#36 Kepler's Law and Black Holes Post-class

Due: 11:00am on Friday, November 16, 2012

Note: You will receive no credit for late submissions. To learn more, read your instructor's Grading Policy

Kepler's 3rd Law

A planet moves in an elliptical orbit around the sun. The mass of the sun is M_s. The minimum and maximum distances of the planet from the sun are R_1 and R_2, respectively.

Part A

Using Kepler's 3rd law and Newton's law of universal gravitation, find the period of revolution P of the planet as it moves around the sun. Assume that the mass of the planet is much smaller than the mass of the sun.

Use G for the gravitational constant.

Express the period in terms of G, M_s, R_1, and R_2.

Hint 1. Kepler's 3rd law

Kepler's 3rd law states that the square of the period of revolution of a planet around the sun is proportional to the cube of the semi-major axis of its orbit. Try finding the period of a circular orbit and then using Kepler's 3rd law (which applies equally to circular and elliptical orbits) to extend your result to an elliptical orbit.

Hint 2. Find the semi-major axis

Find the semi-major axis a.

Express the semi-major axis in terms of R_1 and R_2.

Hint 1. Definition of semi-major axis

The semi-major axis of an ellipse is half of its major axis. The sun is at the focus of the elliptical orbit and the focus lies on the major axis.

ANSWER:

$a = \frac{R_1 + R_2}{2}$

Hint 3. Find the period of a circular orbit

Find the period P of a planet in a circular orbit of semi-major axis a.

Express the period in terms of a, M_s, and G.

Hint 1. Formula for the period

The period is $2\pi r/v$, where r is the radius of the orbit and v is the speed of the object. Note that this is the distance traveled in one orbit divided by the speed.

Hint 2. Find the velocity

Find the velocity v of an object in an orbit of radius r by setting the magnitude of the centripetal acceleration $a_{\rm cent} = v^2/r$ equal to the magnitude of the acceleration due to gravity.

Express your answer in terms of r, M_s, and G.

ANSWER:

$v = \sqrt{\frac{M_s G}{r}}$

Hint 3. Radius of the orbit

For a circle, the semi-major axis is just the radius.

ANSWER:

$$P = 2\pi\sqrt{\frac{a^3}{GM_s}}$$

ANSWER:

$$P = 2\sqrt{\pi\left(\frac{(.5\left(R_1 + R_2\right))^3}{GM_s}\right)}$$

Correct

The Dyson Sphere

The Dyson sphere is an hypothetical spherical structure centered around a star. Inspired by a science fiction story, physicist Freeman Dyson described such a structure for the first time in a scientific paper in 1959. His basic idea consisted of an artificial spherical structure of matter built around a star at a distance comparable to a planetary orbit, with the purpose of capturing the energy radiated by the star and reusing it for industrial purposes. Assume the mass of the sun to be $2.00\times10^{30}\,\mathrm{kg}$.

Part A

Consider a solid, rigid spherical shell with a thickness of $100\,\mathrm{m}$ and a density of $3900\,\mathrm{kg/m^3}$. The sphere is centered around the sun so that its inner surface is at a distance of $1.50\times10^{11}\,\mathrm{m}$ from the center of the sun. What is the net force that the sun would exert on such a Dyson sphere were it to get displaced off-center by some small amount?

Express your answer numerically in newtons.

Hint 1. How to approach the problem

The sun can be modeled as a spherical body with all its mass concentrated at its center. The net force exerted on the Dyson sphere would then be the same as the gravitational force exerted on a spherical shell by a point mass located at the center of the sun. To find this force recall that no work is done on a point mass that moves inside a spherical shell.

ANSWER:

0 N

Correct

Since there is no net attraction between a hollow sphere and a body inside, a Dyson sphere of this kind would be gravitationally unstable. If the sphere were hit by a meteor and were slightly shifted, the sun would exert no force on it to bring it back to its original position. The sphere would simply drift off and eventually hit the sun. Because of this gravitational instability, Dyson himself did not originally suggest a solid spherical shell; rather, he proposed a series of individual plates independently orbiting the sun.

Part B

What is the net gravitational force $F_{\rm out}$ on a unit mass located on the outer surface of the Dyson sphere described in Part A?

Express your answer in newtons.

Hint 1. How to approach the problem

A unit mass on the outer surface of the Dyson sphere is subjected to the gravitational force exerted by the sphere itself and the gravitational force exerted by the sun. Since both the sphere and the sun have a symmetrical distribution of mass, you can calculate the gravitational force that each of these bodies exerts on the unit mass as if their masses were concentrated at their centers.

Hint 2. Find the gravitational force exerted by the Dyson sphere

What is the gravitational force $F_{\rm Dyson}$ exerted by the sphere on a unit mass located on the outer surface of the sphere itself?

Express your answer in newtons.

Hint 1. Find the mass of the Dyson sphere

Recall that density is defined as mass per unit volume. Thus, a body of mass M and volume V has density

$$\rho = \frac{M}{V}.$$

Use this definition to calculate $m_{\rm Dyson}$, the mass of the Dyson sphere.

Express your answer in kilograms.

Hint 1. Volume of a hollow sphere

The volume of a hollow sphere can be calculated by subtracting the volume of the cavity from the volume of the whole sphere. Recall that the volume of a sphere of radius r is given by

$$V = \frac{4}{3}\pi r^3.$$

ANSWER:

$m_{\rm Dyson}$ = 1.10×10^{29} $\rm kg$

Hint 2. Definition of gravitational force

Recall that the force of gravity between a mass M_1 and a mass M_2 is given by the equation

$$F_g = \frac{GM_1M_2}{r^2},$$

where r is the distance between the two masses and G is Newton's gravitational constant.

ANSWER:

$F_{\rm Dyson}$ = 3.27×10^{-4} N

Hint 3. Find the gravitational force exerted by the sun

What is the gravitational force $F_{\rm sun}$ exerted by the sun on a unit mass located on the outer surface of the Dyson sphere?

Express your answer in newtons.

Hint 1. Definition of gravitational force

Recall that the force of gravity between a mass M_1 and a mass M_2 is given by the equation

$$F_g = \frac{GM_1M_2}{r^2},$$

where r is the distance between the two masses and G is Newton's gravitational constant.

Hint 2. Find the outer radius of the Dyson sphere

What is the outer radius r_{out} of the Dyson sphere, that is, the distance between a point on the outer surface of the Dyson sphere and the center of the sun?

Express your answer in meters.

ANSWER:

r_{out} = 1.50×10^{11} m

ANSWER:

F_{sun} = 5.93×10^{-3} N

ANSWER:

F_{out} = 6.26×10^{-3} N

Correct

Part C

What is the net gravitational force $F_{\rm in}$ on a unit mass located on the inner surface of the Dyson sphere described in Part A?

Express your answer in newtons.

Hint 1. How to approach the problem

Recall that there is no net attraction between a spherical shell and a point mass inside it; therefore, the only contribution to the net gravitational force exerted on a unit mass located on the inner surface of the Dyson sphere comes from the sun.

ANSWER:

$F_{\rm in}$ = 5.93×10^{-3} $\rm N$

Correct

The gravitational attraction of the sun would make the inner surface of the Dyson sphere described in Part A uninhabitable, because everything on the inner surface would slowly accelerate toward the sun. One way to solve this problem would be to create artificial gravity through rotation. Assume that the Dyson sphere rotates at a constant angular speed around an axis through its center so that earthlike gravity is re-created along the inner equator of the Dyson sphere. Take the radius of the Earth to be $6.38\times10^{6}\,{\rm m}$ and the mass of the Earth to be $5.97\times10^{24}\,{\rm kg}$.

Part D

What is the linear speed v of a unit mass located at the inner equator of such a sphere?

Express your answer in meters per second.

Hint 1. How to approach the problem

Because of the constant rotation of the sphere, the mass at the inner equator moves along a circular path with constant angular speed; thus it has only a centripetal acceleration. There must be then a net force directed toward the center of the sphere. The only forces acting on the mass are the gravitational force of the sun and the normal force exerted by the surface of the sphere. To create the same gravitational conditions as on earth, the normal force exerted on the mass at the inner equator must be equal to the normal force exerted on a unit mass at earth's equator, since the normal force corresponds to the acceleration felt by a person on the inner surface of the Dyson sphere.

Hint 2. Find the net force at the inner surface of a rotating hollow sphere

Consider a spinning hollow sphere with a particle located at its center. Let $F_{\rm g}$ be the magnitude of the gravitational force that the particle exerts on a unit mass located on the *inner* surface of the sphere and let n be the magnitude of the normal force exerted by the surface of the sphere on the unit mass. What is the magnitude of the net force $F_{\rm net}$ acting on the unit mass?

ANSWER:

- $F_{\rm g} + n$
- $|F_{\rm g} - n|$
- $F_{\rm g} + 2n$
- $|F_{\rm g} - 2n|$
- n

Hint 3. Find the normal force acting on a unit mass on earth's surface

What is the magnitude of the normal force n acting on a unit mass located on the surface of the earth?

Express your answer in newtons.

Hint 1. Acceleration of gravity

Recall that the acceleration of gravity is 9.81 $\rm m/s$.

ANSWER:

n = 9.81 N

Hint 4. Equation for centripetal acceleration

Recall that the equation relating the centripetal acceleration a_c of an object spinning about a point a distance r away at a speed v is given by

$$a_c = \frac{v^2}{r}.$$

ANSWER:

v = 1.21×10^6 $\mathrm{m/s}$

Correct

The stresses generated by such rotation would be so intense that no material would be able to sustain them, another reason for which such a Dyson sphere would not be physically feasible. Nevertheless, it remains popular among many science-fiction authors!

Score Summary:

Your score on this assignment is 107.3%.
You received 21.45 out of a possible total of 20 points.