ECE 345: Introduction to Control Systems
In-Class Exercise #1

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Due Thursday, August 30, 2012 at the end of class

The Mile-High Bungee Jumping Company wants to design a bungee jumping system (e.g., a cord) so that a jumper of mass m (between 50 kg and 100 kg) and height of 2 m: 1) cannot hit the ground, and 2) has a hang time (the time a jumper is moving up and down) of T (between 25 and 50 seconds). The jumper stands on a platform 102 m above the ground, with the cord (affixed to the jumper's feet) secured to the platform. The cord's natural, unweighted length is $L = 50$ m. We define x, the position of the jumper's feet beyond the unweighted length of the cord, as shown in Figure 1, and note that x is positive in the downwards direction. The constant due to gravity is $g = 9.8$ m/s^2.

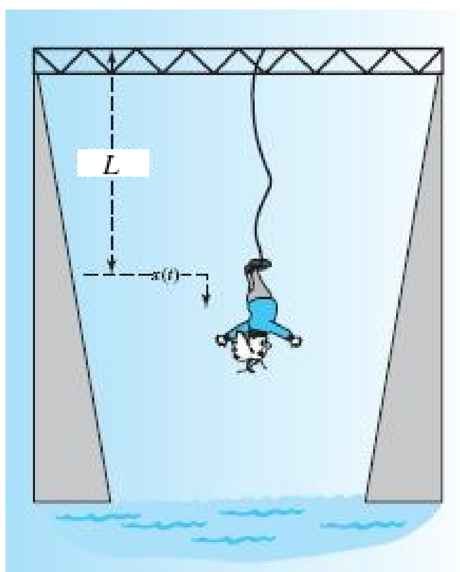

Figure 1: Bungee jumper. Illustration modified (L) from *Differential Equations with Boundary-Value Problems*, D. Zill and M. Cullen, Cengage Learning, 2008.

We will model the jumper attached to the bungee as a modified spring-mass-damper system. Notice that while under tension, a bungee cord will provide a restorative force, but that while under compression, a bungee cord will crumple. Unlike a spring, which would provide a restorative force under both compression and tension, the bungee cord will only provide a restorative force under tension. Hence the spring force from the cord is given by

$$F_{\text{spring}} = \begin{cases} -kx & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \tag{1}$$

with unknown spring constant k. Presume that the cord has no mass.

Damping forces arise due to friction and energy losses in the bungee cord, and drag forces on the jumper at sufficient speeds. We approximate these forces as

$$F_{\text{damping}} = \begin{cases} -c\dot{x} & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \tag{2}$$

with c the unknown damping coefficient.

The goal of this exercise is to choose bungee cord parameters k and c to meet the above design requirements. *Each group should hand in the answers to the asterisk-marked problems (*).*

Pre-lecture

1. Sketch the free body diagram of the jumper when the jumper is in free fall, before the bungee cord has begun to stretch.
2. Now sketch a free body diagram of the jumper after the bungee cord has begun to stretch (e.g., the cord in tension).
3. Show that the equations of motion for the two cases (a) and (b) are:

$$\begin{array}{lrcl} \text{(a)} & m\ddot{x} & = & mg, \\ \text{(b)} & m\ddot{x} & = & mg - c\dot{x} - kx \end{array} \tag{3}$$

4. Consider case (a), when the jumper is in free-fall. Use any method you wish to solve the following.
   i. Determine the velocity of the jumper at the moment that the bungee cord just starts to stretch, that is, when $x = 0$ for the first time. Denote this velocity v_0. Assume that the jumper's speed when leaving the platform is 0.
   ii. How long does it take for the jumper to reach $x = 0$?

In-class

Now consider the second section of flight, when the bungee cord is active. We presume that the remaining time is spent with the bungee cord in tension.

*1. Which of the following describes the relationship between $X(s)$ and speed v_0?
   (a) $X(s) = \frac{1}{ms^2+cs+k}\,(mv_0 + mg/s)$
   (b) $X(s) = \frac{1}{ms^2+cs}\,(mv_0 + mg/s)$

(c) $X(s) = \frac{1}{ms^2+cs+k}\left(mv_0 - mg/s\right)$

(d) $X(s) = \frac{1}{ms^2-cs-k}\left(-mv_0 + mg/s\right)$

*2. Use the Final Value Theorem to determine the steady-state value $x_{ss} = \lim_{t\to\infty} x(t)$.

3. What restrictions (if any) on k, c will assure that x_{ss} meets the height requirement?

To simplify the mathematics in the following analysis, we define $\Delta x = x - x_{ss}$. Hence by removing the steady-state forces that cancel, the equations of motion in case (b) become:

$$m\Delta\ddot{x} = -k\Delta x - c\Delta\dot{x} \tag{4}$$

*4. Use partial fraction expansion and the inverse Laplace transform to solve for $\Delta x(t)$ with initial velocity v_0 and initial position $\Delta x_0 = -x_{ss}$. Which of the following represents $\Delta x(t)$?

(a) $\Delta x(t) = \left(A\cos(bt) + B\sin(bt) + Ct\right)u(t)$

(b) $\Delta x(t) = \begin{cases} e^{-bt}\left(A\cos(at) + B\sin(at)\right) & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases}$

(c) $\Delta x(t) = e^{-at}\left(A\cos(bt) + B\sin(bt)\right)u(t)$

(d) $\Delta x(t) = \big(A\cos(bt) + Be^{-at}\sin(bt) + x_0\big)\,u(t)$

with $A = -mx_{ss}$, $B = \frac{m}{\sqrt{4mk-c^2}}(mv_0 - \frac{cx_{ss}}{2})$, $C = v_0$, and $a = c/(2m)$, $b = \frac{\sqrt{4mk-c^2}}{m}$.

We can approximate the decay in the jumper's oscillations in (4) with a *decay envelope*

$$d(t) = d_0 \cdot \exp\left(-\frac{c}{2m}t\right). \tag{5}$$

Define hang time as the time it takes (starting from the free fall, when the jumper first leaves the platform) for the decay to reach 2% of d_0, where d_0 is some positive constant that depends on x_0, v_0. (You do not need to solve for d_0.)

5. For the hang time to be between 25 and 50 seconds, what value of damping constant c will work for all values of mass m under consideration?

*6. Summarize your design by choosing the statement(s) below which most closely match your results. *All* correct statements must be chosen for full credit.

(a) 8.4 kg/s $\leq c \leq$ 17.9 kg/s, $k \geq$ 9.8 kg/s^2 to meet the hang-time and height requirements

(b) 16.7 kg/s $\leq c \leq$ 17.9 kg/s to meet the hang-time requirement; $k \geq$ 19.6 kg/s^2 to meet the height requirement

(c) A jumper with lighter mass will experience higher frequency oscillations.

(d) Jumping off the platform *does not affect* the height at which the jumper comes to rest.

(e) A stiffer spring will make the jumper come to rest closer to the ground.

If your group finishes early...

Other points to consider (not necessary to hand in):

- Consider the case in which, due to the high volume of customers, hang time needs to be at least 25 seconds and at most 40 seconds. Notice now that not all design requirements can be met with a single value of k, c. If you could change the design requirements, and limit the range of masses that the cord could accommodate, what values of m would you allow, and why?

- Would you increase or decrease allowable values of k, c to account for the jumper potentially launching from the platform with downwards velocity? That is, should the cord be stiffer? Should the cord be more damped? Why or why not?

- Some jumpers may be overly optimistic about their weight. Would your choice of k be adequate for a jumper who underestimate their weight? If not, how would you modify k?

- Numerical values of system parameters may be erroneous, due to measurement error, changes in material properties with excessive use, weathering, or other factors. Which is more dangerous: overestimating c or underestimating L?