ECE-340, Spring 2010
Final Examination, May 10^{th}, 2010; 5:30-7:30 PM

1. Suppose that there are two categories of eggs: large eggs and small eggs, occurring with probabilities 0.7 and 0.3, respectively. For a large egg, the probabilities of having 1, 2, or 3 yolks are 0.95, 0.045 and 0.005, respectively. On the other hand, for a small egg, the probabilities of having 1, 2, or 3 yolks are 0.98, 0.019 and 0.001, respectively. Suppose that an egg is picked at random and let *X* represent the number of yolks in it.

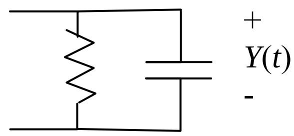
a) Plot the cumulative distribution function of *X*.

b) Calculate the mean and variance of *X*.

c) Suppose that it is known that an egg has more than one yolk, what is the probability that is has three yolks?

d) Suppose that it is known that an egg has three yolks, what is the probability that it is a large egg?

2. Consider the RC circuit shown below. Suppose that the noisy resistor is at temperature T Kelvin and its noise is modeled as Johnson noise.



a) Extract a Thevenin equivalent circuit for the this circuit showing the noise source as the input, as well as a noiseless resistor. Precisely write the mean, autocorrelation function and the power spectral density of the random process representing noise source.

b) Is the noise process wide-sense stationary? Justify your answer.

c) Find the mean and average power of the voltage $Y(t)$ across the capacitor.

d) Find the autocorrelation function of the process $Y(t)$.

e) Find the noise equivalent bandwidth for this RC circuit.

3. Nine resistors are sampled from a large box of resistors. The actual resistance values of these nine elements are: 3.9, 3.7, 3.7, 3.8, 3.5, 3.9, 4.0, 3.7, 4.2, all in the units of kΩ.

a) Find the sample mean. You must show all your work.

b) Write out the formula for calculating the sample variance but do not evaluate it.

c) Suppose that the true mean and true variance of the resistors are 4 kΩ and 0.2 $k\Omega^2$, respectively. Suppose now that 100 resistors are drawn at random from the box, and let $\widehat{\mu_X}$ denote the sample mean corresponding to the 100 samples. Use Chebychev's inequality to estimate the probability that the sample mean (corresponding to the 100 samples) is NOT within $10% of the true mean. You must show all your work.

d) How large must the sample size be in order for the sample mean $\widehat{\mu_X}$ to be **within** 10% of the true mean with probability 0.99? You must show all your work.

e) Suppose that in actuality the value of a randomly selected resistor from the large box is modeled as a Gaussian random variable, X, with true means and variances as given above. Derive an expression for the probability that the sample mean $\widehat{\mu_X}$ (corresponding to the 100 selected samples) is not within 10% of the true mean. You can leave your answer in terms of either the "Φ" function or the "erf" function.

f) Suppose that we are required to find the confidence interval corresponding to a confidence level of q=0.99 for a sample size of 100. Namely, we want the interval $I=[-a+\mu, a+\mu]$ such that $P\{\widehat{\mu_X} \in I\} = q$. Find an expression for the unknown a. Leave your answer in terms of either the "Φ^{-1}" function or the "erf^{-1}" function.

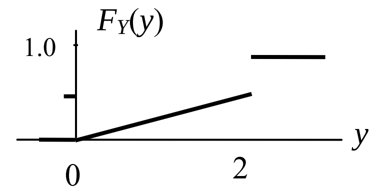
4. Suppose that X is a random variable that is uniformly distributed in [0, 1]. Define the new random variables $Y = 3\,X - 1.5$, $Z = X^5$ and $W = u(X - 0.2)$, where u is the unit-step function.

a) Determine and plot the pdf of Y.

b) Determine the pdf of Z.

c) Determine and plot the cumulative distribution function of W.

5. A random variable Y has the cumulative distribution function shown below. It is known that $P\{Y=2\} = 0.6$. Plot the pdf of Y and calculate the variance of Y.



6. Let $N(t)$ represent the number of hits that a website receives in the interval $[0,t]$, where t , in the units of minutes, is measured starting at 6 AM (namely, 6AM corresponds to t=0). Assume that $N(t)$ is Poisson process with a mean arrival rate of 3 hits/minute.

a) Find the correlation coefficient between $N(2)$ and $N(3)$.

b) Give an example of another physical phenomenon that can be effectively modeled by a Poisson random variable or Poisson process.

7. A pair of random variables, X and Y, have a joint pdf given by $f_{XY}(x,y)=x+y$, for $0 \le x \le 1$, $0 \le y \le 1$, and $f_{XY}(x,y)=0$ otherwise. Calculate $P\{0 < X < 0.1 \mid Y=0.5\}$. Show all your work.

8. Let X be a uniform random variable in the interval [0,1], and let Y be an exponentially distributed random variable with a mean value of 2. Assume that X and Y are independent. Find and plot the probability density function of $Z=X+Y$.

9. A football stadium has 1000 light bulbs used to illuminate the field. The probability that each light bulb is faulty when the switch is closed is 0.05. Use the central-limit theorem to estimate the probability that the total number of working light bulbs (after the switch is closed) is greater than 900? Be precise and state any assumptions you make.