UNM Physics 262, Fall 2006
Midterm Exam 3: Quantum Mechanics

Name and/or CPS number: Dr. Landahl - Solution key

Show all your work for full credit. Remember that quantities have units and vectors have components (or magnitude and direction). **ASK** if anything seems unclear.

CALCULATORS AND CELL PHONES ARE PROHIBITED.
USE OF THESE WILL RESULT IN A ZERO FOR THE EXAM.
Keep any factors of π, e, $\sqrt{2}$, etc. in your answers.

You may use a single 8.5" × 11" paper containing notes you have prepared ahead of time to assist you.

Apportion your time sensibly. Spend about 10–12 minutes per problem.

Please put a box around your final answers.

Useful constants:

$hc = 1240$ eV·nm

Problem 1: 25

Problem 2: 25

Problem 3: 25

Problem 4: 25

1. Short answer [25 points]

[6] *a*) **Rutherford scattering.**

i) What is the important feature in the outcome of the Rutherford experiment and what does it suggest about the structure of the atom? Please limit your answer to one or two sentences.

The α particles scattered off of the gold atoms at <u>much larger angles</u> than was predicted by the Thomson "plum pudding" atomic model.

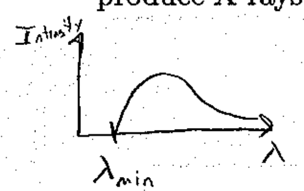
It suggests a "solar system" model of the atom in which positive charge is confined to a compact <u>nucleus</u> about which electrons orbit.

ii) Why does this important feature cease to be observed when the incident particles have a very high energy? Please limit your answer to one or two sentences.

Such α particles have sufficient energy to overcome the nucleus' Coulomb repulsion and actually <u>penetrate the nucleus</u>.

The nuclear force law is different, and results in a different pattern to scattering angles than is predicted by Coulomb scattering from a point charge.

[5] *b*) **Bremsstrahlung.** What is the minimum electron acceleration voltage necessary to produce X-rays with a wavelength of 0.1 nm? (Use $hc = 1240$ eV·nm to do your calculation.)



Energy: e^- Voltage (Potential Energy) $\rightarrow$ e^- kinetic energy

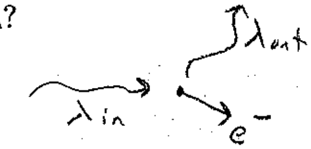
$\rightarrow$ 1 X-ray photon energy

$$\left.\begin{aligned} PE_{init} &= eV \\ E_{X\text{-}ray} &= \frac{hc}{\lambda} \end{aligned}\right\} \quad eV = \frac{hc}{\lambda} \Rightarrow V = \frac{hc}{e\lambda}$$

$$V = \frac{(1240\,\cancel{e}V\cdot\cancel{nm})}{\cancel{e}(0.1\,\cancel{nm})} = 12{,}400\text{ V} = \boxed{12.4\text{ kV}}$$

[Notice that <u>Volts</u> and <u>electron Volts</u> are different things.]

[6] *c*) **Compton scattering.** A photon of wavelength λ strikes an electron at rest and scatters from it elastically at an angle of 90 degrees. The scattered photon has a wavelength that is 2.4 pm longer than what it started with. If a photon of wavelength 3λ similarly strikes an electron at rest and scatters elastically from it at an angle of 180 degrees, how much longer is the wavelength of the scattered photon than 3λ?



Compton formula: $\Delta\lambda = \lambda_c(1-\cos\theta)$

Scattering 1: $\lambda_{out} - \lambda_{in} = \lambda_c(1-\underbrace{\cos 90^\circ}_{0})$

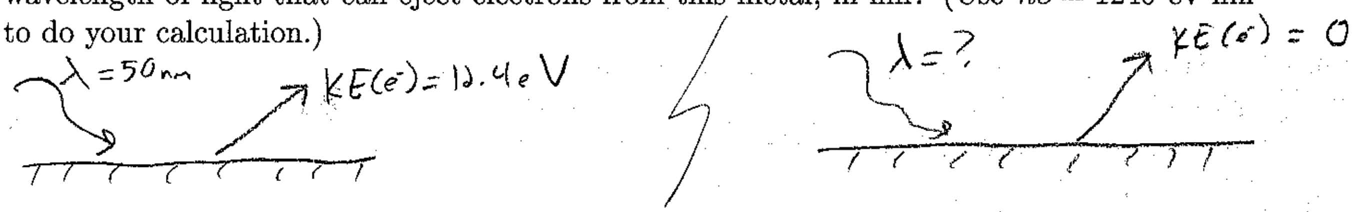
$(\lambda + 2.4\,pm) - \lambda = \lambda_c$

$\Rightarrow \lambda_c = 2.4\,pm$

Scattering 2: $\lambda_{out} - \lambda_{in} = \lambda_c(1-\underbrace{\cos 180^\circ}_{-1}) = \lambda_c(1 - -1) = 2\lambda_c$

$\Rightarrow \lambda_{out} = 3\lambda + 2\lambda_c$, namely $\boxed{2\lambda_c = 4.8\,pm \text{ longer}}$

[8] *d*) **Photoelectric effect.** Light of wavelength 50 nm strikes a clean metal surface in vacuum, emitting electrons of maximum kinetic energy 12.4 eV. What is the maximum wavelength of light that can eject electrons from this metal, in nm? (Use $hc = 1240$ eV·nm to do your calculation.)



If $\lambda = 50\,nm$ produces $K = 12.4\,eV$ electrons, then using longer & longer wavelengths will produce less & less energetic electrons until they have barely zero kinetic energy:

$K = \frac{hc}{\lambda} - \phi_W$ ← work function

$0 = \frac{hc}{\lambda_{max}} - \phi_W$

$\Rightarrow \phi_W = \frac{hc}{\lambda} - K$

$\Rightarrow \lambda_{max} = \frac{hc}{\phi_W}$

$$\lambda_{max} = \frac{hc}{\frac{hc}{\lambda} - K} = \frac{\lambda}{1 - \frac{\lambda K}{hc}} = \frac{50\,nm}{1 - \frac{(50\,nm)(12.4\,eV)}{1240\,eV\,nm}}$$

$$= \frac{50\,nm}{1 - 1/2} = \boxed{100\,nm}$$

2. Bohr quantization [25 points]

A charged particle of mass m moves in a circular orbit in a potential

$$V(r) = -\frac{A}{\sqrt{r}},$$

where A is a positive real constant.

[16] *a*) Use the Bohr quantization condition for angular momentum, $L = n\hbar$, to calculate the allowed (quantized) values for the *radius* of the particle orbit in terms of n, $\hbar$, m, and A.

Force particle experiences: $\vec{F} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r} = -\left(\frac{\partial}{\partial r}\left(-Ar^{-1/2}\right)\right)\hat{r} = -\left(\frac{1}{2}Ar^{-3/2}\right)\hat{r} = -\frac{A}{2r^{3/2}}\hat{r}$

Centripetal acceleration: $\vec{a} = -\frac{v^2}{r}\hat{r}$.

Newton's 2nd Law: $\sum \vec{F} = m\vec{a} \longrightarrow \frac{-A}{2r^{3/2}}\hat{r} = -\frac{mv^2}{r}\hat{r} \Rightarrow r^{1/2} = \frac{A}{2mv^2}$ [1]

Bohr quantization: $L = n\hbar = mvr \longrightarrow v = \frac{n\hbar}{mr}$ [2]

2 equations ([1],[2]) in 2 unknowns (v, r): solve for r

$$r^{1/2} = \frac{A}{2m\left(\frac{n\hbar}{mr}\right)^2} \longrightarrow r^{-3/2} = \frac{mA}{2\hbar^2 n^2} \longrightarrow \boxed{r = \left(\frac{2\hbar^2 n^2}{mA}\right)^{2/3}}$$

Dimensional analysis verification: $\left[\frac{\hbar^2}{mA}\right]^{2/3} = \left[\frac{(J\cdot s)^2}{(kg)(J\,m^{-1/2})}\right]^{2/3} = \left[\frac{J\,s^2 m^{1/2}}{kg}\right]^{2/3} = \left[\frac{\cancel{kg}\,m^2}{\cancel{s^2}}\frac{\cancel{s^2}\,m^{1/2}}{\cancel{kg}}\right]^{2/3} = [m^{3/2}]^{2/3}$ ✓ OK.

[9] *b*) Use the Bohr quantization condition for angular momentum, $L = n\hbar$, to calculate the allowed (quantized) values for the *energy* of the particle in terms of n, $\hbar$, m, and A. **Simplify your answer so that it is a single term that is a function of these variables.**

Kinetic energy: $\frac{1}{2}mv^2$
Potential energy: $\frac{-A}{\sqrt{r}}$

Total Energy: $E = KE + PE = \frac{1}{2}mv^2 - \frac{A}{\sqrt{r}}$

To get final answer in terms of $(n, \hbar, m, A)$, must substitute in for r and v using results of part (a). Haven't solved for v yet, though. Only need v^2 though, so can use equation [1] directly:

$$r^{1/2} = \frac{A}{2mv^2} \Leftrightarrow mv^2 = \frac{A}{2\sqrt{r}} \Rightarrow E = \frac{1}{2}\left(\frac{A}{2\sqrt{r}}\right) - \frac{A}{\sqrt{r}} = -\frac{3}{4}\frac{A}{\sqrt{r}}$$

Plug in r from part (a):

$$\boxed{E = -\frac{3A}{4}\left(\frac{mA}{2\hbar^2 n^2}\right)^{1/3}}$$

3. Heisenberg uncertainty principle [25 points]

For both parts of this problem, you may assume that the motion of the particle is non-relativistic.

[10] *a*) A particle of mass m has a position uncertainty equal to its de Broglie wavelength. What is the minimum fractional uncertainty in its velocity, $\Delta v/v$?

$\left(\hbar = \frac{h}{2\pi} \text{ by definition}\right)$

Heisenberg Uncertainty Principle: $\Delta x \Delta p \geq \hbar/2$

de Broglie Quantum Hypothesis: $p = h/\lambda \Rightarrow \lambda = h/p$

Nonrelativistic momentum: $p = mv \Rightarrow \Delta p = m\Delta v$

Position uncertainty in problem: $\Delta x = \lambda$

Combining results:

$$\Delta x \Delta p \geq \hbar/2 \Rightarrow \lambda(m\Delta v) \geq \hbar/2 \Rightarrow \frac{h}{mv}(m\Delta v) \geq \frac{h}{4\pi} \Rightarrow \boxed{\frac{\Delta v}{v} \geq \frac{1}{4\pi}}$$

[15] *b*) A particle of mass m moves in a one-dimensional potential

$$V(x) = \frac{1}{2}kx^2,$$

where k is a positive constant. Use the Heisenberg uncertainty principle to estimate the minimum total energy (kinetic plus potential) of the particle as a function of m, k, and $\hbar$. (*Hint*: Use the principle to express the minimum energy as a function of either momentum or position and take a derivative.)

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2. \qquad \Delta x \Delta p \geq \hbar/2 \Rightarrow x_{min}p_{min} \approx \frac{\hbar}{2} \text{ at min energy configuration}$$

Method 1: Solve in terms of x_{min}

$$E_{min} = \frac{\hbar^2}{8mx_{min}^2} + \frac{1}{2}kx_{min}^2$$

$$\frac{\partial E_{min}}{\partial x_{min}} = \frac{-2\hbar^2}{8mx_{min}^3} + kx_{min} = 0$$

$$\Rightarrow x_{min} = \pm\left(\frac{\hbar^2}{4mk}\right)^{1/4}$$

$$\Rightarrow E_{min} = \frac{\hbar^2}{8m}\left(\frac{4mk}{\hbar^2}\right)^{1/2} + \frac{1}{2}k\left(\frac{\hbar^2}{4mk}\right)^{1/2}$$

$$\boxed{E_{min} = \frac{\hbar}{2}\sqrt{\frac{k}{m}}}$$

Method 2: Solve in terms of p_{min}

$$E_{min} = \frac{p_{min}^2}{2m} + \frac{\hbar^2 k}{8p_{min}^2}$$

$$\frac{\partial E_{min}}{\partial p_{min}} = \frac{2p_{min}}{2m} - \frac{2\hbar^2 k}{8p_{min}^3} = 0$$

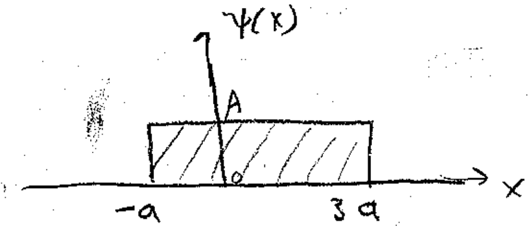
$$\Rightarrow p_{min} = \pm\left(\frac{mk\hbar^2}{4}\right)^{1/4}$$

$$\Rightarrow E_{min} = \frac{1}{2m}\left(\frac{mk\hbar^2}{4}\right)^{1/2} + \frac{\hbar^2 k}{8}\left(\frac{4}{mk\hbar^2}\right)^{1/2}$$

$$\boxed{E_{min} = \frac{\hbar}{2}\sqrt{\frac{k}{m}}}$$

4. Schrödinger's equation [25 points]

The state of a free particle of mass m in one dimension is described by the following quantum wave function:



$$\psi(x) = \begin{cases} 0 & x < -a \\ A & -a \leq x \leq 3a \\ 0 & x > 3a, \end{cases}$$

where A is a positive real constant.

[6] *a*) Determine A using the normalization condition.

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = A^2 \int_{-a}^{3a} dx = 4aA^2 \quad \Rightarrow \quad A = \pm \frac{1}{\sqrt{4a}}.$$

Choose positive solution.

$$\boxed{A = \frac{1}{2\sqrt{a}}}$$

[6] *b*) What is the probability that a measurement of the particle's position will reveal it to be in the range $[0, a]$?

$$P_r\left(x \in [0,a]\right) = \int_0^a |\psi(x)|^2 dx = A^2 \int_0^a dx = A^2 a = \left(\frac{1}{4a}\right)(a) = \boxed{\frac{1}{4}}$$

This also makes geometric sense from the picture above.

[12] *c*) Calculate $\langle x \rangle$ and $\langle x^2 \rangle$ for this state.

$$\langle x \rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 x \, dx.$$

(Note that while x is an odd function, $|\psi(x)|^2$ is neither even nor odd. E.g. $\psi(-2a) = 0 \neq \psi(2a) = A$)

$$= \int_{-a}^{3a} A^2 x \, dx = A^2 \frac{1}{2} x^2 \Big|_{-a}^{3a} = \frac{1}{4a}\left(\frac{1}{2}(3a)^2 - \frac{1}{2}(a)^2\right) = \frac{a}{8}(9-1) = \boxed{a}$$

This also makes geometric sense from the picture above.

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 x^2 dx = A^2 \int_{-a}^{3a} x^2 dx = \frac{1}{4a}\left(\frac{1}{3}(3a)^3 - \frac{1}{3}(-a)^3\right) = \frac{a^2}{12}(27+1)$$

$$= \frac{28}{12} a^2 = \boxed{\frac{7}{3} a^2}$$

[1] *d*) Write down Schrödinger's *time-independent* equation for $\psi(x)$. (*Hint*: Remember, it's a *free* particle.) (*Hint again*: It is a FREE particle.)

Free particle: No force $\Rightarrow$ No potential $\Rightarrow$ $V(x) = 0$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = E\psi}$$