Problem

let X and Y be two statistically independant random variables having probability density functions:

$$f_X(x) = 1 \qquad 0 < x < 1$$
$$= 0 \qquad \text{ow.}$$

$$f_Y(y) = 1 \qquad 0 \leq y \leq 1$$
$$= 0 \qquad \text{ow}$$

For the random variable $Z = X + Y$ find

a) the value of z for which $f_Z(z)$ is a maximum.

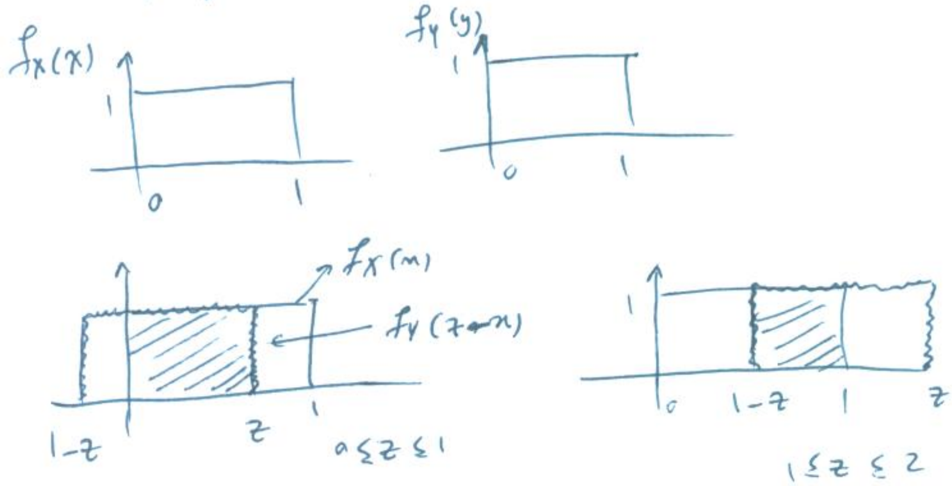
b) the probability that Z is less than 0.5.

we have

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)$$

Since X and Y are statistically independant.

The appropriate diagrams are sketched below:



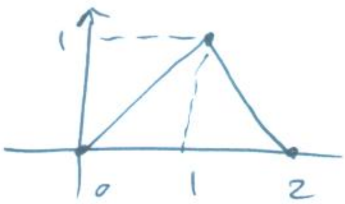
Then

$$0 \leq z \leq 1 \Rightarrow f_Z(z) = \int_0^z 1 \times 1 \, dx = z$$

$$1 \leq z \leq 2 \Rightarrow f_Z(z) = \int_{1-z}^{1} 1 \times 1 \, dz = 2 - z$$

$z \le 0$ or $z \geq 2 \Rightarrow f_Z(z) = 0$

$$\Rightarrow f_Z(z) = \begin{cases} z & 0 \le z \le 1 \\ 2-z & 1 \le z \le 2 \\ 0 & \text{ow} \end{cases}$$



$\therefore f_Z(z)$ has a maximum at $z=1$.

b) $P\{z \le 0.5\} = \int_{-\infty}^{0.5} f_Z(z)\,dz = \int_0^{0.5} z\,dz$

$$= \frac{(0.5)^2}{2} = 0.\cancel{0}125$$

* Two statistically independent random variables have probability density functions as follows:

$$f_X(x) = 5e^{-5x}u(x)$$

$$f_Y(y) = 2e^{-2y}u(y)$$

For the random variable $Z = X + Y$, find:

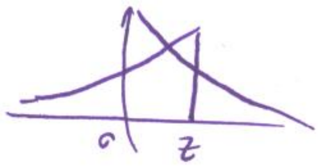
a) $f_Z(0)$

b) the value of z for which $f_Z(z)$ is greater than 1.0.

c) the probability that Z is greater than 0.1.

$$f_Z(z) = 5e^{-5x}u(x) * 2e^{-2y}u(y)$$

$$= \int_0^z 2e^{-2y} \cdot 5e^{-5(z-y)}\,dy$$



$$= 10e^{-5z}\left(e^{3y}/2\right)\Big|_0^z$$

$$= 10/3\left(e^{-2z} - e^{-5z}\right)$$

a) $f_Z(0) = 0$

b) $f_Z(z) > 1$ for $10/3\,(e^{-2z} - e^{-5z}) > 1$

$$\Rightarrow e^{-2z} - e^{-5z} > 3/10 \Rightarrow z > 0.4541$$

c) $\Pr\{Z > 0.1\} = \int_{0.1}^{\infty} f_Z(z)\,dz = \int_{0.1}^{\infty} 10/3\,(e^{-2z} - e^{-5z})\,dz$

$$= \frac{10}{3}\left(\frac{e^{-2z}}{-2} - \frac{e^{-5z}}{-5}\right)\Bigg|_{0.1}^{\infty} = \frac{10}{3}\left(0 + \frac{e^{-0.2}}{2} - \frac{e^{-0.5}}{5}\right) = 0.96$$

* Two random variables X and Y have a joint probability density function of the form:

$$f(x,y) = 1 \qquad 0 \leq x, y \leq 1$$
$$= 0 \qquad \text{elsewhere}$$

Find the probability density function of $Z = XY$.

$$\begin{cases} Z = XY \\ W = X \end{cases}$$

If $Z = \varphi_1(X, Y)$, $W = \varphi_2(X, Y)$

$X = \psi_1(Z, W)$, $Y = \psi_2(Z, W)$

Then $g(z, w) = |J| \, f(\psi_1(z, w), \psi_2(z, w))$

where
$$J = \begin{vmatrix} \partial x/\partial z & \partial x/\partial w \\ \partial y/\partial z & \partial y/\partial w \end{vmatrix}$$

Here:

$$\begin{cases} z = xy \\ w = x \end{cases} \Rightarrow \begin{matrix} x = w \\ y = z/w \end{matrix}$$

$$J = \begin{vmatrix} \partial x/\partial z & \partial x/\partial w \\ \partial y/\partial z & \partial y/\partial w \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{w} & -\frac{z}{w^2} \end{vmatrix} = -\frac{1}{w}$$

$$g(z, w) = \frac{1}{|w|} f(w, z/w)$$

$$g(z) = \int_{-\infty}^{\infty} g(z, w)\, dw = \int_{-\infty}^{\infty} \frac{1}{|w|} f(w, z/w)$$

$$f(w, z/w) = \mathrm{rect}(w - 0.5) \cdot \mathrm{rect}(z/w - 0.5)$$

$g(z) = \int_{-\infty}^{+\infty} \text{rect}(w-0.5) \cdot \text{rect}(z/w - 0.5) \cdot \frac{1}{|w|}$

for the first rect function, it is nonzero over

$0 < w < 1$

for the second rect function, the interval over which it is nonzero is given by:

$0 < z/w < 1 \Rightarrow \quad z < w < \infty$

A sketch will show that the limits of the integral for $0 < z = xy < 1$ is $(z, 1)$.

Then

$$g(z) = \int_z^1 \frac{1}{w} \, dw = -\ln(z) \, , \quad 0 \leq z \leq 1$$

* Show that the random variables X and Y in previous problem are independent and find the expected value of their product. Find $E\{z\}$ by integrating the function $z f(z)$.

$$f(x) = \int_{-\infty}^{+\infty} f(x,y) \, dy = \int_0^1 1 \, dy = 1$$

$$f(y) = \int_{-\infty}^{+\infty} f(x,y) \, dx = \int_0^1 1 \, dx = 1$$

$\Rightarrow f(x,y) = f(x) f(y)$

$\Rightarrow$ X and Y are independant.

$$E(z) = \int_{-\infty}^{+\infty} z\, f(z) = -\int_0^1 z \ln z$$

$$= \tfrac{1}{4} z^2 (1 - 2\ln z) \Big|_0^1 = \tfrac{1}{4}$$

Also

$$E(z) = E(x)\,E(y)$$

$$E(x) = \int_0^1 x f(x)\, dx = \int_0^1 x\, dx = x^2/2 \Big|_0^1 = \frac{1}{2}$$

$$E(y) = \int_0^1 y\, f(y)\, dy = \int_0^1 y\, dy = y^2/2 \Big|_0^1 = \frac{1}{2}$$

$$\Rightarrow E(z) = 1/4$$