ECE340 Spring 2011
Recitation class
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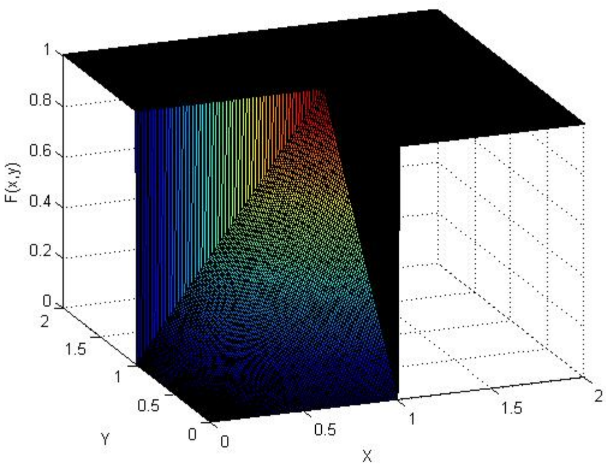
Problem 1

Two random variables have a joint probability distribution function defined by:

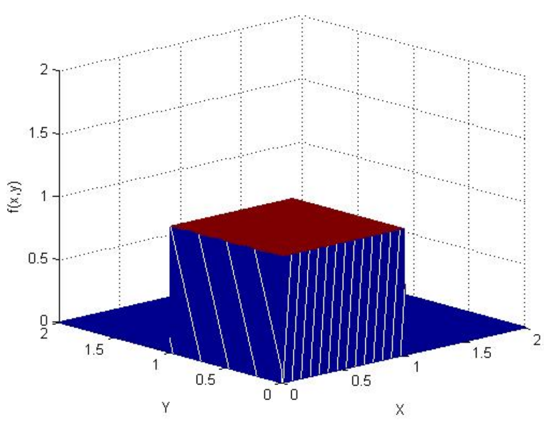
$$f_X(x) = \begin{cases} 0 & x < 0, y < 0 \\ xy & 0 \le x \le 1, 0 \le y \le 1 \\ 1 & x > 1, y > 1 \end{cases}$$

a) Sketch the distribution function.



b) Find the joint probability density function and sketch it.

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \begin{cases} 0 & x < 0, y < 0 \\ 1 & 0 \le x \le 1, 0 \le y \le 1 \\ 1 & x > 1, y > 1 \end{cases}$$



c) Find the joint probability of the event $X \le \frac{3}{4}$ and $Y > \frac{1}{4}$.

$$P\left\{X \le \frac{3}{4}, Y > \frac{1}{4}\right\} = \int_0^{\frac{3}{4}} \int_{\frac{1}{4}}^{1} f(x,y)dydx = \int_0^{\frac{3}{4}} \int_{\frac{1}{4}}^{1} dydx = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

Problem 2

Two random variables X and Y have a joint probability density function

$$f(x,y) = \begin{cases} Ae^{-(2x+3y)} & x \geq 0, y \geq 0 \\ 0 & x < 0, y < 0 \end{cases}$$

Find

a) The value of A for which this is a valid joint probability density function.
b) The probability that $X < 1/2$ and $Y < 1/4$.
c) The expected value of XY.

<u>**Solution**</u>

a) To determine the value A, we need the following condition:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

Then

$$A\int_{0}^{\infty} e^{-2x}\,dx \int_{0}^{\infty} e^{-3y}\,dy = 1$$

$$A\left(-\frac{1}{2}\right)\left(e^{-2x}\big|_0^{\infty}\right)\left(-\frac{1}{3}\right)\left(e^{-3y}\big|_0^{\infty}\right) = 1$$

Which yields to $A = 6$.

b) The probability that $X < 1/2$ and $Y < 1/4$ equals:

$$P\left\{X < \frac{1}{2}, Y < \frac{1}{4}\right\} = \int_{-\infty}^{\frac{1}{2}}\int_{-\infty}^{\frac{1}{4}} 6e^{-(2x+3y)}\,dxdy = 6\int_{0}^{\frac{1}{2}} e^{-2x}\,dx \int_{0}^{\frac{1}{4}} e^{-3y}\,dy$$

$$= 6\left(\frac{1}{6}\right)\left(e^{-2x}\big|_0^{\frac{1}{2}}\right)\ \left(e^{-3y}\big|_0^{\frac{1}{4}}\right) = (e^{-1}-1)\left(e^{-\frac{3}{4}}-1\right) = 0.3335$$

c) The expected value of the product of X and Y is as follows:

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dxdy$$

Then:

$$E[XY] = 6\int_{0}^{\infty} xe^{-2x}\int_{0}^{\infty} ye^{-3y}\,dxdy = 6\left[\left(-\frac{1}{4}(2x+1)e^{-2x}\right)\Big|_0^{\infty} . \left(-\frac{1}{9}(3x+1)e^{-3x}\right)\Big|_0^{\infty}\right]$$

$$= 6\left(\frac{1}{4}\right)\left(\frac{1}{9}\right) = \frac{1}{6} = 0.1667$$

Problem 3

Two random variables X and Y have a joint probability density function of the form:

$$f(x,y) = \begin{cases} k(x+2y) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & elsewhere \end{cases}$$

Find

a) The value of k for which this is a valid joint probability density function
b) The conditional probability that X is greater than 1/2 given that Y=1/2.
c) The conditional probability that Y is less than, or equal to, 1/2 given that X is 1/2.

Solution

a) To determine the value of k, we need the following condition:

$$\int_0^1 \int_0^1 k(x+2y)dxdy = 1$$

$$\int_0^1 \int_0^1 k(x+2y)dxdy = k\int_0^1 [(x^2/2 + 2xy)\mid_0^1]\,dy = k\int_0^1 (1/2+2y)dy = 1/2\ +1 = 3/2$$

Therefore: $k = 2/3$.

b) First we need to find the conditional pdf $f_{X|Y}(x|y)$. Since

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

We should obtain $f_Y(y)$.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \frac{2}{3}\int_0^1 (x+2y)\,dx = \left(\frac{2}{3}\right)[(x^2/2+2xy)\mid_0^1] = \frac{2}{3}\left(2y+\frac{1}{2}\right)$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{(x+2y)}{(2y+1/2)} & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & elsewhere \end{cases}$$

$$f_{X|Y}(x|Y=1/2) = \frac{x+1}{1+\frac{1}{2}} = \frac{2}{3}(x+1)$$

$$P\left\{X > \frac{1}{2}\middle|Y=\frac{1}{2}\right\} = \int_{\frac{1}{2}}^1 f_{X|Y}\left(x\middle|Y=\frac{1}{2}\right)\,dx = \int_{\frac{1}{2}}^1 \frac{2}{3}(x+1)dx$$

$$= \frac{1}{3}(x^2+2x)\Big|_{1/2}^{1} = \frac{7}{12}$$

c) In this case,

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \frac{2}{3}\int_0^1 (x+2y)\,dy = \left(\frac{2}{3}\right)[(xy+y^2)\mid_0^1] = \frac{2}{3}(x+1)$$

$$f_{Y|X}(y|x) = \frac{f_{YX}(y,x)}{f_X(x)} = \begin{cases} \dfrac{(x+2y)}{(x+1)} & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & elsewhere \end{cases}$$

$$f_{Y|X}(y|X=1/2) = \frac{2y+1/2}{\frac{1}{2}+1} = \frac{1}{3}(4y+1)$$

$$P\left\{Y \le \frac{1}{2}\middle|X=\frac{1}{2}\right\} = \int_0^{1/2} f_{Y|X}\left(y\middle|X=\frac{1}{2}\right)\,dx = \int_0^{1/2} \frac{1}{3}(4y+1)dx$$

$$= \frac{1}{3}(2y^2+y)\Big|_0^{1/2} = \frac{1}{3}$$

Problem 4

A random signal X is uniformly distributed between 10 and 20 V. It is observed in the presence of Gaussian noise N having zero mean and a standard division of 5 V.

a) If the observed value of signal plus noise, $(X+N)$, is 5, find the best estimate of the signal amplitude.

b) Repeat (a)if the observed value of the signal plus noise is 12.

Solution

We have the following relationship for the random variables X, N, and Y:

$$Y = X + N,$$

Where X, N, and Y represent the random variables associated with the actual signal, noise, and the measured signal.

If X is given, the only randomness in Y is N. Thus since N=Y-X and $f_N(n)$ is known, we have:

$$f(y|x) = f_N(n = y - x) = f_N(y - x)$$

Therefore:

$$f(x|y) = \frac{f_N(y-x)f_x(x)}{f_Y(y)} = \frac{f_N(y-x)f_X(x)}{\int_{-\infty}^{\infty} f_N(y-x)f_X(x)dx}$$

In this case, we have:

$$f_X(x) = \begin{cases} \frac{1}{10} & 10 \le x \le 20, \\ 0 & elsewhere \end{cases}$$

and

$$f_N(n) = \frac{1}{\sqrt{2\pi}\sigma_N} \exp\left(-\frac{n^2}{2\sigma_N^2}\right) = \frac{1}{25\sqrt{2\pi}} \exp\left(-\frac{n^2}{50}\right)$$

Then the marginal density function of Y, becomes:

$$f_Y(y) = \int_{10}^{20} \frac{1}{\sqrt{2\pi}\sigma_N} \exp\left(-\frac{(y-x)^2}{2\sigma_N^2}\right).\left(\frac{1}{10}\right) dx = 0.1\left[Q\left(\frac{y-20}{2\sigma_N^2}\right) - Q\left(\frac{y-10}{2\sigma_N^2}\right)\right]$$

$$= 0.1\left[Q\left(\frac{y-20}{50}\right) - Q\left(\frac{y-10}{50}\right)\right]$$

Then

$$f(x|y) = \frac{f_N(y-x)f_x(x)}{f_Y(y)} = \begin{cases} \frac{0.1}{25\sqrt{2\pi}f_Y(y)} \exp\left(-\frac{(x-y)^2}{50}\right) & 10 \le x \le 20, \\ 0 & elsewhere \end{cases}$$

When a particular value of Y is observed, a reasonable estimate for the true value of X is that value of x which minimizes $f(x|y)$.

Therefore, if $10 \le y \le 20$ then the appropriate estimate for X is $\hat{X} = Y$. If $y \le 10$ then $\hat{X} = 10$, and If $y \ge 20$ then $\hat{X} = 20$.

a) Since $y = 5 < 10$, the best estimation of the X is $\hat{X} = 10$.
b) Since $10 \le y = 12 \le 20$, the best estimation of the X is $\hat{X} = Y = 12$.
c)

Problem 5

Two random variables, X and Y, have a joint probability density function of the form:

$$f_{XY}(x,y) = \begin{cases} ke^{-(x+y-1)} & 0 \le x \le \infty, 1 \le y \le \infty \\ 0 & elsewhere \end{cases}$$

Find

a) The values of k and a for which the random variables X and Y are statistically independent.
b) The expected value of XY.
c)

Solution

a) If the two random variables are statistically independent, the following equality holds:

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

We use the following to find $f_X(x)$ and $f_Y(y)$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

In this case

$$f_X(x) = \int_1^{\infty} ke^{-(x+y-1)}dy = ke^{-(x-1)}\int_1^{\infty} e^{-y}dy = ke^{-(x-1)}e^{-1} = ke^{-x}$$

$$f_Y(y) = \int_0^{\infty} ke^{-(x+y-1)}dx = ke^{-(y-1)}\int_0^{\infty} e^{-x}dx = ke^{-(y-1)}.1 = ke^{-(y-1)}$$

Then we have:

$$f_X(x)f_Y(y) = ke^{-x}.ke^{-(y-1)} = k^2e^{-(x+y-1)}$$

Therefore we must have $k = 1$ in order to have X and Y statistically independent.

b) If two random variables are statistically independent, we can use the following to compute E[XY]:

$$E[XY] = E[X]E[Y]$$

$$E\{X\} = \int_1^{\infty} xe^{-x}dx = -e^{-x}(x+1)|_0^{\infty} = 2e^{-1}$$

$$E\{Y\} = \int_0^{\infty} ye^{-(y-1)}dy = -e^{-(y-1)}(y+1)\Big|_0^{\infty} = e^{+1}$$

Therefore,

$$E[XY] = E[X]E[Y] = 2$$

Problem 6

Two independent random variables, X and Y, have the following probability density functions.

$$f(x) = 0.5e^{-|x-1|} \qquad -\infty < x < \infty$$

$$f(y) = 0.5e^{-|y-1|} \qquad -\infty < y < \infty$$

Find the probability that XY>0.

<u>Solution</u>

Since two random variables are independent, we have:

$$f(x,y) = f(x)f(y) = 0.25e^{-|x-1|}e^{-|y-1|}$$

Then the probability that XY>0 is calculated as follows:

$$P\{XY > 0\} = P\{X > 0, Y > 0\} + P\{X < 0, Y < 0\}$$

$$= \int_0^{\infty}\int_0^{\infty} 0.25e^{-|x-1|}e^{-|y-1|}dxdy + \int_{-\infty}^{0}\int_{-\infty}^{0} 0.25e^{-|x-1|}e^{-|y-1|}dxdy$$

$$= 0.25\left[\int_0^1\int_0^1 e^{(x-1)}e^{(y-1)}dxdy + \int_1^{\infty}\int_1^{\infty} e^{-(x-1)}e^{-(y-1)}dxdy \right.$$
$$\left. + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{(x-1)}e^{(y-1)}dxdy\right]$$

$$= 0.25[e^{-2}(e-1)^2 + e^2.e^{-2} + e^{-2}] = 0.666$$