#40 Interference and Standing Waves Pre-class

Due: 11:00am on Wednesday, November 28, 2012

Note: You will receive no credit for late submissions. To learn more, read your instructor's Grading Policy

Creating a Standing Wave

Learning Goal:

To see how two traveling waves of the same frequency create a standing wave.

Consider a traveling wave described by the formula

$$y_1(x,t) = A\sin(kx - \omega t).$$

This function might represent the lateral displacement of a string, a local electric field, the position of the surface of a body of water, or any of a number of other physical manifestations of waves.

Part A

Which one of the following statements about the wave described in the problem introduction is correct?

ANSWER:

- (selected) The wave is traveling in the $+x$ direction.
- The wave is traveling in the $-x$ direction.
- The wave is oscillating but not traveling.
- The wave is traveling but not oscillating.

Correct

Part B

Which of the expressions given is a mathematical expression for a wave of the same amplitude that is traveling in the opposite direction? At time $t = 0$

this new wave should have the same displacement as $y_1(x,t)$, the wave described in the problem introduction.

ANSWER:

- $A\cos(kx-\omega t)$
- $A\cos(kx+\omega t)$
- $A\sin(kx-\omega t)$
- $A\sin(kx+\omega t)$

Correct

The principle of *superposition* states that if two functions each separately satisfy the wave equation, then the sum (or difference) also satisfies the wave equation. This principle follows from the fact that every term in the wave equation is linear in the amplitude of the wave.

Consider the sum of two waves $y_1(x,t)+y_2(x,t)$, where $y_1(x,t)$ is the wave described in Part A and $y_2(x,t)$ is the wave described in Part B. These waves have been chosen so that their sum can be written as follows:

$$y_s(x,t)=y_e(x)y_t(t).$$

This form is significant because $y_e(x)$, called the envelope, depends only on position, and $y_t(t)$ depends only on time. Traditionally, the time function is taken to be a trigonometric function with unit amplitude; that is, the overall amplitude of the wave is written as part of $y_e(x)$.

Part C

Find $y_e(x)$ and $y_t(t)$. Keep in mind that $y_t(t)$ should be a trigonometric function of unit amplitude.

Express your answers in terms of A, k, x, ω, and t. Separate the two functions with a comma.

Hint 1. A useful identity

A useful trigonometric identity for this problem is

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B).$$

Hint 2. Applying the identity

Since you really need an identity for $\sin(A-B)$, simply replace B by $-B$ in the identity from Hint C.1, keeping in mind that $\sin(-x) = -\sin(x)$.

ANSWER:

$y_e(x)$, $y_t(t)$ = $2A\sin(kx), \cos(\omega t)$

Correct

Part D

Which one of the following statements about the superposition wave $y_s(x,t)$ is correct?

ANSWER:

- ○ This wave is traveling in the $+x$ direction.
- ○ This wave is traveling in the $-x$ direction.
- ◉ This wave is oscillating but not traveling.
- ○ This wave is traveling but not oscillating.

Correct

A wave that oscillates in place is called a *standing wave*. Because each part of the string oscillates with the same phase, the wave does not appear to move left or right; rather, it oscillates up and down only.

Part E

At the position $x = 0$, what is the displacement of the string (assuming that the standing wave $y_s(x, t)$ is present)?

Express your answer in terms of parameters given in the problem introduction.

ANSWER:

$y_s(x = 0, t)$ = 0

Correct

This could be a useful property of this standing wave, since it could represent a string tied to a post or otherwise constrained at position $x = 0$. Such solutions will be important in treating normal modes that arise when there are two such constraints.

Part F

At certain times, the string will be perfectly straight. Find the first time $t_1 > 0$ when this is true.

Express t_1 in terms of ω, k, and necessary constants.

Hint 1. How to approach the problem

The string can be straight only when $\cos(\omega t) = 0$, for then $y_s(x, t) = 0$ also (for all x). For any other value of $\cos(\omega t)$, $y_s(x, t)$ will be a sinusoidal function of position x.

ANSWER:

t_1 = $\frac{\pi}{2\omega}$

Correct

Part G

From Part F we know that the string is perfectly straight at time t=\frac{\pi}{2\omega}. Which of the following statements does the string's being straight imply about the energy stored in the string?

1. There is no energy stored in the string: The string will remain straight for all subsequent times.
2. Energy will flow into the string, causing the standing wave to form at a later time.
3. Although the string is straight at time t=\frac{\pi}{2\omega}, parts of the string have nonzero velocity. Therefore, there is energy stored in the string.
4. The total mechanical energy in the string oscillates but is constant if averaged over a complete cycle.

ANSWER:

- a
- b
- c (selected)
- d

Correct

Nodes of a Standing Wave (Cosine)

Learning Goal:

To understand the concept of nodes of a standing wave.

The nodes of a standing wave are points where the displacement of the wave is zero at all times. Nodes are important for matching boundary conditions, for example that the point at which a string is tied to a support has zero displacement at all times (i.e., the point of attachment does not move).
Consider a standing wave, where y represents the transverse displacement of a string that extends along the *x* direction. Here is a common mathematical form for such a wave:

$$y(x,t) = A\cos(kx)\sin(\omega t),$$

where A is the maximum transverse displacement of the string (the amplitude of the wave), which is assumed to be nonzero, k is the wavenumber, ω is the angular frequency of the wave, and t is time.

Part A

Which one of the following statements about wave $y(x,t)$ is correct?

ANSWER:

- This wave is traveling toward $+x$
- This wave is traveling toward $-x$.
- (selected) This wave is oscillating but not traveling.
- This wave is traveling but not oscillating.

Correct

Each part of the string oscillates with the same phase, so the wave does not appear to move left or right; rather it oscillates up and down only.

Part B

At time $t = 0$, what is the displacement of the string $y(x,0)$?

Express your answer in terms of A, k, and other previously introduced quantities.

ANSWER:

$y(x, 0) =$ 0

Correct

Part C

What is the displacement of the string as a function of x at time $T/4$, where T is the period of oscillation of the string?

Express the displacement in terms of A, x, and k only. That is, evaluate $\omega \cdot \frac{T}{4}$ and substitute it in the equation for $y(x, t)$.

Hint 1. Period in terms of ω

What is the period T in terms of the angular frequency ω?

Express your answer in terms of ω and constants like π.

ANSWER:

$T = \frac{2\pi}{\omega}$

Hint 2. $\sin\left(\omega \frac{T}{4}\right)$

What is the value of the time-dependent trigonometric function $\sin(\omega t)$ at the time $T/4$?

Express your answer in terms of the previously introduced quantities.

ANSWER:

$\sin\left(\omega\frac{T}{4}\right)$ = 1

ANSWER:

$y\left(x,\frac{T}{4}\right)$ = $A\cos(kx)$

Correct

Part D

At which three points x_1, x_2, and x_3 closest to $x = 0$ but with $x > 0$ will the displacement of the string $y(x,t)$ be zero for all times? These are the first three nodal points.

Express the first three nonzero nodal points as multiples of the wavelength λ, using constants like π. List the factors that multiply λ in increasing order, separated by commas.

Hint 1. What is equal to zero?

If $y(x,t) = 0$, then $A\cos(kx)\sin(\omega t) = 0$. Because $A > 0$, either $\cos(kx) = 0$ and/or $\sin(\omega t) = 0$ if the displacement of the string is to be zero for all times. The frequency ω is a constant for a given wave and t may take any positive value. Therefore $\sin(\omega t)$ cannot equal zero for all times. Hence you need to find solutions to the equation $\cos(kx) = 0$.

Express the first three values of kx (call them $(kx)_1$, $(kx)_2$, and $(kx)_3$) for which $y(x,t) = 0$ as multiples of π. List them in increasing order, separated by commas. You do not need to enter π. It is given on the right. You only need to enter the factor that multiplies π.

ANSWER:

(k x)_1, $(kx)_2$, (k x)_3 = 0.500,1.50,2.50 π

Hint 2. λ in terms of k

What is the wavelength λ of the standing wave in terms of k?

Express your answer in terms of k and constants like π.

ANSWER:

λ = $\frac{2\pi}{k}$

Hint 3. Putting it all together

The nodes of the wave are points where $\cos(kx) = 0$. If you substitute $x_i = \alpha\lambda$ using the previous results, you will find the multiples α of λ where $\cos(kx) = 0$.

ANSWER:

x_1, x_2, x_3 = 0.250,0.750,1.25 λ

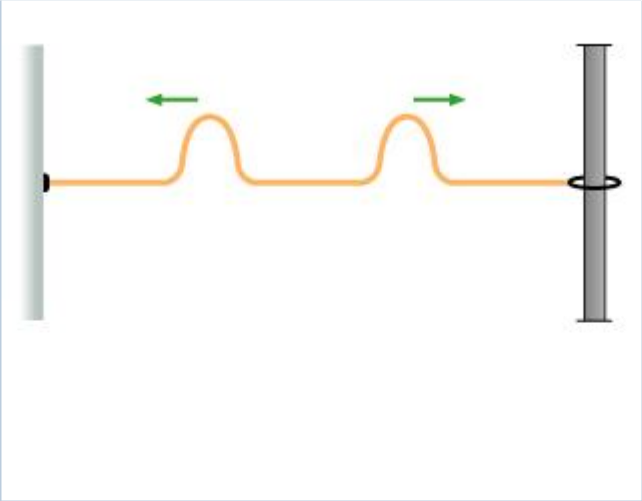
Correct

Note that $x = 0$ is *not* a node of this particular wave; rather, it is an antinode, a point that moves with maximum displacement. If this wave were on a string with one end terminated by being tied to a long, strong thread of negligible mass, that end would behave as an antinode.

Two Identical Pulses along a String

Two identical pulses are moving in opposite directions along a stretched string that has one fixed end and the other movable, as shown in the figure.

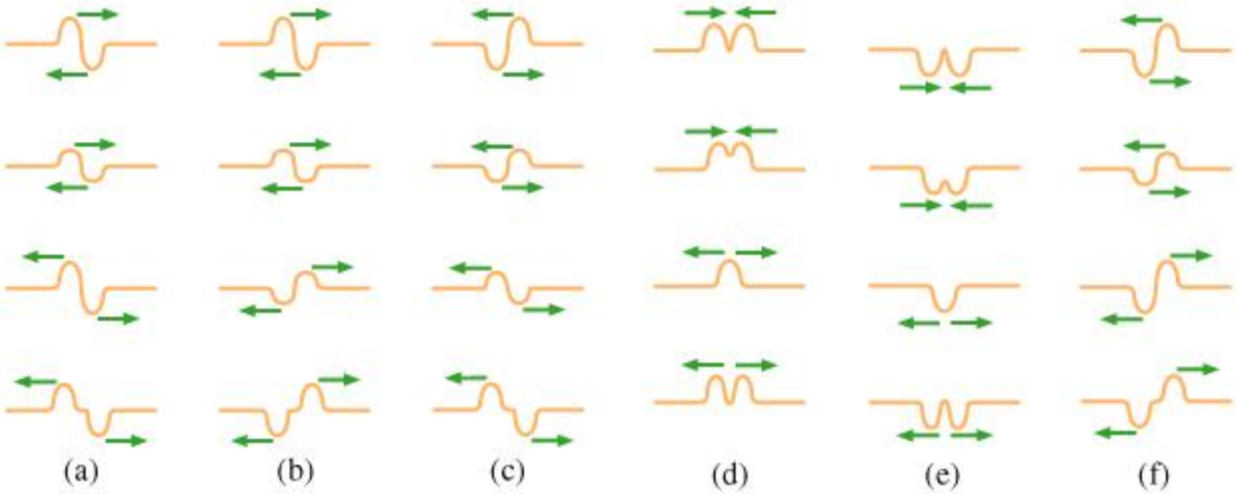
Above each pulse a green arrow indicates the direction of motion of the pulse.



Part A

The two pulses reflect off the boundaries of the string, and at some later time, they pass through the middle of the string and interfere.

Below are six different sequences of snapshots taken as the two pulses meet in the middle of the string. Time increases from top to bottom in each sequence. Which sequence correctly represents the displacement of the string as the pulses interfere?

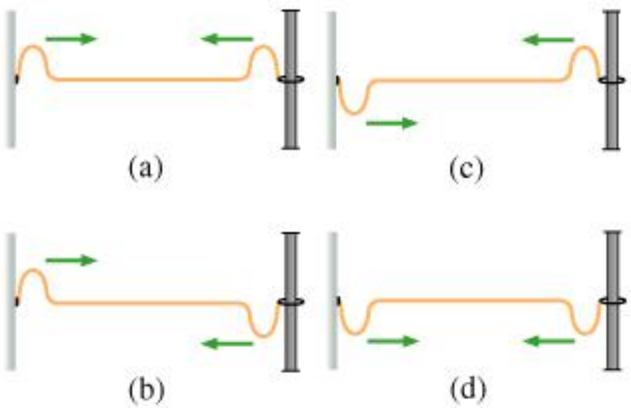


Hint 1. Superposition of waves

When two pulses move through the same section of a string and overlap, you need to add the displacements of the individual pulses at each point to determine the displacement of that section of the string. Note that the pulses pass through the same section of the string only after each reaches a boundary and is reflected.

Hint 2. Reflection of waves at the boundary

When each pulse reaches one end of the string, a reflected pulse forms. Which of the following sketches represents the reflected pulses?



Hint 1. Reflection of a wave pulse at a fixed end

When a wave pulse reaches the end of the string, it is reflected and travels in the opposite direction from the initial pulse. If the end of the string is tied to a support and cannot move, the pulse becomes inverted upon reflection.

Hint 2. Reflection of a wave pulse at a free end

If the end of the string is free to move, the direction of displacement of the reflected pulse is the same as that of the initial pulse.

ANSWER:

- A
- B
- C
- D

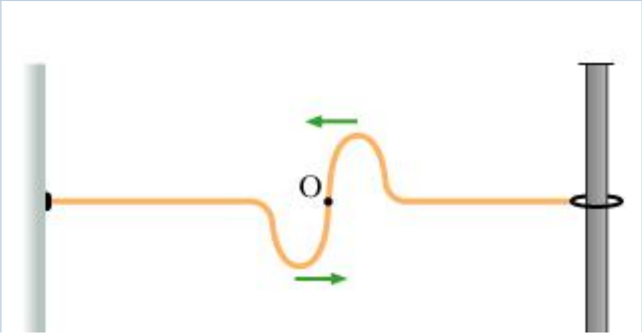
ANSWER:

- A
- B
- C
- D
- E
- F

Correct

Part B

Consider the point where the two pulses start to overlap, point *O* in the figure. What is the displacement of point *O* as these pulses interfere?



Hint 1. How to approach the problem

At any time, the displacement of point *O* is given by the algebraic sum of the displacements of the two pulses at that point.

Hint 2. Displacement of point *O* if only one pulse were present

Let Δt be a short interval of time after the two pulses have begun to overlap. If only the pulse coming from the right were present, the displacement of point *O* at time Δt would be $+y_0$. What would its displacement be if only the other pulse were present?

Express your answer in terms of y_0.

Hint 1. Inverted pulses

Keep in mind that the two pulses are identical but inverted; thus their displacements are equal magnitude but opposite.

ANSWER:

-y_0

ANSWER:

- It varies with time.
- It remains zero.
- It depends on the (identical) amplitude of the pulses.
- It is zero only when the pulses begin to overlap.

Correct

Part C

Why does destructive interference occur when the two pulses overlap instead of constructive interference?

ANSWER:

- because the pulses are traveling in opposite directions
- because a pulse is inverted upon reflection
- because the pulses are identical and cancel each other out
- because constructive interference occurs only when the pulses have the same amplitude

Correct

Part D

As the pulses interfere destructively there is a point in time when the string is perfectly straight. Which of the following statements is true at this moment?

ANSWER:

- The energy of the string is zero.
- The string is not moving either up or down.
- The string has only kinetic energy.

Correct

When you apply a force to a string to produce a pulse, work is done on the string and energy is stored in it. As the pulse travels along the string, this energy is transported. In particular, this energy is converted back and forth between kinetic and potential energy as the particles in the string oscillate. When destructive interference occurs and the string is momentarily straight, it does not mean that the string has zero energy. Rather, the energy transported by the pulses has been completely converted into kinetic energy. A short time later, the pulses will be reconstituted.

± Standing Waves in Action

Consider a string of length 1.0 meter, fixed at both ends, with mass 100 grams and tension 100 Newtons.

Part A

Give the number of nodes and antinodes set up on the string if it is driven at a frequency of 95 hertz. Be sure to include the ends of the string in your count.

Give the number of nodes followed by the number of antinodes, separated by a comma.

Hint 1. How to approach the problem

To solve this problem, you will need to determine the wavelength of the wave on the string. You have enough information to determine this using basic equations about waves on strings. Once you know the wavelength, use your knowledge of standing waves to determine the number of nodes and antinodes. Recall that nodes are points where the string does not move, and antinodes are points of maximal motion. It will be helpful to draw a picture of the standing wave.

Hint 2. Find the wavelength

What is the wavelength λ of the standing wave on the string?

Express your answer in meters.

Hint 1. Relating wavelength and speed

Recall that the wavelength λ, the speed v, and the frequency f of a wave are related by the equation $v = \lambda f$.

Hint 2. Find the speed of the wave

Recall that the speed v of a wave on a string with tension T and linear density μ is given by the equation

v=\sqrt{\frac T \mu}.

What is the speed of a wave on the string described in the introduction?

Express your answer in meters per second to three significant figures.

ANSWER:

v = 31.6 $\mathrm{m/s}$

ANSWER:

λ = 0.333 m

Hint 3. Relating nodes to wavelength

Which of the following equations accurately relates the ratio of the length L of the string to the wavelength λ with the number of antinodes a for a standing wave on a string that is fixed at both ends?

ANSWER:

- L/\lambda=a/2
- L/\lambda=a-1
- L/\lambda=a
- L/\lambda=a+1
- L/\lambda=2a

ANSWER:

7,6

Correct

This applet allows you to simulate the effects of various driving frequencies on different strings.

Part B

Construct a wave in the applet that has four antinodes. Give the value of the expression

\frac{T}{\mu f^2},

where T is the tension, μ is the linear density, and f is the driving frequency.

Express your answer in SI units (meters-kilograms-seconds).

ANSWER:

\frac{T}{\mu f^2} = 0.25

Correct

Note that because this is a ratio, many different values of the quantities in the applet could give the same value (for example, you could multiply the tension and linear density by the same constant value).

Part C

The reason that the ratio in the previous part always comes out the same, regardless of which of the many possible configurations of the string that give four antinodes you chose, is that the ratio actually equals a much simpler quantity that will always be the same for configurations of the string that yield four antinodes. Which of the following gives that quantity? Here E is the energy of the wave, and A is the amplitude.

Hint 1. How to approach the problem

There are two ways of thinking about this problem that may be helpful to you. One way is to look at the dimensions of the quantity. Once you have reduced the dimensions, they should suggest the correct answer. The other way to think about it is to look for familiar quantities. Specifically, you know that f is related to a wave, while T and μ are related to the string itself. If you could turn f into quantities related to the string, then you might be able to use knowledge about strings to reduce the expression down to a simpler one. Similarly, if you can turn T and μ into quantities related to waves, then you might be able to use knowledge about waves to reduce the expression.

ANSWER:

- λ
- \lambda^2
- E
- E^2
- A
- A^2

Correct

Score Summary:

Your score on this assignment is 95.1%.
You received 19.01 out of a possible total of 20 points.