Law of Mass Action The preceding example can be generalized in the following manner. Suppose that a grams of substance A are combined with b grams of substance B. If there are M parts of A and N parts of B formed in the compound, then the amounts of substances A and B remaining at time t are, respectively,

$$a - \frac{M}{M+N}X \quad \text{and} \quad b - \frac{N}{M+N}X.$$

Thus

$$\frac{dX}{dt} \propto \left[a - \frac{M}{M+N}X\right]\left[b - \frac{N}{M+N}X\right]. \tag{14}$$

Proceeding as before, if we factor out $M/(M+N)$ from the first term and $N/(M+N)$ from the second term, the resulting differential equation is the same as (11):

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X), \tag{15}$$

where

$$\alpha = \frac{a(M+N)}{M} \quad \text{and} \quad \beta = \frac{b(M+N)}{N}.$$

Chemists refer to reactions described by equation (15) as the **law of mass action.**

When $\alpha \neq \beta$, it is readily shown (see Problem 9) that a solution of (15) is

$$\frac{1}{\alpha - \beta} \ln\left|\frac{\alpha - X}{\beta - X}\right| = kt + c. \tag{16}$$

When we assume the natural initial condition $X(0) = 0$, equation (16) yields the explicit solution

$$X(t) = \frac{\alpha\beta[1 - e^{(\alpha-\beta)kt}]}{\beta - \alpha e^{(\alpha-\beta)kt}}. \tag{17}$$

Without loss of generality we assume in (17) that $\beta > \alpha$ or $\alpha - \beta < 0$. Since $X(t)$ is an increasing function, we expect $k > 0$, and so it follows immediately from (17) that $X \to \alpha$ as $t \to \infty$.

Escape Velocity In Example 1 of Section 1.2 we saw that the differential equation of a free-falling object of mass m near the surface of the earth is given by

$$m\frac{d^2s}{dt^2} = -mg \quad \text{or simply} \quad \frac{d^2s}{dt^2} = -g,$$

where s represents the distance from the surface of the earth to the object and the positive direction is considered to be upward. In other words, the underlying assumption here is that the distance s to the object is small when compared with the radius R of the earth; put yet another way, the distance y from the center of the earth to the object is approximately the same as R. If, on the other hand, the distance y to an object, such as a rocket or a space probe, is large compared to R, then we combine Newton's second law of motion and his universal law of gravitation to derive a differential equation in the variable y. The solution of this differential equation can be used to determine the minimum velocity, the so-called **escape velocity,** needed by a rocket to break free of the earth's gravitational attraction.