M A T H 2 6 4 — F I N A L R E V I E W T O P I C S

I. Basics

Vectors

add and subtract vectors algebraically and graphically

magnitude of vectors, resultant force

Dot and Cross product

magnitude of dot product, magnitude and direction of cross product

what does $\mathbf{a}\cdot\mathbf{b}=0$ *mean? what does* $\mathbf{a}\times\mathbf{b}=0$ *mean? what is* $\mathbf{a}\cdot\mathbf{a}$*? what is* $\mathbf{a}\times\mathbf{a}$*?*

find projections and orthogonal projections of $\mathbf{b}$ *onto* $\mathbf{a}$

Lines and Planes

find equations for lines/planes

Graphing

graph basic surfaces $F(x,y,z)=const$ *in cartesian, cylindrical, spherical coordinates*

graph surfaces given by functions $z=f(x,y)$

II. Vector functions $\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle$

Sketch elementary curves $\mathbf{r}(t)$ (circles, helices, ellipses, lines)

Compute unit tangent vector, at an arbitrary point $\mathbf{r}(t)$ and at a specific points $\mathbf{r}(t_0)$

Find velocity, speed, acceleration

III. Scalar functions $f(x,y)$ **and** $f(x,y,z)$

Chain Rule

Directional Derivative. Gradient. Level Curves, Level surfaces.

Magnitude and direction of gradient vector, relative to level curves

Tangent Planes

Find Tangent Planes to surfaces $F(x,y,z)=0$ *or* $z=f(x,y)$*.*

Local Extrema

Find local and absolute max/min of $f(x,y)$ *on infinite or bounded domains*

IV. Integrals

Double Integrals $\iint_A f(x,y)dA$

Evaluate over general regions in cartesian or polar coordinates

Triple Integrals $\iiint_V f(x,y,z)dV$

Evaluate over general regions in cartesian, cylindrical or spherical coordinates

Line Integral I : $\int_C f(x,y,z)ds$

Parametrize C *and evaluate (be able to parametrize circles, ellipses, curves* $y=f(x)$*, lines)*

Line Integral II : $\int_C \mathbf{F}\cdot\mathbf{T}ds=\int_C \mathbf{F}\cdot\mathbf{dr}=\int_C Pdx+Qdy+Rdz$

Evaluate when $\mathbf{F}$ *is not conservative (need parametrization of* C*)*

Evaluate when $\mathbf{F}=\nabla f$ *is conservative (do not need parametrization!)*

Applications : Volumes, Centroids; Mass, center of mass.

V. Vector fields $\mathbf{F}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$

Graph vector fields

Compute divergence and curl

Check whether $\mathbf{F}$ is conservative ($\mathbf{F}=\nabla f$). If it is, find $\mathbf{F}$.

VI. Green's Theorem

Understand what it says, then you can remember it (and use it)

STUDY PROBLEMS

Note: These problems are only worthwhile doing if you don't refer to the book as you do them. Work them out as if you were in an exam. If necessary, put them aside and study the relevant section in the book.

I. BASICS

1. (a) Copy the vectors in Fig. 1 and use them to draw each of the following vectors: $\mathbf{a}+\mathbf{b}$, $-\frac{1}{2}\mathbf{a}$, $\mathbf{a}-\mathbf{b}$, $2\mathbf{a}+\mathbf{b}$.
   (b) Is $\mathbf{a}\cdot\mathbf{b}$ positive or negative?
   (c) Does $\mathbf{a}\times\mathbf{b}$ point inside or out of the plane.
   (d) Indicate the length $|\mathbf{a}\cdot\mathbf{b}|/|\mathbf{a}|$ in a figure. Also indicate the length $|\mathbf{a}\cdot\mathbf{b}|/|\mathbf{b}|$.
   (e) Indicate the length $|\mathbf{a}\times\mathbf{b}|/|\mathbf{a}|$ in a figure. Also indicate the length $|\mathbf{a}\times\mathbf{b}|/|\mathbf{b}|$.

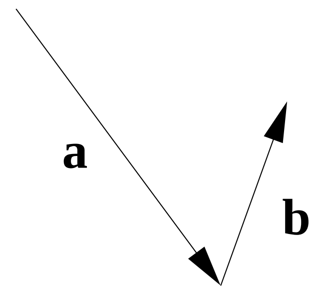


FIG 1

2. (a) If $\mathbf{u}$ and $\mathbf{v}$ are the vectors shown in Fig. 2a, find $\mathbf{u}\cdot\mathbf{v}$ and $|\mathbf{u}\times\mathbf{v}|$. Is $\mathbf{u}\times\mathbf{v}$ directed into the page or out of it?
   (b) Repeat for the vectors shown in Fig. 2b.

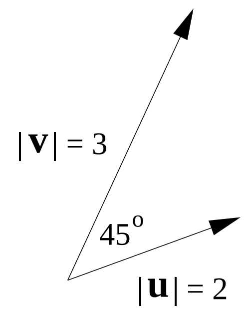


FIG 2a

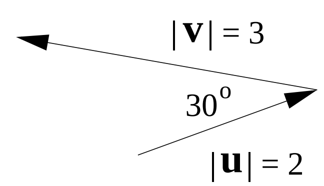


FIG 2b

3. If $\mathbf{u}$ and $\mathbf{v}$ are the vectors shown in Fig. 3, find $\mathbf{u}\cdot\mathbf{v}$ and $|\mathbf{u}\times\mathbf{v}|$. Is $\mathbf{u}\times\mathbf{v}$ directed into the page or out of it? Find $\mathbf{u}\cdot\mathbf{u}$ and $\mathbf{u}\times\mathbf{u}$.

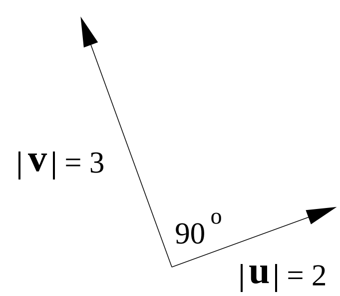


FIG 3

4. Draw two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$. In the same figure, draw $proj_{\mathbf{a}}\mathbf{b}$, the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ and indicate its length. Also indicate the length of the orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$. Write down expressions for the length of the vector projection and the length of the orthogonal projection. How can you use the dot product to differentiate accute and obtuse angles?

5. Given the three points $A(1,0,0)$, $B(2,0,-1)$, $C(1,4,3)$. Let $\mathbf{a}=\overline{AB}$, $\mathbf{b}=\overline{BC}$, $\mathbf{c}=\overline{CA}$.
   (a) Find the length of $proj_{\mathbf{a}}\mathbf{b}$, the vector projection of $\mathbf{b}$ onto $\mathbf{a}$. Find the vector $proj_{\mathbf{a}}\mathbf{b}$.
   (b) Find the length of $proj_{\mathbf{b}}\mathbf{a}$, the vector projection of $\mathbf{a}$ onto $\mathbf{b}$. Find the vector $proj_{\mathbf{b}}\mathbf{a}$.
   (c) Find the length of the orthogonal projection of $\mathbf{a}$ onto $\mathbf{b}$.
   (d) Find the area of the triangle ABC.
   (e) Find the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.
   (f) Draw a sketch of the three points A,B,C and the three vectors $\mathbf{a},\mathbf{b},\mathbf{c}$. From the sketch (ie, without computation) determine $\mathbf{a}+\mathbf{b}+\mathbf{c}$.

   5'. Forces: chain, sum of forces.

6. (a) Find the equation of the plane through the origin, spanned by the two vectors $\mathbf{u}=\langle 0,1,2\rangle$ and $\mathbf{v}=\langle 1,-2,3\rangle$.

(b) Find an equation for the line through the origin that is normal to the plane.

7. Find the equation of the plane that contains the three points $P(1,1,0)$, $Q(0,2,-1)$, $R(3,4,2)$.

8. Find an equation for the plane passing through (-4,1,2) and parallel to the plane $5z = 3-x-2y$.

9. Find parametric equations for the line passing through (-6,-1,0) and (2,-3,5).

10. (a) Describe a method to find the distance from a point to a plane. (Be clear and concise, so that someone else can use this method. Include a sketch. Do not refer to a formula in the book.)
    (b) Use your method to find the distance for the origin to the plane $4x - 6y + z = 5$. (Ans: $\frac{5}{\sqrt{53}}$)

11. Determine whether the following pairs of lines are parallel, skew, or intersecting. If they intersect, find the point of intersection. If they are parallel, determine whether they are identical or distinct lines.
    (a) $\mathbf{r}_1(t) = \langle 1,1,0\rangle + t\langle 1,-1,2\rangle$, $\mathbf{r}_2(t) = \langle 2,0,4\rangle + t\langle -1,1,0\rangle$
    (b) $\mathbf{r}_1(t) = \langle 1,0,2\rangle + t\langle 1,1,-5\rangle$, $\mathbf{r}_2(t) = \langle 1,-2,-1\rangle + t\langle 1,1,-1\rangle$
    (c) $\mathbf{r}_1(t) = \langle 1,0,2\rangle + t\langle 1,1,-5\rangle$, $\mathbf{r}_2(t) = \langle 1,-2,-1\rangle + t\langle 2,2,-10\rangle$

12. Find the points in which the line $x = t, y = 2 - t, z = 2 + t$ intersects the coordinate planes.

13. Describe the graph of the following equations (in $\Re^3$) in words.
    (a) $x = y^2 + z^2$,
    (b) $z = \sqrt{x^2 + y^2}$,
    (c) $z^2 = x^2 + y^2$
    (d) $x = 4$
    (e) $xy = 4$
    (f) $xy = 0$
    (g) $\theta = 3\pi/4$
    (h) $\phi = 3\pi/4$
    (i) $\phi = 0$
    (j) $\rho = 4$
    (k) $\rho = 4\cos\phi$
    (m) $r \sin\theta = 3$
    (n) $r = 3$
    (o) $r = \cos\theta$
    (p) $x = z$
    (q) $y = 3x - 2$
    (r) $y = z^2$
    (s) $4x^2 + y^2 + z^2 = 2z$
    (t) $y^2 + z^2 = 1 + x^2$
    (u) $6x + 4y + 3z = 12$

14. (a) What is the distance from a point (x, y, z) to the x-axis?

(b) What is the distance from a point (x, y, z) to the (y, z)-plane?

(c) Find an equation for the surface consisting of all points (x, y, z) in space for which the distance from (x, y, z) to the x-axis is twice the distance from (x, y, z) to the (y, z) plane. Simplify the equation, removing square roots and absolute values.

II. VECTOR FUNCTIONS

1. Sketch the following curves.
   (a) $\mathbf{r}(t) = \langle \cos 4t, t, \sin 4t \rangle, 0 \le t \le 2\pi$
   (b) $\mathbf{r}(t) = \langle 2\cos t, \frac{1}{2}\sin t, 0 \rangle, 0 \le t \le 2\pi$
   (c) $\mathbf{r}(t) = \langle 2 + t, -t, -1 + 2t \rangle$, $0 \le t \le 2$
   (d) $\mathbf{r}(t) = \langle t^4 + 1, t \rangle$, $-\infty \le t \le \infty$
   (e) $\mathbf{r}(t) = \langle t, t, \cos t \rangle$, $-\infty \le t \le \infty$

2. Sketch the given curves, find the unit tangent vector at $\mathbf{r}(t)$, find the unit tangent vector at the indicated point P.
   (a) $\mathbf{r}(t) = \langle \sqrt{2}\sin t, \sqrt{2}\cos t \rangle$, $0 \le t \le \pi$, P(1,1)
   (b) $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, $-\infty \le t \le \infty$, P(1/4, 1/8)
   (c) $\mathbf{r}(t) = \langle 1 + t, t^2 \rangle$, $-\infty \le t \le \infty$, P(2, 1)

3. The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin.
   (a) Find an equation for the line tangent to $\mathbf{r}_1$ at the origin.
   (b) Find the angle of intersection of the two lines.

4. Find the velocity, acceleration and speed of a particle with position $\mathbf{r}(t) = t^2\mathbf{i} + \ln t\mathbf{j} + t\mathbf{k}$.

5. Find the velocity and position vectors of a particle with acceleration $\mathbf{a}(t) = \mathbf{k}$ that one second into the motion has position and velocity $\mathbf{r}(1) = \mathbf{0}$ and $\mathbf{v}(1) = \mathbf{i} - \mathbf{j}$.

6. Add a projectile problem, or sth like it.

III. SCALAR FUNCTIONS $f(x, y)$, $f(x, y, z)$

1. Sketch the graphs of the following functions $f(x, y)$. In a separate plot, also sketch the level curves.
   (a) $f(x, y) = x^2 + 4y^2$
   (b) $f(x, y) = \sqrt{x^2 + y^2}$
   (c) $f(x, y) = x - 3y$
   (d) $f(x, y) = y^2$
   (e) $f(x, y) = x^2 - y^2$

2. Sketch the level surfaces of the following functions $f(x, y, z)$.
   (a) $f(x, y, z) = x^2 + y^2 + z^2$
   (b) $f(x, y) = x + y + 3z$
   (c) $f(x, y, z) = x^2 - y^2$

3. Find the rate of change of the function $f(x, y, z) = x^2 + y^2 + z$ along the path $x = \cos t, y = \sin t, z = t$.

4. If $u = xy + yz + zx$, $x = st, y = e^{st}, z = t^2$, find $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial t}$ when $s = 0$, $t = 1$.

5. If $f(x, y, z) = \frac{x}{y+z}$, P(4,1,1), $\mathbf{v} = \langle 1, 2, 3 \rangle$.
   (a) Find the rate of change of f at P in direction of the vector $\mathbf{v}$.
   (b) Find the rate of change of f at P in direction of the x-axis.
   (c) In what direction does f increase the fastest at P?
   (d) What is the largest rate of increase of f at P?
   (e) In what direction does f decrease the fastest at P?
   (f) Give a vector normal to the level surface $f = 2$ at P.
   (g) Find an equation for the plane tangent to the level surface $f = 2$ at P.

6. Consider the function $f(x, y) = x - y^2$.
   (a) Draw the level curves $f(x, y) = k$ for $k = -2, -1, 0, 1, 2$. Carefully label the level curves and the axes.
   (b) Find the gradient ∇f at (2,1) and enter it in your sketch.
   (c) If you stand at the point (1,2) and look toward the origin, does f increase or decrease in that direction?
   (d) In what direction does f increase the fastest? What is the rate of maximal increase?

7. Sketch several gradients in the contour map given in Section 15.7, # 38.

8. For each of the following surfaces S
   (i) write down a function $F = F(x, y, z)$ with the property that S is a level surface of F.
   (ii) Write down a vector that is normal to S at the given point
   (iii) write down an equation for the tangent plane to S at the given point.
   (a) $z = x^2 + y^2$, $P(1, 1, 2)$
   (b) $x + y - 2z = 5$, $P(2, 1, -1)$
   (c) $xy + xz + yz = 3$, $P(1, 1, 1)$
   (d) $x^2 = 2y^2 + 3z^2 - xyz + 4$, $P(3, -2, -1)$
   (e) $z = f(x, y)$, where $f(x, y) = xe^y + 3y$ and $(x, y) = (1, 0)$

9. Find and classify all the critical points of
   (a) $f(x, y) = x^2 + y^2 + x^2y + 4$
   (b) $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$

10. Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 + x^2y + 4$ on the square domain D given by $-1 \le x \le 1$, $-1 \le y \le 1$.

11. Consider a function $z = f(x, y)$. Estimate the change Δz in the function values if x and y change from a point (x_0, y_0) by an amount Δx and Δy respectively.

12. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in the table given in Section 15.4, #20.
    (a) Estimate the partial derivatives $\partial h/\partial v$, $\partial h/\partial t$ at the point $(v_o, t_o) = (40, 20)$.
    (b) Estimate the change in the wave height if the wind speed increases from 40 to 43 knots and the length of time t increases from 20 to 24 hours.

(c) Estimate the wave height if $v = 43$ and $t = 24$.

IV. INTEGRALS

1. (a) Write down a formula for the average f_{av} of a function of one variable, $f(x)$, on the interval $[a, b]$.

   (b) Find the average of $f(x) = \int_x^1 \cos(t^2)\, dt$ on the interval $[0, 1]$. (Ans: $\frac{1}{2}\sin 1$)

2. (a) The average of the function $f(x, y)$ on the circle $D : x^2 + y^2 \leq 2$ is estimated to be $f_{av} \approx 5$. Estimate the value of the integral $\int\int_D f(x, y) dA$.

   (b) The average density $\rho(x, y, z)$ of the cone $E : x^2 + y^2 \leq z^2, 0 \leq z \leq 10$ is estimated to be $\rho_{av} \approx 3$ g/cm^3, where x, y, z are measured in cm. Estimate the mass of the cone, $\int\int\int_V \rho(x, y, z) dV$.

3. Evaluate $\iiint_E x dV$ where E is the region above $z = \sqrt{x^2 + y^2}$, below $x^2 + y^2 + z^2 = 2$, with $x \geq 0$. (Ans: $\frac{\pi}{4} - \frac{1}{2}$)

4. (a) Set up an integral for the volume of the sphere of radius a centered at the origin, in cartesian coordinates, in cylindrical coordinates, and in spherical coordinates.

   (b) Evaluate the integral.

5. Evaluate $\iiint_E y dV$ where E is the region below $z = xy$ and above the triangle in the xy-plane with vertices (1,0), (2,1), and (4,0). (Ans: $\frac{11}{20}$)

6. Evaluate $\iiint_E (x + 2y) dV$ where E is bounded by $y = x^2$, $x = z$, $x = y$ and $z = 0$. (Ans: $\frac{2}{15}$)

7. Set up an integral for the volume bounded by the cone $z^2 = a^2(x^2 + y^2)$, $z = 1$, $z = 2$

   (a) by doing the z-integration first

   (b) by doing the r-integration first

   (c) Evaluate the integral. (Ans: $\frac{7\pi}{3a^2}$)

8. Evaluate the integral $\int_0^1 \int_{y^2}^1 y \sin(x^2)\, dxdy$. (Hint: reverse the order of integration.) (Ans: $\frac{1}{4}(1 - \cos 1)$)

9. Find the centroid of the tetrahedron with corners $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, $(0, 1, 1)$. (Ans: Volume $= \frac{1}{6}$, $(\overline{x}, \overline{y}, \overline{z}) = (\frac{1}{4}, \frac{3}{4}, \frac{1}{4})$)

10. (a) Set up an integral for the volume of the region above $z = 1$, below $x^2 + y^2 + z^2 = 4$, with $y \geq 0$ in cartesian, cylindrical and spherical coordinates.

    (b) Evaluate the integral. (Ans: $\frac{7\pi}{3}$)

11. (a) Use cylindrical coordinates to evaluate $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2 + y^2)^{3/2}\, dzdydx$ (Ans: $\frac{8\pi}{35}$)

    (b) Use spherical coordinates to evaluate $\displaystyle\int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} (x^2 + y^2 + z^2)^2\, dzdydx$ (Ans: $\frac{\pi}{14}$)

12. Sketch the region of integration of the following integrals.

    (a) $\displaystyle\int_0^{2\pi} \int_0^2 \int_{2r^2}^8 r\, dzdrd\theta$ (b) $\displaystyle\int_0^{2\pi} \int_0^2 \int_0^8 r\, dzdrd\theta$ (c) $\displaystyle\int_0^{2\pi} \int_0^2 \int_0^8 r\, drdzd\theta$

(d) $\int_0^{2\pi}\int_0^2\int_{2r}^8 r\,dzdrd\theta$ (e) $\int_0^{2\pi}\int_0^2\int_{r^2}^r r\,dzdrd\theta$

13. Evaluate the line integral $\int_C x^3 z\,ds$ where C is given by $x = 2\sin t, y = t, z = 2\cos t, 0 \le t \le pi/2$. (Ans: $4\sqrt{5}$)

14. Evaluate the line integral $\int_C y\,dx + z\,dy + x\,dz$ where C consists of the line segments from $(0,0,0)$ to $(1,1,2)$ and from $(1,1,2)$ to $(3,1,4)$. (Ans: $\frac{17}{2}$)

15. Evaluate the line integral $\int_C x\sqrt{y}\,dx$ where C consists of the shortest arc of the circle $x^2+y^2 = 1$ from (-1,0) to (0,1). (Ans: $-\frac{2}{5}$)

16. Find the work done by the force field $\mathbf{F}(x,y,z) = x\sin y\mathbf{i} + y\mathbf{j}$ on a particle that moves along the parabola $y = x^2$ from (-1,1) to (2,4). (Ans: $\frac{1}{2}(15 + \cos 1 - \cos 4)$)

17. A constant force $\mathbf{F} = 3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$ moves an object along the line segment from (1,0,2) to (5,3,8). Find the work done if the distance is measured in meters and the force in newtons. (Ans: 87 joules)

18. Find the length of the curve $\mathbf{r}(t) = 2t^{3/2}\,\mathbf{i} + \cos 2t\,\mathbf{j} + \sin 2t\,\mathbf{k}$, $0 \le t \le 1$. (Ans: $\frac{2}{27}(13^{3/2} - 8)$)

19. Is the vector field $\mathbf{F} = \langle 3x^2yz - 3y, x^3z - 3x, x^3y + 2z\rangle$ conservative? If so, find the potential function f.

20. Evaluate the line integral $\int_C (3x^2yz - 3y)\,dx + (x^3z - 3x)\,dy + (x^3y + 2z)\,dz$ where C is the curve shown in the Figure 4. (Ans: -4)

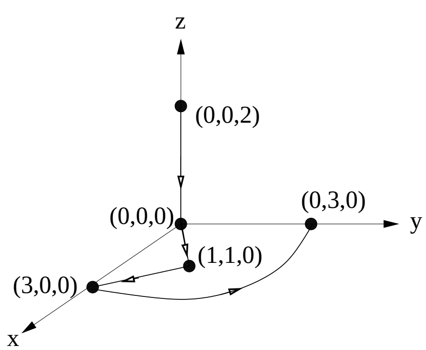


FIG 4

21. Let $\mathbf{F}(x,y) = (2x^3 + 2xy^2 - 2y)\mathbf{i} + (2y^3 + 2x^2y - 2x)\mathbf{j}$. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve shown in Figure 5.

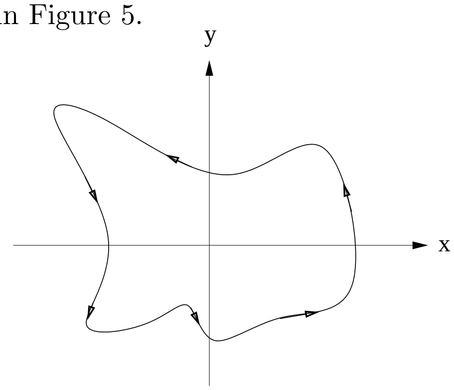


FIG 5

V-VI. VECTOR FIELDS AND GREENS THEOREM

1. For each of the following vector fields, sketch a representative set of vectors in the x-y plane. Make sure the magnitude and direction of the vectors is clear and correct.
   (a) $\mathbf{F}(x, y) = \langle -1, 2\rangle$ (b) $\mathbf{F}(x, y) = \frac{\langle x, y\rangle}{x^2+y^2}$ (c) $\mathbf{F}(x, y) = \langle y, -x\rangle$

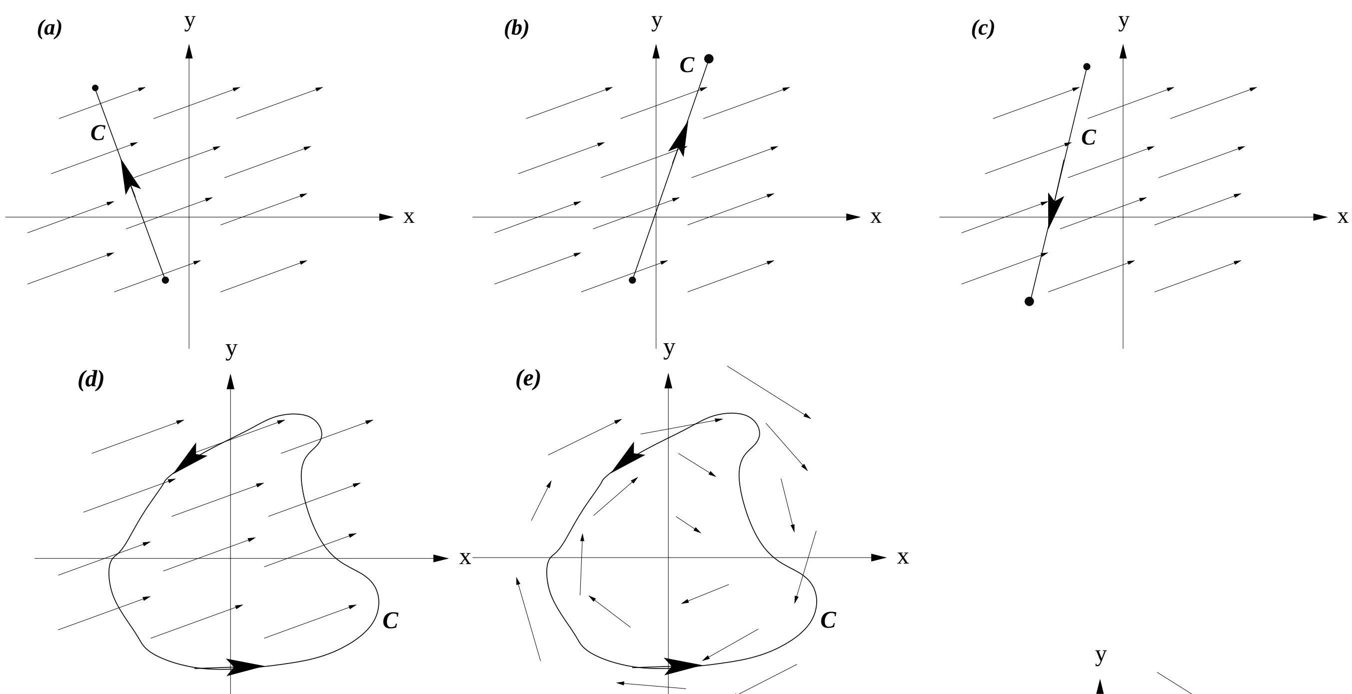
2. Find curl $\mathbf{F}$ and div $\mathbf{F}$ if $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 2x\sin y\mathbf{j} + 2z\cos y\mathbf{k}$.

3. Consider the vector field $\mathbf{u}(x, y, z) = \langle 1, y^2, 0\rangle$
   (a) Sketch a representative set of vectors in the $x - y$ plane.
   (b) Find curl $\mathbf{u}$ and div $\mathbf{u}$.
   (c) Where is the field compressing? Where is it expanding?
   (d) Draw a closed curve C, oriented counterclockwise, on top of your vector field. What is $\int_C \mathbf{u}\cdot\mathbf{T}ds$? Why?

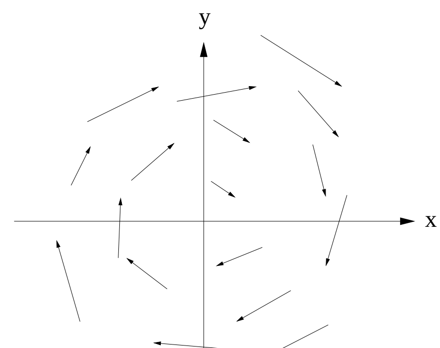
4. (a) Show that if $\mathbf{F}$ is a gradient field (ie conservative), then curl $\mathbf{F} = \mathbf{0}$.
   (b) Is $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 2x\sin y\mathbf{j} + 2z\cos y\mathbf{k}$ a gradient field?

5. (a) Show that if $\mathbf{F}$ is a curl (ie $\mathbf{F} = \text{curl}\mathbf{G}$ for some $\mathbf{G}$), then div $\mathbf{F} = 0$.
   (b) Is $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 2x\sin y\mathbf{j} + 2z\cos y\mathbf{k}$ a curl?

6. The following figures show a vector field $\mathbf{F}$ and an oriented curve C. From the following figures, estimate whether $\int_C \mathbf{F}\cdot d\mathbf{r}$ is positive, negative or zero. Explain your answer.



7. Is the field shown in the figure conservative? Why or why not?



8. Green's Theorem: see homework problems, and Chapter 17 Review, Exercises: 16,17.