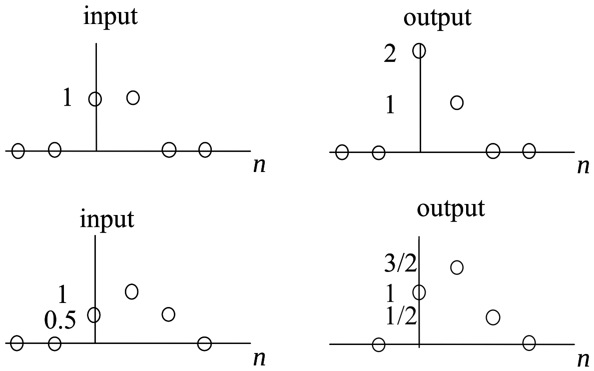
ECE-314 Fall 2012
Midterm I (75 minutes); Closed Book/Notes

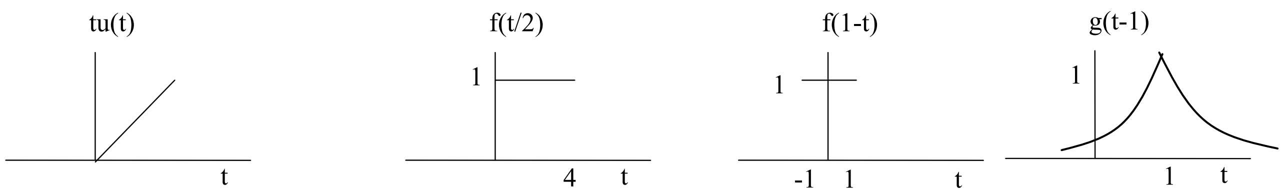
1. A linear time-invariant system has the input-output characteristics shown in the first row of the diagram below. Determine the output for the input shown on the second row of the diagram. Justify your answer carefully.



Solution: Note that $x_2[n]=0.5x_1[n-1]+0.5\,x_1[n]$
As a consequence of the system being LTI, we must have $y_2[n]=0.5\,y_1[n-1]+0.5\,y_1[n]$. Sketch it below.

2. Consider the signals $f(t) = u(t)-u(t-2)$ and $g(t) = \exp(-2|t|)$.

(a) Sketch the signals: $tu(t)$, $f(t/2)$, $f(1-t)$ and $g(t-1)$.



(b) Is g a power or an energy signal? Calculate the appropriate quantity.

Energy signal; E = $2\int_0^\infty \exp(-4t)dt = 1/2.$ Power = 0.

3. Evaluate the following quantities:

a. $\int_{-\infty}^{\infty} \sqrt{s}\,\delta(s-4)ds$ $\boxed{2}$

b. $\int_{-\infty}^{t-1} \delta(s)ds$ $\boxed{u(t-1)}$

c. $\sum_{k=-\infty}^{\infty} \delta(n-k)\sin(k+2n)$ $\boxed{\sin(3n)}$

d. $\sum_{k=-\infty}^{n} \delta(n-k)$ $\boxed{1}$

4. Consider a system described by the following input-output relation:
$y(t) = 2x(e^t)$.

(a) Is the system linear? Justify. It is linear because it satisfied superposition and homogeneity.

Superposition: $O(x_1 + x_2)(t) = 2(x_1 + x_2)(e^t) = 2x_1(e^t) + 2x_2(e^t) = O(x_1)(t) + O(x_2)(t)$
Homogeneity: $O(ax)(t) = 2(ax)(e^t) = 2a\, x(e^t) = a\, O(x)(t)$

(b) Is it time invariant? Justify. No.

$O(x_{t0})(t) = 2(x_{t0})(e^t) = 2x(e^t - t_0)$, but $O(x)(t-t_0) = 2x(e^{t-t0})$

(c) Is it memoryless? Justify.
No. $O(x)(0) = 2\, x(e^0) = 2\, x(1)$ depends on x(1).

(d) Is it causal? Justify.
No. $O(x)(0) = 2\, x(e^0) = 2\, x(1)$ depends on future value of input, namely x(1).

5. Determine whether the following systems are BIBO stable, you must show proper justification (i.e., provide a proof in the case of stable and a counter example otherwise).

(a) $y(t) = 1 + \int_{-\infty}^{t} x(s)ds$.

Pick a bounded input, x(t)=u(t), for example. Now note that for this input, y(t) = 1 + t, which is unbounded. Hence, we found a bounded input whose output is unbounded. Hence the system is not BIBO stable.

(b) $y(t) = x(t) + \int_{t-4}^{t} x(s)ds$

Let an input signal x be bounded by M, namely, |x|<M. Then,

$$|y(t)| \le M + \left|\int_{t-4}^{t} x(s)ds\right| \le M + \int_{t-4}^{t} |x(s)|\, ds \le M + \int_{t-4}^{t} M ds = M + M(4) = 5M$$

Hence, the output remains bounded for any input as long as the input is bounded. Hence, the system is BIBO stable.

(c) $y(n) = 0.5\{x(n-1) - x(n)\}$

Let an input signal x be bounded by M, namely, |x|<M. Then,

$|y(n)| = |0.5\,(x(n-1)+x(n))| \le 0.5\,|x(n-1)| + 0.5|x(n)| \le 0.5(M + M) = M$.
Hence, the output is bounded as long as the input is bounded.
Hence, the system is BIBO stable.

6. Show that the system described by $y(n)=x(3n)$ is not time invariant.

$O(x_{n0})(n) = x_{n0}(3n) = x_{n0}(3n-n_0)$; but $O(x)(n-n_0) = x(3(n-n_0)) = x(3n-3n_0)$.

7. A moving-window system is defined by the following input-output relation:

$$y(n) = M^{-1}\sum_{k=n-M}^{n} x(k).$$

(a) Show that it is LTI and find its impulse response $h(n)$.

Superposition: $O(x_1 + x_2)(n) = M^{-1}\sum_{k=n-M}^{n} (x_1(k) + x_2(k)) =$

$M^{-1}\sum_{k=n-M}^{n} x_1(k) + M^{-1}\sum_{k=n-M}^{n} x_2(k) = O(x_1)(n) + O(x_2)(n)$

Homogeneity: $O(ax)(n) = M^{-1}\sum_{k=n-M}^{n} ax(k) = M^{-1}a\sum_{k=n-M}^{n} x(k) = a\,O(x)(n)$

Impulse response: set $x=\delta$ to obtain

$h(n) = M^{-1}\sum_{k=n-M}^{n} \delta(k) = M^{-1}\left(\sum_{k=-\infty}^{n} \delta(k) - \sum_{k=-\infty}^{n-M-1} \delta(k)\right) = M^{-1}(u(n)-u(n-M-1))$.

(b) Show that this system is stable.

Let an input signal x be bounded by K, namely, $|x|<K$. Then,

$|y(n)| = |M^{-1}(x(n-M)+\ldots+x(n))| \leq M^{-1}(|x(n-M)|+\ldots+|x(n)|) \leq M^{-1}(K+\ldots+K) = M^{-1}(M+1)K$.

Hence, the output remains bounded for any input as long as the input is bounded. Hence, the system is BIBO stable.

8. Show that for an LTI system an identically-zero input signal will result in an identically-zero output signal.
Let *O* be any LTI system, let $\underline{0}$(t) denote the all-zero signal, and let x be an arbitrary signal. Note that $\underline{0}$(t) can be written as 0x(t).
Now $O(\underline{0})(t) = O(0x)(t) = 0O(x)(t) = 0$, where the second equality follows from homogeneity.

9. Consider a system given by $y(t) = 1 + x'(t)$. Prove that this system is not BIBO stable.

Consider the input $x(t) = \sqrt{1-t^2}\,(u(t)-u(t-1))$, which is bounded by 1. However, note that x'(t) is unbounded because x'(t) diverges to $-\infty$ as t approaches 1. Hence, we found a bounded input whose output is unbounded. Hence, the system is not BIBO stable.

10. Find the impulse response of the system described in 5(b)
Replace x with δ in $y(t) = x(t) + \int_{t-4}^{t} x(s)ds$ to obtain

$$h(t) = \delta(t) + \int_{t-4}^{t} \delta(s)ds = \delta(t) + \int_{-\infty}^{t} \delta(s)ds - \int_{-\infty}^{t-4} \delta(s)ds = \delta(t) + u(t) - u(t-4).$$