+3800*s+23.86e6)/(s^2+240*s+2324.8e3);
>> Gdp=4.8782*(s^2-14400*s+106.6e6)/(s^2+240*s+2324.8e3)/(s+1200);
>> step(G,Gdp)
```




Both responses differ because the original non-dominant poles are very close to the complex pair of zeros.

**59.**

M(s) requires at least 4 'far away' poles that are added a decade beyond all original poles and zeros. This gives

$$M(s)=\frac{(s+0.009)^2(s^2+0.018s+0.0001)}{9.72\times10^{-8}(s+0.0001)(1+s/0.1)^4}=\frac{1028.81(s+0.009)^2(s^2+0.018s+0.0001)}{(s+0.0001)(s+0.1)^4}$$

**60.**

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = \frac{-\ln(0.30)}{\sqrt{\pi^2 + \ln^2(0.30)}} = 0.36$$

$$\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}} = \frac{\pi}{127\sqrt{1-0.30^2}} = 0.026$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{0.00067}{s^2 + 0.0187s + 0.00067}$$

**61.**

**a.** Let the impulse response of T(s) be h(t). We have that

$$H(s) = \frac{450}{(s+5)(s+20)} = \frac{A}{s+5} + \frac{B}{s+20}$$

$$A = \frac{450}{s+20}\bigg|_{s=-5} = 30\,;\ B = \frac{450}{s+5}\bigg|_{s=-20} = -30$$

$H(s) = \dfrac{30}{s+5} - \dfrac{30}{s+20}$. Obtaining the inverse Laplace transform we get

$$h(t) = 30e^{-5t} - 30e^{-20t}$$

b. Let the step response of the system be g(t). We have that

$$g(t) = \int_0^t h(t)dt = \int_0^t 30e^{-5t}dt - \int_0^t 30e^{-20t}dt = -\frac{30}{5}e^{-5t}\Big|_0^t - \frac{30}{-20}e^{-20t}\Big|_0^t$$

$$= -6(e^{-5t} - 1) + 1.5(e^{-20t} - 1) = 4.5 - 6e^{-5t} + 1.5e^{-20t}$$

**c.** $G(s) = \dfrac{450}{s(s+5)(s+20)} = \dfrac{A}{s} + \dfrac{B}{s+5} + \dfrac{C}{s+20}$

$$A = \frac{450}{(s+5)(s+20)}\bigg|_{s=0} = 4.5\,;\ B = \frac{450}{s(s+20)}\bigg|_{s=-5} = -6\,;\ C = \frac{450}{s(s+5)}\bigg|_{s=-20} = 1.5$$

Leading $G(s) = \dfrac{4.5}{s} - \dfrac{6}{s+5} + \dfrac{1.5}{s+20}$. After the inverse Laplace we get

$$g(t) = 4.5 - 6e^{-5t} + 1.5e^{-20t}$$

**62.**

**a.** The poles given by $s^2 + 8.99\times10^{-3}s + 3.97\times10^{-3} = 0$ have an $\omega_n = 0.063 rad/\sec$ and $\zeta = 0.0714$

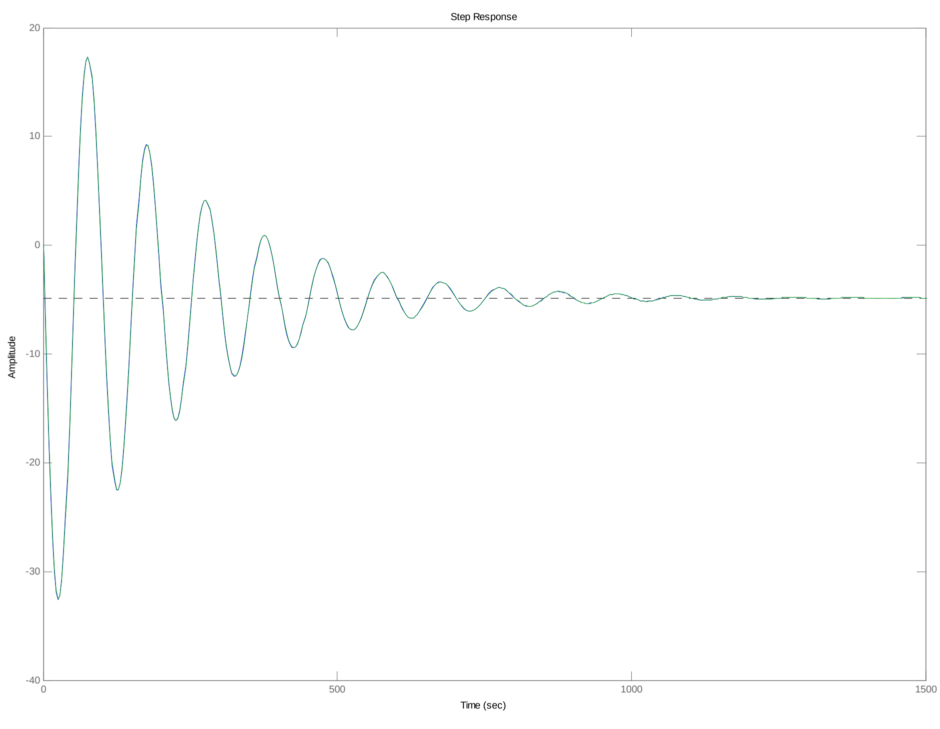
The poles given by $s^2+4.21s+18.23=0$ have an $\omega_n = 4.27 rad/\sec$ and $\zeta = 0.493$ Thus the former represent the Phugoid and the latter the Short Period modes.

**b.** In the original we have $\frac{\theta}{\delta_e}(0) = -4.85$ so the Phugoid approximation is given by:

$$\frac{\theta}{\delta_e} \approx -\frac{1.965(s+0.0098)}{(s^2+8.99\times10^{-3}s+3.97\times10^{-3})}$$

**c.**

```
>> syms s
>> s=tf('s');
>>G=-26.12*(s+0.0098)*(s+1.371)/(s^2+8.99e-3*s+3.97e-3)/(s^2+4.21*s+18.23);
>> Gphug=-1.965*(s+0.0098)/(s^2+8.99e-3*s+3.97e-3);
>> step(G,Gphug)
```




Both responses are indistinguishable.

**63.**

a.

Program
```
numg=[33 202 10061 24332 170704];
deng=[1 8 464 2411 52899 167829 913599 1076555];
G=tf(numg,deng)
[K,p,k]=residue(numg,deng)
```

**Computer Response**
```
K =

   0.0018 + 0.0020i
   0.0018 - 0.0020i
  -0.1155 - 0.0062i
  -0.1155 + 0.0062i
   0.0077 - 0.0108i
   0.0077 + 0.0108i
   0.2119


p =

  -1.6971 +16.4799i
  -1.6971 -16.4799i
  -0.5992 +12.1443i
  -0.5992 -12.1443i
  -1.0117 + 4.2600i
  -1.0117 - 4.2600i
  -1.3839


k =

     []
```
**b.**

Therefore, an approximation to *G(s)/* is:

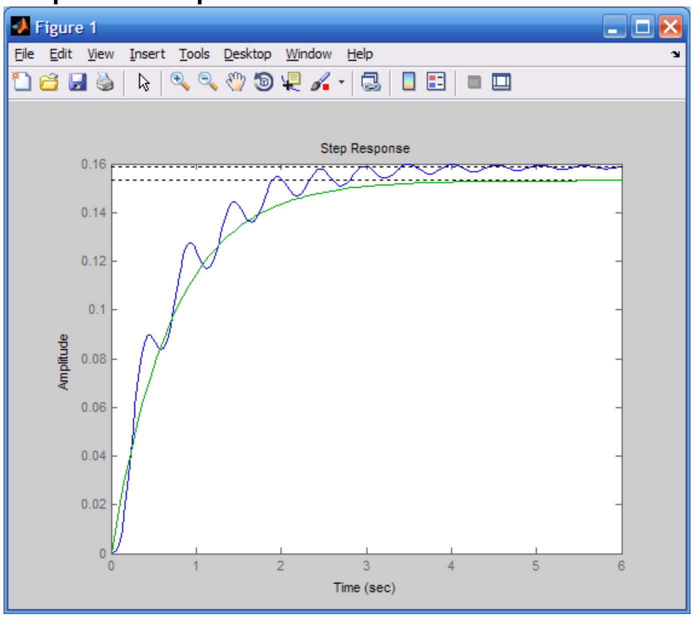
$$G(s) = \frac{0.2119}{s + 1.3839}$$

**c.**

**Program**
```
numg=[33 202 10061 24332 170704];
deng=[1 8 464 2411 52899 167829 913599 1076555];
G=tf(numg,deng);
numga=0.2119;
denga=[1 1.3839];
```

```
Ga=tf(numga,denga);
step(G)
hold on
step(Ga)
```

**Computer Response**

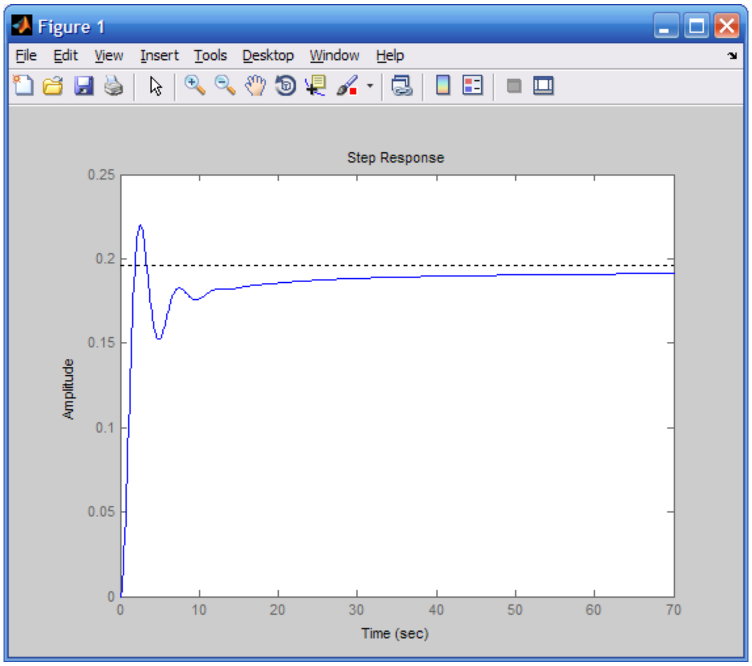



Approximation does not show oscillations and is slightly off of final value.

**64.**

**Computer Response**

```
Transfer function:

 s^15 + 1775 s^14 + 1.104e006 s^13 + 2.756e008 s^12 + 2.272e010 s^11

         + 7.933e011 s^10 + 1.182e013 s^9 + 6.046e013 s^8 + 1.322e014 s^7

         + 1.238e014 s^6 + 3.977e013 s^5 + 5.448e012 s^4 + 3.165e011 s^3

                              + 6.069e009 s^2 + 4.666e007 s + 1.259e005

----------------------------------------------------------------------------
31.62 s^17 + 4.397e004 s^16 + 1.929e007 s^15 + 2.941e009 s^14

        + 1.768e011 s^13 + 4.642e012 s^12 + 5.318e013 s^11 + 2.784e014 s^10

        + 7.557e014 s^9 + 1.238e015 s^8 + 1.356e015 s^7 + 8.985e014 s^6

        + 2.523e014 s^5 + 3.179e013 s^4 + 1.732e012 s^3 + 3.225e010 s^2

                                             + 2.425e008 s + 6.414e005
```




**65.**

a. To find the step responses for these two processes, $y_a(t)$ and $y_p(t)$, we consider first the un-shifted Laplace transform of their outputs for $X_a(s) = X_p(s) = 1/s$:

$$Y_a^*(s) = \frac{14.49}{s(1478.26s+1)} = \frac{9.8\times10^{-3}}{s(s+6.77\times10^{-4})} = \frac{A}{s} + \frac{B}{(s+6.77\times10^{-4})} \quad (1),$$

where $A = \left.\dfrac{9.8\times10^{-3}}{s+6.77\times10^{-4}}\right|_{s=0} = 14.49$ and

$$B = \left.\frac{9.8\times10^{-3}}{s}\right|_{s=-6.77\times10^{-4}} = -14.49 \quad (2)$$

Substituting the values of $A$ and $B$ into equation (1) gives:

$$Y_a^*(s) = \frac{A}{s} + \frac{B}{(s+6.76\times10^{-4})} = 14.49\left(\frac{1}{s} - \frac{1}{(s+6.76\times10^{-4})}\right) \quad (3)$$

Taking the inverse Laplace transform of $Y_a^*(s)$ and delaying the resulting response in the time domain by 4 seconds, we get:

$$y_a(t) = 14.49[1 - e^{-6.76\times10^{-4}(t-4)}]\ u(t-4) \quad (4)$$

Noting that the denominator of $G_p(s)$ can be factored into

$(s+0.174\times10^{-3})(s+6.814\times10^{-3})$, we have:

$$Y_p^*(s)=\frac{1.716\times10^{-5}}{s(s+0.174\times10^{-3})(s+6.814\times10^{-3})}=\frac{C}{s}+\frac{D}{(s+0.174\times10^{-3})}+\frac{E}{(s+6.814\times10^{-3})} \quad (5),$$

where:
$$C=\left.\frac{1.716\times10^{-5}}{(s+0.174\times10^{-3})(s+6.814\times10^{-3})}\right|_{s=0}=14.48;$$

$$D=\left.\frac{1.716\times10^{-5}}{s(s+6.814\times10^{-3})}\right|_{s=-0.174\times10^{-3}}=-14.85;$$

$$E=\left.\frac{1.716\times10^{-5}}{s(s+0.174\times10^{-3})}\right|_{s=-6.814\times10^{-3}}=0.37. \quad (6)$$

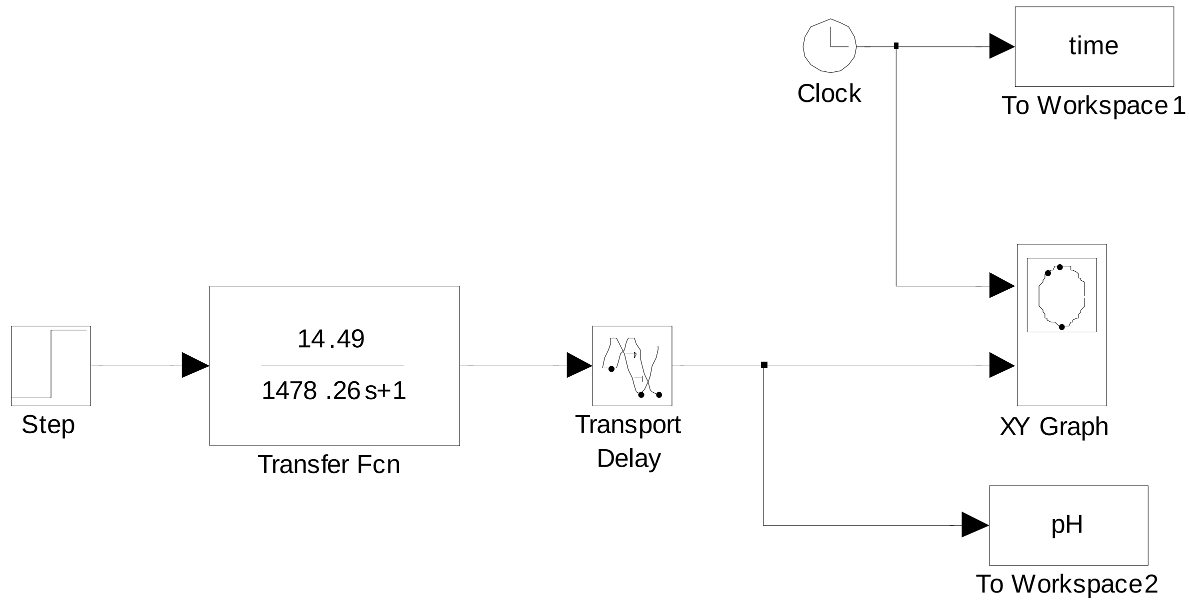
Substituting the values of *C*, *D* and *E* into equation (5) and simplifying gives:

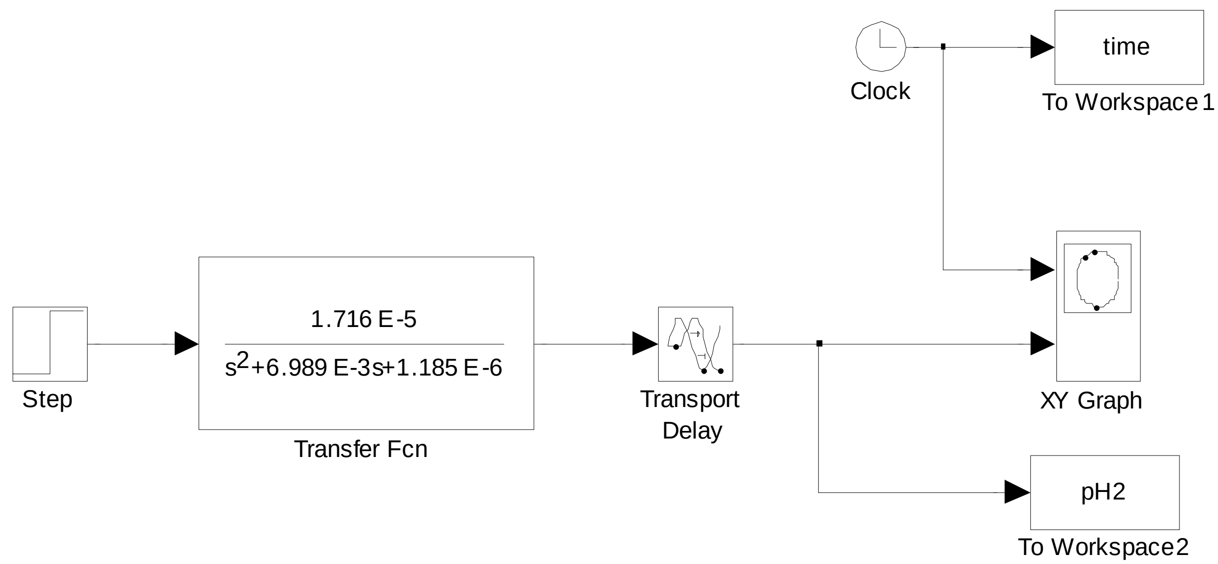
$$Y_p^*(s)=\frac{14.48}{s}-\frac{14.85}{(s+0.174\times10^{-3})}+\frac{0.37}{(s+6.814\times10^{-3})}=14.48\left(\frac{1}{s}-\frac{1.0256}{(s+0.174\times10^{-3})}+\frac{0.0256}{(s+6.814\times10^{-3})}\right) \quad (7)$$

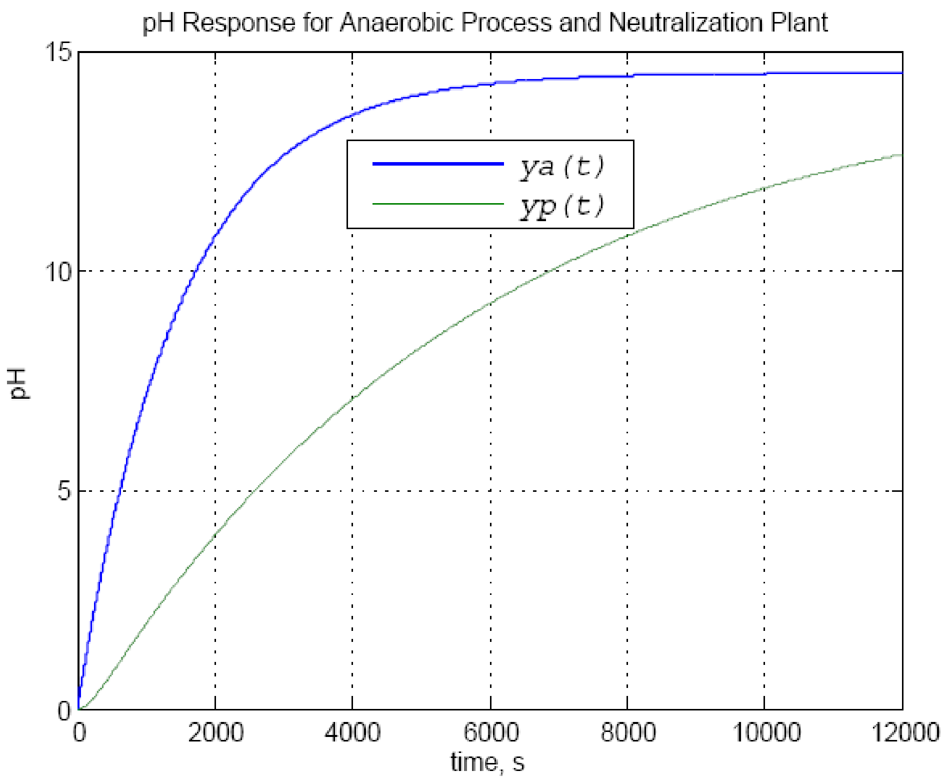
Taking the inverse Laplace transform of $Y_p^*(s)$ and delaying the resulting response in the time domain by 30 seconds, we get:

$$y_p(t)=14.48[1-1.0256e^{-0.174\times10^{-3}(t-30)}+0.0256e^{-6.814\times10^{-3}(t-30)}]u(t-30) \quad (8)$$

b. Using Simulink to model the two processes described above, $y_a(t)$ and $y_p(t)$ were output to the "workspace." Matlab plot commands were then utilized to plot $y_a(t)$ and $y_p(t)$ on a single graph, which is shown below.


Clock
time
To Workspace 1
14.49
1478.26s+1
Step
Transfer Fcn
Transport
Delay
XY Graph
pH
To Workspace2


Clock
time
To Workspace 1
1.716 E-5
s2+6.989 E-3s+1.185 E-6
Step
Transfer Fcn
Transport
Delay
XY Graph
pH2
To Workspace2


pH Response for Anaerobic Process and Neutralization Plant
15
10
5
0
pH
0
2000
4000
6000
8000
10000
12000
time, s
ya(t)
yp(t)

**66.**

**a.**

```
>> A=[-8.792e-3 0.56e-3 -1e-3 -13.79e-3; -0.347e-3 -11.7e-3 -0.347e-3 0; 0.261 -20.8e-3 -96.6e-3
0; 0 0 1 0]

A =

   -0.0088    0.0006   -0.0010   -0.0138
   -0.0003   -0.0117   -0.0003         0
    0.2610   -0.0208   -0.0966         0
         0         0    1.0000         0

>> eig(A)

ans =

  -0.1947
   0.0447 + 0.1284i
   0.0447 - 0.1284i
  -0.0117
```

**b.**

Given the eigenvalues, the state-transition matrix will be of the form

$$\mathbf{\Phi}(t) = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \qquad \text{with}$$

$$K_{ij} = K_{ij1}e^{-0.1947\,t} + K_{ij2}e^{-0.0117\,t} + K_{ij3}e^{+0.0447\,t}\sin(0.1284\,t) + K_{ij4}e^{+0.0447\,t}\cos(0.1284\,t))$$

Thus 64 constants have to be found.

**67.**

**a.** The equations are rewritten as

$$\frac{di_L}{dt} = -\frac{1-d}{L}u_C + \frac{1}{L}E_s$$
$$\frac{du_C}{dt} = \frac{1-d}{C}i_L - \frac{1}{RC}u_C$$

from which we obtain

$$\begin{bmatrix} \dfrac{di_L}{dt} \\ \dfrac{du_C}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{1-d}{L} \\ \dfrac{1-d}{C} & -\dfrac{1}{RC} \end{bmatrix} \begin{bmatrix} i_L \\ u_C \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} E_s$$
$$y = [0 \quad 1] \begin{bmatrix} i_L \\ i_C \end{bmatrix}$$

**b.** To obtain the transfer function we first calculate

$$(s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} s & \dfrac{1-d}{L} \\ -\dfrac{1-d}{C} & s+\dfrac{1}{RC} \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s+\dfrac{1}{RC} & -\dfrac{1-d}{L} \\ \dfrac{1-d}{C} & s \end{bmatrix}}{s(s+\dfrac{1}{RC}) + \dfrac{(1-d)^2}{LC}}$$

So

$$G(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = [0 \quad 1] \frac{\begin{bmatrix} s+\dfrac{1}{RC} & -\dfrac{1-d}{L} \\ \dfrac{1-d}{C} & s \end{bmatrix}}{s(s+\dfrac{1}{RC}) + \dfrac{(1-d)^2}{LC}} \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix}$$
$$= \frac{\begin{bmatrix} \dfrac{1-d}{C} & s \end{bmatrix}}{s^2 + \dfrac{1}{RC}s + \dfrac{(1-d)^2}{LC}} \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} = \frac{\dfrac{1-d}{LC}}{s^2 + \dfrac{1}{RC}s + \dfrac{(1-d)^2}{LC}}$$

**68.**

**a.** We have $(s\mathbf{I}-\mathbf{A})=\begin{bmatrix} s+8.34 & 2.26 \\ -1 & s \end{bmatrix}$ and

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{\begin{bmatrix} s & -2.26 \\ 1 & s+8.34 \end{bmatrix}}{s^2+8.34s+2.26}=\frac{\begin{bmatrix} s & -2.26 \\ 1 & s+8.34 \end{bmatrix}}{(s+0.28)(s+8.06)}$$

We first find $\mathcal{L}^{-1}\left\{\frac{1}{(s+0.28)(s+8.06)}\right\}$

$$\frac{1}{(s+0.28)(s+8.06)}=\frac{K_1}{s+0.28}+\frac{K_2}{s+8.06}$$

$K_1=\frac{1}{s+8.06}\Big|_{s=-0.28}=0.129$; $K_2=\frac{1}{s+0.28}\Big|_{s=-8.06}=-0.129$ so

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+0.28)(s+8.06)}\right\}=0.129e^{-0.28t}-0.129e^{-8.06t}$$

Follows that

$$\mathcal{L}^{-1}\left\{\frac{-2.26}{(s+0.28)(s+8.06)}\right\}=-2.26\mathcal{L}^{-1}\left\{\frac{1}{(s+0.28)(s+8.06)}\right\}=-0.292e^{-0.28t}+0.292e^{-8.06t}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+0.28)(s+8.06)}\right\}=\frac{d}{dt}\mathcal{L}^{-1}\left\{\frac{1}{(s+0.28)(s+8.06)}\right\}=-0.036e^{-0.28t}+1.04e^{-8.06t}$$

And

$$\mathcal{L}^{-1}\left\{\frac{s+8.34}{(s+0.28)(s+8.06)}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{(s+0.28)(s+8.06)}\right\}+8.34\mathcal{L}^{-1}\left\{\frac{1}{(s+0.28)(s+8.06)}\right\}$$

$$=-0.036e^{-0.28t}+1.04e^{-8.06t}+1.076e^{-0.28t}-1.076e^{-8.06t}=1.04e^{-0.28t}-0.036e^{-8.06t}$$

Finally the state transition matrix is given by:

$$\mathbf{\Phi}(t)=\begin{bmatrix} -0.036e^{-0.28t}+1.04e^{-8.06t} & -0.292e^{-0.28t}+0.292e^{-8.06t} \\ 0.129e^{-0.28t}-0.129e^{-8.06t} & 1.04e^{-0.28t}-0.036e^{-8.06t} \end{bmatrix}$$

**b.**

$$\mathbf{\Phi}(t)\mathbf{B} = \begin{bmatrix} -0.036e^{-0.28t} + 1.04e^{-8.06t} \\ 0.129e^{-0.28t} - 0.129e^{-8.06t} \end{bmatrix}$$

$$\mathbf{C\Phi}(t)\mathbf{B} = -0.451e^{-0.28t} + 13.04e^{-8.06t} + 0.292e^{-0.28t} - 0.292e^{-8.06t} = -0.159e^{-0.28t} + 12.748e^{-8.06t}$$

Since $u(t) = 1$

$$y(t) = \int_0^t \mathbf{C\Phi}(t-\tau)\mathbf{B}d\tau = \int_0^t [-0.159e^{-0.28(t-\tau)} + 12.748e^{-8.06(t-\tau)}]d\tau$$

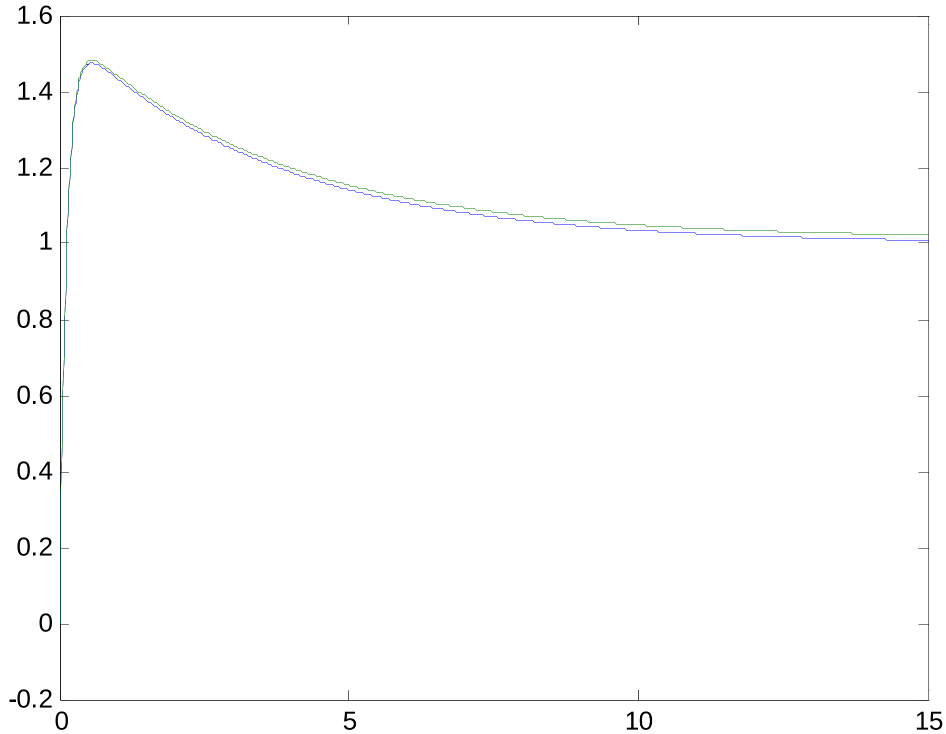
$$= -0.159e^{-0.28t}\int_0^t e^{0.28\tau}d\tau + 12.748e^{-8.06t}\int_0^t e^{8.06\tau}d\tau$$

$$= \frac{-0.159}{0.28}e^{-0.28t}e^{0.28\tau} + \frac{12.748}{8.06}e^{-8.06t}e^{8.06\tau}\Big|_0^t$$

$$= -0.568[1 - e^{-0.28t}] + 1.582[1 - e^{-8.06t}] = 1.014 + 0.568e^{-0.28t} - 1.582e^{-8.06t}$$

**c.**

```
>> A=[-8.34 -2.26; 1 0];

>> B = [1; 0];

>> C = [12.54 2.26];

>> D = 0;

>> t = linspace(0,15,1000);

>> y1 = step(A,B,C,D,1,t);

>> y2 = 1.014+0.568*exp(-0.28.*t)-1.582*exp(-8.06.*t);

>> plot(t,y1,t,y2)
```




## SOLUTIONS TO DESIGN PROBLEMS

**69.**

Writing the equation of motion, $(f_v s+2)X(s)=F(s)$. Thus, the transfer function is $\dfrac{X(s)}{F(s)}=\dfrac{1/f_v}{s+\dfrac{2}{f_v}}$. Hence, $T_s=\dfrac{4}{a}=\dfrac{4}{\dfrac{2}{f_v}}=2f_v$, or $f_v=\dfrac{T_s}{2}$.

**70.**

The transfer function is, $F(s)=\dfrac{1/M}{s^2+\dfrac{1}{M}s+\dfrac{K}{M}}$. Now, $T_s=4=\dfrac{4}{|\mathrm{Re}|}=\dfrac{4}{\dfrac{1}{2M}}=8M$. Thus,

$M=\dfrac{1}{2}$. Substituting the value of M in the denominator of the transfer function yields,

$s^2+2s+2K$. Identify the roots $s_{1,2}=-1\pm j\sqrt{2K-1}$. Using the imaginary part and substituting

into the peak time equation yields $T_p=1=\dfrac{\pi}{|\mathrm{Im}|}=\dfrac{\pi}{\sqrt{2K-1}}$, from which $K=5.43$.

**71.**

Writing the equation of motion, $(Ms^2 + f_v s + 1)X(s) = F(s)$. Thus, the transfer function is $\frac{X(s)}{F(s)} = \frac{1/M}{s^2 + \frac{f_v}{M}s + \frac{1}{M}}$. Since $T_s = 10 = \frac{4}{\zeta\omega_n}$, $\zeta\omega_n = 0.4$. But, $\frac{f_v}{M} = 2\zeta\omega_n = 0.8$. Also, from Eq. (4.39) 17% overshoot implies $\zeta = 0.491$. Hence, $\omega_n = 0.815$. Now, $1/M = \omega_n^2 = 0.664$. Therefore, M 1.51. Since $\frac{f_v}{M} = 2\zeta\omega_n = 0.8$, $f_v = 1.21$.

**72.**

Writing the equation of motion: $(Js^2+s+K)\theta(s) = T(s)$. Therefore the transfer function is

$$\frac{\theta(s)}{T(s)} = \frac{\frac{1}{J}}{s^2+\frac{1}{J}s+\frac{K}{J}}.$$

$$\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.358.$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\frac{1}{2J}} = 8J = 3.$$

Therefore $J = \frac{3}{8}$. Also, $T_s = 3 = \frac{4}{\zeta\omega_n} = \frac{4}{(0.358)\omega_n}$. Hence, $\omega_n = 3.724$. Now, $\frac{K}{J} = \omega_n^2 = 13.868$. Finally, K = 5.2.

**73.**

Writing the equation of motion

$$[s^2+D(5)^2s+\frac{1}{4}(10)^2]\theta(s) = T(s)$$

The transfer function is

$$\frac{\theta(s)}{T(s)} = \frac{1}{s^2+25Ds+25}$$

Also,

$$\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} = 0.358$$

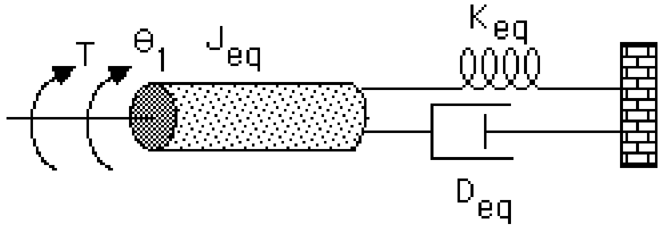
and

$$2\zeta\omega_n = 2(0.358)(5) = 25D$$

Therefore D = 0.14.

**74.**

The equivalent circuit is:




where $J_{eq} = 1+(\frac{N_1}{N_2})^2$ ; $D_{eq} = (\frac{N_1}{N_2})^2$; $K_{eq} = (\frac{N_1}{N_2})^2$. Thus,

$\frac{\theta_1(s)}{T(s)} = \frac{1}{J_{eq}s^2+D_{eq}s+K_{eq}}$ . Letting $\frac{N_1}{N_2}$ = n and substituting the above values into the transfer function,

$\frac{\theta_1(s)}{T(s)} = \frac{\frac{1}{1+n^2}}{s^2+\frac{n^2}{1+n^2}s+\frac{n^2}{1+n^2}}$ . Therefore, $\zeta\omega_n = \frac{n^2}{2(1+n^2)}$ . Finally, $T_s = \frac{4}{\zeta\omega_n} = \frac{8(1+n^2)}{n^2} = 16$. Thus

n = 1.

**75.**

Let the rotation of the shaft with gear $N_2$ be $\theta_L(s)$. Assuming that all rotating load has been reflected to the $N_2$ shaft, $(J_{eqL}s^2 + D_{eqL}s + K)\theta_L(s) + F(s)r = T_{eq}(s)$, where F(s) is the force from the translational system, $r = 2$ is the radius of the rotational member, $J_{eqL}$ is the equivalent inertia at the $N_2$ shaft, and $D_{eqL}$ is the equivalent damping at the $N_2$ shaft. Since $J_{eqL} = 1(2)^2 + 1 = 5$ and $D_{eqL} = 1(2)^2 = 4$, the equation of motion becomes, $(5s^2 + 4s + K)\theta_L(s) + 2F(s) = T_{eq}(s)$. For the translational system $(Ms^2 + s)X(s) = F(s)$. Substituting $F(s)$ into the rotational equation of motion, $(5s^2 + 4s + K)\theta_L(s) + (Ms^2 + s)2X(s) = T_{eq}(s)$.

But, $\theta_L(s) = \frac{X(s)}{r} = \frac{X(s)}{2}$ and $T_{eq}(s) = 2T(s)$. Substituting these quantities in the equation above yields $((5+4M)s^2 + 8s + K)\frac{X(s)}{4} = T(s)$. Thus, the transfer function is

$\frac{X(s)}{T(s)} = \frac{4/(5+4M)}{s^2 + \frac{8}{(5+4M)}s + \frac{K}{(5+4M)}}$. Now, $T_s = 15 = \frac{4}{\mathrm{Re}} = \frac{4}{\frac{8}{2(5+4M)}} = (5+4M)$.

Hence, $M = 5/2$. For 10% overshoot, $\zeta = 0.5912$ from Eq. (4.39). Hence,

$2\zeta\omega_n = \frac{8}{(5+4M)} = 0.5333$. Solving for $\omega_n$ yields $\omega_n$ = 0.4510. But,

$\omega_n = \sqrt{\frac{K}{(5+4M)}} = \sqrt{\frac{K}{15}} = 0.4510$. Thus, K = 3.051.

**76.**

The transfer function for the capacitor voltage is $\frac{V_C(s)}{V(s)} = \frac{\frac{1}{Cs}}{R+Ls+\frac{1}{Cs}} = \frac{10^6}{s^2+Rs+10^6}$.

For 20% overshoot, $\zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2+\ln^2(\frac{\%OS}{100})}}$ = 0.456. Therefore, $2\zeta\omega_n = R = 2(0.456)(10^3)$ = 912Ω.

**77.**

Solving for the capacitor voltage using voltage division, $V_C(s) = V_i(s)\frac{1/(CS)}{R+LS+\frac{1}{CS}}$. Thus, the transfer function is $\frac{V_C(s)}{V_i(s)} = \frac{1/(LC)}{s^2+\frac{R}{L}s+\frac{1}{LC}}$. Since $T_s = \frac{4}{|\mathrm{Re}|} = 10^{-3}$, $|\mathrm{Re}| = \frac{R}{2L} = 4000$. Thus $R = 8\ \mathrm{K\Omega}$. Also, since 20% overshoot implies a damping ratio of 0.46 and $2\zeta\omega_n = 8000$, $\omega_\mathrm{n} = 8695.65 = \frac{1}{\sqrt{LC}}$. Hence, $C = 0.013\ \mu\mathrm{F}$.

**78.**

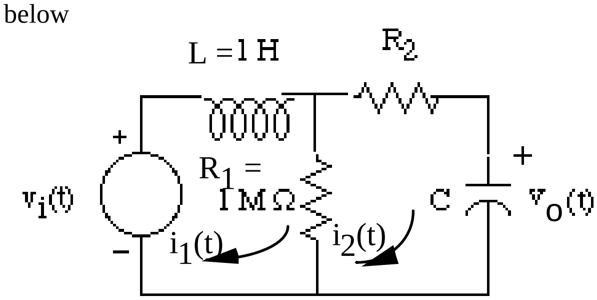
Using voltage division the transfer function is,

$$\frac{V_C(s)}{V_i(s)} = \frac{\frac{1}{Cs}}{R+Ls+\frac{1}{Cs}} = \frac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Also, $T_s = 7x10^{-3} = \frac{4}{\mathrm{Re}} = \frac{4}{\frac{R}{2L}} = \frac{8L}{R}$. Thus, $\frac{R}{L} = 1143$. Using Eq. (4.39) with 15% overshoot, $\zeta$ = 0.5169. But, $2\zeta\omega_n = R/L$. Thus, $\omega_n = 1105.63 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{L(10^{-5})}}$. Therefore, L = 81.8 mH and R = 98.5 Ω.

**79.**

For the circuit shown below




write the loop equations as

$$(R_1+Ls)I_1(s)-R_1I_2(s)=V_i(s)$$
$$-R_1I_1(s)+\left(R_1+R_2+\frac{1}{Cs}\right)I_2(s)=0$$

Solving for $I_2(s)$

$$I_2(s)=\frac{\left|\begin{pmatrix} R_1+Ls & V_i(s) \\ -R_1 & 0 \end{pmatrix}\right|}{\left|\begin{pmatrix} R_1+Ls & -R_1 \\ -R_1 & R_1+R_2+\frac{1}{Cs} \end{pmatrix}\right|}$$

But, $V_o(s)=\frac{1}{Cs}I_2(s)$. Thus,

$$\frac{V_o(s)}{V_i(s)}=\frac{R_1}{(R_2+R_1)CLs^2+(CR_2R_1+L)s+R_1}$$

Substituting component values,

$$\frac{V_o(s)}{V_i(s)}=1000000\frac{\frac{1}{(R_2+1000000)C}}{s^2+\frac{(1000000CR_2+1)}{(R_2+1000000)C}s+1000000\frac{1}{(R_2+1000000)C}}$$

For 8% overshoot, $\zeta = 0.6266$. For $T_s = 0.001$, $\zeta\omega_n = \frac{4}{0.001} = 4000$. Hence, $\omega_n = 6383.66$. Thus,

$$1000000\frac{1}{(R_2+1000000)C}=6383.66^2$$

or,

$$C=0.0245\frac{1}{R_2+1000000} \qquad (1)$$

Also,

$$\frac{1000000\,C\,R_2+1}{(R_2+1000000)\,C}=8000 \qquad (2)$$

Solving (1) and (2) simultaneously, $R_2 = 8023$ Ω, and $C$ = 2.4305 x $10^{-2}$ μF.

**80.**

$$s\mathbf{I}-\mathbf{A}=\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}-\begin{bmatrix} (3.45-14000K_c) & -0.255x10^{-9} \\ 0.499x10^{11} & -3.68 \end{bmatrix}$$

$$=\begin{bmatrix} s-(3.45-14000K_c) & 0.255x10^{-9} \\ -0.499x10^{11} & s+3.68 \end{bmatrix}$$

$$|s\mathbf{I}-\mathbf{A}|=s^2+(0.23+0.14x10^5K_c)s+(51520K_c+0.0285)$$

$$(2\zeta\omega_n)^2=[2*0.9]^2*(51520K_c+0.0285)=(0.23+0.14x10^5K_c)^2$$

or

$$K_c^{\,2}-8.187x10^{-4}K_c-2.0122x10^{-10}=0$$

Solving for $K_c$,

$$K_c=8.189x10^{-4}$$

**81.**

**a.** The transfer function from Chapter 2 is,

$$\frac{Y_h(s)-Y_{cat}(s)}{F_{up}(s)}=\frac{0.7883(s+53.85)}{(s^2+15.47s+9283)(s^2+8.119s+376.3)}$$

The dominant poles come from $s^2+8.119s+376.3$. Using this polynomial, $2\zeta\omega_n=8.119$, and $\omega_n^2=376.3$. Thus, $\omega_n=19.4$ and $\zeta=0.209$. Using Eq. (4.38), %OS = 51.05%. Also, $T_s=\dfrac{4}{\zeta\omega_n}=0.985$ s, and $T_p=\dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}=0.166$ s. To find rise time, use Figure 4.16. Thus, $\omega_n T_r=1.2136$ or $T_r=0.0626$ s.

**b.** The other poles have a real part of 15.47/2 = 7.735. Dominant poles have a real part of 8.119/2 = 4.06. Thus, 7.735/4.06 = 1.91. This is not at least 5 times.

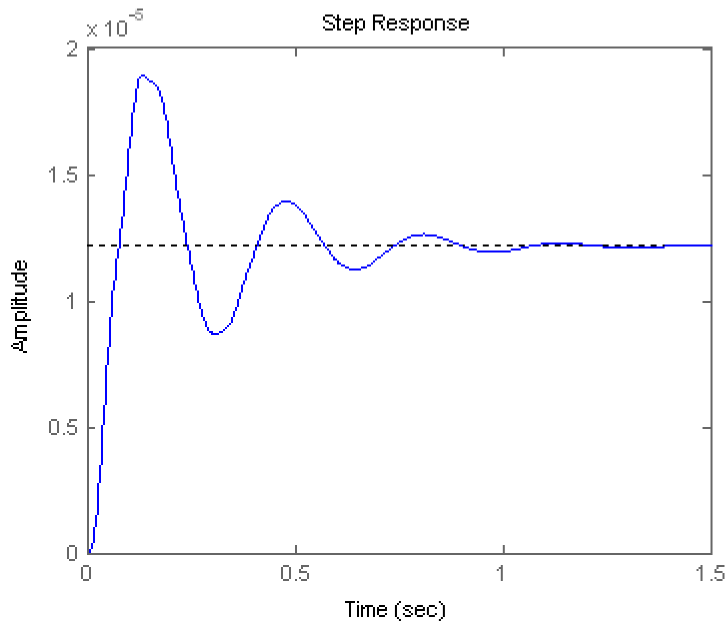
**c.**

**Program:**

```
syms s
numg=0.7883*(s+53.85);
deng=(s^2+15.47*s+9283)*(s^2+8.119*s+376.3);
'G(s) transfer function'
G=vpa(numg/deng,3);
pretty(G)
numg=sym2poly(numg);
deng=sym2poly(deng);
G=tf(numg,deng)
step(G)
```

**Computer response:**

```
ans =
G(s) transfer function

                           .788 s + 42.4
                  -----------------------------------------
                    2                      2
                  (s   + 15.5 s + 9280.) (s   + 8.12 s + 376.)

Transfer function:
                 0.7883 s + 42.45
----------------------------------------------------
s^4 + 23.59 s^3 + 9785 s^2 + 8.119e004 s + 3.493e006
```




The time response shows 58 percent overshoot, $T_s$ = 0.86 s, $T_p$ = 0.13 s, $T_r$ = 0.05 s.

**82.**

**a.** In Problem 3.30 we had

$$\begin{bmatrix} \dot{T} \\ \dot{T}^* \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -(d+\beta v_0) & 0 & -\beta T_0 \\ \beta v_0 & -\mu & \beta T_0 \\ 0 & k & -c \end{bmatrix} \begin{bmatrix} T \\ T^* \\ v \end{bmatrix} + \begin{bmatrix} \beta T_0 v_0 & 0 \\ -\beta T_0 v_0 & 0 \\ 0 & -kT_0^* \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T \\ T^* \\ v \end{bmatrix}$$

When $u_2 = 0$ the equations are equivalent to

$$\begin{bmatrix}\dot{T}\\ \dot{T}^*\\ \dot{v}\end{bmatrix}=\begin{bmatrix}-(d+\beta v_0) & 0 & -\beta T_0\\ \beta v_0 & -\mu & \beta T_0\\ 0 & k & -c\end{bmatrix}\begin{bmatrix}T\\ T^*\\ v\end{bmatrix}+\begin{bmatrix}\beta T_0 v_0\\ -\beta T_0 v_0\\ 0\end{bmatrix}u_1$$

$$y=\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}T\\ T^*\\ v\end{bmatrix}$$

Substituting parameter values one gets

$$\begin{bmatrix}\dot{T}\\ \dot{T}^*\\ \dot{v}\end{bmatrix}=\begin{bmatrix}-0.04167 & 0 & -0.0058\\ 0.0217 & -0.24 & 0.0058\\ 0 & 100 & -2.4\end{bmatrix}\begin{bmatrix}T\\ T^*\\ v\end{bmatrix}+\begin{bmatrix}5.2\\ -5.2\\ 0\end{bmatrix}u_1$$

$$y=\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}T\\ T^*\\ v\end{bmatrix}$$

**b.**

$$(s\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix}s+0.04167 & 0 & 0.0058\\ -0.0217 & s+0.24 & -0.0058\\ 0 & -100 & s+2.4\end{bmatrix}^{-1}=\frac{Adj(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}$$

$$\det(s\mathbf{I}-\mathbf{A})=(s+0.04167)\begin{vmatrix}s+0.24 & -0.0058\\ -100 & s+2.4\end{vmatrix}+0.0058\begin{vmatrix}-0.0217 & s+0.24\\ 0 & -100\end{vmatrix}$$

$$=(s+0.04167)\left[(s+0.24)(s+2.4)-0.58\right]+(0.0058)(2.17)$$

$$=s^3+2.6817s^2+0.11s+0.0126$$

$$=(s+2.6419)(s^2+0.0398s+0.0048)$$

To obtain the adjoint matrix we calculate the cofactors:

$$C_{11}=\begin{vmatrix}s+0.24 & -0.0058\\ -100 & s+2.4\end{vmatrix}=s(s+2.64)$$

$$C_{12}=\begin{vmatrix}-0.0217 & -0.0058\\ 0 & s+2.4\end{vmatrix}=-0.0217(s+2.4)$$

$$C_{13}=\begin{vmatrix}-0.0217 & s+0.24\\ 0 & -100\end{vmatrix}=2.17$$

$$C_{21}=\begin{vmatrix}0 & 0.0058\\ -100 & s+2.4\end{vmatrix}=0.58$$

$$C_{22} = \begin{vmatrix} s+0.04167 & 0.0058 \\ 0 & s+2.4 \end{vmatrix} = (s+0.04167)(s+2.4)$$

$$C_{23} = \begin{vmatrix} s+0.04167 & 0 \\ 0 & -100 \end{vmatrix} = -100(s+0.04167)$$

$$C_{31} = \begin{vmatrix} 0 & 0.0058 \\ s+0.24 & -0.0058 \end{vmatrix} = -0.0058(s+0.24)$$

$$C_{32} = \begin{vmatrix} s+0.04167 & 0.0058 \\ -0.0217 & s+2.4 \end{vmatrix} = s^2 + 2.4117s + 0.1001$$

$$C_{33} = \begin{vmatrix} s+0.04167 & 0 \\ -0.0217 & s+0.24 \end{vmatrix} = s^2 + 0.2817s + 0.01$$

Then we have

$$Adj(s\mathbf{I}-\mathbf{A}) = \begin{bmatrix} s(s+2.64) & -0.58 & -0.0058(s+0.24) \\ 0.0217(s+2.4) & (s+0.04167)(s+2.4) & -(s^2+2.4417s+0.1101) \\ 2.17 & 100(s+0.04167) & s^2+0.2817s+0.01 \end{bmatrix}$$

Finally

$$\frac{Y}{U_1}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \frac{\begin{bmatrix} 2.17 & 100(s+0.04167) & s^2+0.28171s+0.01 \end{bmatrix}}{(s+2.6419)(s^2+0.0398s+0.0048)} \begin{bmatrix} 5.2 \\ -5.2 \\ 0 \end{bmatrix}$$

$$= \frac{-520s - 10.3844}{(s+2.6419)(s^2+0.0398s+0.0048)} = -520\frac{s+0.02}{(s+2.6419)(s^2+0.0398s+0.0048)}$$

**c.** 100% effectiveness means that $u_1 = 1$ or $U_1(s) = \frac{1}{s}$, so by the final value theorem

$$y(\infty) = \lim_{s\to 0} sY(s) = -\lim_{s\to 0} s\frac{520(s+0.02)}{(s+2.6419)(s^2+0.0398s+0.0048)}\frac{1}{s} = -820.1168$$

(virus copies per mL of plasma)

The closest poles to the imaginary axis are $-0.0199 \pm j0.0661$ so the approximate settling time will be $T_s \approx \frac{4}{0.0199} = 210$ days.

**83.**

**a.**

Substituting $\Delta F(s) = \frac{2650}{s}$ into the transfer function and solving for $\Delta V(s)$ gives:

$$\Delta V(s) = \frac{\Delta F(s)}{1908 \cdot s} = \frac{2650}{s(1908 \cdot s + 10)} = \frac{A}{s} + \frac{B}{(1908 \cdot s + 10)}$$

Here: $A = \left.\frac{2650}{(1908 \cdot s + 10)}\right|_{s=0} = 265$ and $B = \left.\frac{2650}{s}\right|_{s=-1/190.8} = -505,620$

Substituting we have:

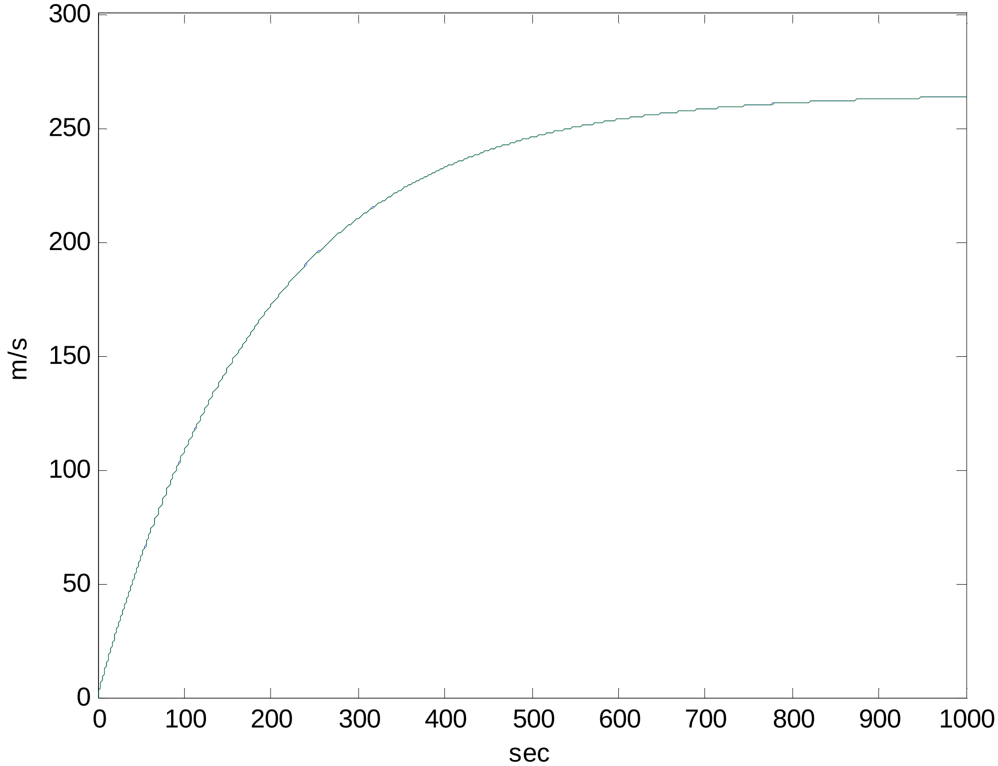
$$\Delta V(s) = \frac{265}{s} - \frac{505620}{(1908 \cdot s + 10)} = 265\left(\frac{1}{s} - \frac{1}{(s + 5.24 \times 10^{-3})}\right)$$

Taking the inverse Laplace transform, we have:

$$\Delta v(t) = 265(1 - e^{-5.24 \times 10^{-3} t}) \cdot u(t), \text{ in m/s}$$

**b.**

```
>> s=tf('s');

>> G=1/(1908*s+10);

>> t=0:0.1:1000;

>> y1=2650*step(G,t);

>> y2=265*(1-exp(-5.24e-3.*t));

>> plot(t,y1,t,y2)

>> xlabel('sec')

>> ylabel('m/s')
```




Both plots are identical.

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