ECE340 spring 2011
Homework-8 Solutions

Problems: 3-4.2, 3.-4.3, 3-5.1, 3-5.5, 3-6.1

3-4.2

a) To find the variances of U and V, we use the following steps:

$$\sigma_U^2 = E[U^2] - (E[U])^2$$
$$= E[(3X+4Y)^2] - (E[3X+4Y])^2$$
$$= E[9X^2+24XY+16Y^2] - (3E[X]+4E[Y])^2$$
$$= 9E[X^2]+24E[XY]+16E[Y^2] - 9(E[X])^2 - 24E[X]E[Y] - 16(E[Y])^2$$
$$= 9\{E[X^2]-(E[X])^2\} + 16\{E[Y^2]-(E[Y])^2\} + 24E[XY] - 24E[X]E[Y]$$
$$= 9\sigma_X^2 + 16\sigma_Y^2 \qquad (since\ X\ and\ Y\ are\ independent)$$
$$= 9*9+16*25 = 481$$

$$\sigma_V^2 = E[V^2] - (E[V])^2$$
$$= E[(5X-2Y)^2] - (E[5X-2Y])^2$$
$$= E[25X^2-20XY+4Y^2] - (5E[X]-2E[Y])^2$$
$$= 25E[X^2]-20E[XY]+4E[Y^2] - 25(E[X])^2 + 20E[X]E[Y] - 4(E[Y])^2$$
$$= 25\{E[X^2]-(E[X])^2\} + 4\{E[Y^2]-(E[Y])^2\} - 20E[XY] + 20E[X]E[Y]$$
$$= 25\sigma_X^2 + 4\sigma_Y^2 \qquad (since\ X\ and\ Y\ are\ independent)$$
$$= 25*9+4*25 = 325$$

b) To find the correlation coefficient of U and V, we follow the definition:

$$\rho = E\left\{\left[\frac{X-E[X]}{\sigma_X}\right]\left[\frac{Y-E[Y]}{\sigma_Y}\right]\right\} = \frac{E[XY]-E[X]E[Y]}{\sigma_X\sigma_Y}$$

In our case:

$$\rho = E\left\{\left[\frac{U-E[U]}{\sigma_U}\right]\left[\frac{V-E[V]}{\sigma_V}\right]\right\} = \frac{E[UV]-E[U]E[V]}{\sigma_U\sigma_V}$$
$$= \frac{E[(3X+4Y)(5X-2Y)] - E[3X+4Y]E[5X-2Y]}{\sigma_U\sigma_V}$$
$$= \frac{E[15X^2+14XY-8Y^2] - (3E[X]+4E[Y])(5E[X]-2E[Y])}{\sigma_U\sigma_V}$$
$$= \frac{15E[X^2]+14E[XY]-8E[Y^2] - (15(E[X])^2+14E[X]E[Y]-8(E[Y])^2)}{\sigma_U\sigma_V}$$
$$= \frac{15\sigma_X^2+14E[XY]-8\sigma_Y^2-14E[X]E[Y]}{\sigma_U\sigma_V}$$
$$= \frac{15\sigma_X^2-8\sigma_Y^2}{\sigma_U\sigma_V} \quad (since\ X\ and\ Y\ are\ independent)$$
$$= \frac{15*9-8*25}{\sqrt{481}\sqrt{325}} = -0.1644$$

3-4.3

a) We have:

$$\rho_{XY} = E\left\{\left[\frac{X-E[X]}{\sigma_X}\right]\left[\frac{Y-E[Y]}{\sigma_Y}\right]\right\} = \frac{E[XY]-E[X]E[Y]}{\sigma_X\sigma_Y}$$

Since the random variables X and Y are statistically independent, we have:

$$E[XY] = E[X]E[Y]$$

Therefore:

$$\rho_{XY} = 0$$

b)

$$\rho_{YZ} = E\left\{\left[\frac{Y - E[Y]}{\sigma_X}\right]\left[\frac{Z - E[Z]}{\sigma_Z}\right]\right\} = E\left\{\left[\frac{Y - E[Y]}{\sigma_X}\right]\left[\frac{X + Y - E[X] - E[Y]}{\sigma_Z}\right]\right\}$$
$$= \frac{E\{(Y - E[Y])(X - E[X])\} - E\{(Y - E[Y])^2\}}{\sigma_Y\sigma_Z}$$

But $E\{(Y - E[Y])(X - E[X])\} = \sigma_X\sigma_Y\rho_{XY}$, and $E\{(Y - E[Y])^2\} = \sigma_Y^2$. Since $\rho_{XY} = 0$ we have:

$$\rho_{YZ} = \frac{0 + E\{(Y - E[Y])^2\}}{\sigma_Y(\sigma_Y^2 + \sigma_X^2)^{1/2}} = \frac{5}{34} = 0.857$$

c) The variance of Z can be calculated as:

$$Var\{z\} = E\{(z - E[z])^2\} = E\{(X + Y - E[X] - E[Y])^2\}$$
$$= E\left\{(X + Y)^2 - 2(X + Y)(E[X] + E[Y]) + \left(E[X] + E(Y)\right)^2\right\}$$
$$= E\{X^2 + Y^2 + 2XY - 2XE[X] - 2YE[Y] - 2XE[Y] - 2YE[X] + (E[X])^2 + (E[Y])^2 + 2E[X]E[Y]\}$$
$$= E\{X^2 - 2XE[X] + (E[X])^2 + Y^2 - 2YE[Y] + (E[Y])^2 + 2XY - 2XE[Y] - 2YE[X] + 2E[X]E[Y]\}$$
$$= E\{X^2 - 2XE[X] + (E[X])^2\} + E\{Y^2 - 2YE[Y] + (E[Y])^2\} + E\{2XY - 2XE[Y] - 2YE[X] + 2E[X]E[Y]\}$$
$$= E\{(X - E[X])^2\} + 2E\{(X - E[X])(Y - E[Y])\} + E\{(Y - E[Y])^2\}$$
$$= \sigma_Y^2 + \sigma_X^2 + 2\rho_{XY} = 34$$

3-5.1

a) Firstly, if we introduce another random variable R such that R = 2Y, then the pdf of R is

$$f_R(r) = \begin{cases} \frac{1}{4} & -2 \le r \le 2 \\ 0 & elsewhere \end{cases}$$

This can be obtained, for example, by first going after the probability distribution function of R ()as we always do in class), relating it to that of X, and then taking the derivative with respect to r. Now we have the following:

$$Z = X + 2Y = X + R$$
$$F_Z(z) = P\{Z \le z\} = P\{X + R \le z\}$$

Now, for every fixed r, x must be such that

$$-\infty < x < z - r$$

Thus,

$$F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-r} f_{XR}(x, r)dxdr$$

Now, since X and Y are statistically independent, we know that X and R are also independent, we have the following:

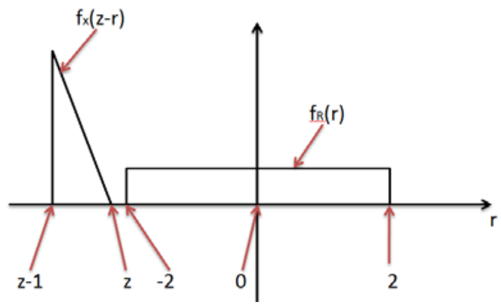
$$F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-r} f_X(x)f_R(r)dxdr = \int_{-\infty}^{\infty} f_R(r)\int_{-\infty}^{z-r} f_X(x)dxdr$$

We also know that the pdf of Z is the following:

$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{d[\int_{-\infty}^{\infty} f_R(r) \int_{-\infty}^{z-r} f_X(x)dxdr]}{dz} = \int_{-\infty}^{\infty} f_R(r) f_X(z-r)dr$, which is a convolution between the pdf of R and that of X. Note that we could have come to this expression by recalling that the pdf of the sum of two independent random variables is the convolution of the respective pdfs.

To evaluate the convolution, we have the following 5 cases:

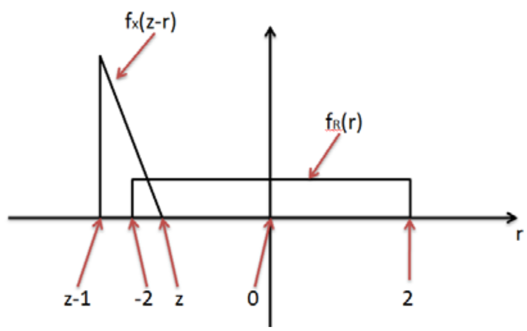
Case 1:



$$z \leq -2$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_R(r) f_X(z-r)dr = 0$$

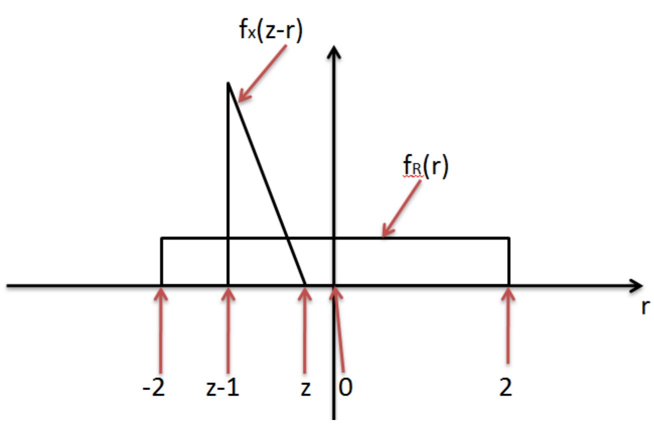
Case 2:



$$-2 < z \leq -1$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_R(r) f_X(z-r)dr = \int_{-2}^{z} f_R(r) f_X(z-r)dr = \int_{-2}^{z} \frac{1}{4} 2(z-r)dr = \frac{z^2}{4} + z + 1$$

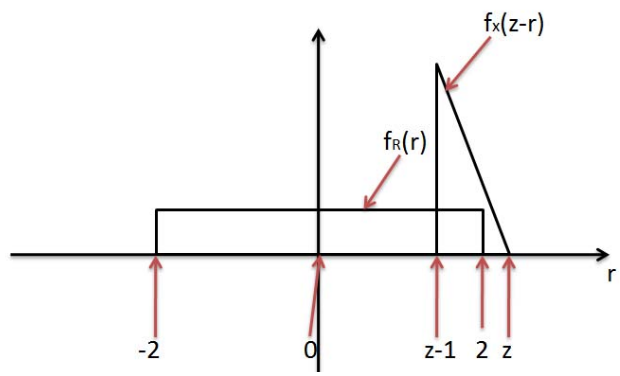
Case 3:



$$-1 < z \le 2$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_R(r) f_X(z-r)dr = \int_{z-1}^{z} f_R(r) f_X(z-r)dr = \int_{z-1}^{z} \frac{1}{4} 2(z-r)dr = \frac{1}{4}$$

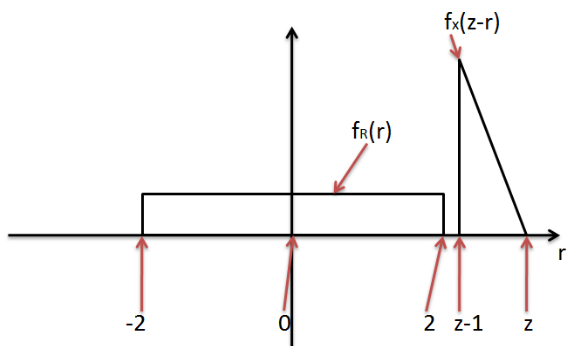
Case 4:



$$2 < z \le 3$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_R(r) f_X(z-r)dr = \int_{z-1}^{2} f_R(r) f_X(z-r)dr = \int_{z-1}^{2} \frac{1}{4} 2(z-r)dr = -\frac{z^2}{4} + z - \frac{3}{4}$$

Case 5:



$$z > 3$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_R(r) f_X(z-r)dr = 0$$

So, to sum up we have the result of the pdf of random variable Z =X + 2Y as the following:

$$f_Z(z) = \begin{cases} \dfrac{z^2}{4} + z + 1 & -2 < z \le -1 \\ \dfrac{1}{4} & -1 < z \le 2 \\ -\dfrac{z^2}{4} + z - \dfrac{3}{4} & 2 < z \le 3 \\ 0 & elsewhere \end{cases}$$

b) To find the probability that $0 < Z \le 1$, we have the following:

$$P\{0 < Z \le 1\} = \int_0^1 f_Z(z)dz = \int_0^1 \frac{1}{4} dz = \frac{1}{4}$$

3-5.5

Since X and Y are statistically independent, the joint density function $f_Z(z)$ can be written as:

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(z-y) dy$$

In this case, for the evaluation of the convolution we have only one case:

$$f_Z(z) = \int_0^z 2e^{-2y}.5\, e^{-5(z-y)} dy$$
$$= 10e^{-5z} \int_0^z e^{3y} dy = \frac{10}{3}\, e^{-5z}\, .\, e^{3y}\ \Big|_0^z$$
$$= \frac{10}{3}\, (e^{-2z} - e^{-5z})$$

a) For $z = 0$ we have:

$$f_Z(0) = \frac{10}{3}\, (1-1) = 0$$

b) For $f_Z(z) > 1$ we must have:

$$\frac{10}{3}\, (e^{-2z} - e^{-5z}) > 1$$

Which yields to

$$(e^{-2z} - e^{-5z}) > \frac{3}{10}$$

Solving the above inequality using MATLAB yields to:

$$z > 0.4541$$

c)

$$P\{z > 0.1\} = \int_{0.1}^{\infty} f_Z(z) dz = \int_{0.1}^{\infty} \frac{10}{3}\, (e^{-2z} - e^{-5z}) dz$$
$$= \frac{10}{3} \left(\frac{e^{-2z}}{-2} - \frac{e^{-5z}}{-5} \right) \Bigg|_{0.1}^{\infty} = \frac{10}{3} \left(0 + \frac{e^{-0.2}}{2} + \frac{e^{-0.5}}{5} \right) = 0.96$$

3-6.1

Firstly, we introduce another random variable W as the following:

$$W = X$$

Now we layout the random variables we have in hand and their connections:

$$Z = \varphi_1(X,Y) = X + Y$$
$$W = \varphi_2(X,Y) = X$$

$$X = \psi_1(Z,W) = W$$
$$Y = \psi_2(Z,W) = Z - W$$

Now

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

So the joint pdf of Z and W is the following:

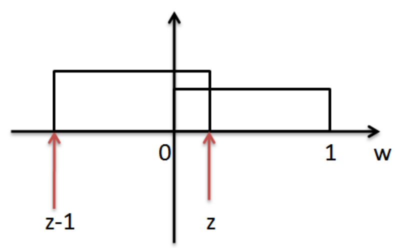
$$f_{ZW}(z,w) = f_{XY}\big(\psi_1(Z,W),\psi_2(Z,W)\big)|J| = f_{XY}\big(\psi_1(Z,W),\psi_2(Z,W)\big)$$
$$= \begin{cases} 4w(z-w) & \mathbf{0} < \boldsymbol{w} < 1,\ 0 < z-w < 1 \\ 0 & elsewhere \end{cases}$$
$$= \begin{cases} 4w(z-w) & \mathbf{0} < \boldsymbol{w} < 1,\ z-1 < w < z \\ 0 & elsewhere \end{cases}$$

Now the marginal pdf for Z is the following:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)dw$$

To evaluate the integral, we have discussions on the value of Z:

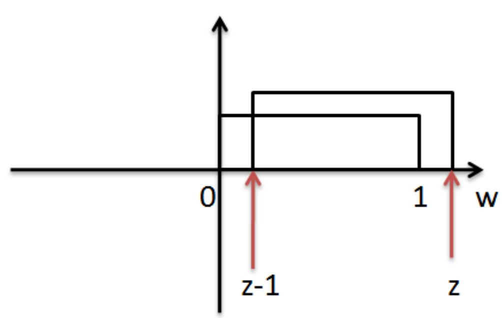
Case 1:



$$0 \le z \le 1$$

$$f_Z(z) = \int_0^z f_{ZW}(z,w)dw = \int_0^z 4w(z-w)dw = \frac{2}{3}z^3$$

Case 2:



$$1 \le z \le 2$$

$$f_Z(z) = \int_{z-1}^{1} f_{ZW}(z,w)dw = \int_0^z 4w(z-w)dw = \frac{2}{3}(-z^3+6z-4)$$

To sum up, we have the following:

$$f_Z(z) = \begin{cases} \frac{2}{3}z^3 & 0 \le z \le 1 \\ \frac{2}{3}(-z^3+6z-4) & 1 \le z \le 2 \\ 0 & elsewhere \end{cases}$$