ECE-314, Fall 2012
Midterm Examination II

Nov. 20, 2012; 6:00PM–8:00PM

Closed book; no calculators

Problem 1 Consider the system represented by the differential equation $y'' + 6y' + 9y = x$.
(a) Is this system stable? Justify your answer.

Find the roots of the characteristic equation without any input: $(r+3)^2 = 0$, so we have a repeated root at -3. Since the root has negative real part, the system is stable.

(b) Find the impulse response for this system.

From class handout: $h(t) = te^{-3t}u(t)$. We also observe that $\int_{-\infty}^{\infty} |h(t)|dt < \infty$, which confirms the answer in part (a).

Problem 2 Consider the system described by the following difference equation: $y[n] = 0.6y[n-1] + 2x[n], n \geq 0$.
(a) Is this system stable? Justify your answer.

Find the roots of the characteristic equation without any input: $(\rho - 0.6) = 0$, so we have a repeated root at 0.6. Since the root is inside the unit circle, the system is stable.

(b) Find the impulse response response $h[n]$.

The impulse response has the same form as the zero-input solution, i.e., $h[n] = d(0.6)^n$, $n \geq 0$, with the proviso that we use a derived forced initial condition at $n = 0$ and set all initial conditions prior to $n = 0$ to zero. To find the derived forced initial condition at zero, we use the difference equation with every x replaced by a delta function, or $h[n] = 0.6h[n-1] + \delta[n] + 2\delta[n-2]$. Therefore, $h[0] = 0.6h[-1] + \delta[0] + 2\delta[0-2]$. Using the fact that $h[-1] = 0$, we obtain $h[0] = 1$,and $h[n] = (0.6)^n u[n]$.

Problem 3 Obtain an analytical solution to the convolution $z[n] = x[n] * y[n]$, where $x[n] = u[n] - u[n-6]$ and $y[n] = nu[n]$.

Write $z[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$. By sketching $x[k]$ and $y[n-k]$, we observe that $z[n] = 0$ when $n \leq 0$. For $1 \leq n \leq 5$, $z[n] = \sum_{k=0}^{n}(n-k) = n(n+1) - 0.5n(n+1) = 0.5n(n+1)$. Finally, for $n \geq 6$, $z[n] = \sum_{k=0}^{5}(n-k) = 6n - 0.5(5)(6) = 6n - 15$.

Problem 4 It is known that if the input $3\delta[n-1]$ is applied to a certain linear time-invariant system, the output is $5(2^n)u[n-1]$.
a) Is this system causal? Justify your answer thoroughly.

To find the impulse response, we exploit homogeneity and time invariance to scale the output $5(2^n)u[n-1]$ by 1/3 and replace every n by $n+1$. Hence, $h[n] = (5/3)(2^{n+1})u[n]$. Since $h[n]$ is a causal signal, the system is causal.

b) Is this system memoryless? Justify your answer thoroughly.
Since the impulse response is not a delta function, the system has memory.

c) Is this system stable? Justify your answer thoroughly.

Since, $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$, the system is unstable. One can also argue directly that since an input that is bounded $(3\delta[n-1])$ resulted in an unbounded output, the system is unstable.

Problem 5 Consider the differential equation $y'' + 4y' + 4y = x$, with initial conditions $y(0^-) = 2$ and $y'(0^-) = 1$.
(a) Find the form of the zero-input solution to this equation and explain how you would go about evaluating any constants therein.

The characteristic equation for the homogenous equation (with the input set to zero) is $(r+2)^2 = 0$, and its solution yields $r = -2$, repeated. Hence, the zero-input response is $y_{0x}(t) = c_1 e^{-2t}u(t) + c_2 te^{-2t}u(t)$. We find the constants c_1 and c_2 by applying the initial conditions to this solution. That is, we set $y_{0x}(0) = 2$ and $y'_{0x}(0) = 1$.

(b) Without evaluating the final answer, write a mathematical expression for the forced output $y_f(t)$ in response to $e^{-t}\cos(0.1\pi t)u(t)$. Your answer should involve no unknown functions.

$h(t) = te^{-2t}u(t)$. Next, $y_f(t) = h(t)*e^{-t}\cos(0.1\pi t)u(t) = \int_0^t e^{-\tau}\cos(0.1\pi\tau)(t-\tau)e^{-2(t-\tau)}d\tau$.

Problem 6. Perform the convolution $e^{-t} * [u(t) - u(t-10)]$.

$e^{-t} * [u(t) - u(t-10)] = \int_{-\infty}^{\infty} e^{-(t-\tau)}[u(\tau) - u(\tau - 10)]\, d\tau = \int_0^{10} e^{-t+\tau}\, d\tau = e^{-t}(e^{10t} - 1)$.

Problem 7 It is known that if the $5\delta(t+1)$ is applied to a certain linear time-invariant system, the output is $e^{-t+1}\cos(0.1\pi t)u(t)$. Prove that this system is stable.

Using homogeneity and superposition, the impulse response is $h(t) = 0.2e^{-t+2}\cos(0.1\pi(t-1))u(t-1)$. Now $\int_{-\infty}^{\infty}|h(t)|dt = \int_1^{\infty} 0.2e^{-t+2}|\cos(0.1\pi(t-1))|dt \leq \int_1^{\infty} 0.2e^{-t+2}dt < 1$. Hence, the system is stable.

Problem 8. A discrete-time system an impulse response whose DTFT over the period $(-\pi, \pi]$ is given by $H(e^{j\Omega}) = \frac{1}{1+0.9e^{-j\Omega}}$. Determine the output in time domain when the input $x[n] = 1 + \cos(0.7\pi n)$ is applied to the system.

$y[n] = |H(e^0)| + |H(e^{j0.7\pi})|\cos(0.7\pi n + \psi)$, where ψ is the phase of $H(e^{j0.7\pi})$.

Problem 9. A periodic continuous-time system has a Fourier series representation given by $x(t) = \sum_{k=-\infty}^{\infty} 0.8^{|k|}\exp(j0.5kt)$.
(a) Find the period of x.

Set $0.5 = 2\pi/T$, so $T = 4\pi$.

(b) Find the power of x.

Note that x is given above in its Fourier series form, where the FS coefficients are given by $C_k = 0.8^{|k|}$. Using Parserval's Theorem, $P = \sum_{k=-\infty}^{\infty}|C_k|^2$.

Problem 10 Consider the difference equation $y[n] = y[n-1] - \frac{1}{3}y[n-2] + 2u[n], n \geq 0$. Determine the particular solution $y_p[n]$.

Recall that the particular solution is always of the form of the input unless the input is of the form of the homogeneous solution (not the case here). Thus, we propose $y_p(n) = a$, a constant, and plu this function into the equation to obtain

$$a = a - \frac{1}{3}a + 2(1)$$

to conclude that $a = 6$. Hence, $y_p(n) = 6$, $n \geq 0$.

Problem 11 When an input $2tu(t)$ is applied to an LTI system the corresponding output is $e^{-2t}u(t)$. What is the output when the input is $\delta(t) + u(t)$.

Differentiating the input, $2tu(t)$, we obtain $2u(t)$. Thus, using the LTI property of the system, we conclude that an input $u(t)$ would result in an output $0.5(e^{-2t}u(t))' = -e^{-2t}u(t) + 0.5e^{-2t}\delta(t)$.

Next, we use the LTIproperty once gain to conclude that the input $u(t)'$ would result in the output $0.5(e^{-2t}u(t))'' = 2e^{-2t}u(t) - e^{-2t}\delta(t) - e^{-2t}\delta(t) + 0.5e^{-2t}\delta'(t)$.

Thus, the input $u(t) + \delta(t)$ results in the output $-e^{-2t}u(t) + 0.5e^{-2t}\delta(t) + 2e^{-2t}u(t) - e^{-2t}\delta(t) - e^{-2t}\delta(t) + 0.5e^{-2t}\delta'(t) = e^{-2t}u(t) - 1.5\delta(t) + 0.5\delta'(t)$. Recall from class notes that $\delta'(t)$ is defined through the integral $\int_{-\infty}^{\infty} \delta'(t)f(t)dt = -\int_{-\infty}^{\infty} \delta(t)f'(t)dt = f'(0)$.

Problem 12 A discrete LTI system has a real-valued impulse response whose DTFT, or the frequency response, is given by $e^{j0.1\pi\Omega}$ over $\Omega \in [0, \pi)$.

(a) Complete the description of the above frequency response for all values of Ω.

Using the symmetry property of the DTFT about $\Omega = \pi$, we conclude that the DTFT is $e^{-j0.1\pi\Omega}$ over $\Omega \in [\pi, 2\pi)$.

(b) Find the DTFT of the output and the output when the input is $x[n] = 0.1^n u[n]$.

From class notes, $X(e^{j\Omega}) = \frac{1}{1-0.1e^{-j\Omega}}$, and thus $Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) = \frac{1}{1-0.1e^{-j\Omega}}H(e^{j\Omega})$. In time domain,

$$y(n) = \frac{1}{2\pi}\int_0^{\pi} \frac{1}{1-0.1e^{-j\Omega}} e^{j0.1\pi\Omega} d\Omega + \frac{1}{2\pi}\int_{\pi}^{2\pi} \frac{1}{1-0.1e^{-j\Omega}} e^{-j0.1\pi\Omega} d\Omega.$$