ECE 340: PROBABILISTIC METHODS IN ENGINEERING

SOLUTIONS TO HOMEWORK #6

3.35

a) By the definition of conditional probability,

$$P(\{X=k\}|\{X>0\}) = \frac{P(\{X=k\}\cap\{X>0\})}{P\{X>0\}} = \frac{P\{X=k\}}{P\{X>0\}} = \frac{P\{X=k\}}{15/16}$$

$$= \begin{cases} \dfrac{\left(\frac{8}{16}\right)}{\left(\frac{15}{16}\right)} = \dfrac{8}{15}, & k=1 \\ \dfrac{\left(\frac{7}{16}\right)}{\left(\frac{15}{16}\right)} = 7/15, & k=2 \end{cases}$$

Note: we used the fact that $\{X=k\} \subset \{X>0\}$ for $k=1, 2$.

b) $P(\{X=k\} \mid \{N_m = 1\}) = \frac{P(\{(X=k\}\cap\{N_m=1)\})}{P\{N_m=1\}} = \frac{P(\{(X=k\}\cap\{N_m=1)\})}{1/2}$

$$= \begin{cases} \dfrac{\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{1/2} = \dfrac{3}{4}, & k=1 \\ \dfrac{\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)}{\frac{1}{2}} = \dfrac{1}{4}, & k=2 \end{cases}$$

3.44

a) $S=\{1,2,3,4,5\}$ $A=\{\zeta > 3\}$.

Let us assume that all outcomes are equally likely. Now,

$P\{I_A=0\} = P(A^c)=P\{\zeta=1,2,3\}=3/5$;

$P\{I_A=1\} = P(A)=P\{\zeta=4,5\}= 2/5$;

$E[I_A] = 0 \times 3/5 + 1 \times 2/5 = 2/5$.

b) $S=[0,1]$ $A=\{0.3 < \zeta < 0.7\}$

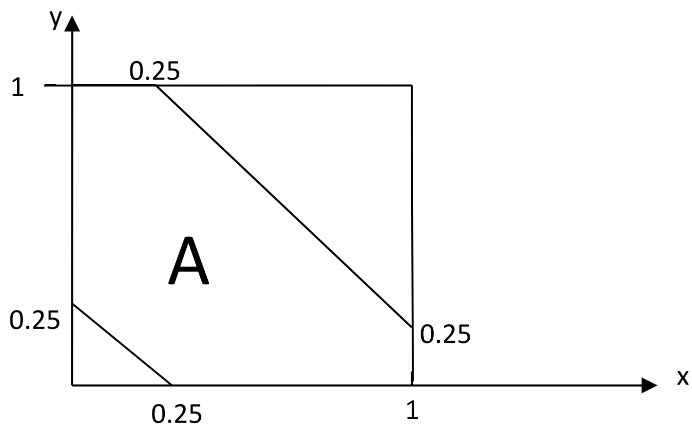
Again let us assume that we have uniform outcomes; that is, if E is an event, the $P(A)=\text{length}(E)$.

$P\{I_A=0\} = P(A^c)=0.3+0.3 = 0.6$

$P\{I_A=1\} = P(A) = 0.4$

$E[I_A] = 0 \times 0.6 + 1 \times 0.4 = 0.4$

c) $S=\{\zeta = (x,y): 0 < x < 1,\ 0 < y < 1\}$ and
$A = \{ \zeta = (x,y): 0.25 < x+y < 1.25\}$



$I_A(\zeta)= 0$ when the pair (x,y) is not in the indicated area.
$I_A(\zeta)= 1$ when the pair (x,y) is in the indicated area.

The probabilities can be calculated by computing the areas
$P\{I_A=0\} = \frac{0.25^2}{2} + \frac{(1-0.25)^2}{2} = \frac{5}{16} = 0.3125$

$P\{I_A=1\} = 1- P\{I_A=0\} = 1 - \frac{5}{16} = \frac{11}{16} = 0.6875$

$E[I_A] = 0 \times 0.3125 + 1 \times 0.6875 = 0.6875$

d) $S=(-\infty , \infty)$ and $A=\{\zeta >a\}$
$I_A(\zeta)=$ if $\zeta > a$, and it is 0 otherwise.
$P\{I_A=1\} =P(A)$ since ω is in A if and only if $I_A=1$.
Similarly, $P\{I_A=0\} = P(A^c)=1-P(A)$

$E[I_A] = 1 \times P(A) + 0 \times P(A^c)= P(A)$

3.45 A and B are events from a random experiment with sample space S.
a) Show that, $I_S = 1$
From the definition of the indicator function we have that,

$$I_S(\zeta) = \begin{cases} 0 & if\ \zeta\ not\ in\ S \\ 1 & if\ \zeta\ in\ S \end{cases}$$

However, by the definition of sample space, all the points ζ have to be in S, so I_S=1 for any $\zeta \in \Omega$.
Likewise, we show that, $I_\varnothing = 0$
From the definition of the indicator function we have that,

$$I_\varnothing(\zeta) = \begin{cases} 0 & if\ \zeta\ not\ in\ \varnothing \\ 1 & if\ \zeta\ in\ \varnothing \end{cases}$$

but by the definition of the null event Ø, contains no outcomes and hence never occurs, so $I_\varnothing$=0 for any $\zeta \in \Omega$.

b) $I_{A \cap B} = I_A I_B$

- $\boldsymbol{I_{A\cap B}(\zeta) = 0}$ if $\zeta \notin A\ and\ \zeta \notin B$, which corresponds to I_A=0 and I_B=0; hence, $I_A I_B$ =0
  or if $\zeta \epsilon A\ and\ \zeta \notin B$, which corresponds to I_A=1 and I_B=0; hence, $I_A I_B$ =0
  or if $\zeta \notin A\ and\ \zeta \notin B$, which corresponds to I_A=0 and I_B=1; hence, $I_A I_B$ =0.
- $\boldsymbol{I_{A\cap B}(\zeta) = 1}$ if $\zeta \epsilon A\ and\ \zeta \epsilon B$, which is corresponds to I_A=1 and I_B=1; hence, $I_A I_B$ =1.

To show that,
$I_{AUB} = I_A + I_B - I_{A\cap B}$ notice that

- $\boldsymbol{I_{AUB}(\zeta) = 0}$ only if $\zeta \notin A\ and\ \zeta \notin B$, which is corresponds to I_A=0 and I_B=0, and $I_A + I_B - I_{AUB} = 0$
- $\boldsymbol{I_{AUB}(\zeta) = 1}$ if $\zeta \epsilon A\ and\ \zeta \epsilon B$, which in this case makes to I_A=1 and I_B=1 and I_{AUB}=1, and $I_A + I_B - I_{AUB} = 1$
  or if $\zeta \epsilon A\ and\ \zeta \notin B$, which corresponds to I_A=1 and I_B=0 and I_{AUB}=0, so $I_A + I_B - I_{AUB} = 1$
  or if $\zeta \notin A\ and\ \zeta \notin B$, which is equivalent to I_A=0 and I_B=1 and I_{AUB}=0, so $I_A + I_B - I_{AUB} = 1$

c) The expected values are
$E[I_S] = 0p_I(0) + 1p_I(1) = p_I(1) = 1$
$E[I_\varnothing] = p_I(1) = 0$
$E[I_{A\cap B}] = p_I(1) = P(A \cap B)$
$E[I_{AUB}] = p_I(1) = P(A \cup B) = P(A) + P(B) – P(A \cap B)$

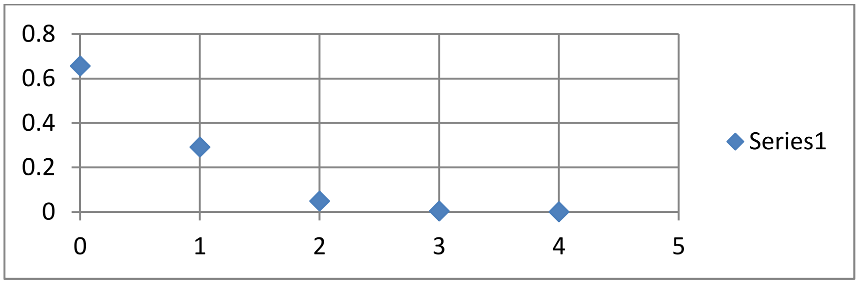
3.48

The pmf of a binomial random variable is given as

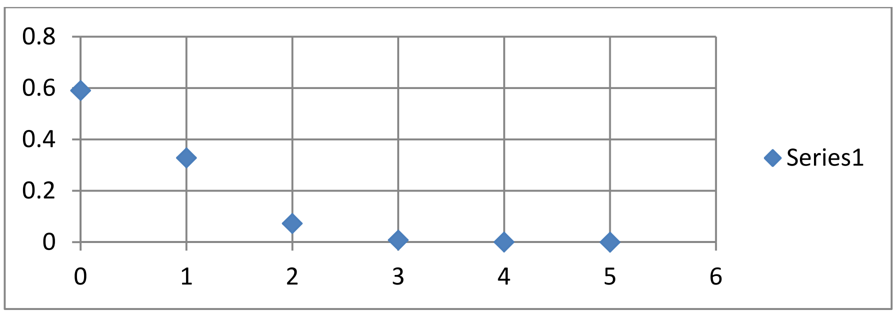
$$p_X(k) = P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \dots, n$$

The pmfs of various values of n and p are given in the following figures

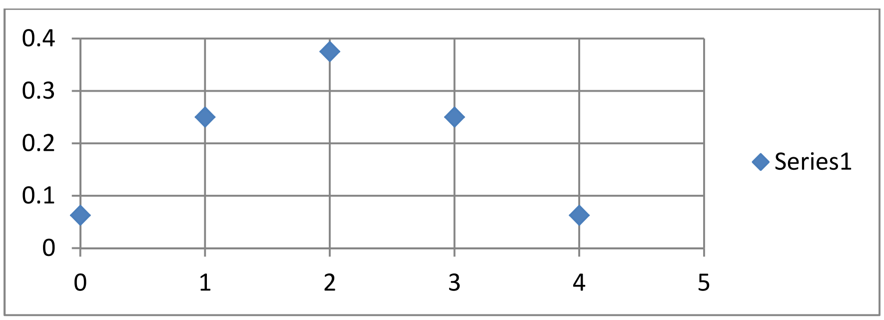
for n = 4 and p = 0.1



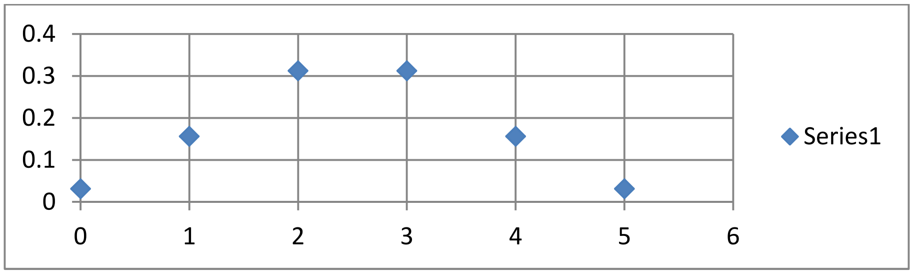
for n = 5 and p = 0.1



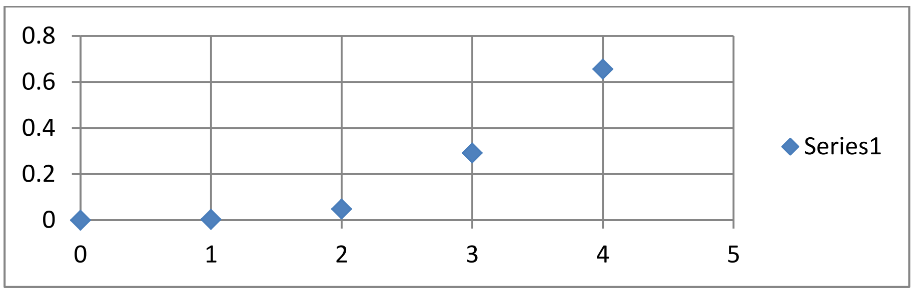
for n = 4 and p = 0.5



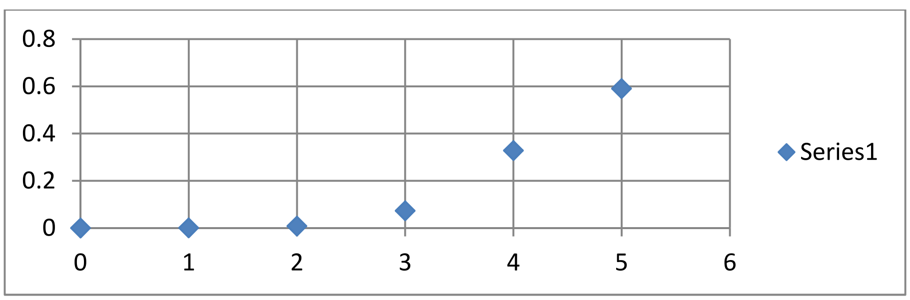
for n = 5 and p = 0.5



for n = 4 and p = 0.9



for n = 5 and p = 0.9



3.49

a) Let I_K denote the outcome of the *k*th Brenoulli trial. The probability that the single event occurred in the *k*th trial is

$$P(\{I_k = 1\}|\{X = 1\}) = \frac{P\left(\{I_k = 1\} \cap \{I_j = 0 \; for \; all \; j \neq k\}\right)}{P\{X = 1\}}$$
$$= \frac{P(\{0, 0, \dots, 1, 0, \dots 0\})}{P\{X = 1\}}$$

$$= \frac{p(1-p)^{n-1}}{\binom{n}{1} p(1-p)^{n-1}} = \frac{1}{n}$$

Thus, the single event is equally likely to have occurred in any of the *n* trials.

b) The probability that the two successes occurred in trials *j* and *k* is

$$P(\{I_j = 1, I_k = 1 \,|X = 2\}) = \frac{P(\{I_j = 1, I_k = 1\} \cap \{\, I_m = 0\, for\ all\ m \neq j,k\})}{P\{X = 2\}}$$

$$= \frac{p^2(1-p)^{n-2}}{\binom{n}{2} p^2(1-p)^{n-2}} = \frac{1}{\binom{n}{2}}$$

c) If *X*=*k*, then the locations of the successes are randomly selected from the $\binom{n}{k}$ possible permutations of possible locations. The concept of completely at random can be interpreted as the concept of 'uniformly distributed' as can be seen from parts a) and b).

3.52

p=0.01 *N*=number of error-free characters until the first error.

a) P({*N*=*k*})=(*1*-*p*)^k*p* *k*=0,1,2,…

b) $E[N] = \sum_{k=0}^{\infty} k(1-p)^k p = (1-p)p\sum_{k=0}^{\infty} k(1-p)^{k-1}$

$$= (1-p)p\frac{1}{\left(1-(1-p)\right)^2} = \frac{1-p}{p} \quad by\ Eq. (3.14)$$

c) $0.99 = P\{N > k_0\} = \sum_{k_0+1}^{\infty}(1-p)^k p = p(1-p)^{k_0+1}\sum_{k=0}^{\infty}(1-p)^k$

$$= (1-p)^{1001} \quad \Rightarrow \quad p = 1 - 0.99^{\frac{1}{1001}} = 1.004x10^{-5}$$

3.56 Let's denote the random variable X as the cost of repair of the audio player in one year period, and denote c as the charge for the one year warranty.

The pmf of r.v. X can be found as

$P(X = x) = \binom{12}{(x/20)}\left(\frac{1}{12}\right)^{\frac{x}{20}}(1-\frac{1}{12})^{12-\frac{x}{20}}$ for $x = 0, 20, 40, 60, \ldots, 240$.

Thus,

$$P(X = 0) = \begin{pmatrix} 12 \\ \frac{0}{20} \end{pmatrix}\left(\frac{1}{12}\right)^{\frac{0}{20}}(1-\frac{1}{12})^{12-\frac{0}{20}} = 0.351995628014137$$

$$P(X = 20) = 0.383995230560877$$

$$P(X = 40) = 0.191997615280438$$
$$P(X = 60) = 0.058181095539527$$
$$P(X = 80) = 0.011900678633085$$
$$P(X = 100) = 0.001731007801176$$
$$P(X = 120) = 0.000183591736488$$
$$P(X = 140) = 0.000014305849596$$
$$P(X = 160) = 0.000000812832363$$
$$P(X = 180) = 0.000000032841712$$
$$P(X = 200) = 0.000000000895683$$
$$P(X = 220) = 0.000000000014805$$
$$P(X = 240) = 0.000000000000112$$

If now we want the probability of losing money on a player is 1% or less, we want $P[X > c] \leq 0.01$. And the question now is 'what value should the c take such that the previous inequality is satisfied?'

Note that P[X>c] is monotonically decreasing as c increases and
P[X>60] = 0.013830430605021 > 0.01;
and P[X>c] = 0.013830430605021 > 0.01 for any c in the interval of (60,80);
P[X>80] = 0.001929751971936 <= 0.01;
P[X>c] <= P[X>80] = 0.001929751971936 <= 0.01 for any c >= 80.

So the manufacturer should charge at least 80 dollars to make sure the probability of losing money on a player is 1% or less (excluding the initial cost of $50.).

The average cost (of repair) per player is $E[X] = \sum_x x\, P[X = x] = 20$ dollars.
This expectation can be calculated by the following matlab code:

```
for x = 0:20:240
cost((x+20)/20) = x*(nchoosek(12,x/20))*((1/12)^(x/20))*((11/12)^(12-(x/20)));
end
>> sum(cost)

ans =

  19.999999999999996

>>
```

3.59 (Please also see Example 3.30 on page 122)

a) We know that the average is 6000 requests per minute, which is equivalent to 0.1 requests per millisecond, which corresponds to having α = *λt* = 0.1*t* in the Poisson distribution (where *t* is in milliseconds). The probability of having no requests in a 100-ms period is

P{*N*=*0*} = $\frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t} = e^{-0.1*100} = e^{-10} = 45.399 * 10^{-6}$

b) The probability that there are between 5 and 10 requests in a 100-ms period

$$P\{5 \le \ N \le \ 10\} = \sum_{k=5}^{10} \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \sum_{k=5}^{10} \frac{(0.1 \times 100)^k}{k!} e^{-0.1\times 100} = 0.5538$$

3.62 Solution:

The Poisson r.v. has the following pmf:

$$P[N = k] = p_N(k) = \frac{\alpha^k}{k!} e^{-\alpha}, \quad for\ k = 0, 1, 2, ...$$

$$\frac{p_k}{p_{k-1}} = \frac{\left(\frac{\alpha^k}{k!} e^{-\alpha}\right)}{\frac{\alpha^{k-1}}{(k-1)!} e^{-\alpha}} = \frac{\alpha}{k}$$

$$If\ \alpha < 1\ \ then\ \ \frac{p_k}{p_{k-1}} = \frac{\alpha}{k} < 1\ \ for\ k \ge 1$$

$\therefore p_k\ decreases\ as\ k\ increases\ from\ 0$

$\therefore p_k\ attains\ its\ maximum\ at\ k = 0$

$If\ \alpha > 1\ then$ (Note that we denote by [x] the largest integer that is smaller than or equal to x.)

If $0 \le k \le [\alpha] < \alpha,\ then\ \frac{p_k}{p_{k-1}} = \frac{\alpha}{k} > 1$

Hence, p_k increases from k=0 to k=[α]

If $[\alpha] \le \alpha \le k,\ \frac{p_k}{p_{k-1}} = \frac{\alpha}{k} < 1$

and p_k decreases as k increases beyond [α]

$\therefore p_k\ attains\ its\ maximum\ at\ k_{max} = [\alpha]$

$If\ \alpha = [\alpha]\ then\ for\ k = [\alpha]$

$$\frac{p_k}{p_{k-1}} = 1, \text{and hence}, p_{k_{max}} = \ p_{k_{max}-1}$$

3.68 Solution:

$$E[X] = \sum_{k=1}^{L} kP\left(\{X = k\}\right) = \sum_{k=1}^{L} k\frac{1}{L} = \frac{1}{L}\sum_{k=1}^{L} k = \frac{L(L+1)}{2L} = \frac{L+1}{2}$$

$$\sigma_X^2 = E[X^2] - E[X]^2 = \sum_{k=1}^{L} k^2 \frac{1}{L} - \left(\frac{L+1}{2}\right)^2 = \frac{L(L+1)(2L+1)}{6L} - \frac{(L+1)^2}{4} = \frac{L^2-1}{12}.$$

Special Problem: [0,1] is uncountable.
Proof:
If the interval [0,1] were countable, then there must be a one-to-one function, h, that maps the set of natural numbers {1,2,3,…} onto [0,1]. That is, for each x in [0,1], there must be a k such that $h(k)=x$. In other words, the range of h is [0,1]. We will show that this assumption will lead to a contradiction, and thus there is no such function, which, in turn, proves that we cannot label *all* the elements of [0,1] using natural numbers.

For i=1, 2, 3, …, let us write $h(1)$, $h(2)$, $h(3)$,…, in their decimal representation as follows:

$h(1)=0.b^{(1)}_1\ b^{(1)}_2\ b^{(1)}_3, \ldots$
$h(2)=0.b^{(2)}_1\ b^{(2)}_2\ b^{(2)}_3 \ldots$
.
.
.
$h(n)=0.b^{(n)}_1\ b^{(n)}_2\ b^{(n)}_3, \ldots, b^{(n)}_n, \ldots$
.
.

Here, the b's are simply integers from the set {0,1, 2 ,…, 9}.

Now construct a number x_0 between 0 and 1 using the decimal representation:
$x_0=0.c_1\ c_2\ c_3 \ldots$, with the proviso that for each i=1, 2, 3, …, $c_i \neq b^{(i)}_i$. Note that the number x_0 has the property $x_0 \neq h(i)$, since for any i=1, 2, 3,…, the ith decimal point of x_0 is different from the ith decimal point of $h(i)$. Hence, we have found a number, namely x_0 , that is not in the range of h. Thus, the range of h is not [0,1], which is a contradiction to the assumption.

One implication of this remarkable fact that we have just proved is that the topic of discrete random variables is not broad enough to cover all random variables of interest. Specifically, we must go beyond discrete random variables to include random variables whose ranges are intervals or the entire real line.