ECE 340: PROBABILISTIC METHODS IN ENGINEERING

SOLUTIONS TO HOMEWORK #7

4.5 *Y* is the difference between the number of heads and the number of tails in the 3 tosses of a fair coin. Let *m* be the number of tails $0 \le m \le 3$. Then 3-*m* is the number of heads and the difference is

$Y=3-m-m = 3-2m$ with $0 \le m \le 3$.

Thus, $S_Y=\{-3,-1,1,3\}$ and the probabilities are:

$P\{Y=-3\} = P\{(T,T,T)\} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

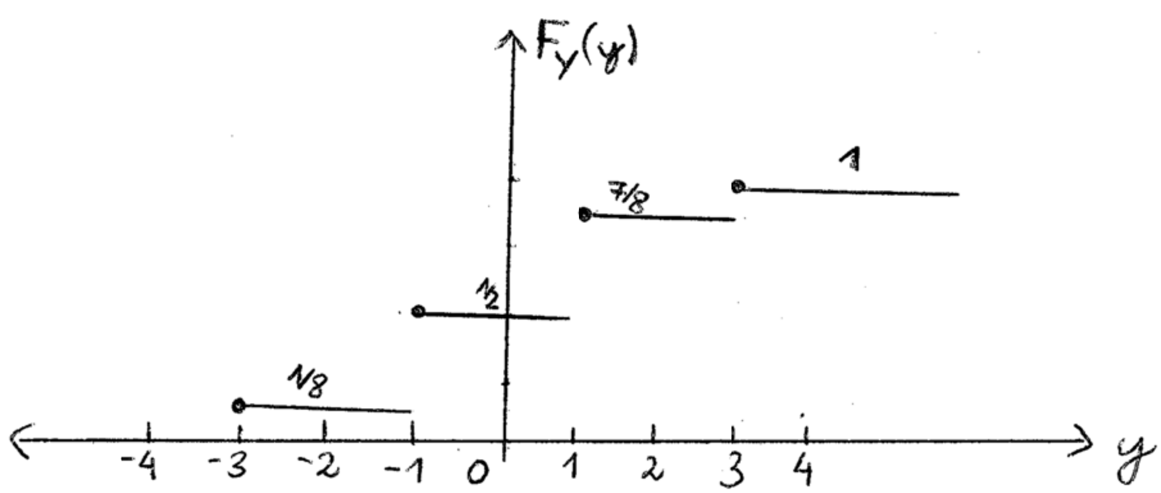
$P\{Y=-1\} = P\{(T,T,H), (T,H,T), (H,T,T)\} = \frac{3}{8}$

$P\{Y=1\} = P\{(H,H,T), (H,T,H), (T,H,H)\} = \frac{3}{8}$

$P\{Y=3\} = P\{(H,H,H)\} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

a) The cdf of Y is:

$$F_Y(y) = P\{Y \le y\} = \begin{cases} 0 & y < -3 \\ \frac{1}{8} & -3 \le y < -1 \\ \frac{1}{8}+\frac{3}{8}=\frac{1}{2} & -1 \le y < 1 \\ \frac{1}{8}+\frac{3}{8}+\frac{3}{8}=\frac{7}{8} & 1 \le y < 3 \\ \frac{7}{8}+\frac{1}{8} = 1 & y \ge 3 \end{cases}$$



b) To express P{|*Y*| < *y*} in terms of the cdf of Y, we have

If $y \geq 0$,

$$P\{|Y| < y\} = P\{-y < Y < y\}$$
$$= P\{-y < Y \leq y\} - P\{Y = y\}$$
$$= (P\{Y \leq y\} - P\{Y \leq -y\}) - P(Y = y)$$
$$= F_Y(y) - F_Y(-y) - P(Y = y)$$

If $y < 0$,

$$P\{|Y| < y\} = 0$$

So,

$$P\{|Y| < y\} = \begin{cases} F_Y(y) - F_Y(-y) - P(Y = y), & if\ y > 0 \\ 0, & if\ \ otherwise \end{cases}$$

4.6 Solution:

a) The sample space is given by all the points with coordinates *x,y* that are at a distance less than or equal to 2 from the origin. In other words, all the points within a circle of radius equal to 2:

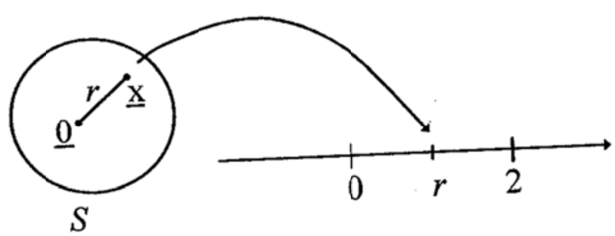
$S=\{(x,y): x^2 + y^2 \leq 4\}$ (of course we assume x and y are real numbers).

R is the distance of the landing point to the origin, so: $R = \sqrt{x^2 + y^2}$

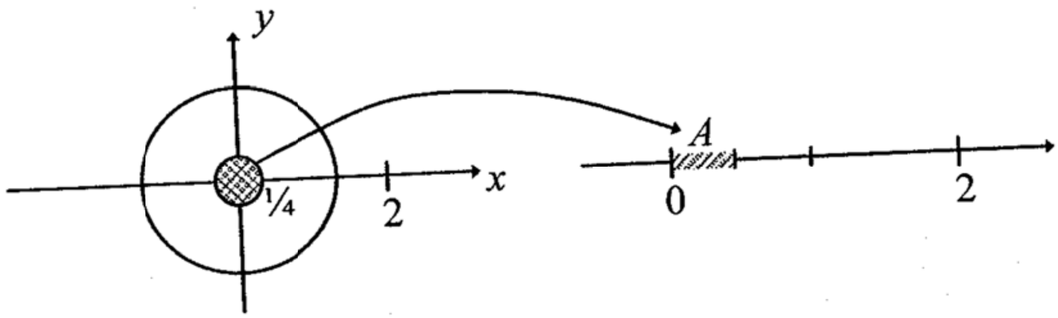
The sample space of *R* is given by all the real numbers between 0 and 2:

$S_R = \{r : 0 \leq r \leq 2\}$

b) The mapping from *S* to S_R:



c) The event *A* (dart hits the bull's eye):



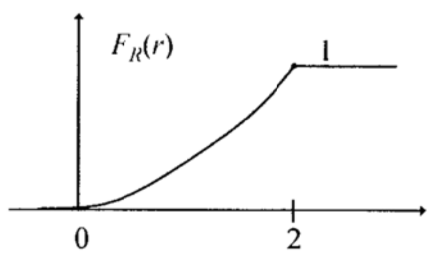
As we can see from the mapping, the equivalent event in S is
A' = {(x,y): x^2+y^2 ≤ (1/4)^2}

$$P(A) = P\left\{R \le \frac{1}{4}\right\} = \frac{\pi\left(\frac{1}{4}\right)^2}{\pi(2)^2} = \frac{1}{64}$$

d) The cdf of *R*:

For 0 ≤ *r* ≤ 2

$$F_R(r) = P\{R \le r\} = \frac{\pi r^2}{\pi 2^2} = \left(\frac{r}{2}\right)^2$$

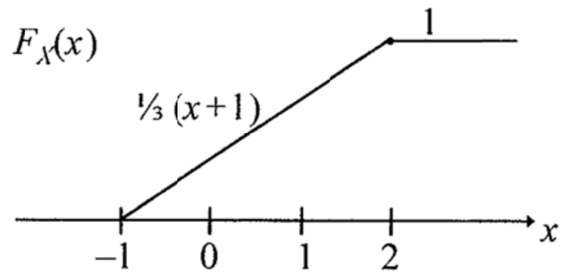


4.11 Solution:

a) *X* is a continuous random variable whose cdf has a linear increase between -1 and 2 (because it is uniformly distributed). We know that the value of the cdf at -1 has to be 0 and 1 at 2. As a result, the cdf of X is given by

$$F_X(x) = \begin{cases} 0, & if\ x \le -1 \\ \dfrac{x+1}{3}, & if -1 < x \le 2 \\ 1, & if\ x > 2 \end{cases}$$

The plot is:



b) $P\{X \le 0\} = F_X(0) = \frac{1}{3}(0+1) = 1/3$

$P\{|X - 0.5| < 1\} = P(\{-1 < X - 0.5 < 1\}) = P\left\{-\frac{1}{2} < X \le \frac{3}{2}\right\}$

$= F_X(1.5) - F_X(-0.5)$

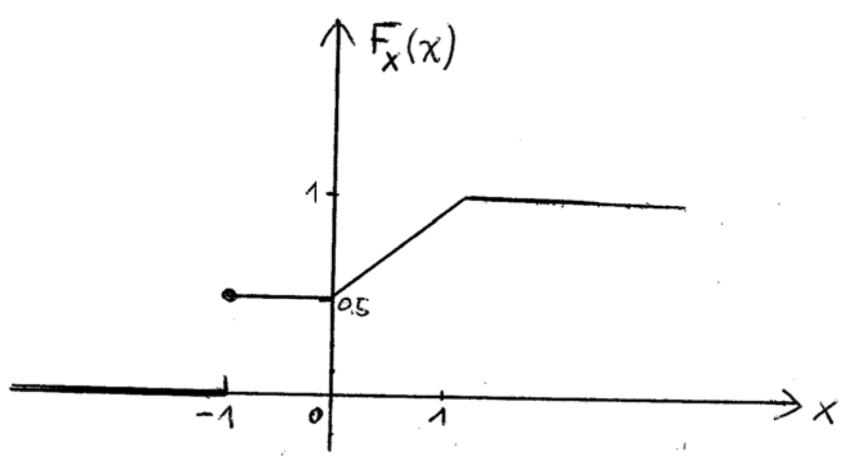
$= \frac{1}{3}(1.5+1) - \frac{1}{3}(-0.5+1)$

$= \frac{2}{3}$

$P\left\{X > -\frac{1}{2}\right\} = 1 - P\left\{X \le -\frac{1}{2}\right\} = 1 - \frac{1}{3}\left(-\frac{1}{2}+1\right) = \frac{5}{6}$

4.12 Solution:

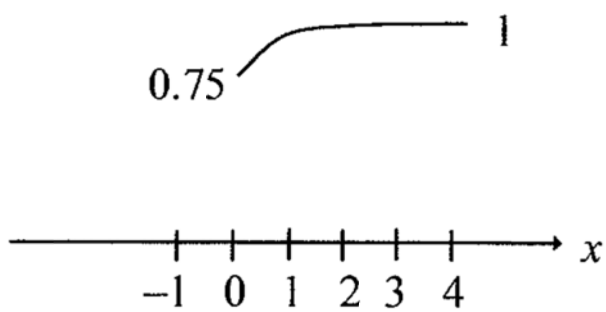
a) *X* is a random variable of mixed type



Note that from the book on page 146 section 4.1.1, the **discrete** r.v. have a cdf that is a right-continuous, staircase function of x, with jumps at a countable set of points. A **continuous** r.v. on the other hand is defined as a r.v. whose cdf is continuous everywhere. A r.v. of **mixed type** is a r.v. with a cdf that has jumps on a countable set of points, but also increases continuously over at least one interval of values of x.

b) $P\{X \leq -1\} = 0.5$
$P\{X = -1\} = F_X(1) - F_X(1^-) = 0.5 - 0 = 0.5$
$P\{X < 0.5\} = P\{X \leq 0.5\} = \frac{1}{2}(1 + 0.5) = 0.75$
$P\{-0.5 < X < 0.5\} = P\{-0.5 < X \leq 0.5\} = F_X(0.5) - F_X(-0.5) = \frac{1}{2}(1 + 0.5) - 0.5 = 0.25$
$P\{X > -1\} = 1 - P(\{X \leq -1\}) = 1 - 0.5 = 0.5$
$P\{X \leq 2\} = F_X(2) = 1$
$P\{X > 3\} = 1 - F_X(3) = 1 - 1 = 0$

4.13 Solution:
a) X is a random variable of mixed type, so we have to watch out for values where the cdf is not continuous.



b) $P\{X \leq 2\} = 1 - \frac{1}{4}e^{-2(2)} = 0.9954$
$P\{X = 0\} = P\{X \leq 0\} - P\{X \leq 0^-\} = (1 - \frac{1}{4}e^{-2(0)}) - 0 = 0.75$ using property (vii) of the cdf.
$P\{X < 0\} = 0$ where we took the limit from the left to exclude the point X=0.

Since the cdf is continuous at x=6
$P\{2 < X < 6\} = P\{2 < X \leq 6\} = P\{X \leq 6\} - P\{X \leq 2\}$
$= 1 - \frac{1}{4}e^{-2(6)} - 1 + \frac{1}{4}e^{-2(2)} = 0.0046$

$P\{X > 10\} = 1 - P\{X \leq 10\}$
$= 1 - (1 - \frac{1}{4}e^{-2(10)}) = 5.15 * 10^{-10}$

4.14 Solution: First of all, note that since F_X is right continuous, $F_X(-1) = \frac{2}{10}$; $F_X(-1/2) = \frac{2}{10}$; $F_X(0) = \frac{6}{10}$; $and\ F_X(1) = 1.$

a) X is a random variable of mixed type

b) $P\{X < -1\} = P\{X \leq -1\} - P\{X = -1\} = F_X(-1) - \left(F_X(-1) - F_X(-1^-)\right) = \frac{2}{10} - \left(\frac{2}{10} - 0\right) = 0$

$P\{X \leq -1\} = \frac{2}{10}$

$P\{-1 < X < -0.75\} = F_X(-0.75^-) - F_X(-1) = 0.2 - 0.2 = 0$

$$P\{-0.5 \leq X < 0\} = F_X(0^-) - F_X(-0.5^-) = \frac{4}{10} - \frac{2}{10} = \frac{2}{10} = \frac{1}{5}$$

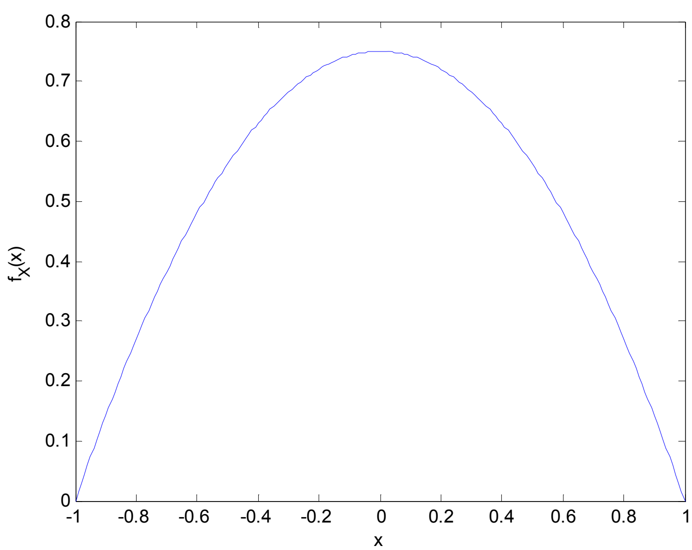
$$P\{-0.5 \leq X \leq 0.5\} = F_X(0.5) - F_X(-0.5^-) = 0.8 - \frac{2}{10} = 0.6$$

(because the equation of the line between 0 and 1 has a slope equal to $k = \frac{\Delta y}{\Delta x} = \frac{1-\frac{6}{10}}{1-0} = \frac{4}{10}$. So $F_X(0.5) = \frac{4}{10} * 0.5 + \frac{6}{10} = 0.8$)

$P\{|X - 0.5| < 0.5\} = P\{-0.5 < X - 0.5 < 0.5\} = P(\{0 < X < 1\} = F_X(1^-) - F_X(0) = 1 - \frac{6}{10} = \frac{4}{10} = 0.4$

4.17

a) To find the constant *c* we need to integrate the pdf over the interval (-∞,∞) and find the value of *c* that makes the value of the integral unity. This yields *c*=3/4.



b)

When x>1, $F_X(x)$ =1; when x<-1, $F_X(x)$ =0; when -1≤x≤1, $F_X(x)$ = $\int_{-1}^{x}(\frac{3}{4})(1 - x^2)dx = (3/4)(\frac{2}{3} + x - \frac{x3}{3})$.

c) P{0<X<0.5} = $\int_{-0}^{0.5}(\frac{3}{4})(1-x^2)dx = 11/24.$

P{X=1} =0 since the cdf has no discontinuity at x=1.

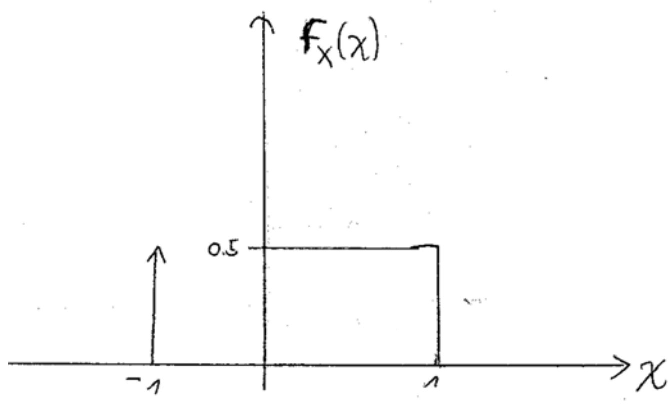
P{0<X<0.5} = $\int_{0.25}^{0.5}(\frac{3}{4})(1-x^2)dx = 0.3477.$

4.19
From class notes, replacing r by 2, we obtain $f_R(d)$ = 0 if d>2 or d<0; and $f_R(d) = 2d/(2)^2 = d/2$ if $0 \leq d \leq 2$.

4.22
a) By differentiating the cdf, we obtain $f_X(x)$ =0 if x<0, and $f_X(x) = 0.5e^{-2x}$ when $x \geq 0$.

b) P{-1<X<0.25} = $\int_{-1}^{0.25} f(x)dx = \int_{0}^{0.25}(\frac{1}{2})e^{-2x}dx = 0.0984.$

4.44 Mean and variance of X in Problem 4.12:
We can plot the pdf of X by observing the cdf found in problem 4.12:



Notice that the delta function for x=-1 corresponds to the discontinuity of the cdf at that point. A jump always leads to a delta function. For the interval [0,1] the cdf increases linearly, which corresponds to a constant value in the pdf.

So, the expected and the variance are:

$$E[X] = 0.5 * (-1) + \int_0^1 \frac{1}{2} x dx = -0.5 + \frac{1}{2} * \frac{1}{2}(1^2 - 0) = -0.25$$

$$VAR[X] = E[X^2] - E[X]^2 = \left(\frac{1}{2} * (-1)^2 + \int_0^1 \frac{1}{2} x^2 dx\right) - 0.25^2 = 29/48.$$

4.46 Perform the change of variable u=(x-μ)/σ in the integral $a = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} x e^{-(x-\mu)^2/2\sigma^2} dx$ to obtain $a = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} (\sigma u + \mu) e^{-u^2/2} du$. The first integral is zero since the integrand is an odd function. The second integral is simply μ since $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$=1.

To find the variance, perform the change of variables u=(x-μ)/σ in the integral $b = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} (x - \mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx$ to obtain $b = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \sigma^2 u^2 e^{-u^2/2} du$. Now integrate by parts and to obtain $b = \sigma^2$.

4.48 Use integration by parts twice to obtain $E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{a} e^{-x/a} dx$=$\frac{2}{a^2}$. Hence, the variance is $E[X^2] - E[X]^2 = \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2}$.