ECE340 Spring 2011
Homework-5 Solutions

Problems: 2-3.4, 2-4.1, 2-4.3, 2-4.5, 2-5.1, 2-5.6, 2-6.2, 2-6.3

2-3.4

a) Random variable Y is related to X by Y = 3X-4; From Problem 2-3.3 we know that

$$f_X(x) = \begin{cases} \exp(2x) & x < 0 \\ \exp(-2x) & x \geq 0 \end{cases}$$

Also,

$$F_X(x) = \begin{cases} 1 - \frac{1}{2}\exp(-2x) & if\ x \geq 0 \\ \frac{1}{2}\exp(2x) & if\ x < 0 \end{cases}$$

Now,

$$F_Y(y) = P\{Y \leq y\} = P\{3X - 4 \leq y\} = P\left\{X \leq \frac{y+4}{3}\right\} = F_X\left(\frac{y+4}{3}\right)$$

$$= \begin{cases} 1 - \frac{1}{2}\exp\left(-2\frac{y+4}{3}\right) & if\ \frac{y+4}{3} \geq 0 \\ \frac{1}{2}\exp\left(2\frac{y+4}{3}\right) & if\ \frac{y+4}{3} < 0 \end{cases}$$

$$= \begin{cases} 1 - \frac{1}{2}\exp\left(-\frac{2(y+4)}{3}\right) & if\ y \geq -4 \\ \frac{1}{2}\exp\left(\frac{2(y+4)}{3}\right) & if\ y < -4 \end{cases}$$

So now,

$$f_Y(y) = \frac{d(F_Y(y))}{dy} = \begin{cases} \dfrac{d[1 - \frac{1}{2}\exp\left(-\frac{2(y+4)}{3}\right)]}{dy} & if\ y \geq -4 \\ \dfrac{d[\frac{1}{2}\exp\left(\frac{2(y+4)}{3}\right)]}{dy} & if\ y < -4 \end{cases}$$

$$= \begin{cases} \frac{1}{3}\exp\left(-\frac{2(y+4)}{3}\right) & if\ y \geq -4 \\ \frac{1}{3}\exp\left(\frac{2(y+4)}{3}\right) & if\ y < -4 \end{cases}$$

b) We know that

$$P\{Y < 0\} = P\{Y < -4\} + P\{-4 \leq Y < 0\} = \int_{-\infty}^{-4} f_Y(y)\,dy + \int_{-4}^{0} f_Y(y)\,dy$$

$$= \int_{-\infty}^{-4} \frac{1}{3}\exp\left(\frac{2(y+4)}{3}\right) dy + \int_{-4}^{0} \frac{1}{3}\exp\left(-\frac{2(y+4)}{3}\right) dy = 0.9653$$

Also,

$$P\{Y<0\}=F_Y(0)=1-\frac{1}{2}\exp\left(-\frac{2(0+4)}{3}\right)=1-\frac{1}{2}\exp\left(-\frac{8}{3}\right)=0.9653$$

c) We know that

$$P\{Y>X\}=P\{3X-4>X\}=P\{X>2\}=1-P\{X\le 2\}=\ 1-F_X(2)$$
$$=1-\left[1-\frac{1}{2}\exp(-2\times 2)\right]=\ \frac{1}{2}\exp(-4)=0.0092$$

2-4.1

a) We know from 2-3.2 that

$$F_X(x)=\begin{cases}1-\exp[-(x-1)] & 1<x<\infty\\ 0 & -\infty<x\le 1\end{cases}$$

And also

$$f_X(x)=\frac{dF_X(x)}{dx}=\begin{cases}\frac{d\{1-\exp[-(x-1)]\}}{dx} & 1<x<\infty\\ 0 & -\infty<x\le 1\end{cases}$$
$$=\begin{cases}\exp[-(x-1)] & 1<x<\infty\\ 0 & -\infty<x\le 1\end{cases}$$

Then we have the mean value of X:

$$E[X]=\int_{-\infty}^{\infty}x\cdot f_X(x)dx=\int_{1}^{\infty}x\cdot\exp[-(x-1)]\,dx=\ 2$$

b) The mean-square value of X:

$$E[X^2]=\int_{-\infty}^{\infty}x^2\cdot f_X(x)dx=\int_{1}^{\infty}x^2\cdot\exp[-(x-1)]\,dx=\ 5$$

c) The variance of X:

$$\sigma_X{}^2=E[(X-E[X])^2]=E[X^2]-(E[X])^2=5-2^2=1$$

2-4.3

a) First notice that

$$f_Y(y)=\begin{cases}Ky & 0<y\le 6\\ 0 & elsewhere\end{cases}$$

In order to make this pdf valid, we need to make sure that the integral of the pdf over all define regions is 1, i.e.:

$$\int_{-\infty}^{\infty}f_Y(y)dy=1$$
$$\int_0^6 Kydy=1$$
$$K\int_0^6\frac{1}{2}dy^2=1=\frac{K}{2}36=18K$$

Then we know

$$K=\frac{1}{18}$$

Now we have the pdf as:

$$f_Y(y)=\begin{cases}\frac{1}{18}y & 0<y\le 6\\ 0 & elsewhere\end{cases}$$

b) The mean value of Y

$$E[Y] = \int_{-\infty}^{\infty} y \cdot f_Y(y)dy = \int_0^6 y \cdot \frac{1}{18} ydy = \frac{1}{18 \times 3}\, 6^3 = 4$$

c) The mean-square value of Y

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 \cdot f_Y(y)dy = \int_0^6 y^2 \cdot \frac{1}{18} ydy = \frac{1}{18 \times 4}\, 6^4 = 18$$

d) The variance of Y

$${\sigma_Y}^2 = E[(Y - E[Y])^2] = E[Y^2] - (E[Y])^2 = 18 - 16 = 2$$

e) The third central moment of Y

$$E[(Y - E[Y])^3] = \int_{-\infty}^{\infty} (y - E[Y])^3 \cdot f_Y(y)dy = \int_0^6 (y-4)^3 \cdot \frac{1}{18} ydy = -1.6$$

f) The nth moment of Y

$$E[Y^n] = \int_{-\infty}^{\infty} y^n \cdot f_Y(y)dy = \int_0^6 y^n \cdot \frac{1}{18} ydy = \frac{1}{18}\int_0^6 y^{n+1}dy = \frac{1}{18}\int_0^6 \frac{dy^{n+2}}{n+2} = \frac{1}{18}\frac{6^{n+2}}{n+2}$$
$$= \frac{6^{n+1}}{3(n+2)}$$

2-4.5

a) First we know that

$$f_X(x) = \begin{cases} ax^2 & 0 < y \le 2 \\ ax & 2 < y \le 3 \end{cases}$$

In order to make this pdf to be valid, we need to make sure the following:

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 = \int_0^2 ax^2dx + \int_2^3 axdx = a\left(\int_0^2 \frac{1}{3}dx^3 + \int_2^3 \frac{1}{2}dx^2\right) = \frac{31a}{6}$$

So, we have

$$a = \frac{6}{31}$$

b) The mean of the random variable X

$$E[X] = \int_0^3 x \cdot f_X(x)dx = \frac{6}{31}\left(\int_0^2 x^3dx + \int_2^3 x^2dx\right) = \frac{6}{31}\left(\int_0^2 \frac{1}{4}dx^4 + \int_2^3 \frac{1}{3}dx^3\right) = 2$$

c) .

$$P\{2 < x \le 3\} = \int_2^3 f_X(x)dx = \int_2^3 \frac{6}{31}xdx = \frac{6}{31}\int_2^3 \frac{1}{2}dx^2 = \frac{6 \times 5}{31 \times 2} = \frac{15}{31} = 0.484$$

2-5.1

a) $\mathrm{E[X]} = \bar{X} = 5$, $\sigma^2 = 16$, so we have the following pdf:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[-\frac{(x - E[X])^2}{2\sigma^2}\right], \qquad -\infty < x < \infty$$

$$f_X(x) = \frac{1}{4\sqrt{2\pi}} exp\left[-\frac{(x-5)^2}{32}\right], \qquad -\infty < x < \infty$$

Now we know that

$$P\{X > 0\} = 1 - P\{X \le 0\} = 1 - F_X(0)$$

Generally we compute $F_X(x)$ using the following method:

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} exp\left[-\frac{(u-\bar{X})^2}{2\sigma^2}\right] du \ = \Phi(\frac{x-\bar{X}}{\sigma})$$

Where,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} exp\left[-\frac{u^2}{2}\right] du$$

An abbreviated table of values for $\Phi(x)$is given in Appendix D. Since only positive values of x are tabulated, it is frequently necessary to use the additional relationship:

$$\Phi(-x) = 1 - \Phi(x)$$

Another function that is closely related to $\Phi(x)$, and is often more convenient to use, is the Q-function defined by

$$\mathrm{Q}(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} exp\left[-\frac{u^2}{2}\right] du = 1 - \ \Phi(x)$$

Also note that

$$\mathrm{Q}(-x) = 1 - \mathrm{Q}(x)$$

Now we have

$$F_X(x) = \Phi\left(\frac{x-\bar{X}}{\sigma}\right) = 1 - Q(\frac{x-\bar{X}}{\sigma})$$

A brief table of values for $\mathrm{Q}(x)$is given in Appendix E for small values of x, in our case

$$x = 0$$

And

$$\frac{x-\bar{X}}{\sigma} = \frac{0-5}{\sqrt{16}} = -\frac{5}{4} = -1.25$$

Look up $\mathrm{Q}(1.25) = 0.1056$, so we know $\mathrm{Q}(-1.25) = 1 - \mathrm{Q}(1.25) = 0.8944$

So now we have

$$P\{X > 0\} = 1 - P\{X \le 0\} = \ 1 - F_X(0) = 1 - [1 - Q(-1.25)] = Q(-1.25) = 0.8944$$

b) $P\{0 < X \le 5\} = P\{X \le 5\} - P\{X \le 0\} = \frac{1}{2} - (1 - 0.8944) = \frac{1}{2} - 0.1056$=0.3944

c) $P\{X > 10\} = 1 - P\{X \le 10\} = 1 - F_X(10) = 1 - [1 - Q(1.25)] = Q(1.25) = 0.1056$

2-5.6

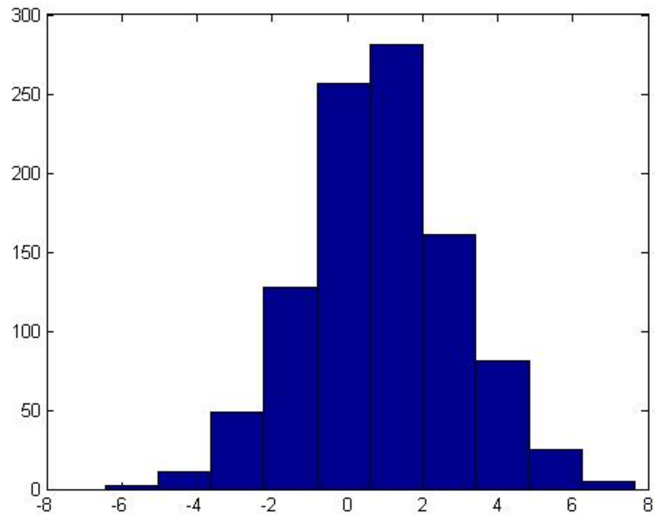
a) A Gaussian random variable has a mean of 1 and a variance of 4. Here is the Matlab code used to generate a histogram of samples of this random variable using 1000 samples:

```
clc
clear all
close all

n=1000;%number of samples
meanx = 1;%mean value
var = 4;%variance
```

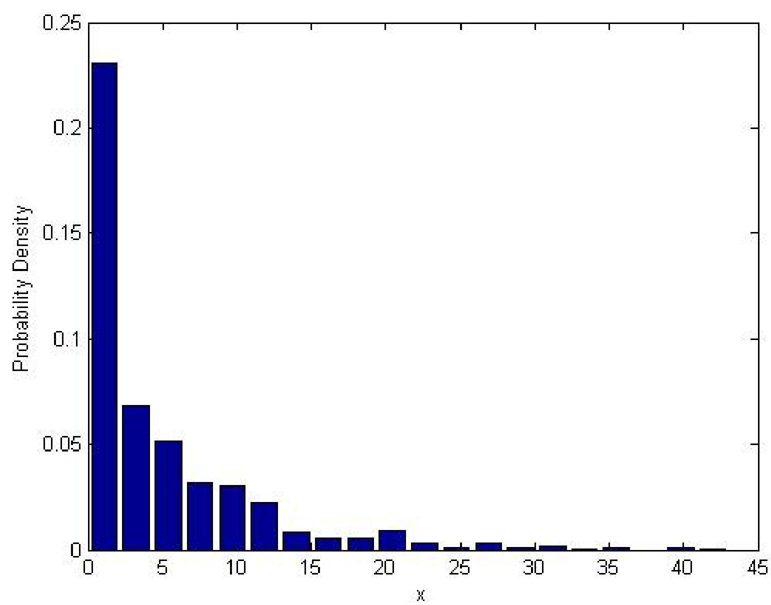
```
sig = sqrt(var);%standard deviation
x = sig*randn(1,n)+ meanx*ones(1,n);%generate vector of samples
hist(x)
```

Here is the histogram result:



b)

```
y = x.^2;% let y be the square of rv x
[m,z] = hist(y,20);%calculate counts in bins and bin coordinates
w = max(z)/20;%calculate bin width
mm = m/(1000*w);%find probability in each bin
figure(2)
bar(z,mm)%plot histogram
xlabel('x');ylabel('Probability Density');
```



2-6.2

a) We know that

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \qquad -\infty < x < \infty$$

Now we have

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du, \qquad -\infty < x < \infty$$

Since $Y = X^3$, we have the following:

$$F_Y(y) = P\{Y \le y\} = P\{X^3 \le y\} = P\left\{X \le y^{\frac{1}{3}}\right\} = F_X\left(y^{\frac{1}{3}}\right) = \int_{-\infty}^{y^{\frac{1}{3}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

Now we have

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\int_{-\infty}^{y^{\frac{1}{3}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du = \frac{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^{\frac{2}{3}}}{2}\right)}{3y^{2/3}} = \frac{1}{3\sqrt{2\pi}} y^{-2/3} \exp\left(-\frac{y^{\frac{2}{3}}}{2}\right)$$

So, the probability density function of the random variable Y is the following:

$$f_Y(y) = \frac{1}{3\sqrt{2\pi}} y^{-2/3} \exp\left(-\frac{y^{\frac{2}{3}}}{2}\right), \qquad -\infty < y < \infty$$

b) Find the mean value of Y

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} y \frac{1}{3\sqrt{2\pi}} y^{-\frac{2}{3}} \exp\left(-\frac{y^{\frac{2}{3}}}{2}\right) dy = \int_{-\infty}^{\infty} \frac{1}{3\sqrt{2\pi}} y^{\frac{1}{3}} \exp\left(-\frac{y^{\frac{2}{3}}}{2}\right) dy$$

Now we set $u = y^{\frac{1}{3}}$, we can compute this integration and the result is 0 (since it is a basic result from Gaussian Distribution).

c) Find the variance of Y

$\sigma_Y{}^2 = E[Y^2] - (E[Y])^2 = E[Y^2] =$

$\int_{-\infty}^{\infty} y^2 f_Y(y) dy = \int_{-\infty}^{\infty} y^2 \frac{1}{3\sqrt{2\pi}} y^{-\frac{2}{3}} \exp\left(-\frac{y^{\frac{2}{3}}}{2}\right) dy = \int_{-\infty}^{\infty} y^{\frac{4}{3}} \frac{1}{3\sqrt{2\pi}} \exp\left(-\frac{y^{\frac{2}{3}}}{2}\right) dy = \frac{1}{3\sqrt{2\pi}} [45\sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{\sqrt[3]{x}}{\sqrt{2}}\right) +$
$(-3x^{\frac{5}{3}} - 15x - 45\sqrt[3]{x}) \exp\left(-\frac{x^{2/3}}{2}\right)] \Big|_{-\infty}^{\infty}$ =15/2 + 15/2 = 15

A reference to calculate and check your integration (http://integrals.wolfram.com/index.jsp)

2-6.3

a) $\overline{W} = E[W] = E[RI^2] = RE[I^2]$. Now since current I is a Rayleigh distributed, and the mean value of a Rayleigh distributed random variable is $\sqrt{\frac{\pi}{2}}\sigma$ and mean-square value is $2\sigma^2$(Page 80-81), we have the following:

$$\sqrt{\frac{\pi}{2}}\sigma_I = 2$$

$$\sigma_I = 2\sqrt{\frac{2}{\pi}}$$

$$E[I^2] = 2\sigma_I{}^2 = 2\frac{8}{\pi} = \frac{16}{\pi}$$

$$\overline{W} = R \times E[I^2] = 2\pi\frac{16}{\pi} = 32\,Watt$$

b) $P\{W \leq 12\} = P\{RI^2 \leq 12\} = P\{I^2 \leq 12/R\} = P\left\{-\sqrt{\frac{12}{R}} \leq I \leq \sqrt{\frac{12}{R}}\right\} = P\left\{0 \leq I \leq \sqrt{\frac{12}{2\pi}}\right\} = P\left\{0 \leq I \leq \sqrt{\frac{6}{\pi}}\right\} = \int_0^{\sqrt{\frac{6}{\pi}}} f_I(u)du = \int_0^{\sqrt{\frac{6}{\pi}}} \frac{u}{\sigma_I{}^2}\exp\left(-\frac{u^2}{2\sigma_I{}^2}\right)du = 1 - \exp\left(-\frac{3}{8}\right) = 0.3127$

c) $P\{W > 72\} = 1 - P\{W \leq 72\} = 1 - P\{RI^2 \leq 72\} = 1 - P\left\{I^2 \leq \frac{72}{R}\right\} = 1 - P\left\{I \leq \frac{6}{\sqrt{\pi}}\right\} = 1 - \int_0^{\frac{6}{\sqrt{\pi}}} \frac{u}{\sigma_I{}^2}\exp\left(-\frac{u^2}{2\sigma_I{}^2}\right)du = 1 - 0.8946 = 0.1054$