ECE 340: PROBABILISTIC METHODS IN ENGINEERING

SOLUTIONS TO HOMEWORK #3

2.39 Here, we have replacement and we do care about the ordering.
Thus, the number of distinct ordered triples is $\mathbf{60 \times 60 \times 60 = 60^3}$

2.43 There are 24 special characters, 10 numbers, 26 lowercase letters and 26 uppercase letters.

Without any restriction, there are $(24 + 10 + 26 + 26)^8 = 86^8 = \mathbf{2.99 \times 10^{15}}$ different passwords of length 8 that combine special characters, numbers and letters.
However, we have the restriction that all the passwords must contain at least one special character, so from the above quantity we need to subtract all the passwords that do not contain any (one or more) special characters, or in other words, all the passwords that are exclusively made up of a combination of numbers and letters.
There are $(10 + 26 + 26)^8 = 62^8 = 2.18 \times 10^{14}$ such passwords.
So, the amount of passwords of length 8 that have at least one special character is:

$$86^8 - 62^8 = \mathbf{2.77 \times 10^{15}}$$

The passwords can have from 8 to 10 characters, so the total amount of passwords that comply with the conditions is:

$$\#P = (86^8 - 62^8) + (86^9 - 62^9) + (86^{10} - 62^{10}) = \mathbf{2.15 \times 10^{19}}$$

If a password can be tested in 1 microsecond, the time it would take to try them all is:

$$t = 2.15 \times 10^{19} * (1\mu s) = 2.15 \times 10^{13} \text{ seconds} = \mathbf{682.95 \times 10^3} \text{ years.}$$

2.45

a) We have 5 choices for a shirt, and for each choice of shirt, we have 3 choices of jeans.

The total number of possible combinations is therefore $\binom{3}{1}\binom{5}{1} = 15$.

b) The maximum amount of time the student can dress without repeating a combination of jeans and t-shirt is at the most 15 days. Many combinations can be found that comply with the restriction that the student cannot wear the same t-shirt on two consecutive days. One such combination is:

(S1, J1), (S2, J1), (S3, J1), (S4, J1), (S5, J1), (S1, J2), (S2, J2), (S3, J2), (S4, J2), (S5, J2), (S1, J3), (S2, J3), (S3, J3), (S4, J3), (S5, J3).

Where S1 represents “shirt 1,” J1 represents “jeans 1”, and so on.

2.46 The order in which the 4 toppings are selected does not matter, so we have sampling **without ordering**.

If toppings may not be repeated (**without replacement**), we will have

$\binom{15}{4} = 1365$ possible deluxe pizzas.

If toppings may be repeated (**with replacement**), we have sampling with replacement and without ordering. The number of such arrangements is (see Example 2.23):
$\binom{15+4-1}{4} = 3060$ possible pizzas.
Another way to solve this problem is consider five categories of pizzas, indexed by *k*=1, *k*=2, …, *k*=4, where the *k*th category is characterized by having *k* distinct types of topping. For the *k*th category, there are $\binom{15}{k}$ ways to select the distinct k toppings from the 15 possible toppings. And, once we select a set of k toppings, we can create $\binom{4-1}{k-1}$ different pizzas. The latter is the number of ways we can divide slices into k (ordered) bins, where each bin represents a topping. Thus, in total we will have $\sum_{i=1}^{4}\binom{15}{k}\binom{3}{i-1}$ choices, which is also equal to 3,060.
Hence, we just discovered the formula:
$\binom{n+k-1}{k} = \sum_{i=1}^{k}\binom{n}{k}\binom{k-1}{i-1}$.

2.47 Imagine the sequence of students selecting their seats. The first student that comes in has 60 choices; the second has 59 and so on. Hence, for 45 students and 60 desks: we have

$$60(59)(58)\ldots(16) = \mathbf{6.36x10^{69}}$$

choices, or $n!/(n-k)!$, where n = 60 and k=45.

2.50 There are 5^5 possible placements of the 5 balls in the 5 cells. Without any restriction each ball has 5 choices of cells, and there are five balls, so we have 5^5 number of ways in total to put 5 balls in 5 cells. Now let's count the number of ways we can end up with exactly one ball per cell. Note that if we have one ball in each cell, we can rearrange the balls in 5! different ways, while maintaining one ball in each cell. Thus, the probability in is

$$\frac{5!}{5^5} = 0.0384$$

See also Example 2.19.

2.62 From problem 2.2 we have A $\supset$ B; therefore,

$$P(A\,|\,B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1 \text{ (which makes intuitive sence)}$$

and

$$P(B\,|\,A) = \frac{P(A \cap B)}{P(A)} = \frac{6/36}{21/36} = 2/7.$$

2.69 First, notice that the event B can be rewritten as the open interval B= (0, 1), A = [-1, 0), and C=(0.75, 2]. It is clear that A and B are disjoint. Therefore,

$$P(A\,|\,B) = \frac{P(A \cap B)}{P(B)} = 0$$

$$P(B\,|\,C) = \frac{P(B \cap C)}{P(C)} = \frac{P(0.75 < x < 1)}{P(x > 0.75)} = \frac{0.25/3}{1.25/3} = 1/5$$

$$P(A\,|\,C^C) = \frac{P(A \cap C^C)}{P(C^C)} = \frac{P(x<0)}{P(x<0.75)} = \frac{1/3}{1.75/3} = 4/7$$

$$P(B\,|\,C^C) = \frac{P(B \cap C^C)}{P(C^C)} = \frac{P(0 < x < 0.75)}{P(x<0.75)} = \frac{0.75/3}{1.75/3} = 3/7$$

2.77

a) Let X0 and X1 denote the events that a 0 and 1 are transmitted, respectively. Similarly, Let Y0 and Y1 denote the events that a 0 and 1 are received, respectively. Note the X0 and X1 form a partition of the sample space Ω={(0S,0R), (1S,0R), (0S,1R), (1S,1R)}.

Now by using the lay of total probability in conjunction with conditional probabilities, we have

$P(Y0) = P(Y0|X0)\ P(X0) + P(Y0|X1)\ P(X1)$

$= (1- \varepsilon_1)\ p + \varepsilon_2\ (1-p)$

By Bayes rule,

b) $P(X0 \mid Y1) = P(Y1 \mid X0)\ P(X0) / P(Y1) = \varepsilon_1\ p / [1-\{(1- \varepsilon_1)\ p + \varepsilon_2\ (1-p)\}] = \varepsilon_1\ p / [\varepsilon_1\ p - (1- \varepsilon_2)\ (1-p)]$

Also,

$P(X1 \mid Y1) = 1 - P(X0 \mid Y1) = 1 - \varepsilon_1\ p / [\varepsilon_1\ p - (1- \varepsilon_2)\ (1-p)]$.

Given that Y1 has occurred, the event X1 is more likely than X0 ***if***

$$P(X1 \mid Y1) > P(X0 \mid Y1)]$$

or equivalently ***if***

$$\frac{\varepsilon_1 p}{\varepsilon_1 p + (1-\varepsilon_2)(1-p)} > \frac{(1-\varepsilon_2)(1-p)}{\varepsilon_1 p + (1-\varepsilon_2)(1-p)}$$

or ***if***

$\varepsilon_1 p > (1-\varepsilon_2)(1-p)$

2.81

Let X0 and X1 be defined as in Problem 2.77; and let Y0, Y1, and Y3 denote the events that a 0, 1, or 2 are received, respectively.

a)

$P(Y0) = P(Y0|X0)\, P(X0) + P(Y0|X1)\, P(X1) + P(Y0|X2)\, P[X2)$

$= (1-\varepsilon) * 1/3 + 0 + \varepsilon * 1/3$

$= 1/3$

$P(Y1) = P(Y1|X0)\, P(X0) + P(Y1|X1)\, P(X1) + P(Y1|X2)\, P(X2)$

$= \varepsilon * 1/3 + (1-\varepsilon) * 1/3$

$= 1/3$

$P(Y2) = P(Y2|X0)\, P(X0) + P(Y2|X1)\, P(X1) + P(Y2|X2)\, P(X2)$

$= \varepsilon * 1/3 + (1-\varepsilon) * 1/3$

$= 1/3$

b) Using Bayes rule:

$P(X0|Y1) = P(Y1|X0)P(X0)/P(Y1) = \varepsilon$

$P(X1|Y1) = P(Y1|X1)P(X1)/P(Y1) = 1-\varepsilon$

$P(X2|Y1) = P(Y1|X2)P(X2)/P(Y1) = 0$

Special Problems:

a) In how many ways can you place n identical balls in k bins numbered as B1, B2, ,,, Bk? Assume that k is less than or equal to n. We require that each bin should contain at least one ball.

We can recast this problem as the number of ways we cane separate or divide n balls in k divisions. To answer this questions, we recognize that we will need k-1 dividers tat we should place among the n balls. Next, we realize that there are n-1 places for these k-1 dividers to be placed. Hence, the total number of ways is $\binom{n-1}{k-1}$.

b) In how many ways can you place k identical balls in n numbered bins? Assume that k is less than or equal to n. Here we allow each bin to have multiple balls. This is the type of problem encountered in Problem 2.46, and the answer is

$\binom{n+k-1}{k}$ or equivalently $\sum_{i=1}^{k}\binom{n}{k}\binom{k-1}{i-1}$.