— Abbreviated —

ECE-340, Spring 2011
Midterm-1, February 21, 2011 **Name:** Key

Problem 1. In a binary communication system, each "1" symbol that is transmitted has a probability of 0.0001 of being misinterpreted and announced as a "0" symbol by the receiver. Similarly, each transmitted "0" symbol can be erroneously interpreted and received as a "1" symbol with probability 0.001. It is know that the probability of transmitting a "0" symbol is 0.51.

a) Calculate the probability that "0" symbol is announced by the receiver.

$$P(E_{0R}) = P(E_{0R} \mid E_{1T})\, P(E_{1T}) + P(E_{0R} \mid E_{0T})\, P(E_{0T})$$
$$= (0.0001)(1-0.51) + (0.001)(0.51)$$

b) What is the probability that a "0" symbol was transmitted given that a "1" symbol was received?

$$P(E_{0T} \mid E_{1R}) = \frac{P(E_{1R} \mid E_{0T})\, P(E_{0T})}{P(E_{1R})}$$
$$= \frac{P(E_{1R} \mid E_{0T})\, P(E_{0T})}{1 - P(E_{0R})}$$

c) Calculate the probability that an error occurs in the reception.

$$P(E_{error}) = P(E_{0R} \mid E_{1T})\, P(E_{1T}) + P(E_{1R} \mid E_{0T})\, P(E_{0T})$$

Problem 2. Consider a probability space $(\Omega, \mathcal{F}, \mathsf{P})$.

a) Mathematically speaking, what does it mean when we say A is an event?

An event is a subset of Ω

if $P(A) > 0$ and $P(B) > 0$, we will have a contradiction since we conclude $0 = P(A)P(B)$

b) Assume that A and B are events. Prove that if A and B are disjoint then they cannot be independent unless $\mathsf{P}(A) = 0$ or $\mathsf{P}(B) = 0$.

Suppose that A & B are disjoint and independent. Then, $P(A \cap B) = P(A)P(B)$ and $A \cap B = \emptyset$. Thus, $P(\emptyset) = P(A)P(B)$, which means either $P(A) = 0$ or $P(B) = 0$, On the other hand

c) Suppose that $\mathsf{P}(A) = 0.2$, $\mathsf{P}(B) = 0.5$ and $\mathsf{P}(A \cup B) = 0.6$. Are A and B disjoint? You must justify your answer to receive credit.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore \quad 0.6 = 0.2 + 0.5 - P(A \cap B)$$

$$\therefore P(A \cap B) = 0.1 \quad \therefore A \cap B \neq \emptyset$$

Problem 3. Consider a class of 30 students.

a) In how many ways can we have a class with no two or more students sharing the same birthday?

In $(365)(364)(363)\cdots(365-30+1)$ ways

b) How many birthday arrangements are there in total in a class of size 30?

$$(365)^{30}$$

c) What is the probability that a randomly selected class of 30 has no two students or more with the same birthday?

$$1 - \frac{(365)\cdots(365-30+1)}{365^{30}}$$

d) What is the probability that a randomly selected class of 30 has exactly two students sharing the same birthday while all the other students having distinct birthdays?

- Let a certain group of 2 students select a specific BD.
- There are 365 choices for this group
- There are $\binom{30}{2}$ choices for selecting the group.

$$\therefore \# \text{ of ways} = \binom{30}{2}(365)(364)\cdots(365-30+2)$$

$$\text{Probability in question} = \frac{\# \text{ of ways}}{(365)^{30}} = \frac{\binom{30}{2}(365)\cdots(337)}{365^{30}}$$

Problem 4. An experiment consists of rolling a fair die twice.
a) Propose an appropriate sample space, Ω, for this experiment?

$\Omega = \{1,2,3,4,5,6\}^2$

b) How many events can there be at most for this experiment?

Ω has 36 element.

Ω has a total of 2^{36} subsets

c) Describe mathematically the event, E, that the sum is greater or equal to 8, and find $\mathsf{P}(E)$.

$E = \{(2,6),(6,2),(3,5),(5,3),(3,6),(6,3),(4,4),(4,5),(\cancel{4},5),$
$(4,6),(6,4),(5,6),(6,5),(6,6),(5,5)\}$

$P(E) = 15/36$

d) Let F be the event that the sum is an even number. Find $\mathsf{P}(F)$.

$F = \{(1,1),(1,3),(3,1),(1,5),(5,1),(3,5),(5,3),(3,3),(5,5)$
$(2,2),(2,4),(4,2),(2,6),(6,2),(6,4),(4,6)$
$(4,4),(6,6)\}$

$P(F) = 18/36$

e) Are the events F and E independent? Justify your answer thoroughly.

$F \cap E = \{(5,3),(3,5),(5,5),(6,2),(2,6),(6,4),(4,6),(4,4),$
$(6,6)\}$

$P(E \cap F) = \frac{9}{36}$

$P(E)P(F) = \frac{15}{36} \cdot \frac{18}{36} \neq \frac{9}{36}$ $\therefore$ E & F are not independent

f) Calculate $\mathsf{P}(E|F)$.

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{9/36}{18/36} = \frac{1}{2}$$

Problem 5. A chandelier has 12 light bulbs formed in a ring, each with a power rating of 100W. When it is switched on, each light bulb has a probability 0.99 of being functional.
a) What is the probability that the chandelier provides 1200W of power when it is switched on?

$$(0.99)^{12}$$

b) What is the probability that the chandelier provides at least 900W of power when it is switched on?

$$\sum_{i=9}^{12} \binom{12}{i} (0.99)^{i} (1-0.99)^{12-i}$$

c) What is the probability that all bulbs are non-working when it is switched on?

$$(1-0.99)^{12}$$

d) It was observed that the chandelier was providing exactly 600W of power. What is the probability that every working bulb is surrounded on each side by a non-working bulb?

There are $\binom{12}{6}$ ways of having 6 bulbs working, and there is only one way for the specified scenario.

The probability that the specified scenario occurs given that we have 6 bulbs working

is $\frac{1}{\binom{12}{6}}$

Problem 6. Circle ALL correct answers:

(i) Suppose that F_1 and F_2 are events, then

a) $\mathsf{P}(F_1 \cup F_2) \leq \mathsf{P}(F_1) + \mathsf{P}(F_2)$.
b) $\mathsf{P}(F_1 \cup F_2) = \mathsf{P}(F_1) + \mathsf{P}(F_2 \setminus F_1)$.
c) $\mathsf{P}(F_1 \cup F_2) = \mathsf{P}(F_1) + \mathsf{P}(F_2 \setminus (F_1 \cap F_2))$.
d) $\mathsf{P}(F_1 \cup F_2) = \mathsf{P}(F_1) + \mathsf{P}(F_2)$ only if F_1 and F_2 are independent and $F_1 \cap F_2 = \emptyset$.
e) $\mathsf{P}(F_1 \cap F_2) = \mathsf{P}(F_1)\mathsf{P}(F_2)$.

(ii) Suppose that F_1 and F_2 are events, then

a) $\mathsf{P}(F_1) \leq \mathsf{P}(F_2)$ if $F_1 \subset F_2$.
b) $\mathsf{P}(F_1) = \mathsf{P}(F_1 \cap F_2) + \mathsf{P}(F_1 \cap F_2^c)$ only if $F_1 \cup F_2 = \Omega$.
c) $\mathsf{P}(F_1) = \mathsf{P}(F_1 \cap F_2) + \mathsf{P}(F_1 \cap F_2^c)$.
d) $\mathsf{P}(F_1 \cup F_2) = \mathsf{P}(F_1) + \mathsf{P}(F_2)$ only if F_1 and F_2 are independent.
e) None of the above.

(iii) Suppose that A and B are events, then

a) $\mathsf{P}(A|B) = \mathsf{P}(A \cap B)/\mathsf{P}(B)$ whenever $\mathsf{P}(B) > 0$.
b) $\mathsf{P}(A|B) = \mathsf{P}(A \cap B)/\mathsf{P}(B)$.
c) $\mathsf{P}(A|B) = \mathsf{P}(B|A)$ whenever $\mathsf{P}(A) > 0$ and $\mathsf{P}(B) > 0$.
d) $\mathsf{P}(A|B) = \mathsf{P}(A)$ when $\mathsf{P}(B) = 0$.
e) $\mathsf{P}(A|B) = \mathsf{P}(A)$ when (1) $\mathsf{P}(B) > 0$ and (2) A and B are independent.
f) $\mathsf{P}(A|B) = \mathsf{P}(B|A)\mathsf{P}(A)/\mathsf{P}(B)$ whenever $\mathsf{P}(A) > 0$ and $\mathsf{P}(B) > 0$.