ECE 314 – Signals and Systems Fall 2012

Solutions to Homework 5

Problem 2.51 Determine the impulse response of the system described by $y(n) = x(n) + ax(n-k)$.

Solution: Replace x by δ to obtain the impulse response: $h(n) = \delta(n) + a\delta(n-k)$.

Problem 2.53 Determine the homogeneous solutions for the systems described by the following differential equations:
(a) $5\frac{d}{dt}y(t) + 10y(t) = 2x(t)$
Solution: The homogeneous counterpart to the equation above is:

$$5\frac{d}{dt}y(t) + 10y(t) = 0.$$

To find the homogeneous solution, let's suppose that $y_h(t) = ae^{bt}$, where a and b are constants. Then, if we plug the proposed solution in the equation above, we will get:

$$5abe^{bt} + 10ae^{bt} = 0,$$

what means that $5ab = -10a$, or $b = -2$. Therefore, the set of all homogeneous solutions to the equation is given by $\{ae^{-2t},\ a \in \mathbb{R}\}$.
(d) $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 2y(t) = x(t)$
Solution: The homogeneous counterpart to the equation above is:

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 2y(t) = 0.$$

Now, if we again suppose that $y_h(t) = ae^{bt}$. Then, if we plug the proposed solution in the equation above, we will get:

$$ab^2e^{bt} + 2abe^{bt} + 2ae^{bt} = 0,$$

what means that,

$$b^2 + 2b + 2 = 0.$$

This second-degree equation has two possible solutions, either $b = -1+j$, or $b = -1-j$ (j corresponds to the complex number $\sqrt{-1}$). So, the set of all possible homogeneous solutions to the equation above is $\{a_1 e^{(-1-j)t} + a_2 e^{(-1+j)t},\ a_1, a_2 \in \mathbb{R}\}$.

Problem 2.54
(a) $y[n] - \alpha y[n-1] = 2x[n]$

Propose $y_{0x}[n] = C\rho^n$, plug it in the homogeneous equation, and obtain the characteristic equation $\rho - \alpha = 0$. Thus, $y_{0x}[n] = C\alpha^n$. To find C, we need to know $y[-1]$ (which is not given in the problem). Set $C\alpha^{-1} = y[-1]$ to obtain $C = \alpha y[-1]$.

(b) $y[n] - 1/4y[n-1] - 1/8y[n-2] = x[n] + x[n-1]$

Propose $y_{0x}[n] = C\rho^n$, plug it in the homogeneous equation and obtain the characteristic equation $\rho^2 - 1/4\rho - 1/8 = 0$. The valid values for ρ are $\rho = 1/2$ or $\rho = -1/4$. Thus, $y_{0x}[n] = C_1(1/2)^n + C_2(-1/4)^n$. To find C, we need to know $y[-1]$ and $y[-2]$ (which is not given in the problem).

Problem 2.55 Determine a particular solution for the systems described by the following differential equations, for the given inputs:

(a) $5\frac{d}{dt}y(t) + 10y(t) = 2x(t)$
(i) $x(t) = 2$
(ii) $x(t) = e^{-t}$
(iii) $x(t) = \cos(3t)$

Solution:
(i) Since $x(t)$ is a constant function, and, as we know, derivatives of constants are also constants, let's suppose that $y(t) = A$ (const). Thus, plugging it into the equation, we get:

$$\begin{aligned} 10A &= 4 \Rightarrow \\ A &= 0.4. \end{aligned}$$

(ii) Now, we suppose that $y(t) = Ae^{-t}$. Then, plugging it into the equation, we get:

$$\begin{aligned} -5Ae^{-t} + 10Ae^{-t} &= 2e^{-t} \Rightarrow \\ -5A + 10A &= 2 \Rightarrow \\ A = 0.4. \end{aligned}$$

Thus, $y(t) = 0.4e^{-t}$.
(iii) Now, let's suppose that $y(t) = A\cos(3t) + B\sin(3t)$. Then:

$$-5A\sin(3t) + 5B\cos(3t) + 10A\cos(3t) + 10B\sin(3t) = 2\cos(3t) \Rightarrow$$
$$\left\{\begin{array}{rcl} -5A + 10B & = & 0 \\ 10A + 5B & = & 2 \end{array}\right. \Rightarrow$$
$$\left\{\begin{array}{rcl} A & = & 0.16 \\ B & = & 0.08 \end{array}\right.$$

Thus, $y(t) = 0.16\cos(3t) + 0.08\sin(3t)$.

(b) $\dfrac{d^2}{dt^2}y(t) + 4y(t) = 3\dfrac{d}{dt}x(t)$
(i) $x(t) = t$
(ii) $x(t) = e^{-t}$
(iii) $x(t) = (\cos(t) + \sin(t))$

Solution:
(i) We suppose $y(t) = c_1 t + c_2$. Plugging it into the equation, we get $4c_1 t + 4c_2 = 3$. Hence, $c_1 = 0$ and $c_2 = 3/4$, and we obtain $y(t) = 3/4$.

(ii) Propose $y(t) = Ae^{-t}$ and plug it into the equation to obtain $Ae^{-t} + 4Ae^{-t} = 3e^{-t}$, or $A = 3/5$. Thus, $y(t) = 3/5e^{-t}$.

(iii) Propose $y(t) = A\cos(t) + B\sin(t)$. Then:

$$-A\cos(t) - B\sin(t) + 4A\cos(t) + 4B\sin(t) = -3\sin(t) + 3\cos(t)$$
$$\left\{\begin{array}{rcl} -A + 4A & = & 3 \\ -B + 4B & = & -3 \end{array}\right.$$
$$\text{or} \left\{\begin{array}{rcl} A & = & 1 \\ B & = & -1 \end{array}\right.$$

Thus, $y(t) = \cos(t) - \sin(t)$.

Problem 2.61

Solution: Let i_1, i_2, i_3 be the downward currents in the resistor, inductor and capacitor, respectively. KCL implies $x(t) = i_1(t)+i_2(t)+i_3(t)$. It follows that $x'(t) = i_1'(t) + i_2'(t) + i_3'(t)$. However, $y(t) = Ri_1(t)$ (or $i_1'(t) = (1/R)y'(t)$), $y(t) = Li_2'(t)$, and $i_3(t) = Cy'(t)$ (or $i_3'(t) = Cy''(t)$). Combining, we obtain

$$x'(t) = Cy''(t) + (1/R)y'(t) + (1/L)y(t).$$

To find the step response, we replace $x(t)$ by $u(t)$, which results in the equation $Cy''(t) + (1/R)y'(t) + (1/L)y(t) = \delta(t)$.

When actual R-L-C values are substituted, we obtain $y''(t) + 5y'(t) + 20y(t) = 5\delta(t)$. The solution to this equation is 5 times the solution to the equation $y''(t) + 5y'(t) + 20y(t) = \delta(t)$ (why?), and the solution to the latter is simply the impulse response of the system represented by $y''(t) + 5y'(t) + 20y(t) = x(t)$. From class notes we know that the impulse response for the latter system is $h(t) = (1/(r_1-r_2))e^{r_1 t} - (1/(r_1-r_2))e^{r_2 t}$, $t \geq 0$, where r_1 and r_2 are the roots of the equation $r^2 + 5r + 20$. They are $r_1 = -5/2 + j\sqrt{55}/2$ and $r_2 = -5/2 - j\sqrt{55}/2 = -2.5 - j3.7081$.

Hence, $h(t) = -j0.1348e^{-2.5t+j3.7081} + j0.1348e^{-2.5t-j3.7081}$, or
$h(t) = 0.1348e^{-2.5t+j3.7081t-j\pi/2} + 0.1348e^{2.5t-j3.7081+j\pi/2}$, or
$h(t) = 0.1348e^{-2.5t}\left(e^{j(3.7081t-\pi/2)} + e^{-j(3.7081t-\pi/2)}\right)$, or
$h(t) = 0.2696e^{-2.5t}\cos(3.7081t - \pi/2)$.

Finally, the step response of the RLC circuit is obtained by multiplying h by 5:
$y_{step}(t) = 1.348e^{-2.5t}\sin(3.7081t)$, $t \geq 0$,

Problem 2.62 The difference equation to be considered is
$y(n) = 1.01y(n-1) - 1,200u(n-1)$, $n \geq 1$, with $y(0) = 100,000$.

The zero-input (or natural) response is of the form $y_{0i}(n) = c(1.01)^n$, $n \geq 0$. By applying the initial condition $y_{0i}(0) = 100,000$, we obtain $y_{0i}(n) = 100,000(1.01)^n$, $n \geq 0$.

The forced response can be computed by convolving the impulse response of the system $y(n) - 1.01y(n-1) = x(n)$ by the input $x(n) = -1,200u(n-1)$. Let us find the impulse response first. To do so, write the difference equation $h(n) - 1.01h(n-1) = \delta(n)$, $n \geq 0$. Now consider $n \geq 1$ to obtain the homogeneous equation $h(n) - 1.01h(n-1) = 0$ (whose solution is of the form

$h(n) = h(0)1.01^n u(n))$, for which we need to find the derived initial condition $h(0)$. But from the difference equation for h, $h(0) = 1.01h(-1) + \delta(0)$; hence, $h(0) = 1$ since $h(-1) = 0$. (Recall that when calculating the impulse response we assume no initial conditions prior to the application of the input.) Hence, $h(n) = 1.01^n u(n)$.

The forced response is then
$y_f(n) = h(n) * (-1, 200u(n-1))$
$= \sum_{k=-\infty}^{\infty} 1.01^k u(k)(-1, 200u(n-1-k))$
$= \sum_{k=0}^{n-1} 1.01^k(-1, 200)$
$= -120000(1.01^n - 1), n \geq 0.$

The total response is the sum of the zero-input response and the forced response:
$y(n) = 100, 000(1.01)^n - 120000(1.01^n - 1)$, $n \geq 0$.

Note that $y(181) = -1115.2$ while $y(180) = 83.96$; thus, the loan is paid off after the 181th payment (n=181). [Useful Matlab command: to find the index for which y drops below zero for the first time consider the command "min(find(y<0))"; read about the command "find"]

Problem 2.65 Find the difference equation for the three systems depicted in Fig. P2.65 (in the textbook).

(a)

Solution: Let's call the signal coming out of the first adder (Σ) $f[n]$. We can see that

$$f[n] = x[n] - 2y[n].$$

Hence the signal coming out of the second adder is

$$\begin{aligned} y[n] &= 2f[n] + f[n-1] \\ &= 2x[n] - 4y[n] + x[n-1] - 2y[n-1] \\ \therefore \quad y[n] &= \frac{2}{5}x[n] + \frac{1}{5}x[n-1] - \frac{2}{5}y[n-1]. \end{aligned}$$

(b)

Solution: The signal coming out of the first adder is

$$f[n] = \frac{1}{4}y[n] + x[n-1]$$

Hence,

$$\begin{aligned} y[n] &= f[n-1] \\ &= \frac{1}{4}y[n-1] + x[n-2]. \end{aligned}$$

(c)

Solution: The output of the first adder is

$$f[n] = x[n] - \frac{1}{8}y[n].$$

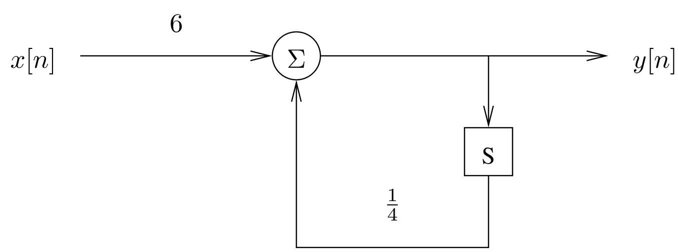
Hence, the output of the second adder is

$$\begin{aligned} y[n] &= \frac{1}{2}x[n-1] + f[n-2] \\ &= \frac{1}{2}x[n-1] + x[n-2] - \frac{1}{8}y[n-2]. \end{aligned}$$

Problem 2.66 Draw direct form I (only) implementation for the following difference equations:

(a) $y[n] - \frac{1}{4}y[n-1] = 6x[n]$

Solution:



(b) $y[n] + \frac{1}{2}y[n-1] - \frac{1}{8}y[n-2] = x[n] + 2x[n-1]$

Solution: