ECE340 Spring 2011

Homework-3 Solutions

Problems: 1-8.1, 1-8.4, 1-9.2, 1-9.3, 1-10.3, 1-10.5 (a, b), 1-10.6, 1-10.7, 1-10.8, 1-10.9

1-8.1

When a pair of dice are rolled, the sum of the two dice can be one of the following: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12.
Here is a list of the outcomes associated with each of the sums, followed by the number of outcomes in each case.

Sum= 2: (1, 1); -----------1
Sum= 3: (1, 2), (2, 1); -----------2
Sum =4: (1, 3), (3, 1), (2, 2); -----------3
Sum =5: (1, 4), (4, 1), (2, 3), (3, 2); -----------4
Sum =6: (1, 5), (5, 1), (2, 4), (4, 2), (3, 3); -----------5
Sum= 7: (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3); -----------6
Sum =8: (2, 6), (6, 2), (3, 5), (5, 3), (4, 4); -----------5
Sum =9: (3, 6), (6, 3), (4, 5), (5, 4); -----------4
Sum =10: (4, 6), (6, 4), (5, 5); -----------3
Sum =11: (5, 6), (6, 5); -----------2
Sum =12: (6, 6); -----------1

There are a total of 6x6=36 outcomes.
As such, P(A) = (5+6+5+4+3+2+1)/36 = 26/36, and P(B)=(1+2+3+4+5)/36 = 15/36
However, P(A∩B) = P(obtaining 6) = 5/36.
Since P(A∩B) ≠ P(A)P(B), events A and B are not independent.

1-8.4 If A is independent of B, prove that:

a) A is independent of $\bar{B}$.
Proof: Since A is independent of B, we know that P(A∩B) = P(A)P(B)---------------------(1)
But, P(A)P(B)=P(A)(1-P($\bar{B}$)) = P(A) – P(A)P($\bar{B}$) --------------------------------------(2)
On the other hand, we know that A - A∩ $\bar{B}$= A∩B, so P(A)- P(A∩ $\bar{B}$) = P(A∩B)--------(3)
From (1), (2) &(3), we have P(A) – P(A)P($\bar{B}$) = P(A)- P(A∩ $\bar{B}$); or P(A)P($\bar{B}$) = P(A∩ $\bar{B}$).
Thus, A is independent of $\bar{B}$.

b) We can now apply the conclusion of Part (a) to the independent events $\bar{B}$ and A to conclude that $\bar{B}$ and $\bar{A}$ are independent.

1-9.2

a) S = S_1 X S_2 X S_3 X S_4 = {(G, F, C, S), (G, F, C, $\bar{S}$), (G, F, $\bar{C}$, S), (G, $\bar{F}$, C, S), ($\bar{G}$, F, C, S),
(G, F, $\bar{C}$, $\bar{S}$), (G, $\bar{F}$, C, $\bar{S}$), ($\bar{G}$, F, C, $\bar{S}$), (G, $\bar{F}$, $\bar{C}$,S),
($\bar{G}$,F , $\bar{C}$,S), ($\bar{G}$, $\bar{F}$, C, S), (G, $\bar{F}$, $\bar{C}$, $\bar{S}$), ($\bar{G}$, F, $\bar{C}$, $\bar{S}$),
($\bar{G}$, $\bar{F}$, C, $\bar{S}$), ($\bar{G}$, $\bar{F}$, $\bar{C}$,S), ($\bar{G}$, $\bar{F}$, $\bar{C}$, $\bar{S}$)}

b) Define E_D as the event that the device will work.

Define E_G as the event that the NAND gate will work.
Define E_F as the event that the flip-flop will work.
Define E_C as the event that the counter will work.
Define E_S as the event that the shift register will work.

Then, $E_D = E_G \cap E_F \cap E_C \cap E_S$,
$P(E_D) = P(E_G)P(E_F)P(E_C)P(E_S) = (1\text{-}0.05)(1\text{-}0.1)(1\text{-}0.03)(1\text{-}0.12) = 0.7298$

c) $P(\bar{E}_F \cap E_G \cap E_C \cap E_S \mid \bar{E}_D) = \frac{P(\bar{E}_F \cap E_G \cap E_C \cap E_S \cap \bar{E}_D)}{P(\bar{E}_D)} = \frac{P(\bar{E}_D \mid \bar{E}_F \cap E_G \cap E_C \cap E_S)P(\bar{E}_F \cap E_G \cap E_C \cap E_S)}{1-P(E_D)} = \frac{1\times P(\bar{E}_F)P(E_G)P(E_C)P(E_S)}{1-0.7298} = \frac{0.1\times 0.95\times 0.97\times 0.88}{0.2702} = 0.3001$

d) $P(\bar{E}_F \cap \bar{E}_C \mid \bar{E}_D) = \frac{P(\bar{E}_F \cap \bar{E}_C \cap \bar{E}_D)}{P(\bar{E}_D)} = \frac{P(\bar{E}_D \mid \bar{E}_F \cap \bar{E}_C)P(\bar{E}_F \cap \bar{E}_C)}{1-P(E_D)} = \frac{1\times P(\bar{E}_F)P(\bar{E}_C)}{1-0.7298} = \frac{0.1\times 0.03}{0.2702} = 0.0111$

1-9.3

a) $S = S_1 \times S_2 \times S_3$ = {(H, H, H), (H, H, T), (H, T, H), (T, H, H), (H, T, T), (T, H, T), (T, T, H), (T, T, T)}.

b) P(two Heads) = $\binom{3}{2}(0.5)^2(1-0.5)^{3-2} = \frac{3!}{2!1!}\frac{1}{8} = \frac{3}{8}$

c) P(more than one head) = P(two Heads) + P(three Heads) = $\frac{3}{8} + \frac{1}{8} = 0.5$

1-10.3 Prove that ${}_nC_r$ is equal to ${}_{(n-1)}C_r$ + ${}_{(n-1)}C_{(r-1)}$

By definition, ${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

Now ${}_{(n-1)}C_r + {}_{(n-1)}C_{(r-1)} = \binom{n-1}{r} + \binom{n-1}{r-1} = \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-1-r+1)!} = \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-r)!} = \frac{(n-r)(n-1)!+r(n-1)!}{r!(n-r)!} = \frac{(n-r+r)(n-1)!}{r!(n-r)!} = \frac{n(n-1)!}{r!(n-r)!} = \frac{n!}{r!(n-r)!} = {}_nC_r = \binom{n}{r}$, which completes the proof.

1-10.5(a, b)

a) Our goal is choose 3 from 7 EEs and choose 2 from 5 MEs.
The total number of ways we can make such selection is $\binom{7}{3} \times \binom{5}{2} = 35 \times 10 = 350$

b) One of the EE is already chosen to be on the committee. Then, we choose another 2 from the remaining 6 EEs and choose 2 from the 5 MEs, so the total number of ways we can make such selection is
$\binom{6}{2} \times \binom{5}{2} = 15 \times 10 = 150$

c) When two particular MEs cannot be on the committee, we just choose 3 from 7 EEs and then choose 2 from the remaining 3 MEs. The total number of ways we can make such selection is

$\binom{7}{3} \times \binom{3}{2} = 35 \times 3 = 105$

1-10.6

Probability of one symbol with error: P(e) =0.4x0.08 + 0.6x0.05 = 0.062 from **1-7.1**.

a) $(1-P(e))^6$

b) $\binom{6}{1}P(e)^1(1-P(e))^{6-1}$

c) $1-(1-P(e))^6 - \binom{6}{1}P(e)^1(1-P(e))^{6-1}$

d) 1-(1-P(e))^6

1-10.7

a) Note that 80% of the 12 subscribers constitute 10 subscribers. We will need **10 channels** to accommodate the 10 users all the time.

b) There are 12 subscribers and the probability that each user occupies a channel is p, where p is 0.2. Note that the number of occupied channels can be thought of as the number of heads in a Bernoulli-trial experiment with size 12 and success probability p.

Suppose that there are k channels available. Now, the event "a user is successful in occupying a channel" is equivalent to the event "the number of occupied channels is less than or equal to k," which, in turn, is equivalent to the event "the number of heads in the Bernoulli trials is less than or equal to k."

Thus, the probability, Q, that a user is successful in occupying a channel is $P(E_0)+...+P(E_k)$, where E_i (i=1, ..., k) is the event that the number of heads in the Bernoulli trials is i. If we set Q=0.8, we must then find the smallest k for which $P(E_0)+...+P(E_k)$ is at least 0.8. By using Matlab we find that such minimal ***k* is 4**, yielding a probability of 0.927. Note that when k=3, the probability is 0.795, which is just below the required 0.8. (Due to close proximity of 0.795 to 0.8, either answers, namely k=3 or 4, will be acceptable for this problem.) Here is my Matlab code and its execution results.

```
clear all
close all
N_max=input('What is the number of Bernoulli trials?')
N=[1:1:N_max];
p=0.2;
P=cdf('bino',N,N_max,p)
N(min(find(P > 0.8)))
```

When you run this script you get

```
>> microwave
What is the number of repitions?12

N_max =

   12

P =

   0.2749    0.5583    0.7946    0.9274    0.9806    0.9961    0.9994    0.9999    1.0000    1.0000    1.0000
1.0000

ans =

   4
```

1-10.8

a) Set P(e) = 0.001, as the probability of any one character being transferred in error. Then the probability that the file can be transferred without any error is: $(1\text{-P(e)})^{10000} = 0.999^{10000} =$ 4.5173e-005

b) Since the 10 errors can happen in any combination within the 10000 characters in term of the positions of errors in the file, the probability that there will be exactly 10 errors in the transferred file is then $\binom{10000}{10}0.001^{10}0.999^{10000\text{-}10}$, in this case, n = 10000, k = 10, p = 0.001, q = 0.999, so npq = 9.99 ≫ 1, $|k\text{-}np|$ = |10-10| = 0, $\sqrt{npq} = \sqrt{9.99}$, so $|k\text{-}np|$ is on the order or less than $\sqrt{npq}$; in this case, we can use the DeMoivre-Laplace approximation to calculate $\binom{10000}{10}0.001^{10}0.999^{10000\text{-}10} \cong \frac{1}{\sqrt{2\pi npq}}e^{-(k-np)^2/2npq} = \frac{1}{\sqrt{2\pi\times 9.99}}e^{-(0)^2/2\times 9.99} = 0.1262$

If we use Matlab to calculate $\binom{10000}{10}0.001^{10}0.999^{10000\text{-}10}$ directly (the command in Matlab to compute this: " >> nchoosek(10000,10)*(0.001^10)*(0.999^(10000-10))"), the result is 0.1252, which is close to the approximation.

c) Similarly, use P(e) to denote the probability of error in transferring one character. What we need is the following:

$$(1-\mathrm{P}\ (e))^{10000} \geq 0.99$$

Or,

$$10000\log\ (1-\mathrm{P}\ (e)) \geq log0.99$$
$$\log\ (1-\mathrm{P}\ (e)) \geq \frac{log0.99}{10000}$$
$$1-\mathrm{P}\ (e) \geq 10^{\frac{log0.99}{10000}}$$
$$\mathrm{P}(e) \leq 1-10^{\frac{log0.99}{10000}}$$

This gives

$$\mathrm{P}(e) \leq 1.0050331 \times 10^{-6}$$

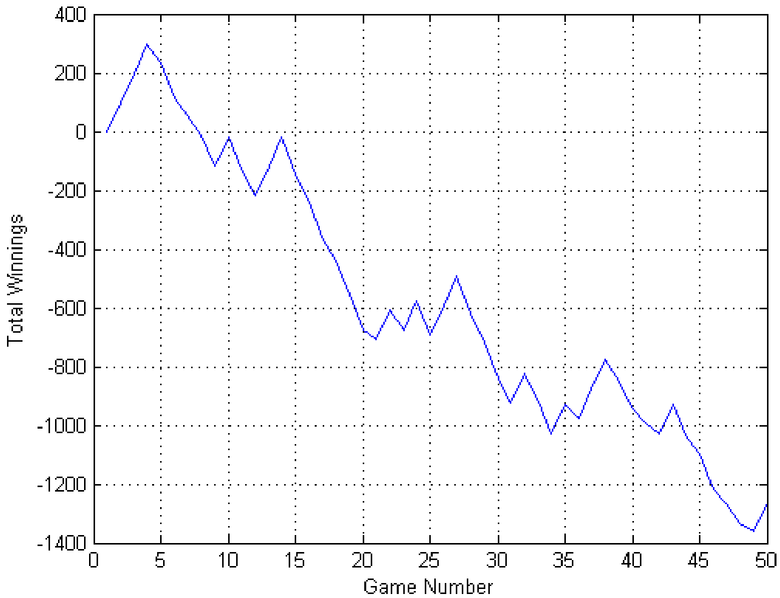
1-10.9

Code:

```
%P110_9.m
clc
clear all
close all
B=1; %size of standard bet
T(1)=0;%initial total winnings
rand('seed',1000)
for m=2:50
    clear y;clear w; y(1)=0;w(1)=B;
    for k=2:10000
        x=rand;
        if x <= 18/38;%18:38 probability of winning
            y(k)=y(k-1)+w(k-1);w(k)=B;
        else y(k)=y(k-1)-w(k-1);w(k)=2*w(k-1);
        end
        if w(k) >= 100*B;break
        elseif y(k) >=100*B;break
        end
    end
    T(m)=T(m-1)+y(k);
```

```
end
plot(T);xlabel('Game Number');ylabel('Total Winnings');grid
```

a) Plot:



b) Notice that the game is not a fair one because the house always has a higher probability to win, namely 20/38, whereas the bettor has a lower probability to win (18/38). That's why the bettor always loses in the long run.

c) If we change the winning probability from 18/38 to 0.5, the game becomes fair. What is shown below is a same plot but after changing the winning probability in the program from 18/38 to 0.5.