ECE 314 – Signals and Systems Fall 2012

Solutions to Homework 1

Problem 1.1 Find the even and odd components of each of the following signals:

(a) $x(t) = \cos(t) + \sin(t) + \sin(t)\cos(t)$

Solution: First, let's determine what is $x(-t)$:

$$\begin{aligned} x(-t) &= \cos(-t) + \sin(-t) + \sin(-t)\cos(-t) \\ &= \cos(t) - \sin(t) - \sin(t)\cos(t). \end{aligned}$$

Now, we can evaluate what are the even and odd components of $x(t)$ using the formulas $x_e(t) = 1/2[x(t) + x(-t)]$ and $x_o(t) = 1/2[x(t) - x(-t)]$.

$$\begin{aligned} x_e(t) &= \frac{1}{2}[\cos(t) + \sin(t) + \sin(t)\cos(t) + \cos(t) - \sin(t) - \sin(t)\cos(t)] \\ &= \cos(t). \end{aligned}$$

$$\begin{aligned} x_o(t) &= \frac{1}{2}[\cos(t) + \sin(t) + \sin(t)\cos(t) - \cos(t) + \sin(t) + \sin(t)\cos(t)] \\ &= \sin(t) + \sin(t)\cos(t). \end{aligned}$$

(c) $x(t) = 1 + t\cos(t) + t^2\sin(t) + t^3\sin(t)\cos(t)$

Solution: We may solve this by inspection, if we consider the following properties:

1. The product of two even signals is also even $(x_1(-t)x_2(-t) = x_1(t)x_2(t))$.
2. The product of two odd signals is even $(x_1(-t)x_2(-t) = (-x_1(t))(-x_2(t)) = x_1(t)x_2(t))$.
3. The product of an even signal and an odd signal is odd $(x_e(-t)x_o(-t) = x_e(t)(-x_o(t)) = -x_e(t)x_o(t))$.

¿From the properties above, we may draw the following conclusions:

- $t\cos(t)$ is odd, since t is odd and $\cos(t)$ is even.
- $t^2\sin(t)$ is odd, since t^2 is even and $\sin(t)$ is odd.
- $t^3\sin(t)\cos(t)$ is even, since t^3 is odd, $\sin(t)$ is odd, and $\cos(t)$ is even.

Therefore, by inspection, we may conclude that the even and odd components of $x(t)$ are, respectively,

$$\begin{aligned} x_e(t) &= 1 + t^3\sin(t)\cos(t), \quad \text{and} \\ x_o(t) &= t\cos(t) + t^2\sin(t). \end{aligned}$$

Problem 1.3 Figure 1.15 shows a triangular wave. What is the fundamental frequency of this wave? Express the fundamental frequency in units of Hz and rad/s.

Solution: Note that

$$x(t) = x(t+0.2) = x(t+0.4) = \ldots, \quad \forall t \in \mathbb{R}$$

Therefore, $\{0.2, 0.4, \ldots\}$ is the set of periods of $x(t)$. The fundamental period is the least value in that set, i. e., $T_o = 0.2$. Hence, the fundamental frequency will be $f_o = 1/T_o = 5$ Hz. In radians per second, the fundamental frequency is $\omega_o = 2\pi f_o = 10\pi$ rad/s.

Problem 1.4 Determine the fundamental frequency of the discrete-time square wave shown in Fig. 1.16.

Solution: As the graph shows,

$$x[n] = x[n+8] = x[n+16] = \ldots, \quad \forall n \in \mathbb{Z}.$$

Therefore, the fundamental period is $N_o = 8$, and, consequently, $\omega_o = 2\pi/N_o = \pi/4$ rad/sample.

Problem 1.5

a) First note that $x(t) = \cos^2(2\pi t) = 0.5\Big(1+\cos(4\pi t)\Big)$. Hence, if x is periodic, then its period should be the same as that of $\cos(4\pi t)$, which is $1/2$.

c) Suppose that x is periodic with period, say, $T > 0$. Then, it must be true that $x(t+T) = x(t)$ for all t. In particular, pick $t = 0$ and obtain

$x(T) = x(0)$, or $\cos(0) = e^{-2T}\cos(2\pi T)$, or $1 = e^{-2T}\cos(2\pi T)$. The latter implies that $\cos(2\pi T) = e^{2T}$, which is greater than 1 (since $T > 0$). Thus, we conclude that $\cos(2\pi T) > 1$, which is a contradition. Hence, our assumption that x is periodic is false. This is an example of what is called *proof by contradiction.*

d) Note that $x[n+2] = (-1)^{(n+2)} = (-1)^n = x[n]$. Hence, x is periodic with period 2.

f) If $\cos[n+N] = \cos[n]$, then we require that $(n+N) - n = 2\pi k$, where k is an integer. (Two sinusoids are identical if and only if their arguments are offset by a multiple of 2π.) Hence, we require that $N = 2\pi k$, but there is no integer k that would satisfy this equation. Thus, x is not periodic.

g) As in part (f), if $\cos[2\pi(n+N)] = \cos[2\pi n]$, then we require that $2\pi(n+N) - 2\pi n = 2\pi k$, where k is an integer. Hence, we require that $2\pi N = 2\pi k$, or $N = 1$. Thus, x is periodic with period 1.

Problem 1.6

(a) What is the total energy of the rectangular pulse shown in Fig. 1.14(b) ?

Solution: The energy of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

In our case, the signal is equal to zero outside the interval $[-T_1/2, T_1/2]$. Therefore,

$$\begin{aligned} E &= \int_{-T_1/2}^{T_1/2} A^2 dt \\ &= A^2 \int_{-T_1/2}^{T_1/2} dt \\ &= A^2 T_1. \end{aligned}$$

(b) What is the average power of the square wave shown in Fig. 1.14(a)?

Solution: The average power of a wave $x(t)$ is defined as:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

So, first we need to determine the average power as a function of T, that is

$$\begin{aligned} P_T &= \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} dt \\ &= 1, \end{aligned}$$

since $|x(t)|^2$ is constant and equal to 1.

Now, taking P_T to the limit, we get

$$P = \lim_{T\to\infty} P_T = \lim_{T\to\infty} 1 = 1.$$

Observation: For a periodic wave, the average power defined above can be shown to correspond exactly to

$$P = \frac{E_o}{T_o},$$

where E_o is the energy of the wave in one period of length T_o.