MATH 264 — R E V I E W   E X A M   2

TOPICS COVERED (Chapter 15 - skip 15.2)

1. Graphs of quadratic equations

Graph quadratic equations in x, y, z by looking a crosssection. You need to be able to do this to graph functions.

2. Functions of several variables

Graphing

2D: Graph surfaces $z = f(x, y)$, graph contour curves $f(x, y)$ in x and y.

3D: Graph level surfaces of $f(x, y, z)$.

Compute Partial Derivatives. Interpret as slopes.

Tangent planes and normals.

Find equations of tangent plane to a surface $z = f(x, y)$ at a point

Find equations of tangent plane to a surface $F(x, y, z) = c$ at a point

Find normal to any surface ($F(x, y, z) = c$, $z = f(x, y)$) at a point

Linearization.

Find the linear approximation of $f(x, y)$.

Find the linear approximation of the change of $f(x, y)$ as (x, y) change from a base point (x_0, y_0) to a nearby point $(x_0 + \Delta x, y_0 + \Delta y)$.

Chain Rule. Implicit differentiation.

Properties of the Gradient: (be able to use them!)

Vector that points in direction of maximal increase.

Magnitude = maximal rate of change (derivative in direction of maximal increase)

2D: ∇f points normal to level curves.

3D: ∇f points normal to level surfaces.

Directional derivative

Compute derivatives in specified direction (directional derivative)

3. Maxima and Minima

Find local max/min

Find critical points. Use second derivative test.

Find absolute max/min

1. Find local max/min

2. Investigate behaviour as $x, y \to \pm\infty$ if function is defined on an infinite domain.

2'. Investigate behaviour on boundary if function is defined on a closed, bounded domain (make sure you can handle domains in the shape of squares, triangles or circles)

Remember: continuous functions on closed, bounded domains always have an absolute max/min

Lagrange multipliers

Solve {max/min $f(x, y)$ such that $g(x, y) = c$} using Lagrange multipliers.

Be able to solve those type of problems using substition as well (sometimes easier).

Understand the picture associated with Lagrange multipliers. (Why is ∇f parallel to ∇g at a local extremum?)

STUDY PROBLEMS

The following problems are a pretty comprehensive set, but make sure to work additional problems out of the homework, specially in those topics listed on the other side in which you feel a little shaky.

Chapter 15 Review, Concept Check: 1,2,5(b,c),7,8,9,11
Chapter 15 Review, True-False: 2,3,4,6,7,11,12
Chapter 15 Review, Exercises: 3-6,12,13-20,23,25,26,28,29,34,35,36,39,42,45,46,47,48,49,51, 52,55,56,59
Section 15.4: 33,34

Other selected problems.

1. Make sure to include some implicit differentiation!
2. Let $z = x^2 + 3xy$
   (a) Write down the equation for the tangent plane at $(1, 1, 4)$ in the form $z = L(x, y)$, where L is the linearization of f, as we learnt it in Section 15.4
   (b) Find a normal to the surface by writing the surface as a level surface of some function. Use the normal to write down the equation of a tangent plane.
   (c) Confirm that the results in (a) and (b) are the same.
3. (a) Graph the surface $r = a$ in 3D.
   (b) Find a unit normal to the surface at an arbitrary point (x_o, y_o, z_o) on the surface (Hint: write $r = \sqrt{x^2 + y^2}$ and proceed as in 2(b)).
   (c) Also sketch the unit normal in your graph, and make sure the answer makes sense.
4. Prove that the directional derivative in the direction of the gradient (the direction of maximal increase) is $|\nabla f|$
5. FIG 1 shows the curve $g(x, y) = k$ (dashed) and several level curves of the function $f(x, y)$ (solid). It also shows the gradient ∇f at several points. Does the absolute maximum of f under the constraint $g(x, y) = k$ occur at P? At Q? At R? Somewhere else?
6. In FIG 2, the solid lines are the indicated level curves of $f(x, y)$. The dashed line is the level curve $g(x, y) = k$. At which points does the maximum of f occur, under the constraint that $g = k$? Indicate these points in the figure.
   Also, on the figure, plot the gradient of f along the curve $g = k$. (Plot several gradients, as in the previous figure.)

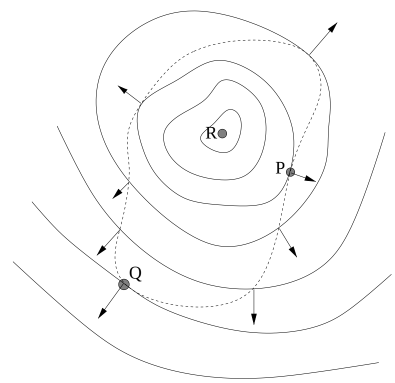

FIG. 1

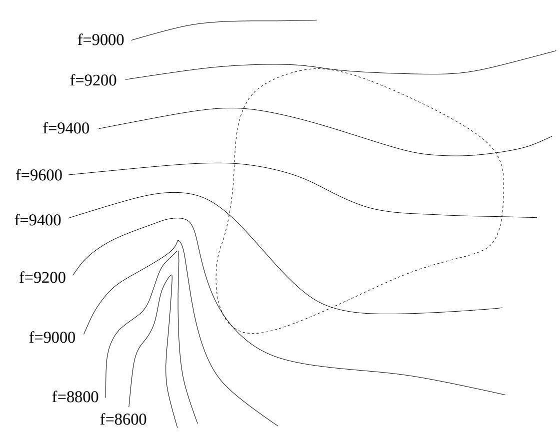

FIG. 2

7. FIG 3 shows the contour level $g(x,y)=1$ and several contour levels for the function $f(x,y)$. From the picture it is clear that the

$$\text{maximum of } f(x,y) \text{ such that } g(x,y)=1$$

is equal to 4, and occurs at the indicated points P and Q. However, it looks like ∇f is not parallel to ∇g at these points, contradicting what we just learnt about Lagrange multipliers. What is going on?? Explain in words. Be clear and concise.

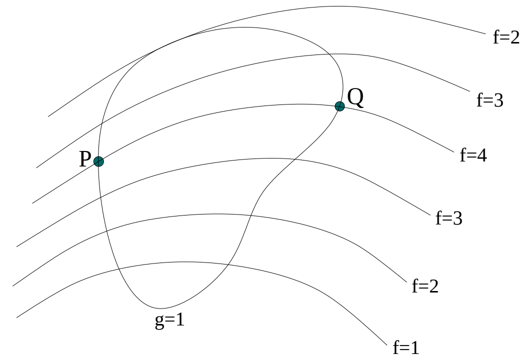

FIG. 3

PARTIAL ANSWERS

Chapter 15 Review, Exercises.

4: upper portion of hyperboloid of two sheets

6: downward parabolas

12: $T(x,y) \approx T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4) \approx 80 + \frac{86-72}{4}(x-6) + \frac{75-87}{4}(y-4) = 80 + \frac{7}{2}(x-6) - 3(y-4)$ $T(5,3.8) \approx 75.9$

18: $\partial C/\partial T(10,35,100) = 3.413$ (sound travels faster as temperature increases)
$\partial C/\partial S(10,35,100) = 1.24$ (sound travels faster as salinity increases)
$\partial C/\partial D(10,35,100) = 0.016$ (sound travels faster with increasing depth)

34: (a) $A = xy/2$, $\Delta A \approx (y/2)\Delta x + (x/2)\Delta y$ $|\Delta A| \le (12/2)0.002 + (5/2)0.002$
(b) $d = \sqrt{x^2+y^2}$, $\Delta d \approx \frac{x\Delta x + y\Delta y}{\sqrt{x^2+y^2}}$ $|\Delta d| \le \frac{17(0.002)}{13}$

36: $z_u = -(3u^2 - 2uv^2 + v)\sin[(u^2+v)(u-v^2)] - 2u(u-v^2)\sin(u^2+v) + \cos(u^2+v)$,
$z_v = -(u - 3v^2 - 2u^2v)\sin[(u^2+v)(u-v^2)] - (u-v^2)\sin(u^2+v) - 2v\cos(u^2+v)$

42: $z_x = \frac{yze^{xyz} - 2xz^3}{4z^2 + x^2 3z^2 - xye^{xyz}}$

46: 12.5/3

48: $\nabla f(P)$, $|\nabla f(P)|$

52: saddle at (0,0), local min at (1,1/2)

56: Critical points in interior $(0,0), (0,\pm 1), (\pm 1, 0)$ Critical points on boundary $(\pm 2, 0), (0, \pm 2)$
Abs max: $2/e$. Abs min: 0.

Section 15.4.

34: $V = \pi r^2 h$, $\Delta V \approx 2\pi r h \Delta r + \pi r^2 \Delta h$, where $\Delta r = 0.05$, $\Delta h = 0.2$. Therefore $\Delta V \approx 2.8\pi$.

Other selected problems:

2: (a) $z = 4 + 5(x-1) + 3(y-1)$ (b) The surface is a level surface of $F(x,y,z) = x^2 + 3xy - z$ (since it is given by $F(x,y,z) = 0$). Therefore, a normal to the surface at P is $\nabla F(P) = \langle 5, 3, -1 \rangle$. The equation for the tangent plane at P is $5(x-1) + 3(y-1) - (z-4)$.

3: (a) Cylinder of radius a. (b) Unit normal $\mathbf{n} = \dfrac{\langle x_o, y_o \rangle}{\sqrt{x_0^2 + y_0^2}} = \dfrac{\langle x_o, y_o \rangle}{a}$

4: See classnotes or book, p966.

5: At Q.

7: The points P and Q lie at a local maximum of f (they lie on a ridge). Therefore $\nabla f = \mathbf{0}$. This does not contradict what we learnt about Lagrange multipliers since at P and Q, the equation $\nabla f = \lambda \nabla g$ still holds, however with $\lambda = 0$.