# #40 Waves Post-class

Due: 11:00am on Wednesday, November 28, 2012

***Note: You will receive no credit for late submissions.*** To learn more, read your instructor's Grading Policy

## Two Velocities in a Traveling Wave

Wave motion is characterized by two velocities: the velocity with which the wave moves in the medium (e.g., air or a string) and the velocity of the medium (the air or the string itself).

Consider a transverse wave traveling in a string. The mathematical form of the wave is

$$y(x,t) = A\sin(kx - \omega t).$$

### Part A

Find the speed of propagation $v_{\rm p}$ of this wave.

**Express the velocity of propagation in terms of some or all of the variables $A$, $k$, and $\omega$.**

**Hint 1.** Perform an intermediate step

Note that the phase of the wave $(kx - \omega t)$, and therefore the displacement of the string, is equal to zero at $(x,t) = (0,0)$.

At what position $x = \Delta x$ is the phase equal to zero a short time $t = \Delta t$ later?

**Express your answer in terms of $\Delta t$, $\omega$, and $k$.**

ANSWER:

$\Delta x$ = $\frac{\omega \Delta t}{k}$

ANSWER:

$v_p = \frac{\omega}{k}$

**Correct**

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**Part B**

Find the *y* velocity $v_y\left(x, t\right)$ of a point on the string as a function of $x$ and $t$.

**Express the *y* velocity in terms of $\omega$, $A$, $k$, $x$, and $t$.**

**Hint 1.** How to approach the problem

In the problem introduction, you are given an expression for $y\left(x, t\right)$, the displacement of the string as a function of $x$ and $t$. To find the *y* velocity, take the partial derivative of $y\left(x, t\right)$ with respect to time. That is, take the time derivative of $y\left(x, t\right)$ while treating $x$ as a constant.

**Hint 2.** A helpful derivative

$$\frac{d}{dt}\sin(at+b) = a\cos(at+b)$$

ANSWER:

$v_y\left(x, t\right) = -A\omega\cos\left(kx - \omega t\right)$

**Correct**

## Part C

Which of the following statements about $v_x(x,t)$, the *x* component of the velocity of the string, is true?

**Hint 1.** How to approach this question

You are given a form for $y(x,t)$. You are not given any information about $\Delta x(x,t)$, but it is assumed that each point on the string only moves in the *y* direction, i.e. $\Delta x(x,t)=0$.

ANSWER:

- $v_x(x,t)=v_{\rm p}$
- $v_x(x,t)=v_y(x,t)$
- $v_x(x,t)$ has the same mathematical form as $v_y(x,t)$ but is $180^\circ$ out of phase.
- $v_x(x,t)=0$

**Correct**

So the wave moves in the *x* direction, even though the string does not. What this means is that even though individual points on the string only move up and down, a given shape or pattern of points (in this sinusoidal) will move to the right as time progresses.

## Part D

Find the slope of the string $\frac{\partial y(x,t)}{\partial x}$ as a function of position $x$ and time $t$.

**Express your answer in terms of $A$, $k$, $\omega$, $x$, and $t$.**

**Hint 1.** A helpful derivative

$$\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$$

ANSWER:

$\frac{\partial y(x,t)}{\partial x}$ = $kA\cos\left(kx-\omega t\right)$

**Correct**

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## Part E

Find the ratio of the *y* velocity of the string to the slope of the string calculated in the previous part.

**Express your answer as a suitable combination of some of the variables $\omega$, $k$, and $v_{\rm p}$.**

ANSWER:

$\frac{v_y(x,t)}{\frac{\partial y(x,t)}{\partial x}}$ = $-\frac{\omega}{k}$

**Correct**

To understand why the ratio of the *y* velocity of the string to its slope is constant, draw the string with a wave running along it at time $t = 0$. In the vicinity of $x = 0$, the string is sloped upward. The bit of string at position $x = 0$ moves downward as the wave moves forward. One-half cycle later, the string in the vicinity of $x = 0$ will be sloped downward, and the string at position $x = 0$ will move upward as the wave moves forward.

In general, if at some particular $(x, t)$ the slope of the string is positive ($\partial y(x,t)/\partial x > 0$), that bit of string will be moving downward ($v_y(x,t) < 0$). If the slope at $(x, t)$ is negative, that bit of string will be moving upward. This explains why the sign of the ratio of string velocity to slope is always negative.

One way of understanding why the ratio has a constant magnitude is to observe that the more steeply the string is sloped, the more quickly it will move up or down.

## Ant on a Tightrope

A large ant is standing on the middle of a circus tightrope that is stretched with tension $T_s$. The rope has mass per unit length $\mu$. Wanting to shake the ant off the rope, a tightrope walker moves her foot up and down near the end of the tightrope, generating a sinusoidal transverse wave of wavelength $\lambda$ and amplitude $A$. Assume that the magnitude of the acceleration due to gravity is $g$.

### Part A

What is the minimum wave amplitude $A_{\rm min}$ such that the ant will become momentarily "weightless" at some point as the wave passes underneath it? Assume that the mass of the ant is too small to have any effect on the wave propagation.

**Express the minimum wave amplitude in terms of $T_s$, $\mu$, $\lambda$, and $g$.**

**Hint 1.** Weight and weightless

Weight is generally defined as being the equal and opposite force to the normal force. On a flat surface in a static situation, the weight is equal to the force due to gravity acting on a mass.

"Weightless" is a more colloquial term meaning that if you stepped on a scale (e.g., in a falling elevator) it would read zero. Think about what happens to the normal force in this situation.

Note that the force due to gravity does not change and would still be the same as when the elevator was static.

### **Hint 2.** How to approach the problem

In the context of this problem, when will the ant become "weightless"?

ANSWER:

- When it has no net force acting on it
- When the normal force of the string equals its weight
- When the normal force of the string equals twice its weight
- When the string has a downward acceleration of magnitude $g$

### **Hint 3.** Find the maximum acceleration of the string

Assume that the wave propagates as $y(x,t) = A\sin(\omega t - kx)$. What is the maximum downward acceleration $a_{max}$ of a point on the string?

**Express the maximum downward acceleration in terms of $\pi$ and any quantities given in the problem introduction.**

#### **Hint 1.** How to approach the problem

Use the formula given for the displacement of the string to find the acceleration of the string as a function of position and time. Then determine what the maximum value of this acceleration is. (At some time, the bit of rope underneath the ant will have this maximum downward acceleration.)

#### **Hint 2.** Acceleration of a point on the string

Find the vertical acceleration $a_y(x,t)$ of an arbitrary point on the string as a function of time.

**Express your answer in terms of $A$, $\omega$, $t$, $k$, and $x$.**

**Hint 1.** How to find the acceleration

Differentiate the expression given for $y(x,t)$, the displacement of a point on the string, twice.

**Hint 2.** The first derivative

Differentiate the given equation for the displacement of the string $y(x,t)$ to find the vertical velocity $v_y(x,t)$ of the rope.

**Express your answer in terms of $A$, $\omega$, $t$, $k$, and $x$.**

ANSWER:

$v_y(x,t) = \omega A\cos(\omega t - kx)$

ANSWER:

$a_y(x,t) = -\omega^2 A\sin(\omega t - kx)$

**Hint 3.** Find the maximum downward acceleration

The maximum downward acceleration $a_{\max}$ is the most negative possible value of $a_y(x,t)$. As the wave passes beneath the ant, at some time or another the ant will be at a point where the acceleration of the string has this most negative value.

What is $a_{\max}$?

**Express your answer in terms of $\omega$ and quantities given in the problem introduction.**

**Hint 1.** When will the acceleration reach its most negative value?

The most negative acceleration occurs when $\sin(\omega t - kx) = 1$.

ANSWER:

$a_{\max}$ = $-\omega^2 A$

**Hint 4.** Determine $\omega$ in terms of given quantities

The angular frequency $\omega$ of the wave in the string was not given in the problem introduction. To solve the problem, find an expression for $\omega$ in terms of given quantities.

**Express the angular frequency in terms of $T_s$, $\mu$, $\lambda$, and $\pi$.**

**Hint 1.** How to approach this question

Combine a general formula for $\omega$, a relationship among frequency, wavelength, and velocity, and a formula for the velocity of a wave on a string to find an expression for $\omega$ in terms of quantities given in the problem introduction.

**Hint 2.** General formula for $\omega$

The angular frequency of a wave is equal to $2\pi$ times the normal frequency: $\omega = 2\pi f$.

**Hint 3.** Relationship among frequency, wavelength, and velocity

The frequency, wavelength, and velocity of a wave are related by $v = \lambda f$.

**Hint 4.** Speed of a wave on a string

What is the speed $v$ of any wave on the string described in the problem introduction?

ANSWER:

$v = \sqrt{\frac{T_s}{\mu}}$

ANSWER:

$\omega = \sqrt{\frac{T_s}{\mu}}\frac{2\pi}{\lambda}$

ANSWER:

$a_{\rm max} = -\frac{T_s}{\mu}\left(\frac{2\pi}{\lambda}\right)^2 A$

**Hint 4.** Putting it all together

Once you have an expression for the maximum acceleration of a point on the string $a_{\rm max}$, determine what amplitude is required such that $a_{\rm max} = -g$. This will be the minimum amplitude $A_{\rm min}$ for which the ant becomes weightless.

ANSWER:

$A_{\rm min} = \frac{g}{\left(2\pi\left(\frac{\sqrt{\frac{T_s}{\mu}}}{\lambda}\right)\right)^2}$

**Correct**

## Wave and Particle Velocity Vector Drawing

A long string is stretched and its left end is oscillated upward and downward. Two points on the string are labeled A and B.

### Part A

Points A and B are indicated on the string. Orient the two vectors, $\vec{v}_{\rm A}$ and $\vec{v}_{\rm B}$, to correctly represent the direction of the *wave* velocity at points A and B.

**Rotate the given vectors to indicate the direction of the wave velocity at the indicated points.**

**Hint 1.** Distinguishing between wave velocity and particle velocity

A wave is a collective disturbance that, typically, travels through some medium, in this case along a string. The velocity of the individual particles of the medium are quite distinct from the velocity of the wave as it passes through the medium. In fact, in a transverse wave such as a wave on a string, the wave velocity and particle velocities are perpendicular.

**Hint 2.** Wave velocity

A wave on a stretched string travels away from the source of the wave along the length of the string.

ANSWER:

**Correct**

## Part B

At the instant shown, orient the given vectors $\vec{v}_A$ and $\vec{v}_B$ to correctly represent the direction of the velocity of points A and B.

**At each of the points A and B, rotate the given vector to indicate the direction of the velocity.**

**Hint 1.** Distinguishing between wave velocity and particle velocity

A wave is a collective disturbance that, typically, travels through some medium, in this case along a string. The velocity of the individual particles of the medium are quite distinct from the velocity of the wave as it passes through the medium. In fact, in a transverse wave such as a wave on a string, the wave velocity and particle velocities are perpendicular.

**Hint 2.** Determining velocity from a snapshot

The diagram represents the position of a small portion of the string at a specific instant of time: a snapshot of the string at this time. Based on only a snapshot, you cannot determine the velocity of an object, such as a point on the string. However, you also know that the left end of the string is the source of the wave disturbance. From this information you can deduce what is about to happen to point A's position, and from this change in position deduce the direction of point A's velocity (and similarly for point B).

**Hint 3.** Find the change in point A's position

Based on the location of the source of the wave (the left end of the string), will the wave crest to the immediate left of point A soon raise or lower point A's position?

ANSWER:

- (●) raise
- ( ) lower

**Hint 4.** Find the change in point B's position

Based on the location of the source of the wave (the left end of the string), will the wave trough to the immediate left of point B soon raise or lower point B's position?

ANSWER:

- ( ) raise
- (●) lower

ANSWER:

**Correct**

## Score Summary:

Your score on this assignment is 104.4%.
You received 31.33 out of a possible total of 30 points.