Lecture 34
(Maxwell’s Equations)

Physics 161-01 Spring 2012
Douglas Fields

Faraday’s Law

- So, even though we have talked about motional EMF, i.e. a conductor moving in a magnetic field, remember that the most general form for Faraday’s Law is:

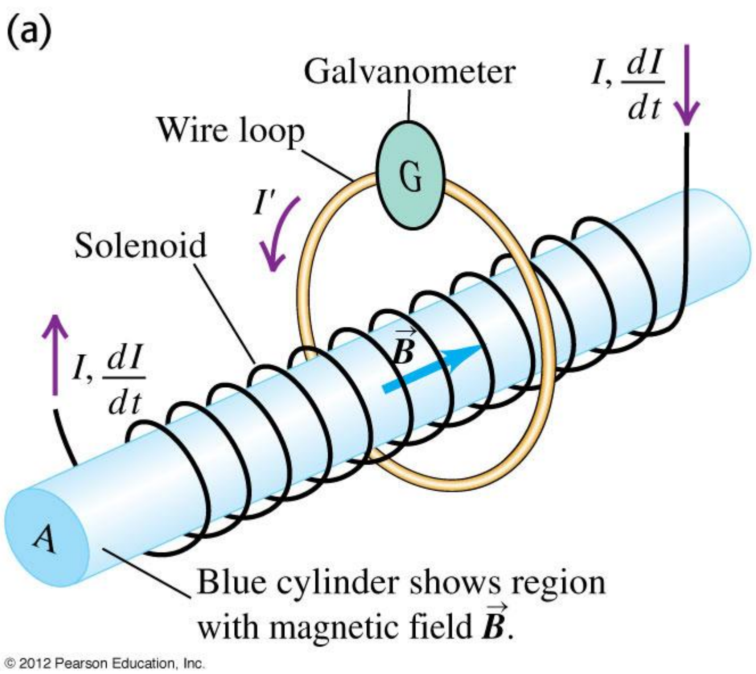
$$\mathcal{E} = -\frac{d\Phi_B}{dt},$$

$$\Phi_B = \int_{\text{Surface}} \vec{B} \cdot d\vec{A}$$

- Now, let’s explore a very important and strange aspect of this.

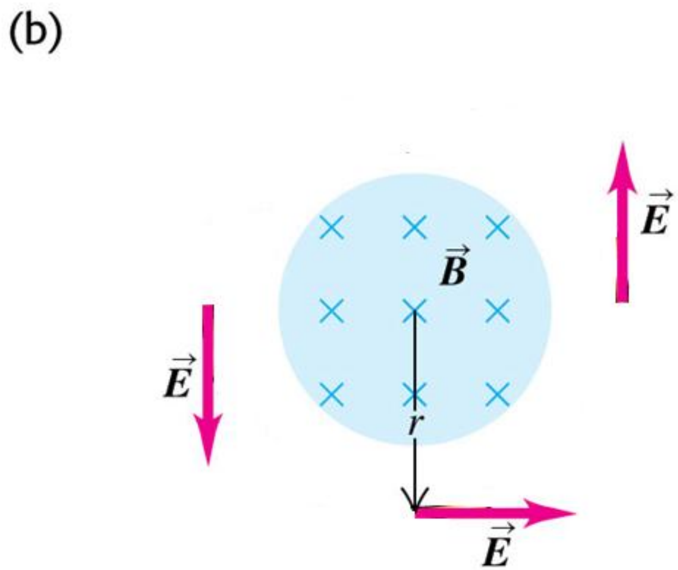
Induced EMF

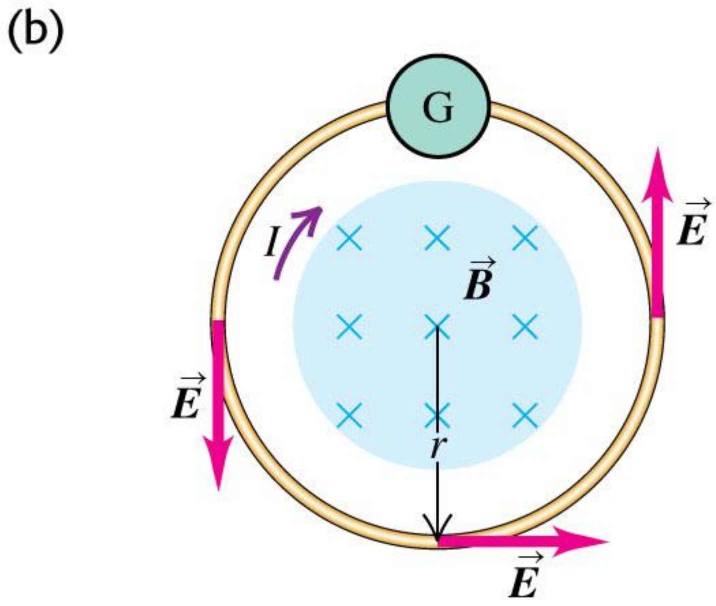
- Let’s take a very long (infinite) solenoid with a changing current.
- Remember that in this case, the magnetic field outside the solenoid is extremely weak.
- Now, outside of the conductor, let’s put a wire loop, with a galvanometer to measure current.
- It is safe to say that the magnetic field at the wire loop is zero.
- But there IS a changing flux through the loop.
- Will there be an induced current in the loop?



Induced Electric Field

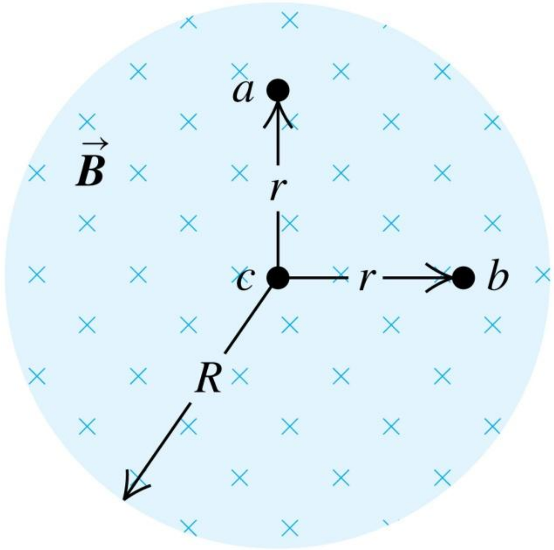
- YES!
- Even though the conductor is not moving, and even isn't in a magnetic field, the changing flux through the surface bounded by the conductor, will create an electric field, which will create a current.
- Even if there is no conductor there, there is an electric field!
- What's more, this electric field is fundamentally different than the electric field caused by charges…





CPS 34-1

The drawing shows the uniform magnetic field inside a long, straight solenoid. The field is directed into the plane of the drawing and is increasing.



What is the direction of the *electric* force on a positive point charge placed at point *b*?

A. to the left

B. to the right

C. straight up

D. straight down

E. misleading question—the electric force at this point is zero

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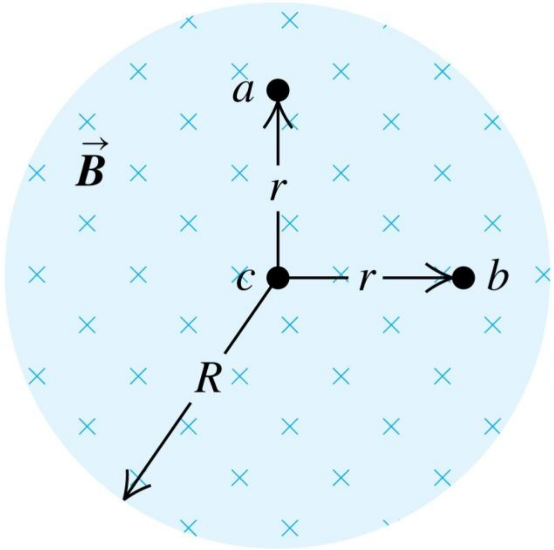

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Faraday's Law

- Let's look at the equation for electric potential change going from some point a to point b in the presence of a charge:

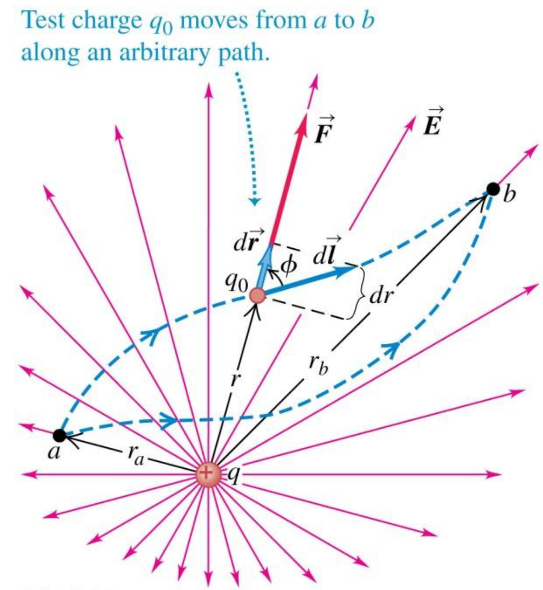
$$\Delta V = V(\text{point b}) - V(\text{point a}) = -\int_a^b \vec{E} \cdot d\vec{l}$$

- Now, what would we get if we turned around and returned to point a? In other words, what is the integral of E dot dl around a closed path?

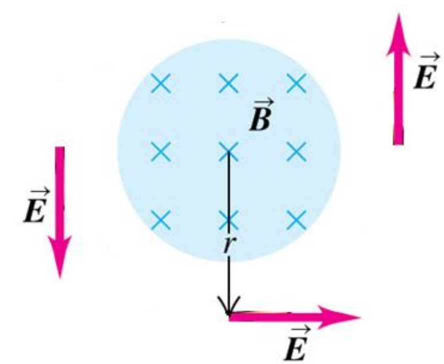
$$\int_{\text{Closed Path}} \vec{E} \cdot d\vec{l} = \Delta V(a \rightarrow b) - \Delta V(b \rightarrow a) = 0$$

- How would this answer change in the presence of a changing magnetic flux?

$$\int_{\text{Closed Path}} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

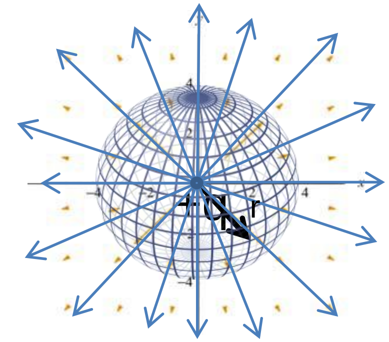


(b)

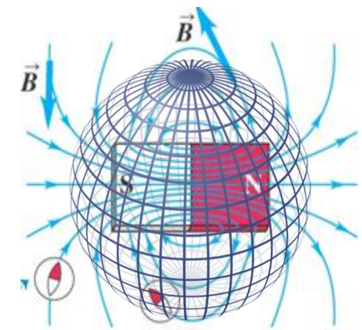


E&M Equations So Far

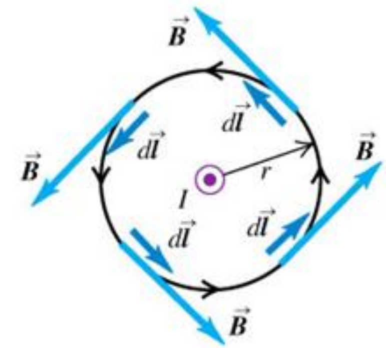
- Gauss's Law for E-Field $$\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$$



- Gauss's Law for B-Field $$\oint \vec{B} \cdot d\vec{A} = 0$$



- Ampere's Law $$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$



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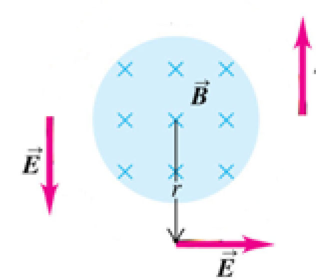
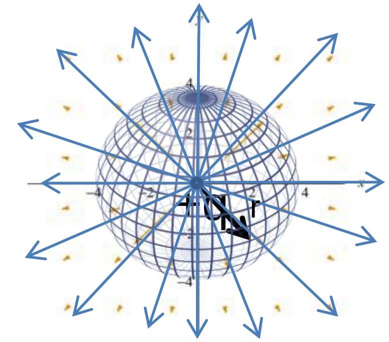


In Vacuum…

- Gauss's Law for E-Field $\oint \vec{E} \cdot d\vec{A} = 0$



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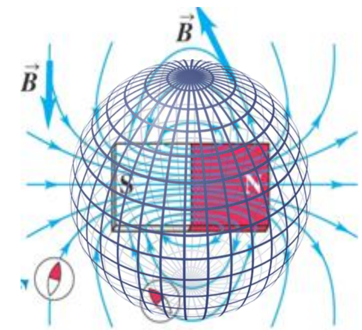

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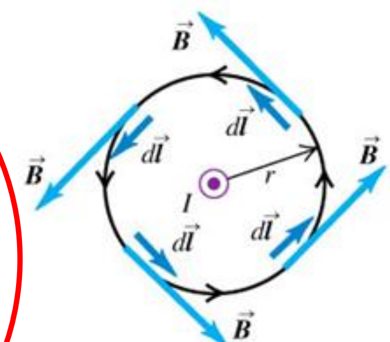

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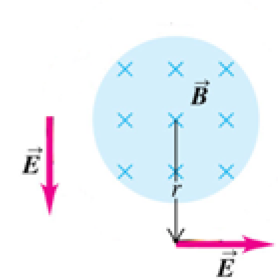


Fixing Ampere’s Law

- In 1865 (right after the American Civil War), James Clerk Maxwell was examining Ampere’s Law and found a fundamental flaw.
- Fixing the flaw led to a fundamental shift in the way we understood nature.
- Because of that, all of the E&M equations were renamed in his honor.

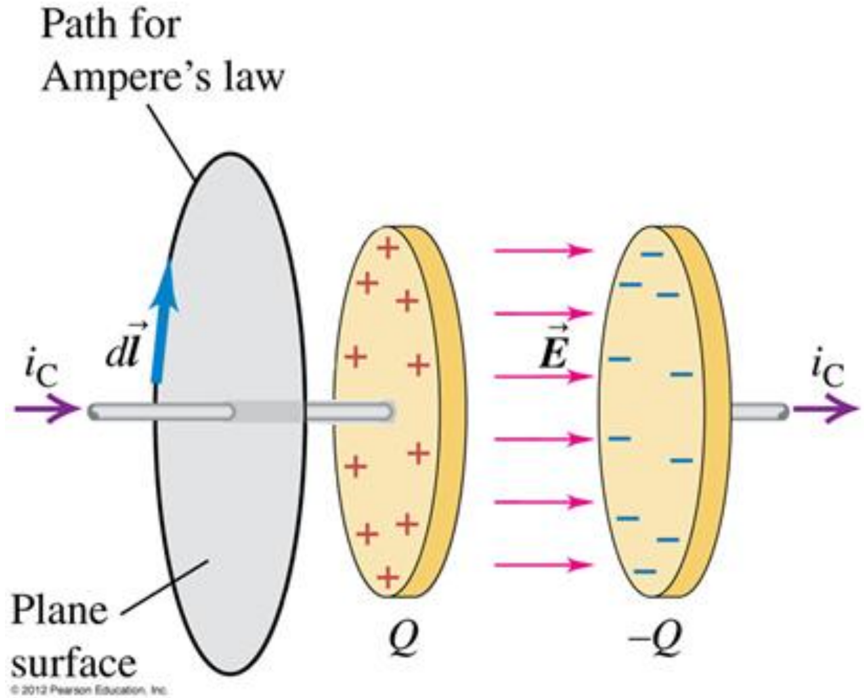
Fixing Ampere's Law

- Let's use Ampere's Law to examine the magnetic field around a wire with a current that is leading to a charging capacitor.
- Remember the English translation of Ampere's Law:
- The integral of the magnetic field components along a path, times the differential path lengths around a closed path bounding a surface is equal to a constant (μ_0) times the current which passes through that surface.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} \Rightarrow$$

$$B 2\pi r = \mu_0 I_{\text{C}} \Rightarrow$$

$$B = \frac{\mu_0 I_{\text{C}}}{2\pi r}$$



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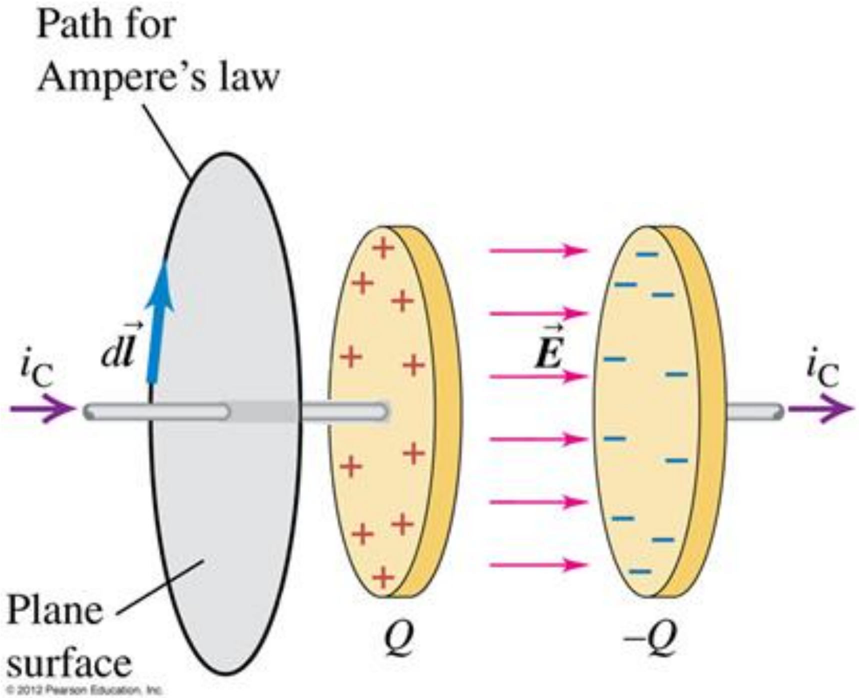
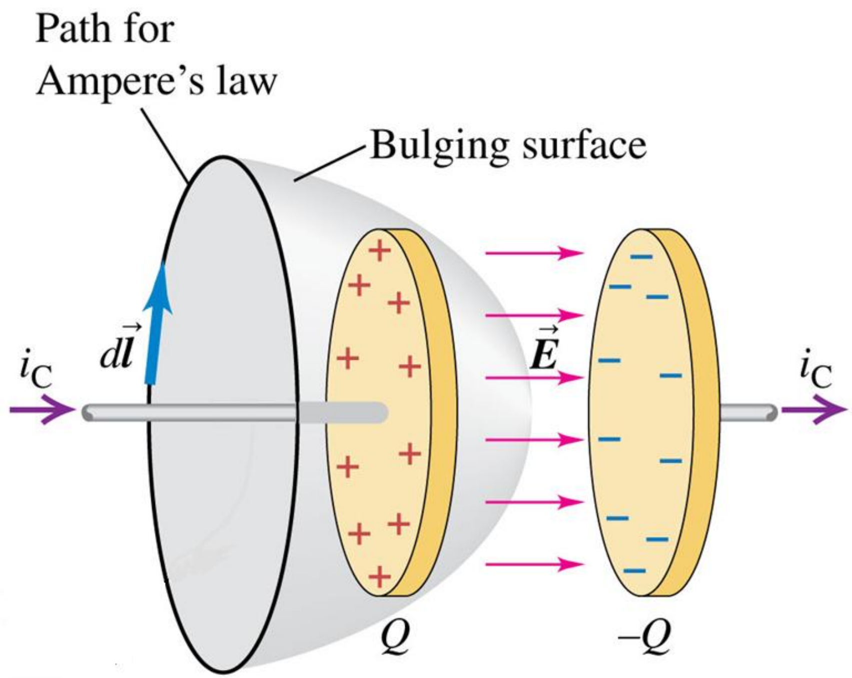


Fixing Ampere’s Law

- Now, Maxwell realized that the surface could be any surface whose bound was still the same closed path.
- So, what about a bulging surface as shown below?
- NO CURRENT passes through this surface, but it has the same bound, so one would expect the same field on the bound as before…



Fixing Ampere's Law

- But, there IS something passing through the surface – a changing electric field:

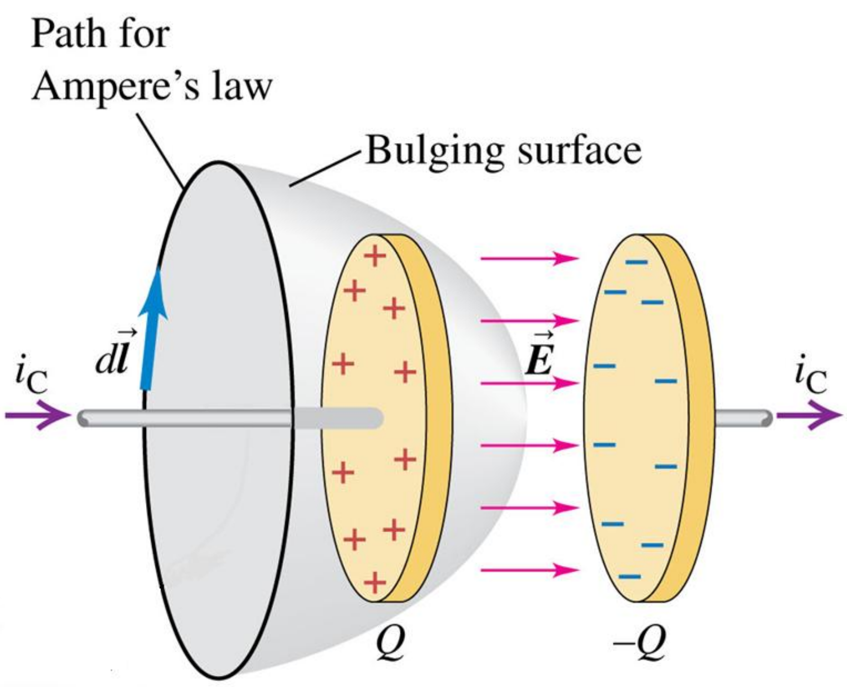
$$q(t) = CV(t) = \varepsilon_0 \frac{A}{d} V(t) = \varepsilon_0 \frac{A}{d} E(t) d \Rightarrow$$

$$q(t) = \varepsilon_0 E(t) A = \varepsilon_0 \Phi_E(t)$$

- Now, let's define a "current" analogous to the current in the wire, i_C, which Maxwell called the displacement current i_D:

$$q(t) = \varepsilon_0 \Phi_E(t) \Rightarrow$$

$$i_D = \frac{dq}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt}$$



Fixing Ampere's Law

- Then we can rescue Ampere's Law by adding another "current" term:

$$q(t) = \varepsilon_0 \Phi_E(t) \Rightarrow$$

$$i_D = \frac{dq}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I_C + I_D \right) \Rightarrow$$

$$B 2\pi r = \mu_0 \left(0 + \varepsilon_0 \frac{d\Phi_E}{dt} \right) \Rightarrow$$

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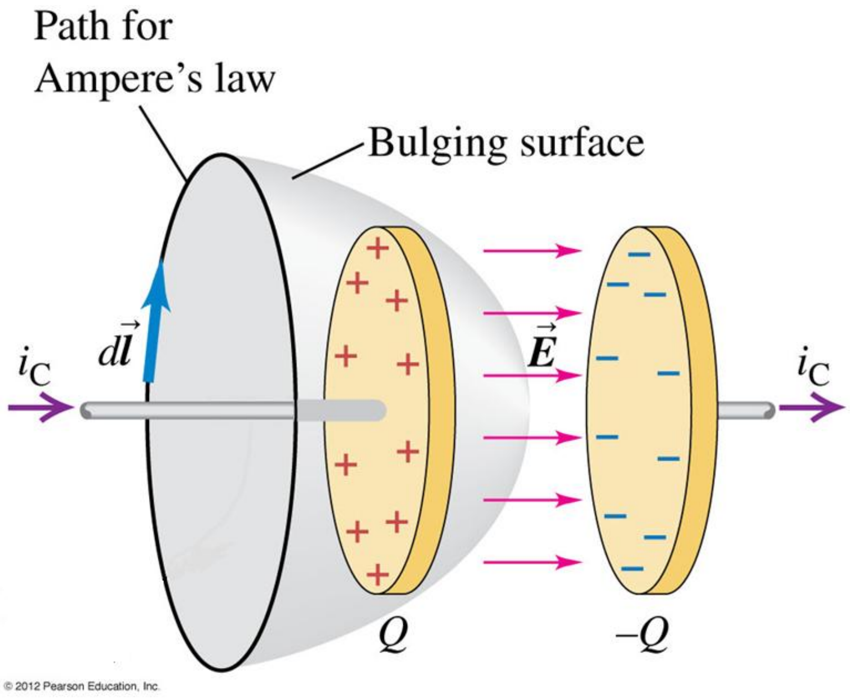


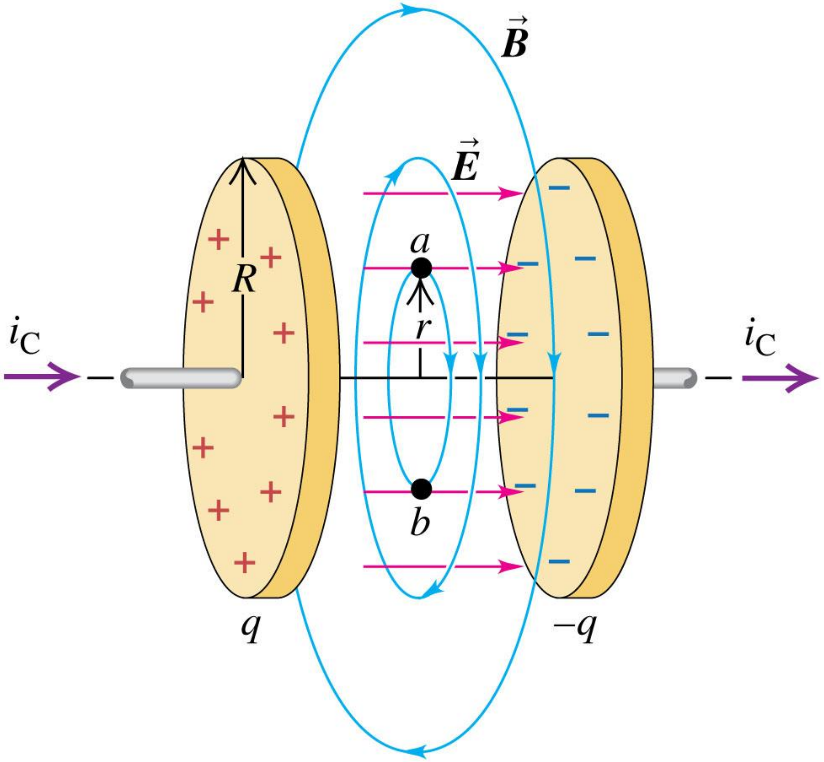
Fixing Ampere's Law

$$i_D = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{dE}{dt} A \Rightarrow$$

$$J_D = \varepsilon_0 \frac{dE}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I_C + I_D \right) \Rightarrow$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \oint \left(J_C + J_D \right) \cdot d\vec{A}$$



No, REALLY!

$$q\left(t\right)=\varepsilon_0 E\left(t\right)A \Rightarrow$$

$$E\left(t\right)=\frac{q\left(t\right)}{\varepsilon_0 A} \Rightarrow \quad \frac{dE}{dt}=\frac{1}{\varepsilon_0 A}i_C$$

$$i_D=\varepsilon_0\frac{d\Phi_E}{dt}=\varepsilon_0\frac{dE}{dt}A \Rightarrow$$

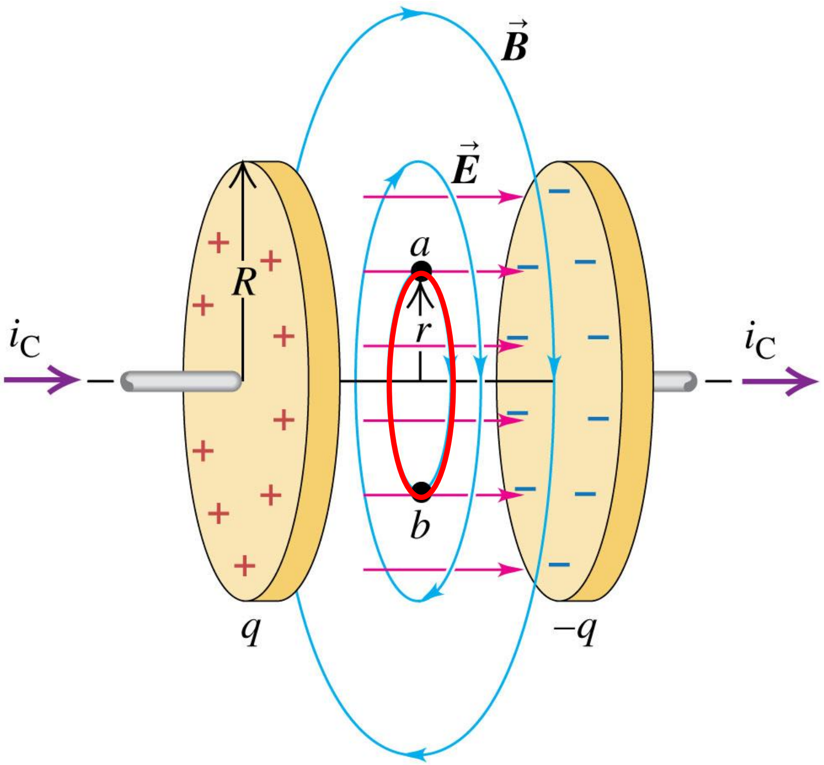
$$J_D=\varepsilon_0\frac{dE}{dt}$$

$$\oint \vec{B}\cdot d\vec{l}=\mu_0\oint\left(J_C+J_D\right)\cdot d\vec{A} \Rightarrow$$

$$B2\pi r=\mu_0\left(0+\varepsilon_0\frac{dE}{dt}\right)\pi r^2 \Rightarrow$$

$$B=\frac{\mu_0\varepsilon_0 r}{2}\frac{dE}{dt}=\frac{\mu_0\varepsilon_0 r}{2}\frac{1}{\varepsilon_0\pi R^2}\frac{dq}{dt} \Rightarrow$$

$$B=\frac{\mu_0 r}{2\pi R^2}i_C$$



No, REALLY!

$$q(t) = \varepsilon_0 E(t) A \Rightarrow$$

$$E(t) = \frac{q(t)}{\varepsilon_0 A} \Rightarrow \quad \frac{dE}{dt} = \frac{1}{\varepsilon_0 A} i_C$$

$$i_D = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{dE}{dt} A \Rightarrow$$

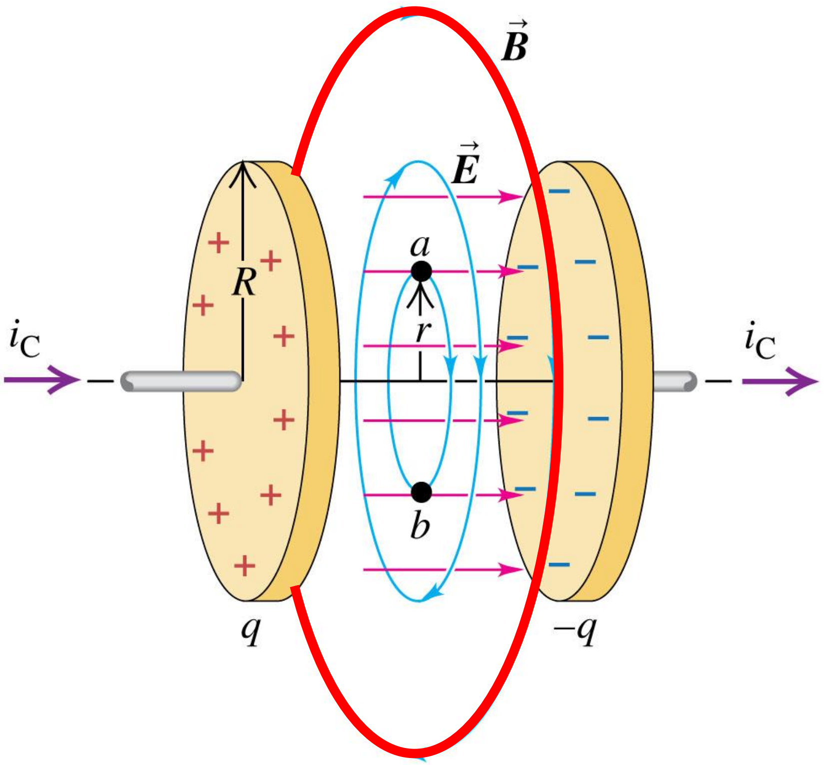
$$J_D = \varepsilon_0 \frac{dE}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \oint \left(J_C + J_D \right) \cdot d\vec{A} \Rightarrow$$

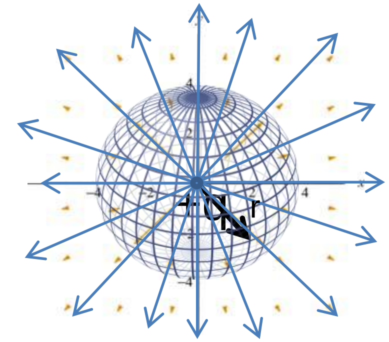
$$B 2\pi r = \mu_0 \left(0 + \varepsilon_0 \frac{dE}{dt} \right) \pi R^2 \Rightarrow$$

$$B = \frac{\mu_0 \varepsilon_0 \pi R^2}{2\pi r} \frac{dE}{dt} = \frac{\mu_0 \varepsilon_0 \pi R^2}{2\pi r} \frac{1}{\varepsilon_0 \pi R^2} \frac{dq}{dt} \Rightarrow$$

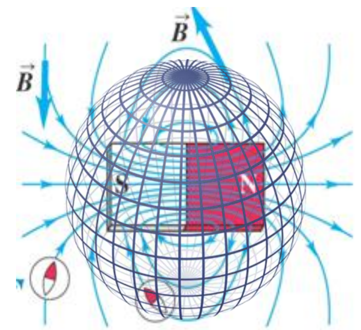
$$B = \frac{\mu_0}{2\pi r} i_C$$



In Vacuum…



- Gauss's Law for E-Field $$\oint \vec{E} \cdot d\vec{A} = 0$$



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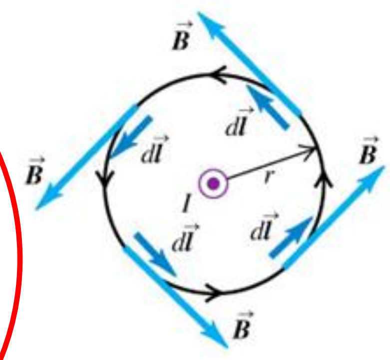

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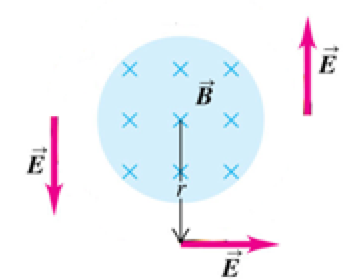

- Faraday's Law $$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

With some math…

- Stoke's Theorem: $\oint \vec{V} \cdot d\vec{l} = \oint \vec{\nabla} \times \vec{V} \cdot d\vec{A}$

$$\oint \vec{B} \cdot d\vec{l} = \oint \vec{\nabla} \times \vec{B} \cdot d\vec{A}$$

$$= \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$= \mu_0 \varepsilon_0 \frac{d}{dt} \oint \vec{E} \cdot d\vec{A} \Rightarrow$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt} \Rightarrow$$

$$\vec{\nabla} \times \left[\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt} \right] \Rightarrow$$

$$\nabla \left(\cancel{\vec{\nabla} \cdot \vec{B}} \right) - \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{d}{dt} \vec{\nabla} \times \vec{E}$$

$$\oint \vec{E} \cdot d\vec{l} = \oint \vec{\nabla} \times \vec{E} \cdot d\vec{A}$$

$$= -\frac{d\Phi_B}{dt}$$

$$= \frac{d}{dt} \oint \vec{B} \cdot d\vec{A} \Rightarrow$$

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt} \Rightarrow$$

$$-\nabla^2 B = \mu_0 \varepsilon_0 \frac{d}{dt} \vec{\nabla} \times \vec{E} = -\mu_0 \varepsilon_0 \frac{d^2 B}{dt^2}$$

$$\nabla^2 B = \mu_0 \varepsilon_0 \frac{d^2 B}{dt^2}$$

And then, a miracle occurs…

- Wave Equation

$$y(x,t) = A\cos(kx - \omega t)$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y$$

$$\frac{\partial^2 y}{\partial x^2} = -k^2 y$$

$$but, \quad \omega = vk \Rightarrow v^2 = \frac{\omega^2}{k^2} \Rightarrow$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\nabla^2 B = \mu_0 \varepsilon_0 \frac{d^2 B}{dt^2}$$

- EM wave velocity

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$