* Let X and Y be two statistically independent random variables having probability density funtions:

$f_X(x) = 1 \quad 0 < x < 1$

$\quad\quad\quad 0 \quad \text{elsewhere}$

$f_Y(y) = 1 \quad 0 < x < 1$

$\quad\quad\quad 0 \quad \text{elsewhere}$

Find the probability density function of $Z = X + Y$ by using the characteristic function,

$$\varphi_X(u) = \int_0^1 (1) e^{jux}\, dx = \frac{e^{jux}}{ju}\Big|_0^1$$

$$= \frac{e^{ju}-1}{ju}$$

$$\varphi_Y(u) = \int_0^1 (1) e^{juy}\, dy = \frac{e^{juy}}{ju}\Big|_0^1$$

$$= \frac{e^{ju}-1}{ju}$$

$\varphi_Z(u) = \varphi_X(u)\, \varphi_Y(u)$ (for $Z = X + Y$ and X and Y statistically independent)

$$\varphi_Z(u) = \frac{e^{ju}-1}{ju} \times \frac{e^{ju}-1}{ju}$$

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{(e^{ju}-1)}{ju} \times \frac{(e^{ju}-1)}{ju}\, e^{-juz}\, \cancel{dz}\, du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{2ju} - 2e^{ju} + 1}{(ju)^2}\, e^{-juz}\, du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-2ju} - 2e^{-ju} + 1}{(ju)^2} e^{juz} \, du$$

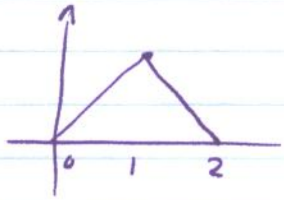
$$= F^{-1}\left\{ \frac{e^{-2ju}}{(ju)^2} - 2\,\frac{e^{-ju}}{(ju)^2} + 1\,\frac{1}{(ju)^2} \right\}$$

$$F^{-1}\left\{ \frac{1}{(ju)^2} \right\} = z\,u(z)$$

$$F^{-1}\left\{ e^{-\alpha ju} F(u) \right\} = f(z-\alpha)$$

$$\Rightarrow \quad f_Z(z) = z\,u(z-2) - 2z\,u(z-1) + z\,u(z)$$

$$= \begin{cases} z & 0 \le z \le 1 \\ 2-z & 1 \le z \le 2 \\ 0 & \text{elsewhere} \end{cases}$$



A random variable x has a probability density function of the form $f(x) = 2e^{-2x} u(x)$

Using the characteristic function, find the first and second moments of this random variable.

$$j\bar{x} = \left. \frac{d\varphi(u)}{du} \right|_{u=0}$$

$$\varphi(u) = \int_0^{\infty} 2e^{-2x} e^{jux} \, dx$$

$$= 2\int_0^{\infty} e^{(-2+ju)x} \, dx$$

$$= \left. \frac{e^{(-2+ju)x}}{(-2+ju)} \right|_0^{\infty} = \frac{1}{-2+ju}$$

$$\frac{d\varphi(u)}{du} = \frac{j}{(ju-2)^2}$$

$$\left.\frac{d\varphi(u)}{du}\right|_{u=0} = j/4 = j\bar{x}$$

$$\Rightarrow \quad \bar{x} = \frac{1}{4}$$

$$\overline{x^2} = E(x^2) = \frac{1}{j^2} \left.\frac{d^2\varphi(u)}{du^2}\right|_{u=0}$$

$$= -\left.\frac{d^2\varphi(u)}{du^2}\right|_{u=0}$$

$$\frac{d^2\varphi(u)}{du^2} = \frac{d}{du}\left(\frac{d\varphi(u)}{du}\right) = \frac{-2j^2}{(ju-2)^3}$$

$$\left.\frac{d^2\varphi(u)}{du^2}\right|_{u=0} = \frac{2}{(-2)^3}$$

$$\Rightarrow \quad \bar{x}^2 = \frac{-2}{(-2)^3} = \frac{1}{4}$$

The random variable Y is defined as:

$$Y = \sum_{i=1}^{n} x_i$$

where x_i, $i = 1, 2, \dots, n$ are statistically independant random variables, with

$$x_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{" \quad " \quad } (1-p) \end{cases}$$

a) Determine the characteristic function of Y.

b) From the characteristic function, determine the moments $E(Y)$ and $E(Y^2)$.

$$\varphi_Y(u) = E[e^{juY}] = E[e^{ju\sum_{i=1}^{n} x_i}] = E\left[\prod_{i=1}^{n} e^{jux_i}\right]$$

$$= \prod_{i=1}^{n} E[e^{jux}]$$

$$= [\varphi_X(u)]^n$$

$$f_X(x) = p\,\delta(x-1) + (1-p)\,\delta(x)$$

$$\Rightarrow \varphi_X(u) = \int_{-\infty}^{+\infty} f_X(x)\, e^{jux}\, dx = \int (p\,\delta(x-1) + (1-p)\,\delta(x))\, e^{jux}\, dx$$

$$= (1-p) + p e^{ju}$$

$$\Rightarrow \varphi_Y(u) = (1-p+pe^{ju})^n$$

(b) $$\left.\frac{d\varphi_Y(u)}{du}\right|_{u=0} = j\bar{Y}$$

$$\frac{d\varphi_Y(u)}{du} = n(1-p+pe^{ju})^{n-1} \times jpe^{ju}$$

$$\Rightarrow \left.\frac{d\varphi_Y(u)}{du}\right|_{u=0} = n(1-p+p)\,jp = jnp = j\bar{Y}$$

$$\Rightarrow \boxed{E(Y) = np}$$

$$\overline{Y^2} = E(Y^2) = -\left.\frac{d^2\varphi_Y(u)}{d^2u}\right|_{u=0}$$

$$\left.\frac{d^2\varphi_Y(u)}{d^2u}\right|_{u=0} = \frac{d}{du}\left(n(1-p+pe^{ju})^{n-1}\, jpe^{ju}\right)\Big|_{u=0}$$

$$= np + np(n-1)p$$