

CHAPTER 1

1.1 to 1.41 - part of text

- 1.42 (a) Periodic:
Fundamental period = 0.5s
- (b) Nonperiodic
- (c) Periodic
Fundamental period = 3s
- (d) Periodic
Fundamental period = 2 samples
- (e) Nonperiodic
- (f) Periodic:
Fundamental period = 10 samples
- (g) Nonperiodic
- (h) Nonperiodic
- (i) Periodic:
Fundamental period = 1 sample

1.43
$$y(t) = \left(3 \cos \left(200t + \frac{\pi}{6} \right) \right)^2$$
$$= 9 \cos^2 \left(200t + \frac{\pi}{6} \right)$$
$$= \frac{9}{2} \left[\cos \left(400t + \frac{\pi}{3} \right) + 1 \right]$$

(a) DC component = $\frac{9}{2}$

(b) Sinusoidal component = $\frac{9}{2} \cos \left(400t + \frac{\pi}{3} \right)$

Amplitude = $\frac{9}{2}$

$$\text{Fundamental frequency} = \frac{200}{\pi} \text{Hz}$$

- 1.44 The RMS value of sinusoidal $x(t)$ is $A/\sqrt{2}$. Hence, the average power of $x(t)$ in a 1-ohm resistor is $(A/\sqrt{2})^2 = A^2/2$.

- 1.45 Let N denote the fundamental period of $x[n]$, which is defined by

$$N = \frac{2\pi}{\Omega}$$

The average power of $x[n]$ is therefore

$$\begin{aligned} P &= \frac{1}{N} \sum_{n=0}^{N-1} x^2[n] \\ &= \frac{1}{N} \sum_{n=0}^{N-1} A^2 \cos^2\left(\frac{2\pi n}{N} + \phi\right) \\ &= \frac{A^2}{N} \sum_{n=0}^{N-1} \cos^2\left(\frac{2\pi n}{N} + \phi\right) \end{aligned}$$

- 1.46 The energy of the raised cosine pulse is

$$\begin{aligned} E &= \int_{-\pi/\omega}^{\pi/\omega} \frac{1}{4} (\cos(\omega t) + 1)^2 dt \\ &= \frac{1}{2} \int_0^{\pi/\omega} (\cos^2(\omega t) + 2\cos(\omega t) + 1) dt \\ &= \frac{1}{2} \int_0^{\pi/\omega} \left(\frac{1}{2} \cos(2\omega t) + \frac{1}{2} + 2\cos(\omega t) + 1 \right) dt \\ &= \frac{1}{2} \left(\frac{3}{2} \right) \left(\frac{\pi}{\omega} \right) = 3\pi/4\omega \end{aligned}$$

- 1.47 The signal $x(t)$ is even; its total energy is therefore

$$E = 2 \int_0^5 x^2(t) dt$$

$$\begin{aligned}
&= 2 \int_0^4 (1)^2 dt + 2 \int_4^5 (5-t)^2 dt \\
&= 2[t]_{t=0}^4 + 2 \left[-\frac{1}{3}(5-t)^3 \right]_{t=4}^5 \\
&= 8 + \frac{2}{3} = \frac{26}{3}
\end{aligned}$$

1.48 (a) The differentiator output is

$$y(t) = \begin{cases} 1 & \text{for } -5 < t < -4 \\ -1 & \text{for } 4 < t < 5 \\ 0 & \text{otherwise} \end{cases}$$

(b) The energy of $y(t)$ is

$$\begin{aligned}
E &= \int_{-5}^{-4} (1)^2 dt + \int_4^5 (-1)^2 dt \\
&= 1 + 1 = 2
\end{aligned}$$

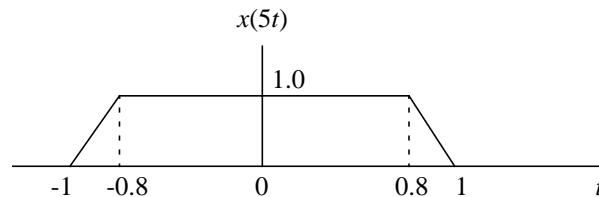
1.49 The output of the integrator is

$$y(t) = A \int_0^t \tau d\tau = At \quad \text{for } 0 \leq t \leq T$$

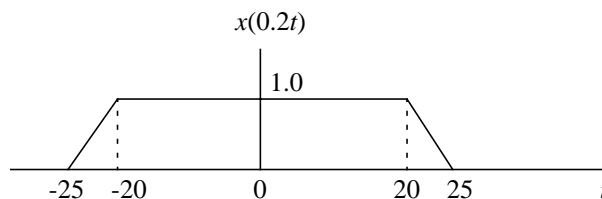
Hence the energy of $y(t)$ is

$$E = \int_0^T A^2 t^2 dt = \frac{A^2 T^3}{3}$$

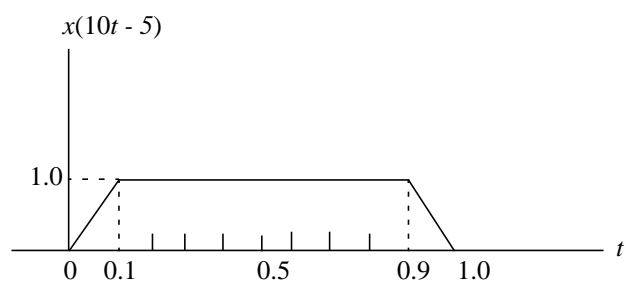
1.50 (a)



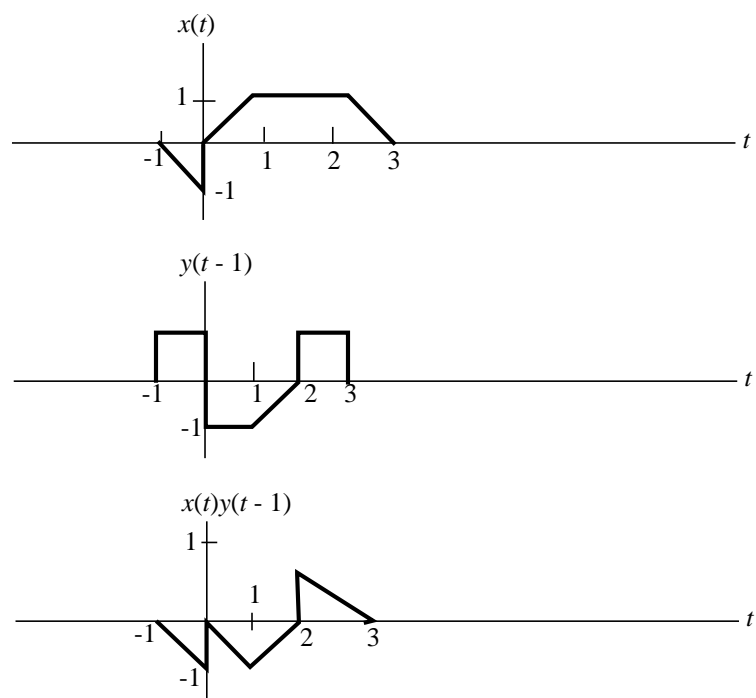
(b)



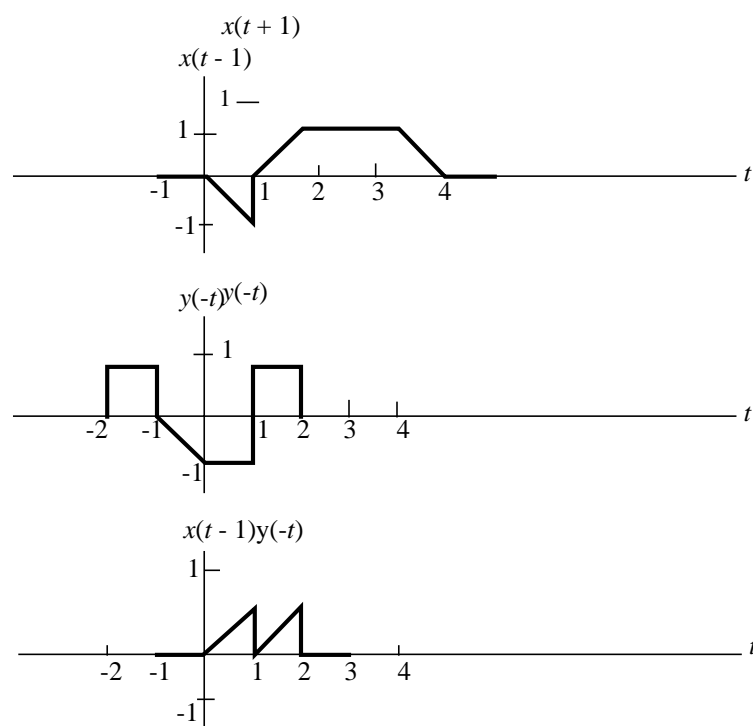
1.51



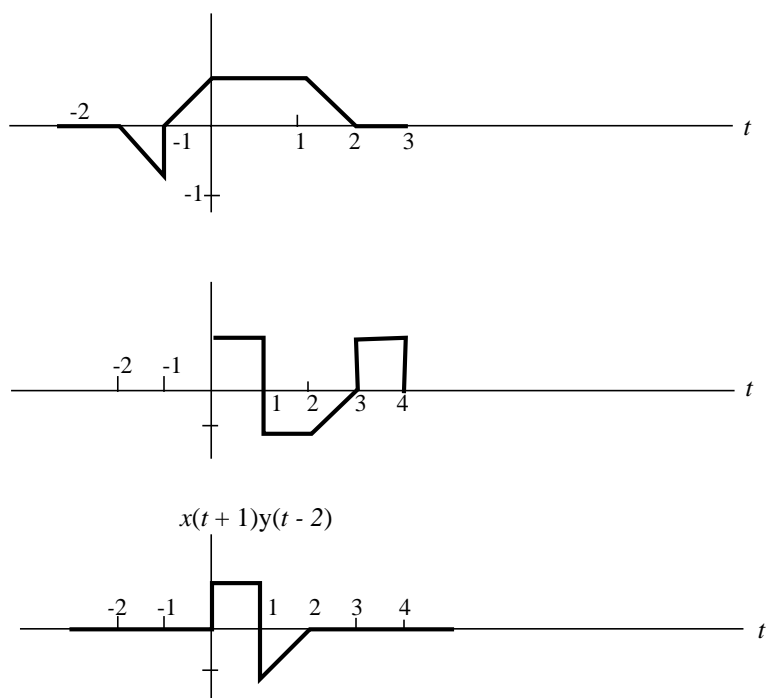
1.52 (a)



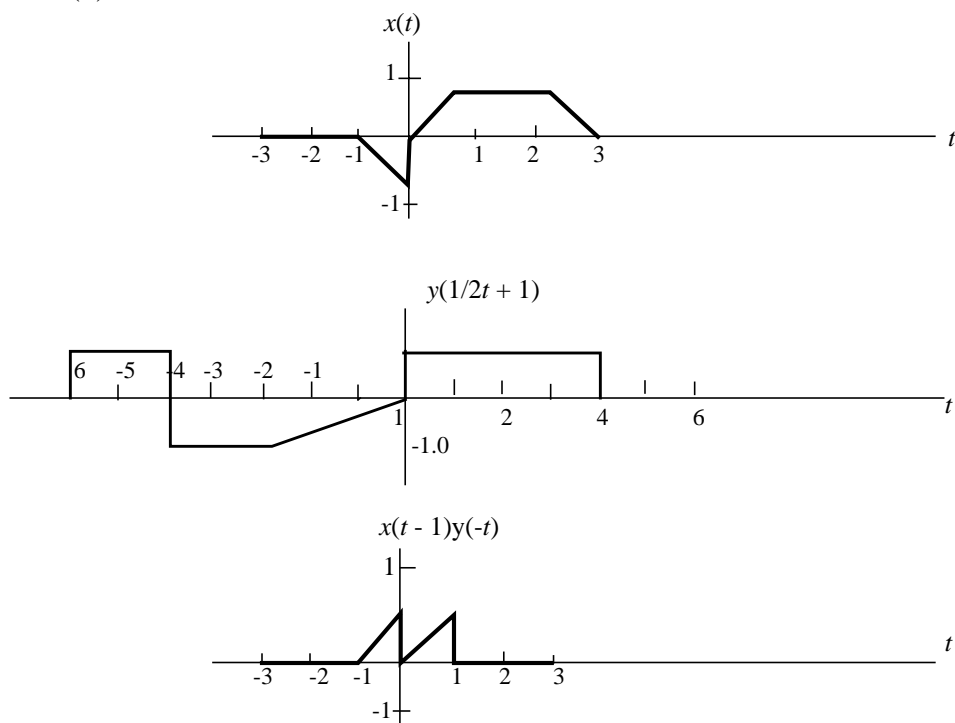
1.52 (b)



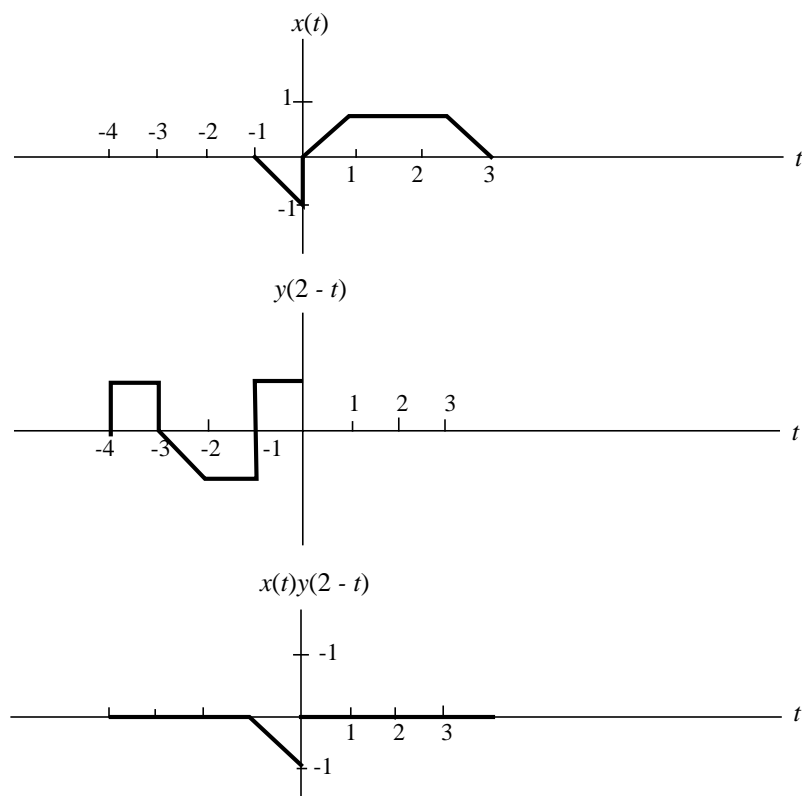
1.52 (c)



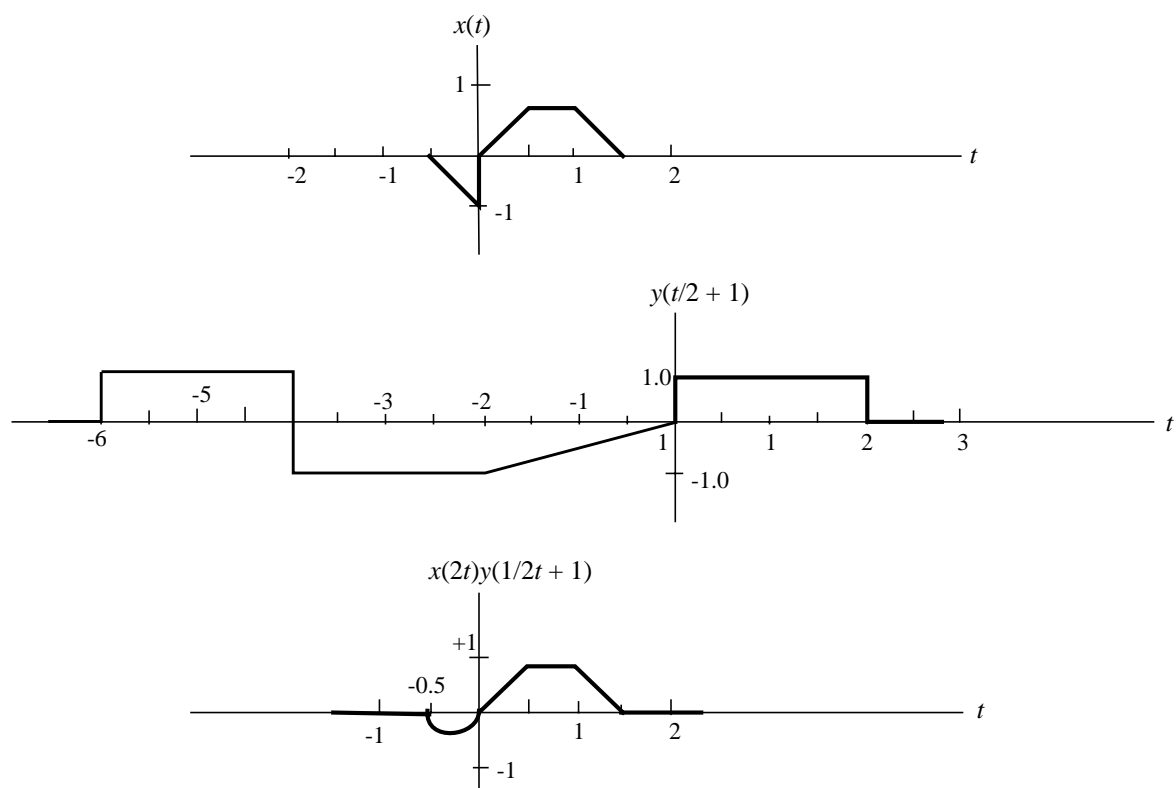
1.52 (d)



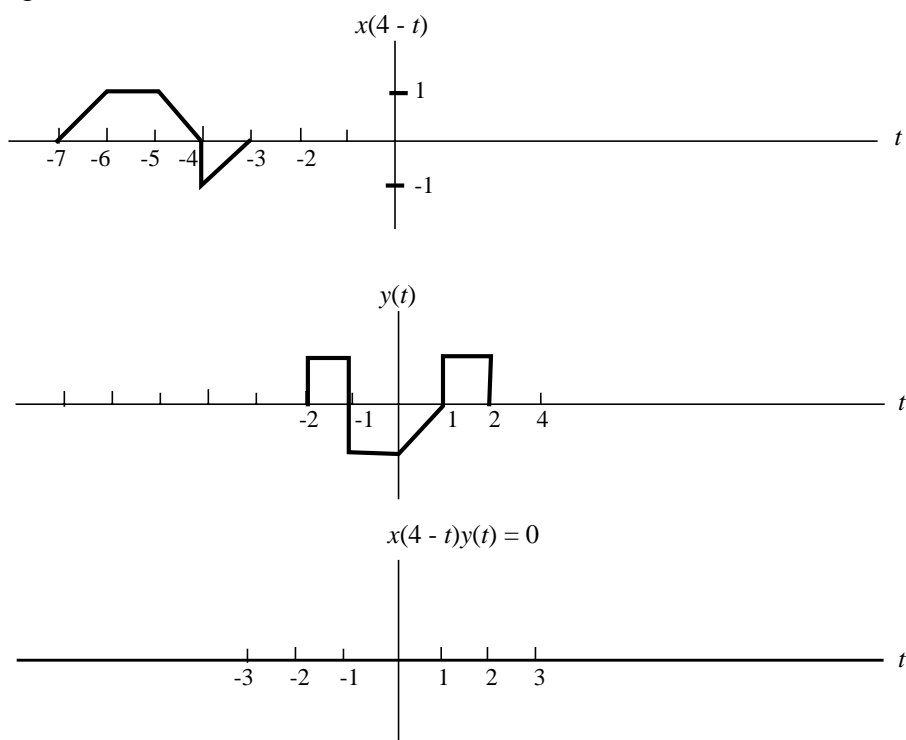
1.52 (e)



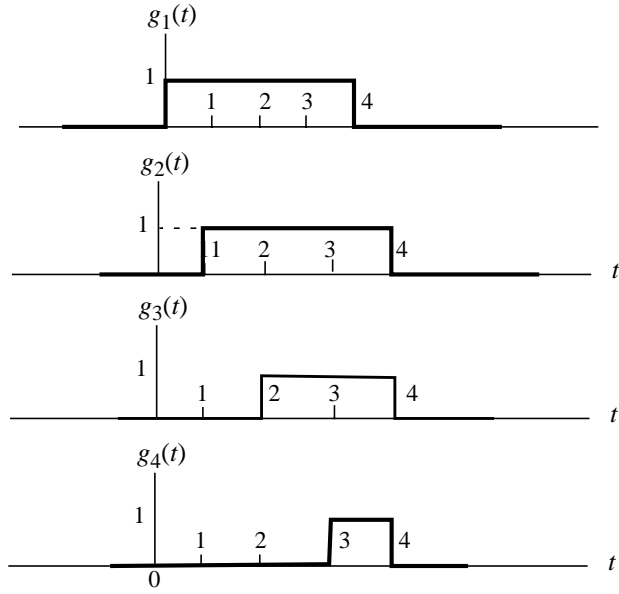
1.52 (f)



1.52 (g)



1.53 We may represent $x(t)$ as the superposition of 4 rectangular pulses as follows:



To generate $g_1(t)$ from the prescribed $g(t)$, we let

$$g_1(t) = g(at - b)$$

where a and b are to be determined. The width of pulse $g(t)$ is 2, whereas the width of pulse $g_1(t)$ is 4. We therefore need to expand $g(t)$ by a factor of 2, which, in turn, requires that we choose

$$a = \frac{1}{2}$$

The mid-point of $g(t)$ is at $t = 0$, whereas the mid-point of $g_1(t)$ is at $t = 2$. Hence, we must choose b to satisfy the condition

$$a(2) - b = 0$$

or

$$b = 2a = 2\left(\frac{1}{2}\right) = 1$$

$$\text{Hence, } g_1(t) = g\left(\frac{1}{2}t - 1\right)$$

Proceeding in a similar manner, we find that

$$g_2(t) = g\left(\frac{2}{3}t - \frac{5}{3}\right)$$

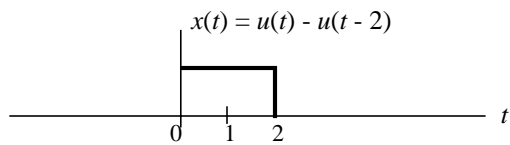
$$g_3(t) = g(t - 3)$$

$$g_4(t) = g(2t - 7)$$

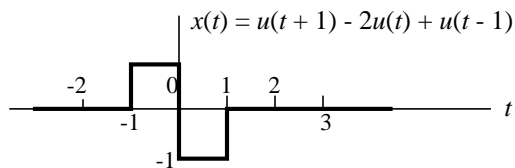
Accordingly, we may express the staircase signal $x(t)$ in terms of the rectangular pulse $g(t)$ as follows:

$$x(t) = g\left(\frac{1}{2}t - 1\right) + g\left(\frac{2}{3}t - \frac{5}{3}\right) + g(t - 3) + g(2t - 7)$$

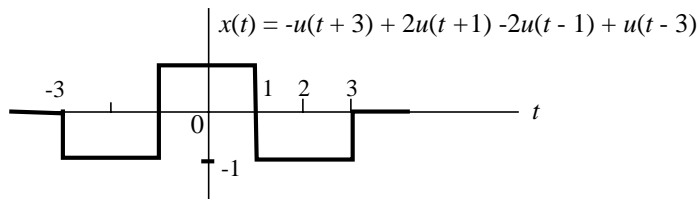
1.54 (a)



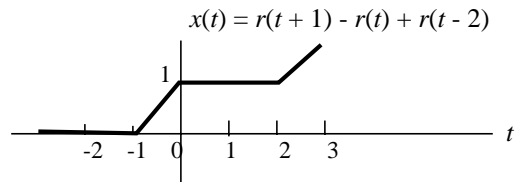
(b)



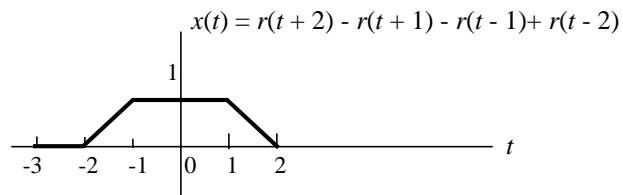
(c)



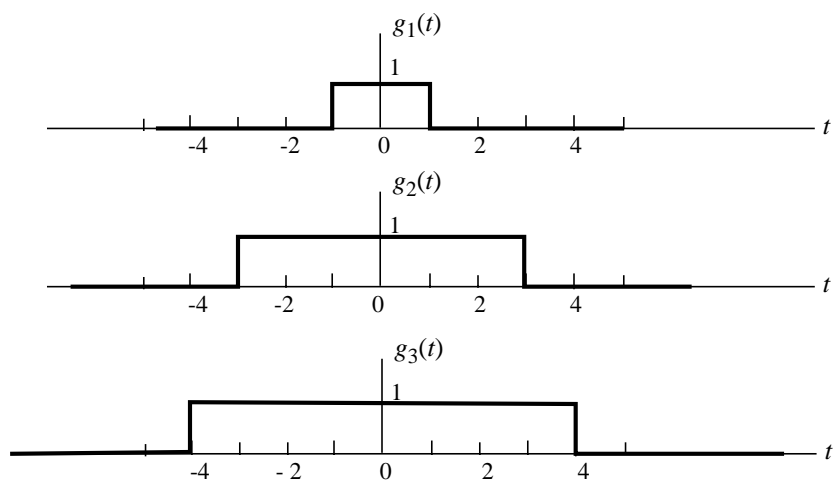
(d)



(e)



1.55 We may generate $x(t)$ as the superposition of 3 rectangular pulses as follows:



All three pulses, $g_1(t)$, $g_2(t)$, and $g_3(t)$, are symmetrically positioned around the origin:

1. $g_1(t)$ is exactly the same as $g(t)$.
2. $g_2(t)$ is an expanded version of $g(t)$ by a factor of 3.
3. $g_3(t)$ is an expanded version of $g(t)$ by a factor of 4.

Hence, it follows that

$$g_1(t) = g(t)$$

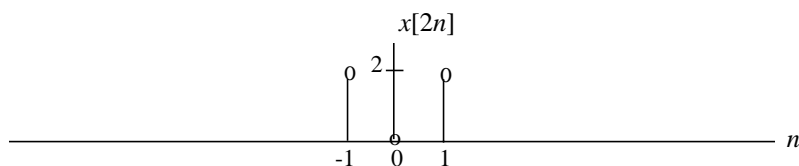
$$g_2(t) = g\left(\frac{1}{3}t\right)$$

$$g_3(t) = g\left(\frac{1}{4}t\right)$$

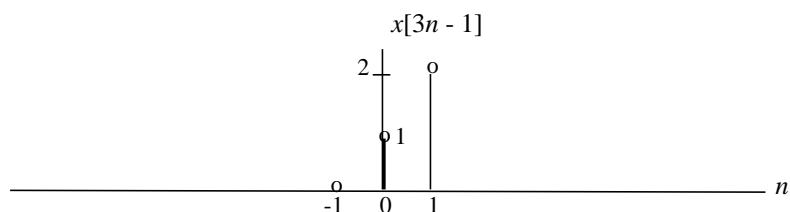
That is,

$$x(t) = g(t) + g\left(\frac{1}{3}t\right) + g\left(\frac{1}{4}t\right)$$

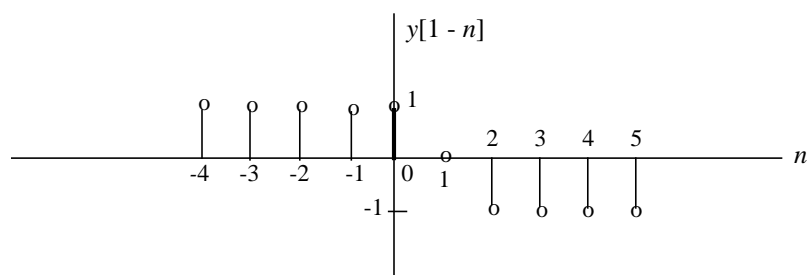
1.56 (a)



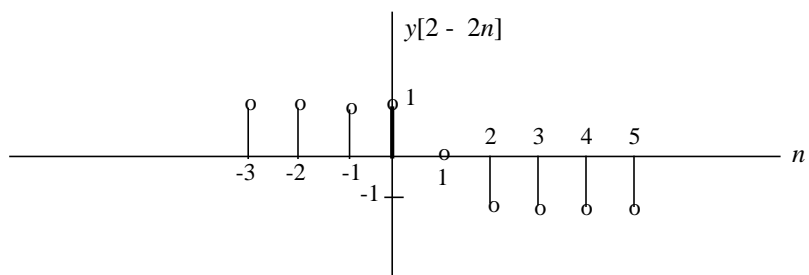
(b)



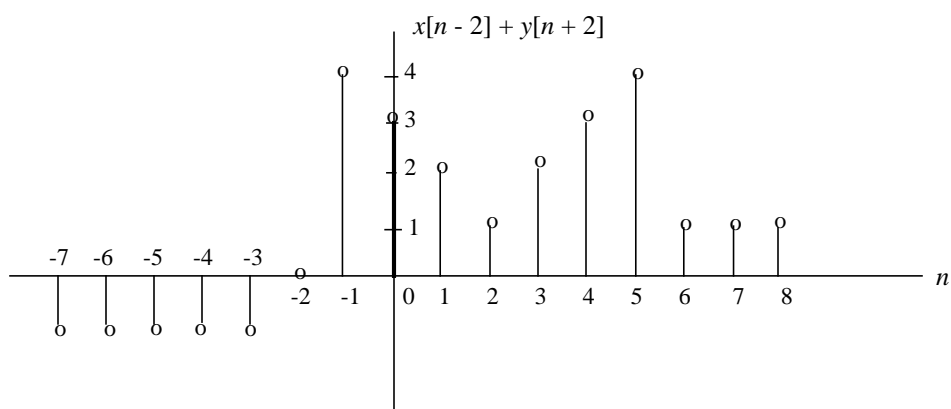
1.56 (c)



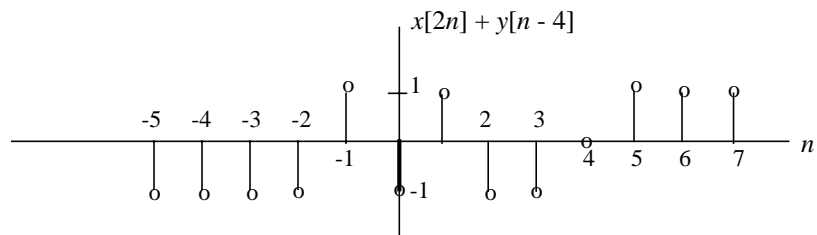
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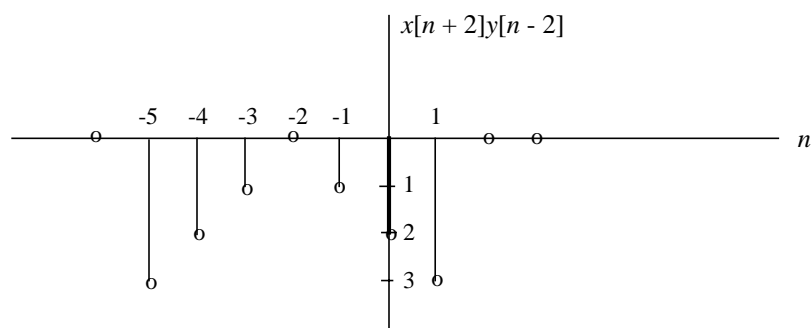
(e)



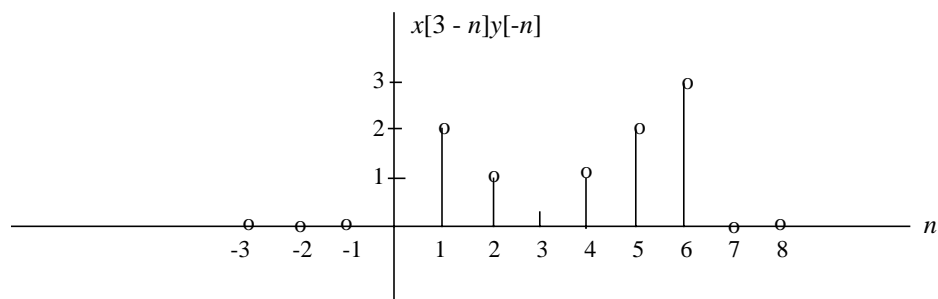
(f)



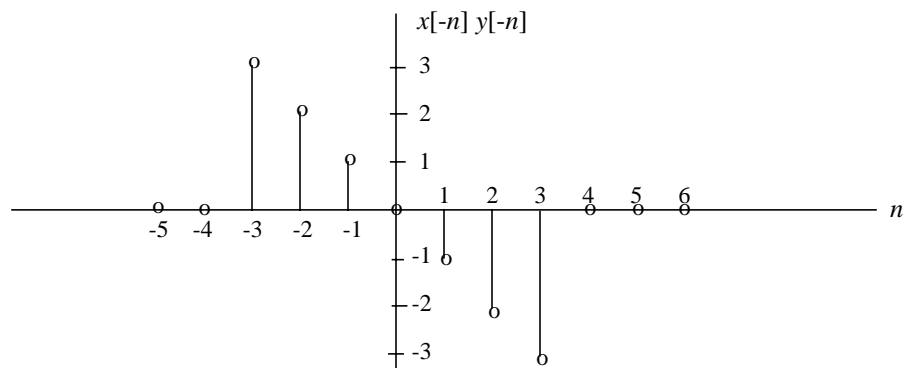
1.56 (g)



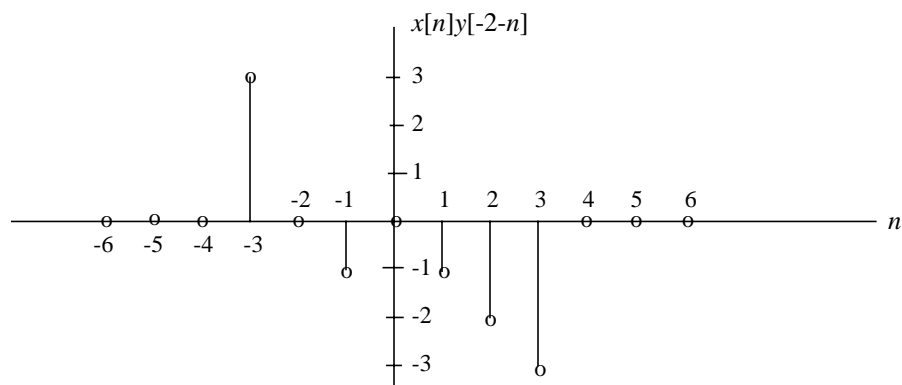
(h)



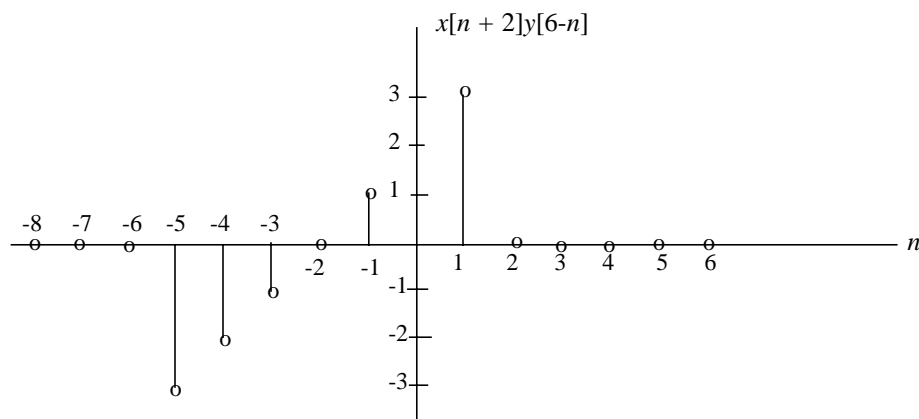
(i)



(j)



1.56 (k)



1.57 (a) Periodic

Fundamental period = 15 samples

(b) Periodic

Fundamental period = 30 samples

(c) Nonperiodic

(d) Periodic

Fundamental period = 2 samples

(e) Nonperiodic

(f) Nonperiodic

(g) Periodic

Fundamental period = 2π seconds

(h) Nonperiodic

(i) Periodic

Fundamental period = 15 samples

1.58 The fundamental period of the sinusoidal signal $x[n]$ is $N = 10$. Hence the angular frequency of $x[n]$ is

$$\Omega = \frac{2\pi m}{N} \quad m: \text{integer}$$

The smallest value of Ω is attained with $m = 1$. Hence,

$$\Omega = \frac{2\pi}{10} = \frac{\pi}{5} \text{ radians/cycle}$$

1.59 The amplitude of complex signal $x(t)$ is defined by

$$\begin{aligned}\sqrt{x_R^2(t) + x_I^2(t)} &= \sqrt{A^2 \cos^2(\omega t + \phi) + A^2 \sin^2(\omega t + \phi)} \\ &= A \sqrt{\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)} \\ &= A\end{aligned}$$

1.60 Real part of $x(t)$ is

$$\text{Re}\{x(t)\} = Ae^{\alpha t} \cos(\omega t)$$

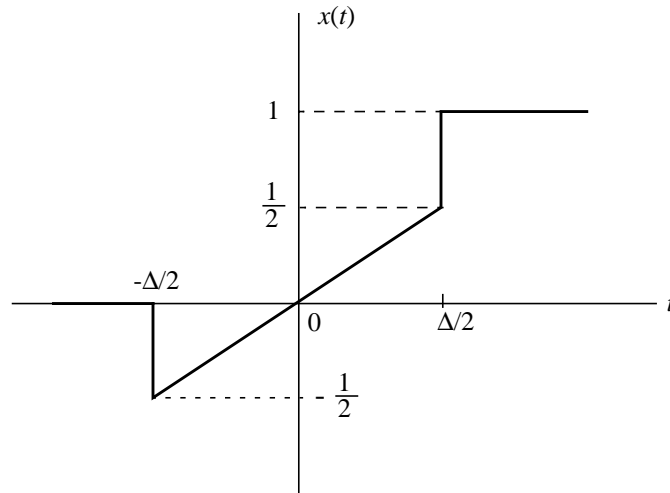
Imaginary part of $x(t)$ is

$$\text{Im}\{x(t)\} = Ae^{\alpha t} \sin(\omega t)$$

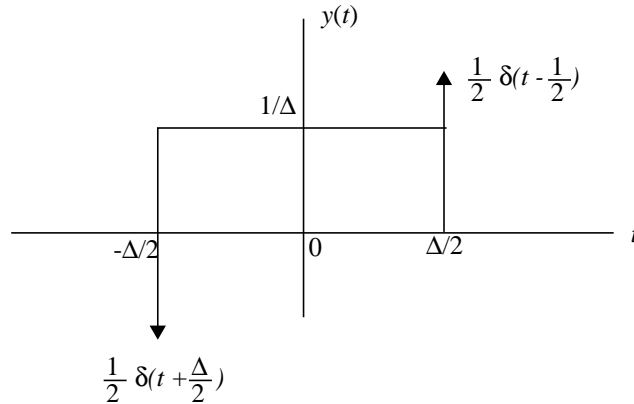
1.61 We are given

$$x(t) = \begin{cases} \frac{t}{\Delta} & \text{for } -\frac{\Delta}{2} \leq t \leq \frac{\Delta}{2} \\ 1 & \text{for } t \geq \frac{\Delta}{2} \\ 2 & \text{for } t < -\frac{\Delta}{2} \end{cases}$$

The waveform of $x(t)$ is as follows



The output of a differentiator in response to $x(t)$ has the corresponding waveform:



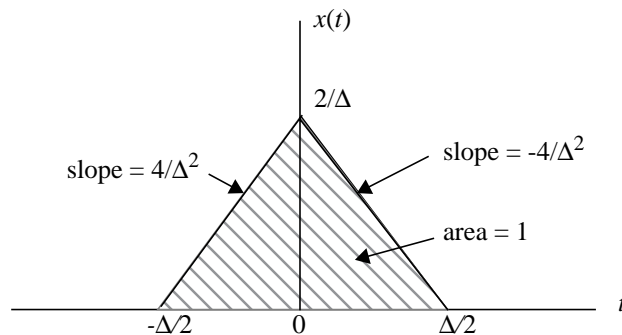
$y(t)$ consists of the following components:

1. Rectangular pulse of duration Δ and amplitude $1/\Delta$ centred on the origin; the area under this pulse is unity.
2. An impulse of strength $1/2$ at $t = \Delta/2$.
3. An impulse of strength $-1/2$ at $t = -\Delta/2$.

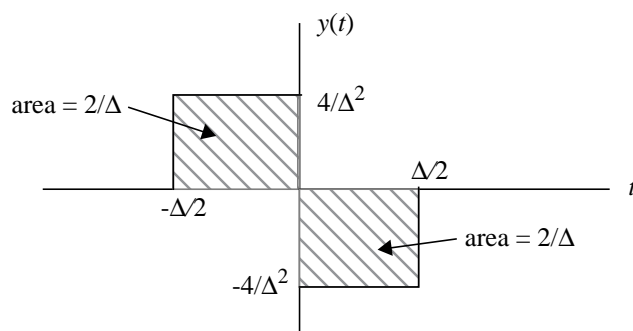
As the duration Δ is permitted to approach zero, the impulses $(1/2)\delta(t-\Delta/2)$ and $-(1/2)\delta(t+\Delta/2)$ coincide and therefore cancel each other. At the same time, the rectangular pulse of unit area (i.e., component 1) approaches a unit impulse at $t = 0$. We may thus state that in the limit:

$$\begin{aligned} \lim_{\Delta \rightarrow 0} y(t) &= \lim_{\Delta \rightarrow 0} \frac{d}{dt} x(t) \\ &= \delta(t) \end{aligned}$$

- 1.62 We are given a triangular pulse of total duration Δ and unit area, which is symmetrical about the origin:



(a) Applying $x(t)$ to a differentiator, we get an output $y(t)$ depicted as follows:



(b) As the triangular pulse duration Δ approaches zero, the differentiator output approaches the combination of two impulse functions described as follows:

- An impulse of positive infinite strength at $t = 0^-$.
- An impulse of negative infinite strength at $t = 0^+$.

(c) The total area under the differentiator output $y(t)$ is equal to $(2/\Delta) + (-2/\Delta) = 0$.

In light of the results presented in parts (a), (b), and (c) of this problem, we may now make the following statement:

When the unit impulse $\delta(t)$ is differentiated with respect to time t , the resulting output consists of a pair of impulses located at $t = 0^-$ and $t = 0^+$, whose respective strengths are $+\infty$ and $-\infty$.

1.63 From Fig. P.1.63 we observe the following:

$$x_1(t) = x_2(t) = x_3(t) = x(t)$$

$$x_4(t) = y_3(t)$$

Hence, we may write

$$y_1(t) = x(t)x(t-1) \tag{1}$$

$$y_2(t) = |x(t)| \tag{2}$$

$$y_4(t) = \cos(y_3(t)) = \cos(1 + 2x(t)) \tag{3}$$

The overall system output is

$$y(t) = y_1(t) + y_2(t) - y_4(t) \tag{4}$$

Substituting Eqs. (1) to (3) into (4):

$$y(t) = x(t)x(t-1) + |x(t)| - \cos(1 + 2x(t)) \tag{5}$$

Equation (5) describes the operator H that defines the output $y(t)$ in terms of the input $x(t)$.

1.64	<u>Memoryless</u>	<u>Stable</u>	<u>Causal</u>	<u>Linear</u>	<u>Time-invariant</u>
(a)	✓	✓	✓	x	✓
(b)	✓	✓	✓	✓	✓
(c)	✓	✓	✓	x	✓
(d)	x	✓	✓	✓	✓
(e)	x	✓	x	✓	✓
(f)	x	✓	✓	✓	✓
(g)	✓	✓	x	x	✓
(h)	x	✓	✓	✓	✓
(i)	x	✓	x	✓	✓
(j)	✓	✓	✓	✓	✓
(k)	✓	✓	✓	✓	✓
(l)	✓	✓	✓	x	✓

1.65 We are given

$$y[n] = a_0x[n] + a_1x[n-1] + a_2x[n-2] + a_3x[n-3] \quad (1)$$

Let

$$S^k\{x(n)\} = x(n-k)$$

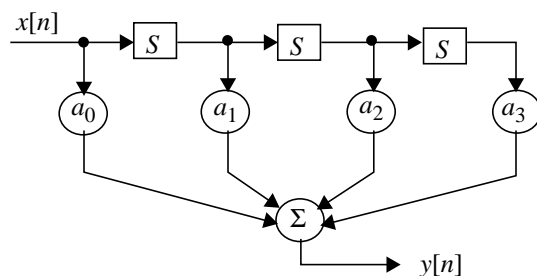
We may then rewrite Eq. (1) in the equivalent form

$$\begin{aligned}
 y[n] &= a_0x[n] + a_1S^1\{x[n]\} + a_2S^2\{x[n]\} + a_3S^3\{x[n]\} \\
 &= (a_0 + a_1S^1 + a_2S^2 + a_3S^3)\{x[n]\} \\
 &= H\{x[n]\}
 \end{aligned}$$

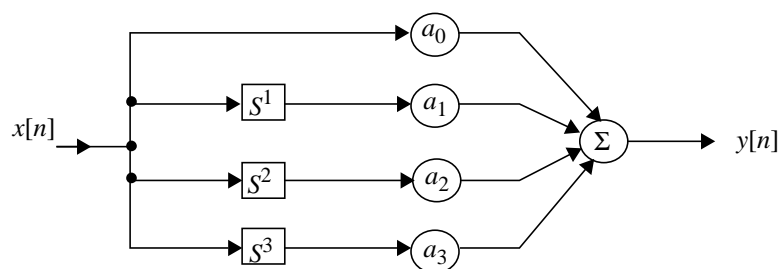
where

$$H = a_0 + a_1S^1 + a_2S^2 + a_3S^3$$

(a) Cascade implementation of operator H :



(b) Parallel implementation of operator H :



1.66 Using the given input-output relation:

$$y[n] = a_0x[n] + a_1x[n-1] + a_2x[n-2] + a_3x[n-3]$$

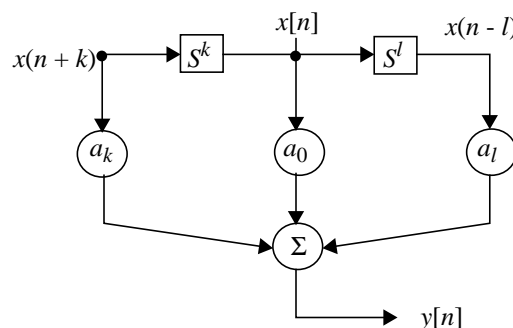
we may write

$$\begin{aligned} |y[n]| &= |a_0x[n] + a_1x[n-1] + a_2x[n-2] + a_3x[n-3]| \\ &\leq |a_0x[n]| + |a_1x[n-1]| + |a_2x[n-2]| + |a_3x[n-3]| \\ &\leq |a_0|M_x + |a_1|M_x + |a_2|M_x + |a_3|M_x \\ &= (|a_0| + |a_1| + |a_2| + |a_3|)M_x \end{aligned}$$

where $M_x = |x(n)|$. Hence, provided that M_x is finite, the absolute value of the output will always be finite. This assumes that the coefficients a_0, a_1, a_2, a_3 have finite values of their own. It follows therefore that the system described by the operator H of Problem 1.65 is stable.

1.67 The memory of the discrete-time described in Problem 1.65 extends 3 time units into the past.

1.68 It is indeed possible for a noncausal system to possess memory. Consider, for example, the system illustrated below:



That is, with $S^l\{x[n]\} = x[n-l]$, we have the input-output relation

$$y[n] = a_0x[n] + a_kx[n+k] + a_lx[n-l]$$

This system is noncausal by virtue of the term $a_kx[n+k]$. The system has memory by virtue of the term $a_lx[n-l]$.

- 1.69 (a) The operator H relating the output $y[n]$ to the input $x[n]$ is

$$H = 1 + S^1 + S^2$$

where

$$S^k \{x[n]\} = x[n - k] \quad \text{for integer } k$$

- (b) The inverse operator H^{inv} is correspondingly defined by

$$H^{\text{inv}} = \frac{1}{1 + S^1 + S^2}$$

Cascade implementation of the operator H is described in Fig. 1. Correspondingly,

feedback implementation of the inverse operator H^{inv} is described in Fig. 2

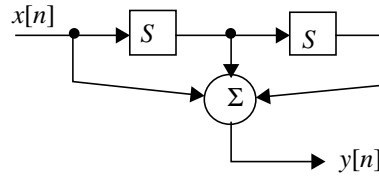


Fig. 1
Operator H

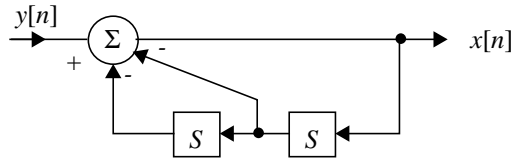


Fig. 2
Inverse Operator H^{inv}

Figure 2 follows directly from the relation:

$$x[n] = y[n] - x[n - 1] - x[n - 2]$$

- 1.70 For the discrete-time system (i.e., the operator H) described in Problem 1.65 to be time-invariant, the following relation must hold

$$S^{n_0} H = H S^{n_0} \quad \text{for integer } n_0 \quad (1)$$

where

$$S^{n_0} \{x[n]\} = x[n - n_0]$$

and

$$H = 1 + S^1 + S^2$$

We first note that

$$\begin{aligned} S^{n_0} H &= S^{n_0} (1 + S^1 + S^2) \\ &= S^{n_0} + S^{n_0+1} + S^{n_0+2} \end{aligned} \quad (2)$$

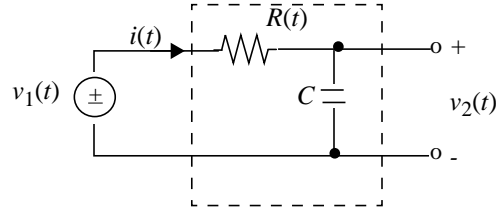
Next we note that

$$H S^{n_0} = (1 + S^1 + S^2) S^{n_0}$$

$$= S^{n_0} + S^{1+n_0} + S^{2+n_0} \quad (3)$$

From Eqs. (2) and (3) we immediately see that Eq. (1) is indeed satisfied. Hence, the system described in Problem 1.65 is time-invariant.

- 1.71 (a) It is indeed possible for a time-variant system to be linear.
 (b) Consider, for example, the resistance-capacitance circuit where the resistive component is time variant, as described here:



This circuit, consisting of the series combination of the resistor $R(t)$ and capacitor C , is time variant because of $R(t)$.

The input of the circuit, $v_1(t)$, is defined in terms of the output $v_2(t)$ by

$$v_1(t) = R(t)C \frac{dv_2(t)}{dt} + v_2(t)$$

Doubling the input $v_1(t)$ results in doubling the output $v_2(t)$. Hence, the property of homogeneity is satisfied.

Moreover, if

$$v_1(t) = \sum_{k=1}^N v_{1,k}(t)$$

then

$$v_2(t) = \sum_{k=1}^N v_{2,k}(t)$$

where

$$v_{1,k}(t) = R(t)C \frac{dv_{2,k}(t)}{dt} + v_{2,k}(t), \quad k = 1, 2, \dots, N$$

Hence, the property of superposition is also satisfied.

We therefore conclude that the time-varying circuit of Fig. P1.71 is indeed linear.

- 1.72 We are given the p th power law device:

$$y(t) = x^p(t) \quad (1)$$

Let $y_1(t)$ and $y_2(t)$ be the outputs of this system produced by the inputs $x_1(t)$ and $x_2(t)$, respectively. Let $x(t) = x_1(t) + x_2(t)$, and let $y(t)$ be the corresponding output. We then note that

$$y(t) = (x_1(t) + x_2(t))^p \neq y_1(t) + y_2(t) \quad \text{for } p \neq 0, 1$$

Hence the system described by Eq. (1) is nonlinear.

- 1.73 Consider a discrete-time system described by the operator H_1 :

$$H_1: \quad y[n] = a_0x[n] + a_kx[n-k]$$

This system is both linear and time invariant. Consider another discrete-time system described by the operator H_2 :

$$H_2: \quad y[n] = b_0x[n] + b_kx[n+k]$$

which is also both linear and time invariant. The system H_1 is causal, but the second system H_2 is noncausal.

- 1.74 The system configuration shown in Fig. 1.56(a) is simpler than the system configuration shown in Fig. 1.56(b). They both involve the same number of multipliers and summer. however, Fig. 1.56(b) requires N replicas of the operator H , whereas Fig. 1.56(a) requires a single operator H for its implementation.

- 1.75 (a) All three systems

- have memory because of an integrating action performed on the input,
- are causal because (in each case) the output does not appear before the input, and
- are time-invariant.

- (b) H_1 is noncausal because the output appears before the input. The input-output relation of H_1 is representative of a differentiating action, which by itself is memoryless. However, the duration of the output is twice as long as that of the input. This suggests that H_1 may consist of a differentiator in parallel with a storage device, followed by a combiner. On this basis, H_1 may be viewed as a time-invariant system with memory.

System H_2 is causal because the output does not appear before the input. The duration of the output is longer than that of the input. This suggests that H_2 must have memory. It is time-invariant.

System H_3 is noncausal because the output appears before the input. Part of the output, extending from $t = -1$ to $t = +1$, is due to a differentiating action performed on the input; this action is memoryless. The rectangular pulse, appearing in the output from $t = +1$ to $t = +3$, may be due to a pulse generator that is triggered by the termination of the input. On this basis, H_3 would have to be viewed as time-varying.

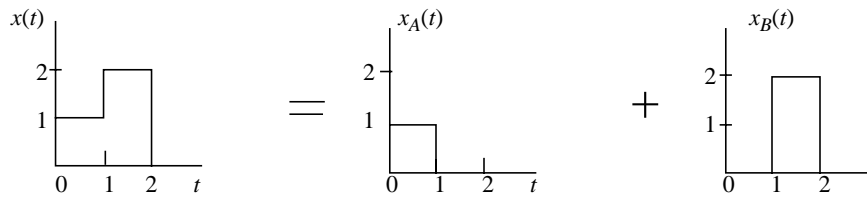
Finally, the output of H_4 is exactly the same as the input, except for an attenuation by a factor of $1/2$. Hence, H_4 is a causal, memoryless, and time-invariant system.

- 1.76 H_1 is representative of an integrator, and therefore has memory. It is causal because the output does not appear before the input. It is time-invariant.

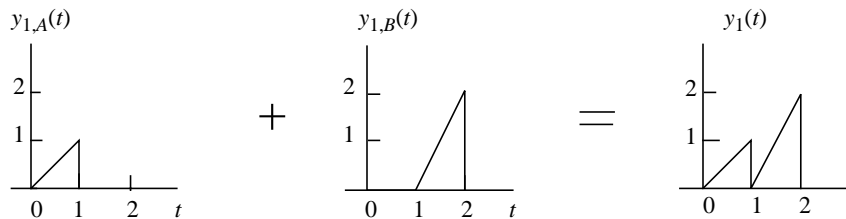
H_2 is noncausal because the output appears at $t = 0$, one time unit before the delayed input at $t = +1$. It has memory because of the integrating action performed on the input. But, how do we explain the constant level of $+1$ at the front end of the output, extending from $t = 0$ to $t = +1$? Since the system is noncausal, and therefore operating in a non real-time fashion, this constant level of duration 1 time unit may be inserted into the output by artificial means. On this basis, H_2 may be viewed as time-varying.

H_3 is causal because the output does not appear before the input. It has memory because of the integrating action performed on the input from $t = 1$ to $t = 2$. The constant level appearing at the back end of the output, from $t = 2$ to $t = 3$, may be explained by the presence of a strong device connected in parallel with the integrator. On this basis, H_3 is time-invariant.

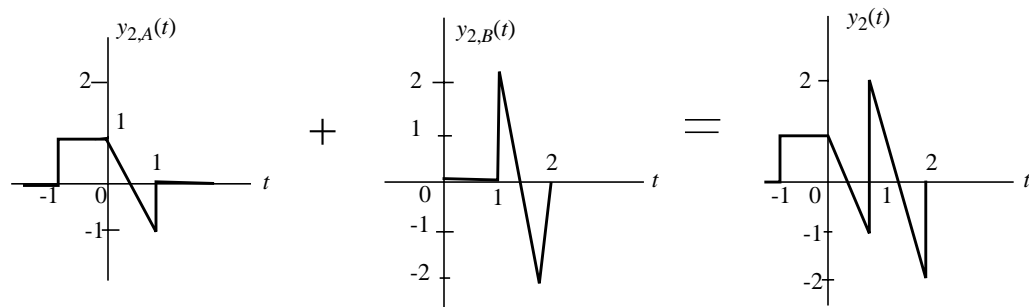
Consider next the input $x(t)$ depicted in Fig. P1.76b. This input may be decomposed into the sum of two rectangular pulses, as shown here:



Response of H_1 to $x(t)$:

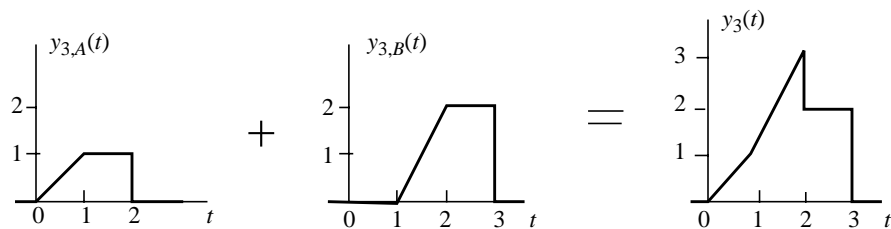


Response of H_2 to $x(t)$:

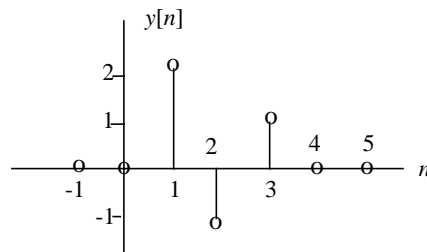


The rectangular pulse of unit amplitude and unit duration at the front end of $y_2(t)$ is inserted in an off-line manner by artificial means

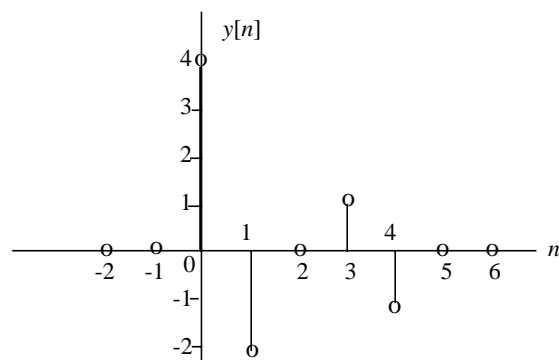
Response of H_3 to $x(t)$:



1.77 (a) The response of the LTI discrete-time system to the input $\delta[n-1]$ is as follows:



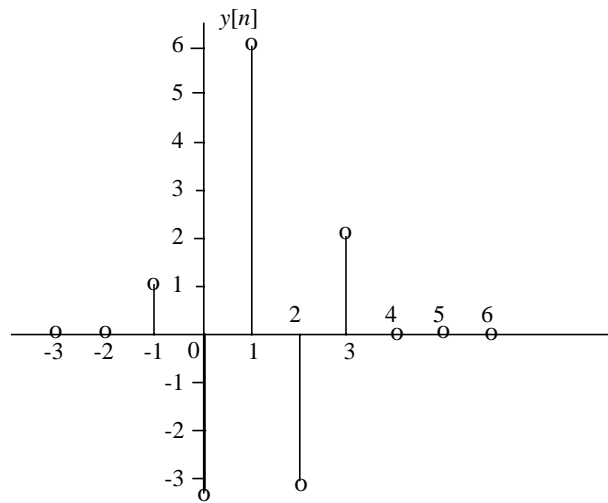
(b) The response of the system to the input $2\delta[n] - \delta[n-2]$ is as follows



- (c) The input given in Fig. P1.77b may be decomposed into the sum of 3 impulse functions: $\delta[n + 1]$, $-\delta[n]$, and $2\delta[n - 1]$. The response of the system to these three components is given in the following table:

Time n	$\delta[n + 1]$	$-\delta[n]$	$2\delta[n - 1]$	Total response
-1	+2			+1
0	-1	-2		-3
1	+1	+1	+4	+6
2		-1	-2	-3
3			+2	2

Thus, the total response $y[n]$ of the system is as shown here:



Advanced Problems

- 1.78 (a) The energy of the signal $x(t)$ is defined by

$$E = \int_{-\infty}^{\infty} x^2(t) dt$$

Substituting

$$x(t) = x_e(t) + x_o(t)$$

into this formula yields

$$\begin{aligned}
 E &= \int_{-\infty}^{\infty} [x_e(t) + x_o(t)]^2 dt \\
 &= \int_{-\infty}^{\infty} [x_e^2(t) + x_o^2(t) + 2x_e(t)x_o(t)] dt \\
 &= \int_{-\infty}^{\infty} x_e^2(t) dt + \int_{-\infty}^{\infty} x_o^2(t) dt + 2 \int_{-\infty}^{\infty} x_e(t)x_o(t) dt
 \end{aligned} \tag{1}$$

With $x_e(t)$ even and $x_o(t)$ odd, it follows that the product $x_e(t)x_o(t)$ is odd, as shown by

$$\begin{aligned} x_e(-t)x_o(-t) &= x_e(t)[-x_o(t)] \\ &= -x_e(t)x_o(t) \end{aligned}$$

Hence,

$$\begin{aligned} \int_{-\infty}^{\infty} (x_e(t)x_o(t))dt &= \int_{-\infty}^0 x_e(t)x_o(t)dt + \int_0^{\infty} x_e(t)x_o(t)dt \\ &= \int_{-\infty}^0 (-x_e(t)x_o(t))(-dt) + \int_0^{\infty} (x_e(t)x_o(t))dt \\ &= -\int_0^{\infty} x_e(t)x_o(t)dt + \int_0^{\infty} x_e(t)x_o(t)dt \\ &= 0 \end{aligned}$$

Accordingly, Eq. (1) reduces to

$$E = \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt$$

(b) For a discrete-time signal $x[n]$, $-\infty \leq n \leq \infty$, we may similarly write

$$\begin{aligned} E &= \sum_{n=-\infty}^{\infty} x^2[n] \\ &= \sum_{n=-\infty}^{\infty} [x_e[n] + x_o[n]]^2 \\ &= \sum_{n=-\infty}^{\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2[n] + 2 \sum_{n=-\infty}^{\infty} x_e[n]x_o[n] \end{aligned} \tag{2}$$

With

$$x_e[-n]x_o[-n] = -x_e[n]x_o[n]$$

it follows that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x_e[n]x_o[n] &= \sum_{n=-\infty}^0 x_e[n]x_o[n] + \sum_{n=-0}^{\infty} x_e[n]x_o[n] \\ &= \sum_{n=-\infty}^0 x_e[-n]x_o[-n] + \sum_{n=-0}^{\infty} x_e[n]x_o[n] \\ &= -\sum_{n=-\infty}^0 x_e[n]x_o[n] + \sum_{n=0}^{\infty} x_e[n]x_o[n] \\ &= 0 \end{aligned}$$

Accordingly, Eq. (2) reduces to

$$E = \sum_{n=-\infty}^{\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2[n]$$

1.79 (a) From Fig. P1.79,

$$i(t) = i_1(t) + i_2(t) \quad (1)$$

$$L \frac{di_1(t)}{dt} + Ri_1(t) = \frac{1}{C} \int_{-\infty}^t i_2(\tau) d\tau \quad (2)$$

Differentiating Eq. (2) with respect to time t :

$$L \frac{d^2 i_1(t)}{dt^2} + R \frac{di_1(t)}{dt} = \frac{1}{C} i_2(t) \quad (3)$$

Eliminating $i_2(t)$ between Eqs. (1) and (2):

$$L \frac{d^2 i_1(t)}{dt^2} + R \frac{di_1(t)}{dt} = \frac{1}{C} [i(t) - i_1(t)]$$

Rearranging terms:

$$\frac{d^2 i_1(t)}{dt^2} + \frac{R}{L} \frac{di_1(t)}{dt} + \frac{1}{LC} i_1(t) = \frac{1}{LC} i(t) \quad (4)$$

(b) Comparing Eqs. (4) with Eq. (1.108) for the MEMS as presented in the text, we may derive the following analogy:

MEMS of Fig. 1.64	LRC circuit of Fig. P1.79
$y(t)$	$i_1(t)$
ω_n	$1/\sqrt{LC}$
Q	$\frac{\omega_n L}{R} = \frac{1}{R} \sqrt{\frac{L}{C}}$
$x(t)$	$\frac{1}{LC} i(t)$

1.80 (a) As the pulse duration Δ approaches zero, the area under the pulse $x_\Delta(t)$ remains equal to unity, and the amplitude of the pulse approaches infinity.

(b) The limiting form of the pulse $x_\Delta(t)$ *violates* the even-function property of the unit impulse:

$$\delta(-t) = \delta(t)$$

1.81 The output $y(t)$ is related to the input $x(t)$ as

$$y(t) = H\{x(t)\} \quad (1)$$

Let T_0 denote the fundamental period of $x(t)$, assumed to be periodic. Then, by definition,

$$x(t) = x(t + T_0) \quad (2)$$

Substituting $t + T_0$ for t into Eq. (1) and then using Eq. (2), we may write

$$\begin{aligned} y(t + T_0) &= H\{x(t + T_0)\} \\ &= H\{x(t)\} \\ &= y(t) \end{aligned} \quad (3)$$

Hence, the output $y(t)$ is also periodic with the same period T_0 .

1.82 (a) For $0 \leq t < \infty$, we have

$$x_\Delta(t) = \frac{1}{\Delta} e^{-t/\tau}$$

At $t = \Delta/2$, we have

$$\begin{aligned} A &= x_\Delta(\Delta/2) \\ &= \frac{1}{\Delta} e^{-\Delta/(2\tau)} \end{aligned}$$

Since $x_\Delta(t)$ is even, then

$$A = x_\Delta(\Delta/2) = x_\Delta(-\Delta/2) = \frac{1}{\Delta} e^{-\Delta/(2\tau)}$$

(b) The area under the pulse $x_\Delta(t)$ must equal unity for

$$\delta(t) = \lim_{\Delta \rightarrow 0} x_\Delta(t)$$

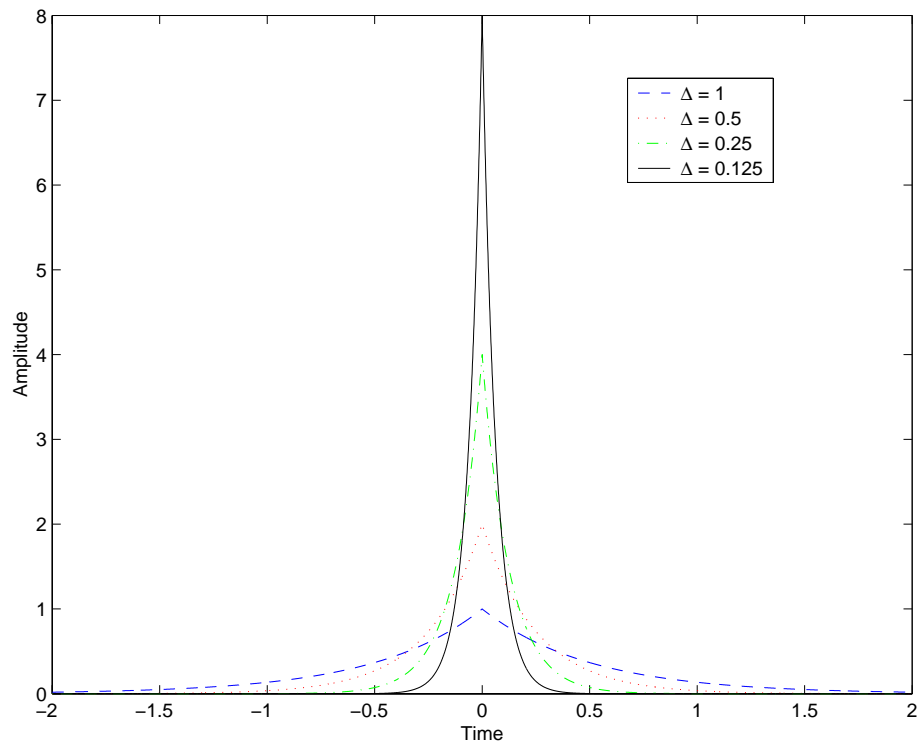
The area under $x_\Delta(t)$ is

$$\begin{aligned}
\int_{-\infty}^{\infty} x_{\Delta}(t) dt &= 2 \int_0^{\infty} x_{\Delta}(t) dt \\
&= 2 \int_0^{\infty} \frac{1}{\Delta} e^{-t/\tau} dt \\
&= \frac{2}{\Delta} (-\tau) e^{-t/\tau} \Big|_0^{\infty} \\
&= \frac{2\tau}{\Delta}
\end{aligned}$$

For this area to equal unity, we require

$$\tau = \frac{\Delta}{2}$$

(c)



1.83 (a) Let the integral of a continuous-time signal $x(t)$, $-\infty < t < \infty$, be defined by

$$\begin{aligned}
y(t) &= \int_{-\infty}^t x(\tau) d\tau \\
&= \int_{-\infty}^0 x(t) dt + \int_0^t x(\tau) d\tau
\end{aligned}$$

The definite integral $\int_{-\infty}^0 x(t)dt$, representing the *initial condition*, is a constant.

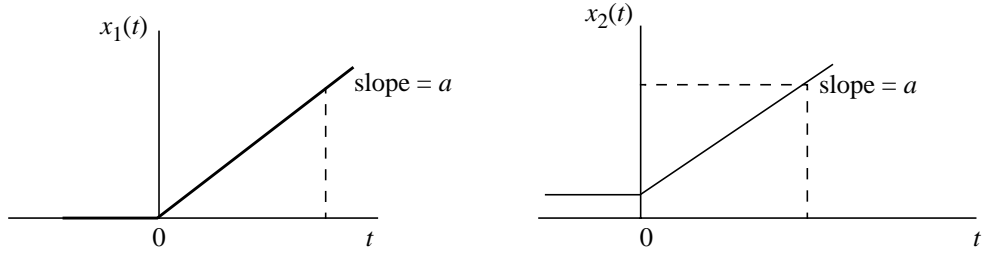
With differentiation as the operation of interest, we may also write

$$x(t) = \frac{dy(t)}{dt}$$

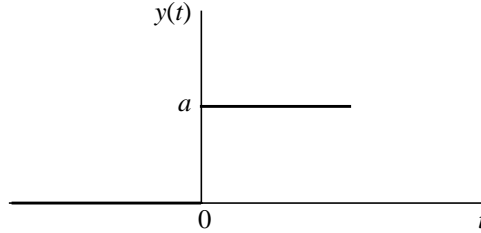
Clearly, the value of $x(t)$ is unaffected by the value assumed by the initial condition

$$\int_{-\infty}^0 x(t)dt$$

It would therefore be wrong to say that differentiation and integration are the inverse of each other. To illustrate the meaning of this statement, consider the two following two waveforms that differ from each other by a constant value for $-\infty < t < \infty$:



Yet, $y(t) = \frac{dx_1(t)}{dt} = \frac{dx_2(t)}{dt}$, as illustrated below:



(b) For Fig. P1.83(a):

$$y(t) + \frac{R}{L} \int_{-\infty}^t y(\tau) d\tau = x(t)$$

For R/L large, we approximately have

$$\frac{R}{L} \int_{-\infty}^t y(\tau) d\tau \approx x(t)$$

Equivalently, we have a differentiator described by

$$y(t) \approx \frac{L}{R} \frac{dx(t)}{dt}, \quad \frac{R}{L} \text{ large}$$

For Fig. P1.83(b):

$$y(t) + \frac{L}{R} \frac{dy(t)}{dt} = x(t)$$

For R/L small, we approximately have

$$\frac{L}{R} \frac{dy(t)}{dt} \approx x(t)$$

Equivalently, we have an integrator described by

$$y(t) \approx \frac{R}{L} \int_{-\infty}^t x(\tau) d\tau \quad \frac{R}{L} \text{ small}$$

(c) Consider the following two scenarios describing the LR circuits of Fig. P1.83

- The input $x(t)$ consists of a voltage source with an average value equal to zero.
- The input $x(t)$ includes a dc component E (exemplified by a battery).

These are two different input conditions. Yet for large R/L , the differentiator of Fig. P1.83(a) produces the same output. On the other hand, for small R/L the integrator of Fig. P1.83(b) produces different outputs. Clearly, on this basis it would be wrong to say that these two LR circuits are the inverse of each other.

1.84 (a) The output $y(t)$ is defined by

$$y(t) = A_0 \cos(\omega_0 t + \phi) x(t) \quad (1)$$

This input-output relation satisfies the following two conditions:

- Homogeneity: If the input $x(t)$ is scaled by an arbitrary factor a , the output $y(t)$ will be scaled by the same factor.
- Superposition: If the input $x(t)$ consists of two additive components $x_1(t)$ and $x_2(t)$, then

$$y(t) = y_1(t) + y_2(t)$$

where

$$y(t) = A_0 \cos(\omega_0 t + \phi) x_k(t), \quad k = 1, 2$$

Hence, the system of Fig. P1.84 is linear.

(b) For the impulse input

$$x(t) = \delta(t),$$

Eq. (1) yields the corresponding output

$$y'(t) = A_0 \cos(\omega_0 t + \phi) \delta(t)$$

$$= \begin{cases} A_0 \cos \phi \delta(0), & t = 0 \\ 0, & \text{otherwise} \end{cases}$$

For $x(t) = \delta(t - t_0)$, Eq. (1) yields

$$\begin{aligned} y''(t) &= A_0 \cos(\omega_0 t + \phi) \delta(t - t_0) \\ &= \begin{cases} A_0 \cos(\omega_0 t_0 + \phi) \delta(0), & t = t_0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Recognizing that $y'(t) \neq y''(t)$, the system of Fig. P1.84 is time-variant.

1.85 (a) The output $y(t)$ is related to the input $x(t)$ as

$$y(t) = \cos\left(2\pi f_c t + k \int_{-\infty}^t x(\tau) d\tau\right) \quad (1)$$

The output is nonlinear as the system violates both the homogeneity and superposition properties:

- Let $x(t)$ be scaled by the factor a . The corresponding value of the output is

$$y_a(t) = \cos\left(2\pi f_c t + ka \int_{-\infty}^t x(\tau) d\tau\right)$$

For $a \neq 1$, we clearly see that $y_a(t) \neq y(t)$.

- Let

$$x(t) = x_1(t) + x_2(t)$$

Then

$$y(t) = \cos\left(2\pi f_c t + k \int_{-\infty}^t x_1(\tau) d\tau + k \int_{-\infty}^t x_2(\tau) d\tau\right)$$

$$\neq y_1(t) + y_2(t)$$

where $y_1(t)$ and $y_2(t)$ are the values of $y(t)$ corresponding to $x_1(t)$ and $x_2(t)$, respectively.

(b) For the impulse input

$$x(t) = \delta(t),$$

Eq. (1) yields

$$y'(t) = \cos\left(2\pi f_c t + k \int_{-\infty}^t \delta(\tau) d\tau\right)$$

$$= \cos k, \quad t = 0^+$$

For the delayed impulse input $x(t) = \delta(t - t_0)$, Eq. (1) yields

$$\begin{aligned} y''(t) &= \cos\left(2\pi f_c t + k \int_{-\infty}^t \delta(\tau - t_0) d\tau\right) \\ &= \cos(2\pi f_c t_0 + k), \quad t = t_0^+ \end{aligned}$$

Recognizing that $y'(t) \neq y''(t)$, it follows that the system is time-variant.

1.86 For the square-law device

$$y(t) = x^2(t),$$

the input

$$x(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)$$

yields the output

$$\begin{aligned} y(t) &= x^2(t) \\ &= [A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)]^2 \\ &= A_1^2 \cos^2(\omega_1 t + \phi_1) + A_2^2 \cos^2(\omega_2 t + \phi_2) \\ &\quad + 2A_1 A_2 \cos(\omega_1 t + \phi_1) \cos(\omega_2 t + \phi_2) \\ &= \frac{A_1^2}{2} [1 + \cos(2\omega_1 t + 2\phi_1)] \\ &\quad + \frac{A_2^2}{2} [1 + \cos(2\omega_2 t + 2\phi_2)] \\ &\quad + A_1 A_2 [\cos((\omega_1 + \omega_2)t + (\phi_1 + \phi_2)) + \cos((\omega_1 - \omega_2)t + (\phi_1 - \phi_2))] \end{aligned}$$

The output $y(t)$ contains the following components:

- DC component of amplitude $(A_1^2 + A_2^2)/2$
- Sinusoidal component of frequency $2\omega_1$, amplitude $A_1^2/2$, and phase $2\phi_1$
- Sinusoidal component of frequency $2\omega_2$, amplitude $A_2^2/2$, and phase $2\phi_2$
- Sinusoidal component of frequency $(\omega_1 - \omega_2)$, amplitude $A_1 A_2$, and phase $(\phi_1 - \phi_2)$
- Sinusoidal component of frequency $(\omega_1 + \omega_2)$, amplitude $A_1 A_2$, and phase $(\phi_1 + \phi_2)$

1.87 The cubic-law device

$$y(t) = x^3(t),$$

in response to the input,

$$x(t) = A \cos(\omega t + \phi),$$

produces the output

$$\begin{aligned} y(t) &= A^3 \cos^3(\omega t + \phi) \\ &= A^3 \cos^3(\omega t + \phi) \cdot \frac{1}{2}(\cos((2\omega t + 2\phi) + 1)) \\ &= \frac{A^3}{2} [\cos((2\omega t + 2\phi) + (\omega t + \phi)) \\ &\quad + \cos((2\omega t + 2\phi) - (\omega t + \phi))] + \frac{A^3}{2} \cos(\omega t + \phi) \\ &= \frac{A^3}{2} [\cos(3\omega t + 3\phi) + \cos(\omega t + \phi)] + \frac{A^3}{2} \cos(\omega t + \phi) \\ &= \frac{A^3}{2} \cos(3\omega t + 3\phi) + A^3 \cos(\omega t + \phi) \end{aligned}$$

The output $y(t)$ consists of two components:

- Sinusoidal component of frequency ω , amplitude A^3 and phase ϕ
- Sinusoidal component of frequency 3ω , amplitude $A^3/2$, and phase 3ϕ

To extract the component with frequency 3ω (i.e., the third harmonic), we need to use a band-pass filter centered on 3ω and a pass-band narrow enough to suppress the fundamental component of frequency ω .

From the analysis presented here we infer that, in order to generate the p th harmonic in response to a sinusoidal component of frequency ω , we require the use of two subsystems:

- Nonlinear device defined by
$$y(t) = x^p(t), \quad p = 2, 3, 4, \dots$$
- Narrowband filter centered on the frequency $p\omega$.

1.88 (a) Following the solution to Example 1.21, we start with the pair of inputs:

$$x_1(t) = \frac{1}{\Delta} u\left(t + \frac{\Delta}{2}\right)$$

$$x_2(t) = \frac{1}{\Delta} u\left(t - \frac{\Delta}{2}\right)$$

The corresponding outputs are respectively given by

$$y_1(t) = \frac{1}{\Delta} \left[1 - e^{-\alpha \left(t + \frac{\Delta}{2} \right)} \cos \omega_n \left(t + \frac{\Delta}{2} \right) \right] u \left(t + \frac{\Delta}{2} \right)$$

$$y_2(t) = \frac{1}{\Delta} \left[1 - e^{-\alpha \left(t - \frac{\Delta}{2} \right)} \cos \omega_n \left(t - \frac{\Delta}{2} \right) \right] u \left(t - \frac{\Delta}{2} \right)$$

The response to the input

$$x_\Delta(t) = x_1(t) - x_2(t)$$

is given by

$$\begin{aligned} y_\Delta(t) = \frac{1}{\Delta} & \left\{ \left[u \left(t + \frac{\Delta}{2} \right) - u \left(t - \frac{\Delta}{2} \right) \right] \right. \\ & - \frac{1}{\Delta} \left\{ \left[e^{-\alpha \left(t + \frac{\Delta}{2} \right)} \right] \cos \left(\omega_n t + \frac{\omega_n \Delta}{2} \right) u \left(t + \frac{\Delta}{2} \right) \right. \\ & \left. \left. - e^{-\alpha \left(t - \frac{\Delta}{2} \right)} \cos \left(\omega_n t - \frac{\omega_n \Delta}{2} \right) u \left(t - \frac{\Delta}{2} \right) \right\} \right\} \end{aligned}$$

As $\Delta \rightarrow 0$, $x_\Delta(t) \rightarrow \delta(t)$. We also note that

$$\frac{d}{dt} z(t) = \lim_{\Delta \rightarrow 0} \left\{ \frac{1}{\Delta} \left[z \left(t + \frac{\Delta}{2} \right) - z \left(t - \frac{\Delta}{2} \right) \right] \right\}$$

Hence, with $z(t) = e^{-\alpha t} \cos(\omega_n t) u(t)$, we find that the impulse response of the system is

$$\begin{aligned} y(t) &= \lim_{\Delta \rightarrow 0} y_\Delta(t) \\ &= \delta(t) - \frac{d}{dt} [e^{-\alpha t} \cos(\omega_n t) u(t)] \\ &= \delta(t) - \frac{d}{dt} [e^{-\alpha t} \cos(\omega_n t)] \cdot u(t) - [e^{-\alpha t} \cos(\omega_n t)] \frac{d}{dt} u(t) \\ &= \delta(t) - [-\alpha e^{-\alpha t} \cos(\omega_n t) - \omega_n e^{-\alpha t} \sin(\omega_n t)] u(t) \\ &\quad - e^{-\alpha t} \cos(\omega_n t) \delta(t) \end{aligned} \tag{1}$$

Since $e^{-\alpha t} \cos(\omega_n t) = 1$ at $t = 0$, Eq. (1) reduces to

$$y(t) = [\alpha e^{-\alpha t} \cos(\omega_n t) + \omega_n e^{-\alpha t} \sin(\omega_n t)] u(t) \quad (2)$$

(b) $\omega_n = j\alpha_n$ where $\alpha_n < \alpha$

Using Euler's formula, we can write

$$\cos(\omega_n t) = \frac{e^{j\omega_n t} + e^{-j\omega_n t}}{2} = \frac{e^{-\alpha_n t} + e^{\alpha_n t}}{2}$$

The step response can therefore be rewritten as

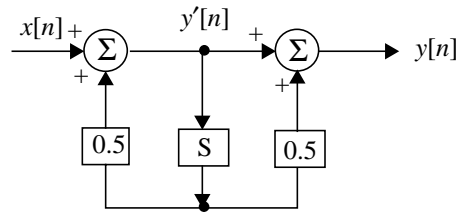
$$y(t) = \left[1 - \frac{1}{2} (e^{-(\alpha + \alpha_n)t} + e^{-(\alpha - \alpha_n)t}) \right] u(t)$$

Again, the impulse response in this case can be obtained as

$$\begin{aligned} h(t) &= \frac{dy(t)}{dt} = \underbrace{\left[1 - \frac{1}{2} (e^{-(\alpha + \alpha_n)t} + e^{-(\alpha - \alpha_n)t}) \right]}_{=0} \delta(t) \\ &\quad + \left[\frac{\alpha + \alpha_n}{2} e^{-(\alpha + \alpha_n)t} + \frac{\alpha - \alpha_n}{2} e^{-(\alpha - \alpha_n)t} \right] u(t) \\ &= \left[\frac{\alpha_2}{2} e^{-\alpha_2 t} + \frac{\alpha_1}{2} e^{-\alpha_1 t} \right] u(t) \end{aligned}$$

where $\alpha_1 = \alpha - \alpha_n$ and $\alpha_2 = \alpha + \alpha_n$.

1.89 Building on the solution described in Fig. 1.69, we may relabel Fig. P1.89 as follows



where (see Eq. (1.117))

$$y'[n] = x[n] + \sum_{k=1}^{\infty} 0.5^k x[n-k]$$

and

$$y[n] = y'[n] + 0.5y'[n-1]$$

$$\begin{aligned}
&= x[n] + \sum_{k=1}^{\infty} 0.5^k x[n-k] + 0.5x[n-1] + \sum_{k=1}^{\infty} 0.5^k x[n-1-k] \\
&= \sum_{k=0}^{\infty} 0.5^k x[n-k] + 0.5 \sum_{k=0}^{\infty} 0.5^k x[n-1-k]
\end{aligned}$$

1.90 According to Eq. (1.108) the MEMS accelerometer is described by the second-order equation

$$\frac{d^2 y(t)}{dt^2} + \frac{\omega_n}{Q} \frac{dy(t)}{dt} + \omega_n^2 y(t) = x(t) \quad (1)$$

Next, we use the approximation (assuming that T_s is sufficiently small)

$$\frac{d}{dt} y(t) \approx \frac{1}{T_s} \left[y\left(t + \frac{T_s}{2}\right) - y\left(t - \frac{T_s}{2}\right) \right] \quad (2)$$

Applying this approximation a second time:

$$\begin{aligned}
\frac{d^2 y(t)}{dt^2} &\approx \frac{1}{T_s} \frac{d}{dt} \left[y\left(t + \frac{T_s}{2}\right) - y\left(t - \frac{T_s}{2}\right) \right] \\
&= \frac{1}{T_s} \frac{d}{dt} y\left(t + \frac{T_s}{2}\right) - \frac{1}{T_s} \frac{d}{dt} y\left(t - \frac{T_s}{2}\right) \\
&\approx \frac{1}{T_s} \left\{ \frac{1}{T_s} [y(t + T_s) - y(t)] \right\} \\
&\quad - \frac{1}{T_s} \left\{ \frac{1}{T_s} [y(t) - y(t - T_s)] \right\} \\
&= \frac{1}{T_s^2} [y(t + T_s) - 2y(t) + y(t - T_s)] \quad (3)
\end{aligned}$$

Substituting Eqs. (2) and (3) into (1):

$$\frac{1}{T_s^2} [y(t + T_s) - 2y(t) + y(t - T_s)] + \frac{\omega_n^2}{QT_s} \left[y\left(t + \frac{T_s}{2}\right) - y\left(t - \frac{T_s}{2}\right) \right] + \omega_n^2 y(t) = x(t) \quad (4)$$

Define

$$\frac{\omega_n^2 T_s}{Q} = a_1,$$

$$\omega_n^2 T_s^2 - 2 = a_2,$$

$$\frac{1}{T_s^2} = b_0,$$

$$t = nT_s/2,$$

$$y[n] = y(nT_s/2),$$

where, in effect, continuous time is normalized with respect to $T_s/2$ to get n .

We may then rewrite Eq. (4) in the form of a noncausal difference equation:

$$y[n+2] + a_1y[n+1] + a_2y[n] - a_1y[n-1] + y[n-2] = b_0x[n] \quad (5)$$

Note: The difference equation (5) is of order 4, providing an approximate description of a second-order continuous-time system. This doubling in order is traced to Eq. (2) as the approximation for a derivative of order 1. We may avoid the need for this order doubling by adopting the alternative approximation:

$$\frac{d}{dt}y(t) \approx \frac{1}{T_s}[y(t+T_s) - y(t)]$$

However, in general, for a given sampling period T_s , this approximation may not be as accurate as that defined in Eq. (2).

1.91 Integration is preferred over differentiation for two reasons:

- (i) Integration tends to attenuate high frequencies. Recognizing that noise contains a broad band of frequencies, integration has a smoothing effect on receiver noise.
- (ii) Differentiation tends to accentuate high frequencies. Correspondingly, differentiation has the opposite effect to integration on receiver noise.

1.92 From Fig. P1.92, we have

$$i(t) = i_1(t) + i_2(t)$$

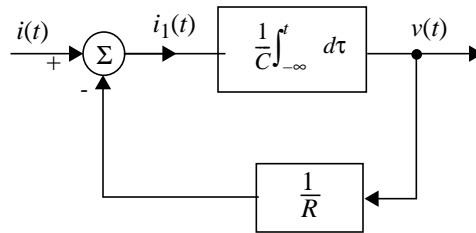
$$v(t) = Ri_2(t) = \frac{1}{C} \int_{-\infty}^t i_1(\tau) d\tau$$

This pair of equations may be rewritten in the equivalent form:

$$i_1(t) = i(t) - \frac{1}{R}v(t)$$

$$v(t) = \frac{1}{C} \int_{-\infty}^t i_1(\tau) d\tau$$

Correspondingly, we may represent the parallel RC circuit of Fig. P1.92 by the block diagram

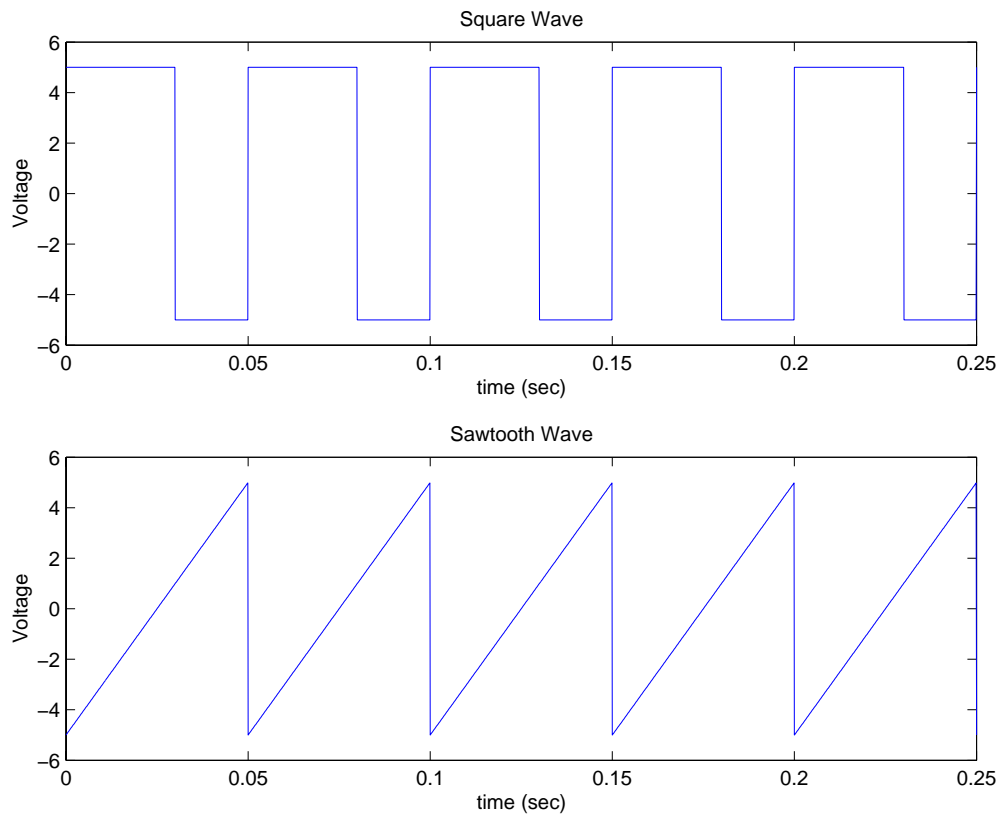


The system described herein is a feedback system with the capacitance C providing the forward path and the conductance $1/R$ providing the feedback path.

%Solution to Matlab Experiment 1.93

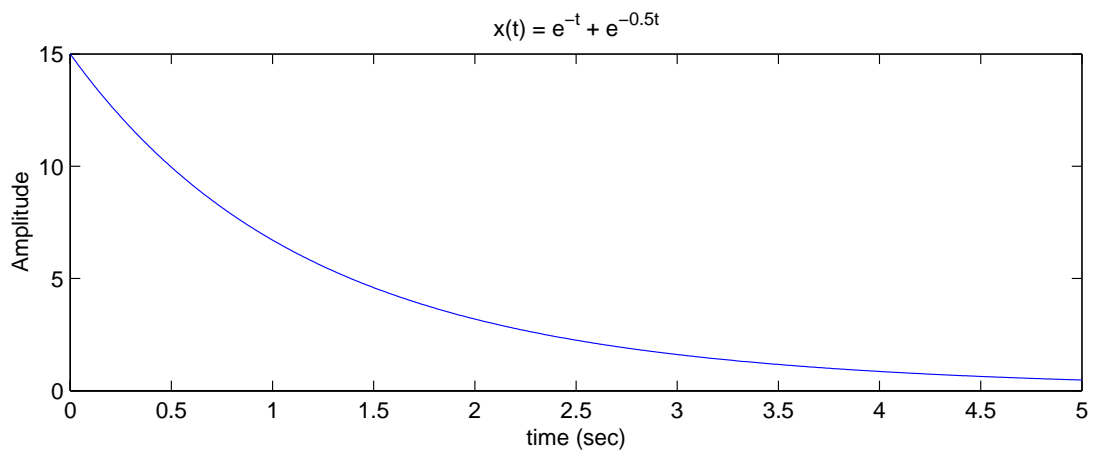
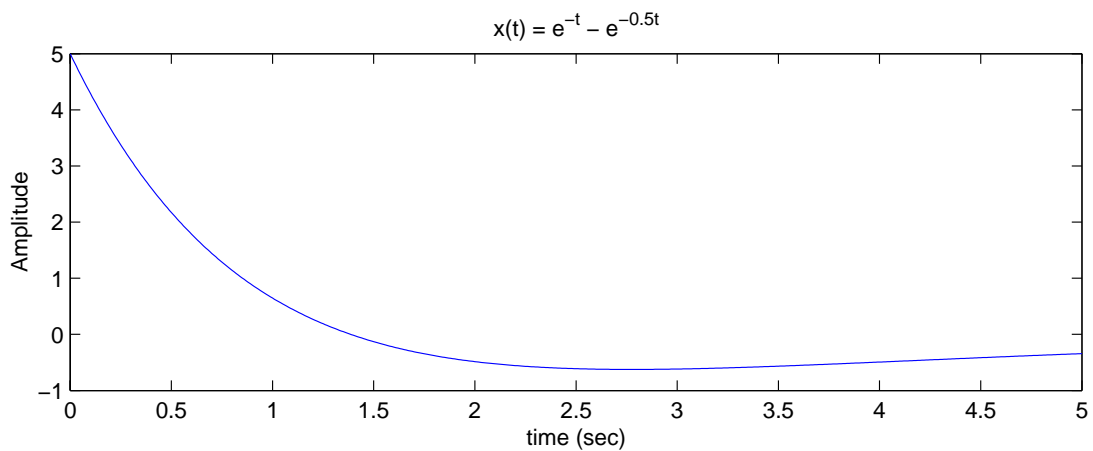
```
f = 20;
k = 0:0.0001:5/20;
amp = 5;
duty = 60;
%Make Square Wave
y1 = amp * square(2*pi*f*k,duty);
%Make Sawtooth Wave
y2 = amp * sawtooth(2*pi*f*k);
%Plot Results
figure(1); clf;
subplot(2,1,1)
plot(k,y1)
xlabel('time (sec)')
ylabel('Voltage')
title('Square Wave')
axis([0 5/20 -6 6])
subplot(2,1,2)
plot(k,y2)
xlabel('time (sec)')
ylabel('Voltage')
title('Sawtooth Wave')
```

axis([0 5/20 -6 6])



% Solution to Matlab Experiment 1.94

```
t = 0:0.01:5;
x1 = 10*exp(-t) - 5*exp(-0.5*t);
x2 = 10*exp(-t) + 5*exp(-0.5*t);
%Plot Figures
figure(1); clf;
subplot(2,1,1)
plot(t,x1)
xlabel('time (sec)')
ylabel('Amplitude')
title('x(t) = e^{-t} - e^{-0.5t}')
subplot(2,1,2);
plot(t,x2)
xlabel('time (sec)')
ylabel('Amplitude')
title('x(t) = e^{-t} + e^{-0.5t}')
```



% Solution to Matlab Experiment 1.95

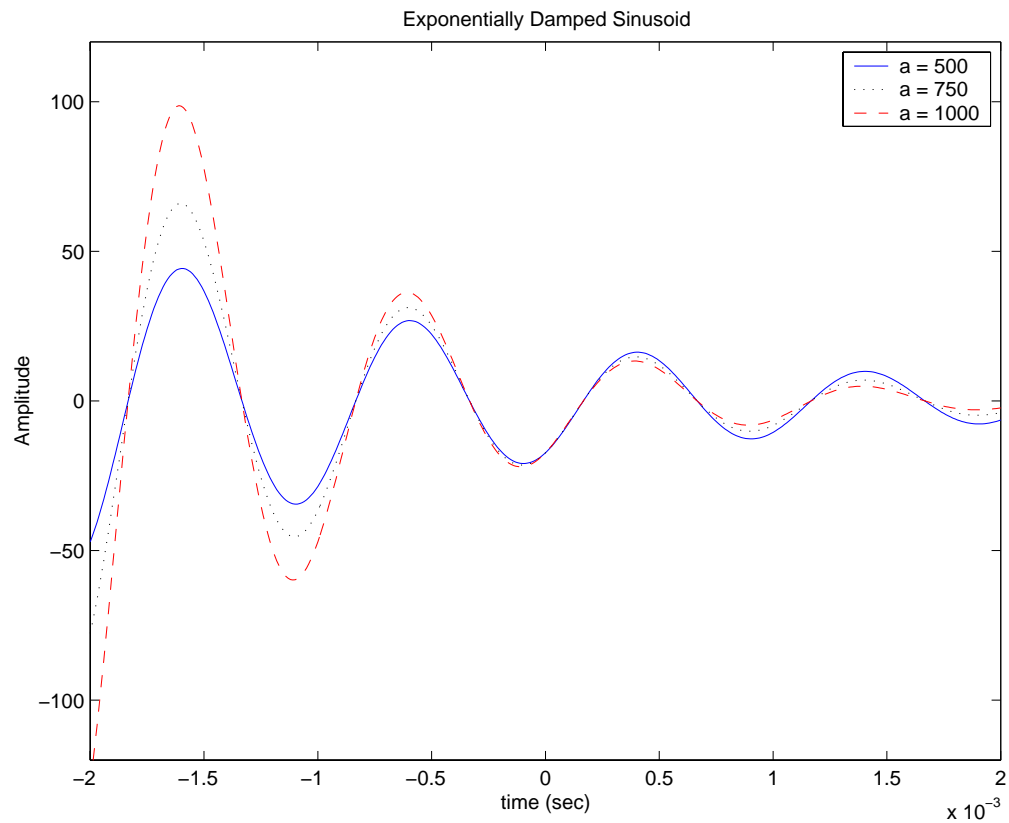
```
t = (-2:0.01:2)/1000;
a1 = 500;
x1 = 20 * sin(2*pi*1000*t - pi/3) .* exp(-a1*t);
a2 = 750;
x2 = 20 * sin(2*pi*1000*t - pi/3) .* exp(-a2*t);
a3 = 1000;
x3 = 20 * sin(2*pi*1000*t - pi/3) .* exp(-a3*t);
%Plot Results
figure(1); clf;
plot(t,x1,'b');
hold on
plot(t,x2,'k:');
plot(t,x3,'r--');
hold off
xlabel('time (sec)')
```



```

ylabel('Amplitude')
title('Exponentially Damped Sinusoid')
axis([-2/1000 2/1000 -120 120])
legend('a = 500', 'a = 750', 'a = 1000')

```

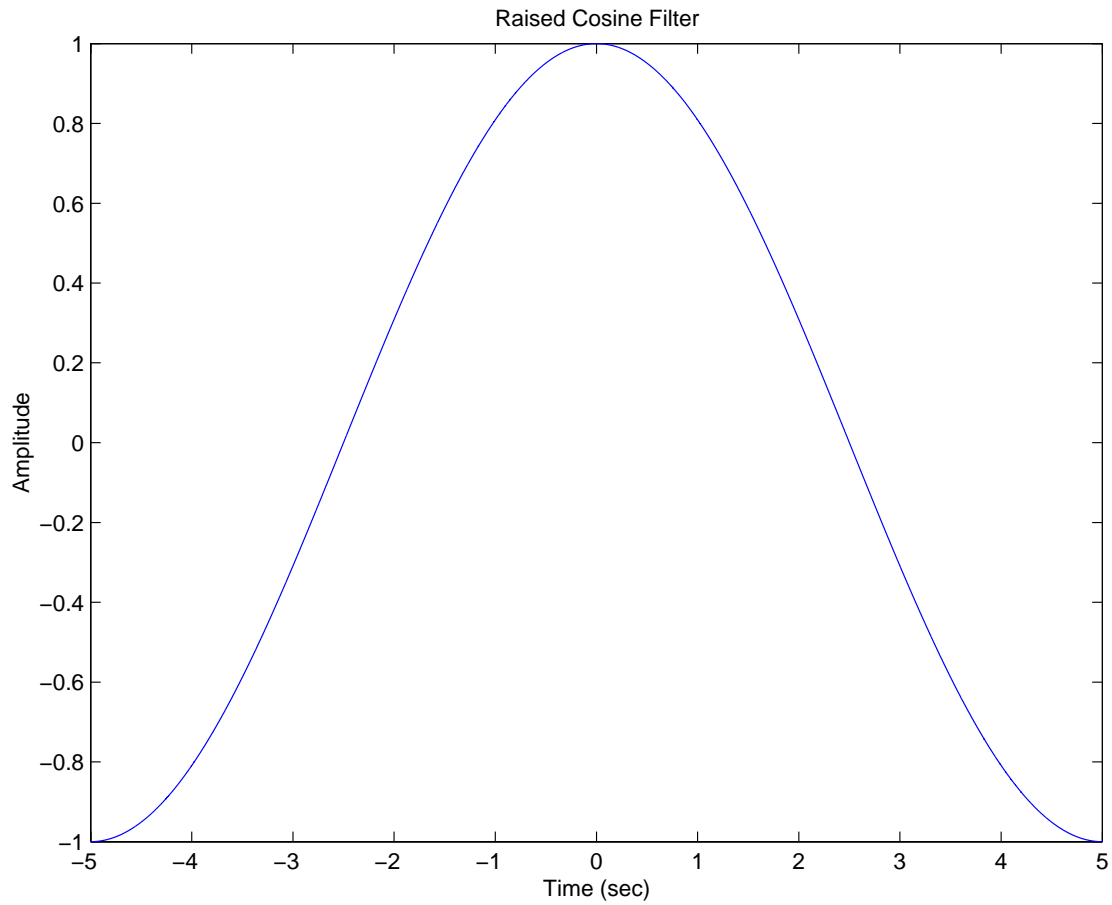


% Solution to Matlab Experiment 1.96

```

F = 0.1;
n = -1/(2*F):0.001:1/(2*F);
w = cos(2*pi*F*n);
%Plot results
figure(1); clf;
plot(n,w)
xlabel('Time (sec)')
ylabel('Amplitude')
title('Raised Cosine Filter')

```



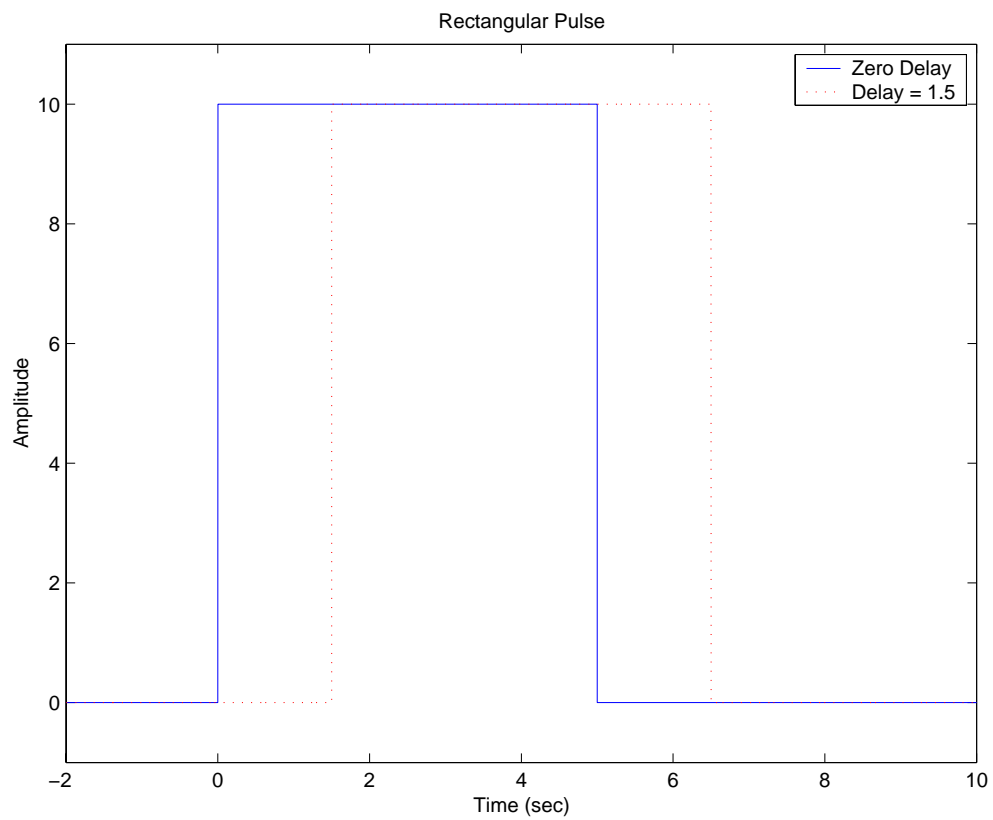
% Solution to Matlab Experiment 1.97

```
t = -2:0.001:10;  
%Generate first step function  
x1 = zeros(size(t));  
x1(t>0)=10;  
x1(t>5)=0;  
%Generate shifted function  
delay = 1.5;  
x2 = zeros(size(t));  
x2(t>(0+delay))=10;  
x2(t>(5+delay))=0;
```

```

%Plot data
figure(1); clf;
plot(t,x1,'b')
hold on
plot(t,x2,'r:')
xlabel('Time (sec)')
ylabel('Amplitude')
title('Rectangular Pulse')
axis([-2 10 -1 11])
legend('Zero Delay', 'Delay = 1.5');

```



Solutions to Additional Problems

2.32. A discrete-time LTI system has the impulse response $h[n]$ depicted in Fig. P2.32 (a). Use linearity and time invariance to determine the system output $y[n]$ if the input $x[n]$ is Use the fact that:

$$\begin{aligned}\delta[n-k] * h[n] &= h[n-k] \\ (ax_1[n] + bx_2[n]) * h[n] &= ax_1[n] * h[n] + bx_2[n] * h[n]\end{aligned}$$

(a) $x[n] = 3\delta[n] - 2\delta[n-1]$

$$\begin{aligned}y[n] &= 3h[n] - 2h[n-1] \\ &= 3\delta[n+1] + 7\delta[n] - 7\delta[n-2] + 5\delta[n-3] - 2\delta[n-4]\end{aligned}$$

(b) $x[n] = u[n+1] - u[n-3]$

$$\begin{aligned}x[n] &= \delta[n] + \delta[n-1] + \delta[n-2] \\ y[n] &= h[n] + h[n-1] + h[n-2] \\ &= \delta[n+1] + 4\delta[n] + 6\delta[n-1] + 4\delta[n-2] + 2\delta[n-3] + \delta[n-5]\end{aligned}$$

(c) $x[n]$ as given in Fig. P2.32 (b)

.

$$\begin{aligned}x[n] &= 2\delta[n-3] + 2\delta[n] - \delta[n+2] \\ y[n] &= 2h[n-3] + 2h[n] - h[n+2] \\ &= -\delta[n+3] - 3\delta[n+2] + 7\delta[n] + 3\delta[n-1] + 8\delta[n-3] + 4\delta[n-4] - 2\delta[n-5] + 2\delta[n-6]\end{aligned}$$

2.33. Evaluate the discrete-time convolution sums given below.

(a) $y[n] = u[n+3] * u[n-3]$

Let $u[n+3] = x[n]$ and $u[n-3] = h[n]$

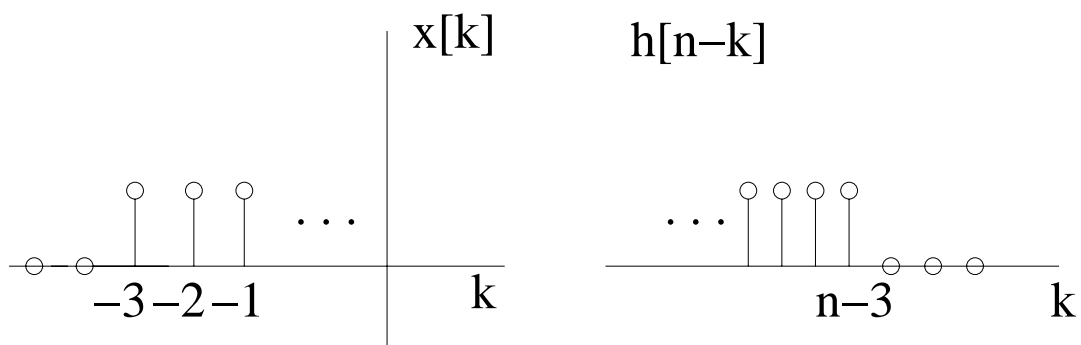


Figure P2.33. (a) Graph of $x[k]$ and $h[n-k]$

$$\text{for } n-3 < -3 \quad n < 0$$

$$y[n] = 0$$

$$\text{for } n-3 \geq -3 \quad n \geq 0$$

$$y[n] = \sum_{k=-3}^{n-3} 1 = n+1$$

$$y[n] = \begin{cases} n+1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(b) $y[n] = 3^n u[-n+3] * u[n-2]$

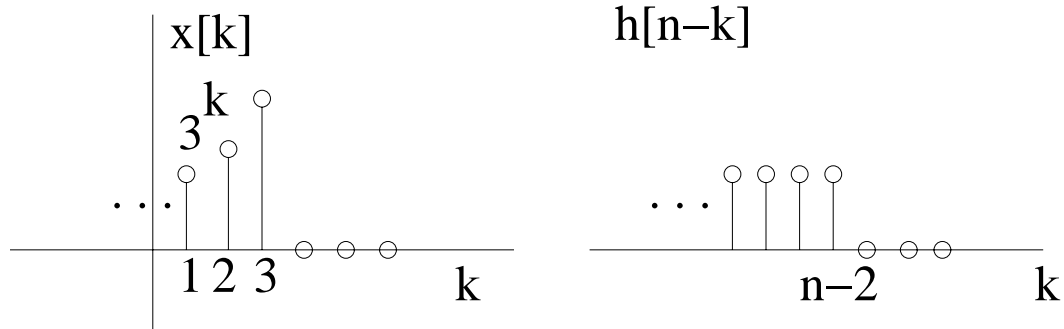


Figure P2.33. (b) Graph of $x[k]$ and $h[n-k]$

$$\text{for } n-2 \leq 3 \quad n \leq 5$$

$$y[n] = \sum_{k=-\infty}^{n-2} 3^k$$

$$y[n] = \frac{1}{6} 3^n$$

$$\text{for } n-2 \geq 4 \quad n \geq 6$$

$$y[n] = \sum_{k=-\infty}^3 3^k$$

$$y[n] = \frac{81}{2}$$

$$y[n] = \begin{cases} \frac{1}{6}3^n & n \leq 5 \\ \frac{81}{2} & n \geq 6 \end{cases}$$

(c) $y[n] = \left(\frac{1}{4}\right)^n u[n] * u[n+2]$

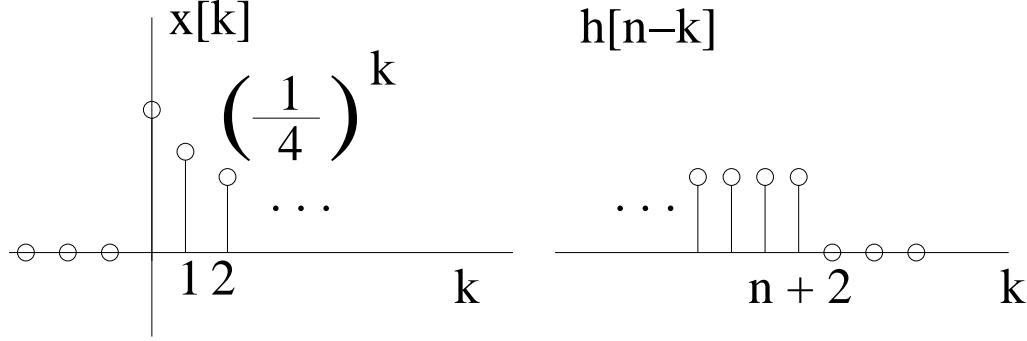


Figure P2.33. (c) Graph of $x[k]$ and $h[n-k]$

for $n+2 < 0$ $n < -2$

$$y[n] = 0$$

for $n+2 \geq 0$ $n \geq -2$

$$y[n] = \sum_{k=0}^{n+2} \left(\frac{1}{4}\right)^k$$

$$y[n] = \frac{4}{3} - \frac{1}{12} \left(\frac{1}{4}\right)^n$$

$$y[n] = \begin{cases} \frac{4}{3} - \frac{1}{12} \left(\frac{1}{4}\right)^n & n \geq -2 \\ 0 & n < -2 \end{cases}$$

(d) $y[n] = \cos\left(\frac{\pi}{2}n\right)u[n] * u[n-1]$

for $n-1 < 0$ $n < 1$

$$y[n] = 0$$

for $n-1 \geq 0$ $n \geq 1$

$$y[n] = \sum_{k=0}^{n-1} \cos\left(\frac{\pi}{2}k\right)$$

$$y[n] = \begin{cases} 1 & n = 4v+1, \quad 4v+2 \\ 0 & n = 4v, \quad 4v+3 \end{cases}$$

$$y[n] = u[n-1]f[n]$$

where

$$f[n] = \begin{cases} 1 & n = 4v+1, \quad 4v+2 \\ 0 & n = 4v, \quad 4v+3 \end{cases}$$

(e) $y[n] = (-1)^n * 2^n u[-n + 2]$

$$\begin{aligned}
y[n] &= \sum_{k=n-2}^{\infty} (-1)^k 2^{n-k} \\
&= 2^n \sum_{k=n-2}^{\infty} \left(-\frac{1}{2}\right)^k \\
&= 2^n \frac{\left(-\frac{1}{2}\right)^{n-2}}{1 - \left(-\frac{1}{2}\right)} \\
&= \frac{8}{3} (-1)^n
\end{aligned}$$

(f) $y[n] = \cos\left(\frac{\pi}{2}n\right) * \left(\frac{1}{2}\right)^n u[n - 2]$

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{n-2} \cos\left(\frac{\pi}{2}k\right) \left(\frac{1}{2}\right)^{n-k} \\
&\quad \text{substituting } p = -k \\
y[n] &= \sum_{p=-(n-2)}^{\infty} \cos\left(\frac{\pi}{2}p\right) \left(\frac{1}{2}\right)^{n+p} \\
y[n] &= \begin{cases} \sum_{p=-(n-2)}^{\infty} (-1)^{\frac{p}{2}} \left(\frac{1}{2}\right)^{n+p} & n \text{ even} \\ \sum_{p=-(n-3)}^{\infty} (-1)^{\frac{p}{2}} \left(\frac{1}{2}\right)^{n+p} & n \text{ odd} \end{cases} \\
y[n] &= \begin{cases} \frac{1}{5} (-1)^n & n \text{ even} \\ \frac{1}{10} (-1)^{n+1} & n \text{ odd} \end{cases}
\end{aligned}$$

(g) $y[n] = \beta^n u[n] * u[n - 3], \quad |\beta| < 1$

$$\begin{aligned}
&\text{for } n - 3 < 0 && n < 3 \\
&&& y[n] = 0 \\
&\text{for } n - 3 \geq 0 && n \geq 3 \\
&&& y[n] = \sum_{k=0}^{n-3} \beta^k \\
&&& y[n] = \left(\frac{1 - \beta^{n-2}}{1 - \beta} \right) \\
y[n] &= \begin{cases} \left(\frac{1 - \beta^{n-2}}{1 - \beta} \right) & n \geq 3 \\ 0 & n < 3 \end{cases}
\end{aligned}$$

(h) $y[n] = \beta^n u[n] * \alpha^n u[n - 10], \quad |\beta| < 1, \quad |\alpha| < 1$

for $n - 10 < 0 \quad n < 10$

$$\begin{aligned}
& y[n] = 0 \\
\text{for } n - 10 \geq 0 & \quad n \geq 10 \\
& y[n] = \sum_{k=0}^{n-10} \left(\frac{\beta}{\alpha}\right)^k \alpha^n \\
y[n] &= \begin{cases} \alpha^n \left(\frac{1 - \left(\frac{\beta}{\alpha}\right)^{n-9}}{1 - \frac{\beta}{\alpha}}\right) & \alpha \neq \beta \\ \alpha(n-9) & \alpha = \beta \end{cases}
\end{aligned}$$

$$(i) \ y[n] = (u[n+10] - 2u[n] + u[n-4]) * u[n-2]$$

$$\begin{aligned}
& \text{for } n - 2 < -10 & n < -8 \\
& & y[n] = 0 \\
& \text{for } n - 2 < 0 & -8 \leq n < 2 \\
& & y[n] = \sum_{k=-10}^{n-2} 1 = n + 9 \\
& \text{for } n - 2 \leq 3 & 2 \leq n \leq 5 \\
& & y[n] = \sum_{k=-10}^{-1} 1 - \sum_{k=0}^{n-2} 1 = 11 - n \\
& \text{for } n - 2 \geq 4 & n \geq 6 \\
& & y[n] = \sum_{k=-10}^{-1} 1 - \sum_{k=0}^3 1 = 6 \\
y[n] &= \begin{cases} 0 & n < -8 \\ n + 9 & -8 \leq n < 2 \\ 11 - n & -2 \leq n \leq 5 \\ 6 & n > 5 \end{cases}
\end{aligned}$$

$$(j) \ y[n] = (u[n+10] - 2u[n] + u[n-4]) * \beta^n u[n], \quad |\beta| < 1$$

$$\begin{aligned}
& \text{for } n < -10 \\
& & y[n] = 0 \\
& \text{for } n < 0 \\
& & y[n] = \beta^n \sum_{k=-10}^n \left(\frac{1}{\beta}\right)^k \\
& & y[n] = \frac{\beta^{n+11} - 1}{\beta - 1} \\
& \text{for } n \leq 3 \\
& & y[n] = \beta^n \sum_{k=-10}^{-1} \left(\frac{1}{\beta}\right)^k - \beta^n \sum_{k=0}^n \left(\frac{1}{\beta}\right)^k \\
& & y[n] = \frac{\beta^{n+11} - \beta^{n+1}}{\beta - 1} - \frac{\beta^{n+1} - 1}{\beta - 1}
\end{aligned}$$

for $n > 3$

$$y[n] = \beta^n \sum_{k=-10}^{-1} \left(\frac{1}{\beta}\right)^k - \beta^n \sum_{k=0}^3 \left(\frac{1}{\beta}\right)^k$$

$$y[n] = \frac{\beta^{n+11} - \beta^{n+1}}{\beta - 1} - \frac{\beta^{n+1} - \beta^{n-3}}{\beta - 1}$$

$$y[n] = \begin{cases} 0 & n < -10 \\ \frac{\beta^{n+11}-1}{\beta-1} & -10 \leq n < 0 \\ \frac{\beta^{n+11}-\beta^{n+1}}{\beta-1} - \frac{\beta^{n+1}-1}{\beta-1} & 0 \leq n \leq 3 \\ \frac{\beta^{n+11}-\beta^{n+1}}{\beta-1} - \frac{\beta^{n+1}-\beta^{n-3}}{\beta-1} & n > 3 \end{cases}$$

(k) $y[n] = (u[n+10] - 2u[n+5] + u[n-6]) * \cos(\frac{\pi}{2}n)$

There are four different cases:

(i) $n = 4v$ v is any integer

$$y[n] = (1)[-1+0+1+0-1] + (-1)[0+1+0-1+0+1+0-1+0+1+0] = -2$$

(ii) $n = 4v + 2$

$$y[n] = (1)[1+0-1+0+1] + (-1)[0-1+0+1+0-1+0+1+0-1+0] = 2$$

(iii) $n = 4v + 3$

$$y[n] = (1)[0-1+0+1+0] + (-1)[-1+0+1+0-1+0+1+0-1+0+1] = 0$$

(iv) $n = 4v + 1$

$$y[n] = \begin{cases} 0 & \\ -2 & n = 4v \\ 2 & n = 4v + 2 \\ 0 & \text{otherwise} \end{cases}$$

(l) $y[n] = u[n] * \sum_{p=0}^{\infty} \delta[n-4p]$

for $n < 0$

$$y[n] = 0$$

for $n \geq 0$ $n = 0, 4, 8, \dots$

$$y[n] = \frac{n}{4} + 1$$

for $n \geq 0$ $n \neq 0, 4, 8, \dots$

$$y[n] = \left\lceil \frac{n}{4} \right\rceil$$

where $\lceil x \rceil$ is the smallest integer larger than x . Ex. $\lceil 3.2 \rceil = 4$

(m) $y[n] = \beta^n u[n] * \sum_{p=0}^{\infty} \delta[n-4p], \quad |\beta| < 1$

for $n < 0$

$$\begin{aligned}
& y[n] = 0 \\
\text{for } n \geq 0 \quad & n = 0, 4, 8, \dots \\
& y[n] = \sum_{k=0}^{\frac{n}{4}} \beta^{4k} \\
& y[n] = \frac{1 - \beta^{4(\frac{n}{4}+1)}}{1 - \beta^4} \\
\text{for } n \geq 0 \quad & n = 1, 5, 9, \dots \\
& y[n] = \sum_{k=0}^{\frac{n-1}{4}} \beta^{4k-1} \\
& y[n] = \frac{1}{\beta} \left(\frac{1 - \beta^{4(\frac{n-1}{4}+1)}}{1 - \beta^4} \right) \\
\text{for } n \geq 0 \quad & n = 2, 6, 10, \dots \\
& y[n] = \sum_{k=0}^{\frac{n-2}{4}} \beta^{4k-2} \\
& y[n] = \frac{1}{\beta^2} \left(\frac{1 - \beta^{4(\frac{n-2}{4}+1)}}{1 - \beta^4} \right) \\
\text{for } n \geq 0 \quad & n = 3, 7, 11, \dots \\
& y[n] = \sum_{k=0}^{\frac{n-3}{4}} \beta^{4k-3} \\
& y[n] = \frac{1}{\beta^3} \left(\frac{1 - \beta^{4(\frac{n-3}{4}+1)}}{1 - \beta^4} \right)
\end{aligned}$$

$$(n) \quad y[n] = \left(\frac{1}{2}\right)^n u[n+2] * \gamma^{|n|}$$

$$\begin{aligned}
\text{for } n+2 \leq 0 \quad & n \leq -2 \\
& y[n] = \sum_{k=-\infty}^{n+2} \left(\frac{1}{2}\right)^{n-k} \gamma^{-k} \\
& y[n] = \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{n+2} \left(\frac{\gamma}{2}\right)^{-k} \\
& \text{let } l = -k \\
& y[n] = \left(\frac{1}{2}\right)^n \sum_{l=-(n+2)}^{\infty} \left(\frac{\gamma}{2}\right)^l \\
& y[n] = \left(\frac{1}{2}\right)^n \frac{\left(\frac{\gamma}{2}\right)^{-(n+2)}}{1 - \frac{\gamma}{2}} \\
& y[n] = \left(\frac{2}{\gamma}\right)^2 \frac{\left(\frac{1}{\gamma}\right)^n}{1 - \frac{\gamma}{2}} \\
\text{for } n+2 \geq 0 \quad & n > -2
\end{aligned}$$

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^0 \left(\frac{1}{2}\right)^{n-k} \gamma^{-k} + \sum_{k=1}^{n+2} \left(\frac{1}{2}\right)^{n-k} \gamma^k \\
y[n] &= \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^0 \left(\frac{\gamma}{2}\right)^{-k} + \left(\frac{1}{2}\right)^n \sum_{k=1}^{n+2} (2\gamma)^k \\
y[n] &= \left(\frac{1}{2}\right)^n \left[\frac{1}{1-\frac{\gamma}{2}} + \left(\frac{1-(2\gamma)^{n+3}}{1-2\gamma} - 1 \right) \right]
\end{aligned}$$

2.34. Consider the discrete-time signals depicted in Fig. P2.34. Evaluate the convolution sums indicated below.

(a) $m[n] = x[n] * z[n]$

$$\begin{aligned}
&\text{for } n+5 < 0 && n < -5 \\
&&& m[n] = 0 \\
&\text{for } n+5 < 4 && -5 \leq n < -1 \\
&&& m[n] = \sum_{k=0}^{n+5} 1 = n+6 \\
&\text{for } n-1 < 1 && -1 \leq n < 2 \\
&&& m[n] = \sum_{k=0}^3 1 + 2 \sum_{k=4}^{n+5} 1 = 2n+8 \\
&\text{for } n+5 < 9 && 2 \leq n < 4 \\
&&& m[n] = \sum_{k=n-1}^3 1 + 2 \sum_{k=4}^{n+5} 1 = 9+n \\
&\text{for } n-1 < 4 && 4 \leq n < 5 \\
&&& m[n] = \sum_{k=n-1}^3 1 + 2 \sum_{k=4}^8 1 = 15-n \\
&\text{for } n-1 < 9 && 5 \leq n < 10 \\
&&& m[n] = 2 \sum_{k=n-1}^8 1 = 20-2n \\
&\text{for } n-1 \geq 9 && n \geq 10 \\
&&& m[n] = 0
\end{aligned}$$

$$m[n] = \begin{cases} 0 & n < -5 \\ n+6 & -5 \leq n < -1 \\ 2n+8 & -1 \leq n < 2 \\ 9+n & 2 \leq n < 4 \\ 15-n & 4 \leq n < 5 \\ 20-2n & 5 \leq n < 10 \\ 0 & n \geq 10 \end{cases}$$

(b) $m[n] = x[n] * y[n]$

for $n+5 < -3$ $n < -8$

$$\begin{aligned}
& m[n] = 0 \\
\text{for } n+5 < 1 & \quad -8 \leq n < -4 \\
& m[n] = \sum_{k=-3}^{n+5} 1 = n+9 \\
\text{for } n-1 < -2 & \quad -4 \leq n < -1 \\
& m[n] = \sum_{k=-3}^0 1 - \sum_{k=1}^{n+5} 1 = -n-1 \\
\text{for } n+5 < 5 & \quad -1 \leq n < 0 \\
& m[n] = \sum_{k=n-1}^0 1 - \sum_{k=1}^{n+5} 1 = -2n-4 \\
\text{for } n-1 < 1 & \quad 0 \leq n < 2 \\
& m[n] = \sum_{k=n-1}^0 1 - \sum_{k=1}^4 1 = -n-2 \\
\text{for } n-1 < 5 & \quad 2 \leq n < 6 \\
& m[n] = - \sum_{k=n-1}^4 1 = n-6 \\
\text{for } n-1 \geq 5 & \quad n \geq 6 \\
& m[n] = 0
\end{aligned}$$

$$m[n] = \begin{cases} 0 & n < -8 \\ n+3 & -8 \leq n < -4 \\ -n-1 & -4 \leq n < -1 \\ -2n-4 & -1 \leq n < 0 \\ -n-2 & 0 \leq n < 2 \\ n-6 & 2 \leq n < 6 \\ 0 & n \geq 6 \end{cases}$$

(c) $m[n] = x[n] * f[n]$

$$\begin{aligned}
\text{for } n+5 < -5 & \quad n < -10 \\
& m[n] = 0 \\
\text{for } n-1 < -5 & \quad -10 \leq n < -4 \\
& m[n] = \frac{1}{2} \sum_{k=-5}^{n+5} k = -5n - 55 + \frac{1}{2}(n+10)(n+11) \\
\text{for } n+5 < 6 & \quad -4 \leq n < 1 \\
& m[n] = \frac{1}{2} \sum_{k=n-1}^{n+5} k = \frac{7}{2}(n-1) + \frac{21}{2} \\
\text{for } n-1 < 6 & \quad 1 \leq n < 7 \\
& m[n] = \frac{1}{2} \sum_{k=n-1}^5 k = \frac{1}{2}(7-n) \left[(n-1) + \frac{1}{2}(6-n) \right] \\
\text{for } n-1 \geq 6 & \quad n \geq 7
\end{aligned}$$

$$m[n] = 0$$

$$m[n] = \begin{cases} 0 & n < -10 \\ -5n - 55 + \frac{1}{2}(n+10)(n+11) & -10 \leq n < -4 \\ \frac{7}{2}(n-1) + \frac{21}{2} & -4 \leq n < 1 \\ \frac{1}{2}(7-n) \left[(n-1) + \frac{1}{2}(6-n) \right] & 1 \leq n < 7 \\ 0 & n \geq 7 \end{cases}$$

(d) $m[n] = x[n] * g[n]$

$$\begin{aligned} \text{for } n+5 < -8 & \quad n < -13 \\ & \quad m[n] = 0 \\ \text{for } n-1 < -7 & \quad -14 \leq n < -6 \\ & \quad m[n] = \sum_{k=-8}^{n+5} 1 = n+14 \\ \text{for } n+5 < 4 & \quad -6 \leq n < -1 \\ & \quad m[n] = \sum_{k=n-1}^{-2} 1 = -n \\ \text{for } n-1 < -1 & \quad -1 \leq n < 0 \\ & \quad m[n] = \sum_{k=n-1}^{-2} 1 + \sum_{k=4}^{n+5} 1 = -2 \\ \text{for } n-1 < 4 & \quad 0 \leq n < 5 \\ & \quad m[n] = \sum_{k=4}^{n+5} 1 = n+2 \\ \text{for } n-1 < 11 & \quad 5 \leq n < 12 \\ & \quad m[n] = \sum_{k=n-1}^{10} 1 = 12-n \\ \text{for } n-1 \geq 11 & \quad n \geq 12 \\ & \quad m[n] = 0 \end{aligned}$$

$$m[n] = \begin{cases} 0 & n < -13 \\ n+14 & -13 \leq n < -6 \\ -n & -6 \leq n < -1 \\ -2 & -1 \leq n < 0 \\ n+2 & 0 \leq n < 5 \\ 12-n & 5 \leq n < 12 \\ 0 & n \geq 12 \end{cases}$$

(e) $m[n] = y[n] * z[n]$

The remaining problems will not show all of the steps of convolution, instead figures and intervals will be given for the solution.

Intervals

$n < -3$
 $-3 \leq n < 1$
 $1 \leq n < 5$
 $5 \leq n < 6$
 $6 \leq n < 9$
 $9 \leq n < 13$
 $n \geq 13$

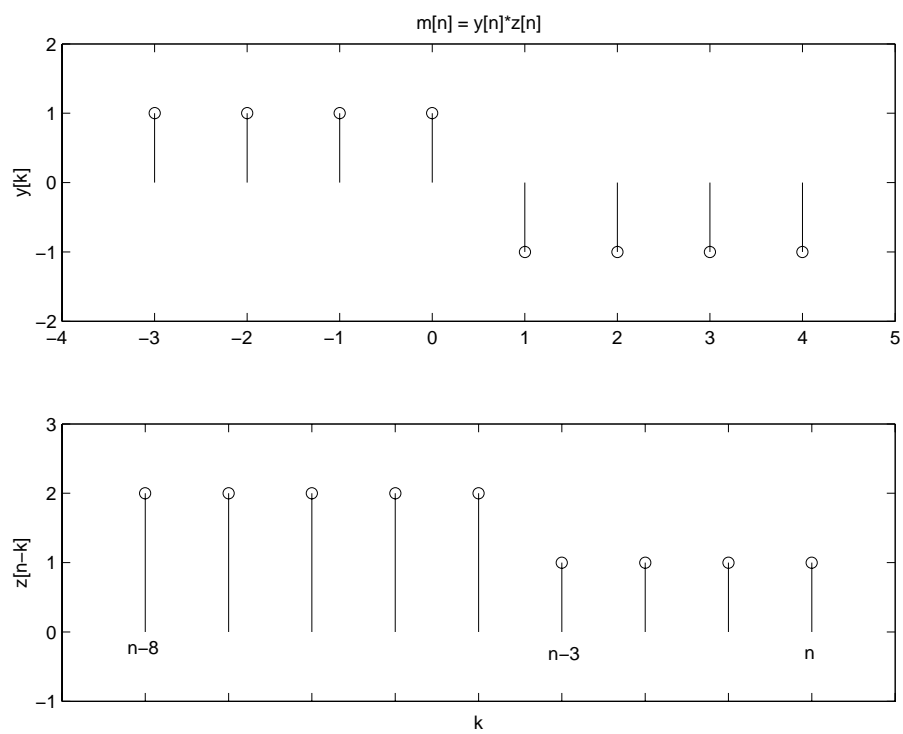


Figure P2.34. Figures of $y[n]$ and $z[n - k]$

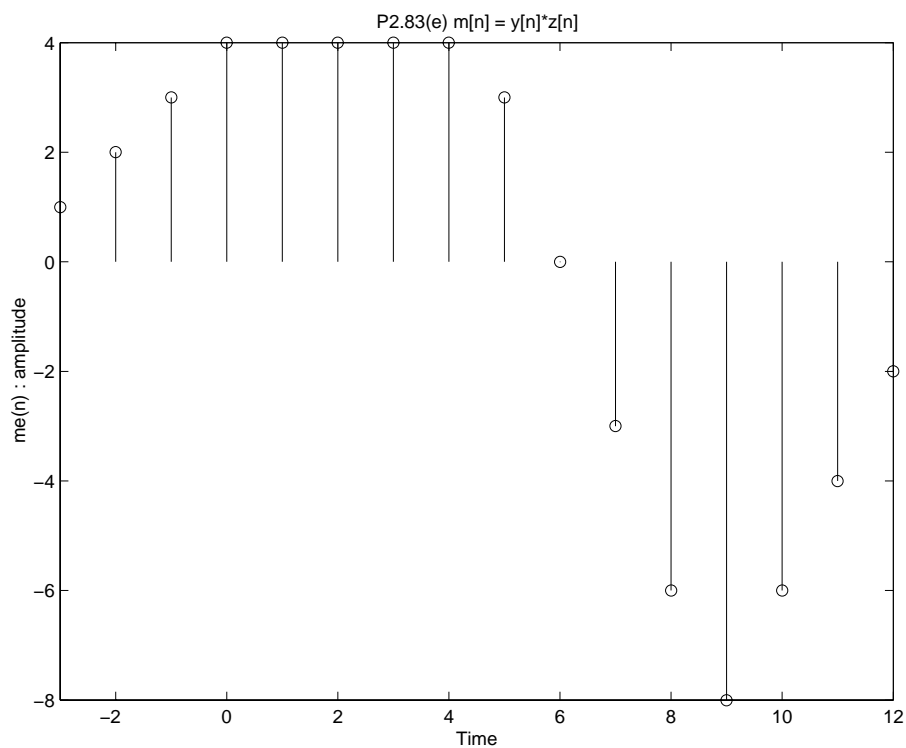


Figure P2.34. $m[n] = y[n] * z[n]$

(f) $m[n] = y[n] * g[n]$

Intervals

$$n < -11$$

$$-11 \leq n < -7$$

$$-7 \leq n \leq -5$$

$$-4 \leq n < -3$$

$$-3 \leq n < -1$$

$$-1 \leq n < 1$$

$$1 \leq n < 3$$

$$3 \leq n < 5$$

$$5 \leq n < 7$$

$$7 \leq n < 9$$

$$9 \leq n < 11$$

$$11 \leq n < 15$$

$$n \geq 15$$

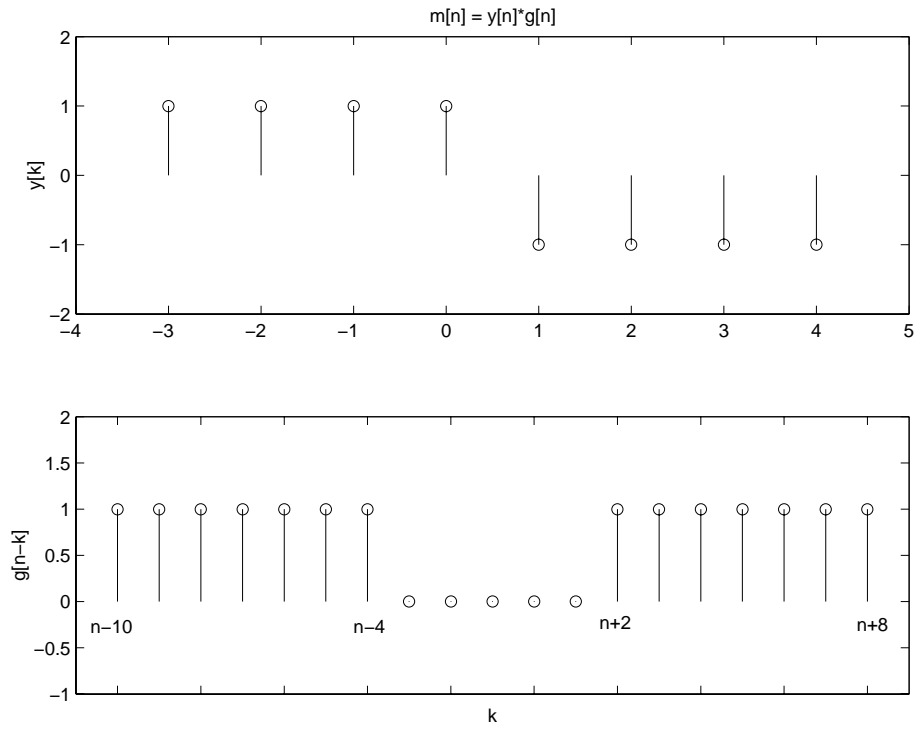


Figure P2.34. Figures of $y[n]$ and $g[n - k]$

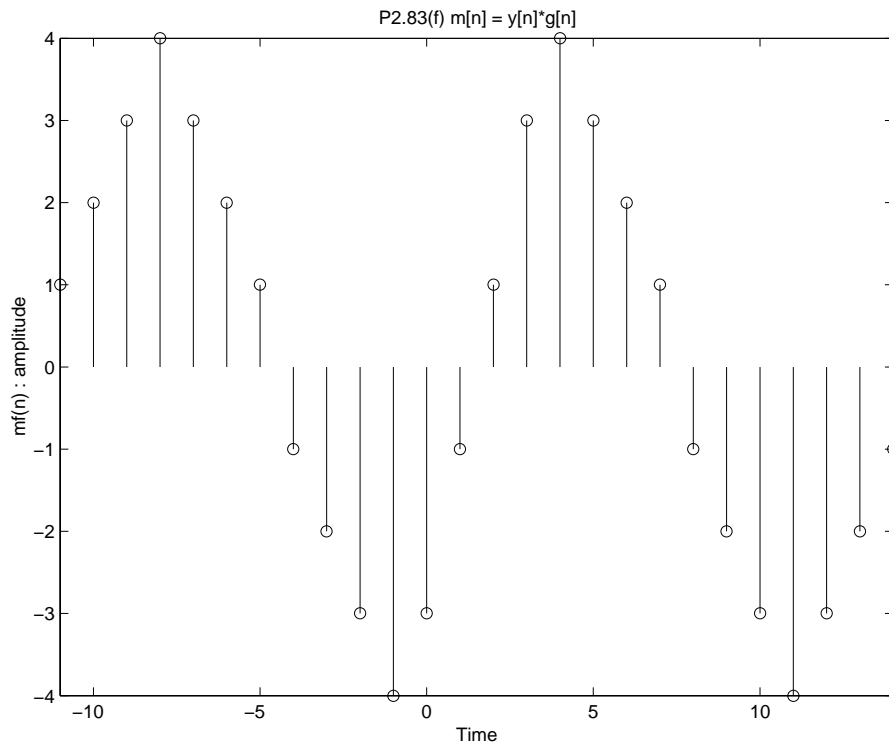


Figure P2.34. $m[n] = y[n] * g[n]$

(g) $m[n] = y[n] * w[n]$

Intervals

$$n < -7$$

$$-7 \leq n < -3$$

$$-3 \leq n < -2$$

$$-2 \leq n < 1$$

$$1 \leq n < 2$$

$$2 \leq n < 5$$

$$5 \leq n < 9$$

$$n \geq 9$$

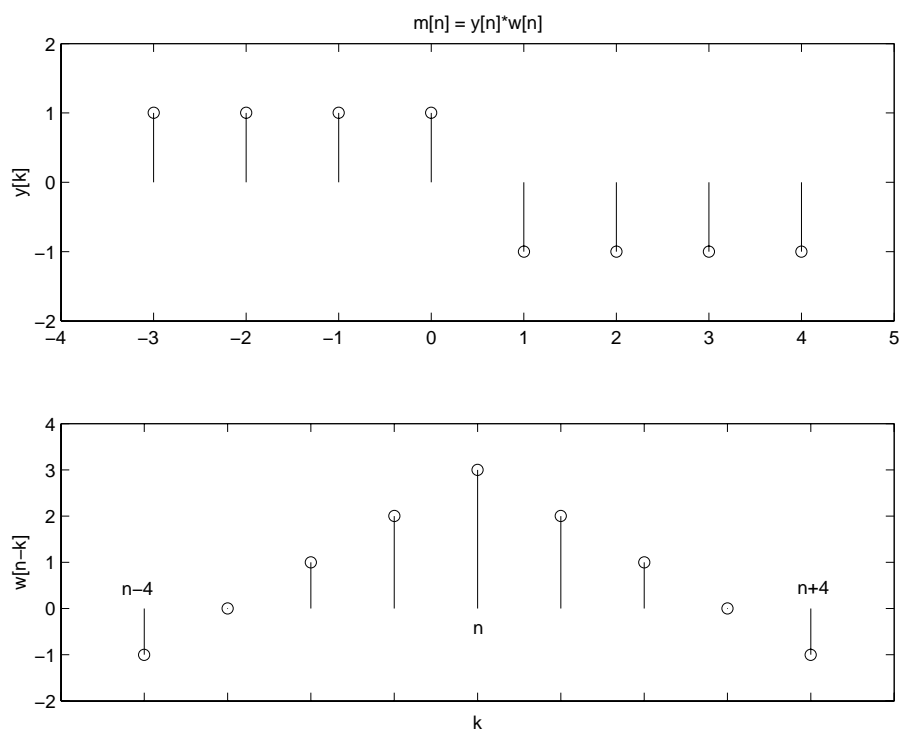


Figure P2.34. Figures of $y[n]$ and $w[n - k]$

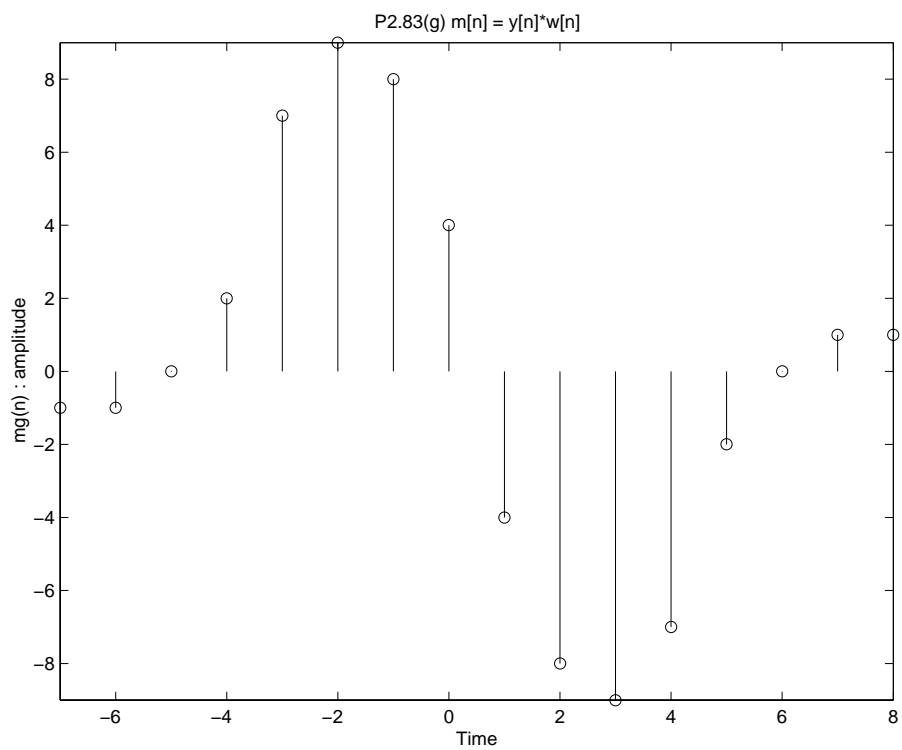


Figure P2.34. $m[n] = y[n] * w[n]$

(h) $m[n] = y[n] * f[n]$

Intervals

$$n < -8$$

$$-8 \leq n < -4$$

$$-4 \leq n < 0$$

$$0 \leq n < 2$$

$$2 \leq n < 6$$

$$6 \leq n < 10$$

$$n \geq 10$$

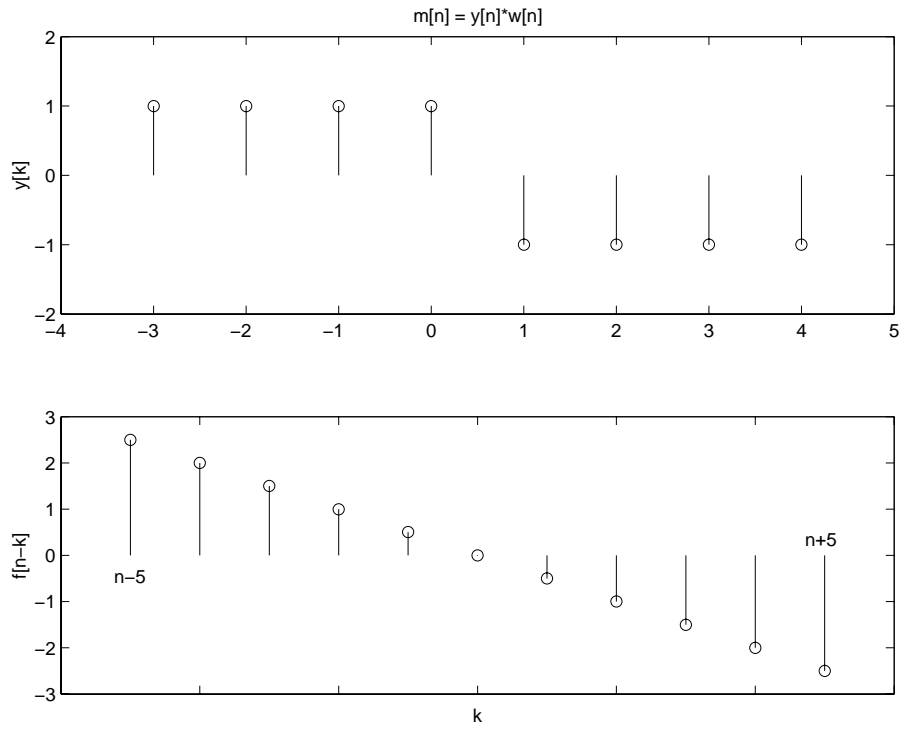


Figure P2.34. Figures of $y[n]$ and $f[n - k]$

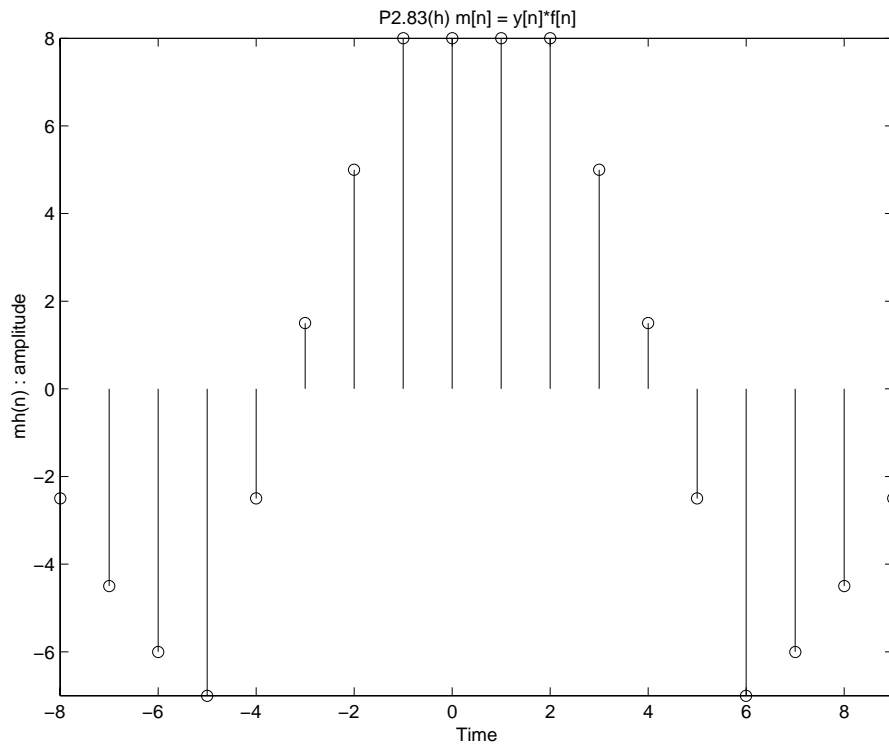


Figure P2.34. $m[n] = y[n] * f[n]$

(i) $m[n] = z[n] * g[n]$

Intervals

$n < -8$
 $-8 \leq n < -4$
 $-4 \leq n < -1$
 $-1 \leq n < 1$
 $1 \leq n < 2$
 $2 \leq n < 4$
 $4 \leq n < 7$
 $7 \leq n < 8$
 $8 \leq n < 11$
 $11 \leq n < 13$
 $13 \leq n < 14$
 $14 \leq n < 19$
 $n \geq 19$

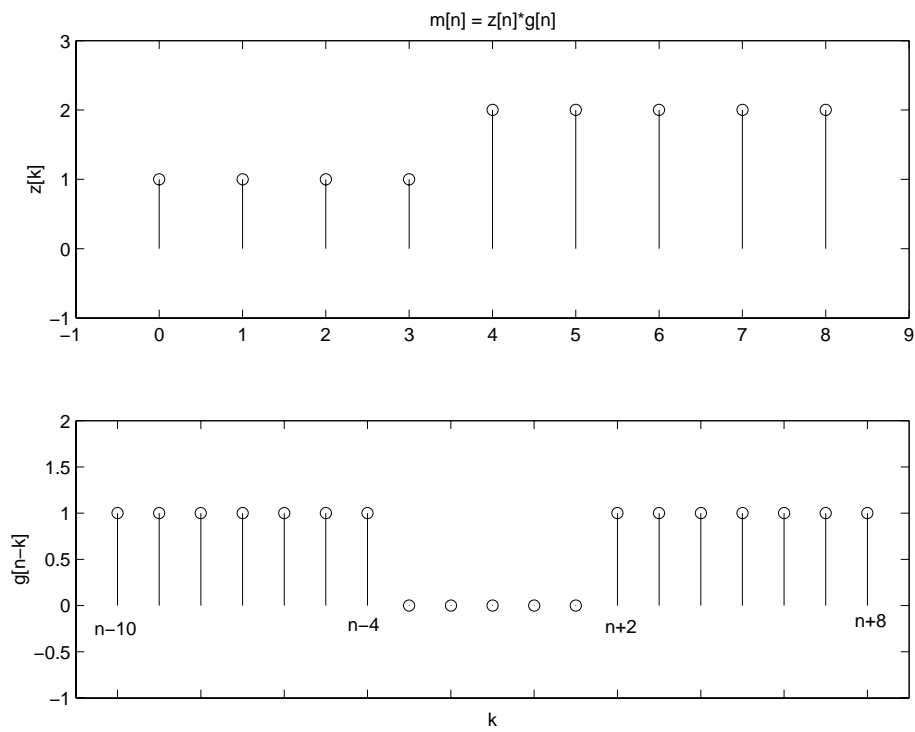


Figure P2.34. Figures of $z[n]$ and $g[n - k]$

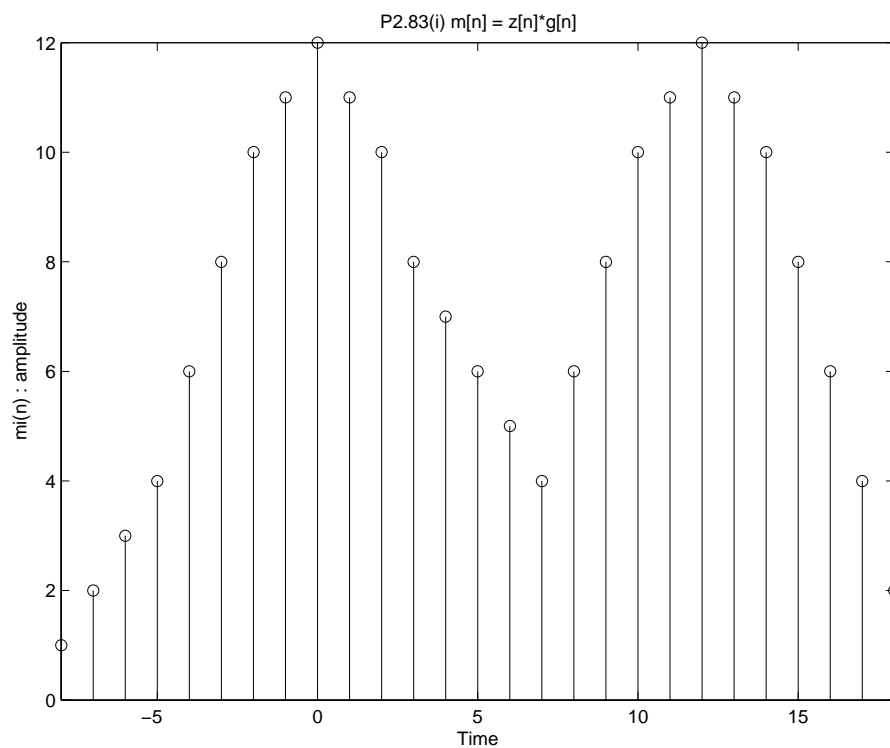


Figure P2.34. $m[n] = z[n] * g[n]$

(j) $m[n] = w[n] * g[n]$

Intervals

$$n < -12$$

$$-12 \leq n < -7$$

$$-7 \leq n < -6$$

$$-6 \leq n < -3$$

$$-3 \leq n < -1$$

$$-1 \leq n < 0$$

$$0 \leq n < 3$$

$$3 \leq n < 5$$

$$5 \leq n < 7$$

$$7 \leq n < 9$$

$$9 \leq n < 11$$

$$11 \leq n < 15$$

$$n \geq 15$$

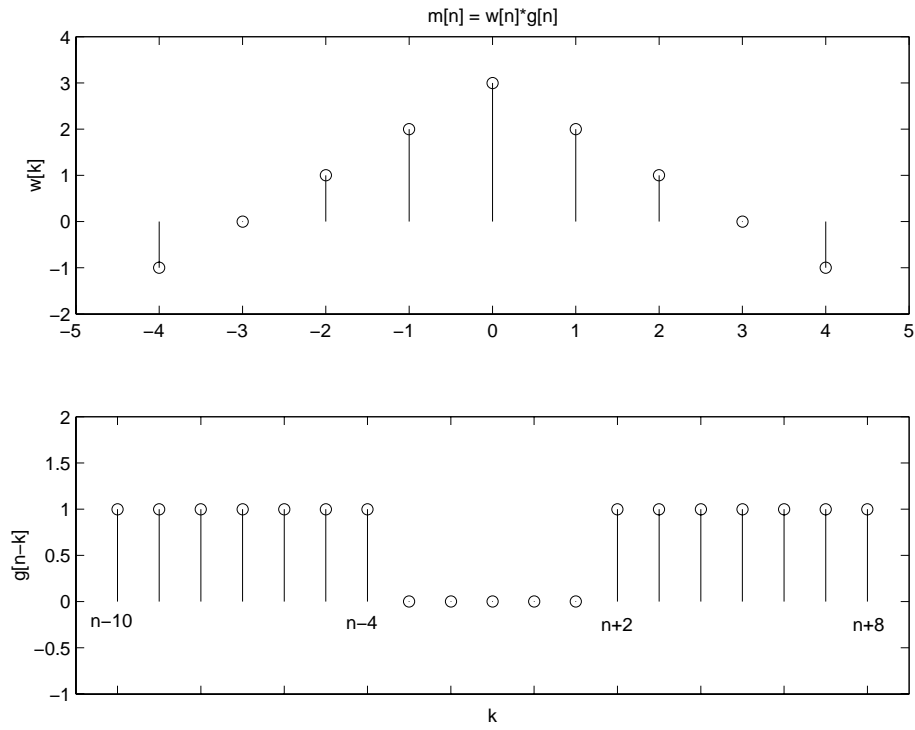


Figure P2.34. Figures of $w[n]$ and $g[n - k]$

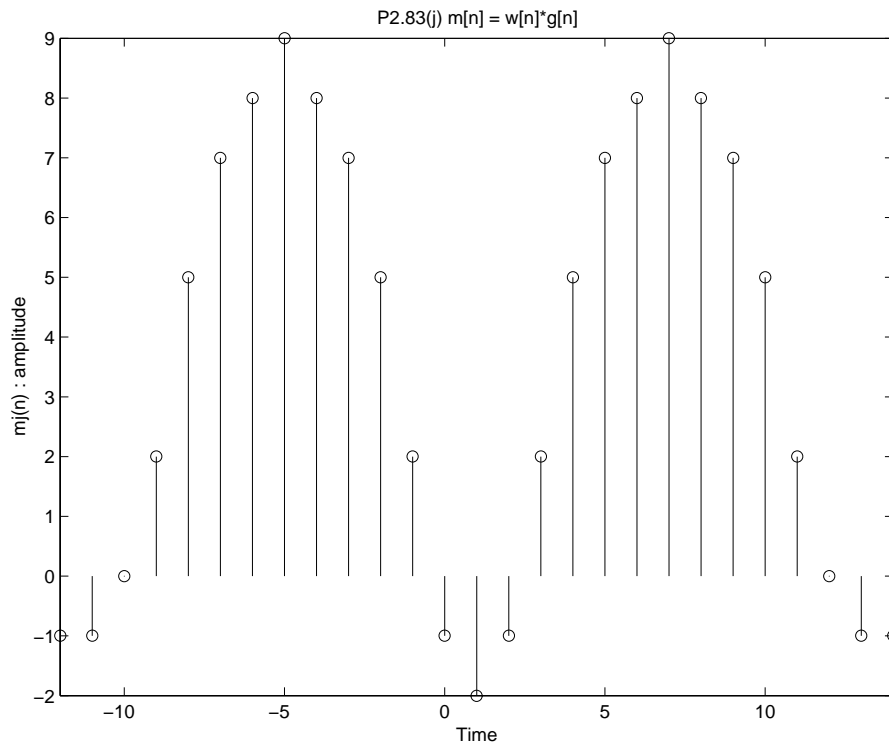


Figure P2.34. $m[n] = w[n] * g[n]$

(k) $m[n] = f[n] * g[n]$

Intervals

$$n < -13$$

$$-13 \leq n < -7$$

$$-7 \leq n < -2$$

$$-2 \leq n < -1$$

$$-1 \leq n < 4$$

$$4 \leq n < 5$$

$$5 \leq n < 10$$

$$10 \leq n < 16$$

$$n \geq 16$$

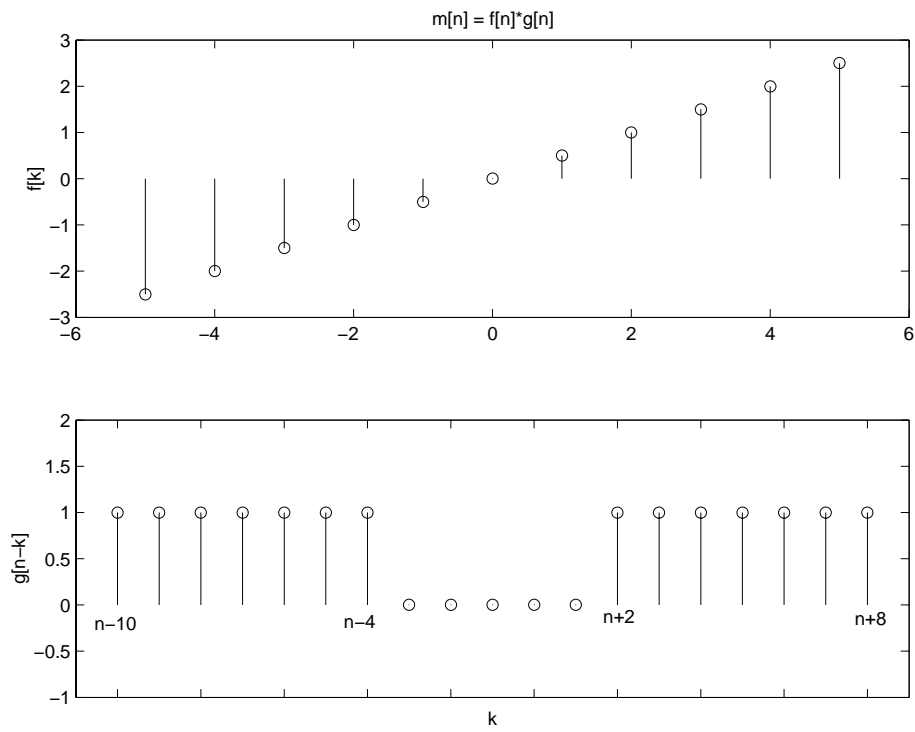


Figure P2.34. Figures of $f[n]$ and $g[n - k]$

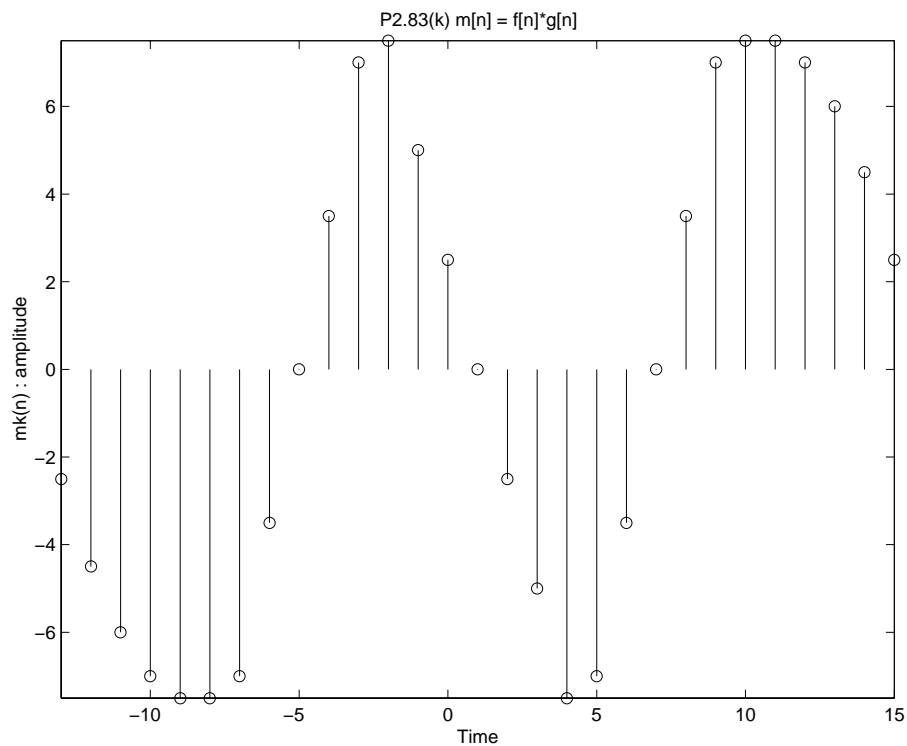


Figure P2.34. $m[n] = f[n] * g[n]$

2.35. At the start of the first year \$10,000 is deposited in a bank account earning 5% per year. At the start of each succeeding year \$1000 is deposited. Use convolution to determine the balance at the start of each year (after the deposit). Initially \$10000 is invested.

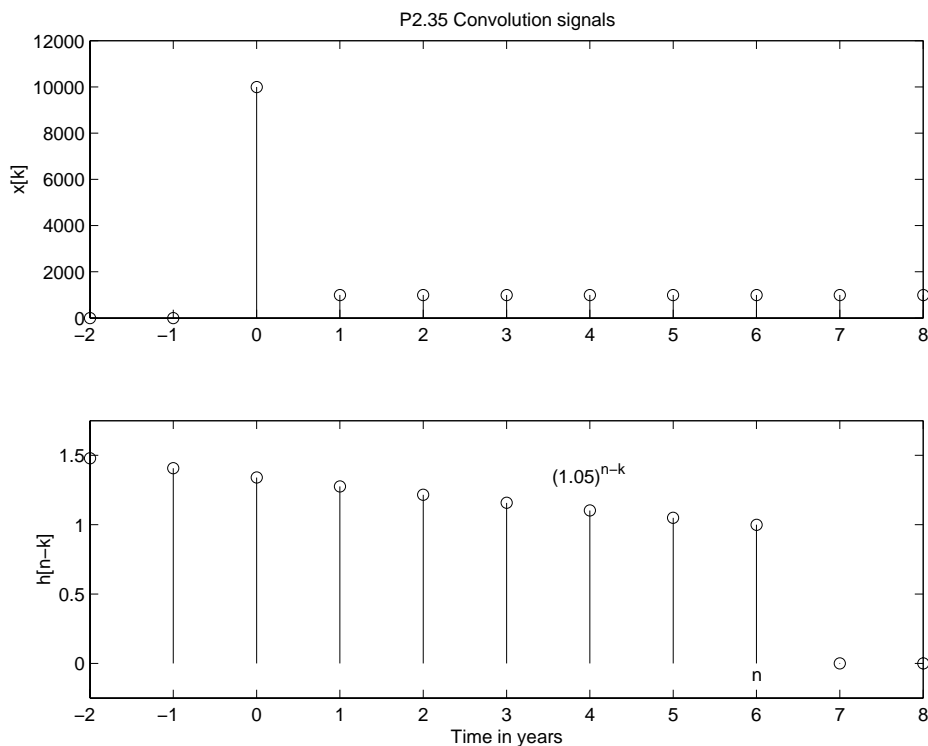


Figure P2.35. Graph of $x[k]$ and $h[n - k]$

for $n = -1$

$$y[-1] = \sum_{k=-1}^{-1} 10000(1.05)^{n-k} = 10000(1.05)^{n+1}$$

\$1000 is invested annually, similar to example 2.5

for $n \geq 0$

$$y[n] = 10000(1.05)^{n+1} + \sum_{k=0}^n 1000(1.05)^{n-k}$$

$$y[n] = 10000(1.05)^{n+1} + 1000(1.05)^n \sum_{k=0}^n (1.05)^{-k}$$

$$y[n] = 10000(1.05)^{n+1} + 1000(1.05)^n \frac{1 - \left(\frac{1}{1.05}\right)^{n+1}}{1 - \frac{1}{1.05}}$$

$$y[n] = 10000(1.05)^{n+1} + 20000 [1.05^{n+1} - 1]$$

The following is a graph of the value of the account.

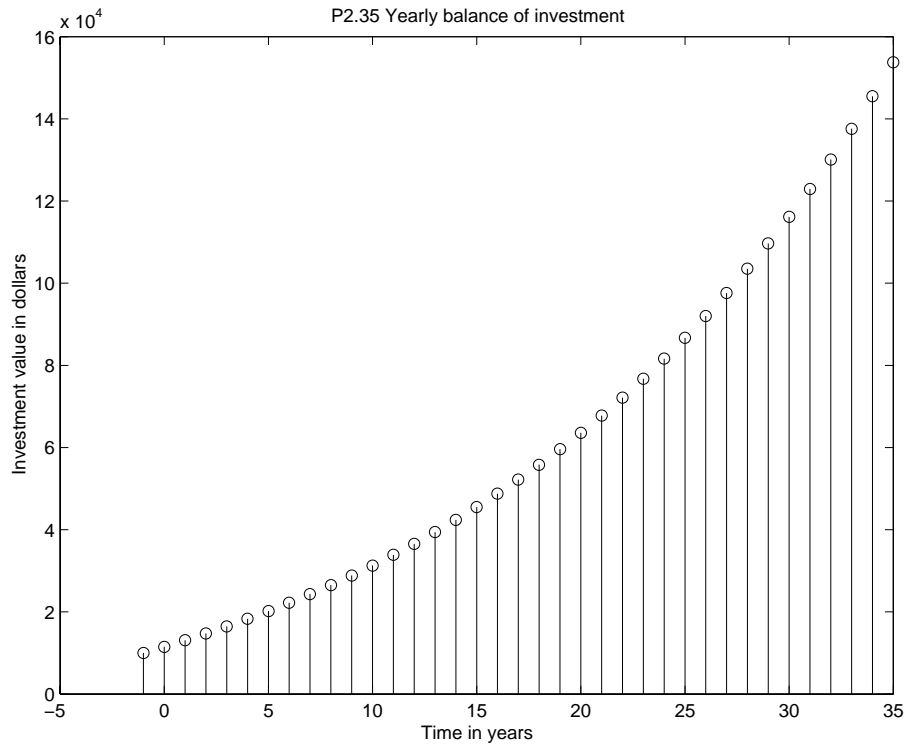


Figure P2.35. Yearly balance of the account

2.36. The initial balance of a loan is \$20,000 and the interest rate is 1% per month (12% per year). A monthly payment of \$200 is applied to the loan at the start of each month. Use convolution to calculate the loan balance after each monthly payment.

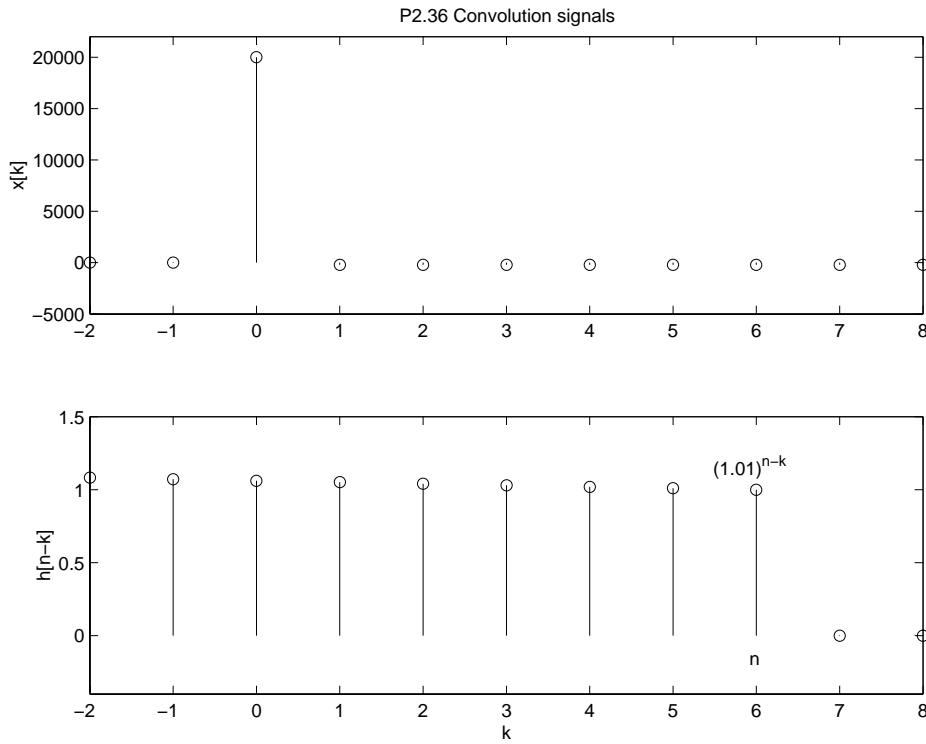


Figure P2.36. Plot of $x[k]$ and $h[n-k]$

for $n = -1$

$$y[n] = \sum_{k=-1}^{-1} 20000(1.01)^{n-k} = 20000(1.01)^{n+1}$$

for ≥ 0

$$y[n] = 20000(1.01)^{n+1} - \sum_{k=0}^n 200(1.01)^{n-k}$$

$$y[n] = 20000(1.01)^{n+1} - 200(1.01)^n \sum_{k=0}^n (1.01)^{-k}$$

$$y[n] = 20000(1.01)^{n+1} - 20000[(1.01)^{n+1} - 1]$$

The following is a plot of the monthly balance.

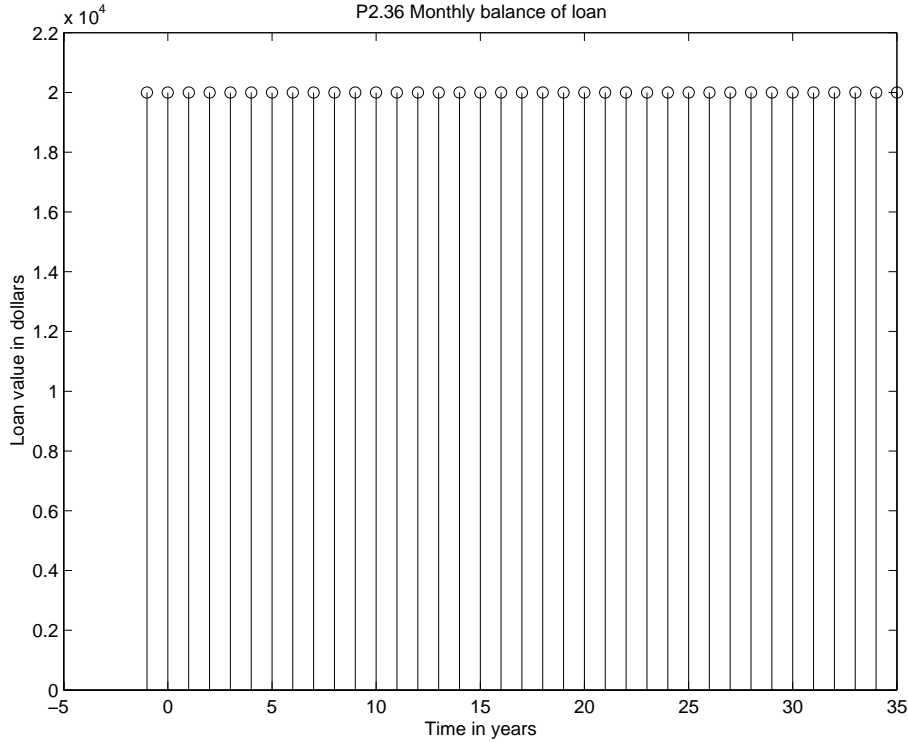


Figure P2.36. Monthly loan balance

Paying \$200 per month only takes care of the interest, and doesn't pay off any of the principle of the loan.

2.37. The convolution sum evaluation procedure actually corresponds to a formal statement of the well-known procedure for multiplication of polynomials. To see this, we interpret polynomials as signals by setting the value of a signal at time n equal to the polynomial coefficient associated with monomial z^n . For example, the polynomial $x(z) = 2 + 3z^2 - z^3$ corresponds to the signal $x[n] = 2\delta[n] + 3\delta[n-2] - \delta[n-3]$. The procedure for multiplying polynomials involves forming the product of all polynomial coefficients that result in an n -th order monomial and then summing them to obtain the polynomial coefficient of the n -th order monomial in the product. This corresponds to determining $w_n[k]$ and summing over k to obtain $y[n]$.

Evaluate the convolutions $y[n] = x[n] * h[n]$ using both the convolution sum evaluation procedure and as a product of polynomials.

(a) $x[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$, $h[n] = u[n] - u[n-3]$

$$\begin{aligned}
 x(z) &= 1 - 2z + z^2 \\
 h(z) &= 1 + z + z^2 \\
 y(z) &= x(z)h(z) \\
 &= 1 - z - z^3 + z^4 \\
 y[n] &= \delta[n] - \delta[n-1] - \delta[n-3] + \delta[n-4]
 \end{aligned}$$

$$\begin{aligned}
y[n] &= x[n] * h[n] = h[n] - 2h[n-1] + h[n-2] \\
&= \delta[n] - \delta[n-1] - \delta[n-3] - \delta[n-4]
\end{aligned}$$

(b) $x[n] = u[n-1] - u[n-5]$, $h[n] = u[n-1] - u[n-5]$

$$x(z) = z + z^2 + z^3 + z^4$$

$$h(z) = z + z^2 + z^3 + z^4$$

$$y(z) = x(z)h(z)$$

$$= z^2 + 2z^3 + 3z^4 + 4z^5 + 3z^6 + 2z^7 + z^8$$

$$y[n] = \delta[n-2] + 2\delta[n-3] + 3\delta[n-4] + 4\delta[n-5] + 3\delta[n-6] + 2\delta[n-7] + \delta[n-8]$$

for $n-1 \leq 0$ $n \leq 1$

$$y[n] = 0$$

for $n-1 \leq 4$ $n \leq 5$

$$y[n] = \sum_{k=1}^{n-1} 1 = n-1$$

for $n-4 \leq 4$ $n \leq 8$

$$y[n] = \sum_{k=n-4}^4 1 = 9-n$$

for $n-4 \geq 5$ $n \geq 9$

$$y[n] = 0$$

$$y[n] = \delta[n-2] + 2\delta[n-3] + 3\delta[n-4] + 4\delta[n-5] + 3\delta[n-6] + 2\delta[n-7] + \delta[n-8]$$

2.38. An LTI system has impulse response $h(t)$ depicted in Fig. P2.38. Use linearity and time invariance to determine the system output $y(t)$ if the input $x(t)$ is

(a) $x(t) = 2\delta(t+2) + \delta(t-2)$

$$y(t) = 2h(t+2) + h(t-2)$$

(b) $x(t) = \delta(t-1) + \delta(t-2) + \delta(t-3)$

$$y(t) = h(t-1) + h(t-2) + h(t-3)$$

(c) $x(t) = \sum_{p=0}^{\infty} (-1)^p \delta(t-2p)$

$$y(t) = \sum_{p=0}^{\infty} (-1)^p h(t-2p)$$

2.39. Evaluate the continuous-time convolution integrals given below.

(a) $y(t) = (u(t) - u(t-2)) * u(t)$

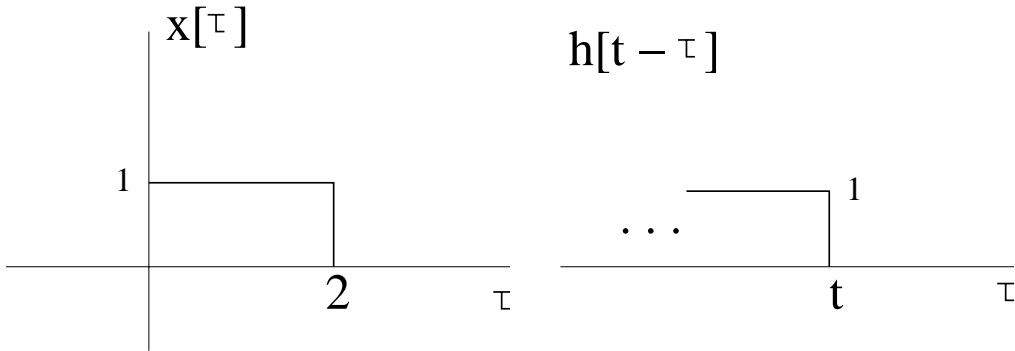


Figure P2.39. (a) Graph of $x[\tau]$ and $h[t - \tau]$

for $t < 0$

$$y(t) = 0$$

for $0 \leq t < 2$

$$y(t) = \int_0^t d\tau = t$$

for $t \geq 2$

$$y(t) = \int_0^2 d\tau = 2$$

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 2 \\ 2 & t \geq 2 \end{cases}$$

(b) $y(t) = e^{-3t}u(t) * u(t+3)$

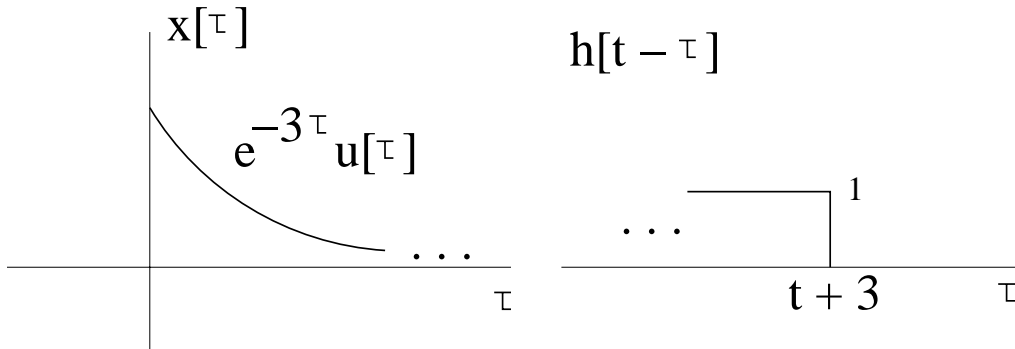


Figure P2.39. (b) Graph of $x[\tau]$ and $h[t - \tau]$

for $t + 3 < 0$ $t < -3$

$$y(t) = 0$$

for $t \geq -3$

$$y(t) = \int_0^{t+3} e^{-3\tau} d\tau$$

$$y(t) = \frac{1}{3} [1 - e^{-3(t+3)}]$$

$$y(t) = \begin{cases} 0 & t < -3 \\ \frac{1}{3} [1 - e^{-3(t+3)}] & t \geq -3 \end{cases}$$

(c) $y(t) = \cos(\pi t)(u(t+1) - u(t-1)) * u(t)$

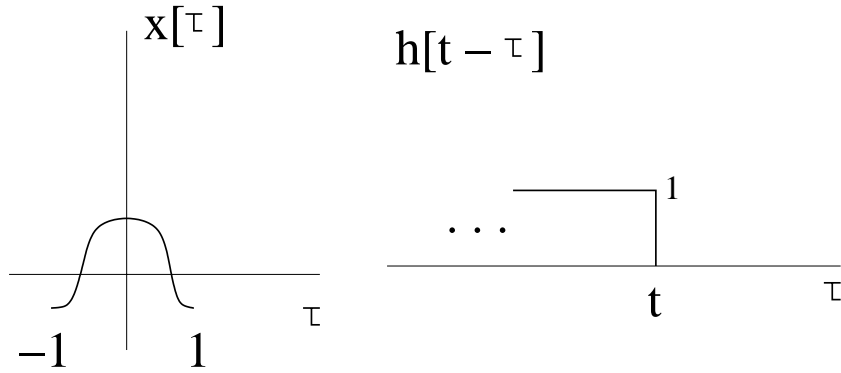


Figure P2.39. (c) Graph of $x[\tau]$ and $h[t - \tau]$

for $t < -1$

$$y(t) = 0$$

for $t < 1$

$$y(t) = \int_{-1}^t \cos(\pi \tau) d\tau$$

$$y(t) = \frac{1}{\pi} \sin(\pi t)$$

for $t > 1$

$$y(t) = \int_{-1}^1 \cos(\pi \tau) d\tau$$

$$y(t) = 0$$

$$y(t) = \begin{cases} \frac{1}{\pi} \sin(\pi t) & -1 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

(d) $y(t) = (u(t+3) - u(t-1)) * u(-t+4)$

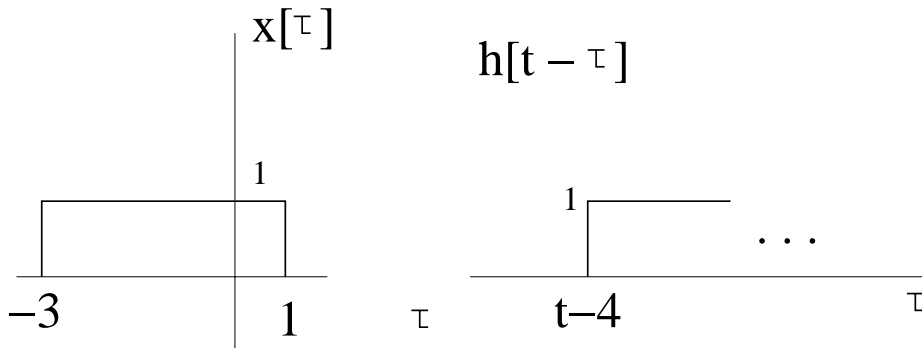


Figure P2.39. (d) Graph of $x[\tau]$ and $h[t - \tau]$

$$\begin{aligned}
 &\text{for } t - 4 < -3 && t < 1 \\
 &&& y(t) = \int_{-3}^1 d\tau = 4 \\
 &\text{for } t - 4 < 1 && t < 5 \\
 &&& y(t) = \int_{t-4}^1 d\tau = 5 - t \\
 &\text{for } t - 4 \geq 1 && t \geq 5 \\
 &&& y(t) = 0 \\
 y(t) &= \begin{cases} 4 & t < 1 \\ 5 - t & 1 \leq t < 5 \\ 0 & t \geq 5 \end{cases}
 \end{aligned}$$

(e) $y(t) = (tu(t) + (10 - 2t)u(t - 5) - (10 - t)u(t - 10)) * u(t)$

$$\begin{aligned}
 &\text{for } t < 0 \\
 &&& y(t) = 0 \\
 &\text{for } 0 \leq t < 5 \\
 &&& y(t) = \int_0^t \tau d\tau = \frac{1}{2}t^2 \\
 &\text{for } 5 \leq t < 10 \\
 &&& y(t) = \int_0^5 \tau d\tau + \int_5^t (10 - \tau) d\tau = -\frac{1}{2}t^2 + 10t - 25 \\
 &\text{for } t \geq 10 \\
 &&& y(t) = \int_0^5 \tau d\tau + \int_5^{10} (10 - \tau) d\tau = 25 \\
 y(t) &= \begin{cases} 0 & t < 0 \\ \frac{1}{2}t^2 & 0 \leq t < 5 \\ -\frac{1}{2}t^2 + 10t - 25 & 5 \leq t < 10 \\ 25 & t \geq 10 \end{cases}
 \end{aligned}$$

(f) $y(t) = 2t^2(u(t + 1) - u(t - 1)) * 2u(t + 2)$

$$\begin{aligned}
 &\text{for } t + 2 < -1 && t < -3 \\
 &&& y(t) = 0 \\
 &\text{for } t + 2 < 1 && -3 \leq t < -1 \\
 &&& y(t) = 2 \int_{-1}^{t+2} 2\tau^2 d\tau = \frac{4}{3} [(t + 2)^3 + 1] \\
 &\text{for } t + 2 \geq -1 && t \geq -1 \\
 &&& y(t) = 2 \int_{-1}^1 2\tau^2 d\tau = \frac{8}{3}
 \end{aligned}$$

$$y(t) = \begin{cases} 0 & t < -3 \\ \frac{4}{3}[(t+2)^3 + 1] & -3 \leq t < -1 \\ \frac{8}{3} & t \geq -1 \end{cases}$$

(g) $y(t) = \cos(\pi t)(u(t+1) - u(t-1)) * (u(t+1) - u(t-1))$

$$\begin{aligned} \text{for } t+1 < -1 & \quad t < -2 \\ & \quad y(t) = 0 \\ \text{for } t+1 < 1 & \quad -2 \leq t < 0 \\ & \quad y(t) = \int_{-1}^{t+1} \cos(\pi \tau) d\tau = \frac{1}{\pi} \sin(\pi(t+1)) \\ \text{for } t-1 < 1 & \quad 0 \leq t < 2 \\ & \quad y(t) = \int_{t-1}^1 \cos(\pi \tau) d\tau = -\frac{1}{\pi} \sin(\pi(t-1)) \\ \text{for } t-1 \geq 1 & \quad t \geq 2 \\ & \quad y(t) = 0 \end{aligned}$$

$$y(t) = \begin{cases} 0 & t < -2 \\ \frac{1}{\pi} \sin(\pi(t+1)) & -2 \leq t < 0 \\ -\frac{1}{\pi} \sin(\pi(t-1)) & 0 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

(h) $y(t) = \cos(2\pi t)(u(t+1) - u(t-1)) * e^{-t}u(t)$

$$\begin{aligned} \text{for } t < -1 & \\ & \quad y(t) = 0 \\ \text{for } t < 1 & \quad -1 \leq t < 1 \\ & \quad y(t) = \int_{-1}^t e^{-(t-\tau)} \cos(2\pi \tau) d\tau \\ & \quad y(t) = e^{-t} \left[\frac{e^{\tau}}{1+4\pi^2} (\cos(2\pi \tau) + 2\pi \sin(2\pi \tau)) \right]_{-1}^t \\ & \quad y(t) = \frac{\cos(2\pi t) + 2\pi \sin(2\pi t) - e^{-(t+1)}}{1+4\pi^2} \\ \text{for } t \geq 1 & \quad t \geq 1 \\ & \quad y(t) = \int_{-1}^1 e^{-(t-\tau)} \cos(2\pi \tau) d\tau \\ & \quad y(t) = e^{-t} \left[\frac{e^{\tau}}{1+4\pi^2} (\cos(2\pi \tau) + 2\pi \sin(2\pi \tau)) \right]_{-1}^1 \\ & \quad y(t) = \frac{e^{-(t-1)} - e^{-(t+1)}}{1+4\pi^2} \end{aligned}$$

$$y(t) = \begin{cases} 0 & t < -1 \\ \frac{\cos(2\pi t) + 2\pi \sin(2\pi t) - e^{-(t+1)}}{1+4\pi^2} & -1 \leq t < 1 \\ \frac{e^{-(t-1)} - e^{-(t+1)}}{1+4\pi^2} & t \geq 1 \end{cases}$$

$$(i) \ y(t) = (2\delta(t+1) + \delta(t-5)) * u(t-1)$$

$$\begin{aligned}
& \text{for } t-1 < -1 && t < 0 \\
& && y(t) = 0 \\
& \text{for } t-1 < 5 && 0 \leq t < 6 \\
& && \text{By the sifting property.} \\
& && y(t) = \int_{-\infty}^{t-1} 2\delta(\tau+1) d\tau = 2 \\
& \text{for } t-1 \geq 5 && t \geq 6 \\
& && y(t) = \int_{-\infty}^{t-1} (2\delta(\tau+1) + \delta(\tau-5)) d\tau = 3 \\
& && y(t) = \begin{cases} 0 & t < 0 \\ 2 & 0 \leq t < 6 \\ 3 & t \geq 6 \end{cases}
\end{aligned}$$

$$(j) \ y(t) = (\delta(t+2) + \delta(t-2)) * (tu(t) + (10-2t)u(t-5) - (10-t)u(t-10))$$

$$\begin{aligned}
& \text{for } t < -2 && y(t) = 0 \\
& \text{for } t < 2 && -2 \leq t < 2 \\
& && y(t) = \int_{t-10}^t (t-\tau)\delta(\tau+2) d\tau = t+2 \\
& \text{for } t-5 < -2 && 2 \leq t < 3 \\
& && y(t) = \int_{t-10}^t (t-\tau)\delta(\tau+2) d\tau + \int_{t-10}^t (t-\tau)\delta(\tau-2) d\tau = 2t \\
& \text{for } t-5 < 2 && 3 \leq t < 7 \\
& && y(t) = \int_{t-10}^t [10-(t-\tau)]\delta(\tau+2) d\tau + \int_{t-10}^t (t-\tau)\delta(\tau-2) d\tau = 6 \\
& \text{for } t-10 < -2 && 7 \leq t < 8 \\
& && y(t) = \int_{t-10}^t [10-(t-\tau)]\delta(\tau+2) d\tau + \int_{t-10}^t [10-(t-\tau)]\delta(\tau-2) d\tau = 20-2t \\
& \text{for } t-10 < 2 && 8 \leq t < 12 \\
& && y(t) = \int_{t-10}^t [10-(t-\tau)]\delta(\tau-2) d\tau = 12-t \\
& \text{for } t-10 \geq 2 && t \geq 12 \\
& && y(t) = 0 \\
& && y(t) = \begin{cases} 0 & t < -2 \\ t+2 & -2 \leq t < 2 \\ 2t & 2 \leq t < 3 \\ 6 & 3 \leq t < 7 \\ 20-2t & 7 \leq t < 8 \\ 12-t & 8 \leq t < 12 \\ 0 & t \geq 12 \end{cases}
\end{aligned}$$

$$(k) \ y(t) = e^{-\gamma t} u(t) * (u(t+2) - u(t))$$

$$\text{for } t < -2$$

$$y(t) = 0$$

$$\text{for } t < 0$$

$$-2 \leq t < 0$$

$$y(t) = \int_{-2}^t e^{-\gamma(t-\tau)} d\tau$$

$$y(t) = \frac{1}{\gamma} [1 - e^{-\gamma(t+2)}]$$

$$\text{for } t \geq 0$$

$$y(t) = \int_{-2}^0 e^{-\gamma(t-\tau)} d\tau$$

$$y(t) = \frac{1}{\gamma} [e^{-\gamma t} - e^{-\gamma(t+2)}]$$

$$y(t) = \begin{cases} 0 & t < -2 \\ \frac{1}{\gamma} [1 - e^{-\gamma(t+2)}] & -2 \leq t < 0 \\ \frac{1}{\gamma} [e^{-\gamma t} - e^{-\gamma(t+2)}] & t \geq 0 \end{cases}$$

$$(l) \ y(t) = e^{-\gamma t} u(t) * \sum_{p=0}^{\infty} \left(\frac{1}{4}\right)^p \delta(t-2p)$$

$$\text{for } t < 0$$

$$y(t) = 0$$

$$\text{for } t \geq 0$$

$$y(t) = \int_0^t e^{-\gamma\tau} \sum_{p=0}^{\infty} \left(\frac{1}{4}\right)^p \delta(t-2p-\tau) d\tau$$

$$y(t) = \sum_{p=0}^{\infty} \left(\frac{1}{4}\right)^p \int_0^t e^{-\gamma\tau} \delta(t-2p-\tau) d\tau$$

Using the sifting property yields

$$y(t) = \sum_{p=0}^{\infty} \left(\frac{1}{4}\right)^p e^{-\gamma(t-2p)} u(t-2p)$$

$$\text{for } 0 \leq t < 2$$

$$y(t) = e^{-\gamma t}$$

$$\text{for } 2 \leq t < 4$$

$$y(t) = e^{-\gamma t} + \frac{1}{4} e^{-\gamma(t-2)}$$

$$\text{for } 4 \leq t < 6$$

$$y(t) = e^{-\gamma t} + \frac{1}{4} e^{-\gamma(t-2)} + \frac{1}{16} e^{-\gamma(t-4)}$$

$$\text{for } 2l \leq t < 2l+2$$

$$y(t) = \left(\sum_{p=0}^l \left(\frac{1}{4}\right)^p e^{2p\gamma} \right) e^{-\gamma t}$$

$$(m) \ y(t) = (2\delta(t) + \delta(t-2)) * \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t-p)$$

$$\text{let } x_1(t) = \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t-p)$$

for $t < 0$

$$y(t) = 0$$

for $t < 2$

$$y(t) = 2\delta(t) * x_1(t) = 2x_1(t)$$

for $t \geq 2$

$$y(t) = 2\delta(t) * x_1(t) + \delta(t-2) * x_1(t) = 2x_1(t) + x_1(t-2)$$

$$y(t) = \begin{cases} 0 & t < 0 \\ 2 \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t-p) & 0 \leq t < 2 \\ 2 \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t-p) + \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t-p-2) & t \geq 2 \end{cases}$$

$$(n) \ y(t) = e^{-\gamma t} u(t) * e^{\beta t} u(-t)$$

for $t < 0$

$$y(t) = \int_0^{\infty} e^{\beta t} e^{-(\beta+\gamma)\tau} d\tau$$

$$y(t) = \frac{e^{\beta t}}{\beta + \gamma}$$

for $t \geq 0$

$$y(t) = \int_t^{\infty} e^{\beta t} e^{-(\beta+\gamma)\tau} d\tau$$

$$y(t) = \frac{e^{\beta t}}{\beta + \gamma} e^{-(\beta+\gamma)t}$$

$$y(t) = \frac{e^{-\gamma t}}{\beta + \gamma}$$

$$y(t) = \begin{cases} \frac{e^{\beta t}}{\beta + \gamma} & t < 0 \\ \frac{e^{-\gamma t}}{\beta + \gamma} & t \geq 0 \end{cases}$$

$$(o) \ y(t) = u(t) * h(t) \text{ where } h(t) = \begin{cases} e^{2t} & t < 0 \\ e^{-3t} & t \geq 0 \end{cases}$$

for $t < 0$

$$y(t) = \int_{-\infty}^t e^{2\tau} d\tau$$

$$y(t) = \frac{1}{2} e^{2t}$$

for $t \geq 0$

$$y(t) = \int_{-\infty}^0 e^{2\tau} d\tau + \int_0^t e^{-3\tau} d\tau$$

$$y(t) = \frac{1}{2} + \frac{1}{3} [1 - e^{-3t}]$$

$$y(t) = \begin{cases} \frac{1}{2} e^{2t} & t < 0 \\ \frac{1}{2} + \frac{1}{3} [1 - e^{-3t}] & t \geq 0 \end{cases}$$

2.40. Consider the continuous-time signals depicted in Fig. P2.40. Evaluate the following convolution integrals:.

(a) $m(t) = x(t) * y(t)$

$$\begin{aligned}
 &\text{for } t+1 < 0 && t < -1 \\
 &&& m(t) = 0 \\
 &\text{for } t+1 < 2 && -1 \leq t < 1 \\
 &&& m(t) = \int_0^{t+1} d\tau = t+1 \\
 &\text{for } t+1 < 4 && 1 \leq t < 3 \\
 &&& m(t) = \int_{t-1}^2 d\tau + \int_2^{t+1} 2d\tau = t+1 \\
 &\text{for } t-1 < 4 && 3 \leq t < 5 \\
 &&& m(t) = \int_{t-1}^4 2d\tau = 10-2t \\
 &\text{for } t-1 \geq 4 && t \geq 5 \\
 &&& m(t) = 0
 \end{aligned}$$

$$m(t) = \begin{cases} 0 & t < -1 \\ t+1 & -1 \leq t < 1 \\ t+1 & 1 \leq t < 3 \\ 10-2t & 3 \leq t < 5 \\ 0 & t \geq 5 \end{cases}$$

(b) $m(t) = x(t) * z(t)$

$$\begin{aligned}
 &\text{for } t+1 < -1 && t < -2 \\
 &&& m(t) = 0 \\
 &\text{for } t+1 < 0 && -2 \leq t < -1 \\
 &&& m(t) = -\int_{-1}^{t+1} d\tau = -t-2 \\
 &\text{for } t+1 < 1 && -1 \leq t < 0 \\
 &&& m(t) = -\int_{-1}^0 d\tau + \int_0^{t+1} d\tau = t \\
 &\text{for } t-1 < 0 && 0 \leq t < 1 \\
 &&& m(t) = -\int_{t-1}^0 d\tau + \int_0^1 d\tau = t \\
 &\text{for } t-1 < 1 && 1 \leq t < 2 \\
 &&& m(t) = \int_{t-1}^1 d\tau = 2-t \\
 &\text{for } t-1 \geq 1 && t \geq 2 \\
 &&& m(t) = 0
 \end{aligned}$$

$$m(t) = \begin{cases} 0 & t < -2 \\ -t-2 & -2 \leq t < -1 \\ t & -1 \leq t < 0 \\ t & 0 \leq t < 1 \\ 2-t & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

(c) $m(t) = x(t) * f(t)$

for $t < -1$

$$m(t) = 0$$

for $t < 0$ $-1 \leq t < 0$

$$m(t) = \int_{-1}^t e^{-(t-\tau)} d\tau = 1 - e^{-(t+1)}$$

for $t < 1$ $0 \leq t < 1$

$$m(t) = \int_{t-1}^t e^{-(t-\tau)} d\tau = 1 - e^{-1}$$

for $t < 2$ $1 \leq t < 2$

$$m(t) = \int_{t-1}^1 e^{-(t-\tau)} d\tau = e^{1-t} - e^{-1}$$

for $t \geq 2$

$$m(t) = 0$$

$$m(t) = \begin{cases} 0 & t < -1 \\ 1 - e^{-(t+1)} & -1 \leq t < 0 \\ 1 - e^{-1} & 0 \leq t < 1 \\ e^{1-t} - e^{-1} & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

(d) $m(t) = x(t) * a(t)$

By inspection, since $x(\tau)$ has a width of 2 and $a(t-\tau)$ has the period 2 and duty cycle $\frac{1}{2}$, the area under the overlapping signals is always 1, thus $m(t) = 1$ for all t .

(e) $m(t) = y(t) * z(t)$

for $t < -1$

$$m(t) = 0$$

for $t < 0$ $-1 \leq t < 0$

$$m(t) = - \int_{-1}^t d\tau = -t - 1$$

for $t < 1$ $0 \leq t < 1$

$$m(t) = - \int_{-1}^0 d\tau + \int_0^t d\tau = -1 + t$$

for $t < 2$ $1 \leq t < 2$

$$\begin{aligned}
& \text{for } t - 2 < 1 & m(t) &= -2 \int_{-1}^{t-2} d\tau - \int_{t-2}^0 d\tau + \int_0^1 d\tau = -t + 1 \\
& & 2 \leq t < 3 & \\
& \text{for } t - 4 < 0 & m(t) &= -2 \int_{-1}^0 d\tau + 2 \int_0^{t-2} d\tau + \int_{t-2}^1 d\tau = t - 3 \\
& & 3 \leq t < 4 & \\
& \text{for } t - 4 < 1 & m(t) &= -2 \int_{t-4}^0 d\tau + 2 \int_0^1 d\tau = 2t - 6 \\
& & 4 \leq t < 5 & \\
& \text{for } t \geq 5 & m(t) &= 2 \int_{t-4}^1 d\tau = 10 - 2t
\end{aligned}$$

$$\begin{aligned}
& m(t) = 0 \\
& m(t) = \begin{cases} 0 & t < -1 \\ -t - 1 & -1 \leq t < 0 \\ -1 + t & 0 \leq t < 1 \\ -t + 1 & 1 \leq t < 2 \\ t - 3 & 2 \leq t < 3 \\ 2t - 6 & 3 \leq t < 4 \\ 10 - 2t & 4 \leq t < 5 \\ 0 & t \geq 5 \end{cases}
\end{aligned}$$

(f) $m(t) = y(t) * w(t)$

$$\begin{aligned}
& \text{for } t < 0 & m(t) &= 0 \\
& \text{for } t < 1 & 0 \leq t < 1 & \\
& & m(t) &= \int_0^t d\tau = t \\
& \text{for } t < 2 & 1 \leq t < 2 & \\
& & m(t) &= \int_0^1 d\tau - \int_1^t d\tau = 2 - t \\
& \text{for } t < 3 & 2 \leq t < 3 & \\
& & m(t) &= 2 \int_0^{t-2} d\tau + \int_{t-2}^1 d\tau - \int_1^t d\tau = 0 \\
& \text{for } t - 4 < 0 & 3 \leq t < 4 & \\
& & m(t) &= 2 \int_0^1 d\tau - 2 \int_1^{t-2} d\tau - \int_{t-2}^3 d\tau = 3 - t \\
& \text{for } t - 4 < 1 & 4 \leq t < 5 & \\
& & m(t) &= 2 \int_{t-4}^1 d\tau - 2 \int_1^{t-2} d\tau - \int_{t-2}^3 d\tau = 11 - 3t \\
& \text{for } t - 4 < 3 & 5 \leq t < 7 & \\
& & m(t) &= -2 \int_{t-4}^3 d\tau = 2t - 14 \\
& \text{for } t \geq 7 & &
\end{aligned}$$

$$m(t) = 0$$

$$m(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2-t & 1 \leq t < 2 \\ 0 & 2 \leq t < 3 \\ 3-t & 3 \leq t < 4 \\ 11-3t & 4 \leq t < 5 \\ 2t-14 & 5 \leq t < 7 \\ 0 & t \geq 7 \end{cases}$$

(g) $m(t) = y(t) * g(t)$

for $t < -1$

$$m(t) = 0$$

for $t < 1$ $-1 \leq t < 1$

$$m(t) = \int_{-1}^t \tau d\tau = 0.5[t^2 - 1]$$

for $t - 2 < 1$ $1 \leq t < 3$

$$m(t) = \int_{-1}^{t-2} 2\tau d\tau + \int_{t-2}^1 \tau d\tau = 0.5t^2 + 0.5(t-2)^2 - 1$$

for $t - 4 < 1$ $3 \leq t < 5$

$$m(t) = \int_{t-4}^1 2\tau d\tau = 1 - (t-4)^2$$

for $t \geq 5$

$$m(t) = 0$$

$$m(t) = \begin{cases} 0 & t < -1 \\ 0.5[t^2 - 1] & -1 \leq t < 1 \\ 0.5t^2 + 0.5(t-2)^2 - 1 & 1 \leq t < 3 \\ 1 - (t-4)^2 & 3 \leq t < 5 \\ 0 & t \geq 5 \end{cases}$$

(h) $m(t) = y(t) * c(t)$

for $t + 2 < 0$ $t < -2$

$$m(t) = 0$$

for $t + 2 < 2$ $-2 \leq t < 0$

$$m(t) = 1$$

for $t - 1 < 0$ $0 \leq t < 1$

$$m(t) = \int_0^t d\tau + 2 = t + 2$$

for $t < 2$ $1 \leq t < 2$

$$m(t) = \int_{t-1}^t d\tau + 2 = 3$$

$$\begin{aligned}
&\text{for } t < 3 && 2 \leq t < 3 \\
&&& m(t) = -1 + \int_{t-1}^2 d\tau + 2 \int_2^t d\tau = t - 2 \\
&\text{for } t < 4 && 3 \leq t < 4 \\
&&& m(t) = -1 + 2 \int_{t-2}^t d\tau = 1 \\
&\text{for } t < 5 && 4 \leq t < 5 \\
&&& m(t) = -2 + 2 \int_{t-1}^4 d\tau = 8 - 2t \\
&\text{for } t - 2 < 4 && 5 \leq t < 6 \\
&&& m(t) = -2 \\
&\text{for } t \geq 6 && m(t) = 0 \\
&&& m(t) = \begin{cases} 0 & t < -2 \\ 1 & -2 \leq t < 0 \\ t+2 & 0 \leq t < 1 \\ 3 & 1 \leq t < 2 \\ t-2 & 2 \leq t < 3 \\ 1 & 3 \leq t < 4 \\ 8-2t & 5 \leq t < 5 \\ -2 & 5 \leq t < 6 \\ 0 & t \geq 6 \end{cases}
\end{aligned}$$

(i) $m(t) = z(t) * f(t)$

$$\begin{aligned}
&\text{for } t+1 < 0 && t < -1 \\
&&& m(t) = 0 \\
&\text{for } t+1 < 1 && -1 \leq t < 0 \\
&&& m(t) = - \int_0^{t+1} e^{-\tau} d\tau = e^{-(t+1)} - 1 \\
&\text{for } t < 1 && 0 \leq t < 1 \\
&&& m(t) = \int_0^t e^{-\tau} d\tau - \int_t^1 e^{-\tau} d\tau = 1 + e^{-1} - 2e^{-t} \\
&\text{for } t-1 < 1 && 1 \leq t < 2 \\
&&& m(t) = \int_{t-1}^1 e^{-\tau} d\tau = -e^{-1} + e^{-(t-1)} \\
&\text{for } t \geq 2 && m(t) = 0 \\
&&& m(t) = \begin{cases} 0 & t < -1 \\ e^{-(t+1)} - 1 & -1 \leq t < 0 \\ 1 + e^{-1} - 2e^{-t} & 0 \leq t < 1 \\ -e^{-1} + e^{-(t-1)} & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases}
\end{aligned}$$

$$(j) \ m(t) = z(t) * g(t)$$

$$\begin{aligned}
& \text{for } t+1 < -1 && t < -2 \\
& && m(t) = 0 \\
& \text{for } t+1 < 0 && -2 \leq t < -1 \\
& && m(t) = -\int_{-1}^{t+1} \tau d\tau = -0.5[(t+1)^2 - 1] \\
& \text{for } t < 0 && -1 \leq t < 0 \\
& && m(t) = -\int_t^{t+1} \tau d\tau + \int_{-1}^t \tau d\tau = -\frac{1}{2}((t+1)^2 - t^2) + \frac{1}{2}(t^2 - 1) \\
& \text{for } t-1 < 0 && 0 \leq t < 1 \\
& && m(t) = \int_{t-1}^t \tau d\tau - \int_t^1 \tau d\tau = 0.5[t^2 - (t-1)^2] - 0.5[1 - t^2] \\
& \text{for } t-1 < 1 && 1 \leq t < 2 \\
& && m(t) = \int_{t-1}^1 \tau d\tau = 0.5[1 - (t-1)^2] \\
& \text{for } t \geq 2 && m(t) = 0
\end{aligned}$$

$$m(t) = \begin{cases} 0 & t < -2 \\ -0.5[(t+1)^2 - 1] & -2 \leq t < -1 \\ -\frac{1}{2}((t+1)^2 - t^2) + \frac{1}{2}(t^2 - 1) & -1 \leq t < 0 \\ 0.5[t^2 - (t-1)^2] - 0.5[1 - t^2] & 0 \leq t < 1 \\ 0.5[1 - (t-1)^2] & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

$$(k) \ m(t) = z(t) * b(t)$$

$$\begin{aligned}
& \text{for } t+1 < -3 && t < -4 \\
& && m(t) = 0 \\
& \text{for } t+1 < -2 && -4 \leq t < -3 \\
& && m(t) = -\int_{-3}^{t+1} (\tau+3) d\tau = -0.5(t+1)^2 + \frac{9}{2} - 3(t+1) - 9 \\
& \text{for } t < -2 && -3 \leq t < -2 \\
& && m(t) = \int_{-3}^t (\tau+3) d\tau - \int_t^{-2} (\tau+3) d\tau - \int_{-2}^t d\tau = t^2 + 5t + \frac{11}{2} \\
& \text{for } t-1 < -2 && -2 \leq t < -1 \\
& && m(t) = \int_{t-1}^{-2} (\tau+3) d\tau + \int_{-2}^t d\tau - \int_t^{t+1} d\tau = 12 - \frac{1}{2}(t-1)^2 - 2t \\
& \text{for } t-1 \geq -2 && t \geq -1 \\
& && m(t) = 0
\end{aligned}$$

$$m(t) = \begin{cases} 0 & t < -4 \\ -0.5(t+1)^2 - 3(t+1) - \frac{9}{2} & -4 \leq t < -3 \\ t^2 + 5t + \frac{11}{2} & -3 \leq t < -2 \\ 12 - \frac{1}{2}(t-1)^2 - 2t & -2 \leq t < -1 \\ 0 & t \geq 2 \end{cases}$$

(l) $m(t) = w(t) * g(t)$

for $t < -1$

$$m(t) = 0$$

for $t < 0$

$$-1 \leq t < 0$$

$$m(t) = - \int_{-1}^t \tau d\tau = -0.5t^2 + 0.5$$

for $t < 1$

$$0 \leq t < 1$$

$$m(t) = \int_{t-1}^t \tau d\tau - \int_{-1}^{t-1} \tau d\tau = 0.5t^2 - (t-1)^2 + 0.5$$

for $t-1 < 1$

$$1 \leq t < 2$$

$$m(t) = - \int_{-1}^{t-1} \tau d\tau + \int_{t-1}^1 \tau d\tau = 1 - (t-1)^2$$

for $t-3 < 1$

$$2 \leq t < 4$$

$$m(t) = - \int_{t-3}^1 \tau d\tau = 0.5(t-3)^2 - 0.5$$

for $t-3 \geq 1$

$$t \geq 4$$

$$m(t) = 0$$

$$m(t) = \begin{cases} 0 & t < -1 \\ -0.5t^2 + 0.5 & -1 \leq t < 0 \\ 0.5t^2 - (t-1)^2 + 0.5 & 0 \leq t < 1 \\ 1 - (t-1)^2 & 1 \leq t < 2 \\ 0.5(t-3)^2 - 0.5 & 2 \leq t < 4 \\ 0 & t \geq 4 \end{cases}$$

(m) $m(t) = w(t) * a(t)$

$$\text{let } a'(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{then } a(t) = \sum_{k=-\infty}^{\infty} a'(t-2k)$$

$$\text{consider } m'(t) = w(t) * a'(t)$$

for $t < 0$

$$m'(t) = 0$$

for $t < 1$

$$0 \leq t < 1$$

$$m'(t) = \int_0^t d\tau = t$$

$$\begin{aligned}
&\text{for } t - 1 < 1 && 1 \leq t < 2 \\
&&& m'(t) = - \int_0^{t-1} d\tau + \int_{t-1}^1 d\tau = 3 - 2t \\
&\text{for } t - 3 < 0 && 2 \leq t < 3 \\
&&& m'(t) = - \int_0^1 d\tau = -1 \\
&\text{for } t - 3 < 1 && 3 \leq t < 4 \\
&&& m'(t) = - \int_{t-3}^1 d\tau = t - 4 \\
&\text{for } t - 3 \geq 1 && t \geq 4 \\
&&& m'(t) = 0 \\
m'(t) &= \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 3 - 2t & 1 \leq t < 2 \\ -1 & 2 \leq t < 3 \\ t - 4 & 3 \leq t < 4 \\ 0 & t \geq 4 \end{cases} \\
m(t) &= \sum_{k=-\infty}^{\infty} m'(t - 2k)
\end{aligned}$$

(n) $m(t) = f(t) * g(t)$

$$\begin{aligned}
&\text{for } t < -1 && m(t) = 0 \\
&\text{for } t < 0 && -1 \leq t < 0 \\
&&& m(t) = - \int_{-1}^t \tau e^{-(t-\tau)} d\tau = t - 1 + 2e^{-(t+1)} \\
&\text{for } t < 1 && 0 \leq t < 1 \\
&&& m(t) = \int_{t-1}^t \tau e^{-(t-\tau)} d\tau = t - 1 - (t - 2)e^{-1} \\
&\text{for } t - 1 < 1 && 1 \leq t < 2 \\
&&& m(t) = \int_{t-1}^1 \tau e^{-(t-\tau)} d\tau = -e^{-1}(t - 2) \\
&\text{for } t - 2 \geq 1 && m(t) = 0 \\
m(t) &= \begin{cases} 0 & t < -1 \\ t - 1 + 2e^{-(t+1)} & -1 \leq t < 0 \\ t - 1 - (t - 2)e^{-1} & 0 \leq t < 1 \\ -e^{-1}(t - 2) & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases}
\end{aligned}$$

(o) $m(t) = f(t) * d(t)$

$$\begin{aligned}
d(t) &= \sum_{k=-\infty}^{\infty} f(t-k) \\
m'(t) &= f(t) * f(t) \\
m(t) &= \sum_{k=-\infty}^{\infty} m'(t-k) \\
\text{for } t < 0 & \\
& m'(t) = 0 \\
\text{for } t < 1 & \quad 0 \leq t < 1 \\
& m'(t) = \int_0^t e^{-(t-\tau)} e^{-\tau} d\tau = te^{-t} \\
\text{for } t < 2 & \quad 1 \leq t < 2 \\
& m'(t) = e^{-t} \int_{t-1}^1 e^{\tau} e^{-\tau} d\tau = (2-t)e^{-t} \\
\text{for } t \geq 2 & \\
& m'(t) = 0 \\
m'(t) &= \begin{cases} 0 & t < 0 \\ te^{-t} & 0 \leq t < 1 \\ (2-t)e^{-t} & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases} \\
m(t) &= \sum_{k=-\infty}^{\infty} m'(t-k)
\end{aligned}$$

(p) $m(t) = z(t) * d(t)$

$$\begin{aligned}
\text{let } d'(t) &= \begin{cases} e^{-t} & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \\
\text{then } d(t) &= \sum_{k=-\infty}^{\infty} d'(t-k) \\
\text{consider } m'(t) &= z(t) * d'(t) \\
\text{for } t < -1 & \\
& m'(t) = 0 \\
\text{for } t < 0 & \quad -1 \leq t < 0 \\
& m'(t) = \int_{-1}^t -e^{-(t-\tau)} d\tau = e^{-(t+1)} - 1 \\
\text{for } t < 1 & \quad 0 \leq t < 1 \\
& m'(t) = -\int_{t-1}^0 -e^{-(t-\tau)} d\tau + \int_0^t e^{-(t-\tau)} d\tau = 1 + e^{-1} - 2e^{-t} \\
\text{for } t-1 < 1 & \quad 1 \leq t < 2 \\
& m'(t) = \int_{t-1}^1 e^{-(t-\tau)} d\tau = e^{-(t-1)} - e^{-1}
\end{aligned}$$

$$\begin{aligned}
& \text{for } t-1 \geq 1 \quad t \geq 2 \\
& m'(t) = 0 \\
& m'(t) = \begin{cases} 0 & t < -1 \\ e^{-(t+1)} - 1 & -1 \leq t < 0 \\ 1 + e^{-1} - 2e^{-t} & 0 \leq t < 1 \\ e^{-(t-1)} - e^{-1} & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases} \\
& m(t) = \sum_{k=-\infty}^{\infty} m'(t-k)
\end{aligned}$$

2.41. Suppose we model the effect of imperfections in a communication channel as the RC circuit depicted in Fig. P2.41(a). Here the input $x(t)$ is the transmitted signal and the output $y(t)$ is the received signal. Assume the message is represented in binary format and that a “1” is communicated in an interval of length T by transmitting the symbol $p(t)$ depicted in Fig. P2.41 (b) in the pertinent interval and that a “0” is communicated by transmitting $-p(t)$ in the pertinent interval. Figure P2.41 (c) illustrates the transmitted waveform for communicating the sequence “1101001”. Assume that $T = 1/(RC)$.

(a) Use convolution to calculate the received signal due to transmission of a single “1” at time $t = 0$. Note that the received waveform extends beyond time T and into the interval allocated for the next bit, $T < t < 2T$. This contamination is called intersymbol interference (ISI), since the received waveform at any time is interfered with by previous symbols.

The impulse response of an RC circuit is $h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$. The output of the system is the convolution of the input, $x(t)$ with the impulse response, $h(t)$.

$$\begin{aligned}
y_p(t) &= h(t) * p(t) \\
&\text{for } t < 0 \\
& y_p(t) = 0 \\
&\text{for } t < T \quad 0 \leq t < T \\
& y_p(t) = \int_0^t \frac{1}{RC} e^{-\frac{(t-\tau)}{RC}} d\tau \\
& y_p(t) = 1 - e^{-\frac{t}{RC}} \\
&\text{for } t \geq T \\
& y_p(t) = \int_0^T \frac{1}{RC} e^{-\frac{(t-\tau)}{RC}} d\tau \\
& y_p(t) = e^{-\frac{(t-T)}{RC}} - e^{-\frac{t}{RC}} \\
y_p(t) &= \begin{cases} 0 & t < 0 \\ 1 - e^{-\frac{t}{RC}} & 0 \leq t < T \\ e^{-\frac{(t-T)}{RC}} - e^{-\frac{t}{RC}} & t \geq T \end{cases}
\end{aligned}$$

(b) Use convolution to calculate the received signal due to transmission of the sequences “1110” and “1000”. Compare the received waveforms to the output of an ideal channel ($h(t) = \delta(t)$) to evaluate the

effect of ISI for the following choices of RC :

- (i) $RC = 1/T$
- (ii) $RC = 5/T$
- (iii) $RC = 1/(5T)$

Assuming $T = 1$

$$(1) \quad \begin{aligned} x(t) &= p(t) + p(t-1) + p(t-2) - p(t-3) \\ y(t) &= y_p(t) + y_p(t-1) + y_p(t-2) + y_p(t-3) \end{aligned}$$

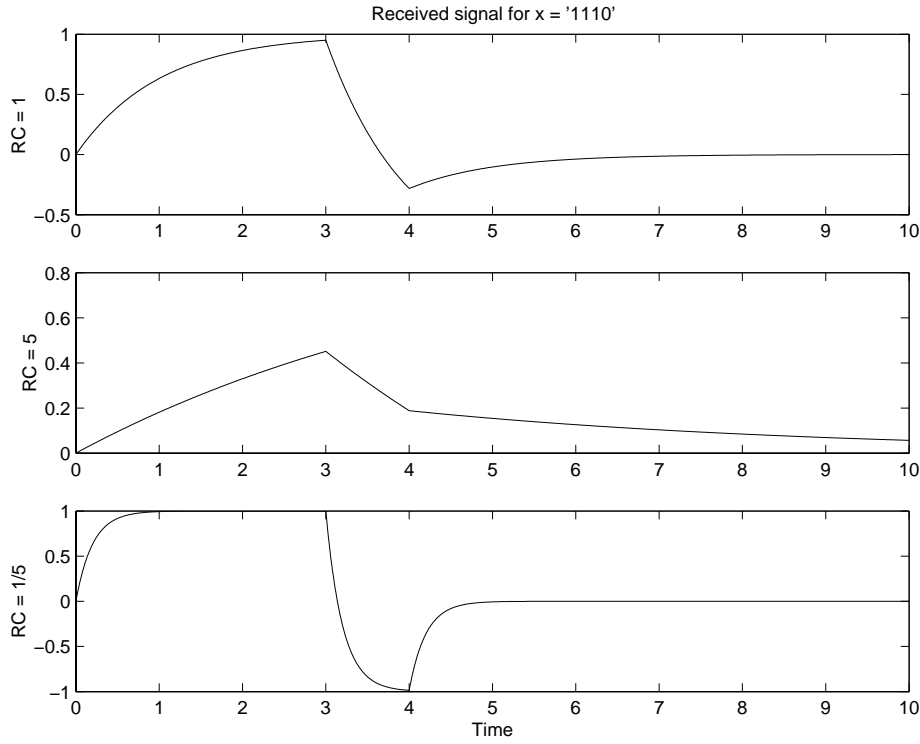


Figure P2.41. $x = \text{"1110"}$

$$(2) \quad \begin{aligned} x(t) &= p(t) - p(t-1) - p(t-2) - p(t-3) \\ y(t) &= y_p(t) - y_p(t-1) - y_p(t-2) + y_p(t-3) \end{aligned}$$

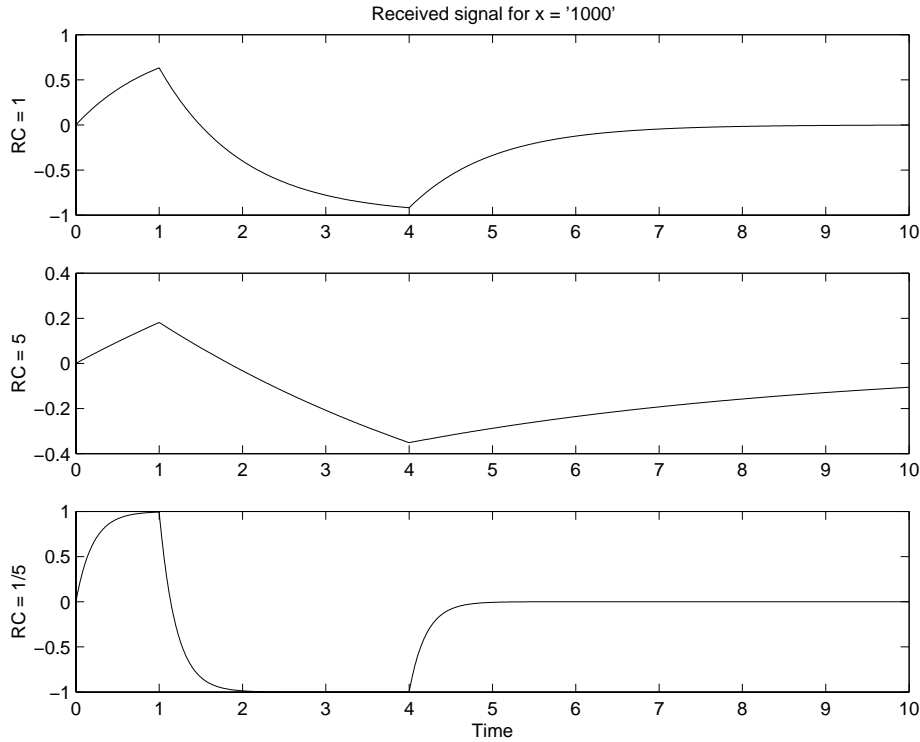


Figure P2.41. $x = \text{"1000"}$

From the two graphs it becomes apparent that as T becomes smaller, ISI becomes a larger problem for the communication system. The bits blur together and it becomes increasingly difficult to determine if a '1' or '0' is transmitted.

2.42. Use the definition of the convolution sum to prove the following properties

(a) Distributive: $x[n] * (h[n] + g[n]) = x[n] * h[n] + x[n] * g[n]$

$$\begin{aligned}
 \text{LHS} &= x[n] * (h[n] + g[n]) \\
 &= \sum_{k=-\infty}^{\infty} x[k] (h[n-k] + g[n-k]): \text{The definition of convolution.} \\
 &= \sum_{k=-\infty}^{\infty} (x[k]h[n-k] + x[k]g[n-k]): \text{the dist. property of mult.} \\
 &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] + \sum_{k=-\infty}^{\infty} x[k]g[n-k] \\
 &= x[n] * h[n] + x[n] * g[n] \\
 &= \text{RHS}
 \end{aligned}$$

(b) Associative: $x[n] * (h[n] * g[n]) = (x[n] * h[n]) * g[n]$

$$\text{LHS} = x[n] * (h[n] * g[n])$$

$$\begin{aligned}
&= x[n] * \left(\sum_{k=-\infty}^{\infty} h[k]g[n-k] \right) \\
&= \sum_{l=-\infty}^{\infty} x[l] \left(\sum_{k=-\infty}^{\infty} h[k]g[n-k-l] \right) \\
&= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (x[l]h[k]g[n-k-l]) \\
&\quad \text{Use } v = k + l \text{ and exchange the order of summation} \\
&= \sum_{v=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} x[l]h[v-l] \right) g[n-v] \\
&= \sum_{v=-\infty}^{\infty} (x[v] * h[v]) g[n-v] \\
&= (x[n] * h[n]) * g[n] \\
&= \text{RHS}
\end{aligned}$$

(c) Commutative: $x[n] * h[n] = h[n] * x[n]$

$$\begin{aligned}
\text{LHS} &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&\quad \text{Use } k = n - l \\
&= \sum_{l=-\infty}^{\infty} x[n-l]h[l] \\
&= \sum_{l=-\infty}^{\infty} h[l]x[n-l] \\
&= \text{RHS}
\end{aligned}$$

2.43. An LTI system has the impulse response depicted in Fig. P2.43.

(a) Express the system output $y(t)$ as a function of the input $x(t)$.

$$\begin{aligned}
y(t) &= x(t) * h(t) \\
&= x(t) * \left(\frac{1}{\Delta} \delta(t) - \frac{1}{\Delta} \delta(t - \Delta) \right) \\
&= \frac{1}{\Delta} (x(t) - x(t - \Delta))
\end{aligned}$$

(b) Identify the mathematical operation performed by this system in the limit as $\Delta \rightarrow 0$.

When $\Delta \rightarrow 0$

$$\lim_{\Delta \rightarrow 0} y(t) = \lim_{\Delta \rightarrow 0} \frac{x(t) - x(t - \Delta)}{\Delta}$$

is nothing but $\frac{d}{dt}x(t)$, the first derivative of $x(t)$ with respect to time t .

(c) Let $g(t) = \lim_{\Delta \rightarrow 0} h(t)$. Use the results of (b) to express the output of an LTI system with impulse response $h^n(t) = \underbrace{g(t) * g(t) * \dots * g(t)}_{n \text{ times}}$ as a function of the input $x(t)$.

$$\begin{aligned} g(t) &= \lim_{\Delta \rightarrow 0} h(t) \\ h^n(t) &= \underbrace{g(t) * g(t) * \dots * g(t)}_{n \text{ times}} \\ y^n(t) &= x(t) * h^n(t) \\ &= (x(t) * g(t)) * \underbrace{g(t) * g(t) * \dots * g(t)}_{(n-1) \text{ times}} \\ \text{let } x^{(1)}(t) &= x(t) * g(t) = x(t) * \lim_{\Delta \rightarrow 0} h(t) \\ &= \lim_{\Delta \rightarrow 0} (x(t) * h(t)) \\ &= \frac{d}{dt}x(t) \text{ from (b)} \\ \text{then } y^n(t) &= \left(x^{(1)}(t) * g(t) \right) * \underbrace{g(t) * g(t) * \dots * g(t)}_{(n-2) \text{ times}} \end{aligned}$$

Similarly

$$\begin{aligned} x^{(1)}(t) * g(t) &= \frac{d^2}{dt^2}x(t) \\ \text{Doing this repeatedly, we find that} \\ y^n(t) &= x^{(n-1)}(t) * g(t) \\ &= \frac{d^{n-1}}{dt^{n-1}}x(t) * g(t) \end{aligned}$$

Therefore

$$y^n(t) = \frac{d^n}{dt^n}x(t)$$

2.44. If $y(t) = x(t) * h(t)$ is the output of an LTI system with input $x(t)$ and impulse response $h(t)$, then show that

$$\frac{d}{dt}y(t) = x(t) * \left(\frac{d}{dt}h(t) \right)$$

and

$$\frac{d}{dt}y(t) = \left(\frac{d}{dt}x(t) \right) * h(t)$$

$$\begin{aligned} \frac{d}{dt}y(t) &= \frac{d}{dt}x(t) * h(t) \\ &= \frac{d}{dt} \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \end{aligned}$$

Assuming that the functions are sufficiently smooth,

the derivative can be pulled through the integral

$$\frac{d}{dt}y(t) = \int_{-\infty}^{\infty} x(\tau) \frac{d}{dt}h(t-\tau)d\tau$$

Since $x(\tau)$ is independent of t

$$\frac{d}{dt}y(t) = x(t) * \left(\frac{d}{dt}h(t) \right)$$

The convolution integral can also be written as

$$\begin{aligned} \frac{d}{dt}y(t) &= \frac{d}{dt} \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) \frac{d}{dt}x(t-\tau)d\tau \\ &= \left(\frac{d}{dt}x(t) \right) * h(t) \end{aligned}$$

2.45. If $h(t) = H\{\delta(t)\}$ is the impulse response of an LTI system, express $H\{\delta^{(2)}(t)\}$ in terms of $h(t)$.

$$\begin{aligned} H\{\delta^{(2)}(t)\} &= h(t) * \delta^{(2)}(t) \\ &= \int_{-\infty}^{\infty} h(\tau)\delta^{(2)}(t-\tau)d\tau \\ &\quad \text{By the doublet sifting property} \\ &= \frac{d}{dt}h(t) \end{aligned}$$

2.46. Find the expression for the impulse response relating the input $x[n]$ or $x(t)$ to the output $y[n]$ or $y(t)$ in terms of the impulse response of each subsystem for the LTI systems depicted in

(a) Fig. P2.46 (a)

$$y(t) = x(t) * \{h_1(t) - h_4(t) * [h_2(t) + h_3(t)]\} * h_5(t)$$

(b) Fig. P2.46 (b)

$$y[n] = x[n] * \{-h_1[n] * h_2[n] * h_4[n] + h_1[n] * h_3[n] * h_5[n]\} * h_6[n]$$

(c) Fig. P2.46 (c)

$$y(t) = x(t) * \{[-h_1(t) + h_2(t)] * h_3(t) * h_4(t) + h_2(t)\}$$

2.47. Let $h_1(t), h_2(t), h_3(t)$, and $h_4(t)$ be impulse responses of LTI systems. Construct a system with impulse response $h(t)$ using $h_1(t), h_2(t), h_3(t)$, and $h_4(t)$ as subsystems. Draw the interconnection of systems required to obtain

(a) $h(t) = \{h_1(t) + h_2(t)\} * h_3(t) * h_4(t)$

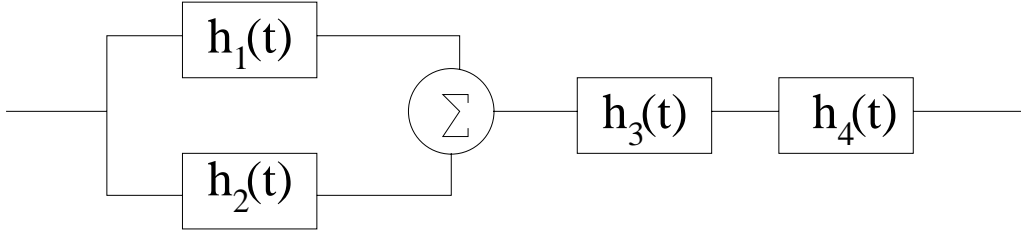


Figure P2.47. (a) Interconnections between systems

(b) $h(t) = h_1(t) * h_2(t) + h_3(t) * h_4(t)$

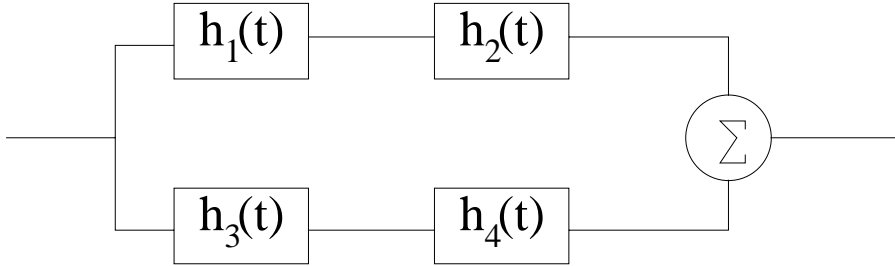


Figure P2.47. (b) Interconnections between systems

(c) $h(t) = h_1(t) * \{h_2(t) + h_3(t) * h_4(t)\}$

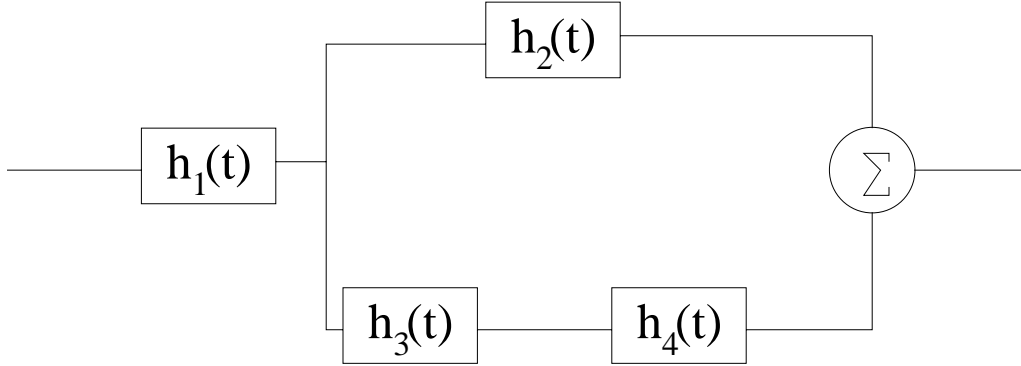


Figure P2.47. (c) Interconnections between systems

2.48. For the interconnection of LTI systems depicted in Fig. P2.46 (c) the impulse responses are $h_1(t) = \delta(t - 1)$, $h_2(t) = e^{-2t}u(t)$, $h_3(t) = \delta(t - 1)$ and $h_4(t) = e^{-3(t+2)}u(t + 2)$. Evaluate $h(t)$, the impulse response of the overall system from $x(t)$ to $y(t)$.

$$\begin{aligned}
 h(t) &= [-\delta(t - 1) + e^{-2t}u(t)] * \delta(t - 1) * e^{-3(t+2)}u(t + 2) + \delta(t - 1) \\
 &= [-\delta(t - 2) + e^{-2(t-1)}u(t - 1)] * e^{-3(t+2)}u(t + 2) + \delta(t - 1) \\
 &= -e^{-3t}u(t) + e^{-2(t-1)}u(t - 1) * e^{-3(t+2)}u(t + 2) + \delta(t - 1)
 \end{aligned}$$

$$= -e^{-3t}u(t) + \left(e^{-2(t-1)} - e^{-3(t+3)}\right)u(t+1) + \delta(t-1)$$

2.49. For each impulse response listed below, determine whether the corresponding system is (i) memoryless, (ii) causal, and (iii) stable.

- (i) Memoryless if and only if $h(t) = c\delta(t)$ or $h[n] = c\delta[k]$
 - (ii) Causal if and only if $h(t) = 0$ for $t < 0$ or $h[n] = 0$ for $n < 0$
 - (iii) Stable if and only if $\int_{-\infty}^{\infty} |h(t)|dt < \infty$ or $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$
- (a) $h(t) = \cos(\pi t)$
 - (i) has memory
 - (ii) not causal
 - (iii) not stable
 - (b) $h(t) = e^{-2t}u(t-1)$
 - (i) has memory
 - (ii) causal
 - (iii) stable
 - (c) $h(t) = u(t+1)$
 - (i) has memory
 - (ii) not causal
 - (iii) not stable
 - (d) $h(t) = 3\delta(t)$
 - (i) memoryless
 - (ii) causal
 - (iii) stable
 - (e) $h(t) = \cos(\pi t)u(t)$
 - (i) has memory
 - (ii) causal
 - (iii) not stable
 - (f) $h[n] = (-1)^n u[-n]$
 - (i) has memory
 - (ii) not causal
 - (iii) not stable
 - (g) $h[n] = (1/2)^{|n|}$
 - (i) has memory
 - (ii) not causal
 - (iii) stable
 - (h) $h[n] = \cos(\frac{\pi}{8}n)\{u[n] - u[n-10]\}$
 - (i) has memory
 - (ii) causal
 - (iii) stable
 - (i) $h[n] = 2u[n] - 2u[n-5]$
 - (i) has memory

- (ii) causal
- (iii) stable
- (j) $h[n] = \sin(\frac{\pi}{2}n)$
 - (i) has memory
 - (ii) not causal
 - (iii) not stable
- (k) $h[n] = \sum_{p=-1}^{\infty} \delta[n - 2p]$
 - (i) has memory
 - (ii) not causal
 - (iii) not stable

2.50. Evaluate the step response for the LTI systems represented by the following impulse responses:

(a) $h[n] = (-1/2)^n u[n]$

for $n < 0$

$$s[n] = 0$$

for $n \geq 0$

$$s[n] = \sum_{k=0}^n \left(-\frac{1}{2}\right)^k$$

$$s[n] = \frac{1}{3} \left(2 + \left(-\frac{1}{2}\right)^n \right)$$

$$s[n] = \begin{cases} \frac{1}{3} \left(2 + \left(-\frac{1}{2}\right)^n \right) & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(b) $h[n] = \delta[n] - \delta[n - 2]$

for $n < 0$

$$s[n] = 0$$

for $n = 0$

$$s[n] = 1$$

for $n \geq 1$

$$s[n] = 0$$

$$s[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

(c) $h[n] = (-1)^n \{u[n + 2] - u[n - 3]\}$

for $n < -2$

$$s[n] = 0$$

for $-2 \leq n \leq 2$

$$s[n] = \begin{cases} 1 & n = \pm 2, 0 \\ 0 & n = \pm 1 \end{cases}$$

for $n \geq 3$

$$s[n] = 1$$

(d) $h[n] = nu[n]$

for $n < 0$

$$s[n] = 0$$

for $n \geq 0$

$$s[n] = \sum_{k=0}^n k$$

$$s[n] = \begin{cases} \frac{n(n+1)}{2} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(e) $h(t) = e^{-|t|}$

for $t < 0$

$$s(t) = \int_{-\infty}^t e^{\tau} d\tau = e^t$$

for $t \geq 0$

$$s(t) = \int_{-\infty}^0 e^{\tau} d\tau + \int_0^t e^{-\tau} d\tau = 2 - e^{-t}$$

$$s(t) = \begin{cases} e^t & t < 0 \\ 2 - e^{-t} & t \geq 0 \end{cases}$$

(f) $h(t) = \delta^{(2)}(t)$

for $t < 0$

$$s(t) = 0$$

for $t \geq 0$

$$s(t) = \int_{-\infty}^t \delta^{(2)}(\tau) d\tau = \delta(t)$$

$$s(t) = \delta(t)$$

(g) $h(t) = (1/4)(u(t) - u(t-4))$

for $t < 0$

$$s(t) = 0$$

for $t < 4$

$$\begin{aligned}
& s(t) = \frac{1}{4} \int_0^t d\tau = \frac{1}{4}t \\
& \text{for } t \geq 4 \\
& s(t) = \frac{1}{4} \int_0^4 d\tau = 1 \\
& s(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{4}t & 0 \leq t < 4 \\ 1 & t \geq 4 \end{cases}
\end{aligned}$$

(h) $h(t) = u(t)$

$$\begin{aligned}
& \text{for } t < 0 \\
& s(t) = 0 \\
& \text{for } t \geq 0 \\
& s(t) = \int_0^t d\tau = t \\
& s(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}
\end{aligned}$$

2.51. Suppose the multipath propagation model is generalized to a k -step delay between the direct and reflected paths as shown by the input-output equation

$$y[n] = x[n] + ax[n - k]$$

Find the impulse response of the inverse system.

The inverse system must satisfy

$$\begin{aligned}
h^{inv}[n] + ah^{inv}[n - k] &= \delta[n] \\
h^{inv}[0] + ah^{inv}[-k] &= 1
\end{aligned}$$

Implies

$$h^{inv}[0] = 1$$

For the system to be causal

$$h^{inv}[n] = -ah^{inv}[n - k]$$

Which means $h^{inv}[n]$ is nonzero

only for positive multiples of k

$$h^{inv}[n] = \sum_{p=0}^{\infty} (-a)^p \delta[n - pk]$$

2.52. Write a differential equation description relating the output to the input of the following electrical circuits.

(a) Fig. P2.52 (a)

Writing node equations, assuming node A is the node the resistor, inductor and capacitor share.

$$(1) \quad y(t) + \frac{v_A(t) - x(t)}{R} + C \frac{d}{dt} v_A(t) = 0$$

For the inductor L:

$$(2) \quad v_A(t) = L \frac{d}{dt} y(t)$$

Combining (1) and (2) yields

$$\frac{1}{RLC} x(t) = \frac{d^2}{dt^2} y(t) + \frac{1}{RC} \frac{d}{dt} y(t) + \frac{1}{LC} y(t)$$

(b) Fig. P2.52 (b)

Writing node equations, assuming node A is the node the two resistors and C_2 share. $i(t)$ is the current through R_2 .

$$(1) \quad 0 = C_2 \frac{d}{dt} y(t) + \frac{y(t) - x(t)}{R_1} + i(t)$$

Implies

$$i(t) = -C_2 \frac{d}{dt} y(t) - \frac{y(t) - x(t)}{R_1}$$

For capacitor C_1

$$\begin{aligned} \frac{d}{dt} V_{C1}(t) &= \frac{i(t)}{C_1} \\ y(t) &= i(t) R_2 + V_{C1}(t) \end{aligned}$$

$$(2) \quad \frac{d}{dt} y(t) = R_2 \frac{d}{dt} i(t) + \frac{i(t)}{C_1}$$

Combining (1) and (2)

$$\frac{d^2}{dt^2} y(t) + \left(\frac{1}{C_2 R_2} + \frac{1}{C_2 R_1} + \frac{1}{C_1 R_2} \right) \frac{d}{dt} y(t) + \frac{1}{C_1 C_2 R_1 R_2} y(t) = \frac{1}{C_1 C_2 R_1 R_2} x(t) + \frac{1}{C_2 R_1} \frac{d}{dt} x(t)$$

2.53. Determine the homogeneous solution for the systems described by the following differential equations:

(a) $5 \frac{d}{dt} y(t) + 10 y(t) = 2 x(t)$

$$5r + 10 = 0$$

$$r = -2$$

$$y^{(h)}(t) = c_1 e^{-2t}$$

(b) $\frac{d^2}{dt^2} y(t) + 6 \frac{d}{dt} y(t) + 8 y(t) = \frac{d}{dt} x(t)$

$$r^2 + 6r + 8 = 0$$

$$r = -4, -2$$

$$y^{(h)}(t) = c_1 e^{-4t} + c_2 e^{-2t}$$

$$(c) \quad \frac{d^2}{dt^2}y(t) + 4y(t) = 3\frac{d}{dt}x(t)$$

$$\begin{aligned} r^2 + 4 &= 0 \\ r &= \pm j2 \\ y^{(h)}(t) &= c_1 e^{j2t} + c_2 e^{-j2t} \end{aligned}$$

$$(d) \quad \frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 2y(t) = x(t)$$

$$\begin{aligned} r^2 + 2r + 2 &= 0 \\ r &= -1 \pm j \\ y^{(h)}(t) &= c_1 e^{(-1+j)t} + c_2 e^{-(1+j)t} \end{aligned}$$

$$(e) \quad \frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t)$$

$$\begin{aligned} r^2 + 2r + 1 &= 0 \\ r &= -1, -1 \\ y^{(h)}(t) &= c_1 e^{-t} + c_2 t e^{-t} \end{aligned}$$

2.54. Determine the homogeneous solution for the systems described by the following difference equations:

$$(a) \quad y[n] - \alpha y[n-1] = 2x[n]$$

$$\begin{aligned} r - \alpha &= 0 \\ y^{(h)}[n] &= c_1 \alpha^n \end{aligned}$$

$$(b) \quad y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + x[n-1]$$

$$\begin{aligned} r^2 - \frac{1}{4}r - \frac{1}{8} &= 0 \\ r &= \frac{1}{2}, -\frac{1}{4} \\ y^{(h)}[n] &= c_1 \left(\frac{1}{2}\right)^n + c_2 \left(-\frac{1}{4}\right)^n \end{aligned}$$

$$(c) \quad y[n] + \frac{9}{16}y[n-2] = x[n-1]$$

$$r^2 + \frac{9}{16} = 0$$

$$\begin{aligned}
r &= \pm j \frac{3}{4} \\
y^{(h)}[n] &= c_1 \left(j \frac{3}{4} \right)^n + c_2 \left(-j \frac{3}{4} \right)^n
\end{aligned}$$

$$(d) \ y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1]$$

$$\begin{aligned}
r^2 + r + \frac{1}{4} &= 0 \\
r &= -\frac{1}{2}, -\frac{1}{2} \\
y^{(h)}[n] &= c_1 \left(-\frac{1}{2} \right)^n + c_2 n \left(-\frac{1}{2} \right)^n
\end{aligned}$$

2.55. Determine a particular solution for the systems described by the following differential equations for the given inputs:

$$(a) \ 5 \frac{d}{dt}y(t) + 10y(t) = 2x(t)$$

$$(i) \ x(t) = 2$$

$$\begin{aligned}
y^{(p)}(t) &= k \\
10k &= 2(2) \\
k &= \frac{2}{5} \\
y^{(p)}(t) &= \frac{2}{5}
\end{aligned}$$

$$(ii) \ x(t) = e^{-t}$$

$$\begin{aligned}
y^{(p)}(t) &= ke^{-t} \\
-5ke^{-t} + 10ke^{-t} &= 2e^{-t} \\
k &= \frac{2}{5} \\
y^{(p)}(t) &= \frac{2}{5}e^{-t}
\end{aligned}$$

$$(iii) \ x(t) = \cos(3t)$$

$$\begin{aligned}
y^{(p)}(t) &= A \cos(3t) + B \sin(3t) \\
\frac{d}{dt}y^{(p)}(t) &= -3A \sin(3t) + 3B \cos(3t) \\
5(-3A \sin(3t) + 3B \cos(3t)) + 10A \cos(3t) + 10B \sin(3t) &= 2 \cos(3t) \\
-15A + 10B &= 0 \\
10A + 15B &= 2
\end{aligned}$$

$$\begin{aligned}
A &= \frac{4}{65} \\
B &= \frac{6}{65} \\
y^{(p)}(t) &= \frac{4}{65} \cos(3t) + \frac{6}{65} \sin(3t)
\end{aligned}$$

(b) $\frac{d^2}{dt^2}y(t) + 4y(t) = 3\frac{d}{dt}x(t)$

(i) $x(t) = t$

$$\begin{aligned}
y^{(p)}(t) &= k_1 t + k_2 \\
4k_1 t + 4k_2 &= 3 \\
k_1 &= 0 \\
k_2 &= \frac{3}{4} \\
y^{(p)}(t) &= \frac{3}{4}
\end{aligned}$$

(ii) $x(t) = e^{-t}$

$$\begin{aligned}
y^{(p)}(t) &= ke^{-t} \\
ke^{-t} + 4ke^{-t} &= -3e^{-t} \\
k &= -\frac{3}{5} \\
y^{(p)}(t) &= -\frac{3}{5}e^{-t}
\end{aligned}$$

(iii) $x(t) = (\cos(t) + \sin(t))$

$$\begin{aligned}
y^{(p)}(t) &= A \cos(t) + B \sin(t) \\
\frac{d}{dt}y^{(p)}(t) &= -A \sin(t) + B \cos(t) \\
\frac{d^2}{dt^2}y^{(p)}(t) &= -A \cos(t) - B \sin(t) \\
-A \cos(t) - B \sin(t) + 4A \cos(t) + 4B \sin(t) &= -3 \sin(t) + 3 \cos(t) \\
-A + 4A &= 3 \\
-B + 4B &= -3 \\
A &= 1 \\
B &= -1 \\
y^{(p)}(t) &= \cos(t) - \sin(t)
\end{aligned}$$

(c) $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t)$

(i) $x(t) = e^{-3t}$

$$\begin{aligned}
y^{(p)}(t) &= ke^{-3t} \\
9ke^{-3t} - 6ke^{-3t} + ke^{-3t} &= -3e^{-3t} \\
k &= -\frac{3}{4} \\
y^{(p)}(t) &= -\frac{3}{4}e^{-3t}
\end{aligned}$$

(ii) $x(t) = 2e^{-t}$

Since e^{-t} and te^{-t} are in the natural response, the particular solution takes the form of

$$\begin{aligned}
y^{(p)}(t) &= kt^2e^{-t} \\
\frac{d}{dt}y^{(p)}(t) &= 2kte^{-t} - kt^2e^{-t} \\
\frac{d^2}{dt^2}y^{(p)}(t) &= 2ke^{-t} - 4kte^{-t} + kt^2e^{-t} \\
-2e^{-t} &= 2ke^{-t} - 4kte^{-t} + kt^2e^{-t} + 2(2kte^{-t} - kt^2e^{-t}) + kt^2e^{-t} \\
k &= -1 \\
y^{(p)}(t) &= -t^2e^{-t}
\end{aligned}$$

(iii) $x(t) = 2\sin(t)$

$$\begin{aligned}
y^{(p)}(t) &= A\cos(t) + B\sin(t) \\
\frac{d}{dt}y^{(p)}(t) &= -A\sin(t) + B\cos(t) \\
\frac{d^2}{dt^2}y^{(p)}(t) &= -A\cos(t) - B\sin(t) \\
-A\cos(t) - B\sin(t) - 2A\sin(t) + 2B\cos(t) + A\cos(t) + B\sin(t) &= 2\cos(t) \\
-A - 2B + A &= 2 \\
-B - 2A + B &= 0 \\
A &= 0 \\
B &= -1 \\
y^{(p)}(t) &= -\sin(t)
\end{aligned}$$

2.56. Determine a particular solution for the systems described by the following difference equations for the given inputs:

- (a) $y[n] - \frac{2}{5}y[n-1] = 2x[n]$
(i) $x[n] = 2u[n]$

$$\begin{aligned}
y^{(p)}[n] &= ku[n] \\
k - \frac{2}{5}k &= 4
\end{aligned}$$

$$\begin{aligned}k &= \frac{20}{3} \\ y^{(p)}[n] &= \frac{20}{3}u[n]\end{aligned}$$

$$(ii) \ x[n] = -(\frac{1}{2})^n u[n]$$

$$\begin{aligned}y^{(p)}[n] &= k \left(\frac{1}{2}\right)^n u[n] \\ k \left(\frac{1}{2}\right)^n - \frac{2}{5} \left(\frac{1}{2}\right)^{n-1} k &= -2 \left(\frac{1}{2}\right)^n \\ k &= -10 \\ y^{(p)}[n] &= -10 \left(\frac{1}{2}\right)^n u[n]\end{aligned}$$

$$(iii) \ x[n] = \cos(\frac{\pi}{5}n)$$

$$\begin{aligned}y^{(p)}[n] &= A \cos(\frac{\pi}{5}n) + B \sin(\frac{\pi}{5}n) \\ 2 \cos(\frac{\pi}{5}n) &= A \cos(\frac{\pi}{5}n) + B \sin(\frac{\pi}{5}n) - \frac{2}{5} \left[A \cos(\frac{\pi}{5}(n-1)) + B \sin(\frac{\pi}{5}(n-1)) \right] \\ \text{Using the trig identities} \\ \sin(\theta \pm \phi) &= \sin \theta \cos \phi \pm \cos \theta \sin \phi \\ \cos(\theta \pm \phi) &= \cos \theta \cos \phi \mp \sin \theta \sin \phi \\ y^{(p)}[n] &= 2.6381 \cos(\frac{\pi}{5}n) + 0.9170 \sin(\frac{\pi}{5}n)\end{aligned}$$

$$(b) \ y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + x[n-1]$$

$$(i) \ x[n] = nu[n]$$

$$\begin{aligned}y^{(p)}[n] &= k_1 nu[n] + k_2 u[n] \\ k_1 n + k_2 - \frac{1}{4}[k_1(n-1) + k_2] - \frac{1}{8}[k_1(n-2) + k_2] &= n + n - 1 \\ k_1 &= \frac{16}{5} \\ k_2 &= -\frac{104}{5} \\ y^{(p)}[n] &= \frac{16}{5}nu[n] - \frac{104}{5}u[n]\end{aligned}$$

$$(ii) \ x[n] = (\frac{1}{8})^n u[n]$$

$$y^{(p)}[n] = k \left(\frac{1}{8}\right)^n u[n]$$

$$\begin{aligned}
k \left(\frac{1}{8}\right)^n - \frac{1}{4} \left(\frac{1}{8}\right)^{n-1} k - \frac{1}{8} \left(\frac{1}{8}\right)^{n-2} k &= \left(\frac{1}{8}\right)^n + \left(\frac{1}{8}\right)^{n-1} \\
k &= -1 \\
y^{(p)}[n] &= -1 \left(\frac{1}{8}\right)^n u[n]
\end{aligned}$$

(iii) $x[n] = e^{j\frac{\pi}{4}n} u[n]$

$$\begin{aligned}
y^{(p)}[n] &= k e^{j\frac{\pi}{4}n} u[n] \\
k e^{j\frac{\pi}{4}n} - \frac{1}{4} k e^{j\frac{\pi}{4}(n-1)} - \frac{1}{8} k e^{j\frac{\pi}{4}(n-2)} &= e^{j\frac{\pi}{4}n} + e^{j\frac{\pi}{4}(n-1)} \\
k &= \frac{1 + e^{-j\frac{\pi}{4}}}{1 - \frac{1}{4} e^{-j\frac{\pi}{4}} - \frac{1}{8} k e^{-j\frac{\pi}{2}}}
\end{aligned}$$

(iv) $x[n] = \left(\frac{1}{2}\right)^n u[n]$

Since $\left(\frac{1}{2}\right)^n u[n]$ is in the natural response, the particular solution takes the form of:

$$\begin{aligned}
y^{(p)}[n] &= kn \left(\frac{1}{2}\right)^n u[n] \\
kn \left(\frac{1}{2}\right)^n - k \frac{1}{4} (n-1) \left(\frac{1}{2}\right)^{n-1} - k \frac{1}{8} (n-2) \left(\frac{1}{2}\right)^{n-2} &= \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} \\
k &= 2 \\
y^{(p)}[n][n] &= 2n \left(\frac{1}{2}\right)^n u[n]
\end{aligned}$$

(c) $y[n] + y[n-1] + \frac{1}{2}y[n-2] = x[n] + 2x[n-1]$

(i) $x[n] = u[n]$

$$\begin{aligned}
y^{(p)}[n][n] &= ku[n] \\
k + k + \frac{1}{2}k &= 2 + 2 \\
k &= \frac{8}{5} \\
y^{(p)}[n] &= \frac{8}{5} u[n]
\end{aligned}$$

(ii) $x[n] = \left(\frac{-1}{2}\right)^n u[n]$

$$\begin{aligned}
y^{(p)}[n] &= k \left(-\frac{1}{2}\right)^n u[n] \\
k \left(-\frac{1}{2}\right)^n + k \left(-\frac{1}{2}\right)^{n-1} + \frac{1}{2} \left(-\frac{1}{2}\right)^{n-2} k &= \left(-\frac{1}{2}\right)^n + 2 \left(-\frac{1}{2}\right)^{n-1} \\
k &= -3 \\
y^{(p)}[n] &= -3 \left(-\frac{1}{2}\right)^n u[n]
\end{aligned}$$

2.57. Determine the output of the systems described by the following differential equations with input and initial conditions as specified:

(a) $\frac{d}{dt}y(t) + 10y(t) = 2x(t)$, $y(0^-) = 1$, $x(t) = u(t)$

$$\begin{aligned} t \geq 0 & \quad \text{natural: characteristic equation} \\ r + 10 &= 0 \\ r &= -10 \\ y^{(n)}(t) &= ce^{-10t} \\ & \quad \text{particular} \\ y^{(p)}(t) &= ku(t) = \frac{1}{5}u(t) \\ y(t) &= \frac{1}{5} + ce^{-10t} \\ y(0^-) = 1 &= \frac{1}{5} + c \\ c &= \frac{4}{5} \\ y(t) &= \frac{1}{5} [1 + 4e^{-10t}] u(t) \end{aligned}$$

(b) $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 4y(t) = \frac{d}{dt}x(t)$, $y(0^-) = 0$, $\frac{d}{dt}y(t)|_{t=0^-} = 1$, $x(t) = \sin(t)u(t)$

$$\begin{aligned} t \geq 0 & \quad \text{natural: characteristic equation} \\ r^2 + 5r + 4 &= 0 \\ r &= -4, -1 \\ y^{(n)}(t) &= c_1e^{-4t} + c_2e^{-t} \\ & \quad \text{particular} \\ y^{(p)}(t) &= A\sin(t) + B\cos(t) \\ &= \frac{5}{34}\sin(t) + \frac{3}{34}\cos(t) \\ y(t) &= \frac{5}{34}\sin(t) + \frac{3}{34}\cos(t) + c_1e^{-4t} + c_2e^{-t} \\ y(0^-) = 0 &= \frac{3}{34} + c_1 + c_2 \\ \frac{d}{dt}y(t)\Big|_{t=0^-} = 1 &= \frac{5}{34} - 4c_1 - c_2 \\ c_1 &= -\frac{13}{51} \\ c_2 &= \frac{1}{6} \\ y(t) &= \frac{5}{34}\sin(t) + \frac{3}{34}\cos(t) - \frac{13}{51}e^{-4t} + \frac{1}{6}e^{-t} \end{aligned}$$

(c) $\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 8y(t) = 2x(t)$, $y(0^-) = -1$, $\frac{d}{dt}y(t)|_{t=0^-} = 1$, $x(t) = e^{-t}u(t)$

$$\begin{aligned}
t &\geq 0 && \text{natural: characteristic equation} \\
r^2 + 6r + 8 &= 0 \\
r &= -4, -2 \\
y^{(n)}(t) &= c_1 e^{-2t} + c_2 e^{-4t} \\
&\text{particular} \\
y^{(p)}(t) &= k e^{-t} u(t) \\
&= \frac{2}{3} e^{-t} u(t) \\
y(t) &= \frac{2}{3} e^{-t} u(t) + c_1 e^{-2t} + c_2 e^{-4t} \\
y(0^-) = -1 &= \frac{2}{3} + c_1 + c_2 \\
\left. \frac{d}{dt} y(t) \right|_{t=0^-} = 1 &= -\frac{2}{3} - 2c_1 - 4c_2 \\
c_1 &= -\frac{5}{2} \\
c_2 &= \frac{5}{6} \\
y(t) &= \frac{2}{3} e^{-t} u(t) - \frac{5}{2} e^{-2t} + \frac{5}{6} e^{-4t}
\end{aligned}$$

(d) $\frac{d^2}{dt^2} y(t) + y(t) = 3 \frac{d}{dt} x(t), \quad y(0^-) = -1, \quad \left. \frac{d}{dt} y(t) \right|_{t=0^-} = 1, \quad x(t) = 2te^{-t} u(t)$

$$\begin{aligned}
t &\geq 0 && \text{natural: characteristic equation} \\
r^2 + 1 &= 0 \\
r &= \pm j \\
y^{(n)}(t) &= A \cos(t) + B \sin(t) \\
&\text{particular} \\
y^{(p)}(t) &= k t e^{-t} u(t) \\
\frac{d^2}{dt^2} y^{(p)}(t) &= -2k e^{-t} + k t e^{-t} \\
-2k e^{-t} + k t e^{-t} + k t e^{-t} &= 3[2e^{-t} - 2t e^{-t}] \\
k &= -3 \\
y^{(p)}(t) &= -3t e^{-t} u(t) \\
y(t) &= -3t e^{-t} u(t) + A \cos(t) + B \sin(t) \\
y(0^-) &= -1 = 0 + A + 0 \\
\left. \frac{d}{dt} y(t) \right|_{t=0^-} &= 1 = -3 + 0 + B \\
y(t) &= -3t e^{-t} u(t) - \cos(t) + 4 \sin(t)
\end{aligned}$$

2.58. Identify the natural and forced response for the systems in Problem 2.57.
See problem 2.57 for the derivations of the natural and forced response.

(a)

(i) Natural Response

$$\begin{aligned} r + 10 &= 0 \\ r &= -10 \\ y^{(n)}(t) &= c_1 e^{-10t} \\ y(0^-) = 1 &= c_1 \\ y^{(n)}(t) &= e^{-10t} \end{aligned}$$

(ii) Forced Response

$$\begin{aligned} y^{(f)}(t) &= \frac{1}{5} + k e^{-10t} \\ y(0) = 0 &= \frac{1}{5} + k \\ y^{(f)}(t) &= \frac{1}{5} - \frac{1}{5} e^{-10t} \end{aligned}$$

(b)

(i) Natural Response

$$\begin{aligned} y^{(n)}(t) &= c_1 e^{-4t} + c_2 e^{-t} \\ y(0^-) = 0 &= c_1 + c_2 \\ \left. \frac{d}{dt} y(t) \right|_{t=0^-} = 1 &= -4c_1 - c_2 \\ y^{(n)}(t) &= -\frac{1}{3} e^{-4t} + \frac{1}{3} e^{-t} \end{aligned}$$

(ii) Forced Response

$$\begin{aligned} y^{(f)}(t) &= \frac{5}{34} \sin(t) + \frac{3}{34} \cos(t) + c_1 e^{-4t} + c_2 e^{-t} \\ y(0) = 0 &= \frac{3}{34} + c_1 + c_2 \\ \left. \frac{d}{dt} y(t) \right|_{t=0^-} = 0 &= \frac{5}{34} - 4c_1 - c_2 \\ y^{(f)}(t) &= \frac{5}{34} \sin(t) + \frac{3}{34} \cos(t) + \frac{4}{51} e^{-4t} - \frac{1}{6} e^{-t} \end{aligned}$$

(c)

(i) Natural Response

$$\begin{aligned}y^{(n)}(t) &= c_1 e^{-4t} + c_2 e^{-2t} \\y(0^-) = -1 &= c_1 + c_2 \\ \left. \frac{d}{dt} y(t) \right|_{t=0^-} = 1 &= -4c_1 - 2c_2 \\y^{(n)}(t) &= \frac{1}{2} e^{-4t} - \frac{3}{2} e^{-2t}\end{aligned}$$

(ii) Forced Response

$$\begin{aligned}y^{(f)}(t) &= \frac{2}{3} e^{-t} u(t) + c_1 e^{-2t} u(t) + c_2 e^{-4t} u(t) \\y(0) = 0 &= \frac{2}{3} + c_1 + c_2 \\ \left. \frac{d}{dt} y(t) \right|_{t=0^-} = 0 &= -\frac{2}{3} - 2c_1 - 4c_2 \\y^{(f)}(t) &= \frac{2}{3} e^{-t} u(t) - e^{-2t} u(t) + \frac{1}{3} e^{-4t} u(t)\end{aligned}$$

(d)

(i) Natural Response

$$\begin{aligned}y^{(n)}(t) &= c_1 \cos(t) + c_2 \sin(t) \\y(0^-) = -1 &= c_1 \\ \left. \frac{d}{dt} y(t) \right|_{t=0^-} = 1 &= c_2 \\y^{(n)}(t) &= -\cos(t) + \sin(t)\end{aligned}$$

(ii) Forced Response

$$\begin{aligned}y^{(f)}(t) &= -3te^{-t} u(t) + c_1 \cos(t) u(t) + c_2 \sin(t) u(t) \\y(0) = 0 &= c_1 + \\ \left. \frac{d}{dt} y(t) \right|_{t=0^-} = 0 &= -3 + c_2 \\y^{(f)}(t) &= -3te^{-t} u(t) + 3 \sin(t) u(t)\end{aligned}$$

2.59. Determine the output of the systems described by the following difference equations with input and initial conditions as specified:

$$(a) \ y[n] - \frac{1}{2}y[n-1] = 2x[n], \quad y[-1] = 3, x[n] = \left(\frac{-1}{2}\right)^n u[n]$$

$$\begin{aligned}
n &\geq 0 && \text{natural: characteristic equation} \\
r - \frac{1}{2} &= 0 \\
y^{(n)}[n] &= c \left(\frac{1}{2}\right)^n \\
&&& \text{particular} \\
y^{(p)}[n] &= k \left(-\frac{1}{2}\right)^n u[n] \\
k \left(-\frac{1}{2}\right)^n - \frac{1}{2}k \left(-\frac{1}{2}\right)^{n-1} &= 2 \left(-\frac{1}{2}\right)^n \\
k &= 1 \\
y^{(p)}[n] &= \left(-\frac{1}{2}\right)^n u[n] \\
&&& \text{Translate initial conditions} \\
y[n] &= \frac{1}{2}y[n-1] + 2x[n] \\
y[0] &= \frac{1}{2}3 + 2 = \frac{7}{2} \\
y[n] &= \left(-\frac{1}{2}\right)^n u[n] + c \left(\frac{1}{2}\right)^n u[n] \\
\frac{7}{2} &= 1 + c \\
c &= \frac{5}{2} \\
y[n] &= \left(-\frac{1}{2}\right)^n u[n] + \frac{5}{2} \left(\frac{1}{2}\right)^n u[n]
\end{aligned}$$

$$(b) \ y[n] - \frac{1}{9}y[n-2] = x[n-1], y[-1] = 1, y[-2] = 0, x[n] = u[n]$$

$$\begin{aligned}
n &\geq 0 && \text{natural: characteristic equation} \\
r^2 - \frac{1}{9} &= 0 \\
r &= \pm \frac{1}{3} \\
y^{(n)}[n] &= c_1 \left(\frac{1}{3}\right)^n + c_2 \left(-\frac{1}{3}\right)^n \\
&&& \text{particular} \\
y^{(p)}[n] &= ku[n] \\
k - \frac{1}{9}k &= 1 \\
k &= \frac{9}{8} \\
y^{(p)}[n] &= \frac{9}{8}u[n]
\end{aligned}$$

$$\begin{aligned}
y[n] &= \frac{9}{8}u[n] + c_1 \left(\frac{1}{3}\right)^n + c_2 \left(-\frac{1}{3}\right)^n \\
&\text{Translate initial conditions} \\
y[n] &= \frac{1}{9}y[n-2] + x[n-1] \\
y[0] &= \frac{1}{9}0 + 0 = 0 \\
y[1] &= \frac{1}{9}1 + 1 = \frac{10}{9} \\
0 &= \frac{9}{8} + c_1 + c_2 \\
\frac{10}{9} &= \frac{9}{8} + \frac{1}{3}c_1 - \frac{1}{3}c_2 \\
y[n] &= \frac{9}{8}u[n] - \frac{7}{12}\left(\frac{1}{3}\right)^n - \frac{13}{24}\left(-\frac{1}{3}\right)^n
\end{aligned}$$

(c) $y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + x[n-1], \quad y[-1] = 4, y[-2] = -2, x[n] = (-1)^n u[n]$

$$\begin{aligned}
n &\geq 0 && \text{natural: characteristic equation} \\
r^2 + \frac{1}{4}r - \frac{1}{8} &= 0 \\
r &= -\frac{1}{4}, \frac{1}{2} \\
y^{(n)}[n] &= c_1 \left(\frac{1}{2}\right)^n + c_2 \left(-\frac{1}{4}\right)^n \\
&\text{particular} \\
y^{(p)}[n] &= k(-1)^n u[n] \\
&\text{for } n \geq 1 \\
k(-1)^n + k\frac{1}{4}(-1)^{n-1} - k\frac{1}{8}(-1)^{n-2} &= (-1)^n + (-1)^{n-1} = 0 \\
k &= 0 \\
y^{(p)}[n] &= 0 \\
y[n] &= c_1 \left(\frac{1}{2}\right)^n + c_2 \left(-\frac{1}{4}\right)^n \\
&\text{Translate initial conditions} \\
y[n] &= -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n] + x[n-1] \\
y[0] &= \frac{1}{4}4 + \frac{1}{8}(-2) + 1 + 0 = -\frac{1}{4} \\
y[1] &= -\frac{1}{4}\left(-\frac{1}{4}\right) + \frac{1}{8}4 + -1 + 1 = \frac{9}{16} \\
-\frac{1}{4} &= c_1 + c_2 \\
\frac{9}{16} &= \frac{1}{2}c_1 - \frac{1}{4}c_2 \\
y[n] &= \frac{2}{3}\left(\frac{1}{2}\right)^n - \frac{11}{12}\left(-\frac{1}{4}\right)^n
\end{aligned}$$

$$(d) \ y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n], \quad y[-1] = 1, y[-2] = -1, x[n] = 2u[n]$$

$$\begin{aligned} n &\geq 0 && \text{natural: characteristic equation} \\ r^2 - \frac{3}{4}r + \frac{1}{8} &= 0 \\ r &= \frac{1}{4}, \frac{1}{2} \\ y^{(n)}[n] &= c_1 \left(\frac{1}{2}\right)^n + c_2 \left(\frac{1}{4}\right)^n \\ &\text{particular} \\ y^{(p)}[n] &= ku[n] \\ k - k\frac{3}{4} + k\frac{1}{8} &= 4 \\ k &= \frac{32}{3} \\ y^{(p)}[n] &= \frac{32}{3}u[n] \\ y[n] &= \frac{32}{3}u[n] + c_1 \left(\frac{1}{2}\right)^n + c_2 \left(\frac{1}{4}\right)^n \\ &\text{Translate initial conditions} \\ y[n] &= \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2] + 2x[n] \\ y[0] &= \frac{3}{4}1 - \frac{1}{8}(-1) + 2(2) = \frac{39}{8} \\ y[1] &= \frac{3}{4}\left(\frac{39}{8}\right) - \frac{1}{8}1 + 2(2) = \frac{241}{32} \\ \frac{39}{8} &= \frac{32}{3} + c_1 + c_2 \\ \frac{241}{32} &= \frac{32}{3} + \frac{1}{2}c_1 + \frac{1}{4}c_2 \\ y[n] &= \frac{32}{3}u[n] - \frac{27}{4}\left(\frac{1}{2}\right)^n + \frac{23}{24}\left(\frac{1}{4}\right)^n \end{aligned}$$

2.60. Identify the natural and forced response for the systems in Problem 2.59. (a)

(i) Natural Response

$$\begin{aligned} y^{(n)}[n] &= c \left(\frac{1}{2}\right)^n \\ y[-1] = 3 &= c \left(\frac{1}{2}\right)^{-1} \\ c &= \frac{3}{2} \\ y^{(n)}[n] &= \frac{3}{2} \left(\frac{1}{2}\right)^n \end{aligned}$$

(ii) Forced Response

$$\begin{aligned}
 y^{(f)}[n] &= k \left(\frac{1}{2} \right)^n + \left(-\frac{1}{2} \right)^n \\
 &\quad \text{Translate initial conditions} \\
 y[n] &= \frac{1}{2}y[n-1] + 2x[n] \\
 y[0] &= \frac{1}{2}(0) + 2 = 2 \\
 y[0] = 2 &= k + 1 \\
 k &= 1 \\
 y^{(f)}[n] &= \left(\frac{1}{2} \right)^n + \left(-\frac{1}{2} \right)^n
 \end{aligned}$$

(b)

(i) Natural Response

$$\begin{aligned}
 y^{(n)}[n] &= c_1 \left(\frac{1}{3} \right)^n u[n] + c_2 \left(-\frac{1}{3} \right)^n u[n] \\
 y[-2] = 0 &= c_1 \left(\frac{1}{3} \right)^{-2} + c_2 \left(-\frac{1}{3} \right)^{-2} \\
 y[-1] = 1 &= c_1 \left(\frac{1}{3} \right)^{-1} + c_2 \left(-\frac{1}{3} \right)^{-1} \\
 c_1 &= \frac{1}{6} \\
 c_2 &= -\frac{1}{6} \\
 y^{(n)}[n] &= \frac{1}{6} \left(\frac{1}{3} \right)^n u[n] - \frac{1}{6} \left(-\frac{1}{3} \right)^n u[n]
 \end{aligned}$$

(ii) Forced Response

$$\begin{aligned}
 y^{(f)}[n] &= \frac{9}{8}u[n] + c_1 \left(\frac{1}{3} \right)^n u[n] + c_2 \left(-\frac{1}{3} \right)^n u[n] \\
 &\quad \text{Translate initial conditions} \\
 y[n] &= \frac{1}{9}y[n-2] + x[n-1] \\
 y[0] &= \frac{1}{9}0 + 0 = 0 \\
 y[1] &= \frac{1}{9}0 + 1 = 1 \\
 y[0] = 0 &= \frac{9}{8} + c_1 + c_2 \\
 y[1] = 1 &= \frac{9}{8} + \frac{1}{3}c_1 - \frac{1}{3}c_2 \\
 y^{(f)}[n] &= \frac{9}{8}u[n] - \frac{3}{4} \left(\frac{1}{3} \right)^n u[n] - \frac{3}{8} \left(-\frac{1}{3} \right)^n u[n]
 \end{aligned}$$

(c)

(i) Natural Response

$$\begin{aligned}y^{(n)}[n] &= c_1 \left(-\frac{1}{2}\right)^n u[n] + c_2 \left(\frac{1}{4}\right)^n u[n] \\y[-2] = -2 &= c_1 \left(-\frac{1}{2}\right)^{-2} + c_2 \left(\frac{1}{4}\right)^{-2} \\y[-1] = 4 &= c_1 \left(-\frac{1}{2}\right)^{-1} + c_2 \left(\frac{1}{4}\right)^{-1} \\c_1 &= -\frac{3}{2} \\c_2 &= \frac{1}{4} \\y^{(n)}[n] &= -\frac{3}{2} \left(-\frac{1}{2}\right)^n u[n] - \frac{3}{8} \left(\frac{1}{4}\right)^n u[n]\end{aligned}$$

(ii) Forced Response

$$\begin{aligned}y^{(f)}[n] &= \frac{16}{5}(-1)^n u[n] + c_1 \left(-\frac{1}{2}\right)^n u[n] + c_2 \left(\frac{1}{4}\right)^n u[n] \\&\text{Translate initial conditions} \\y[n] &= -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n] + x[n-1] \\y[0] &= -\frac{1}{4}0 + \frac{1}{8}0 + 1 + 0 = 1 \\y[1] &= -\frac{1}{4}1 + \frac{1}{8}0 - 1 + 1 = -\frac{1}{4} \\y[0] = 1 &= \frac{16}{5} + c_1 + c_2 \\y[1] = -\frac{1}{4} &= -\frac{16}{5} - \frac{1}{2}c_1 + \frac{1}{4}c_2 \\y^{(f)}[n] &= \frac{16}{5}(-1)^n u[n] - \frac{14}{3} \left(-\frac{1}{2}\right)^n u[n] + \frac{37}{15} \left(\frac{1}{4}\right)^n u[n]\end{aligned}$$

(d)

(i) Natural Response

$$\begin{aligned}y^{(n)}[n] &= c_1 \left(\frac{1}{2}\right)^n u[n] + c_2 \left(\frac{1}{4}\right)^n u[n] \\y[-2] = -1 &= c_1 \left(\frac{1}{2}\right)^{-2} + c_2 \left(\frac{1}{4}\right)^{-2} \\y[-1] = 1 &= c_1 \left(\frac{1}{2}\right)^{-1} + c_2 \left(\frac{1}{4}\right)^{-1} \\y^{(n)}[n] &= \frac{5}{4} \left(\frac{1}{2}\right)^n u[n] - \frac{3}{8} \left(\frac{1}{4}\right)^n u[n]\end{aligned}$$

(ii) Forced Response

$$\begin{aligned}
 y^{(f)}[n] &= \frac{32}{3}u[n] + c_1 \left(\frac{1}{2}\right)^n u[n] + c_2 \left(\frac{1}{4}\right)^n u[n] \\
 &\text{Translate initial conditions} \\
 y[n] &= \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2] + 2x[n] \\
 y[0] &= \frac{3}{4}0 - \frac{1}{8}0 + 2(2) = 4 \\
 y[1] &= \frac{3}{4}4 - \frac{1}{8}0 + 2(2) = 7 \\
 y[0] = 4 &= \frac{32}{3} + c_1 + c_2 \\
 y[1] = 7 &= \frac{32}{3} + \frac{1}{2}c_1 + \frac{1}{4}c_2 \\
 y^{(f)}[n] &= \frac{32}{3}u[n] - 8\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3}\left(\frac{1}{4}\right)^n u[n]
 \end{aligned}$$

2.61. Write a differential equation relating the output $y(t)$ to the circuit in Fig. P2.61 and find the step response by applying an input $x(t) = u(t)$. Next use the step response to obtain the impulse response. *Hint:* Use principles of circuit analysis to translate the $t = 0^-$ initial conditions to $t = 0^+$ before solving for the undetermined coefficients in the homogeneous component of the complete solution.

$$\begin{aligned}
 x(t) &= i_R(t) + i_L(t) + i_c(t) \\
 y(t) &= i_R(t) \\
 y(t) &= L \frac{d}{dt} i_L(t) \\
 i_L(t) &= \frac{1}{L} \int_0^t y(\tau) d\tau + i_L(0^-) \\
 i_c(t) &= C \frac{d}{dt} y(t) \\
 x(t) &= y(t) + \frac{1}{L} \int_0^t y(\tau) d\tau + i_L(0^-) + C \frac{d}{dt} y(t) \\
 \frac{d}{dt} x(t) &= C \frac{d^2}{dt^2} y(t) + \frac{d}{dt} y(t) + \frac{1}{L} y(t)
 \end{aligned}$$

Given $y(0^-) = 0$, $\frac{d}{dt} y(t)|_{t=0^-} = 0$, find $y(0^+)$, $\frac{d}{dt} y(t)|_{t=0^+}$.

Since current cannot change instantaneously through an inductor and voltage cannot change instantaneously across a capacitor,

$$\begin{aligned}
 i_L(0^+) &= 0 \\
 y(0^+) &= y(0^-) \\
 \text{Implies} \\
 i_R(0^+) &= 0
 \end{aligned}$$

$$\begin{aligned}
i_C(0^+) &= 1 \quad \text{since} \\
i_C(t) &= C \frac{d}{dt} y(t) \\
\left. \frac{d}{dt} y(t) \right|_{t=0^+} &= \frac{i_C(0^+)}{C} = \frac{1}{C}
\end{aligned}$$

Solving for the step response,

$$5 \frac{d}{dt} x(t) = \frac{d^2}{dt^2} y(t) + 5 \frac{d}{dt} y(t) + 20 y(t)$$

Finding the natural response

$$\begin{aligned}
r^2 + 5r + 20 &= 0 \\
r &= \frac{-5 \pm j\sqrt{55}}{2} \\
y^{(n)}(t) &= c_1 e^{-\frac{5}{2}t} \cos(\omega_o t) + c_2 e^{-\frac{5}{2}t} \sin(\omega_o t) \\
\omega_o &= \frac{\sqrt{55}}{2}
\end{aligned}$$

Particular

$$\begin{aligned}
y^{(p)}(t) &= k \\
x(t) &= u(t) \\
0 &= 0 + 0 + 20k \\
k &= 0
\end{aligned}$$

Step Response

$$y(t) = y^{(n)}(t)$$

Using the initial conditions to solve for the constants

$$\begin{aligned}
y(0^+) = 0 &= c_1 \\
\left. \frac{d}{dt} y(t) \right|_{t=0^+} = 5 &= c_2 \omega_o \\
y(t) &= \frac{5}{\omega_o} e^{-\frac{5}{2}t} \sin(\omega_o t) \\
&= \frac{10}{\sqrt{55}} e^{-\frac{5}{2}t} \sin\left(\frac{\sqrt{55}}{2}t\right)
\end{aligned}$$

2.62. Use the first-order difference equation to calculate the monthly balance of a \$100,000 loan at 1% per month interest assuming monthly payments of \$1200. Identify the natural and forced response. In this case the natural response represents the loan balance assuming no payments are made. How many payments are required to pay off the loan?

$$\begin{aligned}
y[n] - 1.01y[n-1] &= x[n] \\
x[n] &= -1200
\end{aligned}$$

Natural

$$r - 1.01 = 0$$

$$\begin{aligned}
y^{(n)}[n] &= c_n(1.01)^n \\
y[-1] = 100000 &= c_n(1.01)^{-1} \\
y^{(n)}[n] &= 101000(1.01)^n \\
\text{Particular} \\
y^{(p)}[n] &= c_p \\
c_p - 1.01c_p &= -1200 \\
y^{(p)}[n] &= 120000 \\
\text{Forced} \\
&\text{Translate initial conditions} \\
y[0] &= 1.01y[-1] + x[0] \\
&= 1.01(100000) - 1200 = -1200 \\
y^{(f)}[n] &= 120000 + c_f(1.01)^n \\
y^{(f)}[0] = -1200 &= 120000 + c_f \\
y^{(f)}[n] &= 120000 - 121200(1.01)^n
\end{aligned}$$

To solve for the required payments to pay off the loan, add the natural and forced response to obtain the complete solution, and solve for the number of payments to where the complete response equals zero.

$$\begin{aligned}
y^{(c)}[n] &= 120000 - 20200(1.01)^n \\
y^{(c)}[\tilde{n}] = 0 &= 120000 - 20200(1.01)^{\tilde{n}} \\
n &\cong 179
\end{aligned}$$

Since the first payment is made at $n = 0$, 180 payments are required to pay off the loan.

2.63. Determine the monthly payments required to pay off the loan in Problem 2.62 in 30 years (360 payments) and 15 years (180 payments).

$$\begin{aligned}
p &= 1.01 \\
y[-1] &= 100000 \\
x[n] &= b \\
&\text{The homogeneous solution is} \\
y^{(h)}[n] &= c_h(1.01)^n \\
&\text{The particular solution is} \\
y^{(p)}[n] &= c_p \\
y[n] - 1.01y[n-1] &= x[n] \\
&\text{Solving for } c_p \\
c_p - 1.01c_p &= b
\end{aligned}$$

$$c_p = -100b$$

The complete solution is of the form

$$y[n] = c_h(1.01)^n - 100b$$

Translating the initial conditions

$$\begin{aligned} y[0] &= 1.01y[-1] + x[0] \\ &= 101000 + b \end{aligned}$$

$$101000 + b = c_h - 100b$$

$$c_h = 101000 + 101b$$

$$y[n] = (101000 + 101b)1.01^n - 100b$$

Now solving for b by setting $y[359] = 0$

$$b = \frac{-101000(1.01)^{359}}{(101)1.01^{359} - 100} = -1028.61$$

Hence a monthly payment of \$1028.61 will pay off the loan in 30 years.

Now solving for b by setting $y[179] = 0$

$$b = \frac{-101000(1.01)^{179}}{(101)1.01^{179} - 100} = -1200.17$$

A monthly payment of \$1200.17 will pay off the loan in 15 years.

2.64. The portion of a loan payment attributed to interest is given by multiplying the balance after the previous payment was credited times $\frac{r}{100}$ where r is the rate per period expressed in percent. Thus if $y[n]$ is the loan balance after the n^{th} payment, the portion of the n^{th} payment required to cover the interest cost is $y[n-1](r/100)$. The cumulative interest paid over payments for period n_1 through n_2 is thus

$$I = (r/100) \sum_{n=n_1}^{n_2} y[n-1]$$

Calculate the total interest paid over the life of the 30-year and 15-year loans described in Problem 2.63.

For the 30-year loan:

$$I = (1/100) \sum_{n=0}^{359} y[n-1] = \$270308.77$$

For the 15-year loan:

$$I = (1/100) \sum_{n=0}^{179} y[n-1] = \$116029.62$$

2.65. Find difference-equation descriptions for the three systems depicted in Fig. P2.65. (a)

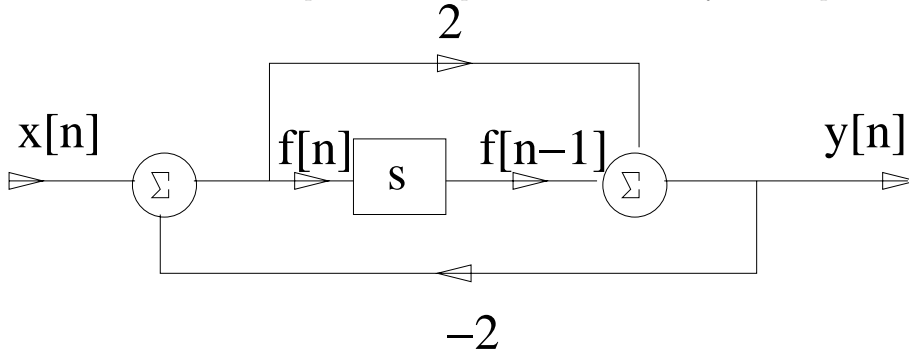


Figure P2.65. (a) Block diagram

$$\begin{aligned}
 f[n] &= -2y[n] + x[n] \\
 y[n] &= f[n-1] + 2f[n] \\
 &= -2y[n-1] + x[n-1] - 4y[n] + 2x[n] \\
 5y[n] + 2y[n-1] &= x[n-1] + 2x[n]
 \end{aligned}$$

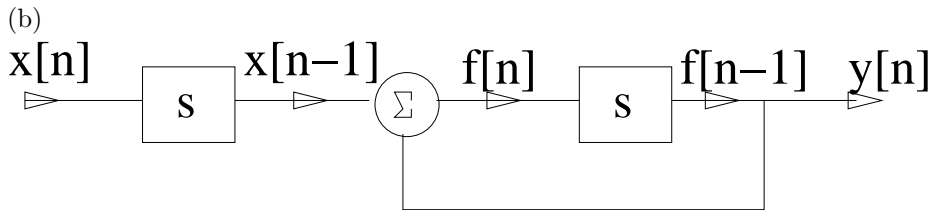


Figure P2.65. (b) Block diagram

$$\begin{aligned}
 f[n] &= y[n] + x[n-1] \\
 y[n] &= f[n-1] \\
 &= y[n-1] + x[n-2]
 \end{aligned}$$

(c)

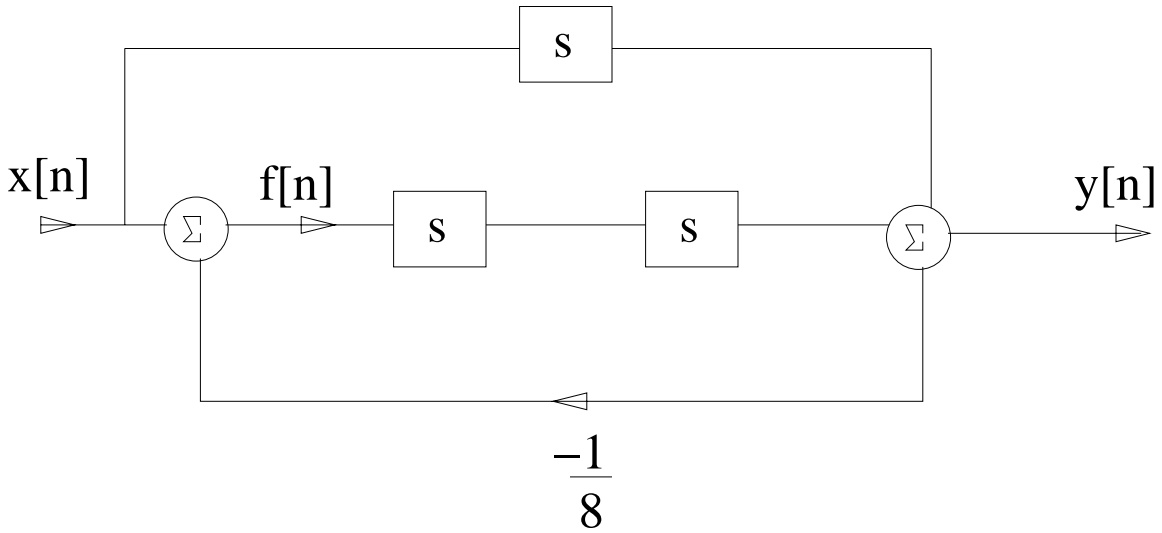


Figure P2.65. (c) Block diagram

$$\begin{aligned} f[n] &= x[n] - \frac{1}{8}y[n] \\ y[n] &= x[n-1] + f[n-2] \\ y[n] + \frac{1}{8}y[n-2] &= x[n-1] + x[n-2] \end{aligned}$$

2.66. Draw direct form I and direct form II implementations for the following difference equations.

(a) $y[n] - \frac{1}{4}y[n-1] = 6x[n]$

(i) Direct Form I

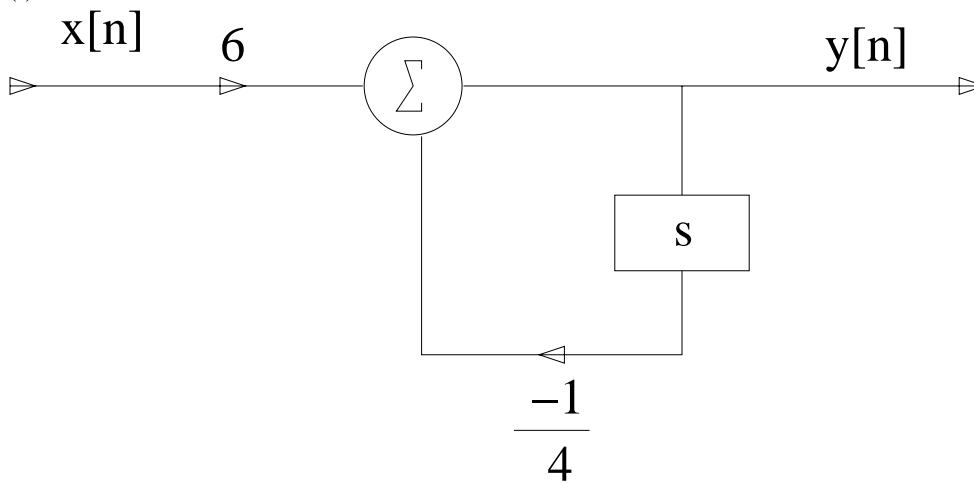
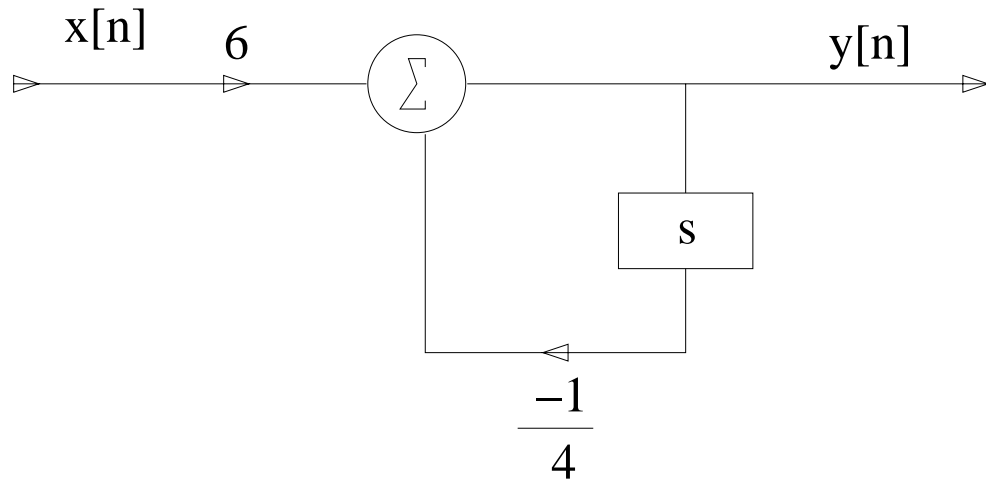


Figure P2.66. (a) Direct form I



(ii) Direct form II

Figure P2.66. (a) Direct form II

(b) $y[n] + \frac{1}{2}y[n-1] - \frac{1}{8}y[n-2] = x[n] + 2x[n-1]$

(i) Direct Form I

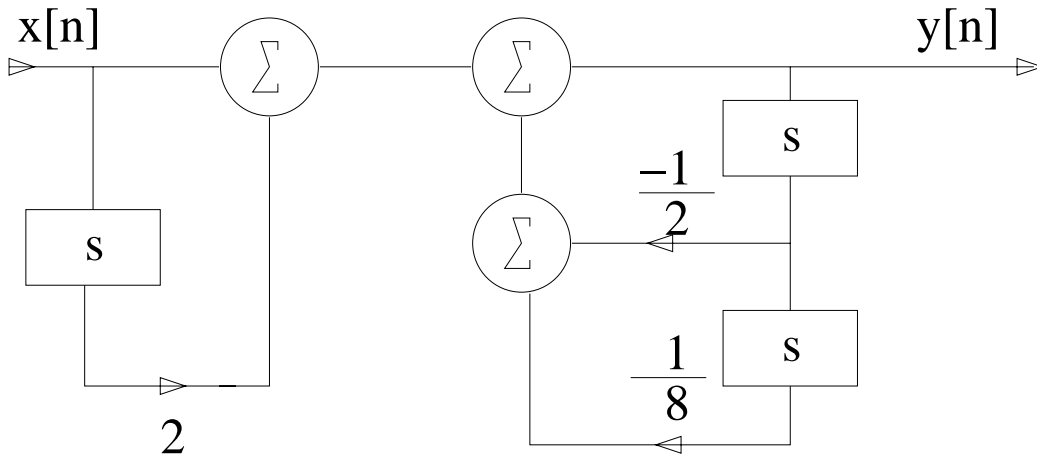
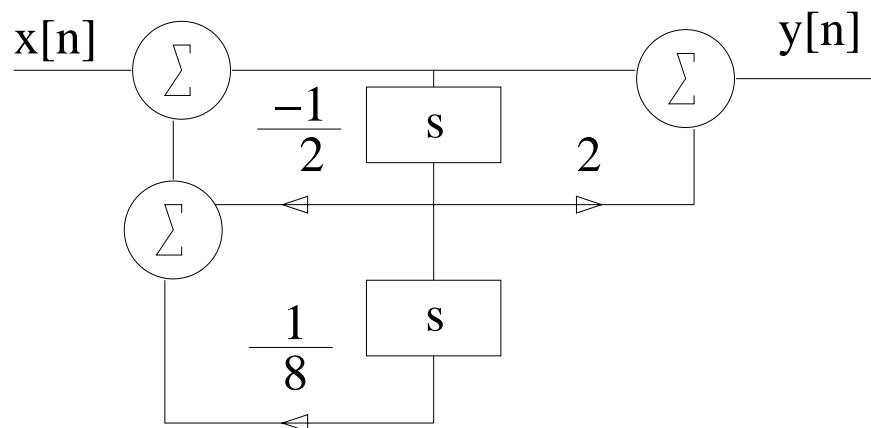


Figure P2.66. (b) Direct form I



(ii) Direct form II

Figure P2.66. (b) Direct form II

(c) $y[n] - \frac{1}{9}y[n-2] = x[n-1]$

(i) Direct Form I

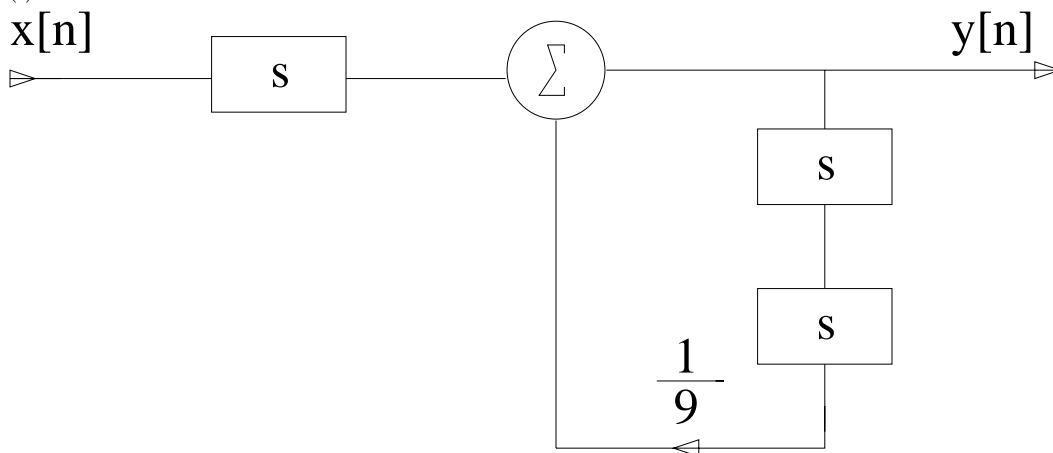
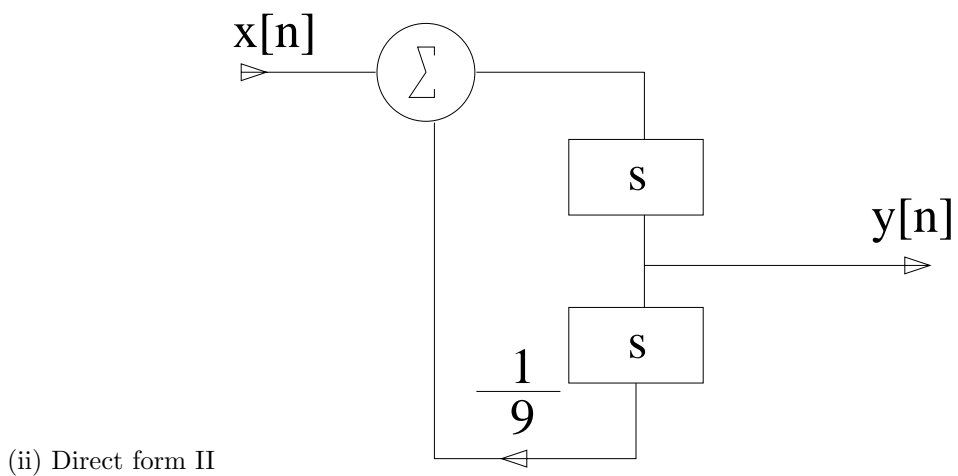


Figure P2.66. (c) Direct form I



(ii) Direct form II

Figure P2.66. (c) Direct form II

(d) $y[n] + \frac{1}{2}y[n-1] - y[n-3] = 3x[n-1] + 2x[n-2]$

(i) Direct Form I

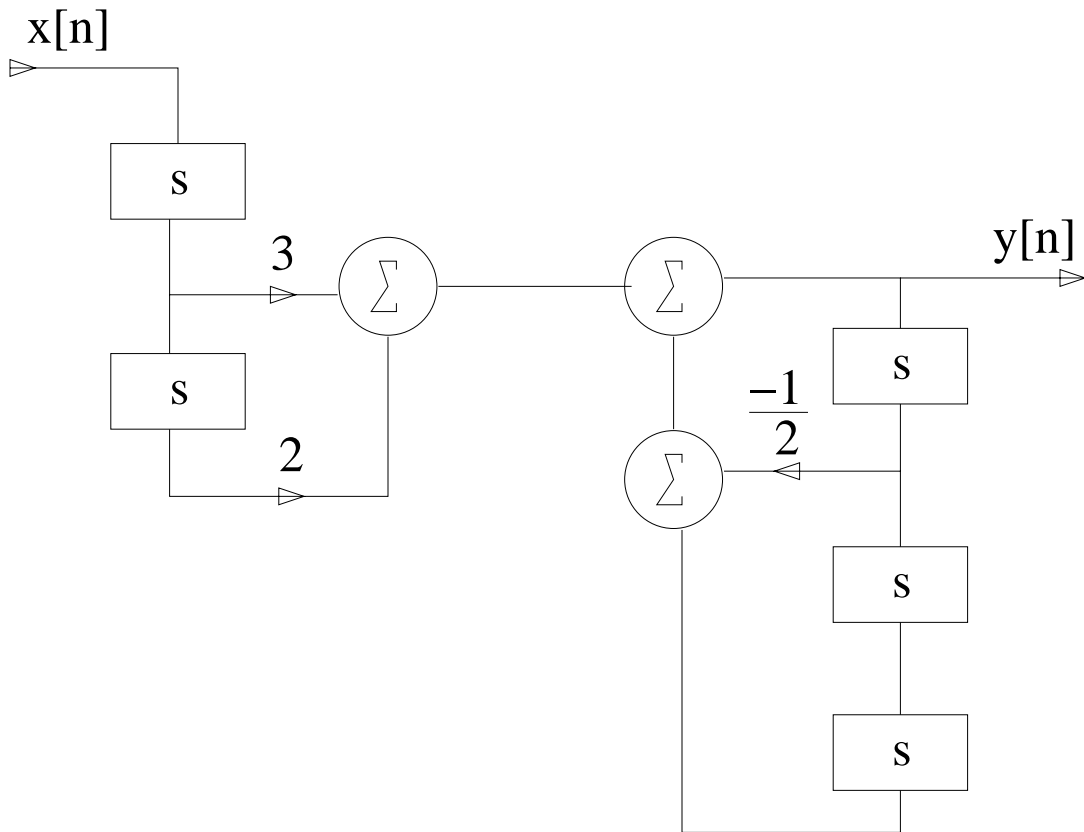
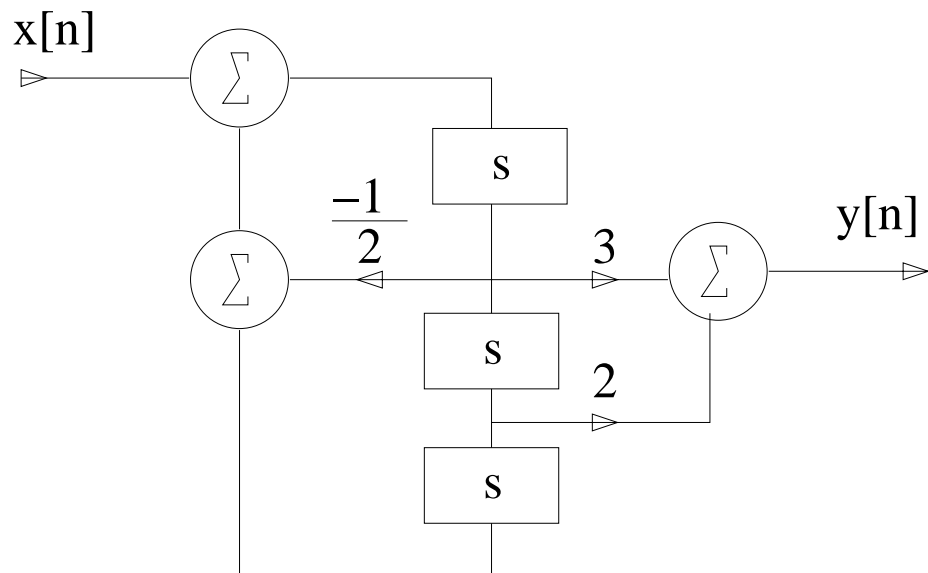


Figure P2.66. (d) Direct form I



(ii) Direct form II

Figure P2.66. (d) Direct form II

2.67. Convert the following differential equations to integral equations and draw direct form I and direct form II implementations of the corresponding systems.

(a) $\frac{d}{dt}y(t) + 10y(t) = 2x(t)$

$$\begin{aligned} y(t) + 10y^{(1)}(t) &= 2x^{(1)}(t) \\ y(t) &= 2x^{(1)}(t) - 10y^{(1)}(t) \end{aligned}$$

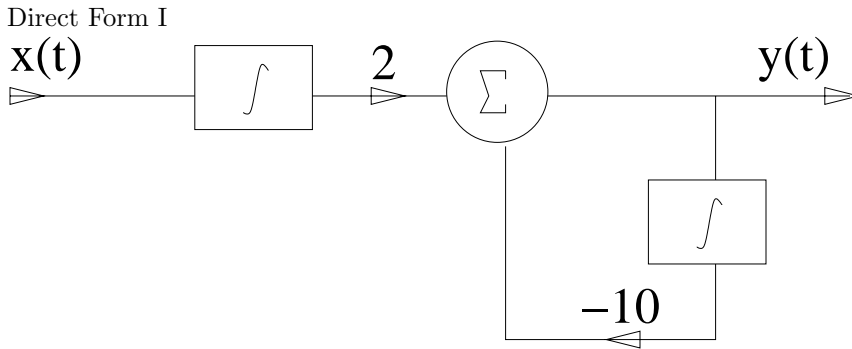


Figure P2.67. (a) Direct Form I

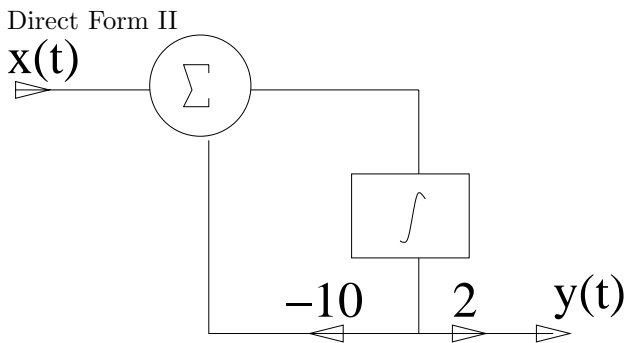


Figure P2.67. (a) Direct Form II

(b) $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 4y(t) = \frac{d}{dt}x(t)$

$$\begin{aligned} y(t) + 5y^{(1)}(t) + 4y^{(2)}(t) &= x^{(1)}(t) \\ y(t) &= x^{(1)}(t) - 5y^{(1)}(t) - 4y^{(2)}(t) \end{aligned}$$

Direct Form I

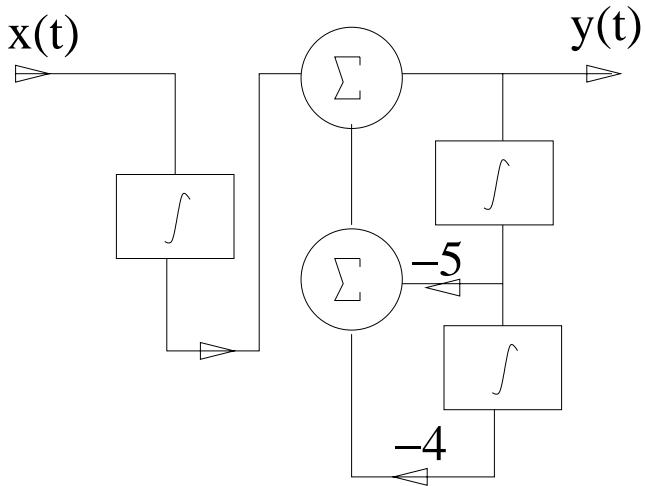


Figure P2.67. (b) Direct Form I

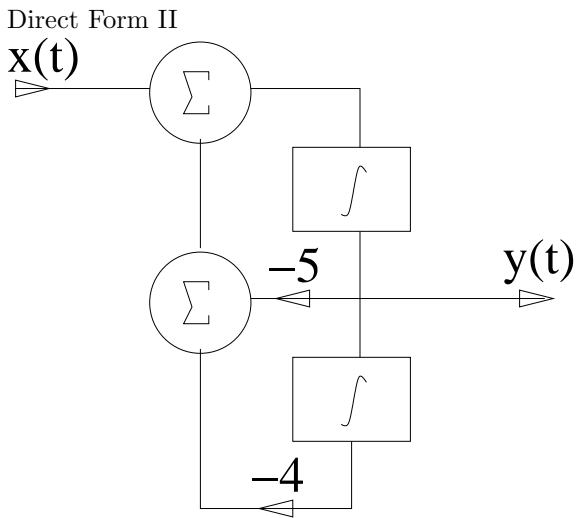


Figure P2.67. (b) Direct Form II

(c) $\frac{d^2}{dt^2}y(t) + y(t) = 3\frac{d}{dt}x(t)$

$$\begin{aligned} y(t) + y^{(2)}(t) &= 3x^{(1)}(t) \\ y(t) &= 3x^{(1)}(t) - y^{(2)}(t) \end{aligned}$$

Direct Form I

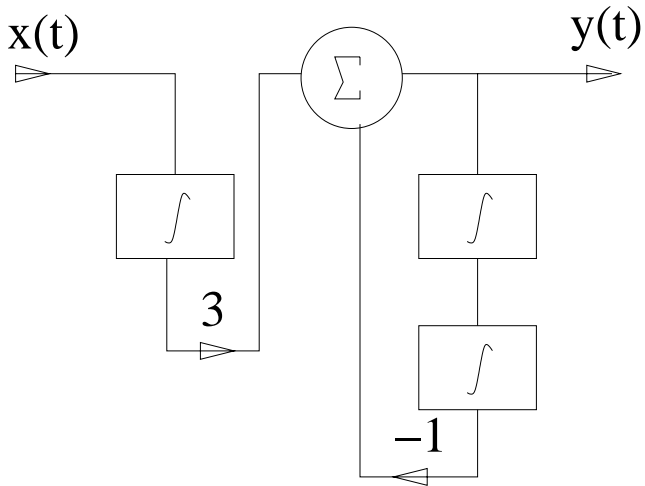


Figure P2.67. (c) Direct Form I

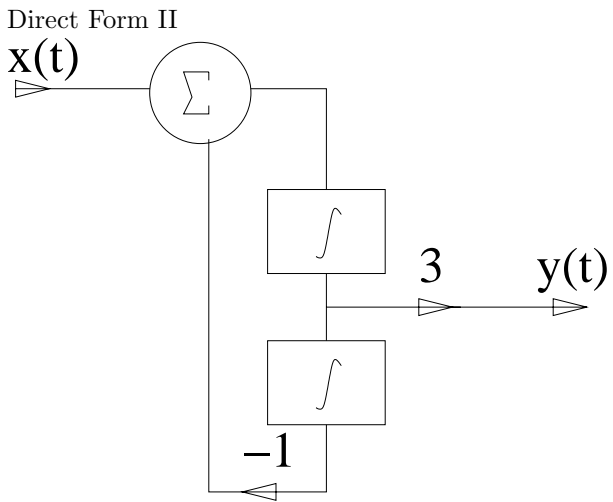


Figure P2.67. (c) Direct Form II

(d) $\frac{d^3}{dt^3}y(t) + 2\frac{d}{dt}y(t) + 3y(t) = x(t) + 3\frac{d}{dt}x(t)$

$$\begin{aligned} y(t) + 2y^{(2)}(t) + 3y^{(3)}(t) &= x^{(3)}(t) + 3x^{(2)}(t) \\ y(t) &= x^{(3)}(t) + 3x^{(2)}(t) - 2y^{(2)}(t) - 3y^{(3)}(t) \end{aligned}$$

Direct Form I

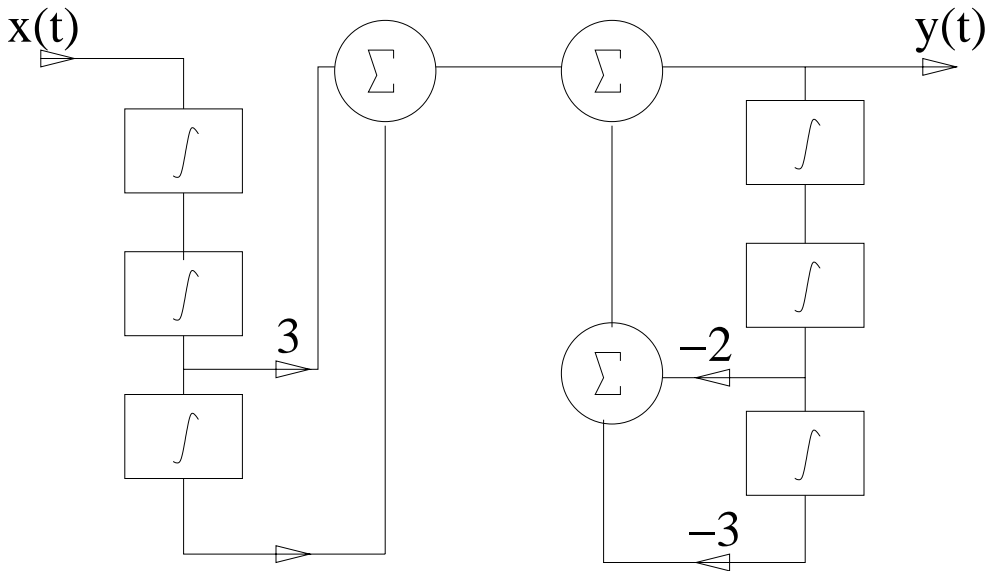


Figure P2.67. (d) Direct Form I

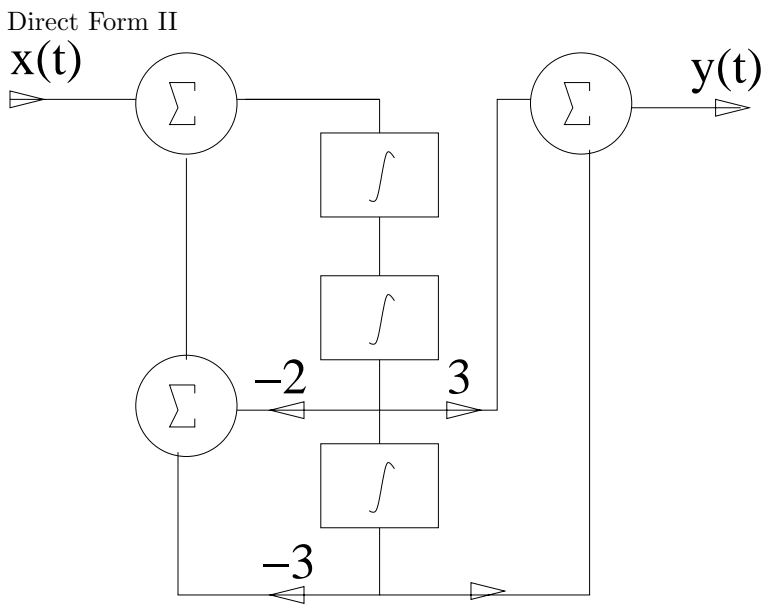


Figure P2.67. (d) Direct Form II

2.68. Find differential-equation descriptions for the two systems depicted in Fig. P2.68.
(a)

$$y(t) = x^{(1)}(t) + 2y^{(1)}(t)$$

$$\frac{d}{dt}y(t) - 2y(t) = x(t)$$

(b)

$$\begin{aligned}y(t) &= x^{(1)}(t) + 2y^{(1)}(t) - y^{(2)}(t) \\ \frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + y(t) &= \frac{d}{dt}x(t)\end{aligned}$$

2.69. Determine a state variable description for the four discrete-time systems depicted in Fig. P2.69.

(a)

$$\begin{aligned}q[n+1] &= -2q[n] + x[n] \\ y[n] &= 3x[n] + q[n]\end{aligned}$$

$$\mathbf{A} = [-2], \quad \mathbf{b} = [1], \quad \mathbf{c} = [1], \quad D = [3]$$

(b)

$$\begin{aligned}q_1[n+1] &= -q_2[n] + 2x[n] \\ q_2[n+1] &= \frac{1}{4}q_1[n] + \frac{1}{2}q_2[n] - x[n] \\ y[n] &= -2q_2[n]\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -2 & 0 \end{bmatrix}, \quad D = [0]$$

(c)

$$\begin{aligned}q_1[n+1] &= -\frac{1}{8}q_3[n] + x[n] \\ q_2[n+1] &= q_1[n] + \frac{1}{4}q_3[n] + 2x[n] \\ q_3[n+1] &= q_2[n] - \frac{1}{2}q_3[n] + 3x[n] \\ y[n] &= q_3[n]\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -\frac{1}{8} \\ 1 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad D = [0]$$

(d)

$$\begin{aligned}q_1[n+1] &= -\frac{1}{4}q_1[n] + \frac{1}{6}q_2[n] + x[n] \\ q_2[n+1] &= q_1[n] + q_2[n] + 2x[n] \\ y[n] &= q_2[n] - x[n]\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{6} \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = [-1]$$

2.70. Draw block-diagram representations corresponding to the discrete-time state-variable descriptions of the following LTI systems:

(a) $\mathbf{A} = \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = [0]$

$$q_1[n+1] = q_1[n] - \frac{1}{2}q_2[n] + x[n]$$

$$q_2[n+1] = \frac{1}{3}q_1[n] + 2x[n]$$

$$y[n] = q_1[n] + q_2[n]$$

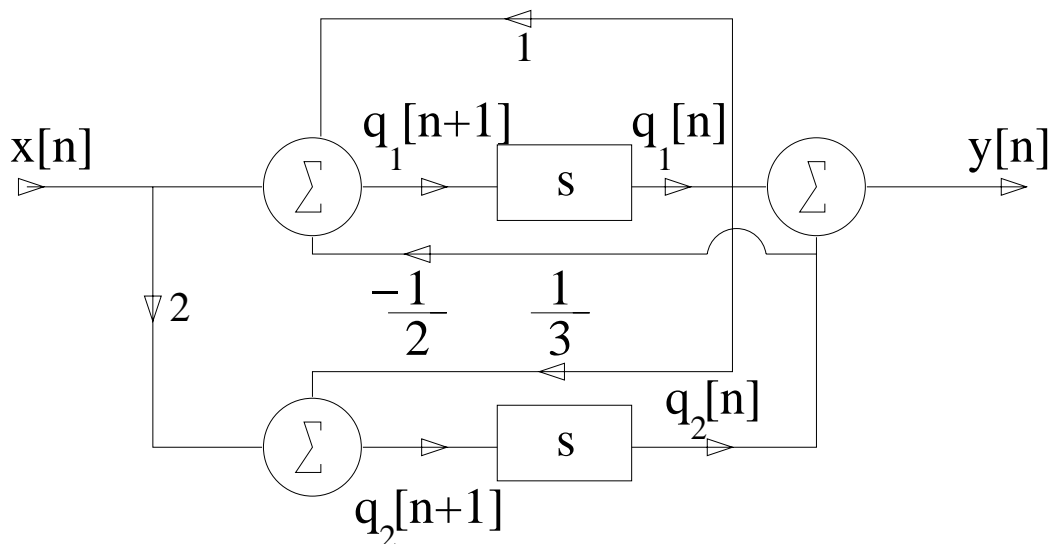


Figure P2.70. (a) Block Diagram

(b) $\mathbf{A} = \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad D = [0]$

$$q_1[n+1] = q_1[n] - \frac{1}{2}q_2[n] + x[n]$$

$$q_2[n+1] = \frac{1}{3}q_1[n] + 2x[n]$$

$$y[n] = q_1[n] - q_2[n]$$

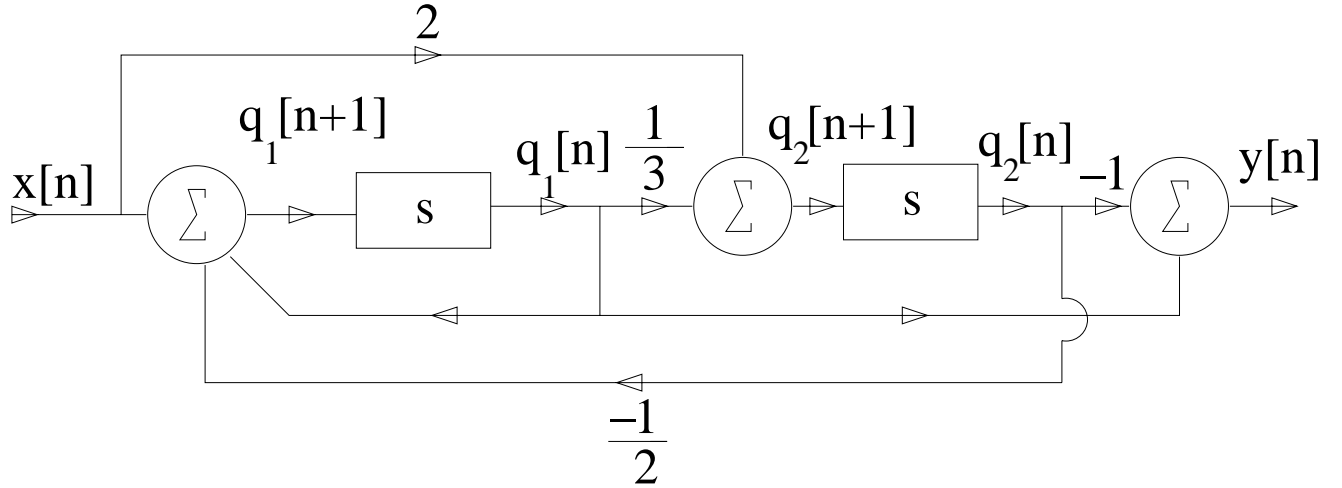


Figure P2.70. (b) Block Diagram

(c) $\mathbf{A} = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{3} & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = [1]$

$$\begin{aligned} q_1[n+1] &= -\frac{1}{2}q_2[n] \\ q_2[n+1] &= \frac{1}{3}q_1[n] - q_2[n] + x[n] \\ y[n] &= q_1[n] + x[n] \end{aligned}$$

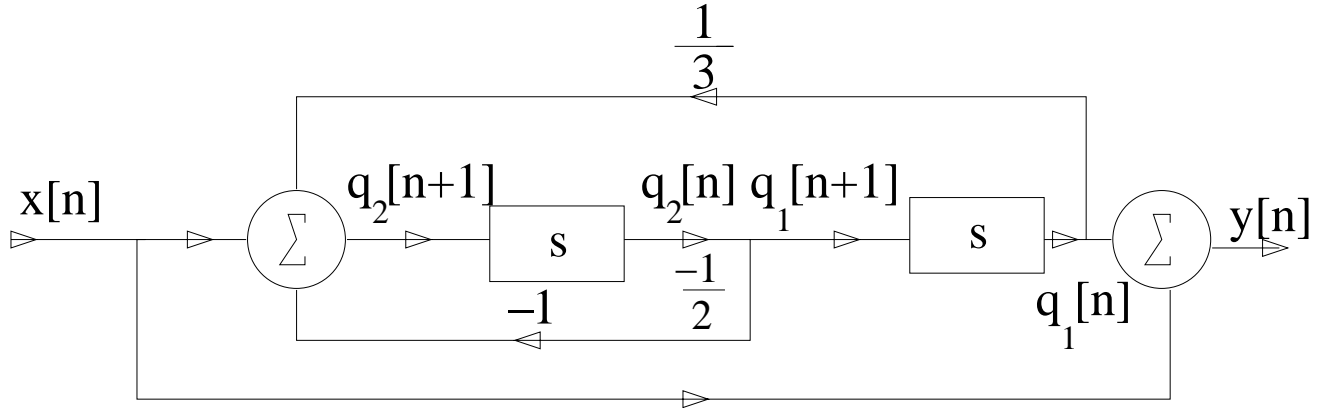


Figure P2.70. (c) Block Diagram

(d) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $D = [0]$

$$\begin{aligned} q_1[n+1] &= 2x[n] \\ q_2[n+1] &= q_2[n] + 3x[n] \\ y[n] &= q_1[n] - q_2[n] \end{aligned}$$

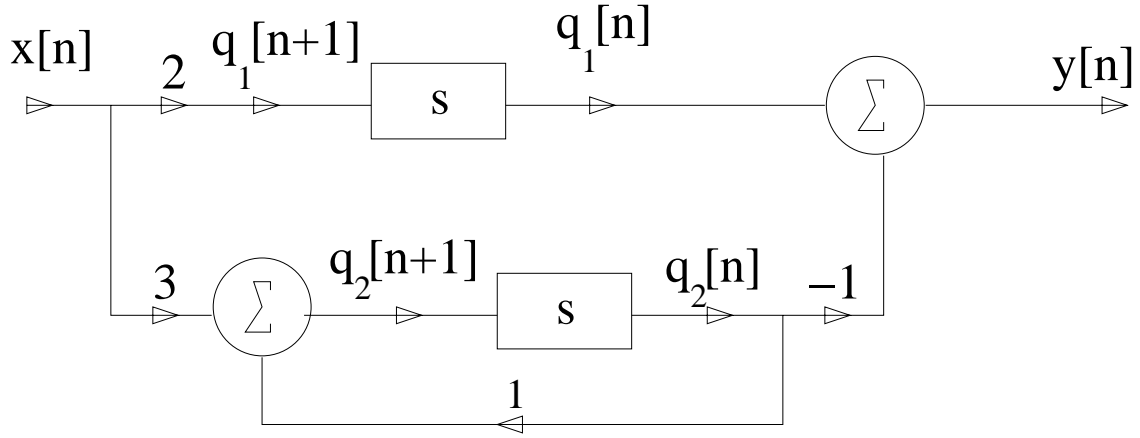


Figure P2.70. (d) Block Diagram

2.71. Determine a state-variable description for the five continuous-time LTI systems depicted in Fig. P2.71.

(a)

$$\begin{aligned}\frac{d}{dt}q(t) &= -q(t) + x(t) \\ y(t) &= 2q(t) + 6x(t)\end{aligned}$$

$\mathbf{A} = [-1], \quad \mathbf{b} = [1], \quad \mathbf{c} = [2], \quad D = [6]$

(b)

$$\begin{aligned}\frac{d}{dt}q_1(t) &= q_2(t) + 2x(t) \\ \frac{d}{dt}q_2(t) &= q_1(t) + q_2(t) \\ y(t) &= q_1(t)\end{aligned}$$

$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = [0]$

(c)

$$\begin{aligned}\frac{d}{dt}q_1(t) &= -8q_2(t) - 3q_3(t) + x(t) \\ \frac{d}{dt}q_2(t) &= q_1(t) + 4q_2(t) + 3x(t) \\ \frac{d}{dt}q_3(t) &= 2q_1(t) + q_2(t) - q_3(t) \\ y(t) &= q_3(t)\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 0 & -8 & -3 \\ 1 & 4 & 0 \\ 2 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad D = [0]$$

(d)

$$\begin{aligned} x(t) &= q_1(t)R + L \frac{d}{dt} q_1(t) + q_2(t) \\ (1) \quad \frac{d}{dt} q_1(t) &= -\frac{R}{L} q_1(t) - \frac{1}{L} q_2(t) + \frac{1}{L} x(t) \\ q_1(t) &= C \frac{d}{dt} q_2(t) \\ (2) \quad \frac{d}{dt} q_2(t) &= \frac{1}{C} q_1(t) \\ x(t) &= q_1(t)R + y(t) + q_2(t) \\ (3) \quad y(t) &= -Rq_1(t) - q_2(t) + x(t) \end{aligned}$$

Combining (1), (2), (3)

$$\mathbf{A} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -R & -1 \end{bmatrix}, \quad D = [1]$$

(e)

$$\begin{aligned} x(t) &= y(t)R + q_1(t) \\ (3) \quad y(t) &= -\frac{1}{R} q_1(t) + \frac{1}{R} x(t) \\ y(t) &= C \frac{d}{dt} q_1(t) + q_2(t) \\ -\frac{1}{R} q_1(t) + \frac{1}{R} x(t) &= C \frac{d}{dt} q_1(t) + q_2(t) \\ (1) \quad \frac{d}{dt} q_1(t) &= -\frac{1}{RC} q_1(t) - \frac{1}{C} q_2(t) + \frac{1}{RC} x(t) \\ q_1(t) &= L \frac{d}{dt} q_2(t) \\ (2) \quad \frac{d}{dt} q_2(t) &= \frac{1}{L} q_1(t) \end{aligned}$$

Combining (1), (2), (3)

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -\frac{1}{R} & 0 \end{bmatrix}, \quad D = [\frac{1}{R}]$$

2.72. Draw block-diagram representations corresponding to the continuous-time state variable descriptions of the following LTI systems:

(a) $\mathbf{A} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = [0]$

$$\frac{d}{dt} q_1(t) = \frac{1}{3} q_1(t) - x(t)$$

$$\begin{aligned}\frac{d}{dt}q_2(t) &= -\frac{1}{2}q_2(t) + 2x(t) \\ y(t) &= q_1(t) + q_2(t)\end{aligned}$$

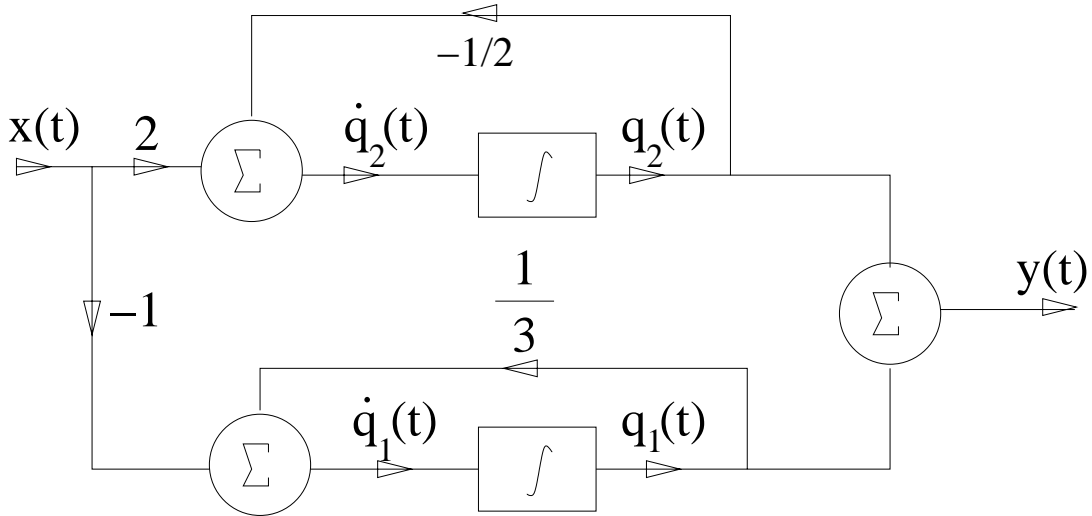


Figure P2.72. (a) Block Diagram

(b) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 & -1 \end{bmatrix}$, $D = [0]$

$$\begin{aligned}\frac{d}{dt}q_1(t) &= q_1(t) + q_2(t) - x(t) \\ \frac{d}{dt}q_2(t) &= q_1(t) + 2x(t) \\ y(t) &= -q_2(t)\end{aligned}$$

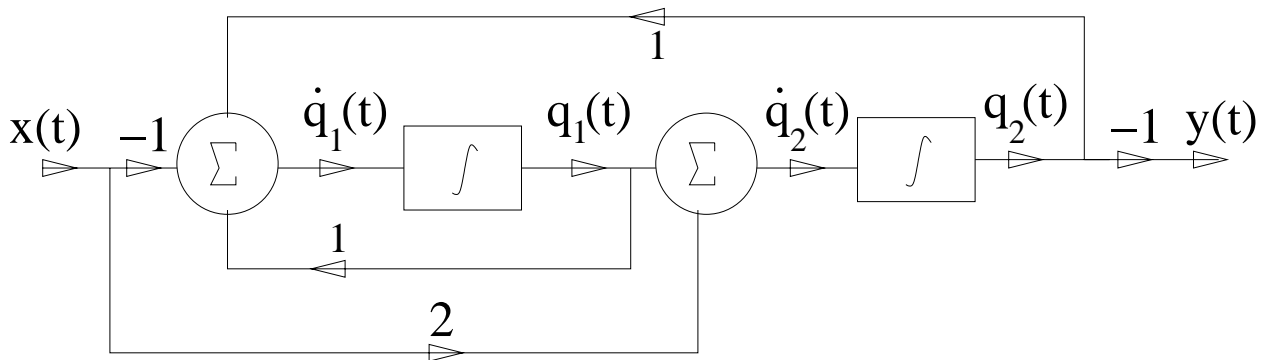


Figure P2.72. (b) Block Diagram

(c) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = [0]$

$$\frac{d}{dt}q_1(t) = q_1(t) - q_2(t)$$

$$\begin{aligned}\frac{d}{dt}q_2(t) &= -q_2(t) + 5x(t) \\ y(t) &= q_1(t)\end{aligned}$$

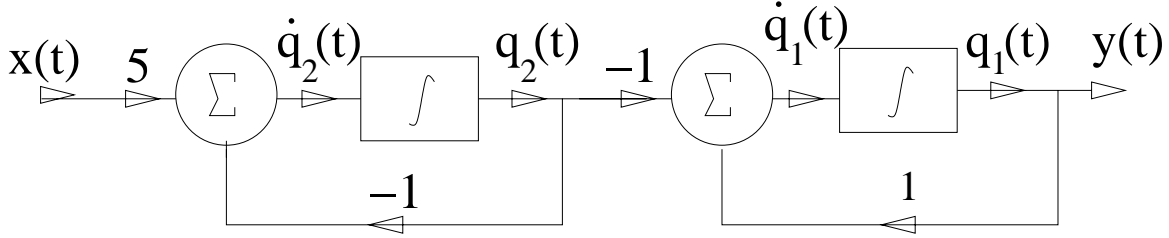


Figure P2.72. (c) Block Diagram

(d) $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $D = [0]$

$$\begin{aligned}\frac{d}{dt}q_1(t) &= q_1(t) - 2q_2(t) + 2x(t) \\ \frac{d}{dt}q_2(t) &= q_1(t) + q_2(t) + 3x(t) \\ y(t) &= q_1(t) + q_2(t)\end{aligned}$$

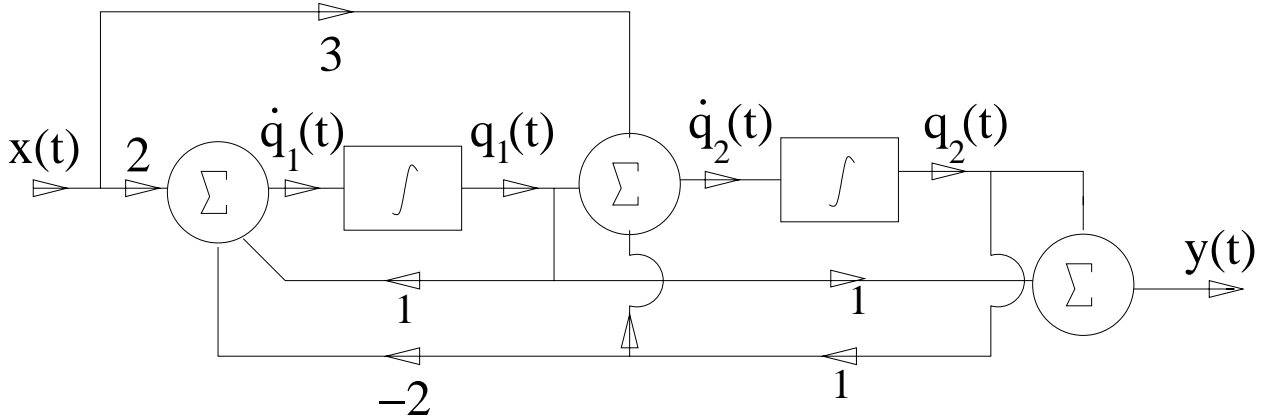


Figure P2.72. (d) Block Diagram

2.73. Let a discrete-time system have the state-variable description

$$\mathbf{A} = \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad D = [0]$$

(a) Define new states $q'_1[n] = 2q_1[n]$, $q'_2[n] = 3q_2[n]$. Find the new state-variable description \mathbf{A}' , \mathbf{b}' , \mathbf{c}' , D' .

$$\mathbf{q}'_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{q}_1$$

Thus the transformation matrix \mathbf{T} is

$$\begin{aligned}
\mathbf{T} &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \\
\mathbf{T}^{-1} &= \frac{1}{6} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \\
\mathbf{A}' = \mathbf{TAT}^{-1} &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -\frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix} \\
\mathbf{b}' = \mathbf{Tb} &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 6 \end{bmatrix} \\
\mathbf{c}' = \mathbf{cT}^{-1} &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \end{bmatrix} \\
D' = D &= 0
\end{aligned}$$

(b) Define new states $q'_1[n] = 3q_2[n]$, $q'_2[n] = 2q_1[n]$. Find the new state-variable description \mathbf{A}' , \mathbf{b}' , \mathbf{c}' , D' .

$$\mathbf{q}'_1 = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \mathbf{q}_1$$

Thus the transformation matrix \mathbf{T} is

$$\begin{aligned}
\mathbf{T} &= \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \\
\mathbf{T}^{-1} &= -\frac{1}{6} \begin{bmatrix} 0 & -3 \\ -2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix} \\
\mathbf{A}' = \mathbf{TAT}^{-1} &= \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{3} & 1 \end{bmatrix} \\
\mathbf{b}' = \mathbf{Tb} &= \begin{bmatrix} 6 \\ 2 \end{bmatrix} \\
\mathbf{c}' = \mathbf{cT}^{-1} &= \begin{bmatrix} -\frac{1}{3} & \frac{1}{2} \end{bmatrix} \\
D' = D &= 0
\end{aligned}$$

(c) Define new states $q'_1[n] = q_1[n] + q_2[n]$, $q'_2[n] = q_1[n] - q_2[n]$. Find the new state-variable description \mathbf{A}' , \mathbf{b}' , \mathbf{c}' , D' .

The transformation matrix \mathbf{T} is

$$\begin{aligned}\mathbf{T} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ \mathbf{T}^{-1} &= -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\ \mathbf{A}' = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} &= \begin{bmatrix} \frac{5}{12} & \frac{11}{12} \\ \frac{1}{12} & \frac{7}{12} \end{bmatrix} \\ \mathbf{b}' = \mathbf{T}\mathbf{b} &= \begin{bmatrix} 3 \\ -1 \end{bmatrix} \\ \mathbf{c}' = \mathbf{c}\mathbf{T}^{-1} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \\ D' = D &= 0\end{aligned}$$

2.74. Consider the continuous-time system depicted in Fig. P2.74.

(a) Find the state-variable description for this system assuming the states $q_1(t)$ and $q_2(t)$ are as labeled.

$$\begin{aligned}\frac{d}{dt}q_1(t) &= \alpha_1 q_1(t) + b_1 x(t) \\ \frac{d}{dt}q_2(t) &= \alpha_2 q_2(t) + b_2 x(t) \\ y(t) &= c_1 q_1(t) + c_2 q_2(t)\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}, \quad D = [0]$$

(b) Define new states $q'_1(t) = q_1(t) - q_2(t)$, $q'_2(t) = 2q_1(t)$. Find the new state-variable description \mathbf{A}' , \mathbf{b}' , \mathbf{c}' , D' .

$$\begin{aligned}\mathbf{T} &= \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \\ \mathbf{T}^{-1} &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}\end{aligned}$$

$$\begin{aligned}
\mathbf{A}' = \mathbf{TAT}^{-1} &= \begin{bmatrix} \alpha_2 & \frac{1}{2}(\alpha_1 - \alpha_2) \\ 0 & \alpha_1 \end{bmatrix} \\
\mathbf{b}' = \mathbf{Tb} &= \begin{bmatrix} b_1 - b_2 \\ 2b_1 \end{bmatrix} \\
\mathbf{c}' = \mathbf{cT}^{-1} &= \begin{bmatrix} -c_2 & \frac{1}{2}(c_1 + c_2) \end{bmatrix} \\
D = D' &= 0
\end{aligned}$$

(c) Draw a block diagram corresponding the new state-variable description in (b).
The corresponding differential equations are:

$$\begin{aligned}
\frac{d}{dt}q_1(t) &= \alpha_2 q_1(t) + \frac{1}{2}(\alpha_1 - \alpha_2)q_2(t) + (b_1 - b_2)x(t) \\
\frac{d}{dt}q_2(t) &= \alpha_1 q_2(t) + 2b_1 x(t) \\
y(t) &= -c_1 q_1(t) + \frac{1}{2}c_1 c_2 q_2(t)
\end{aligned}$$

'a' will replace α in the following figures.

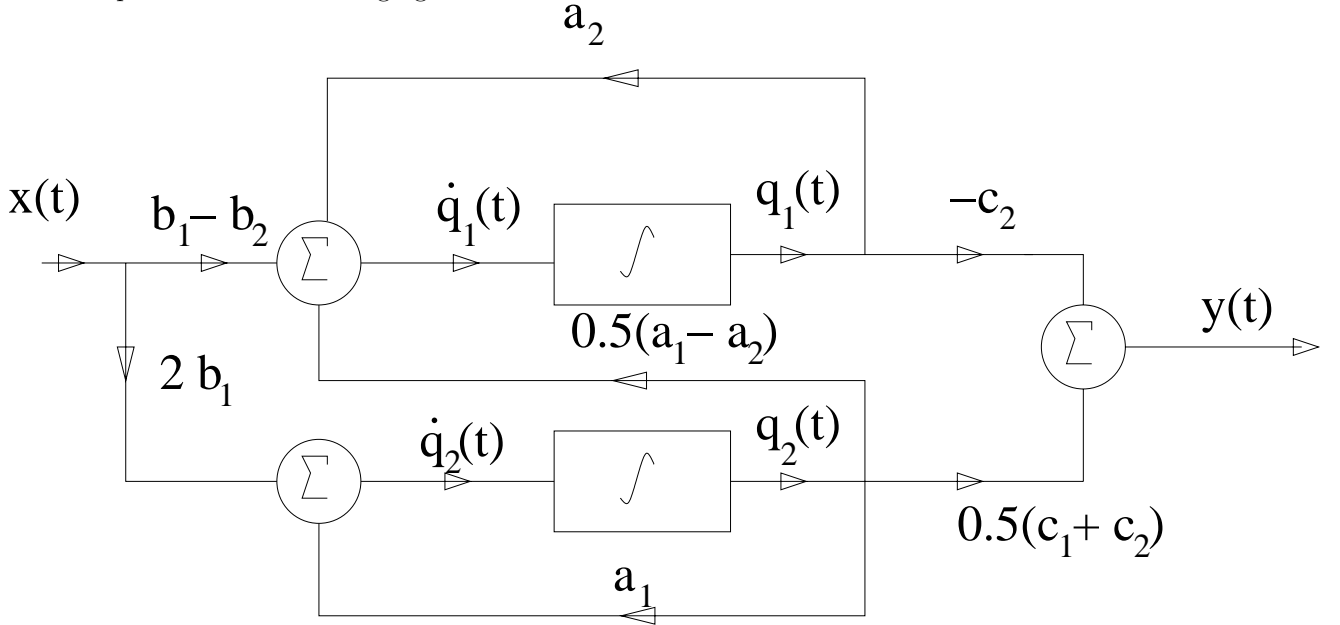


Figure P2.74. (c) Block Diagram

(d) Define new states $q'_1(t) = \frac{1}{b_1}q_1(t)$, $q'_2(t) = b_2q_1(t) - b_1q_2(t)$. Find the new state-variable description \mathbf{A}' , \mathbf{b}' , \mathbf{c}' , D' .

$$\mathbf{T} = \begin{bmatrix} \frac{1}{b_1} & 0 \\ b_2 & -b_1 \end{bmatrix}$$

$$\begin{aligned}
\mathbf{T}^{-1} &= \begin{bmatrix} b_1 & 0 \\ b_2 & -\frac{1}{b_1} \end{bmatrix} \\
\mathbf{A}' = \mathbf{TAT}^{-1} &= \begin{bmatrix} \alpha_1 & 0 \\ b_1 b_2 (\alpha_1 - \alpha_2) & \alpha_2 \end{bmatrix} \\
\mathbf{b}' = \mathbf{Tb} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\mathbf{c}' = \mathbf{cT}^{-1} &= \begin{bmatrix} c_1 b_1 + c_2 b_2 & -\frac{c_2}{b_1} \end{bmatrix} \\
D = D' &= 0
\end{aligned}$$

(e) Draw a block diagram corresponding the new state-variable description in (d).
The corresponding differential equations are:

$$\begin{aligned}
\frac{d}{dt}q_1(t) &= \alpha_1 q_1(t) + x(t) \\
\frac{d}{dt}q_2(t) &= b_1 b_2 (\alpha_1 - \alpha_2) q_1(t) + \alpha_2 q_2(t) \\
y(t) &= c_1 (b_1 + b_2) q_1(t) - \frac{c_2}{b_1} q_2(t)
\end{aligned}$$

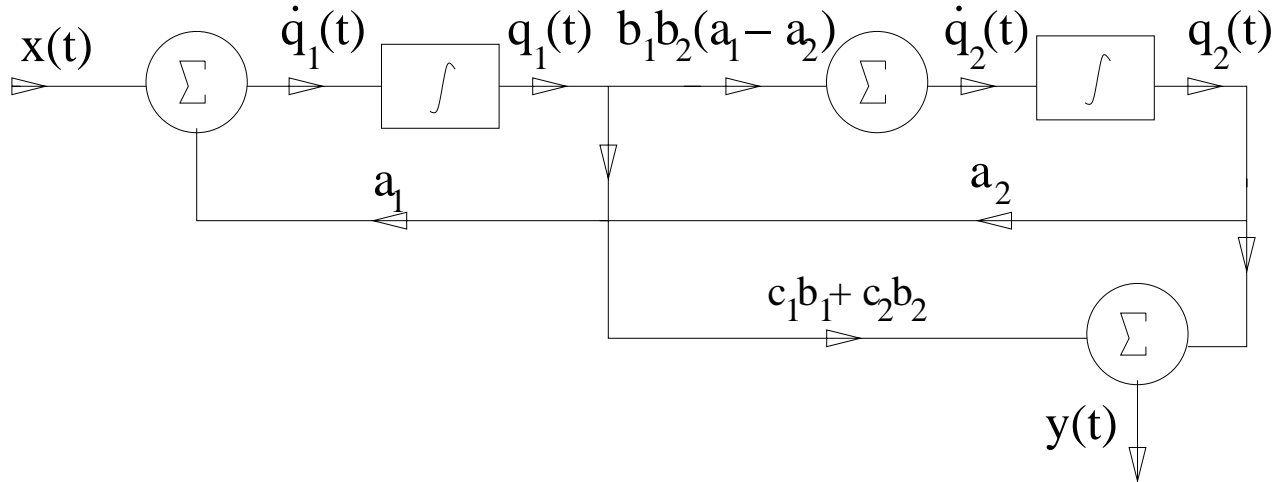


Figure P2.74. (e) Block Diagram

Solutions to Advanced Problems

2.75. We may develop the convolution integral using linearity, time invariance, and the limiting form of a stairstep approximation to the input signal. Define $g_{\Delta}(t)$ as the unit area rectangular pulse depicted in Fig. P2.75 (a).

(a) A stairstep approximation to a signal $x(t)$ is depicted in Fig. P2.75 (b). Express $\tilde{x}(t)$ as a weighted

sum of shifted pulses $g_\Delta(t)$. Does the approximation quality improve as Δ decreases?

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)g_\Delta(t - k\Delta)\Delta$$

As Δ decreases and approaches zero, the approximation quality improves.

(b) Let the response of an LTI system to an input $g_\Delta(t)$ be $h_\Delta(t)$. If the input to this system is $\tilde{x}(t)$, find an expression for the the output of this system in terms of $h_\Delta(t)$.

Let the system be represented by $H\{.\}$ such that $H\{g_\Delta(t)\} = h_\Delta(t)$.

$$\begin{aligned} H\{\tilde{x}(t)\} &= H\left\{\sum_{k=-\infty}^{\infty} x(k\Delta)g_\Delta(t - k\Delta)\Delta\right\} \\ &\quad \text{By the linearity of } H \\ &= \sum_{k=-\infty}^{\infty} (H\{x(k\Delta)g_\Delta(t - k\Delta)\Delta\}) \\ &= \sum_{k=-\infty}^{\infty} x(k\Delta)H\{g_\Delta(t - k\Delta)\}\Delta \\ &\quad \text{By the time-invariance of } H \\ H\{\tilde{x}(t)\} &= \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t - k\Delta)\Delta \end{aligned}$$

(c) In the limit as Δ goes to zero, $g_\Delta(t)$ satisfies the properties of an impulse and we may interpret $h(t) = \lim_{\Delta \rightarrow 0} h_\Delta(t)$ as the impulse response of the system. Show that the expression for the system output derived in (b) reduces to $x(t) * h(t)$ in the limit as Δ goes to zero.

When $\Delta \rightarrow 0$

$$\begin{aligned} h(t) &= \lim_{\Delta \rightarrow 0} h_\Delta(t) \\ \lim_{\Delta \rightarrow 0} H\{\tilde{x}(t)\} &= \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t - k\Delta)\Delta \end{aligned}$$

As $\Delta \rightarrow 0$, the limit is a Riemann sum, which represents an integral.

$$\begin{aligned} y(t) &\cong \int_{-\infty}^{\infty} x(\tau)h_\Delta(t - \tau)d\tau \\ &\quad \text{Using the fact that } h(t) = \lim_{\Delta \rightarrow 0} h_\Delta(t) \\ y(t) &\cong \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \\ y(t) &\cong x(t) * h(t) \end{aligned}$$

2.76. Convolution of finite-duration discrete-time signals may be expressed as the product of a matrix and a vector. Let the input $x[n]$ be zero outside of $n = 0, 1, \dots, L - 1$ and the impulse response $h[n]$ zero outside $n = 0, 1, \dots, M - 1$. The output $y[n]$ is then zero outside $n = 0, 1, \dots, L + M - 1$. Define column vectors $\mathbf{x} = [x[0], x[1], \dots, x[L - 1]]^T$ and $\mathbf{y} = [y[0], y[1], \dots, y[L + M - 1]]^T$. Use the definition of the convolution sum to find a matrix \mathbf{H} such that $\mathbf{y} = \mathbf{H}\mathbf{x}$.

$$y[n] = \sum_{k=0}^{\infty} x[k]h[n-k]$$

Applying the appropriate range for $x[n]$ and $h[n]$, starting with $n = 0$

$$y[0] = x[0]h[0]$$

Since all other values of $x[n]$ and $h[n]$ are 0, similarly

$$y[1] = x[1]h[0] + x[0]h[1]$$

$$y[2] = x[2]h[0] + x[1]h[1] + x[0]h[2]$$

\vdots

Which can be written in matrix form as

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[M-1] \\ \vdots \\ y[L+M-1] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & \dots & 0 \\ h[1] & h[0] & 0 & \dots & 0 \\ & \vdots & & & \\ h[M-1] & h[M-2] & \dots & h[0] & 0 & \dots & 0 \\ & \vdots & & & & & \\ 0 & \dots & 0 & h[M-1] & \dots & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[L-1] \end{bmatrix}$$

Yields a solution of the form

$$\mathbf{y} = \mathbf{H}\mathbf{x}$$

Where \mathbf{y} is an $(L + M - 1)$ by 1 matrix, \mathbf{H} is an $(L + M - 1)$ by L matrix, and \mathbf{x} is L by 1.

2.77. Assume the impulse response of a continuous-time system is zero outside the interval $0 < t < T_o$. Use a Riemann sum approximation to the convolution integral to convert the convolution integral to a convolution sum that relates uniformly spaced samples of the output signal to uniformly spaced samples of the input signal.

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_0^{T_o} h(\tau)x(t-\tau)d\tau \end{aligned}$$

Since $h(t)$ is zero outside of that interval.

The Riemann sum approximation is

$$\int_0^{T_o} f(\tau) d\tau \approx \sum_{k=0}^{N-1} f(k\Delta) \Delta$$

$$\text{Where } \Delta = \frac{T_o}{N}$$

Using this approximation

$$y(t) = \sum_{k=0}^{N-1} h(k\Delta) x(t - k\Delta) \Delta$$

Evaluate at $t = n\Delta$

$$y(n\Delta) = \sum_{k=0}^{N-1} h(k\Delta) x(n\Delta - k\Delta) \Delta$$

Setting

$$y[n] = y(n\Delta)$$

$$h[k] = h(k\Delta)$$

$$x[k] = x(k\Delta)$$

Implies

$$y(n\Delta) = y[n] = \sum_{k=0}^{N-1} h[k] x[n - k]$$

Which is the discrete time convolution sum.

2.78. The cross-correlation between two real signals $x(t)$ and $y(t)$ is defined as

$$r_{xy}(t) = \int_{-\infty}^{\infty} x(\tau) y(\tau - t) d\tau$$

This is the area under the product of $x(t)$ and a shifted version of $y(t)$. Note that the independent variable $\tau - t$ is the negative of that found in the definition of convolution. The autocorrelation, $r_{xx}(t)$, of a signal $x(t)$ is obtained by replacing $y(t)$ with $x(t)$.

(a) Show that $r_{xy}(t) = x(t) * y(-t)$

$$\begin{aligned} r_{xy}(t) &= \int_{-\infty}^{\infty} x(\tau) y(\tau - t) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) y(-(t - \tau)) d\tau \quad (1) \end{aligned}$$

First assume $r_{xy}(t)$ can be expressed in terms of a convolution integral, i.e.,

$$\begin{aligned} r_{xy}(t) &= \int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau) d\tau \\ &= f_1(t) * f_2(t) \quad (2) \end{aligned}$$

By (1), we can see that (1) and (2) are equivalent if:

$$f_1(v_1) = x(v_1), \text{ and}$$

$$f_2(v_2) = y(-v_2)$$

where v_1 , and v_2 are arguments, for this case $v_1 = \tau$ and $v_2 = t - \tau$, then

$$\begin{aligned} r_{xy}(t) &= f_1(t) * f_2(t) \\ &= x(t) * y(-t) \end{aligned}$$

(b) Derive a step-by-step procedure for evaluating the cross-correlation analogous to the one for evaluating convolution integral given in Section 2.2.

1. Graph both $x(\tau)$ and $y(\tau - t)$ as a function of τ . To obtain $y(\tau - t)$, shift $y(\tau)$ by t .
2. Shift t to $-\infty$.
3. Write a mathematical representation for $x(\tau)y(\tau - t)$.
4. Increase the shift until the mathematical representation for $x(\tau)y(\tau - t)$ changes. The value t at which the change occurs defines the end of the current set and begins a new one.
5. Let t be in the new set. Repeat (3) and (4) until all sets of the shifts by t and the corresponding representations for $x(\tau)$ and $y(\tau - t)$ are identified, i.e., shift t until it reaches ∞ .
6. For each set of shifts for t , integrate $x(\tau)$ and $y(\tau - t)$ from $\tau = -\infty$ to $\tau = \infty$ to obtain $r_{xy}(t)$ on each set.

(c) Evaluate the cross-correlation between the following signals:

(i) $x(t) = e^{-t}u(t), y(t) = e^{-3t}u(t)$

for $t < 0$

$$\begin{aligned} r_{xy}(t) &= \int_0^{\infty} e^{3t} e^{-4\tau} d\tau \\ &= \frac{1}{4} e^{3t} \end{aligned}$$

for $t \geq 0$

$$\begin{aligned} r_{xy}(t) &= \int_t^{\infty} e^{3t} e^{-4\tau} d\tau \\ &= \frac{1}{4} e^{-t} \end{aligned}$$

$$r_{xy}(t) = \begin{cases} \frac{1}{4} e^{3t} & t < 0 \\ \frac{1}{4} e^{-t} & t \geq 0 \end{cases}$$

(ii) $x(t) = \cos(\pi t)[u(t+2) - u(t-2)], y(t) = \cos(2\pi t)[u(t+2) - u(t-2)]$

for $t < -4$

$$r_{xy}(t) = 0$$

for $-4 \leq t < 0$

$$\begin{aligned}
r_{xy}(t) &= \int_{-2}^{t+2} \cos(\pi\tau) \cos(2\pi\tau - 2\pi t) d\tau \\
&= \int_{-2}^{t+2} \cos(\pi\tau) [\cos(2\pi\tau) \cos(2\pi t) + \sin(2\pi\tau) \sin(2\pi t)] d\tau \\
&= \frac{1}{2} \cos(2\pi t) \int_{-2}^{t+2} (\cos(\pi\tau) + \cos(3\pi\tau)) d\tau + \frac{1}{2} \sin(2\pi t) \int_{-2}^{t+2} (\sin(\pi\tau) + \sin(3\pi\tau)) d\tau \\
&= \frac{1}{2\pi} \cos(2\pi t) \left[\sin(\pi t) + \frac{1}{3} \sin(3\pi t) \right] - \frac{1}{2\pi} \sin(2\pi t) \left[\cos(\pi t) + \frac{1}{3} \cos(3\pi t) - \frac{4}{3} \right]
\end{aligned}$$

for $0 \leq t < 4$

$$\begin{aligned}
r_{xy}(t) &= \int_{t-2}^2 \cos(\pi\tau) \cos(2\pi\tau - 2\pi t) d\tau \\
&= \int_{t-2}^2 \cos(\pi\tau) [\cos(2\pi\tau) \cos(2\pi t) + \sin(2\pi\tau) \sin(2\pi t)] d\tau \\
&= \frac{1}{2} \cos(2\pi t) \int_{t-2}^2 (\cos(\pi\tau) + \cos(3\pi\tau)) d\tau + \frac{1}{2} \sin(2\pi t) \int_{t-2}^2 (\sin(\pi\tau) + \sin(3\pi\tau)) d\tau \\
&= -\frac{1}{2\pi} \cos(2\pi t) \left[\sin(\pi t) + \frac{1}{3} \sin(3\pi t) \right] + \frac{1}{2\pi} \sin(2\pi t) \left[\cos(\pi t) + \frac{1}{3} \cos(3\pi t) - \frac{4}{3} \right]
\end{aligned}$$

for $t \geq 4$

$$\begin{aligned}
r_{xy}(t) &= 0 \\
r_{xy}(t) &= \begin{cases} 0 & t < -4 \\ \frac{1}{2\pi} \cos(2\pi t) \left[\sin(\pi t) + \frac{1}{3} \sin(3\pi t) \right] - \frac{1}{2\pi} \sin(2\pi t) \left[\cos(\pi t) + \frac{1}{3} \cos(3\pi t) - \frac{4}{3} \right] & -4 \leq t < 0 \\ -\frac{1}{2\pi} \cos(2\pi t) \left[\sin(\pi t) + \frac{1}{3} \sin(3\pi t) \right] + \frac{1}{2\pi} \sin(2\pi t) \left[\cos(\pi t) + \frac{1}{3} \cos(3\pi t) - \frac{4}{3} \right] & 0 \leq t < 4 \\ 0 & t \geq 4 \end{cases}
\end{aligned}$$

$$(iii) \ x(t) = u(t) - 2u(t-1) + u(t-2), y(t) = u(t+1) - u(t)$$

for $t < 0$

$$r_{xy}(t) = 0$$

for $0 \leq t < 1$

$$r_{xy}(t) = t$$

for $1 \leq t < 2$

$$r_{xy}(t) = 3 - 2t$$

for $2 \leq t < 3$

$$r_{xy}(t) = t - 3$$

for $t \geq 3$

$$r_{xy}(t) = 0$$

$$r_{xy}(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 3 - 2t & 1 \leq t < 2 \\ t - 3 & 2 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$

$$(iv) \ x(t) = u(t-a) - u(t-a-1), y(t) = u(t) - u(t-1)$$

$$\begin{aligned}
& \text{for } t < a - 1 \\
& \quad r_{xy}(t) = 0 \\
& \text{for } a - 1 \leq t < a \\
& \quad r_{xy}(t) = t + 1 - a \\
& \text{for } a \leq t < a + 1 \\
& \quad r_{xy}(t) = a + 1 - t \\
& \text{for } t \geq a + 1 \\
& \quad r_{xy}(t) = 0 \\
& r_{xy}(t) = \begin{cases} 0 & t < a - 1 \\ t + 1 - a & a - 1 \leq t < a \\ a + 1 - t & a \leq t < a + 1 \\ 0 & t \geq a + 1 \end{cases}
\end{aligned}$$

(d) Evaluate the autocorrelation of the following signals:

(i) $x(t) = e^{-t}u(t)$

$$\begin{aligned}
& \text{for } t < 0 \\
& \quad r_{xx}(t) = \int_0^\infty e^t e^{-2\tau} d\tau \\
& \quad = \frac{1}{2} e^t \\
& \text{for } t \geq 0 \\
& \quad r_{xx}(t) = \int_t^\infty e^t e^{-2\tau} d\tau \\
& \quad = \frac{1}{2} e^{-t} \\
& \quad r_{xx}(t) = \begin{cases} \frac{1}{2} e^t & t < 0 \\ \frac{1}{2} e^{-t} & t \geq 0 \end{cases}
\end{aligned}$$

(ii) $x(t) = \cos(\pi t)[u(t+2) - u(t-2)]$

$$\begin{aligned}
& \text{for } t < -4 \\
& \quad r_{xx}(t) = 0 \\
& \text{for } -4 \leq t < 0 \\
& \quad r_{xx}(t) = \int_{-2}^{t+2} \cos(\pi\tau) \cos(\pi\tau - \pi t) d\tau \\
& \quad = \int_{-2}^{t+2} \cos(\pi\tau) (\cos(\pi\tau) \cos(\pi t) + \sin(\pi\tau) \sin(\pi t)) d\tau \\
& \quad = \frac{1}{2} \cos(\pi t) \int_{-2}^{t+2} (1 + \cos(2\pi\tau)) d\tau + \frac{1}{2} \sin(\pi t) \int_{-2}^{t+2} \sin(2\pi\tau) d\tau \\
& \quad = \frac{1}{2} \cos(\pi t) \left[t + 4 + \frac{1}{2\pi} \sin(2\pi t) \right] - \frac{1}{4\pi} \sin(\pi t) [\cos(2\pi t) + 2]
\end{aligned}$$

for $0 \leq t < 4$

$$\begin{aligned}
r_{xx}(t) &= \int_{t-2}^2 \cos^2(\pi\tau) d\tau \\
&= \int_{t-2}^2 \cos(\pi\tau) (\cos(\pi\tau) \cos(\pi t) + \sin(\pi\tau) \sin(\pi t)) d\tau \\
&= \frac{1}{2} \cos(\pi t) \int_{t-2}^2 (1 + \cos(2\pi\tau)) d\tau + \frac{1}{4\pi} \sin(\pi t) \int_{t-2}^2 \sin(2\pi\tau) d\tau \\
&= \frac{1}{2} \cos(\pi t) \left[4 - t - \frac{1}{2\pi} \sin(2\pi t) \right] + \frac{1}{4\pi} \sin(\pi t) [2 + \cos(2\pi t)]
\end{aligned}$$

for $t \geq 4$

$$\begin{aligned}
r_{xx}(t) &= 0 \\
r_{xx}(t) &= \begin{cases} 0 & t < -4 \\ \frac{1}{2} \cos(\pi t) \left[t + 4 + \frac{1}{2\pi} \sin(2\pi t) \right] - \frac{1}{4\pi} \sin(\pi t) [\cos(2\pi t) + 2] & -4 \leq t < 0 \\ \frac{1}{2} \cos(\pi t) \left[4 - t - \frac{1}{2\pi} \sin(2\pi t) \right] + \frac{1}{4\pi} \sin(\pi t) [2 + \cos(2\pi t)] & 0 \leq t < 4 \\ 0 & t \geq 4 \end{cases}
\end{aligned}$$

(iii) $x(t) = u(t) - 2u(t-1) + u(t-2)$

for $t < -2$

$$r_{xx}(t) = 0$$

for $-2 \leq t < -1$

$$r_{xx}(t) = -t - 2$$

for $-1 \leq t < 0$

$$r_{xx}(t) = 3t + 2$$

for $0 \leq t < 1$

$$r_{xx}(t) = 2 - 3t$$

for $1 \leq t < 2$

$$r_{xx}(t) = t - 2$$

for $t \geq 2$

$$r_{xx}(t) = 0$$

$$r_{xx}(t) = \begin{cases} 0 & t < -2 \\ -t - 2 & -2 \leq t < -1 \\ 3t + 2 & -1 \leq t < 0 \\ 2 - 3t & 0 \leq t < 1 \\ t - 2 & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

(iv) $x(t) = u(t-a) - u(t-a-1)$

for $t < -1$

$$r_{xx}(t) = 0$$

for $-1 \leq t < 0$

$$r_{xx}(t) = t + 1$$

for $0 \leq t < 1$

$$r_{xx}(t) = 1 - t$$

for $t \geq 1$

$$r_{xx}(t) = 0$$

$$r_{xx}(t) = \begin{cases} 0 & t < -1 \\ t + 1 & -1 \leq t < 0 \\ 1 - t & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases}$$

(e) Show that $r_{xy}(t) = r_{yx}(-t)$

$$r_{xy}(t) = \int_{-\infty}^{\infty} x(\tau)y(\tau - t)d\tau$$

Let $u = \tau - t$

$$r_{xy}(t) = \int_{-\infty}^{\infty} y(u)x(u + t)du$$

$$r_{xy}(t) = \int_{-\infty}^{\infty} y(\tau)x(\tau + t)d\tau$$

$$r_{xy}(t) = r_{yx}(-t)$$

(f) Show that $r_{xx}(t) = r_{xx}(-t)$

$$r_{xx}(t) = \int_{-\infty}^{\infty} x(\tau)x(\tau - t)d\tau$$

Let $u = \tau - t$

$$r_{xx}(t) = \int_{-\infty}^{\infty} x(u)x(u + t)du$$

$$r_{xx}(t) = \int_{-\infty}^{\infty} x(\tau)x(\tau + t)d\tau$$

$$r_{xx}(t) = r_{xx}(-t)$$

2.79. Prove that absolute summability of the impulse response is a necessary condition for stability of a discrete-time system. *Hint:* find a bounded input $x[n]$ such that the output at some time n_o satisfies $|y[n_o]| = \sum_{k=-\infty}^{\infty} |h[k]|$.

let $n_o = 0$

$$y[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k]$$

let $x[-k] = \text{sign}\{h[k]\}$

then $h[k]x[-k] = |h[k]|$

$$\begin{aligned}
y[0] &= \sum_{k=-\infty}^{\infty} |h[k]| \\
|y[0]| &= \left| \sum_{k=-\infty}^{\infty} |h[k]| \right| \\
&= \sum_{k=-\infty}^{\infty} |h[k]|
\end{aligned}$$

Hence there exists an input for which $|y[n_o]| = \sum_{k=-\infty}^{\infty} |h[k]|$, and $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ is a necessary condition for stability.

2.80. Using the Fresnel approximation, light with a complex amplitude $f(x, y)$ in the xy -plane propagating over a distance d along the z -axis in free space generates a complex amplitude $g(x, y)$ given by

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') h(x - x', y - y') dx' dy'$$

where

$$h(x, y) = h_0 e^{-jk(x^2 + y^2)/2d}$$

Here $k = 2\pi/\lambda$ is the wavenumber, λ the wavelength, and $h_0 = j/(\lambda d) e^{-jkd}$.

(a) Determine whether free-space propagation represents a linear system.

Suppose $f(x', y') = af_1(x', y') + bf_2(x', y')$. The system is linear if $g(x, y) = ag_1(x, y) + bg_2(x, y)$

$$\begin{aligned}
g(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') h(x - x', y - y') dx' dy' \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [af_1(x', y') + bf_2(x', y')] h(x - x', y - y') dx' dy' \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} af_1(x', y') h(x - x', y - y') dx' dy' + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} bf_2(x', y') h(x - x', y - y') dx' dy' \\
&= ag_1(x, y) + bg_2(x, y)
\end{aligned}$$

(b) Is this system space invariant? That is, does a spatial shift of the input, $f(x - x_0, y - y_0)$ lead to the identical spatial shift in the output?

$$\begin{aligned}
g'(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x' - x_0, y' - y_0) h(x - x', y - y') dx' dy' \\
&\quad \text{let } z = x' - x_0, \quad w = y' - y_0 \\
g'(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z, w) h(x - x_0 - z, y - y_0 - w) dw dz \\
&= g(x - x_0, y - y_0)
\end{aligned}$$

This shows that the system is space invariant since a shift in the input produces a similar shift in the output.

(c) Evaluate the result of a point source located at (x_1, y_1) propagating a distance d . In this case $f(x, y) = \delta(x - x_1, y - y_1)$ where $\delta(x, y)$ is the two-dimensional version of the impulse. Find the “impulse response” of this system.

$$\begin{aligned}
 g(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') h(x - x', y - y') dx' dy' \\
 g(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - x_1, y - y_1) h(x - x', y - y') dx' dy' \\
 &\quad \text{Using the sifting property} \\
 g(x, y) &= \int_{-\infty}^{\infty} \delta(y - y_1) h(x - x_1, y - y') dy' \\
 &= h(x - x_1, y - y_1)
 \end{aligned}$$

(d) Evaluate the result of two point sources located at (x_1, y_1) and (x_2, y_2) propagating a distance d . To account for the two point sources, $f(x, y)$ is of the form $f(x, y) = f_1(x_1, y_1) + f_2(x_2, y_2) = \delta(x - x_1, y - y_1) + \delta(x - x_2, y - y_2)$

$$\begin{aligned}
 g(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') h(x - x', y - y') dx' dy' \\
 g(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\delta(x - x_1, y - y_1) + \delta(x - x_2, y - y_2)] h(x - x', y - y') dx' dy' \\
 &\quad \text{Using linearity and the sifting property yields} \\
 g(x, y) &= h(x - x_1, y - y_1) + h(x - x_2, y - y_2)
 \end{aligned}$$

2.81. The motion of a vibrating string depicted in Fig. P2.81 may be described by the partial differential equation

$$\frac{\partial^2}{\partial l^2} y(l, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} y(l, t)$$

where $y(l, t)$ is the displacement expressed as a function of position l and time t , and c is a constant determined by the material properties of the string. The initial conditions may be specified as follows:

$$\begin{aligned}
 y(0, t) &= 0, \quad y(a, t) = 0, \quad t > 0 \\
 y(l, 0) &= x(l), \quad 0 < l < a \\
 \left. \frac{\partial}{\partial t} y(l, t) \right|_{t=0} &= g(l), \quad 0 < l < a
 \end{aligned}$$

Here $x(l)$ is the displacement of the string at $t = 0$ while $g(l)$ describes the velocity at $t = 0$. One approach to solving this equation is to assume the solution is separable, that is, $y(l, t) = \phi(l)f(t)$, in which case the partial differential equation becomes

$$f(t) \frac{d^2}{dl^2} \phi(l) = \phi(l) \frac{1}{c^2} \frac{d^2}{dt^2} f(t)$$

This implies

$$\frac{\frac{d^2}{dl^2}\phi(l)}{\phi(l)} = \frac{\frac{d^2}{dt^2}f(t)}{c^2 f(t)}, \quad 0 < l < a, \quad 0 < t$$

For this equality to hold, both sides of the equation must be constant. Let the constant be $-\omega^2$ and separate the partial differential equation into two ordinary second-order differential equations linked by the common parameter ω^2 .

$$\begin{aligned} \frac{d^2}{dt^2}f(t) + \omega^2 c^2 f(t) &= 0, \quad 0 < t \\ \frac{d^2}{dl^2}\phi(l) + \omega^2 \phi(l) &= 0 \quad 0 < l < a \end{aligned}$$

(a) Find the form of the solution for $f(t)$ and $\phi(l)$.

The homogeneous solution for $f(t)$ is of the form:

$$\begin{aligned} r^2 + \omega^2 c^2 &= 0 \\ r &= \pm j c \omega \\ f(t) &= a_1 e^{j \omega c t} + a_2 e^{-j \omega c t} \end{aligned}$$

The homogeneous solution for $\phi(l)$ is of the form:

$$\begin{aligned} r^2 + \omega^2 &= 0 \\ r &= \pm j \omega \\ \phi(l) &= b_1 e^{j \omega l} + b_2 e^{-j \omega l} \\ \phi(l) &= d_1 \cos(\omega l) + d_2 \sin(\omega l) \end{aligned}$$

(b) The boundary conditions at the end points of the string are

$$\phi(0)f(t) = 0 \quad \phi(a)f(t) = 0$$

and, since $f(t) = 0$ gives a trivial solution for $y(l, t)$, we must have $\phi(0) = 0$ and $\phi(a) = 0$. Determine how these constraints restrict the permissible values for ω and the form of the solution for $\phi(l)$.

$$\begin{aligned} \phi(0) = 0 &= d_1 \cos(\omega 0) + d_2 \sin(\omega 0) \\ &\text{implies} \\ d_1 &= 0 \\ \phi(a) = 0 &= d_2 \sin(\omega l) \\ &\text{implies} \\ \omega &= \frac{k\pi}{a} \quad \text{where } k \text{ is any integer.} \end{aligned}$$

(c) Use the boundary conditions in (b) to show that constant $(-\omega^2)$ used to separate the partial differential equation into two ordinary second-order differential equations must be negative.

If $(-\omega^2)$ is positive:

$$\begin{aligned}
 r^2 - \omega^2 &= 0 \\
 r &= \pm \omega \\
 \phi(l) &= b_1 e^{\omega t} + b_2 e^{-\omega t} \\
 \phi(0) = 0 &= b_1 + b_2 \\
 \phi(a) = 0 &= b_1 e^{j\omega a} + b_2 e^{-\omega a} \\
 &\text{implies} \\
 b_1 = b_2 &= 0
 \end{aligned}$$

$(-\omega^2)$ being positive is the trivial solution, hence $(-\omega^2)$ must be negative.

(d) Assume the initial position of the string is $y(l, 0) = x(l) = \sin(\pi l/a)$ and that the initial velocity is $g(l) = 0$. Find $y(l, t)$.

$$\begin{aligned}
 y(l, t) &= \phi(l)f(t) \\
 &= d_2 \sin\left(\frac{k\pi l}{a}\right) \left[e_1 \sin\left(\frac{k\pi ct}{a}\right) + e_2 \cos\left(\frac{k\pi ct}{a}\right) \right] \\
 y(l, 0) = \sin\left(\frac{\pi l}{a}\right) &= d_2 e_2 \sin\left(\frac{k\pi l}{a}\right) \\
 \text{Implies} \\
 k &= 1 \\
 d_2 e_2 &= 1 \\
 \left. \frac{d}{dt} y(l, t) \right|_{t=0} = 0 &= d_2 \sin\left(\frac{k\pi l}{a}\right) e_1 \frac{\pi c}{a} \cos\left(\frac{k\pi ct}{a}\right) \\
 \text{Implies} \\
 e_1 &= 0 \\
 y(l, t) &= \sin\left(\frac{\pi l}{a}\right) \cos\left(\frac{\pi ct}{a}\right)
 \end{aligned}$$

2.82. Suppose the N -by- N matrix \mathbf{A} in a state-variable description has N linearly independent eigenvectors $\mathbf{e}_i, i = 1, 2, \dots, N$ and corresponding distinct eigenvalues λ_i . Thus $\mathbf{A}\mathbf{e}_i = \lambda_i \mathbf{e}_i, i = 1, 2, \dots, N$.

(a) Show that we may decompose \mathbf{A} as $\mathbf{A} = \mathbf{E} \mathbf{\Lambda} \mathbf{E}^{-1}$ where $\mathbf{\Lambda}$ is a diagonal matrix with i -th diagonal element λ_i .

$$\begin{aligned}
 \mathbf{A}\mathbf{e}_1 &= \lambda_1 \mathbf{e}_1 \\
 \mathbf{A}\mathbf{e}_2 &= \lambda_2 \mathbf{e}_2
 \end{aligned}$$

This can be written as

$$\mathbf{A}[\mathbf{e}_1 \ \mathbf{e}_2] = [\mathbf{e}_1 \ \mathbf{e}_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Define

$$\begin{aligned} \mathbf{E} &= [\mathbf{e}_1 \ \mathbf{e}_2] \\ \mathbf{\Lambda} &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\ \mathbf{AE} &= \mathbf{E}\mathbf{\Lambda} \end{aligned}$$

Which can be rewritten as

$$\begin{aligned} \mathbf{AEE}^{-1} &= \mathbf{E}\mathbf{\Lambda E}^{-1} \\ \mathbf{A} &= \mathbf{E}\mathbf{\Lambda E}^{-1} \end{aligned}$$

(b) Find a transformation of the state that will diagonalize \mathbf{A} .

The transformation $\mathbf{A}' = \mathbf{TAT}^{-1}$ will diagonalize \mathbf{A} . Setting $\mathbf{T} = \mathbf{E}^{-1}$.

$$\begin{aligned} \mathbf{A}' &= \mathbf{E}^{-1}\mathbf{AE} \\ \mathbf{A}' &= \mathbf{E}^{-1}[\mathbf{E}\mathbf{\Lambda E}^{-1}]\mathbf{E} \\ \mathbf{A}' &= \mathbf{\Lambda} \end{aligned}$$

(c) Assume $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = [0]$.

Find a transformation that converts this system to diagonal form.

The eigenvalues and eigenvectors are $\lambda_{1,2} = -1, -2$ and $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ respectively. The transformation is

$$\begin{aligned} \mathbf{A}' = \mathbf{E}^{-1}\mathbf{AE} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \\ \mathbf{b}' = \mathbf{E}^{-1}\mathbf{b} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \mathbf{c}' = \mathbf{cE} &= \begin{bmatrix} 1 & 1 \end{bmatrix} \\ D' = D &= 0 \end{aligned}$$

(d) Sketch the block-diagram representation for a discrete-time system corresponding to part (c).

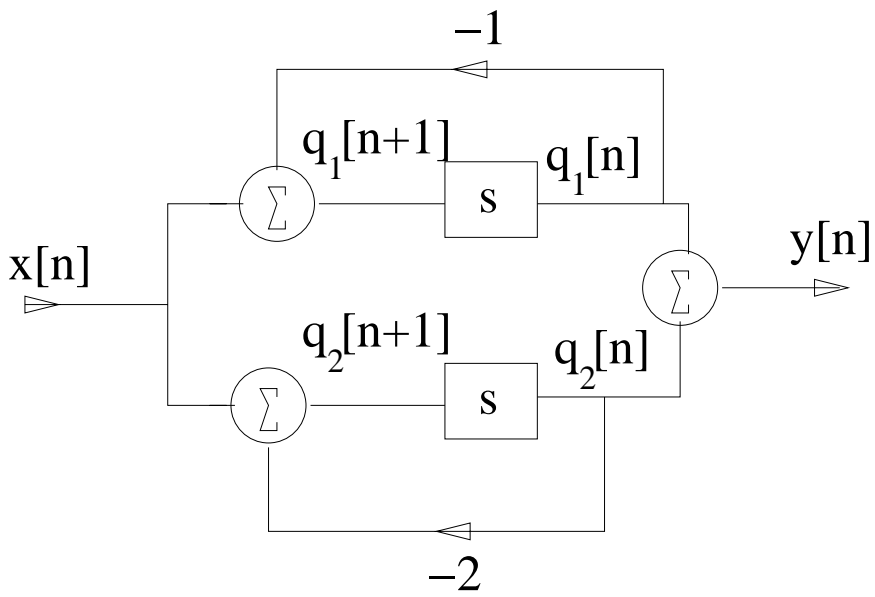


Figure P2.82. Block diagram of diagonal form

Solutions to Computer Experiments

2.83. Repeat Problem 2.34 using MATLABs conv command.

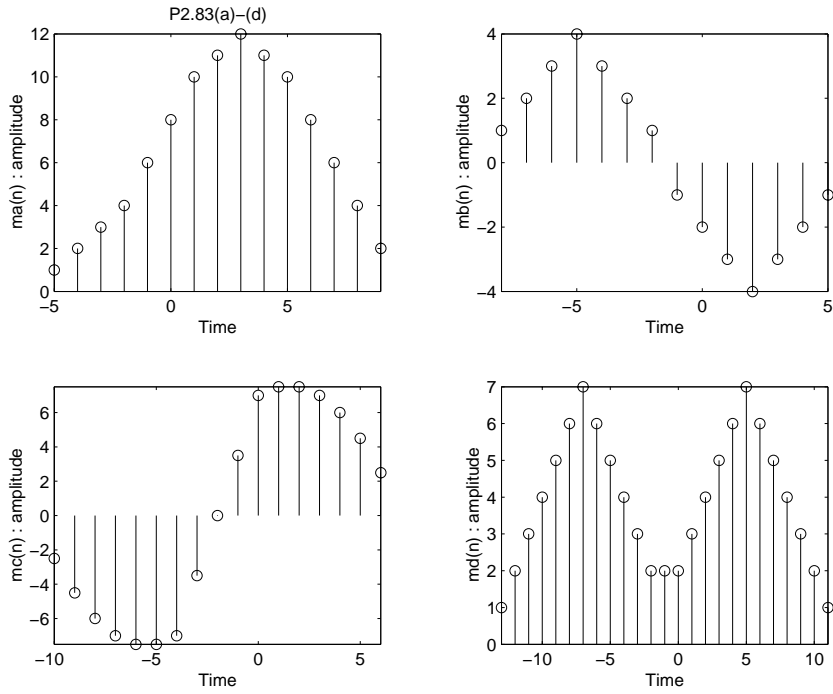


Figure P2.83. Convolution of (a)-(d) using MATLABs conv command

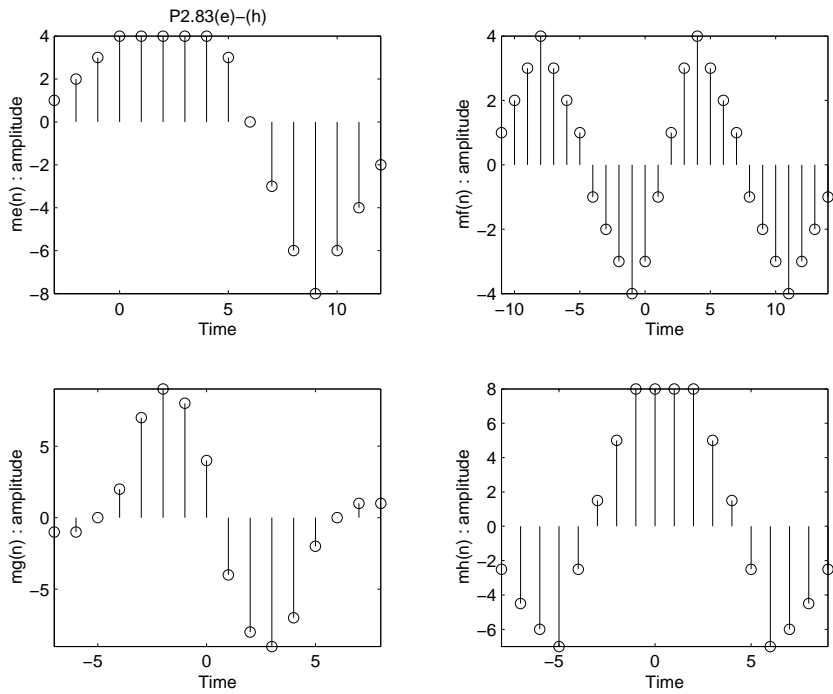


Figure P2.83. Convolution of (e)-(h) using MATLABs conv command

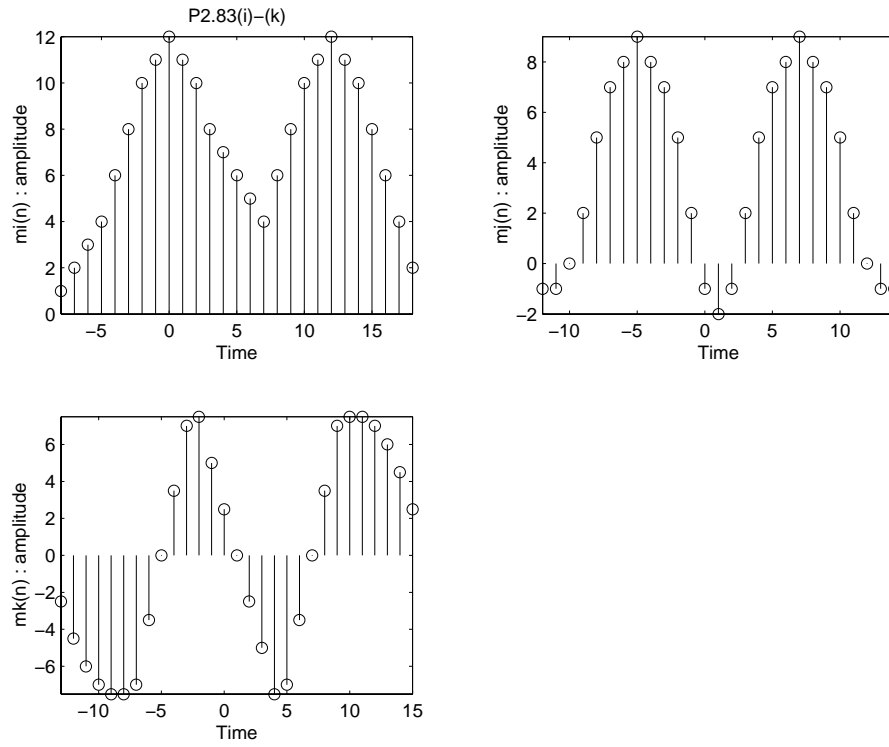


Figure P2.83. Convolution of (i)-(k) using MATLABs `conv` command

2.84. Use MATLAB to repeat Example 2.5.

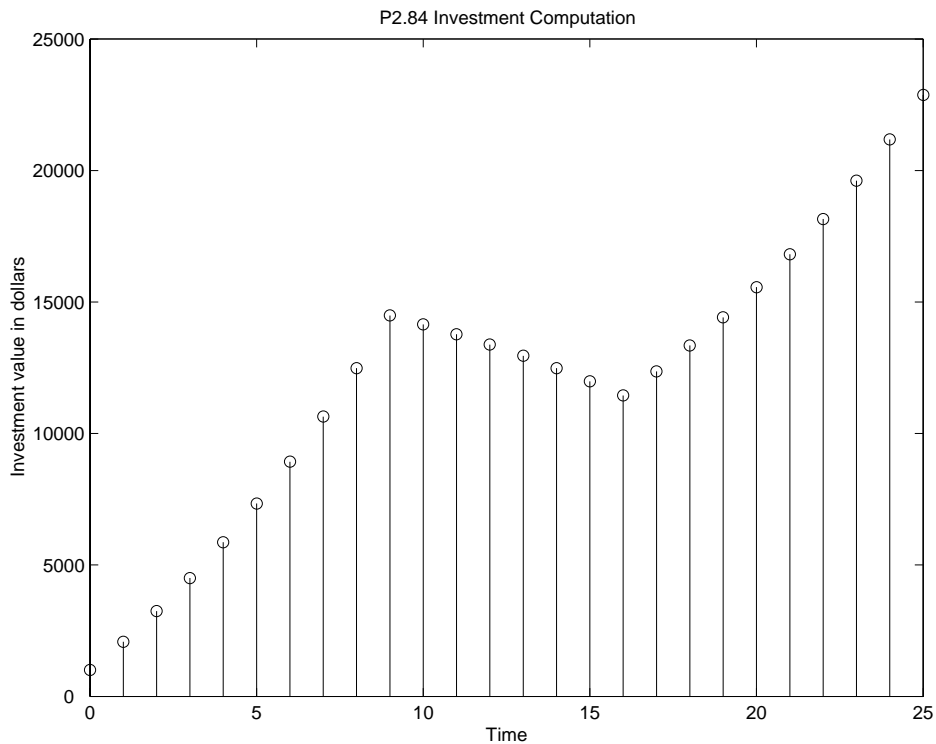


Figure P2.84. Investment computation

2.85. Use MATLAB to evaluate the first twenty values of the step response for Problem 2.50 (a) - (d).

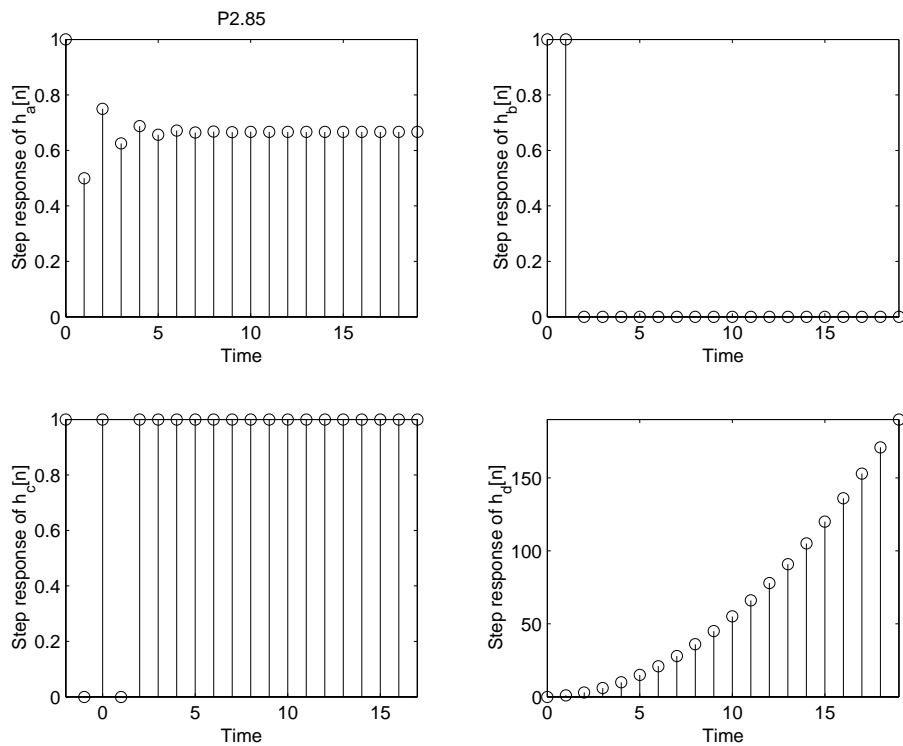


Figure P2.85. Step response for P2.50 (a)-(d)

2.86. Consider the two systems having impulse responses

$$h_1[n] = \begin{cases} \frac{1}{4}, & 0 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

$$h_2[n] = \begin{cases} \frac{1}{4}, & n = 0, 2 \\ -\frac{1}{4}, & n = 1, 3 \\ 0, & \text{otherwise} \end{cases}$$

Use the MATLAB command `conv` to plot the first 20 values of the step response.

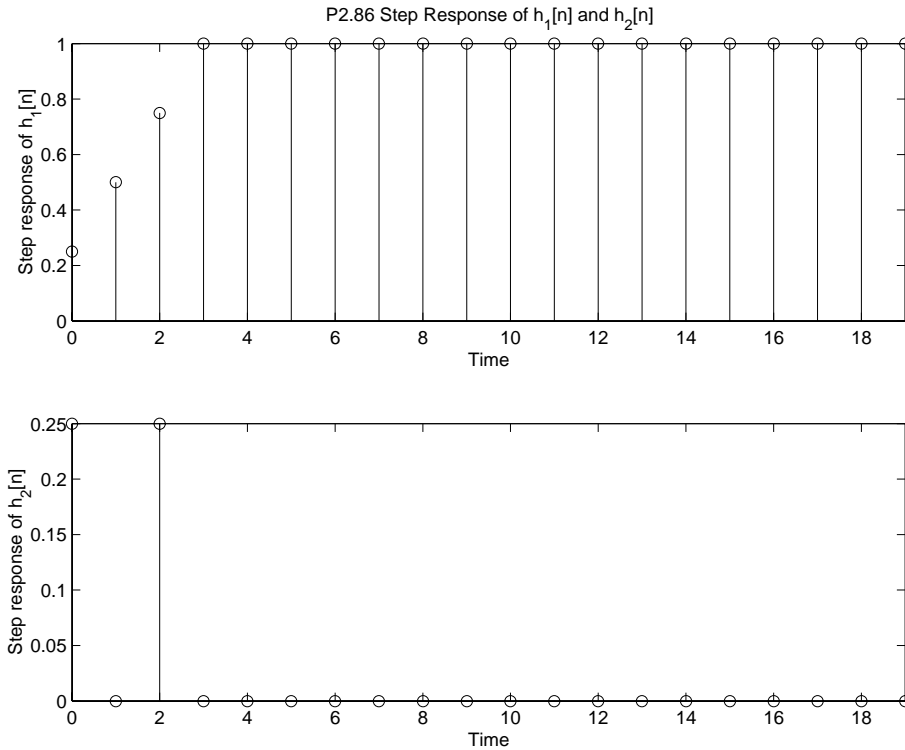


Figure P2.86. Step Response of the two systems

2.87. Use the MATLAB commands `filter` and `filtic` to repeat Example 2.16.

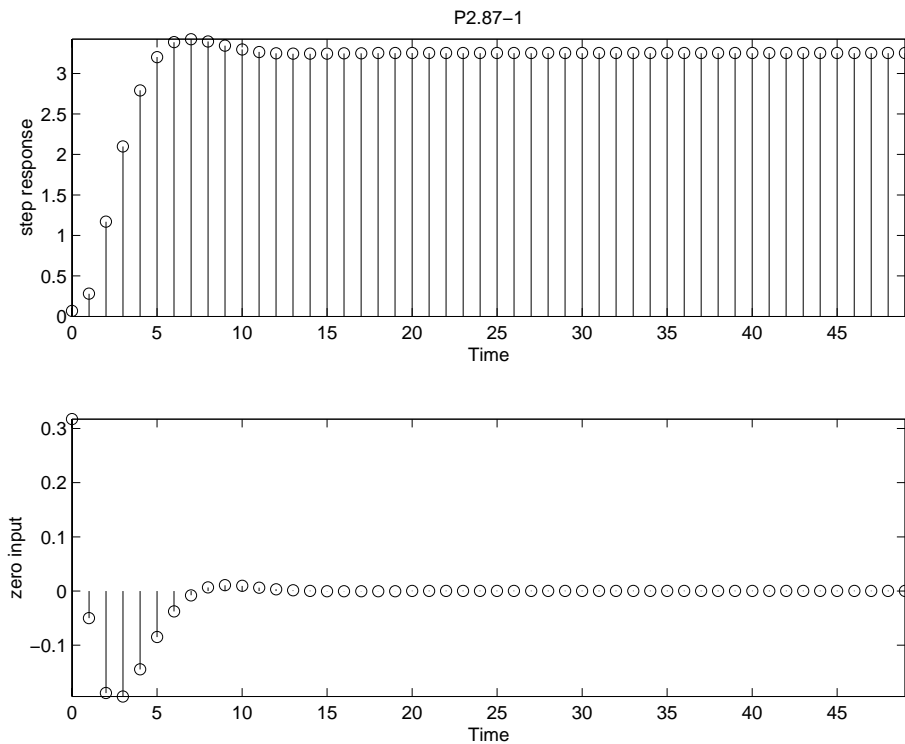


Figure P2.87. Step response and output due to no initial conditions.

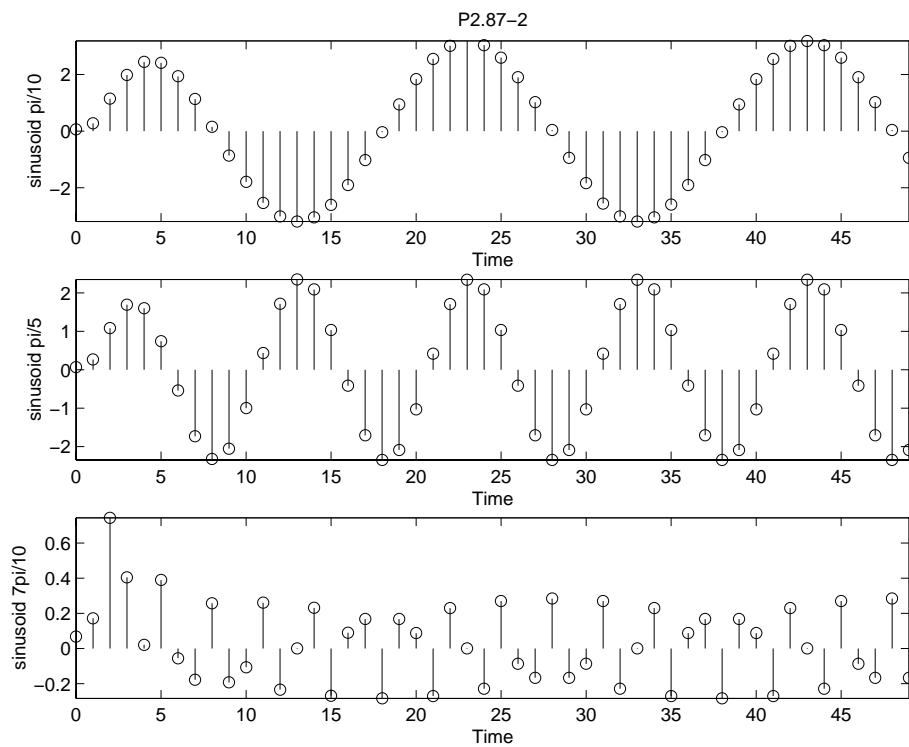


Figure P2.87. Output due to sinusoids of different frequencies.

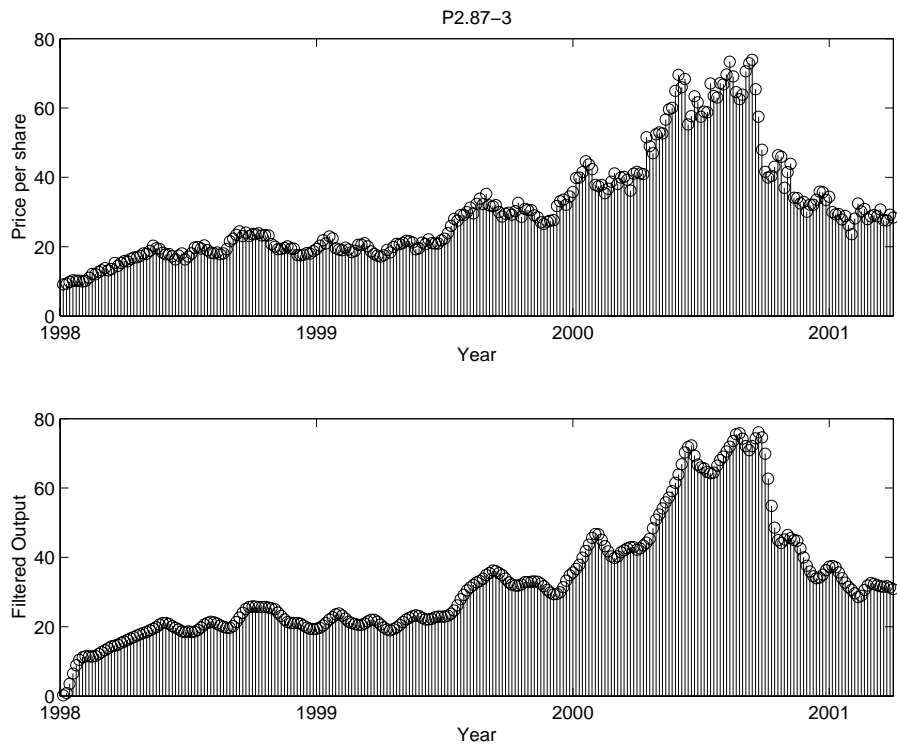


Figure P2.87. Input and filtered signals for Intel Stock Price.

2.88. Use the MATLAB commands `filter` and `fitic` to verify the loan balance in Example 2.23

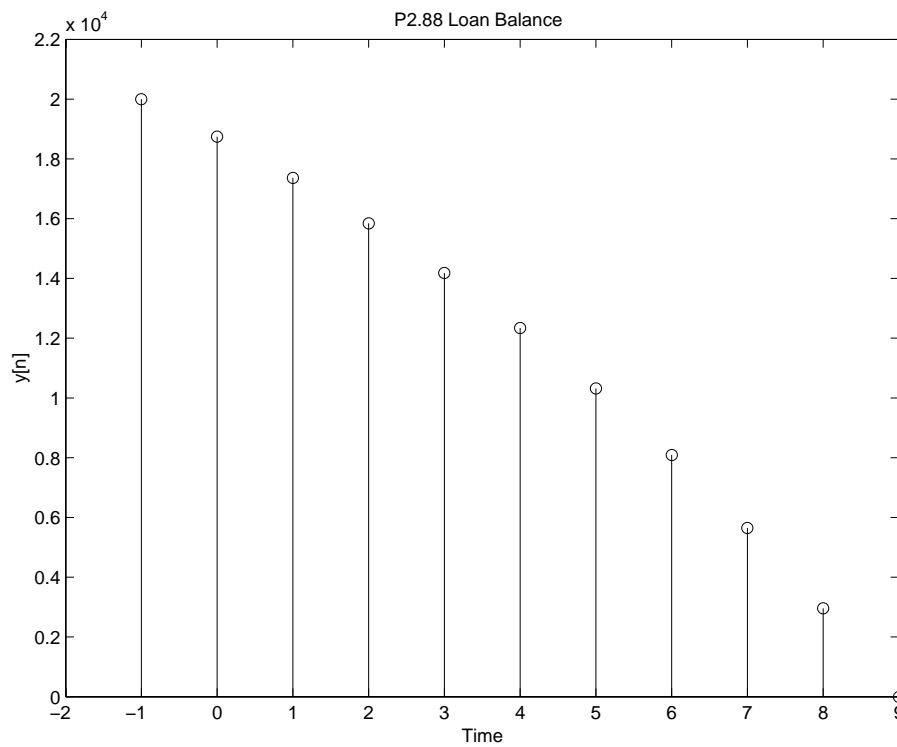


Figure P2.88. Loan Balance for Ex. 2.23

2.89. Use the MATLAB commands `filter` and `filtic` to determine the first fifty output values in Problem 2.59.

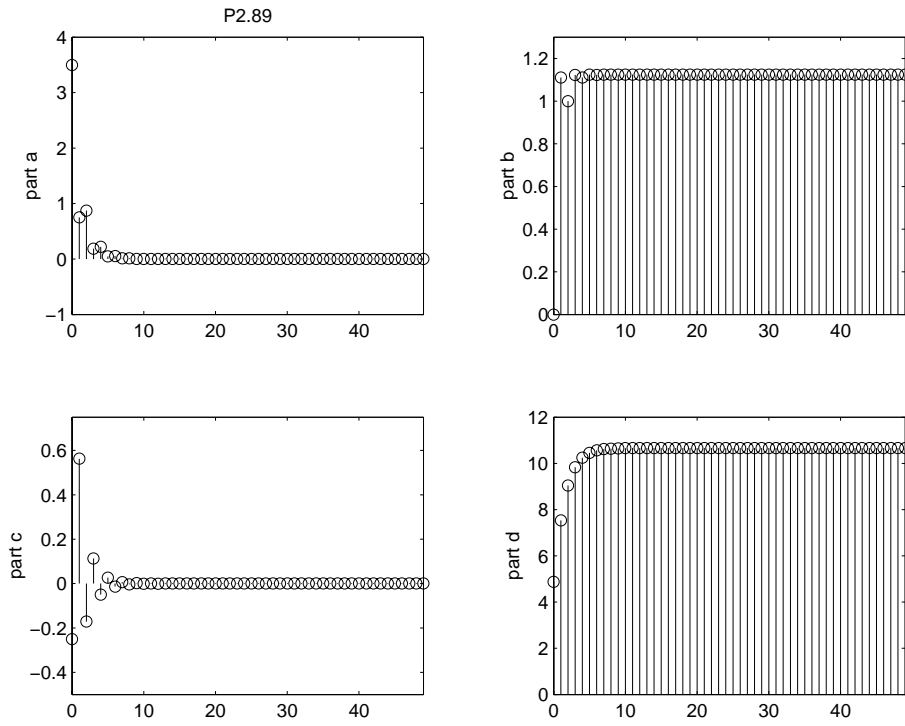


Figure P2.89. Output of first 50 values for Problem 2.59

2.90. Use the MATLAB command `impz` to determine the first 30 values of the impulse response for the systems described in Problem 2.59.

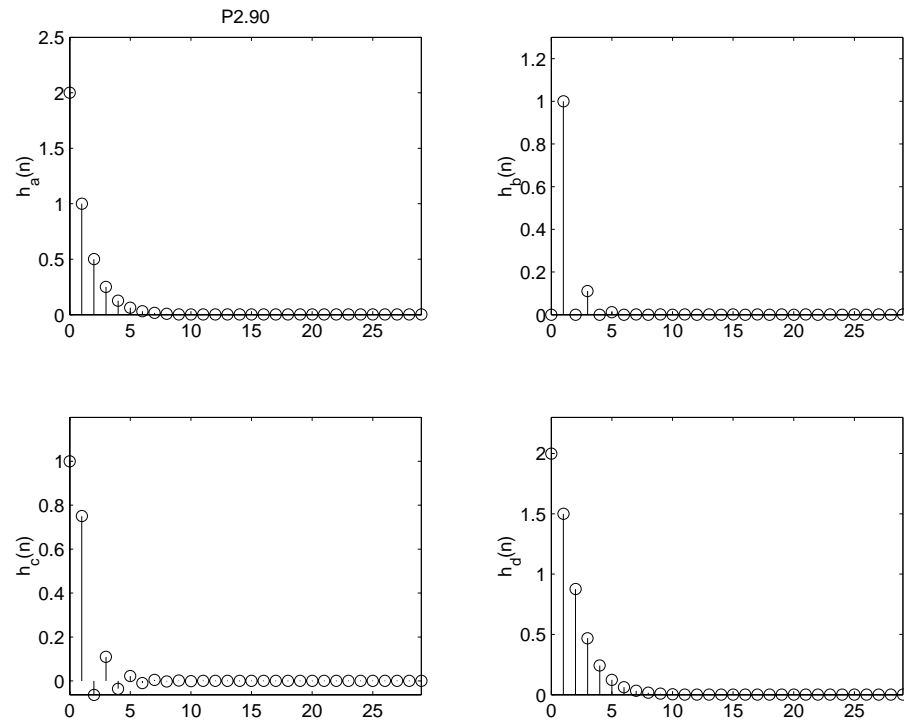


Figure P2.90. The first 30 values of the impulse response.

2.91. Use MATLAB to solve Problem 2.62.

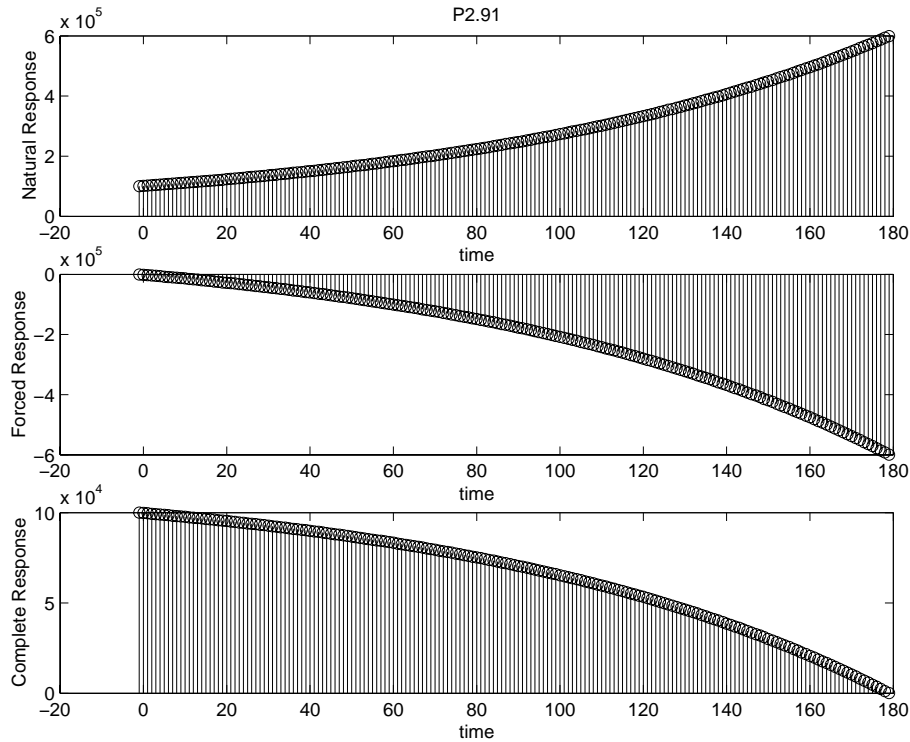


Figure P2.91. Using Matlab to solve Problem 2.62

2.92. Use MATLAB to solve Problem 2.63.

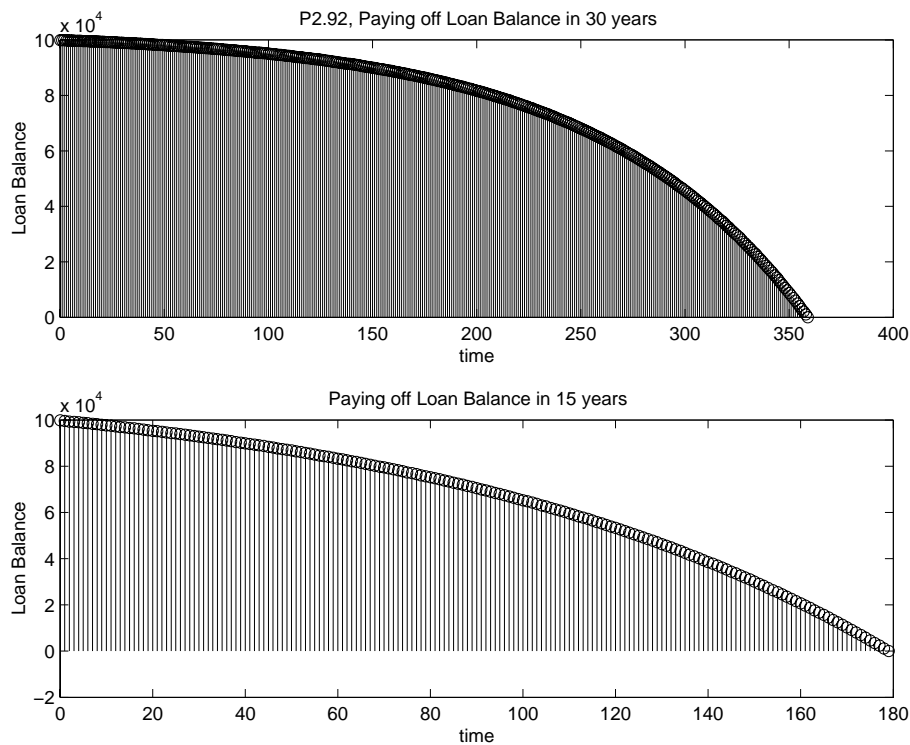


Figure P2.92. Plots of different loan payments, 30 vs. 15 years

2.93. Use the MATLAB command `ss2ss` to solve Problem 2.73.

P2.93 :

=====

Part (a) :

=====

a =

 x1 x2
x1 1 -0.3333
x2 0.5 0

b =

 u1
x1 2
x2 6

c =

 x1 x2
y1 0.5 -0.3333

d =

 u1
y1 0

Sampling time: unspecified

Discrete-time model.

Part (b) :

=====

a =

 x1 x2
x1 0 0.5
x2 -0.3333 1

b =

 u1
x1 6
x2 2


```

c =
    x1 x2
y1 -0.3333 0.5

```

```

d =
    u1
y1 0

```

Sampling time: unspecified
Discrete-time model.
Part (c) :
=====

```

a =
    x1 x2
x1 0.4167 0.9167
x2 0.08333 0.5833

```

```

b =
    u1
x1 3
x2 -1

```

```

c =
    x1 x2
y1 0 1

```

```

d =
    u1
y1 0

```

Sampling time: unspecified
Discrete-time model.

2.94. A system has the state-variable description

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad D = [0]$$

- (a) Use the MATLAB commands `lsim` and `impz` to determine the first 30 values of the step and impulse responses of this system.
- (b) Define new states $q_1[n] = q_1[n] + q_2[n]$ and $q_2[n] = 2q_1[n] - q_2[n]$. Repeat part (a) for the transformed system.

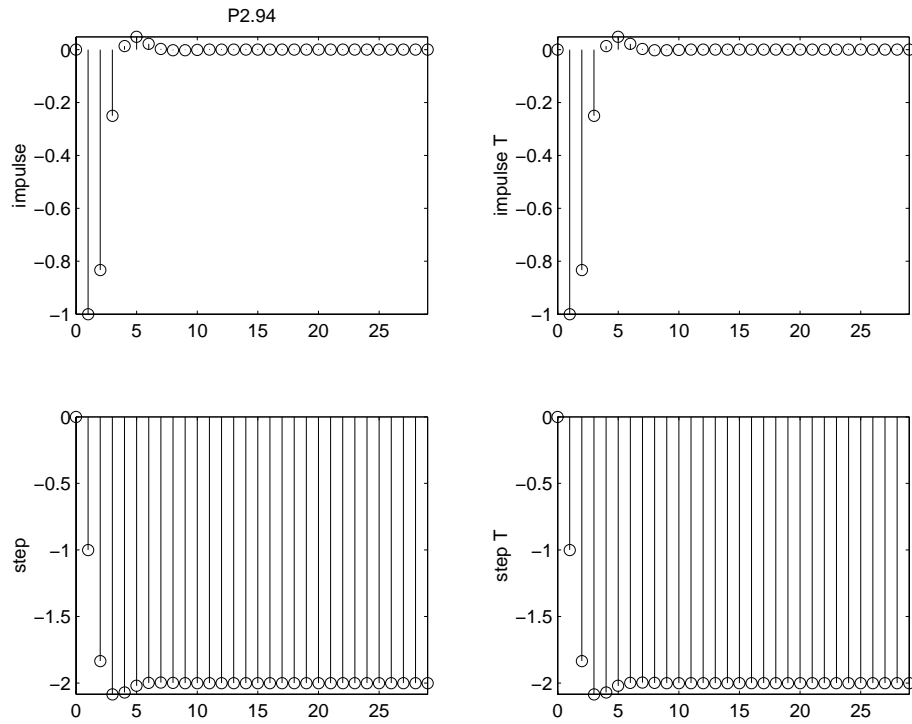


Figure P2.94. System step and impulse response

Solutions to Additional Problems

3.48. Use the defining equation for the DTFS coefficients to evaluate the DTFS representation for the following signals.

(a) $N = 17, \Omega_o = \frac{2\pi}{17}$

$$\begin{aligned} x[n] &= \cos\left(\frac{6\pi}{17}n + \frac{\pi}{3}\right) \\ &= \frac{1}{2} \left(e^{j(\frac{6\pi}{17}n + \frac{\pi}{3})} + e^{-j(\frac{6\pi}{17}n + \frac{\pi}{3})} \right) \\ &= \frac{1}{2} \left(e^{j\frac{\pi}{3}} e^{j(3)\frac{2\pi}{17}n} + e^{-j\frac{\pi}{3}} e^{j(-3)\frac{2\pi}{17}n} \right) \end{aligned}$$

By inspection

$$X[k] = \begin{cases} \frac{1}{2}e^{j\frac{\pi}{3}} & k = 3 \\ \frac{1}{2}e^{-j\frac{\pi}{3}} & k = -3 \\ 0 & \text{otherwise on } k = \{-8, -7, \dots, 8\} \end{cases}$$

(b) $N = 19, \Omega_o = \frac{2\pi}{19}$

$$\begin{aligned} x[n] &= 2\sin\left(\frac{4\pi}{19}n\right) + \cos\left(\frac{10\pi}{19}n\right) + 1 \\ &= \frac{1}{j} [e^{j\frac{4\pi}{19}n} - e^{-j\frac{4\pi}{19}n}] + \frac{1}{2} [e^{j\frac{10\pi}{19}n} + e^{-j\frac{10\pi}{19}n}] + 1 \\ &= -je^{j(2)\frac{2\pi}{19}n} + je^{j(-2)\frac{2\pi}{19}n} + \frac{1}{2} [e^{j(5)\frac{2\pi}{19}n} + e^{j(-5)\frac{2\pi}{19}n}] + e^{j(0)\frac{2\pi}{19}n} \end{aligned}$$

By inspection

$$X[k] = \begin{cases} \frac{1}{2} & k = -5 \\ j & k = -2 \\ 1 & k = 0 \\ -j & k = 2 \\ \frac{1}{2} & k = 5 \\ 0 & \text{otherwise on } k = \{9, -8, \dots, 9\} \end{cases}$$

(c)

$$x[n] = \sum_{m=-\infty}^{\infty} [(-1)^m (\delta[n-2m] + \delta[n+3m])]$$

Graph to find $N = 12, \Omega_o = \frac{\pi}{6}$

$$\begin{aligned} X[k] &= \frac{1}{12} \sum_{n=-5}^6 x[n] e^{-jk\frac{\pi}{6}n} \\ &= \frac{1}{12} \left[e^{-j(-4)\frac{\pi}{6}k} + e^{-j(-3)\frac{\pi}{6}k} - e^{-j(-2)\frac{\pi}{6}k} + 2 - e^{-j(2)\frac{\pi}{6}k} + e^{-j(3)\frac{\pi}{6}k} + e^{-j(4)\frac{\pi}{6}k} \right] \\ &= \frac{1}{6} \cos\left(\frac{2\pi}{3}k\right) + \frac{1}{6} \cos\left(\frac{\pi}{2}k\right) - \frac{1}{6} \cos\left(\frac{\pi}{3}k\right) + \frac{1}{6} \end{aligned}$$

(d) $x[n]$ as depicted in Figure P3.48(a)

$$N = 8, \Omega_o = \frac{\pi}{4}$$

$$\begin{aligned} X[k] &= \frac{1}{8} \sum_{n=-3}^4 x[n] e^{-jk\frac{\pi}{4}n} \\ &= \frac{1}{8} [-e^{-j(-2)\frac{\pi}{4}k} + e^{-j(2)\frac{\pi}{4}k}] \\ &= \frac{-j}{4} \sin(\frac{\pi}{2}k) \quad k \in \{-3, -2, \dots, 4\} \end{aligned}$$

(e) $x[n]$ as depicted in Figure P3.48(b)

$$N = 10, \Omega_o = \frac{\pi}{5}$$

$$\begin{aligned} X[k] &= \frac{1}{10} \sum_{n=-5}^4 x[n] e^{-jk\frac{\pi}{5}n} \\ &= \frac{1}{10} \left[\frac{1}{4} e^{-j(-4)\frac{\pi}{5}k} + \frac{1}{2} e^{-j(-3)\frac{\pi}{5}k} + \frac{3}{4} e^{-j(-2)\frac{\pi}{5}k} + e^{-j(-1)\frac{\pi}{5}k} \right] \quad k \in \{-5, -4, \dots, 4\} \end{aligned}$$

3.49. Use the definition of the DTFS to determine the time-domain signals represented by the following DTFS coefficients.

(a) $N = 21, \Omega_o = \frac{2\pi}{21}$

$$\begin{aligned} X[k] &= \cos(\frac{8\pi}{21}k) \\ &= \frac{1}{2} [e^{-j(-4)\frac{2\pi}{21}k} + \frac{1}{2} e^{-j(4)\frac{2\pi}{21}k}] \end{aligned}$$

By inspection

$$x[n] = \begin{cases} \frac{21}{2} & n = \pm 4 \\ 0 & \text{otherwise on } n \in \{-10, -9, \dots, 10\} \end{cases}$$

(b) $N = 19, \Omega_o = \frac{2\pi}{19}$

$$\begin{aligned} X[k] &= \cos(\frac{10\pi}{19}k) + 2j \sin(\frac{4\pi}{19}k) \\ &= \frac{1}{2} [e^{-j(-5)\frac{2\pi}{19}k} + e^{-j(5)\frac{2\pi}{19}k}] + e^{-j(-2)\frac{2\pi}{19}k} - e^{-j(2)\frac{2\pi}{19}k} \end{aligned}$$

By inspection

$$x[n] = \begin{cases} \frac{19}{2} & n = \pm 5 \\ -19 & n = 2 \\ 19 & n = -2 \\ 0 & \text{otherwise on } n \in \{-9, -8, \dots, 9\} \end{cases}$$

(c)

$$X[k] = \sum_{m=-\infty}^{\infty} [(-1)^m (\delta[k-2m] - 2\delta[k+3m])]$$

Graph to find $N = 12$, $\Omega_o = \frac{\pi}{6}$

$$\begin{aligned} x[n] &= \sum_{k=-5}^6 X[k] e^{jk\frac{\pi}{6}n} \\ &= e^{j(-4)\frac{\pi}{6}n} + 2e^{j(-3)\frac{\pi}{6}n} - e^{j(-2)\frac{\pi}{6}n} - 1 - e^{j(2)\frac{\pi}{6}n} \\ &\quad + 2e^{j(3)\frac{\pi}{6}n} + e^{j(4)\frac{\pi}{6}n} - 3e^{j(6)\frac{\pi}{6}n} \\ &= 2\cos\left(\frac{2\pi}{3}n\right) + 4\cos\left(\frac{\pi}{2}n\right) - 2\cos\left(\frac{\pi}{3}n\right) - 1 - 3(-1)^n \end{aligned}$$

(d) $X[k]$ as depicted in Figure P3.49(a)

$N = 14$, $\Omega_o = \frac{\pi}{7}$

$$\begin{aligned} x[n] &= \sum_{k=-6}^7 X[k] e^{jk\frac{\pi}{7}n} \\ &= e^{-j\frac{\pi}{2}} e^{j(-4)\frac{\pi}{7}n} + e^{j\frac{\pi}{2}} e^{j(-3)\frac{\pi}{7}n} + e^{j\frac{\pi}{2}} e^{j(3)\frac{\pi}{7}n} + e^{-j\frac{\pi}{2}} e^{j(4)\frac{\pi}{7}n} \\ &= 2\cos\left(\frac{3\pi}{7}n + \frac{\pi}{2}\right) + 2\cos\left(\frac{4\pi}{7}n - \frac{\pi}{2}\right) \end{aligned}$$

(e) $X[k]$ as depicted in Figure P3.49(b)

$N = 7$, $\Omega_o = \frac{2\pi}{7}$

$$\begin{aligned} x[n] &= \sum_{k=-3}^3 X[k] e^{jk\frac{2\pi}{7}n} \\ &= e^{j(-1)\frac{2\pi}{7}n} + e^{j(1)\frac{2\pi}{7}n} - \frac{1}{2} \\ &= 2\cos\left(\frac{2\pi}{7}n\right) - \frac{1}{2} \end{aligned}$$

(f) $X[k]$ as depicted in Figure P3.49(c)

$N = 15$, $\Omega_o = \frac{2\pi}{15}$

$$\begin{aligned} x[n] &= \sum_{k=-5}^9 X[k] e^{jk\frac{2\pi}{15}n} \\ &= \sum_{k=-4}^4 e^{-j\frac{\pi}{6}k} e^{jk\frac{2\pi}{15}n} \\ &\quad \text{let } l = k + 4 \\ &= \sum_{l=0}^8 e^{j\pi(l-4)(\frac{2n}{15} - \frac{1}{6})} \\ &= e^{-j4\pi(\frac{2n}{15} - \frac{1}{6})} \sum_{l=0}^8 (e^{j\pi(\frac{2n}{15} - \frac{1}{6})})^l \\ &= e^{-j4\pi(\frac{2n}{15} - \frac{1}{6})} \frac{1 - e^{j9\pi(\frac{2n}{15} - \frac{1}{6})}}{1 - e^{j\pi(\frac{2n}{15} - \frac{1}{6})}} \end{aligned}$$

$$= \frac{\sin(\frac{9\pi}{2}(\frac{2n}{15} - \frac{1}{6}))}{\sin((\frac{\pi}{2}(\frac{2n}{15} - \frac{1}{6}))}$$

3.50. Use the defining equation for the FS coefficients to evaluate the FS representation for the following signals.

(a) $T_1 = \frac{2}{3}$, $T_2 = \frac{1}{2}$, $T = \text{lcm}(T_1, T_2) = 2$, $\omega_o = \pi$
 lcm is the least common multiple.

$$\begin{aligned} x(t) &= \sin(3\pi t) + \cos(4\pi t) \\ &= \frac{1}{2j}e^{j(3)\pi t} - \frac{1}{2j}e^{j(-3)\pi t} + \frac{1}{2}e^{j(4)\pi t} + \frac{1}{2}e^{j(-4)\pi t} \end{aligned}$$

By inspection

$$X[k] = \begin{cases} \frac{1}{2} & k = \pm 4 \\ \frac{1}{2j} & k = 3 \\ \frac{-1}{2j} & k = -3 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$x(t) = \sum_{m=-\infty}^{\infty} (-1)^m [\delta(t - \frac{m}{3}) + \delta(t + \frac{2m}{3})]$$

Graph to find $T = \frac{4}{3}$, $\omega_o = \frac{3\pi}{2}$

$$X[k] = \frac{3}{4} \int_{-\frac{2}{3}}^{\frac{2}{3}} x(t) e^{-jk\pi t} dt = \int_{-\frac{2}{3}}^{\frac{2}{3}} [2\delta(t) - \delta(t-1) - \delta(t+1)] e^{-jk\frac{3\pi}{2}t} dt$$

By the sifting property

$$\begin{aligned} X[k] &= \frac{6}{4} - \frac{3}{4}e^{-jk\frac{3\pi}{2}} - \frac{3}{4}e^{jk\frac{3\pi}{2}} \\ &= \frac{6}{4} - \frac{3}{2}\cos(\frac{3\pi}{2}k) \end{aligned}$$

(c)

$$x(t) = \sum_{m=-\infty}^{\infty} [e^{\frac{2\pi}{7}m}\delta(t-2m)]$$

Graph to find $T = 14$, $\omega_o = \frac{\pi}{7}$

$$X[k] = \frac{1}{14} \int_{-7}^7 x(t) e^{-jk\frac{\pi}{7}t} dt$$

By the sifting property

$$\begin{aligned} &= \frac{1}{14} \left[e^{j(k-1)\frac{6\pi}{7}} + e^{j(k-1)\frac{4\pi}{7}} + e^{j(k-1)\frac{2\pi}{7}} + 1 + e^{j(1-k)\frac{2\pi}{7}} + e^{j(1-k)\frac{4\pi}{7}} + e^{j(1-k)\frac{6\pi}{7}} \right] \\ &= \frac{1}{7} \left[\cos((k-1)\frac{6\pi}{7}) + \cos((k-1)\frac{4\pi}{7}) + \cos((k-1)\frac{2\pi}{7}) + \frac{1}{2} \right] \end{aligned}$$

(d) $x(t)$ as depicted in Figure P3.50(a)

$$x(t) = |\sin(\pi t)|$$

$$T = 1, \omega_o = 2\pi$$

$$\begin{aligned} X[k] &= \frac{1}{2j} \int_0^1 [e^{j\pi t} - e^{-j\pi t}] e^{-jk2\pi t} dt \\ &= \frac{1}{2j} \left[\frac{1}{j\pi(1-2k)} \{e^{j\pi(1-2k)} - 1\} - \frac{1}{j\pi(1+2k)} \{e^{-j\pi(1+2k)} - 1\} \right] \\ &= \frac{4k}{\pi(1-4k)} \end{aligned}$$

(e) $x(t)$ as depicted in Figure P3.50(b)

$$T = 2, \omega_o = \pi$$

$$\begin{aligned} X[k] &= \frac{1}{2} \int_0^1 e^{-t} e^{-jk\pi t} dt \\ &= \frac{1}{2} \int_0^1 e^{-t(1+jk\pi)} dt \\ &= \frac{1 - e^{-1(1+jk\pi)}}{2(jk\pi + 1)} \end{aligned}$$

(f) $x(t)$ as depicted in Figure P3.50(c)

$$T = 3, \omega_o = \frac{2\pi}{3}$$

$$\begin{aligned} X[k] &= \frac{1}{3} \int_{-1}^1 t e^{-jk\frac{2\pi}{3}t} dt + \frac{1}{3} \int_1^2 (3-2t) e^{-jk\frac{2\pi}{3}t} dt \\ &= \frac{j2 \sin(\frac{2\pi}{3}k) - \frac{4\pi}{3} \cos(\frac{2\pi}{3}k) + \frac{4\pi}{3} e^{-\pi k} \cos(\frac{\pi}{3}k) - j4e^{-\pi k} \sin(\frac{\pi}{3}k)}{\frac{4\pi^2}{3}} \end{aligned}$$

By L'Hopital's rule, $X[k] = 0$ for $k = 0$.

3.51. Use the definition of the FS to determine the time-domain signals represented by the following FS coefficients.

(a) $X[k] = j\delta[k-1] - j\delta[k+1] + \delta[k-3] + \delta[k+3], \omega_o = 2\pi$

$$\begin{aligned} x(t) &= \sum_{m=-\infty}^{\infty} X[k] e^{j2\pi kt} \\ &= j e^{j(1)2\pi t} - j e^{j(-1)2\pi t} + e^{j(3)2\pi t} + e^{j(-3)\pi t} \\ &= -2 \sin(2\pi t) + 2 \cos(6\pi t) \end{aligned}$$

(b) $X[k] = j\delta[k-1] - j\delta[k+1] + \delta[k-3] + \delta[k+3], \omega_o = 4\pi$

$$\begin{aligned} x(t) &= j e^{j(1)4\pi t} - j e^{j(-1)4\pi t} + e^{j(3)4\pi t} + e^{j(-3)4\pi t} \\ &= -2 \sin(4\pi t) + 2 \cos(12\pi t) \end{aligned}$$

(c) $X[k] = (-\frac{1}{3})^{|k|}$, $\omega_o = 1$

$$\begin{aligned}
 x(t) &= \sum_{m=-\infty}^{\infty} (-\frac{1}{3})^{|k|} e^{jkt} \\
 &= \sum_{m=0}^{\infty} (-\frac{1}{3} e^{jt})^k + \sum_{m=1}^{\infty} (-\frac{1}{3} e^{-jt})^k \\
 &= \frac{1}{1 + \frac{1}{3} e^{jt}} - \frac{\frac{1}{3} e^{-jt}}{1 + \frac{1}{3} e^{-jt}} \\
 &= \frac{8}{10 + 6 \cos(t)}
 \end{aligned}$$

(d) $X[k]$ as depicted in Figure P3.51(a).

$\omega_o = \pi$

$$\begin{aligned}
 x(t) &= \sum_{m=-\infty}^{\infty} X[k] e^{j\pi k t} \\
 &= 2e^{-j0.25\pi} e^{j(-4)\pi t} + e^{j0.25\pi} e^{j(-3)\pi t} + e^{-j0.25\pi} e^{j(3)\pi t} \\
 &\quad + 2e^{j0.25\pi} e^{j(4)\pi t} \\
 &= 4 \cos(4\pi t + 0.25\pi) + 2 \cos(3\pi t - 0.25\pi)
 \end{aligned}$$

(e) $X[k]$ as depicted in Figure P3.51(b).

$\omega_o = 2\pi$.

$$X[k] = e^{-j2\pi k} \quad -4 \leq k < 4$$

$$\begin{aligned}
 x(t) &= \sum_{m=-4}^4 e^{j2\pi k(t-1)} \\
 &= \frac{\sin(9\pi t)}{\sin(\pi t)}
 \end{aligned}$$

(f) $X[k]$ as depicted in Figure P3.51(b).

$\omega_o = \pi$.

$$X[k] = |k| \quad -4 \leq k < 4$$

$$x(t) = 8 \cos(4\pi t) + 6 \cos(3\pi t) + 4 \cos(2\pi t) + 2 \cos(\pi t)$$

3.52. Use the defining equation for the DTFT to evaluate the frequency-domain representations for the following signals. Sketch the magnitude and phase spectra.

(a) $x[n] = (\frac{3}{4})^n u[n-4]$

$$X(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$\begin{aligned}
&= \sum_{m=4}^{\infty} \left(\frac{3}{4}\right)^n e^{-j\Omega n} \\
&= \sum_{m=4}^{\infty} \left(\frac{3}{4}e^{-j\Omega}\right)^n \\
&= \frac{\left(\frac{3}{4}e^{-j\Omega}\right)^4}{1 - \frac{3}{4}e^{-j\Omega}}
\end{aligned}$$

$$\begin{aligned}
|X(e^{j\Omega})| &= \frac{\left(\frac{3}{4}\right)^4}{\left(\frac{25}{16} - \frac{3}{2}\cos(\Omega)\right)^{0.5}} \\
\angle X(e^{j\Omega}) &= -4\Omega + \arctan\left(\frac{3\sin(\Omega)}{4 - 3\cos(\Omega)}\right)
\end{aligned}$$

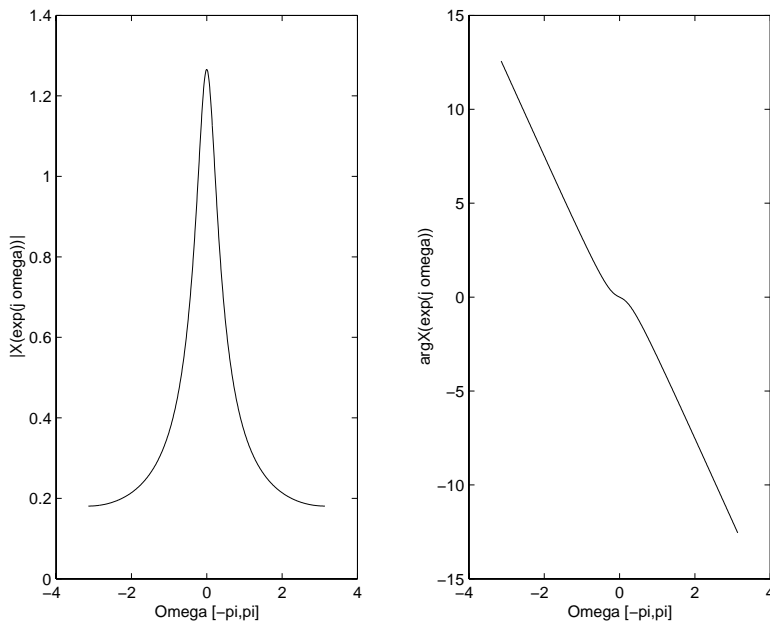


Figure P3.52. (a) Graph of the magnitude and phase

(b) $x[n] = a^{|n|}$ $|a| < 1$

$$\begin{aligned}
X(e^{j\Omega}) &= \sum_{n=0}^{\infty} (ae^{-j\Omega})^n + \sum_{n=-1}^{-\infty} (ae^{j\Omega})^{-n} \\
&= \frac{1}{1 - ae^{-j\Omega}} + \frac{ae^{j\Omega}}{1 - ae^{j\Omega}} \\
&= \frac{1 - a^2}{1 + a^2 - 2a\cos(\Omega)}
\end{aligned}$$

$X(e^{j\Omega})$ is completely real and nonnegative, thus:

$$\begin{aligned}
|X(e^{j\Omega})| &= \frac{1 - a^2}{1 + a^2 - 2a\cos(\Omega)} \\
\angle X(e^{j\Omega}) &= 0
\end{aligned}$$

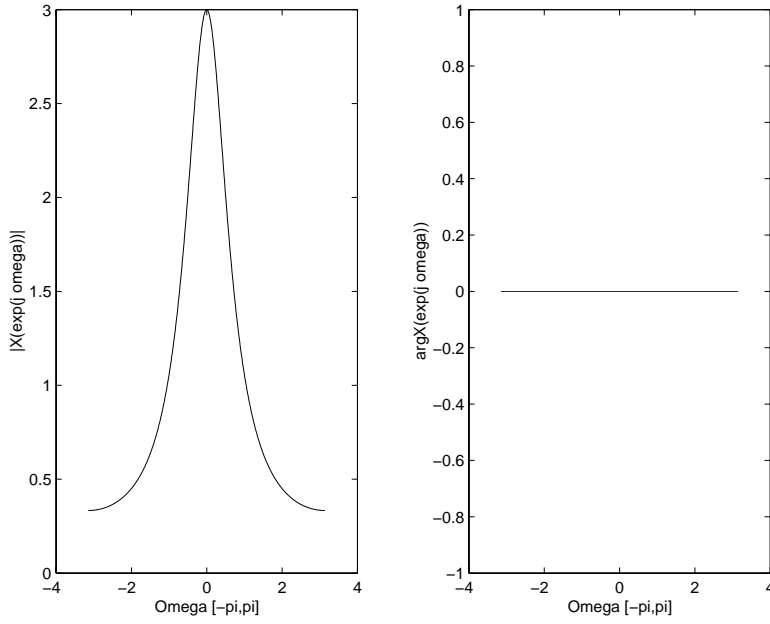


Figure P3.52. (b) Graph of the magnitude and phase for $a = 0.5$

$$(c) \ x[n] = \begin{cases} \frac{1}{2} + \frac{1}{2} \cos(\frac{\pi}{N}n), & |n| \leq N \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} X(e^{j\Omega}) &= \frac{1}{2} \sum_{n=-N}^N \left(1 + \frac{e^{j\frac{\pi}{N}n} + e^{-j\frac{\pi}{N}n}}{2}\right) e^{-j\Omega n} \\ &= \frac{1}{2} \frac{\sin(\frac{2N+1}{2}\Omega)}{\sin(\frac{1}{2}\Omega)} + \frac{1}{2} \frac{\sin(\frac{2N+1}{2}(\Omega - \frac{\pi}{N}))}{\sin(\frac{1}{2}(\Omega - \frac{\pi}{N}))} + \frac{1}{2} \frac{\sin(\frac{2N+1}{2}(\Omega + \frac{\pi}{N}))}{\sin(\frac{1}{2}(\Omega + \frac{\pi}{N}))} \end{aligned}$$

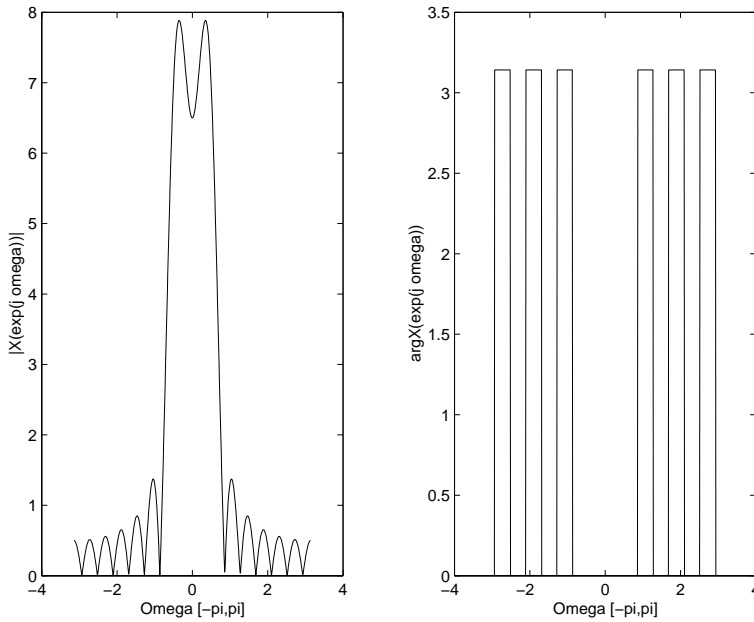


Figure P3.52. (c) Graph of the magnitude and phase for $N = 7$

(d) $x[n] = 2\delta[4 - 2n]$

$$\begin{aligned}
 X(e^{j\Omega}) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} 2\delta[4 - 2n]e^{-j\Omega n} \\
 &= 2e^{-j2\Omega} \\
 |X(e^{j\Omega})| &= 2 \\
 \angle X(e^{j\Omega}) &= -2\Omega
 \end{aligned}$$

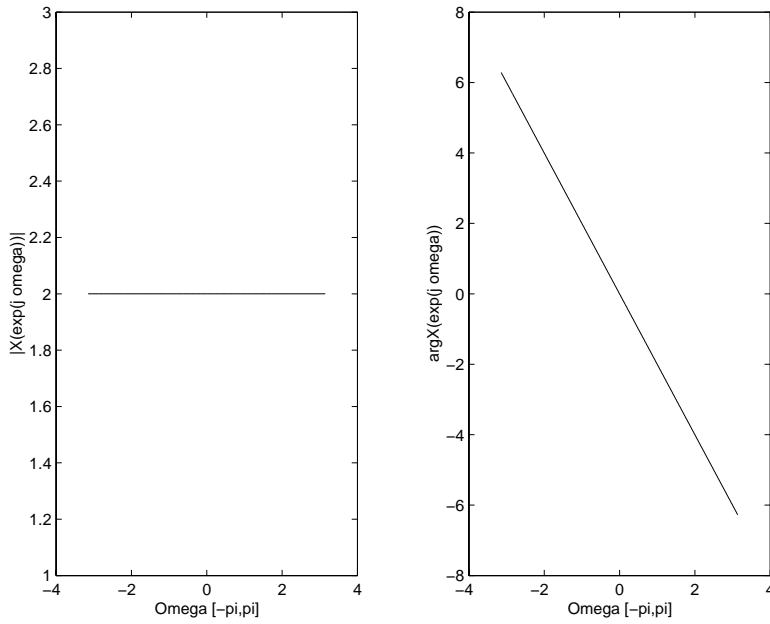


Figure P3.52. (d) Graph of the magnitude and phase

(e) $x[n]$ as depicted in Figure P3.52(a)

$$\begin{aligned}
 X(e^{j\Omega}) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \\
 &= e^{j4\Omega} + e^{j2\Omega} + e^{-j2\Omega} - e^{-j4\Omega} \\
 &= 2\cos(2\Omega) + 2j\sin(4\Omega) \\
 |X(e^{j\Omega})| &= (4\cos^2(2\Omega) + 4\sin^2(4\Omega))^{\frac{1}{2}} \\
 \angle X(e^{j\Omega}) &= \arctan\left(\frac{\sin(4\Omega)}{\cos(2\Omega)}\right)
 \end{aligned}$$

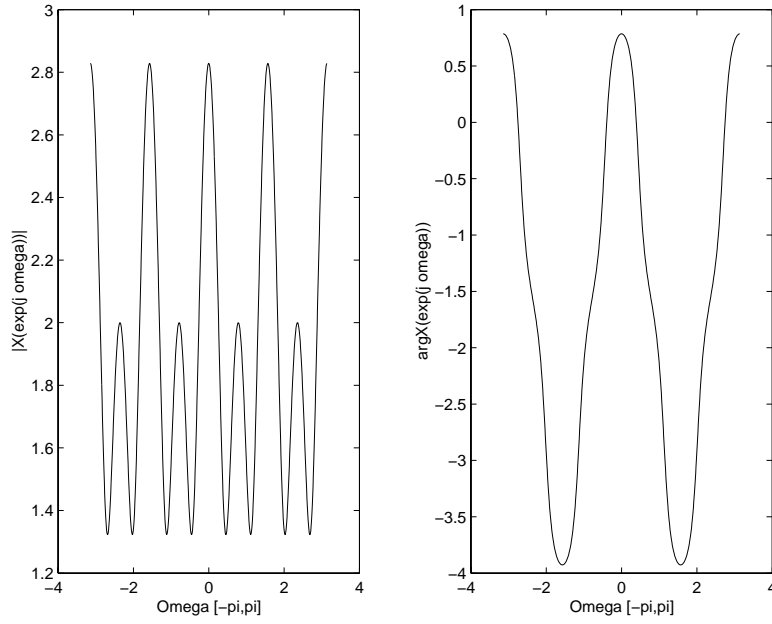


Figure P3.52. (e) Graph of the magnitude and phase

(f) $x[n]$ as depicted in Figure P3.52(b)

$$\begin{aligned}
 X(e^{j\Omega}) &= -j2[\sin(\Omega) + \sin(2\Omega) + \sin(3\Omega) + \sin(4\Omega) + \sin(5\Omega)] \\
 |X(e^{j\Omega})| &= 2|(\sin(\Omega) + \sin(2\Omega) + \sin(3\Omega) + \sin(4\Omega) + \sin(5\Omega))| \\
 \angle X(e^{j\Omega}) &= -\frac{\pi}{2} \text{sgn}(\sin(\Omega) + \sin(2\Omega) + \sin(3\Omega) + \sin(4\Omega) + \sin(5\Omega))
 \end{aligned}$$

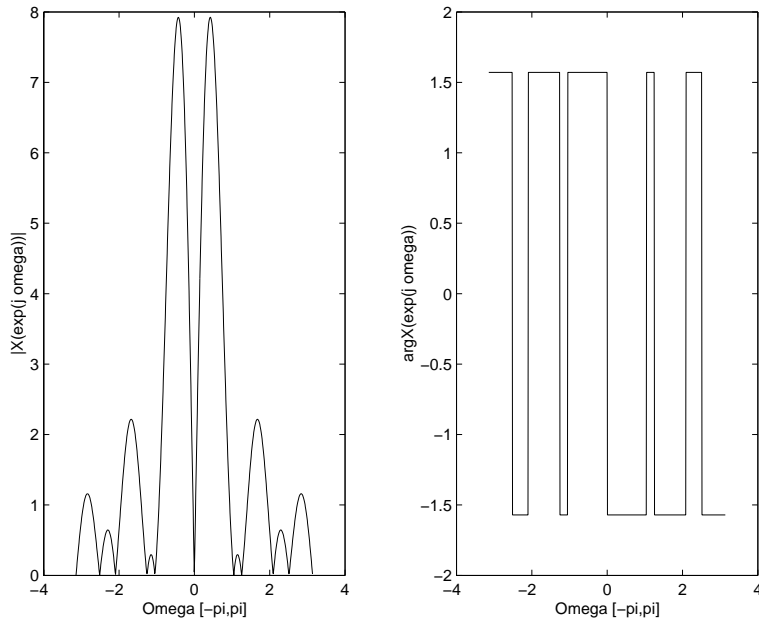


Figure P3.52. (f) Graph of the magnitude and phase

3.53. Use the equation describing the DTFT representation to determine the time-domain signals corresponding to the following DTFT's.

(a) $X(e^{j\Omega}) = \cos(2\Omega) + j \sin(2\Omega)$

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega(2+n)} d\Omega \\ &\quad \text{by orthogonality} \\ &= \delta[n+2] \end{aligned}$$

(b) $X(e^{j\Omega}) = \sin(\Omega) + \cos(\frac{\Omega}{2})$

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{e^{j\Omega} - e^{-j\Omega}}{2j} + \frac{e^{j\frac{\Omega}{2}} + e^{-j\frac{\Omega}{2}}}{2} \right) e^{j\Omega n} d\Omega \\ &= \frac{1}{2j} \delta[n+1] - \frac{1}{2j} \delta[n-1] + \frac{1}{2\pi} \frac{\cos(\pi n)}{n+0.5} - \frac{1}{2\pi} \frac{\cos(\pi n)}{n-0.5} \end{aligned}$$

(c) $|X(e^{j\Omega})| = \begin{cases} 1, & \pi/4 < |\Omega| < 3\pi/4, \\ 0 & \text{otherwise} \end{cases} \quad \arg\{X(e^{j\Omega})\} = -4\Omega$

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{0.25\pi}^{0.75\pi} e^{j\Omega(n-4)} d\Omega + \frac{1}{2\pi} \int_{-0.75\pi}^{-0.25\pi} e^{j\Omega(n-4)} d\Omega \\ &= \frac{\sin(0.75\pi(n-4)) - \sin(0.25\pi(n-4))}{\pi(n-4)} \end{aligned}$$

(d) $X(e^{j\Omega})$ as depicted in Figure P3.53 (a)

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi}^0 e^{j\Omega(jn+1)} d\Omega + \frac{1}{2\pi} \int_0^{\pi} e^{j\Omega(jn-1)} d\Omega \\ &= \frac{1 + e^{-\pi}(-1)^n}{\pi(n^2 + 1)} \end{aligned}$$

(e) $X(e^{j\Omega})$ as depicted in Figure P3.53 (b)

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^0 -\sin(\Omega) e^{j2\Omega} e^{j\Omega n} d\Omega + \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \sin(\Omega) e^{j2\Omega} e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \sin(\Omega) \left[e^{-j(2+n)\Omega} + e^{j(2+n)\Omega} \right] d\Omega \\ &= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \sin(\Omega) \cos((2+n)\Omega) d\Omega \\ &= \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \sin(\Omega(-1-n)) + \sin(\Omega(3+n)) d\Omega \\ &= \frac{1}{2\pi} \frac{1 - \cos(\frac{\pi}{2}(n+1))}{n+1} + \frac{1}{2\pi} \frac{\cos(\frac{\pi}{2}(n+3)) - 1}{n+3} \end{aligned}$$

$$x[n] = \begin{cases} \frac{1}{2\pi} \frac{1 - \cos(\frac{\pi}{2}(n+1))}{n+1} + \frac{1}{2\pi} \frac{\cos(\frac{\pi}{2}(n+3)) - 1}{n+3} & n \neq -1, -3 \\ -\frac{1}{2\pi} & n = -1, -3 \end{cases}$$

(f) $X(e^{j\Omega})$ as depicted in Figure P3.53 (c)

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^0 e^{-j\pi} e^{j\Omega n} d\Omega + \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} e^{j\Omega n} d\Omega \\ &= \frac{\cos(\frac{\pi}{2}n) - 1}{j\pi n} \\ x[n] &= \begin{cases} \frac{\cos(\frac{\pi}{2}n) - 1}{j\pi n} & n \neq 0 \\ 0 & n = 0 \end{cases} \end{aligned}$$

3.54. Use the defining equation for the FT to evaluate the frequency-domain representations for the following signals.

(a) $x(t) = e^{-2t}u(t-3)$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \\ &= \int_3^{\infty} e^{-2t}e^{-j\omega t} dt \\ &= \frac{e^{-3(2+j\omega)}}{2+j\omega} \end{aligned}$$

(b) $x(t) = e^{-4|t|}$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-4|t|}e^{-j\omega t} dt \\ &= \int_0^{\infty} e^{-4t}e^{-j\omega t} dt + \int_{-\infty}^0 e^{4t}e^{-j\omega t} dt \\ &= \frac{8}{16 + \omega^2} \end{aligned}$$

(c) $x(t) = te^{-t}u(t)$

$$\begin{aligned} X(j\omega) &= \int_0^{\infty} te^{-t}e^{-j\omega t} dt \\ &= \frac{1}{(1+j\omega)^2} \end{aligned}$$

(d) $x(t) = \sum_{m=0}^{\infty} a^m \delta(t-m)$, $|a| < 1$

$$\begin{aligned} X(j\omega) &= \int_0^{\infty} \left(\sum_{m=0}^{\infty} a^m \delta(t-m) \right) e^{-j\omega t} dt \\ &= \sum_{m=0}^{\infty} (ae^{-j\omega})^m \\ &= \frac{1}{1 - ae^{-j\omega}} \end{aligned}$$

(e) $x(t)$ as depicted in Figure P3.54 (a)

$$\begin{aligned} X(j\omega) &= \int_{-1}^0 e^{-j\omega t} dt - \int_0^1 e^{-j\omega t} dt \\ &= \frac{2 \cos(\omega) - 2}{j\omega} \end{aligned}$$

$$X(j\omega) = \begin{cases} \frac{2 \cos(\omega) - 2}{j\omega} & \omega \neq 0 \\ 0 & \omega = 0 \end{cases}$$

(f) $x(t)$ as depicted in Figure P3.54 (b)

$$\begin{aligned} X(j\omega) &= \int_{-2}^0 e^t e^{-j\omega t} dt + \int_0^2 e^{-t} e^{-j\omega t} dt \\ &= \frac{1 - e^{-(1-j\omega)2}}{1 - j\omega} + \frac{1 - e^{-(1+j\omega)2}}{1 + j\omega} \\ &= \frac{2 - 2e^{-2} \cos(2\omega) + 2\omega e^{-2} \sin(2\omega)}{1 + \omega^2} \end{aligned}$$

3.55. Use the equation describing the FT representation to determine the time-domain signals corresponding to the following FT's.

(a) $X(j\omega) = \begin{cases} \cos(2\omega), & |\omega| < \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-0.25\pi}^{0.25\pi} \frac{e^{j2\omega} + e^{-j2\omega}}{2} e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-0.25\pi}^{0.25\pi} \frac{1}{2} e^{j(t+2)\omega} dt + \frac{1}{2\pi} \int_{-0.25\pi}^{0.25\pi} \frac{1}{2} e^{j(t-2)\omega} d\omega \\ &= \frac{\sin(0.25\pi(t+2))}{2\pi(t+2)} + \frac{\sin(0.25\pi(t-2))}{2\pi(t-2)} \end{aligned}$$

$$x(t) = \begin{cases} \frac{\sin(0.25\pi(t+2))}{2\pi(t+2)} + \frac{\sin(0.25\pi(t-2))}{2\pi(t-2)} & t \neq 2, -2 \\ \frac{1}{8} & t = \pm 2 \end{cases}$$

(b) $X(j\omega) = e^{-2\omega} u(\omega)$

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_0^{\infty} e^{-2\omega} e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_0^{\infty} e^{(jt-2)\omega} d\omega \\ &= \frac{1}{2\pi(2-jt)} \end{aligned}$$

(c) $X(j\omega) = e^{-2|\omega|}$

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2|\omega|} e^{-j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_0^{\infty} e^{-2\omega} e^{-j\omega t} d\omega + \frac{1}{2\pi} \int_{-\infty}^0 e^{2\omega} e^{-j\omega t} d\omega \\ &= \frac{2}{\pi(4+t^2)} \end{aligned}$$

(d) $X(j\omega)$ as depicted in Figure P3.55 (a)

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-2}^2 e^{-j2\omega} e^{-j\omega t} d\omega \\ &= \frac{\sin(2(t-2))}{\pi(t-2)} \\ x(t) &= \begin{cases} \frac{\sin(2(t-2))}{\pi(t-2)} & t \neq 2 \\ \frac{2}{\pi} & t = 2 \end{cases} \end{aligned}$$

(e) $X(j\omega)$ as depicted in Figure P3.55 (b)

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-3}^3 \frac{2}{3} j\omega e^{j\omega t} d\omega \\ &= \frac{2 \cos(3t)}{2\pi t} - \frac{2 \sin(3t)}{3\pi t^2} \\ x(t) &= \begin{cases} \frac{2 \cos(3t)}{\pi t} - \frac{2 \sin(3t)}{3\pi t^2} & t \neq 0 \\ 0 & t = 0 \end{cases} \end{aligned}$$

(f) $X(j\omega)$ as depicted in Figure P3.55 (c)

$$\begin{aligned} x(t) &= \frac{j}{2\pi} \int_{-2}^0 e^{j\omega t} d\omega - \frac{j}{2\pi} \int_0^2 e^{j\omega t} d\omega \\ &= \frac{1 - \cos(2t)}{\pi t} \\ x(t) &= \begin{cases} \frac{1 - \cos(2t)}{\pi t} & t \neq 0 \\ 0 & t = 0 \end{cases} \end{aligned}$$

3.56. Determine the appropriate Fourier representation for the following time-domain signals, using the defining equations.

(a) $x(t) = e^{-t} \cos(2\pi t) u(t)$

Continuous and Nonperiodic, use FT.

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \frac{1}{2} \int_0^{\infty} e^{-t} (e^{j2\pi t} + e^{-j2\pi t}) e^{-j\omega t} dt \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^\infty e^{-t(1-j2\pi+j\omega)} dt + \frac{1}{2} \int_0^\infty e^{-t(1+j2\pi+j\omega)} dt \\
&= \frac{1}{2} \left[\frac{1}{1-j(2\pi-\omega)} + \frac{1}{1+j(2\pi+\omega)} \right]
\end{aligned}$$

(b) $x[n] = \begin{cases} \cos(\frac{\pi}{10}n) + j \sin(\frac{\pi}{10}n), & |n| < 10 \\ 0, & \text{otherwise} \end{cases}$

Discrete and Nonperiodic, use DTFT.

$$\begin{aligned}
X(e^{j\Omega}) &= \sum_{n=-9}^9 x[n] e^{-j(\frac{\pi}{10}-\Omega)n} \\
&= e^{-j9(\frac{\pi}{10}-\Omega)} \frac{1 - e^{-j9(\frac{\pi}{10}-\Omega)}}{1 - e^{-j(\frac{\pi}{10}-\Omega)}} \\
&= \frac{\sin(\frac{19}{2}(\frac{\pi}{10}-\Omega))}{\sin(\frac{1}{2}(\frac{\pi}{10}-\Omega))}
\end{aligned}$$

(c) $x[n]$ as depicted in Figure P3.56 (a)

Discrete and Periodic, use DTFS.

$$N = 7, \Omega_o = \frac{2\pi}{7}$$

choose $n, k \in \{0, \dots, 6\}$

$$X[k] = \frac{1}{7} (1 + e^{-j\frac{6\pi}{7}k} - e^{-j\frac{8\pi}{7}k})$$

(d) $x(t) = e^{1+t}u(-t+2)$

Continuous and Nonperiodic, use FT.

$$\begin{aligned}
X(j\omega) &= e^1 \int_{-\infty}^2 e^{(1-j\omega)t} dt \\
&= \frac{e^{3-j2\omega}}{1-j\omega}
\end{aligned}$$

(e) $x(t) = |\sin(2\pi t)|$

Continuous and Periodic, use FS.

$$T = \frac{1}{2}, \omega_o = 4\pi$$

$$\begin{aligned}
X[k] &= 2 \int_0^{0.5} \frac{e^{j2\pi t} - e^{-j2\pi t}}{2j} e^{-j4\pi kt} dt \\
&= -j \int_0^{0.5} e^{j2\pi(1-2k)t} dt - j \int_0^{0.5} e^{-j2\pi(1+2k)t} dt \\
&= \frac{1 - (-1)^{1-2k}}{2\pi(1-2k)} + \frac{1 - (-1)^{(1+2k)}}{2\pi(1+2k)}
\end{aligned}$$

(f) $x[n]$ as depicted in Figure P3.56 (b)

Discrete and Nonperiodic, use DTFT.

$$X(e^{j\Omega}) = \frac{-j}{2} (\sin(\Omega) + 2\sin(2\Omega) + 3\sin(3\Omega) + 4\sin(4\Omega))$$

(g) $x(t)$ as depicted in Figure P3.56 (c)

Continuous and Periodic, use FS.

$$T = 4, \omega_o = \frac{\pi}{2}$$

$$\begin{aligned} X[k] &= \frac{1}{4} \int_1^2 e^{-j\frac{\pi}{2}kt} dt + \frac{1}{4} \int_2^3 3e^{-j\frac{\pi}{2}kt} dt \\ &= \frac{2(-1)^k + e^{-j\frac{\pi}{2}k} - 3e^{-j\frac{3\pi}{2}k}}{j2\pi k} \\ x[k] &= \begin{cases} \frac{2(-1)^k + e^{-j\frac{\pi}{2}k} - 3e^{-j\frac{3\pi}{2}k}}{j2\pi k} & k \neq 0 \\ 1 & k = 0 \end{cases} \end{aligned}$$

3.57. The following are frequency-domain representations for signals. Determine the time-domain signal corresponding to each.

$$(a) X[k] = \begin{cases} e^{-jk\pi/2}, & |k| < 10 \\ 0, & \text{otherwise} \end{cases} \quad \text{Fundamental period of time domain signal is } T = 1.$$

Discrete and Nonperiodic $\xleftrightarrow{FS; 2\pi}$ Periodic and Continuous, use FS.

$$\begin{aligned} x(t) &= \sum_{k=-9}^9 e^{-j\frac{\pi}{2}k} e^{-j2\pi k t} \\ &= \sum_{k=-9}^9 (e^{-j\pi(2t-0.5)})^k \\ &= \frac{\sin(\frac{19}{2}\pi(2t-0.5))}{\sin(\frac{1}{2}\pi(2t-0.5))} \end{aligned}$$

(b) $X[k]$ as depicted in Figure P3.57 (a)

Discrete and Periodic $\xleftrightarrow{DTFS; \frac{2\pi}{5}}$ Discrete and Periodic, use DTFS.

$$N = 5, \Omega_o = \frac{2\pi}{5} \text{ Choose } n, k \in \{-2, -1, \dots, 2\}$$

$$\begin{aligned} x[n] &= \sum_{k=-2}^2 X[k] e^{-j\frac{2\pi}{5}kn} \\ &= -e^{j\frac{4\pi}{5}n} + e^{-j\frac{4\pi}{5}n} + 1 - e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n} \\ &= 1 - 2j \sin(\frac{2\pi}{5}n) - 2j \sin(\frac{4\pi}{5}n) \end{aligned}$$

$$(c) X(j\omega) = \begin{cases} \cos(\frac{\omega}{4}) + j \sin(\frac{\omega}{4}), & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}$$

Continuous and Nonperiodic \xleftrightarrow{FT} Nonperiodic and Continuous, use FT.

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\frac{\omega}{4}} e^{j\omega t} d\omega \\ &= \frac{\sin(\pi(t+0.25))}{\pi(t+0.25)} \end{aligned}$$

$$x(t) = \begin{cases} \frac{\sin(\pi(t+0.25))}{\pi(t+0.25)} & k \neq -\frac{1}{4} \\ 1 & k = -\frac{1}{4} \end{cases}$$

(d) $X(j\omega)$ as depicted in Figure P3.57 (b)

Continuous and Nonperiodic \xleftrightarrow{FT} Nonperiodic and Continuous, use FT.

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-2}^0 -e^{\omega} e^{j\omega t} d\omega + \frac{1}{2\pi} \int_0^2 e^{-\omega} e^{j\omega t} d\omega \\ &= \frac{e^{-2(1+jt)} - 1}{2\pi(1+jt)} + \frac{e^{2(jt-1)} - 1}{2\pi(jt-1)} \\ &= \frac{e^{-2t}(\cos(2t) + 1)}{j\pi(1+t^2)} \end{aligned}$$

(e) $X(e^{j\Omega})$ as depicted in Figure P3.57 (c)

Continuous and Periodic \xleftrightarrow{DTFT} Periodic and Discrete, use DTFT.

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2\Omega}{\pi} e^{j\Omega n} d\Omega \\ &\quad \text{Use integration by parts.} \\ &= \frac{2 \cos(\pi n)}{j\pi n} \\ x[n] &= \begin{cases} \frac{2 \cos(\pi n)}{j\pi n} & n \neq 0 \\ 0 & n = 0 \end{cases} \end{aligned}$$

(f) $X[k]$ as depicted in Figure P3.57 (d)

Discrete and Nonperiodic $\xleftrightarrow{FS; \pi}$ Periodic and Continuous, use FS, $T = 2$, $\omega_o = \pi$

$$x(t) = 2[\sin(3\pi t) + \sin(4\pi t) + \sin(5\pi t) + \sin(6\pi t) + \sin(7\pi t)]$$

(g) $X(e^{j\Omega}) = |\sin(\Omega)|$

Continuous and Periodic \xleftrightarrow{DTFT} Nonperiodic and Discrete, use DTFT.

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin(\Omega)| e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_{-\pi}^0 -\sin(\Omega) e^{j\Omega n} d\Omega + \frac{1}{2\pi} \int_0^{\pi} \sin(\Omega) e^{j\Omega n} d\Omega \\ &= \frac{\cos(\pi(n-1)) - 1}{2\pi(n-1)} + \frac{\cos(\pi(n+1)) - 1}{2\pi(n+1)} \\ x[n] &= \begin{cases} \frac{\cos(\pi(n-1)) - 1}{2\pi(n-1)} + \frac{\cos(\pi(n+1)) - 1}{2\pi(n+1)} & k \neq 2, -2 \\ 0 & k = 2, -2 \end{cases} \end{aligned}$$

3.58. Use the tables of transforms and properties to find the FT's of the following signals.

(a) $x(t) = \sin(2\pi t)e^{-t}u(t)$

$$\begin{aligned} x(t) &= \sin(2\pi t)e^{-t}u(t) \\ &= \frac{1}{2j}e^{j2\pi t}e^{-t}u(t) - \frac{1}{2j}e^{-j2\pi t}e^{-t}u(t) \end{aligned}$$

$$\begin{aligned} e^{-t}u(t) &\xleftrightarrow{FT} \frac{1}{1+j\omega} \\ e^{j2\pi t}s(t) &\xleftrightarrow{FT} S(j(\omega - 2\pi)) \\ X(j\omega) &= \frac{1}{2j} \left[\frac{1}{1+j(\omega - 2\pi)} - \frac{1}{1+j(\omega + 2\pi)} \right] \end{aligned}$$

(b) $x(t) = te^{-3|t-1|}$

$$\begin{aligned} e^{-3|t|} &\xleftrightarrow{FT} \frac{6}{9+\omega^2} \\ s(t-1) &\xleftrightarrow{FT} e^{-j\omega}S(j\omega) \\ tw(t) &\xleftrightarrow{FT} j\frac{d}{d\omega}W(j\omega) \\ X(j\omega) &= j\frac{d}{d\omega} \left[e^{-j\omega} \frac{6}{9+\omega^2} \right] \\ &= \frac{6e^{-j\omega}}{9+\omega^2} - \frac{12j\omega e^{-j\omega}}{(9+\omega^2)^2} \end{aligned}$$

(c) $x(t) = \left[\frac{2\sin(3\pi t)}{\pi t} \right] \left[\frac{\sin(2\pi t)}{\pi t} \right]$

$$\begin{aligned} \frac{\sin(Wt)}{\pi t} &\xleftrightarrow{FT} \begin{cases} 1 & \omega \leq W \\ 0, & \text{otherwise} \end{cases} \\ s_1(t)s_2(t) &\xleftrightarrow{FT} \frac{1}{2\pi}S_1(j\omega) * S_2(j\omega) \\ X(j\omega) &= \begin{cases} 5 - \frac{|\omega|}{\pi} & \pi < |\omega| \leq 5\pi \\ 4 & |\omega| \leq \pi \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(d)

$$\begin{aligned} x(t) &= \frac{d}{dt}te^{-2t}\sin(t)u(t) \\ &= \frac{d}{dt}te^{-2t}u(t)\frac{e^{jt} - e^{-jt}}{2j} \end{aligned}$$

$$te^{-2t}u(t) \xleftrightarrow{FT} \frac{1}{(2+j\omega)^2}$$

$$\begin{aligned}
e^{jt} s(t) &\xleftrightarrow{FT} S(j(\omega - 1)) \\
\frac{d}{dt} s(t) &\xleftrightarrow{FT} j\omega S(j\omega) \\
X(j\omega) &= j\omega \frac{1}{2j} \left[\frac{1}{(2 + j(\omega - 1))^2} - \frac{1}{(2 + j(\omega + 1))^2} \right]
\end{aligned}$$

(e) $x(t) = \int_{-\infty}^t \frac{\sin(2\pi\tau)}{\pi\tau} d\tau$

$$\begin{aligned}
\frac{\sin(2\pi t)}{\pi t} &\xleftrightarrow{FT} \begin{cases} 1 & \omega \leq 2\pi \\ 0, & \text{otherwise} \end{cases} \\
\int_{-\infty}^t s(\tau) d\tau &\xleftrightarrow{FT} \frac{S(j\omega)}{j\omega} + \pi S(j0)\delta(\omega) \\
X(j\omega) &= \begin{cases} \pi\delta(\omega) & \omega = 0 \\ \frac{1}{j\omega} & |\omega| \leq 2\pi, \omega \neq 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

(f) $x(t) = e^{-t+2}u(t-2)$

$$\begin{aligned}
e^{-t}u(t) &\xleftrightarrow{FT} \frac{1}{1+j\omega} \\
s(t-2) &\xleftrightarrow{FT} e^{-j2\omega}S(j\omega) \\
X(j\omega) &= e^{-j2\omega} \frac{1}{1+j\omega}
\end{aligned}$$

(g) $x(t) = \left(\frac{\sin(t)}{\pi t} \right) * \frac{d}{dt} \left[\left(\frac{\sin(2t)}{\pi t} \right) \right]$

$$\begin{aligned}
x(t) = a(t) * b(t) &\xleftrightarrow{FT} X(j\omega) = A(j\omega)B(j\omega) \\
\frac{\sin(Wt)}{\pi t} &\xleftrightarrow{FT} \begin{cases} 1 & \omega \leq W \\ 0, & \text{otherwise} \end{cases} \\
\frac{d}{dt} s(t) &\xleftrightarrow{FT} j\omega S(j\omega) \\
X(j\omega) &= \begin{cases} j\omega & |\omega| \leq 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

3.59. Use the tables of transforms and properties to find the inverse FT's of the following signals.

(a) $X(j\omega) = \frac{j\omega}{(1+j\omega)^2}$

$$\begin{aligned}
\frac{1}{(1+j\omega)^2} &\xleftrightarrow{FT} te^{-t}u(t) \\
j\omega S(j\omega) &\xleftrightarrow{FT} \frac{d}{dt} s(t)
\end{aligned}$$

$$\begin{aligned}
x(t) &= \frac{d}{dt}[te^{-t}u(t)] \\
&= (1-t)e^{-t}u(t)
\end{aligned}$$

$$(b) X(j\omega) = \frac{4 \sin(2\omega-4)}{2\omega-4} - \frac{4 \sin(2\omega+4)}{2\omega+4}$$

$$\begin{aligned}
\frac{2 \sin(\omega)}{\omega} &\xleftrightarrow{FT} \text{rect}(t) = \begin{cases} 1 & |t| \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
S(j2\omega) &\xleftrightarrow{FT} \frac{1}{2}s\left(\frac{t}{2}\right) \\
S(j(\omega-2)) &\xleftrightarrow{FT} e^{j2t}s(t)
\end{aligned}$$

$$\begin{aligned}
x(t) &= \text{rect}\left(\frac{t}{2}\right)e^{j2t} - \text{rect}\left(\frac{t}{2}\right)e^{-j2t} \\
&= 2j\text{rect}\left(\frac{t}{2}\right)\sin(2t)
\end{aligned}$$

$$(c) X(j\omega) = \frac{1}{j\omega(j\omega+2)} - \pi\delta(\omega)$$

$$\begin{aligned}
\frac{1}{j\omega} + \pi\delta(j\omega) &\xleftrightarrow{FT} u(t) \\
\frac{1}{2+j\omega} &\xleftrightarrow{FT} e^{-2t}u(t) \\
2\pi\delta(\omega) &\xleftrightarrow{FT} 1 \\
X(j\omega) &= -0.5\frac{1}{(j\omega+2)} + 0.5\frac{1}{j\omega} + 0.5\pi\delta(\omega) - 1.5\pi\delta(\omega) \\
X(j\omega) &\xleftrightarrow{FT} x(t) = -0.5e^{-2t}u(t) + 0.5u(t) - \frac{3}{4}
\end{aligned}$$

$$(d) X(j\omega) = \frac{d}{d\omega} \left[4 \sin(4\omega) \frac{\sin(2\omega)}{\omega} \right]$$

$$\begin{aligned}
S(j\omega) = \frac{2 \sin(2\omega)}{\omega} &\xleftrightarrow{FT} s(t) = \text{rect}(2t) = \begin{cases} 1 & |t| \leq 2 \\ 0, & \text{otherwise} \end{cases} \\
S_1(j\omega) = 2 \sin(4\omega)S(j\omega) &\xleftrightarrow{FT} s_1(t) = -js(t+4) + js(t-4) \\
X(j\omega) = \frac{d}{d\omega}S_1(j\omega) &\xleftrightarrow{FT} x(t) = -jts_1(t) \\
x(t) &= -t\text{rect}(2(t+4)) + t\text{rect}(2(t-4))
\end{aligned}$$

$$(e) X(j\omega) = \frac{2 \sin(\omega)}{\omega(j\omega+2)}$$

$$S_1(j\omega) = \frac{2 \sin(\omega)}{\omega} \xleftrightarrow{FT} s_1(t) = \begin{cases} 1 & |t| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
S_2(j\omega) &= \frac{1}{(j\omega + 2)} \xleftrightarrow{FT} s_2(t) = e^{-2t}u(t) \\
x(t) &= s_1(t) * s_2(t)
\end{aligned}$$

$$x(t) = \begin{cases} 0 & t < -1 \\ \frac{1}{2}[1 - e^{-2(t+1)}] & -1 \leq t < 1 \\ \frac{e^{-2t}}{2}[e^2 - e^{-2}] & t \geq 1 \end{cases}$$

$$(f) X(j\omega) = \frac{4 \sin^2(\omega)}{\omega^2}$$

$$\begin{aligned}
S(j\omega) = \frac{2 \sin(\omega)}{\omega} &\xleftrightarrow{FT} s(t) = \begin{cases} 1 & |t| \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
x(t) &= s(t) * s(t)
\end{aligned}$$

$$x(t) = \begin{cases} 2 - |t| & t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

3.60. Use the tables of transforms and properties to find the DTFT's of the following signals.

$$(a) x[n] = \left(\frac{1}{3}\right)^n u[n+2]$$

$$\begin{aligned}
x[n] &= \left(\frac{1}{3}\right)^n u[n+2] \\
&= \left(\frac{1}{3}\right)^{-2} \left(\frac{1}{3}\right)^{n+2} u[n+2] \\
\left(\frac{1}{3}\right)^n u[n] &\xleftrightarrow{DTFT} \frac{1}{1 - \frac{1}{3}e^{-j\Omega}} \\
s[n+2] &\xleftrightarrow{DTFT} e^{j2\Omega} S(e^{j\Omega}) \\
X(e^{j\Omega}) &= \frac{9e^{j2\Omega}}{1 - \frac{1}{3}e^{-j\Omega}}
\end{aligned}$$

$$(b) x[n] = (n-2)(u[n+4] - u[n-5])$$

$$\begin{aligned}
u[n+4] - u[n-5] &\xleftrightarrow{DTFT} \frac{\sin(\frac{9\Omega}{2})}{\sin(\frac{\Omega}{2})} \\
ns[n] &\xleftrightarrow{DTFT} j \frac{d}{d\Omega} S(e^{j\Omega}) \\
x[n] &= j \frac{d}{d\Omega} \frac{\sin(\frac{9\Omega}{2})}{\sin(\frac{\Omega}{2})} - 2 \frac{\sin(\frac{9\Omega}{2})}{\sin(\frac{\Omega}{2})}
\end{aligned}$$

(c) $x[n] = \cos(\frac{\pi}{4}n)(\frac{1}{2})^n u[n-2]$

$$\begin{aligned}
x[n] &= \cos(\frac{\pi}{4}n)(\frac{1}{2})^n u[n-2] \\
&= \cos(\frac{\pi}{4}n)\frac{1}{4}(\frac{1}{2})^{n-2} u[n-2] \\
a[n] = (\frac{1}{2})^n u[n] &\xleftrightarrow{DTFT} A(e^{j\Omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} \\
b[n] = a[n-2] &\xleftrightarrow{DTFT} B(e^{j\Omega}) = e^{-j2\Omega} A(e^{j\Omega}) \\
x[n] = \cos(\frac{\pi}{4}n)a[n-2] &\xleftrightarrow{DTFT} X(e^{j\Omega}) = \frac{1}{2}B(e^{j(\Omega - \frac{\pi}{4})}) + \frac{1}{2}B(e^{j(\Omega + \frac{\pi}{4})}) \\
X(e^{j\Omega}) &= \frac{1}{8} \left[\frac{e^{-j2(\Omega - \frac{\pi}{4})}}{1 - \frac{1}{2}e^{-j(\Omega - \frac{\pi}{4})}} + \frac{e^{-j2(\Omega + \frac{\pi}{4})}}{1 - \frac{1}{2}e^{-j(\Omega + \frac{\pi}{4})}} \right]
\end{aligned}$$

(d) $x[n] = \left[\frac{\sin(\frac{\pi}{4}n)}{\pi n} \right] * \left[\frac{\sin(\frac{\pi}{4}(n-8))}{\pi(n-8)} \right]$

$$\begin{aligned}
s[n] = \frac{\sin(\frac{\pi}{4}n)}{\pi n} &\xleftrightarrow{FT} S(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\Omega| \leq \pi \end{cases} \\
b[n] = s[n-8] &\xleftrightarrow{DTFT} B(e^{j\Omega}) = e^{-j8\Omega} S(e^{j\Omega}) \\
X(e^{j\Omega}) &= B(e^{j\Omega})S(e^{j\Omega}) = \begin{cases} e^{-j8\Omega} & |\Omega| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\Omega| \leq \pi \end{cases}
\end{aligned}$$

(e) $x[n] = \left[\frac{\sin(\frac{\pi}{2}n)}{\pi n} \right]^2 * \frac{\sin(\frac{\pi}{2}n)}{\pi n}$

$$\begin{aligned}
s[n] = \frac{\sin(\frac{\pi}{4}n)}{\pi n} &\xleftrightarrow{DTFT} S(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\Omega| \leq \pi \end{cases} \\
s_2[n] = s[n]s[n] &\xleftrightarrow{DTFT} S_2(e^{j\Omega}) = \frac{1}{2\pi} S(e^{j\Omega}) * S(e^{j\Omega}) = \begin{cases} \frac{1}{2} - \frac{|\Omega|}{2} & |\Omega| \leq \pi, 2\pi \text{ periodic.} \end{cases} \\
X(e^{j\Omega}) &= S_2(e^{j\Omega})S(e^{j\Omega}) = \begin{cases} \frac{1}{2} - \frac{|\Omega|}{2} & |\Omega| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\Omega| \leq \pi \end{cases}
\end{aligned}$$

3.61. Use the tables of transforms and properties to find the inverse DTFT's of the following signals.

(a)

$$\begin{aligned}
X(e^{j\Omega}) &= j \sin(4\Omega) - 2 \\
&= \frac{1}{2}e^{j4\Omega} - \frac{1}{2}e^{-j4\Omega} - 2 \\
x[n] &= \frac{1}{2}\delta[n+4] - \frac{1}{2}\delta[n-4] - 2\delta[n]
\end{aligned}$$

(b) $X(e^{j\Omega}) = \left[e^{-j2\Omega} \frac{\sin(\frac{15}{2}\Omega)}{\sin(\frac{\Omega}{2})} \right] \otimes \left[\frac{\sin(\frac{7}{2}\Omega)}{\sin(\frac{\Omega}{2})} \right]$

Let the first part be $A((e^{j\Omega}))$, and the second be $B(e^{j\Omega})$.

$$\begin{aligned}
a[n] &= \begin{cases} 1 & |n-2| \leq 7 \\ 0, & \text{otherwise} \end{cases} \\
b[n] &= \begin{cases} 1 & |n| \leq 3 \\ 0, & \text{otherwise} \end{cases} \\
X((e^{j\Omega})) &= A((e^{j\Omega})) \otimes B(e^{j\Omega}) \xleftrightarrow{DTFT} x[n] = 2\pi a[n]b[n] \\
x[n] &= \begin{cases} 2\pi & |n| \leq 3 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

$$(c) X(e^{j\Omega}) = \cos(4\Omega) \left[\frac{\sin(\frac{3}{2}\Omega)}{\sin(\frac{\Omega}{2})} \right]$$

$$\begin{aligned}
A(e^{j\Omega}) &= \frac{\sin(\frac{3}{2}\Omega)}{\sin(\frac{\Omega}{2})} \xleftrightarrow{DTFT} a[n] = \begin{cases} 1 & |n| \leq 1 \\ 0, & \text{otherwise} \end{cases} \\
X(e^{j\Omega}) = \cos(4\Omega)A((e^{j\Omega})) &\xleftrightarrow{DTFT} x[n] = \frac{1}{2}a[n+4] + \frac{1}{2}a[n-4] \\
x[n] &= \begin{cases} \frac{1}{2} & |n+4| \leq 1, |n-4| \leq 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

$$(d) X(e^{j\Omega}) = \begin{cases} e^{-j4\Omega} & \frac{\pi}{4} < |\Omega| < \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases}, \text{ for } |\Omega| < \pi$$

$$\begin{aligned}
X(e^{j\Omega}) &= \begin{cases} e^{-j4\Omega} & |\Omega - \frac{\pi}{2}| < \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}, \text{ for } |\Omega| < \pi \\
x[n] &= e^{j\frac{\pi}{2}n} \frac{\sin(\frac{\pi}{4}(n-4))}{\pi(n-4)}
\end{aligned}$$

$$(e) X(e^{j\Omega}) = e^{-j(4\Omega + \frac{\pi}{2})} \frac{d}{d\Omega} \left[\frac{2}{1 + \frac{1}{4}e^{-j(\Omega - \frac{\pi}{4})}} + \frac{2}{1 + \frac{1}{4}e^{-j(\Omega + \frac{\pi}{4})}} \right]$$

$$\begin{aligned}
S_1(e^{j\Omega}) &= \frac{2}{1 + \frac{1}{4}e^{-j\omega}} \xleftrightarrow{DTFT} s_1[n] = 2(-\frac{1}{4})^n u[n] \\
S_1(e^{j(\Omega - \frac{\pi}{4})}) &\xleftrightarrow{DTFT} e^{j\frac{\pi}{4}n} s_1[n] \\
S_1(e^{j(\Omega + \frac{\pi}{4})}) &\xleftrightarrow{DTFT} e^{-j\frac{\pi}{4}n} s_1[n] \\
S(e^{j\Omega}) = S_1(e^{j(\Omega - \frac{\pi}{4})}) + S_1(e^{j(\Omega + \frac{\pi}{4})}) &\xleftrightarrow{DTFT} s[n] = 2 \cos(\frac{\pi}{4}n) s_1[n] \\
-j e^{j4\Omega} \frac{d}{d\Omega} S(e^{j\Omega}) &\xleftrightarrow{DTFT} -(n-4)s[n-4] \\
x[n] &= -4(n-4) \cos(\frac{\pi}{4}(n-4)) (-\frac{1}{4})^{n-4} u[n-4]
\end{aligned}$$

3.62. Use the FT pair $x(t) = \begin{cases} 1 & |t| < 1 \\ 0 & \text{otherwise} \end{cases} \xleftrightarrow{FT} X(j\omega) = \frac{2\sin(\omega)}{\omega}$

and the FT properties to evaluate the frequency domain representations for the signals depicted in Figure P3.62 (a) – (g).

(a) $y(t) = x(\frac{t-2}{2})$

$$Y(j\omega) = e^{-j2\omega} 2X(j2\omega) = e^{-j2\omega} \frac{2\sin(\omega)}{2\omega}$$

(b) $y(t) = \sin(\pi t)x(t)$

$$\begin{aligned} Y(j\omega) &= \frac{1}{2j}X(j(\omega - \pi)) - \frac{1}{2j}X(j(\omega + \pi)) \\ &= \frac{\sin(\omega - \pi)}{j(\omega - \pi)} - \frac{\sin(\omega + \pi)}{j(\omega + \pi)} \end{aligned}$$

(c) $y(t) = x(t+1) - x(2(t - \frac{1}{2}))$

$$\begin{aligned} Y(j\omega) &= e^{j\omega}X(j\omega) - \frac{1}{2}e^{-j\frac{\omega}{2}}X(j\frac{\omega}{2}) \\ &= e^{j\omega} \frac{2\sin(\omega)}{\omega} - e^{-j\frac{\omega}{2}} \frac{2\sin(\frac{\omega}{2})}{\omega} \end{aligned}$$

(d) $y(t) = 2tx(t)$

$$\begin{aligned} -jtx(t) &\xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega) \\ Y(j\omega) &= 2j \frac{d}{d\omega} \frac{2\sin(\omega)}{\omega} \\ Y(j\omega) &= 2j \left(\frac{2\cos(\omega)}{\omega} - \frac{2\sin(\omega)}{\omega^2} \right) \end{aligned}$$

(e) $y(t) = x(t) * x(t)$

$$Y(j\omega) = \frac{4\sin^2(\omega)}{\omega^2}$$

(f) $y(t) = \int_{-\infty}^t x(\tau) d\tau$

$$\begin{aligned} Y(j\omega) &= 2 \frac{\sin(\omega)}{\omega} \frac{1}{j\omega} + \pi(2)\delta(\omega) \\ &= 2 \frac{\sin(\omega)}{j\omega^2} + 2\pi\delta(\omega) \end{aligned}$$

(g) $y(t) = \frac{d}{dt}x(t)$

$$\begin{aligned} Y(j\omega) &= j\omega 2 \frac{\sin(\omega)}{\omega} \\ &= j2 \sin(\omega) \end{aligned}$$

3.63. We have $x[n] = n \left(\frac{3}{4}\right)^{|n|} \xrightarrow{DTFT} X(e^{j\Omega})$. Without evaluating $X(e^{j\Omega})$, find $y[n]$ if $Y(e^{j\Omega})$ is given by:

(a) $Y(e^{j\Omega}) = e^{-j4\Omega} X(e^{j\Omega})$

$$y[n] = x[n-4] = (n-4) \left(\frac{3}{4}\right)^{|n-4|}$$

(b) $Y(e^{j\Omega}) = \text{Re}\{X(e^{j\Omega})\}$

Since $x[n]$ is real and odd,
 $X(e^{j\Omega})$ is purely imaginary, thus $y[n] = 0$

(c) $Y(e^{j\Omega}) = \frac{d}{d\Omega} X(e^{j\Omega})$

$$y[n] = -jnx[n] = -jn^2 \left(\frac{3}{4}\right)^{|n|}$$

(d) $Y(e^{j\Omega}) = X(e^{j\Omega}) \oplus X(e^{j(\Omega-\pi/2)})$

$$\begin{aligned} y[n] &= 2\pi x_1[n]x[n] \\ x_1[n] &= e^{j\frac{\pi}{2}n}x[n] \\ y[n] &= 2\pi n^2 \left(\frac{3}{4}\right)^{2|n|} e^{j\frac{\pi}{2}n} \end{aligned}$$

(e) $Y(e^{j\Omega}) = \frac{d}{d\Omega} X(e^{j2\Omega})$

$$\begin{aligned} X_z(e^{j\Omega}) = X(e^{j2\Omega}) &\xrightarrow{DTFT} x_z[n] = \begin{cases} x[n] & n \text{ even} \\ 0 & \text{otherwise} \end{cases} \\ y[n] &= \begin{cases} -jn^2 \left(\frac{3}{4}\right)^{|n|} & n \text{ even} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(f) $Y(e^{j\Omega}) = X(e^{j\Omega}) + X(e^{-j\Omega})$

$$y[n] = x[n] + x[-n] = n\left(\frac{3}{4}\right)^{|n|} - n\left(\frac{3}{4}\right)^{|-n|} = 0$$

(g) $Y(e^{j\Omega}) = \frac{d}{d\Omega} \{ e^{-j4\Omega} [X(e^{j(\Omega+\frac{\pi}{4})}) + X(e^{j(\Omega-\frac{\pi}{4})})] \}$

$$\begin{aligned} y[n] &= -jn \left[e^{-j\frac{\pi}{4}(n-4)}x[n-4] + e^{j\frac{\pi}{4}(n-4)}x[n-4] \right] \\ &= -jn \left[2 \cos\left(\frac{\pi}{4}(n-4)\right)(n-4)\left(\frac{3}{4}\right)^{|n-4|} \right] \end{aligned}$$

3.64. A periodic signal has FS representation $x(t) \xrightarrow{FS; \pi} X[k] = -k2^{-|k|}$. Without determining $x(t)$, find the FS representation ($Y[k]$ and ω_o) if $y(t)$ is given by:

(a) $y(t) = x(3t)$

$$Y[k] = -k2^{-|k|}, \omega'_o = 3\pi.$$

(b) $y(t) = \frac{d}{dt}x(t)$

$$Y[k] = jk\omega_o X[k] = jk^2\pi 2^{-|k|}, \omega'_o = \pi.$$

(c) $y(t) = x(t-1)$

$$Y[k] = e^{-jk\pi} X[k] = e^{-jk\pi} k2^{-|k|}, \omega'_o = \pi.$$

(d) $y(t) = \text{Re}\{x(t)\} \xrightarrow{FS; \pi} Y[k]$ conjugate symmetric

$$\begin{aligned} \text{Re}\{Y[k]\} &= \text{even}\{\text{Re}\{X[k]\}\} \\ \text{Re}\{Y[k]\} &= \frac{X[k] + X[-k]}{2} \\ &= \frac{-k2^{-|k|} + k2^{-|k|}}{2} \\ &= 0 \\ \text{Im}\{Y[k]\} &= \text{odd}\{\text{Im}\{X[k]\}\} \\ &= 0 \end{aligned}$$

(e) $y(t) = \cos(4\pi t)x(t)$, $\omega_o = \pi$

$$\begin{aligned} y(t) &= \frac{1}{2}e^{j4\pi t}x(t) + \frac{1}{2}e^{-j4\pi t}x(t) \\ Y[k] &= \frac{1}{2}X[k-4] + \frac{1}{2}X[k+4] \\ &= -\frac{1}{2}(k-4)2^{-|k-4|} - \frac{1}{2}(k+4)2^{-|k+4|} \end{aligned}$$

(f) $y(t) = x(t) \otimes x(t-1)$

$$\begin{aligned} T &= 2, & \omega_o &= \pi \\ x(t) \otimes z(t) &\xleftrightarrow{FS; \pi} TX[k]Z[k] \\ x(t-1) &\xleftrightarrow{FS; \pi} e^{-jk\pi}X[k] \\ Y[k] &= 2e^{-jk\pi}(-k2^{-|k|}) \end{aligned}$$

3.65. Given $x[n] = \frac{\sin(\frac{11\pi}{20}n)}{\sin(\frac{\pi}{20}n)} \xleftrightarrow{DTFS; \frac{\pi}{10}} X[k]$, evaluate the time signal $y[n]$ with the following DTFS coefficients using only DTFS properties.

(a) $Y[k] = X[k-5] + X[k+5] \xleftrightarrow{DTFS; \frac{\pi}{10}} [e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}]x[n]$

$$y[n] = 2 \cos\left(\frac{\pi}{2}n\right) \frac{\sin(\frac{11\pi}{20}n)}{\sin(\frac{\pi}{20}n)}$$

(b) $Y[k] = \cos\left(k\frac{\pi}{5}\right) X[k] \xleftrightarrow{DTFS; \frac{\pi}{10}} \frac{1}{2}[x[n-2] + x[n+2]]$

$$y[n] = \frac{1}{2} \left[\frac{\sin(\frac{11\pi}{20}(n-2))}{\sin(\frac{\pi}{20}(n-2))} + \frac{\sin(\frac{11\pi}{20}(n+2))}{\sin(\frac{\pi}{20}(n+2))} \right]$$

(c) $Y[k] = X[k] \otimes X[k] \xleftrightarrow{DTFS; \frac{\pi}{10}} (x[n])^2$

$$y[n] = \frac{\sin^2(\frac{11\pi}{20}n)}{\sin^2(\frac{\pi}{20}n)}$$

(d) $Y[k] = \text{Re}\{X[k]\} \xleftrightarrow{DTFS; \frac{\pi}{10}} \text{even}(x[n])$

$$\begin{aligned} y[n] &= \frac{x[n] + x[-n]}{2} \\ &= \frac{\sin(\frac{11\pi}{20}n)}{\sin(\frac{\pi}{20}n)} \end{aligned}$$

3.66. Sketch the frequency response of the systems described by the following impulse responses. Characterize each system as lowpass, bandpass, or highpass.

(a) $h(t) = \delta(t) - 2e^{-2t}u(t)$

$$\begin{aligned} Y(j\omega) &= 1 - \frac{2}{2+j\omega} \\ &= \frac{j\omega}{2+j\omega} \end{aligned}$$

Highpass filter

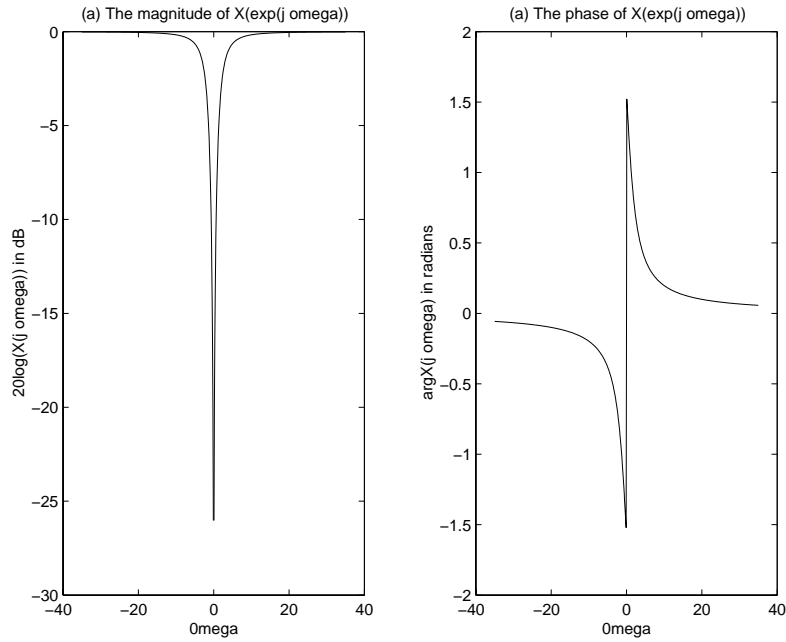


Figure P3.66. (a) Graph of the magnitude and phase

(b) $h(t) = 4e^{-2t} \cos(50t)$

$$\begin{aligned}
 a(t) &= e^{-2t} u(t) \xrightarrow{FT} A(j\omega) = \frac{j\omega}{2 + j\omega} \\
 h(t) &= 4 \cos(50t) a(t) \xrightarrow{FT} H(j\omega) = 2A(j(\omega - 50)) + 2A(j(\omega + 50)) \\
 H(j\omega) &= \frac{2}{2 + j(\omega - 50)} + \frac{2}{2 + j(\omega + 50)}
 \end{aligned}$$

Bandpass filter with maximum gain at $\omega = 50, -50$.

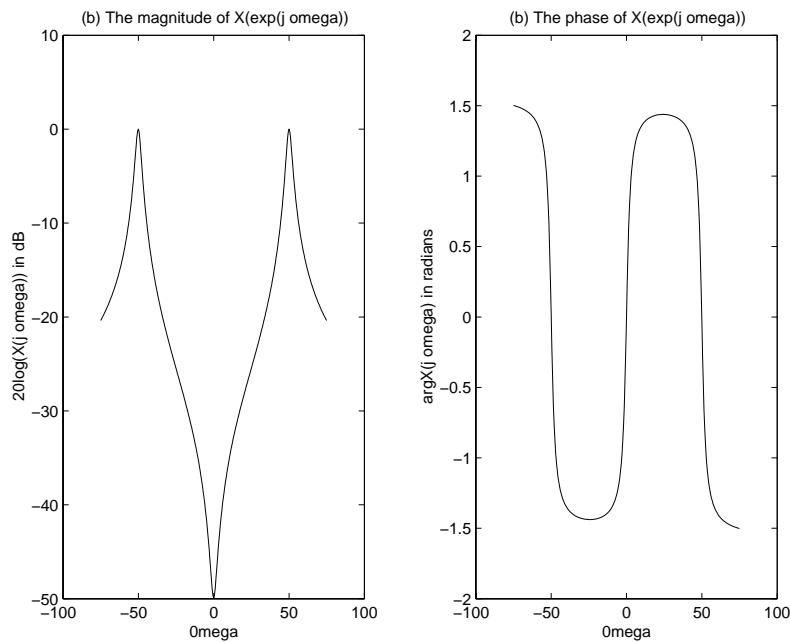


Figure P3.66. (b) Graph of the magnitude and phase

(c) $h[n] = \frac{1}{8} \left(\frac{7}{8}\right)^n u[n]$

$$H(e^{j\Omega}) = \frac{\frac{1}{8}}{1 - \frac{7}{8}e^{-j\Omega}}$$

Low Pass filter.

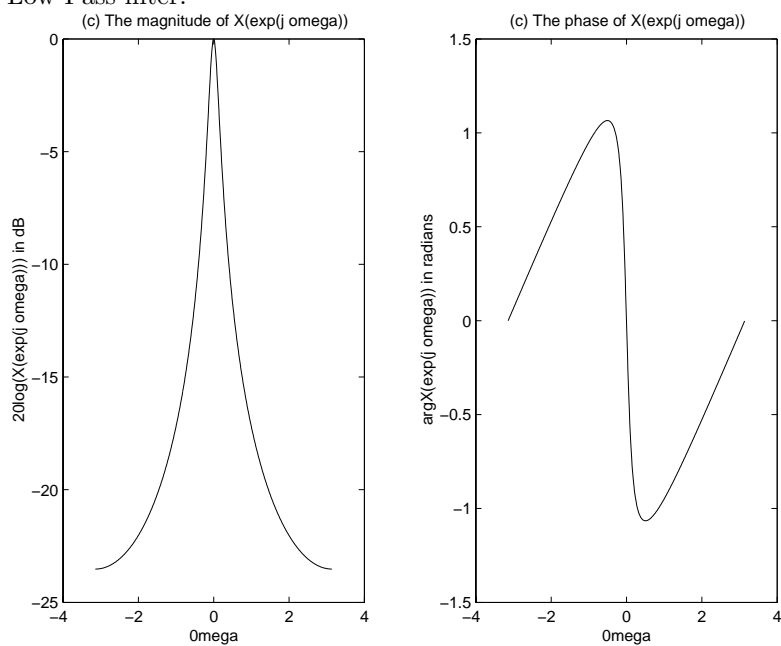


Figure P3.66. (c) Graph of the magnitude and phase

$$(d) \ h[n] = \begin{cases} (-1)^n & |n| \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} e^{j\pi n} & |n| \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

$$H(e^{j\Omega}) = \frac{\sin(\frac{21}{2}(\Omega - \pi))}{\sin(\frac{1}{2}(\Omega - \pi))}$$

High pass filter.

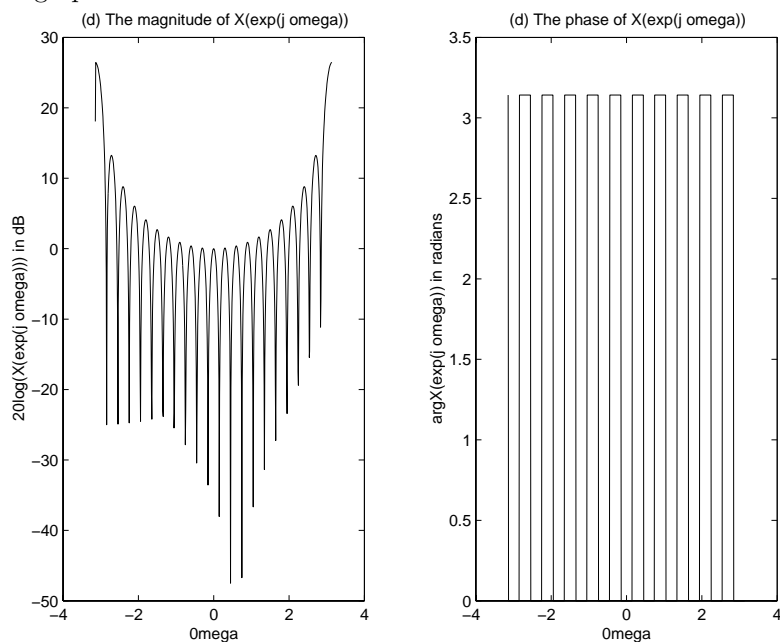


Figure P3.66. (d) Graph of the magnitude and phase

3.67. Find the frequency response and the impulse response of the systems having the output $y(t)$ for the input $x(t)$.

(a) $x(t) = e^{-t}u(t)$, $y(t) = e^{-2t}u(t) + e^{-3t}u(t)$

$$X(j\omega) = \frac{1}{1 + j\omega}$$

$$Y(j\omega) = \frac{1}{2 + j\omega} + \frac{1}{3 + j\omega}$$

$$= \frac{5 + 2j\omega}{(2 + j\omega)(3 + j\omega)}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$= \frac{5 + 7j\omega + 2(j\omega)^2}{(2 + j\omega)(3 + j\omega)}$$

$$\begin{aligned}
&= 2 - \frac{1}{2+j\omega} - \frac{2}{3+j\omega} \\
h(t) &= 2\delta(t) - (e^{-2t} + 2e^{-3t})u(t)
\end{aligned}$$

(b) $x(t) = e^{-3t}u(t)$, $y(t) = e^{-3(t-2)}u(t-2)$

$$\begin{aligned}
X(j\omega) &= \frac{1}{3+j\omega} \\
Y(j\omega) &= e^{-j2\omega} \frac{1}{3+j\omega}
\end{aligned}$$

$$\begin{aligned}
H(j\omega) &= e^{-j2\omega} \\
h(t) &= \delta(t-2)
\end{aligned}$$

(c) $x(t) = e^{-2t}u(t)$, $y(t) = 2te^{-2t}u(t)$

$$\begin{aligned}
X(j\omega) &= \frac{1}{2+j\omega} \\
Y(j\omega) &= \frac{2}{(2+j\omega)^2}
\end{aligned}$$

$$\begin{aligned}
H(j\omega) &= \frac{2}{(2+j\omega)} \\
h(t) &= 2e^{-2t}u(t)
\end{aligned}$$

(d) $x[n] = \left(\frac{1}{2}\right)^n u[n]$, $y[n] = \frac{1}{4} \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n]$

$$\begin{aligned}
X(e^{j\Omega}) &= \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} \\
Y(e^{j\Omega}) &= \frac{1}{4} \left(\frac{1}{1 - \frac{1}{2}e^{-j\Omega}} \right) + \frac{1}{1 - \frac{1}{4}e^{-j\Omega}}
\end{aligned}$$

$$\begin{aligned}
H(e^{j\Omega}) &= \frac{1}{4} + \frac{1 - \frac{1}{2}e^{-j\Omega}}{1 - \frac{1}{4}e^{-j\Omega}} \\
h[n] &= \frac{1}{4}\delta[n] + \left(\frac{1}{4}\right)^n u[n] - \frac{1}{2}\left(\frac{1}{4}\right)^{n-1} u[n-1]
\end{aligned}$$

(e) $x[n] = \left(\frac{1}{4}\right)^n u[n]$, $y[n] = \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{4}\right)^{n-1} u[n-1]$

$$X(e^{j\Omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\Omega}}$$

$$Y(e^{j\Omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\Omega}} - \frac{e^{-j\Omega}}{1 - \frac{1}{4}e^{-j\Omega}}$$

$$\begin{aligned} H(e^{j\Omega}) &= 1 - e^{-j\Omega} \\ h[n] &= \delta[n] - \delta[n-1] \end{aligned}$$

3.68. Determine the frequency response and the impulse response for the systems described by the following differential and difference equations.

(a) $\frac{d}{dt}y(t) + 3y(t) = x(t)$

$$\begin{aligned} j\omega Y(j\omega) + 3Y(j\omega) &= X(j\omega) \\ H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} \\ &= \frac{1}{j\omega + 3} \\ h(t) &= e^{-3t}u(t) \end{aligned}$$

(b) $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = -\frac{d}{dt}x(t)$

$$\begin{aligned} (j\omega)^2 Y(j\omega) + 5j\omega Y(j\omega) + 6Y(j\omega) &= -j\omega X(j\omega) \\ H(j\omega) &= \frac{-j\omega}{(j\omega)^2 + 5j\omega + 6} \\ &= -\frac{3}{3 + j\omega} + \frac{2}{2 + j\omega} \\ h(t) &= (-3e^{-3t} + 2e^{-2t})u(t) \end{aligned}$$

(c) $y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 3x[n] - \frac{3}{4}x[n-1]$

$$\begin{aligned} (1 - \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega})Y(e^{j\Omega}) &= (3 - \frac{3}{4}e^{-j\Omega})X(e^{j\Omega}) \\ H(e^{j\Omega}) &= \frac{3 - \frac{3}{4}e^{-j\Omega}}{1 - \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}} \\ &= \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{2}{1 + \frac{1}{4}e^{-j\Omega}} \\ h[n] &= \left[\left(\frac{1}{2}\right)^n + 2\left(-\frac{1}{4}\right)^n \right] u[n] \end{aligned}$$

(d) $y[n] + \frac{1}{2}y[n-1] = x[n] - 2x[n-1]$

$$(1 + \frac{1}{2}e^{-j\Omega})Y(e^{j\Omega}) = (1 - 2e^{-j\Omega})X(e^{j\Omega})$$

$$\begin{aligned}
H(e^{j\Omega}) &= \frac{1 - 2e^{-j\Omega}}{1 + \frac{1}{2}e^{-j\Omega}} \\
h[n] &= \left(-\frac{1}{2}\right)^n u[n] - 2\left(-\frac{1}{2}\right)^{n-1} u[n-1]
\end{aligned}$$

3.69. Determine the differential or difference equation descriptions for the systems with the following impulse responses.

(a) $h(t) = \frac{1}{a}e^{-\frac{t}{a}}u(t)$

$$\begin{aligned}
H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} \\
&= \frac{1}{a} \frac{1}{\frac{1}{a} + j\omega} \\
\left(\frac{1}{a} + j\omega\right)Y(j\omega) &= \frac{1}{a}X(j\omega) \\
\left(\frac{1}{a} + j\omega\right)Y(j\omega) &= \frac{1}{a}X(j\omega) \\
\frac{1}{a}y(t) + \frac{d}{dt}y(t) &= \frac{1}{a}x(t)
\end{aligned}$$

(b) $h(t) = 2e^{-2t}u(t) - 2te^{-2t}u(t)$

$$\begin{aligned}
H(j\omega) &= \frac{2}{2 + j\omega} - \frac{2}{(2 + j\omega)^2} \\
&= \frac{2 + j2\omega}{(2 + j\omega)^2} \\
(4 + 4j\omega + (j\omega)^2)Y(j\omega) &= (2 + j2\omega)X(j\omega) \\
4y(t) + 4\frac{d}{dt}y(t) + \frac{d^2}{dt^2}y(t) &= 2x(t) + 2\frac{d}{dt}x(t)
\end{aligned}$$

(c) $h[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$\begin{aligned}
H(e^{j\Omega}) &= \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} \\
&= \frac{1}{1 - \alpha e^{-j\Omega}} \\
Y(e^{j\Omega})(1 - \alpha e^{-j\Omega}) &= X(e^{j\Omega}) \\
y[n] - \alpha y[n-1] &= x[n]
\end{aligned}$$

(d) $h[n] = \delta[n] + 2\left(\frac{1}{2}\right)^n u[n] + \left(\frac{-1}{2}\right)^n u[n]$

$$H(e^{j\Omega}) = 1 + \frac{2}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{1}{1 + \frac{1}{2}e^{-j\Omega}}$$

$$\begin{aligned}
&= \frac{4 + \frac{1}{2}e^{-j\Omega} - \frac{1}{4}e^{-j2\Omega}}{1 - \frac{1}{4}e^{-j2\Omega}} \\
Y(e^{j\Omega})(1 - \frac{1}{4}e^{-j2\Omega}) &= (4 + \frac{1}{2}e^{-j\Omega} - \frac{1}{4}e^{-j2\Omega})X(e^{j\Omega}) \\
y[n] - \frac{1}{4}y[n-2] &= 4x[n] + \frac{1}{2}x[n-1] - \frac{1}{4}x[n-2]
\end{aligned}$$

3.70. Determine the differential or difference equation descriptions for the systems with the following frequency responses.

(a) $H(j\omega) = \frac{2+3j\omega-3(j\omega)^2}{1+2j\omega}$

$$\begin{aligned}
H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} \\
&= \frac{2 + 3j\omega - 3(j\omega)^2}{1 + 2j\omega} \\
(1 + 2j\omega)Y(j\omega) &= (2 + 3j\omega - 3(j\omega)^2)X(j\omega) \\
y(t) + 2\frac{d}{dt}y(t) &= 2x(t) + 3\frac{d}{dt}x(t) - 3\frac{d^2}{dt^2}x(t)
\end{aligned}$$

(b) $H(j\omega) = \frac{1-j\omega}{-\omega^2-4}$

$$\begin{aligned}
H(j\omega) &= \frac{1 - j\omega}{(j\omega)^2 - 4} \\
\frac{d^2}{dt^2}y(t) - 4y(t) &= x(t) - \frac{d}{dt}x(t)
\end{aligned}$$

(c) $H(j\omega) = \frac{1+j\omega}{(j\omega+2)(j\omega+1)}$

$$\begin{aligned}
H(j\omega) &= \frac{1 + j\omega}{(j\omega)^2 + 3j\omega + 2} \\
\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) &= x(t) + \frac{d}{dt}x(t)
\end{aligned}$$

(d) $H(e^{j\Omega}) = \frac{1+e^{-j\Omega}}{e^{-j2\Omega}+3}$

$$\begin{aligned}
H(e^{j\Omega}) &= \frac{1 + e^{-j\Omega}}{e^{-j2\Omega} + 3} \\
3y[n] + y[n-2] &= x[n] + x[n-1]
\end{aligned}$$

(e) $H(e^{j\Omega}) = 1 + \frac{e^{-j\Omega}}{(1 - \frac{1}{2}e^{-j\Omega})(1 + \frac{1}{4}e^{-j\Omega})}$

$$\begin{aligned}
 H(e^{j\Omega}) &= 1 + \frac{e^{-j\Omega}}{(1 - \frac{1}{2}e^{-j\Omega})(1 + \frac{1}{4}e^{-j\Omega})} \\
 &= \frac{1 + \frac{3}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}}{1 - \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}} \\
 y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] &= x[n] + \frac{3}{4}x[n-1] - \frac{1}{8}x[n-2]
 \end{aligned}$$

3.71. Consider the RL circuit depicted in Fig. P3.71.

(a) Let the output be the voltage across the inductor, $y_L(t)$. Write a differential equation description for this system and find the frequency response. Characterize this system as a filter.

$$\begin{aligned}
 x(t) &= y_R(t) + y_L(t) \\
 &= i(t)R + y_L(t) \\
 &= \frac{1}{L} \int_{-\infty}^t y_L(\tau) d\tau + y_L(t) \\
 L \frac{d}{dt} x(t) &= y_L(t) + L \frac{d}{dt} y_L(t) \\
 H(j\omega) &= \frac{j\omega L}{1 + j\omega L}
 \end{aligned}$$

High pass filter.

(b) Determine and plot the voltage across the inductor if the input is the square wave depicted in Fig. 3.21 with $T = 1$ and $T_o = 1/4$.

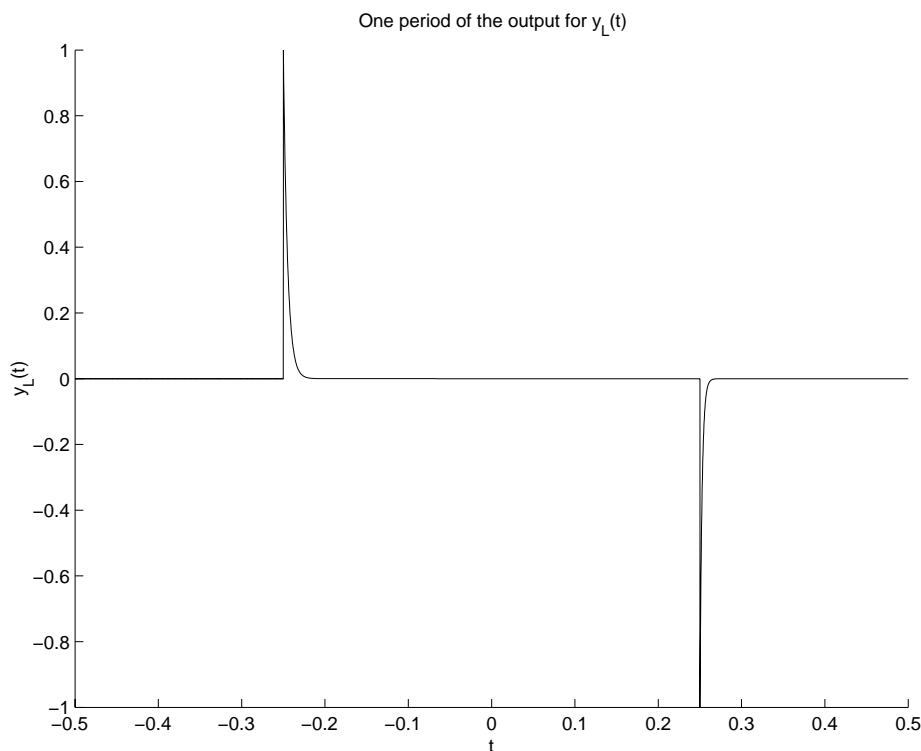


Figure P3.71. Plot 1 of 2

(c) Let the output be the voltage across the resistor, $y_R(t)$. Write a differential equation description for this system and find the frequency response. Characterize this system as a filter.

$$\begin{aligned}
 x(t) &= y_R(t) + y_L(t) \\
 &= y_R(t) + L \frac{d}{dt} y_R(t) \\
 H(j\omega) &= \frac{1}{1 + j\omega L}
 \end{aligned}$$

Low pass filter

(d) Determine and plot the voltage across the resistor if the input is the square wave depicted in Fig. 3.21 with $T = 1$ and $T_o = 1/4$.

$$\begin{aligned}
 h(t) &= \frac{1}{L} e^{-\frac{1}{L}t} \\
 v(t) = u(t) * h(t) &= \frac{1}{L} e^{-\frac{1}{L}t} \left(\int_0^t e^{\frac{\tau}{L}} d\tau \right) u(t) \\
 &= e^{-\frac{1}{L}t} \left(e^{\frac{t}{L}} - 1 \right) u(t) \\
 &= (1 - e^{-\frac{t}{L}}) u(t) \\
 y_R(t) &= v(t + 0.25) - v(t + 0.25)
 \end{aligned}$$

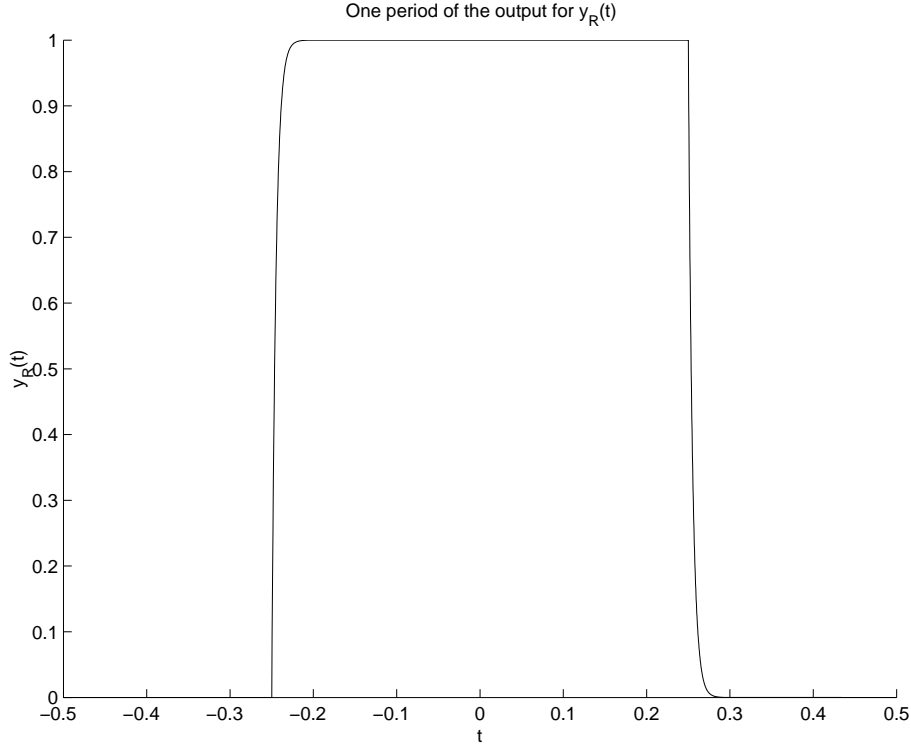


Figure P3.71. Plot 2 of 2

3.72. Consider the *RLC* circuit depicted in Fig. P3.72 with input $x(t)$ and output $y(t)$.

(a) Write a differential equation description for this system and find the frequency response. Characterize this system as a filter.

$$\begin{aligned}
 x(t) &= y(t) + L \frac{d}{dt} y(t) + \frac{1}{C} \int_{-\infty}^t y(\tau) d\tau \\
 \frac{d}{dt} x(t) &= \frac{d}{dt} y(t) + L \frac{d^2}{dt^2} y(t) + \frac{1}{C} y(t) \\
 H(j\omega) &= \frac{j\omega}{\frac{1}{C} + j\omega + L(j\omega)^2}
 \end{aligned}$$

Bandpass filter centered at $\omega_c = \sqrt{\frac{10^4}{L}}$.

(b) Determine and plot the output if the input is the square wave depicted in Fig. 3.21 with $T = 2\pi * 10^{-3}$ and $T_o = \frac{\pi}{2} * 10^{-3}$.

Assume $L = 10\text{mH}$.

This filter picks off the first harmonic of the signal, thus ignoring the smaller terms, $y(t) \approx \frac{2}{\pi} \sin(1000t)$.

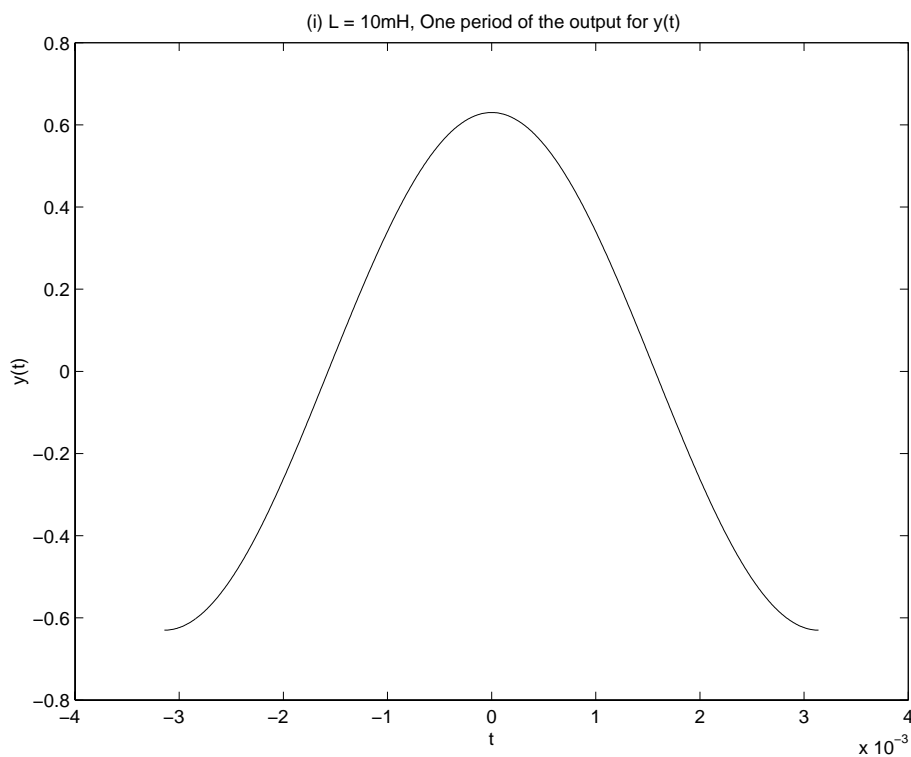


Figure P3.72. Plot 1 of 1

3.73. Use partial fraction expansions to determine the inverse FT for the following signals.

(a)

$$\begin{aligned}
 X(j\omega) &= \frac{6j\omega + 16}{(j\omega)^2 + 5j\omega + 6} \\
 &= \frac{A}{3 + j\omega} + \frac{B}{2 + j\omega} \\
 6 &= A + B \\
 16 &= 2A + 3B \\
 X(j\omega) &= \frac{2}{3 + j\omega} + \frac{4}{2 + j\omega} \\
 x(t) &= (2e^{-3t} + 4e^{-2t})u(t)
 \end{aligned}$$

(b)

$$\begin{aligned}
 X(j\omega) &= \frac{j\omega - 2}{-\omega^2 + 5j\omega + 4} \\
 &= \frac{A}{4 + j\omega} + \frac{B}{1 + j\omega} \\
 1 &= A + B \\
 -2 &= A + 4B \\
 X(j\omega) &= \frac{2}{4 + j\omega} - \frac{1}{1 + j\omega} \\
 x(t) &= (2e^{-4t} - e^{-t})u(t)
 \end{aligned}$$

(c)

$$\begin{aligned}X(j\omega) &= \frac{j\omega}{(j\omega)^2 + 6j\omega + 8} \\&= \frac{A}{4 + j\omega} + \frac{B}{2 + j\omega} \\1 &= A + B \\0 &= 2A + 4B \\X(j\omega) &= \frac{2}{4 + j\omega} - \frac{1}{2 + j\omega} \\x(t) &= (2e^{-4t} - e^{-2t})u(t)\end{aligned}$$

(d)

$$\begin{aligned}X(j\omega) &= \frac{-(j\omega)^2 - 4j\omega - 6}{((j\omega)^2 + 3j\omega + 2)(j\omega + 4)} \\&= \frac{A}{2 + j\omega} + \frac{B}{1 + j\omega} + \frac{C}{4 + j\omega} \\-1 &= A + B + C \\-4 &= 5A + 6B + 3C \\-6 &= 4A + 8B + 2C \\X(j\omega) &= \frac{1}{2 + j\omega} - \frac{1}{1 + j\omega} - \frac{1}{4 + j\omega} \\x(t) &= (e^{-2t} - e^{-t} - e^{-4t})u(t)\end{aligned}$$

(e)

$$\begin{aligned}X(j\omega) &= \frac{2(j\omega)^2 + 12j\omega + 14}{(j\omega)^2 + 6j\omega + 5} \\&= 2 + \frac{4}{(5 + j\omega)(1 + j\omega)} \\&= 2 + \frac{A}{5 + j\omega} + \frac{B}{1 + j\omega} \\0 &= A + B \\4 &= A + 5B \\X(j\omega) &= 2 - \frac{1}{5 + j\omega} + \frac{1}{1 + j\omega} \\x(t) &= 2\delta(t) + (e^{-t} - e^{-5t})u(t)\end{aligned}$$

(f)

$$\begin{aligned}X(j\omega) &= \frac{j\omega + 3}{(j\omega + 1)^2} \\&= \frac{A}{1 + j\omega} + \frac{B}{(1 + j\omega)^2} \\1 &= A\end{aligned}$$

$$\begin{aligned}
3 &= A + B \\
X(j\omega) &= \frac{1}{1 + j\omega} + \frac{2}{(1 + j\omega)^2} \\
x(t) &= (e^{-t} + 2te^{-t})u(t)
\end{aligned}$$

3.74. Use partial fraction expansions to determine the inverse DTFT for the following signals.

(a)

$$\begin{aligned}
X(e^{j\Omega}) &= \frac{2e^{-j\Omega}}{1 - \frac{1}{4}e^{-j2\Omega}} \\
&= \frac{A}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{B}{1 + \frac{1}{2}e^{-j\Omega}} \\
2 &= \frac{1}{2}A - \frac{1}{2}B \\
0 &= A + B \\
X(e^{j\Omega}) &= \frac{2}{1 - \frac{1}{2}e^{-j\Omega}} - \frac{2}{1 + \frac{1}{2}e^{-j\Omega}} \\
x[n] &= \left(2\left(\frac{1}{2}\right)^n - 2\left(-\frac{1}{2}\right)^n \right) u[n]
\end{aligned}$$

(b)

$$\begin{aligned}
X(e^{j\Omega}) &= \frac{2 + \frac{1}{4}e^{-j\Omega}}{-\frac{1}{8}e^{-j2\Omega} + \frac{1}{4}e^{-j\Omega} + 1} \\
&= \frac{A}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{B}{1 - \frac{1}{4}e^{-j\Omega}} \\
\frac{1}{4} &= -\frac{1}{4}A + \frac{1}{2}B \\
2 &= A + B \\
X(e^{j\Omega}) &= \frac{1}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{1}{1 - \frac{1}{4}e^{-j\Omega}} \\
x[n] &= \left(\left(-\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n \right) u[n]
\end{aligned}$$

(c)

$$\begin{aligned}
X(e^{j\Omega}) &= \frac{2}{-e^{-j2\Omega} + e^{-j\Omega} + 6} \\
&= \frac{A}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{B}{1 - \frac{1}{3}e^{-j\Omega}} \\
0 &= -\frac{1}{3}A + \frac{1}{2}B \\
\frac{1}{3} &= A + B \\
X(e^{j\Omega}) &= \frac{\frac{1}{5}}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{\frac{2}{15}}{1 - \frac{1}{3}e^{-j\Omega}} \\
x[n] &= \left(\frac{1}{5}\left(-\frac{1}{2}\right)^n + \frac{2}{15}\left(\frac{1}{3}\right)^n \right) u[n]
\end{aligned}$$

(d)

$$\begin{aligned}
X(e^{j\Omega}) &= \frac{6 - 2e^{-j\Omega} + \frac{1}{2}e^{-j2\Omega}}{(-\frac{1}{4}e^{-j2\Omega} + 1)(1 - \frac{1}{4}e^{-j\Omega})} \\
&= \frac{A}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{B}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{C}{1 - \frac{1}{4}e^{-j\Omega}} \\
6 &= A + B + C \\
-2 &= -\frac{3}{4}A + \frac{1}{4}B \\
\frac{1}{2} &= \frac{1}{8}A - \frac{1}{8}B - \frac{1}{4}C \\
X(e^{j\Omega}) &= \frac{4}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{4}{1 - \frac{1}{2}e^{-j\Omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\Omega}} \\
x[n] &= \left(4\left(-\frac{1}{2}\right)^n + 4\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{4}\right)^n\right) u[n]
\end{aligned}$$

(e)

$$\begin{aligned}
X(e^{j\Omega}) &= \frac{6 - \frac{2}{3}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}}{-\frac{1}{6}e^{-j2\Omega} + \frac{1}{6}e^{-j\Omega} + 1} \\
&= 1 + \frac{A}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{B}{1 - \frac{1}{3}e^{-j\Omega}} \\
5 &= A + B \\
-\frac{1}{2} &= -\frac{1}{3}A + \frac{1}{2}B \\
X(e^{j\Omega}) &= 1 + \frac{4}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{1}{1 - \frac{1}{3}e^{-j\Omega}} \\
x[n] &= \left(4\left(-\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right) u[n] + \delta[n]
\end{aligned}$$

3.75. Evaluate the following quantities.

(a)

$$\begin{aligned}
\frac{2}{1 - \frac{1}{3}e^{-j\Omega}} &\xleftrightarrow{DTFT} 2\left(\frac{1}{3}\right)u[n] \\
\int_{-\pi}^{\pi} \left|\frac{2}{1 - \frac{1}{3}e^{-j\Omega}}\right|^2 d\Omega &= 2\pi \sum_{n=-\infty}^{\infty} \left|2\left(\frac{1}{3}\right)^n u[n]\right|^2 \\
&= 8\pi \sum_{n=0}^{\infty} \left|\left(\frac{1}{3}\right)^n\right| \\
&= \frac{8\pi}{1 - \frac{1}{9}} \\
&= 9\pi
\end{aligned}$$

(b)

$$X[k] = \frac{\sin(k\frac{\pi}{8})}{\pi k} \xleftrightarrow{FS; \pi} x(t) = \begin{cases} 1 & |t| \leq \frac{\pi}{8\omega_o} \\ 0, & \frac{\pi}{8\omega_o} < |t| \leq \frac{2\pi}{\omega_o} \end{cases}$$

$$\begin{aligned}
\pi^2 \sum_{k=-\infty}^{\infty} \frac{\sin^2(k\pi/8)}{\pi^2 k^2} &= \frac{\pi^2}{T} \int_{-0.5T}^{0.5T} |x(t)|^2 dt \\
&= \frac{\pi\omega_o}{2} \int_{-\frac{\pi}{8\omega_o}}^{\frac{\pi}{8\omega_o}} |1|^2 dt \\
&= \frac{\omega_o 2\pi^2}{2(8)\omega_o} \\
&= \frac{\pi^2}{8}
\end{aligned}$$

(c)

$$\begin{aligned}
X(j\omega) = \frac{2(2)}{\omega^2 + 2^2} &\xleftrightarrow{FT} x(t) = e^{-2|t|} \\
\frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{4}{\omega^2 + 2^2} \right)^2 d\omega &= \pi \int_{-\infty}^{\infty} |x(t)|^2 dt \\
&= 2\pi \int_0^{\infty} e^{-4t} dt \\
&= \frac{\pi}{2}
\end{aligned}$$

(d)

$$\begin{aligned}
x(t) = \frac{\sin(\pi t)}{\pi t} &\xleftrightarrow{FT} X(j\omega) = \begin{cases} 1 & |\omega| \leq \pi \\ 0, & \text{otherwise} \end{cases} \\
\pi \int_{-\infty}^{\infty} \left(\frac{\sin(\pi t)}{\pi t} \right)^2 dt &= \frac{1}{2} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \\
&= \frac{1}{2} \int_{-\pi}^{\pi} 1 d\omega \\
&= \pi
\end{aligned}$$

3.76. Use the duality property to evaluate

(a)

$$\begin{aligned}
x(t) &\xleftrightarrow{FT} e^{-2\omega} u(j\omega) \\
e^{2t} u(-t) &\xleftrightarrow{FT} \frac{1}{2 - jt} \\
\text{thus:} & \\
\frac{1}{2 - jt} &\xleftrightarrow{FT} 2\pi e^{-2\omega} u(\omega) \\
x(t) &= \frac{1}{2\pi} \frac{1}{2 - jt}
\end{aligned}$$

(b)

$$X(j\omega) \xleftrightarrow{FT} \frac{1}{(2 + jt)^2}$$

$$\begin{aligned}
te^{-2t}u(t) &\xleftrightarrow{FT} \frac{1}{(2+j\omega)^2} \\
\text{thus:} \\
\frac{1}{(2+jt)^2} &\xleftrightarrow{FT} -2\pi\omega e^{2\omega}u(-\omega) \\
X(j\omega) &= -2\pi\omega e^{2\omega}u(-\omega)
\end{aligned}$$

(c)

$$\begin{aligned}
x[n] = \frac{\sin(\frac{11\pi}{20}n)}{\sin(\frac{\pi}{20}n)} &\xleftrightarrow{DTFS; \frac{\pi}{10}} X[k] \\
\omega_o = \frac{\pi}{10} &\text{ implies } N = 20 \\
\begin{cases} 1 & |n| \leq 5 \\ 0, & 5 < |n| \leq 10 \end{cases} &\xleftrightarrow{DTFS; \frac{\pi}{10}} \frac{\sin(\frac{11\pi}{20}k)}{\sin(\frac{\pi}{20}k)} \\
&\text{ implies } \\
\frac{\sin(\frac{11\pi}{20}k)}{\sin(\frac{\pi}{20}k)} &\xleftrightarrow{DTFS; \frac{\pi}{10}} \frac{1}{20} \begin{cases} 1 & |k| \leq 5 \\ 0, & 5 < |k| \leq 10 \end{cases} \\
X[k] &= \begin{cases} \frac{1}{20} & |k| \leq 5 \\ 0, & 5 < |k| \leq 10 \end{cases} \\
X[k + iN] = X[k + i20] &= X[k] \text{ where } k, i \text{ are integers.}
\end{aligned}$$

3.77. For the FT $X(j\omega)$ shown in Figure P3.77, evaluate the following quantities without explicitly computing $x(t)$

(a)

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)dt &= X(j0) \\
&= 1
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\
&= \frac{1}{2\pi} \left[\int_{-5}^{-3} (\omega + 5)^2 d\omega + \int_{-3}^{-1} (-\omega - 1)^2 d\omega + \int_{-1}^1 (\omega + 1)^2 d\omega + \int_1^3 (-\omega + 3)^2 d\omega \right] \\
&= \frac{16}{3\pi}
\end{aligned}$$

(c)

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)e^{j3t}dt &= X(j(-3)) \\
&= 2
\end{aligned}$$

(d) $\arg\{x(t)\}$

$X(j\omega)$ is a real and even function shifted by 1 to the left, i.e. $X(j\omega) = X_e(j(\omega - 1))$. Since $X_e(j\omega)$ is real and even, so is $x_e(t)$, thus $x(t) = x_e(t)e^{-j(1)t} = |x_e(t)|e^{-j(1)t}$ which means,

$$\arg[x(t)] = -t$$

(e)

$$\begin{aligned} x(0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) d\omega \\ &= \frac{1}{2\pi} \left[\int_{-5}^{-3} (\omega + 5) d\omega + \int_{-3}^{-1} (-\omega - 1) d\omega + \int_{-1}^1 (\omega + 1) d\omega + \int_1^3 (-\omega + 3) d\omega \right] \\ &= \frac{4}{\pi} \end{aligned}$$

3.78. Let $x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$ where $x[n]$ is depicted in Figure P3.78. Evaluate the following without explicitly computing $X(e^{j\Omega})$.

(a) $X(e^{j0})$

$$\begin{aligned} X(e^{j0}) &= \sum_{n=-\infty}^{\infty} x[n] \\ &= 0 \end{aligned}$$

(b) $\arg\{X(e^{j\Omega})\}$

$x[n]$ is a real and odd function shifted by 2 to the right, i.e. $x[n] = x_o[n - 2]$. Since $x_o[n]$ is a real and odd, $X_o(e^{j\Omega})$ is purely imaginary, thus $X(e^{j\Omega}) = |X_o(e^{j\Omega})|e^{-j2\Omega}e^{j\frac{\pi}{2}}$, which means,

$$\arg\{X(e^{j\Omega})\} = \frac{\pi}{2} - 2\Omega$$

(c)

$$\begin{aligned} \int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega &= 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 \\ &= 28\pi \end{aligned}$$

(d)

$$\begin{aligned} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j3\Omega} d\Omega &= 2\pi x(3) \\ &= -2\pi \end{aligned}$$

(e) $y[n] \xleftrightarrow{DTFT} \text{Re}\{e^{j2\Omega} X(e^{j\Omega})\}$

$$\begin{aligned}
y[n] &= \operatorname{Re}\{e^{j2\Omega} X(e^{j\Omega})\} \\
&= 0
\end{aligned}$$

Since $X(e^{j\Omega})$ is purely imaginary, see part (b).

3.79. Prove the following properties.

(a) The FS symmetry properties for

i) Real valued time signals.

If $x(t)$ is real-valued: $x(t) = x^*(t)$

$$\begin{aligned}
X[k] &= \frac{1}{T} \int_0^T x(t) e^{-jk\omega_o t} dt \\
X^*[k] &= \frac{1}{T} \int_0^T x^*(t) e^{jk\omega_o t} dt \\
X^*[k] &= \frac{1}{T} \int_0^T x(t) e^{-j(-k)\omega_o t} dt \\
&= X[-k]
\end{aligned}$$

ii) Real and even time signals.

Further, if $x(t)$ is even: $x(-t) = x(t)$

$$\begin{aligned}
X[-k] &= \frac{1}{T} \int_0^T x(t) e^{jk\omega_o t} dt \\
X[-k] &= \frac{1}{T} \int_0^T x(-t) e^{jk\omega_o t} dt \\
X[-k] &= -\frac{1}{T} \int_0^{-T} x(\tau) e^{-jk\omega_o \tau} d\tau \\
&\quad \tau = -t, \quad d\tau = -dt
\end{aligned}$$

Then flip order of integration.

$$\begin{aligned}
X[-k] &= X[k] \\
X[-k] &= X^*[k]
\end{aligned}$$

Therefore, $X[k]$ is real valued or $\operatorname{Im}\{X[k]\} = 0$

(b) The DTFT time shift property.

$$\begin{aligned}
x[n - n_o] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega(n - n_o)} d\Omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{X(e^{j\Omega}) e^{j\Omega n_o}}_{X(e^{j\Omega})} e^{j\Omega n} d\Omega
\end{aligned}$$

$$x[n - n_o] \xleftrightarrow{DTFT} X(e^{j\Omega})e^{-j\Omega n_o}$$

(c) The DTFS frequency shift property.

$$\begin{aligned} x[n]e^{jk_o\Omega_o n} &= \sum_{k=0}^{N-1} x[n]e^{jk_o\Omega_o n} e^{jk\Omega_o n} \\ &= \sum_{k=0}^{N-1} x[n]e^{j(k-k_o)\Omega_o n} \\ &= X[k - k_o] \end{aligned}$$

(d) Linearity for the FT.

$$\text{Let } ax(t) + by(t) \xleftrightarrow{FT} S(j\omega)$$

$$\begin{aligned} S(j\omega) &= \int_{-\infty}^{\infty} [ax(t) + by(t)]e^{-j\omega t} dt \\ &= a \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt + b \int_{-\infty}^{\infty} y(t)e^{-j\omega t} dt \\ &= aX(j\omega) + bY(j\omega) \end{aligned}$$

$$\text{Therefore } ax(t) + by(t) \xleftrightarrow{FT} aX(j\omega) + bY(j\omega)$$

(e) The DTFT convolution property.

$$\text{Let } x[n] * y[n] \xleftrightarrow{DTFT} C(e^{j\Omega})$$

$$\begin{aligned} C(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x[l]y[n-l]e^{-j\Omega n} \\ &= \sum_{l=-\infty}^{\infty} x[l] \underbrace{\sum_{n=-\infty}^{\infty} y[n-l]e^{-j\Omega(n-l)}}_{Y(e^{j\Omega_o})} e^{-j\Omega l} \\ &= \sum_{l=-\infty}^{\infty} x[l]e^{-j\Omega l} Y(e^{j\Omega}) \\ &= X(e^{j\Omega})Y(e^{j\Omega}) \end{aligned}$$

$$\text{Therefore } x[n] * y[n] \xleftrightarrow{DTFT} X(e^{j\Omega})Y(e^{j\Omega})$$

(f) The DTFT multiplication property.

$$\text{Let } x[n]y[n] \xleftrightarrow{DTFT} M(e^{j\Omega})$$

$$M(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]y[n]e^{-j\Omega n}$$

$$\begin{aligned}
x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Gamma}) e^{j\Gamma n} d\Gamma \\
M(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} y[n] \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Gamma}) e^{j\Gamma n} d\Gamma e^{-j\Omega n} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Gamma}) \sum_{n=-\infty}^{\infty} y[n] e^{j\Gamma n} e^{-j\Omega n} d\Gamma \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Gamma}) \underbrace{\sum_{n=-\infty}^{\infty} y[n] e^{-j(\Omega-\Gamma)n}}_{Y(e^{j(\Omega-\Gamma)})} d\Gamma \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Gamma}) Y(e^{j(\Omega-\Gamma)}) d\Gamma \\
&= \frac{1}{2\pi} X(e^{j\Omega}) * Y(e^{j\Omega})
\end{aligned}$$

Therefore $x[n]y[n] \xrightarrow{DTFT} \frac{1}{2\pi} X(e^{j\Omega}) * Y(e^{j\Omega})$

(g) The DTFS convolution property.

Let $x[n] \otimes y[n] \xrightarrow{DTFS; \Omega_o} C[k]$, $N = \frac{2\pi}{\Omega_o}$

$$\begin{aligned}
C[k] &= \frac{1}{N} \sum_{n=0}^{N-1} \left(\sum_{l=0}^{N-1} x[l] y[n-l] \right) e^{-jk\Omega_o n} \\
&= \sum_{l=0}^{N-1} x[l] \underbrace{\sum_{n-l=0}^{N-1} \frac{1}{N} y[n-l] e^{-jk\Omega_o(n-l)}}_{Y[k]} e^{-jk\Omega_o l} \\
&= N \underbrace{\frac{1}{N} \sum_{l=0}^{N-1} x[l] e^{-jk\Omega_o l}}_{X[k]} Y[k] \\
&= NX[k]Y[k]
\end{aligned}$$

Therefore $x[n] \otimes y[n] \xrightarrow{DTFS; \frac{2\pi}{N}} NX[k]Y[k]$

(h) The FS multiplication property.

Let $x(t)y(t) \xrightarrow{FS; \omega_o} M[k]$

$$\begin{aligned}
M[k] &= \frac{1}{T} \int_0^T x(t)y(t) e^{-jk\omega_o t} dt \\
x(t) &= \sum_{l=-\infty}^{\infty} X[l] e^{jl\omega_o t} \\
M[k] &= \frac{1}{T} \int_0^T \sum_{l=-\infty}^{\infty} X[l] e^{jl\omega_o t} y(t) e^{-jk\omega_o t} dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{T} \sum_{l=-\infty}^{\infty} X[l] \int_0^T y(t) e^{-j(k-l)\omega_o t} dt \\
&= \sum_{l=-\infty}^{\infty} X[l] Y[k-l] \\
&= X[k] * Y[k]
\end{aligned}$$

Therefore $x(t)y(t) \xrightarrow{FS; \omega_o} X[k] * Y[k]$

(i) The Parseval relationship for the FS.

$$\begin{aligned}
\frac{1}{T} \int_0^T |x(t)|^2 dt &= \frac{1}{T} \int_0^T x(t) \sum_{k=-\infty}^{\infty} X^*[k] e^{-jk\omega_o t} dt \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} X^*[k] \underbrace{\frac{1}{T} \int_0^T x(t) e^{-jk\omega_o t} dt}_{X[k]} \\
&= \sum_{k=-\infty}^{\infty} X^*[k] X[k] \\
&= \sum_{k=-\infty}^{\infty} |X[k]|^2
\end{aligned}$$

3.80. Define a signal that is zero except at integer multiples of the scaling parameter p . That is, let

$$x_z[n] = 0, \text{ unless } n/p \text{ is integer}$$

Figure P3.80 (a) illustrates such a signal for $p = 3$.

(a) Show that the DTFT of $z[n] = x_z[pn]$ is given by $Z(e^{j\Omega}) = X_z(e^{j\Omega/p})$.

$$\begin{aligned}
X_z(e^{j\Omega/p}) &= \sum_{n=-\infty}^{\infty} x_z[n] e^{-j\frac{\Omega}{p}n} \\
\text{let } n &= pr \quad \text{sum over } r. \\
&= \sum_{r=-\infty}^{\infty} x_z[pr] e^{-j\Omega r} \\
&= \sum_{n=-\infty}^{\infty} x_z[pn] e^{-j\Omega n} \\
&= Z(e^{j\Omega})
\end{aligned}$$

(b) Use the DTFT of the signal $w[n]$ depicted in Fig. P3.80(b) and the scaling property to determine the DTFT of the signal $f[n]$ depicted in Fig. P3.80 (b).

$$w[n] = (0.9)^n u[n] \xrightarrow{DTFT} W(e^{j\Omega}) = \frac{1}{1 - 0.9e^{-j\Omega}}$$

$$f[n] = \begin{cases} w[\frac{n}{2}] & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \quad \xleftrightarrow{DTFT} \quad F(e^{j\Omega}) = W(e^{j2\Omega}) = \frac{1}{1 - 0.9e^{-j2\Omega}}$$

(c) Assume $x_z[n]$ is periodic with fundamental period N so $z[n] = x_z[pn]$ has fundamental period N/p where N/p is a positive integer. Show that the DTFS of $z[n]$ satisfies $Z[k] = pX_z[k]$.

$$\begin{aligned} Z[k] &= \frac{1}{\frac{N}{p}} \sum_{n=0}^{\frac{N}{p}-1} z[n] e^{-jkn\Omega'_o} \\ \Omega'_o &= \frac{2\pi}{\frac{N}{p}} = p\Omega_o \\ Z[k] &= p \frac{1}{N} \sum_{n=0}^{\frac{N}{p}-1} x_z[pn] e^{-jkn p \Omega_o} \\ &\quad \text{let } l = pn \\ Z[k] &= p \frac{1}{N} \sum_{l=0}^{N-1} x_z[l] e^{-jkl\Omega_o} \\ &= pX[k] \end{aligned}$$

3.81. In this problem we show that Gaussian pulses achieve the lower bound in the time-bandwidth product. *Hint:* Use the definite integrals in Appendix A.4.

(a) Let $x(t) = e^{-\frac{t^2}{2}}$, $X(j\omega) = e^{-\frac{\omega^2}{2}}$. Find the effective duration, T_d , bandwidth, B_w , and evaluate the time-bandwidth product.

$$\begin{aligned} T_d &= \left\{ \frac{\int_{-\infty}^{\infty} t^2 e^{-t^2} dt}{\int_{-\infty}^{\infty} e^{-t^2} dt} \right\}^{\frac{1}{2}} \\ &= \left\{ \frac{(\frac{1}{\sqrt{2}})^3 \sqrt{2\pi}}{(\frac{1}{\sqrt{2}}) \sqrt{2\pi}} \right\}^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}} \\ B_w &= \left\{ \frac{\int_{-\infty}^{\infty} \omega^2 e^{-\omega^2} d\omega}{\int_{-\infty}^{\infty} e^{-\omega^2} d\omega} \right\}^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}} \\ T_d B_w &= \frac{1}{2} \end{aligned}$$

(b) Let $x(t) = e^{-\frac{t^2}{2a^2}}$. Find the effective duration, T_d , bandwidth, B_w , and evaluate the time-bandwidth product. What happens to T_d , B_w , and $T_d B_w$ as a increases?

$$\begin{aligned} f\left(\frac{t}{a}\right) &\xleftrightarrow{FT} aF(j\omega a) \\ \text{so } X(j\omega) &= ae^{-\frac{\omega^2 a^2}{2}} \end{aligned}$$

$$\begin{aligned}
T_d &= \left\{ \frac{\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{a^2}} dt}{\int_{-\infty}^{\infty} e^{-\frac{t^2}{a^2}} dt} \right\}^{\frac{1}{2}} \\
&= \left\{ \frac{(\frac{a}{\sqrt{2}})^3 \sqrt{2\pi}}{(\frac{a}{\sqrt{2}}) \sqrt{2\pi}} \right\}^{\frac{1}{2}} \\
&= \frac{a}{\sqrt{2}} \\
B_w &= \left\{ \frac{\int_{-\infty}^{\infty} \omega^2 e^{-\omega^2 a^2} d\omega}{\int_{-\infty}^{\infty} e^{-\omega^2 a^2} d\omega} \right\}^{\frac{1}{2}} \\
&= \left\{ \frac{(\frac{1}{\sqrt{2}a})^3 \sqrt{2\pi}}{(\frac{1}{\sqrt{2}a}) \sqrt{2\pi}} \right\}^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}a} \\
T_d B_w &= \frac{1}{2}
\end{aligned}$$

If a increases:

- (1) T_d increases
- (2) B_w decreases
- (3) $T_d B_w$ stays the same

3.82. Let

$$x(t) = \begin{cases} 1, & |t| < T \\ 0, & \text{otherwise} \end{cases}$$

Use the uncertainty principle to bound the effective bandwidth of $x(t) * x(t)$.

$$x(t) * x(t) = \begin{cases} 2T - |t| & |t| < 2T \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\int_{-2T}^{2T} t^2 (2T - |t|)^2 dt &= 2 \int_0^{2T} 4T^2 t^2 - 4T t^3 + t^4 dt \\
&= \frac{32}{15} T^5 \\
\int_{-2T}^{2T} (2T - |t|)^2 dt &= 2 \int_0^{2T} 4T^2 - 4T t + t^2 dt \\
&= \frac{16}{3} T^3 \\
T_d &= \left\{ \frac{\frac{32}{15} T^5}{\frac{16}{3} T^3} \right\}^{\frac{1}{2}} \\
&= \sqrt{\frac{2}{5}} T \\
B_w &\geq \frac{\sqrt{5}}{\sqrt{8} T}
\end{aligned}$$

3.83. Use the uncertainty principle to bound the effective bandwidth of $x(t) = e^{-|t|}$.

$$\begin{aligned}
T_d B_w &\geq \frac{1}{2} \\
B_w &\geq \frac{1}{2T_d} \\
T_d &= \left\{ \frac{\int_{-\infty}^{\infty} t^2 e^{-2|t|} dt}{\int_{-\infty}^{\infty} e^{-2|t|} dt} \right\}^{\frac{1}{2}} \\
&= \left(\frac{1}{2} \right)^{\frac{1}{2}} \\
&= \frac{\sqrt{2}}{2} \\
B_w &\geq \frac{\sqrt{2}}{2}
\end{aligned}$$

3.84. Show that the time-bandwidth product, $T_d B_w$, of a signal $x(t)$ is invariant to scaling. That is, use the definitions of T_d and B_w to show that $x(t)$ and $x(at)$ have the same time-bandwidth product.

$$\begin{aligned}
T_d &= \left\{ \frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \right\}^{\frac{1}{2}} \\
T_d^{(s)} &= \left\{ \frac{\int_{-\infty}^{\infty} t^2 |x(at)|^2 dt}{\int_{-\infty}^{\infty} |x(at)|^2 dt} \right\}^{\frac{1}{2}} \\
\text{Let } u &= at \\
T_d^{(s)} &= \left\{ \frac{\frac{1}{a^3} \int_{-\infty}^{\infty} u^2 |x(u)|^2 du}{\frac{1}{a} \int_{-\infty}^{\infty} |x(u)|^2 du} \right\}^{\frac{1}{2}} \\
&= \frac{1}{a} T_d \\
x(at) &\xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right) \\
B_w &= \left\{ \frac{\int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega} \right\}^{\frac{1}{2}} \\
B_w^{(s)} &= \left\{ \frac{\frac{1}{a^2} \int_{-\infty}^{\infty} \omega^2 |X(\frac{j\omega}{a})|^2 d\omega}{\frac{1}{a^2} \int_{-\infty}^{\infty} |X(\frac{j\omega}{a})|^2 d\omega} \right\}^{\frac{1}{2}} \\
\text{Let } v &= \frac{\omega}{a} \\
B_w^{(s)} &= \left\{ \frac{a^3 \int_{-\infty}^{\infty} v^2 |X(jv)|^2 dv}{a \int_{-\infty}^{\infty} |X(jv)|^2 dv} \right\}^{\frac{1}{2}} \\
&= a B_w \\
T_d^{(s)} B_w^{(s)} &= T_d B_w
\end{aligned}$$

Thus the time-bandwidth product is invariant to scaling.

3.85. A key property of the complex sinusoids used in the DTFS and FS expansions is orthogonality. By orthogonality we mean that the inner product of two harmonically related sinusoids is zero. The inner product is defined as the sum or integral of the product of one signal times the conjugate of the other over a fundamental period.

(a) Show orthogonality for discrete-time complex sinusoids, that is, prove

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{jk\frac{2\pi}{N}n} e^{-jl\frac{2\pi}{N}n} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$$

Here we assume that $|k - l| < N$.

Use finite-term geometric series formulas.

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j(k-l)\frac{2\pi}{N}n} = \frac{1}{N} \begin{cases} N & k = l \\ \frac{1 - e^{j2\pi(k-l)}}{1 - e^{j\frac{2\pi}{N}(k-l)}} & k \neq l \end{cases}$$

k, l are integers, thus $e^{j2\pi(k-l)} = 1$

this implies

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j(k-l)\frac{2\pi}{N}n} = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$$

(b) Show orthogonality for harmonically related continuous-time complex sinusoids, that is, prove

$$\frac{1}{T} \int_0^T e^{jk\frac{2\pi}{T}t} e^{-jl\frac{2\pi}{T}t} dt = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$$

If $k = l$

$$\begin{aligned} \frac{1}{T} \int_0^T e^{j\frac{2\pi}{T}(k-l)t} dt &= \frac{1}{T} \int_0^T 1 dt \\ &= 1 \end{aligned}$$

If $k \neq l$

$$\begin{aligned} \frac{1}{T} \int_0^T e^{j\frac{2\pi}{T}(k-l)t} dt &= \left. \frac{1}{T} \frac{e^{j\frac{2\pi}{T}(k-l)t}}{j\frac{2\pi}{T}(k-l)} \right|_0^T \\ &= \frac{e^{j2\pi(k-l)} - 1}{j2\pi(k-l)} \\ &= 0 \end{aligned}$$

(c) Show orthogonality of harmonically related sines and cosines, that is, prove

$$\frac{1}{T} \int_0^T \sin(k\frac{2\pi}{T}t) \sin(l\frac{2\pi}{T}t) dt = \begin{cases} 1/2, & k = l \\ 0, & k \neq l \end{cases}$$

If $k = l$

$$\frac{1}{T} \int_0^T \sin^2(k\frac{2\pi}{T}t) dt = \frac{1}{T} \int_0^T \frac{1}{2} \left(1 - \cos(2k\frac{2\pi}{T}t) \right) dt$$

$$\begin{aligned}
&= \frac{1}{2T} \left[t - \frac{T}{4\pi k} \sin(k \frac{4\pi}{T} t) \right] \Big|_0^T \\
&= \frac{1}{2}
\end{aligned}$$

If $k \neq l$

$$\begin{aligned}
\frac{1}{T} \int_0^T \sin(k \frac{2\pi}{T} t) \sin(l \frac{2\pi}{T} t) dt &= \frac{1}{T} \int_0^T \frac{1}{2} \left[\cos(\frac{2\pi}{T} t(k-l)) - \cos(\frac{2\pi}{T} t(k+l)) \right] dt \\
&= \frac{1}{2T} \left[\frac{T}{2\pi(k-l)} \sin(\frac{2\pi}{T} t(k-l)) - \frac{T}{2\pi(k+l)} \sin(\frac{2\pi}{T} t(k+l)) \right] \Big|_0^T \\
&= 0
\end{aligned}$$

$$\frac{1}{T} \int_0^T \cos(k \frac{2\pi}{T} t) \cos(l \frac{2\pi}{T} t) dt = \begin{cases} 1/2, & k = l \\ 0, & k \neq l \end{cases}$$

If $k = l$

$$\begin{aligned}
\frac{1}{T} \int_0^T \cos^2(k \frac{2\pi}{T} t) dt &= \frac{1}{T} \int_0^T \frac{1}{2} (1 + \cos(2k \frac{2\pi}{T} t)) dt \\
&= \frac{1}{2T} \left[t + \frac{T}{4\pi k} \sin(k \frac{4\pi}{T} t) \right] \Big|_0^T \\
&= \frac{1}{2}
\end{aligned}$$

If $k \neq l$

$$\begin{aligned}
\frac{1}{T} \int_0^T \cos(k \frac{2\pi}{T} t) \cos(l \frac{2\pi}{T} t) dt &= \frac{1}{T} \int_0^T \frac{1}{2} \left[\cos(\frac{2\pi}{T} t(k-l)) + \cos(\frac{2\pi}{T} t(k+l)) \right] dt \\
&= \frac{1}{2T} \left[\frac{T}{2\pi(k-l)} \sin(\frac{2\pi}{T} t(k-l)) + \frac{T}{2\pi(k+l)} \sin(\frac{2\pi}{T} t(k+l)) \right] \Big|_0^T \\
&= 0
\end{aligned}$$

$$\frac{1}{T} \int_0^T \cos(k \frac{2\pi}{T} t) \sin(l \frac{2\pi}{T} t) dt = 0$$

If $k = l$

$$\begin{aligned}
\frac{1}{T} \int_0^T \cos(k \frac{2\pi}{T} t) \sin(l \frac{2\pi}{T} t) dt &= \frac{1}{T} \int_0^T \frac{1}{2} \sin(2k \frac{2\pi}{T} t) dt \\
&= \frac{1}{2T} \left[-\frac{T}{4\pi k} \cos(k \frac{4\pi}{T} t) \right] \Big|_0^T \\
&= 0
\end{aligned}$$

If $k \neq l$

$$\begin{aligned}
\frac{1}{T} \int_0^T \cos(k \frac{2\pi}{T} t) \sin(l \frac{2\pi}{T} t) dt &= \frac{1}{T} \int_0^T \frac{1}{2} \left[\sin(\frac{2\pi}{T} t(k-l)) + \sin(\frac{2\pi}{T} t(k+l)) \right] dt \\
&= \frac{1}{2T} \left[-\frac{T}{2\pi(k-l)} \cos(\frac{2\pi}{T} t(k-l)) - \frac{T}{2\pi(k+l)} \cos(\frac{2\pi}{T} t(k+l)) \right] \Big|_0^T
\end{aligned}$$

$$= 0$$

3.86. The form of the FS representation presented in this chapter,

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_o t}$$

is termed the exponential FS. In this problem we explore several alternative, yet equivalent, ways of expressing the FS representation for real valued periodic signals.

(a) Trigonometric form.

(i) Show that the FS for a real valued signal $x(t)$ can be written as

$$x(t) = B[0] + \sum_{k=1}^{\infty} B[k] \cos(k\omega_o t) + A[k] \sin(k\omega_o t)$$

where $B[k]$ and $A[k]$ are real valued coefficients.

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] (\cos(k\omega_o t) + j \sin(k\omega_o t))$$

$$x(t) = \sum_{k=-\infty}^{\infty} (X[k] \cos(k\omega_o t) + X[k] j \sin(k\omega_o t))$$

$x(t)$ is real, which implies: $X[-k] = X^*[k]$

$$\cos(-u) = \cos(u)$$

$$\sin(-u) = -\sin(u)$$

Thus

$$x(t) = X[0] + \sum_{k=1}^{\infty} [(X[k] + X^*[k]) \cos(k\omega_o t) + j(X[k] - X^*[k]) \sin(k\omega_o t)]$$

$$x(t) = X[0] + \sum_{k=1}^{\infty} [2\operatorname{Re}\{X[k]\} \cos(k\omega_o t) - 2\operatorname{Im}\{X[k]\} \sin(k\omega_o t)]$$

compare with

$$x(t) = B[0] + \sum_{k=1}^{\infty} B[k] \cos(k\omega_o t) + A[k] \sin(k\omega_o t)$$

(ii) Express $X[k]$ in terms of $B[k]$ and $A[k]$.

$$B[0] = X[0]$$

$$B[k] = 2\operatorname{Re}\{X[k]\}$$

$$A[k] = -2\operatorname{Im}\{X[k]\}$$

since

$$X[k] + X^*[k] = B[k]$$

$$X[k] - X^*[k] = -jA[k]$$

$$\begin{aligned} X[0] &= B[0] \\ X[k] &= \frac{B[k] - jA[k]}{2} \end{aligned}$$

(iii) Use the orthogonality of harmonically related sines and cosines (See Problem 3.85) to show

$$\begin{aligned} B[0] &= \frac{1}{T} \int_0^T x(t) dt \\ B[k] &= \frac{2}{T} \int_0^T x(t) \cos k\omega_o t dt \\ A[k] &= \frac{2}{T} \int_0^T x(t) \sin k\omega_o t dt \end{aligned}$$

$$\begin{aligned} B[0] = X[0] &= \frac{1}{T} \int_0^T x(t) e^{-j(0)\omega_o t} dt \\ &= \frac{1}{T} \int_0^T x(t) dt \\ B[k] &= \frac{2}{T} \operatorname{Re} \left\{ \int_0^T x(t) e^{-jk\omega_o t} dt \right\} \\ &= \frac{2}{T} \int_0^T x(t) \operatorname{Re} \{ e^{-jk\omega_o t} \} dt \\ &= \frac{2}{T} \int_0^T x(t) \cos(k\omega_o t) dt \\ A[k] &= -\frac{2}{T} \int_0^T x(t) \operatorname{Im} \{ e^{-jk\omega_o t} \} dt \\ &= \frac{2}{T} \int_0^T x(t) \sin(k\omega_o t) dt \end{aligned}$$

(iv) Show that $A[k] = 0$ if $x(t)$ is even and $B[k] = 0$ if $x(t)$ is odd.

If $x(t)$ is even: $X[k] = X^*[k]$

Thus: $X[k]$ is real valued, i.e. $\operatorname{Im}\{X[k]\} = 0$

Therefore,

$$\begin{aligned} A[k] &= -2(0) \\ A[k] &= 0 \end{aligned}$$

If $x(t)$ is odd: $X[k] = -X^*[k]$

Thus: $X[k]$ is imaginary, i.e. $\operatorname{Re}\{X[k]\} = 0$

Therefore,

$$\begin{aligned} B[k] &= 2(0) \\ B[k] &= 0 \end{aligned}$$

(b) Polar form.

(i) Show that the FS for a real valued signal $x(t)$ can be written as

$$x(t) = C[0] + \sum_{k=1}^{\infty} C[k] \cos(k\omega_o t + \theta[k])$$

where $C[k]$ is the magnitude (positive) and $\theta[k]$ is the phase of the k^{th} harmonic.

Write $X[k]$ in polar form as $X[k] = |X[k]|e^{j\arg\{X[k]\}} = C[k]e^{j\theta[k]}$

$$\text{From (a): } x(t) = B[0] + \sum_{k=1}^{\infty} B[k] \cos(k\omega_o t) + A[k] \sin(k\omega_o t)$$

$$\text{Let: } B[k] = \cos(\theta[k])C[k]$$

$$A[k] = -\sin(\theta[k])C[k]$$

where:

$$C[k] = \sqrt{B[k]^2 + A[k]^2}$$

since

$$\cos(\theta[k]) \cos(k\omega_o t) - \sin(\theta[k]) \sin(k\omega_o t) = \cos(\theta[k] + k\omega_o t)$$

$$\text{Thus: } x(t) = B[0] + \sum_{k=1}^{\infty} \sqrt{B[k]^2 + A[k]^2} \cos(\theta[k] + k\omega_o t)$$

$$\text{Compare with: } x(t) = C[0] + \sum_{k=1}^{\infty} C[k] \cos(k\omega_o t + \theta[k])$$

(ii) Express $C[k]$ and $\theta[k]$ as a function of $X[k]$.

$$\begin{aligned} C[k] &= \sqrt{B[k]^2 + A[k]^2} \\ &= 2\sqrt{\text{Re}\{X[k]\}^2 + \text{Im}\{X[k]\}^2} \\ \theta[k] &= -\arctan\left(\frac{A[k]}{B[k]}\right) \\ &= -\arctan\left(\frac{\text{Im}\{X[k]\}}{\text{Re}\{X[k]\}}\right) \end{aligned}$$

(iii) Express $C[k]$ and $\theta[k]$ as a function of $B[k]$ and $A[k]$ from (a).

$$\begin{aligned} C[k] &= \sqrt{B[k]^2 + A[k]^2} \\ \theta[k] &= -\arctan\left(\frac{A[k]}{B[k]}\right) \end{aligned}$$

3.87. In this problem we derive the frequency response of the continuous- and discrete-time LTI systems described by the state-variable representations.

(a) Define $\mathbf{q}(j\omega)$ as the FT of each element of the state vector in the state-variable representation for a

system. That is,

$$\mathbf{q}(j\omega) = \begin{bmatrix} Q_1(j\omega) \\ Q_2(j\omega) \\ \vdots \\ Q_N(j\omega) \end{bmatrix}$$

where the i th entry in $\mathbf{q}(j\omega)$ is the FT of the i th state variable, $q_i(t) \xrightarrow{FT} Q_i(j\omega)$. Take the FT of the state equation $\frac{d}{dt}\mathbf{q}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)$ using the differentiation property to express $\mathbf{q}(j\omega)$ as a function of ω , \mathbf{A} , \mathbf{b} , and $X(j\omega)$. Next take the FT of the output equation $y(t) = \mathbf{c}\mathbf{q}(t) + Dx(t)$ and substitute for $\mathbf{q}(j\omega)$ to show that:

$$H(j\omega) = \mathbf{c}(\mathbf{j}\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \mathbf{D}$$

$$\begin{aligned} \frac{d}{dt}\mathbf{q}(\mathbf{t}) &= \mathbf{A}\mathbf{q}(\mathbf{t}) + \mathbf{b}\mathbf{x}(\mathbf{t}) \\ j\omega\mathbf{Q}(\mathbf{j}\omega) &= \mathbf{A}\mathbf{Q}(\mathbf{j}\omega) + \mathbf{b}\mathbf{X}(\mathbf{j}\omega) \\ \mathbf{Q}(\mathbf{j}\omega)(\mathbf{j}\omega\mathbf{I} - \mathbf{A}) &= \mathbf{b}\mathbf{X}(\mathbf{j}\omega) \\ \mathbf{Q}(\mathbf{j}\omega) &= (\mathbf{j}\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\mathbf{X}(\mathbf{j}\omega) \end{aligned}$$

$$\begin{aligned} y(t) &= \mathbf{c}\mathbf{q}(\mathbf{t}) + \mathbf{D}\mathbf{x}(\mathbf{t}) \\ Y(j\omega) &= \mathbf{c}\mathbf{Q}(\mathbf{j}\omega) + \mathbf{D}\mathbf{X}(\mathbf{j}\omega) \\ &= \mathbf{c}(\mathbf{j}\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\mathbf{X}(\mathbf{j}\omega) + \mathbf{D}\mathbf{X}(\mathbf{j}\omega) \\ H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} \\ &= \mathbf{c}(\mathbf{j}\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \mathbf{D} \end{aligned}$$

(b) Use the time shift properties to express the frequency response of a discrete-time LTI system in terms of the state-variable representation as:

$$H(e^{j\Omega}) = \mathbf{c}(\mathbf{e}^{j\Omega}\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \mathbf{D}$$

$$\begin{aligned} \mathbf{q}[n+1] &= \mathbf{A}\mathbf{q}[n] + \mathbf{n}\mathbf{x}[n] \\ e^{j\Omega}\mathbf{Q}(\mathbf{e}^{j\Omega}) &= \mathbf{A}\mathbf{Q}(\mathbf{e}^{j\Omega}) + \mathbf{b}\mathbf{X}(\mathbf{e}^{j\Omega}) \\ \mathbf{Q}(\mathbf{e}^{j\Omega}) &= (\mathbf{e}^{j\Omega} - \mathbf{A})^{-1}\mathbf{b}\mathbf{X}(\mathbf{e}^{j\Omega}) \\ y[n] &= \mathbf{c}\mathbf{q}[n] + \mathbf{D}\mathbf{x}[n] \\ Y(e^{j\Omega}) &= \mathbf{c}\mathbf{Q}(\mathbf{e}^{j\Omega}) + \mathbf{D}\mathbf{X}(\mathbf{e}^{j\Omega}) \\ &= \mathbf{c}(\mathbf{e}^{j\Omega} - \mathbf{A})^{-1}\mathbf{b}\mathbf{X}(\mathbf{e}^{j\Omega}) + \mathbf{D}\mathbf{X}(\mathbf{e}^{j\Omega}) \\ H(e^{j\Omega}) &= \mathbf{c}(\mathbf{e}^{j\Omega}\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \mathbf{D} \end{aligned}$$

3.88. Use the result of Problem 3.87 to determine the frequency response, impulse response, and differential equation descriptions for the continuous-time systems described by the following state variable descriptions.

$$(a) \mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = [0]$$

$$\begin{aligned} H(j\omega) &= \mathbf{c}(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D \\ &= \frac{2}{j\omega + 1} \\ h(t) &= 2e^{-t}u(t) \\ &= \frac{Y(j\omega)}{X(j\omega)} \\ Y(j\omega)(j\omega + 1) &= 2X(j\omega) \\ \frac{d}{dt}y(t) + y(t) &= 2x(t) \end{aligned}$$

$$(b) \mathbf{A} = \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = [0]$$

$$\begin{aligned} H(j\omega) &= \mathbf{c}(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D \\ &= \frac{2j\omega - 5}{(j\omega)^2 + 3j\omega + 2} \\ &= \frac{Y(j\omega)}{X(j\omega)} \\ Y(j\omega)((j\omega)^2 + 3j\omega + 2) &= X(j\omega)(2j\omega - 5) \\ \frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) &= 2\frac{d}{dt}x(t) - 5x(t) \\ H(j\omega) &= \frac{A}{2 + j\omega} + \frac{B}{1 + j\omega} \\ 2 &= A + B \\ -5 &= A + 2B \\ &= \frac{9}{2 + j\omega} - \frac{7}{1 + j\omega} \\ h(t) &= (9e^{-2t} - 7e^{-t})u(t) \end{aligned}$$

3.89. Use the result of Problem 3.87 to determine the frequency response, impulse response, and difference equation descriptions for the discrete-time systems described by the following state variable descriptions.

$$(a) \mathbf{A} = \begin{bmatrix} -\frac{1}{2} & 1 \\ 0 & \frac{1}{4} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = [1]$$

$$\begin{aligned} H(e^{j\Omega}) &= \mathbf{c}(e^{j\Omega}\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D \\ &= 1 + \frac{1}{(e^{j\Omega} + \frac{1}{2})(e^{j\Omega} - \frac{1}{4})} \end{aligned}$$

$$\begin{aligned}
&= \frac{1 + \frac{1}{4}e^{-j\Omega} + \frac{7}{8}e^{-j2\Omega}}{1 + \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}} \\
h[n] &= \left(\frac{8}{3} \left(-\frac{1}{2}\right)^n + \frac{16}{3} \left(\frac{1}{4}\right)^n \right) u[n] - 7\delta[n] \\
H(e^{j\Omega}) &= \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} \\
Y(e^{j\Omega}) \left(1 + \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega} \right) &= X(e^{j\Omega}) \left(1 + \frac{1}{4}e^{-j\Omega} + \frac{7}{8}e^{-j2\Omega} \right) \\
y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] &= x[n] + \frac{1}{4}x[n-1] + \frac{7}{8}x[n-2]
\end{aligned}$$

$$(b) \mathbf{A} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = [0]$$

$$\begin{aligned}
H(e^{j\Omega}) &= \mathbf{c}(e^{j\Omega}\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D \\
&= \frac{e^{j\Omega}}{e^{j2\Omega} - \frac{1}{4}} \\
&= \frac{e^{-j\Omega}}{1 - \frac{1}{4}e^{-j2\Omega}} \\
&= \frac{A}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{B}{1 - \frac{1}{2}e^{-j\Omega}} \\
0 &= A + B \\
1 &= -\frac{1}{2}A + \frac{1}{2}B \\
H(e^{j\Omega}) &= -\frac{1}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} \\
h[n] &= \left[-\left(-\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n \right] u[n] \\
H(e^{j\Omega}) &= \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} \\
Y(e^{j\Omega}) \left(1 - \frac{1}{4}e^{-j2\Omega} \right) &= X(e^{j\Omega})e^{-j\Omega} \\
y[n] - \frac{1}{4}y[n-2] &= x[n-1]
\end{aligned}$$

3.90. A continuous-time system is described by the state variable description

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = [0]$$

Transform the state vector associated with this system using the matrix

$$\mathbf{T} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

to find a new state variable description for the system. Show that the frequency response of the original and transformed systems are equal.

$$H(j\omega) = \mathbf{c}(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D$$

$$\begin{aligned}
H'(j\omega) &= \mathbf{c}'(j\omega\mathbf{I} - \mathbf{A}')^{-1}\mathbf{b}' + D' \\
\mathbf{A}' &= \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \\
&= \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \\
\mathbf{b}' &= \mathbf{T}\mathbf{b} \\
&= \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\
\mathbf{c}' &= \mathbf{c}\mathbf{T}^{-1} \\
&= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
D' &= D \\
&= 0
\end{aligned}$$

Use these two equations to verify $H(j\omega) = H'(j\omega)$.

$$\begin{aligned}
H(j\omega) &= \mathbf{c}(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D \\
H'(j\omega) &= \mathbf{c}'(j\omega\mathbf{I} - \mathbf{A}')^{-1}\mathbf{b}' + D' \\
&= \frac{2}{3 + j\omega}
\end{aligned}$$

Solutions to Advanced Problems

3.91. A signal with fundamental period T is said to possess halfwave symmetry if it satisfies $x(t) = -x(t - \frac{T}{2})$. That is, half of one period of the signal is the negative of the other half. Show that the FS coefficients associated with even harmonics, $X[2k]$, are zero for all signals with halfwave symmetry.

$$\begin{aligned}
X[k] &= \frac{1}{T} \int_{-0.5T}^{0.5T} x(t) e^{-jk\omega_o t} dt \\
&= \frac{1}{T} \left\{ \int_0^{0.5T} x(t) e^{-jk\omega_o t} dt + \int_{-0.5T}^0 x(t) e^{-jk\omega_o t} dt \right\} \\
&= \frac{1}{T} \left\{ \int_0^{0.5T} x(t) e^{-jk\omega_o t} dt + \int_{-0.5T}^0 -x(\tau) e^{-jk\omega_o \tau} e^{-jk\pi} dt \right\} \\
&= \frac{1}{T} \left\{ \int_0^{0.5T} x(t) e^{-jk\omega_o t} (1 - (-1)^k) dt \right\} \\
&\text{when } k \text{ is even } (= 0, \pm 2, \pm 4, \dots), \text{ the integrand is } 0
\end{aligned}$$

Thus:

$$X[k] = \begin{cases} \frac{1}{T} \int_0^{0.5T} 2x(t) e^{-jk\omega_o t} dt & k = \text{odd} \\ 0, & k = \text{even} \end{cases}$$

3.92. The FS of piecewise constant signals may be determined using the differentiation and time-shift properties from the FS of the impulse train as follows. Differentiate the time-domain signal to obtain a sum of time-shifted impulse trains. Note that differentiation introduces an impulse at each discontinuity in the time-domain signal. Next use the time-shift property and the FS of the impulse train to find the FS of the differentiated signal. Finally, use the differentiation property to obtain the FS coefficient of the original signal from the differentiated signal.

Assume $x(t)$ has discontinuities at times T_l , $l = 1, 2, \dots, L$, and at these discontinuities the height difference is represented by $c_l = \lim_{t \rightarrow T_l^+} x(t) - \lim_{t \rightarrow T_l^-} x(t)$, then

$$a(t) = \sum_{p=-\infty}^{\infty} \delta(t - pT) \xleftrightarrow{FS; \omega_o} A[k] = \frac{1}{T}$$

$$x(t) = a(t - T_l) \xleftrightarrow{FS; \omega_o} X[k] = \frac{1}{T} e^{-jk\omega_o T_l}$$

$$y(t) = \frac{d}{dt} x(t)$$

$$= \sum_{l=1}^L c_l a(t - T_l)$$

Whose Fourier Series representation is

$$Y[k] = \sum_{l=1}^L c_l A[k] e^{-jk\omega_o T_l}$$

(a) Can this method be used to determine the FS coefficient for $k = 0$? How can you find it?

$X[k]$ is the average value of $x(t)$.

(b) Use this method to find the FS coefficients for the piecewise constant waveforms in Fig. P3.92.

(i)

In one period, $\frac{d}{dt}x(t)$ has a delta function of height 2 at $t = -1$, and a height of -2 at $t = 1$. $T = 4$, so $\omega_o = \frac{\pi}{2}$.

$$\frac{d}{dt}x(t) = \sum_{p=-\infty}^{\infty} 2\delta(t + 1 - pT) - 2\delta(t - 1 - pT)$$

$$X[k] = \frac{2e^{-jk\frac{\pi}{2}(-1)}}{jk2\pi} - \frac{2e^{-jk\frac{\pi}{2}(1)}}{jk2\pi}$$

$$= \frac{2}{k\pi} \sin\left(\frac{\pi}{2}k\right)$$

$$X[k] = \begin{cases} \frac{2}{k\pi} \sin\left(\frac{\pi}{2}k\right) & k \neq 0 \\ \frac{1}{2} & k = 0 \end{cases}$$

(ii)

In one period, $\frac{d}{dt}x(t)$ has 4 delta functions, one of height -2 at $t = -2$, another of height of -1 at $t = -1$, another of height of 4 at $t = 0$, and one of height of -1 at $t = 1$. $T = 4$, so $\omega_o = \frac{\pi}{2}$.

$$\frac{d}{dt}x(t) = \sum_{p=-\infty}^{\infty} [-2\delta(t + 2 - pT) - \delta(t + 1 - pT) + 4\delta(t - pT) - \delta(t - 1 - pT)]$$

$$\begin{aligned}
X[k] &= -\frac{2e^{-jk\frac{\pi}{2}(-2)}}{jk2\pi} - \frac{e^{-jk\frac{\pi}{2}(-1)}}{jk2\pi} + \frac{4}{jk2\pi} - \frac{e^{-jk\frac{\pi}{2}(1)}}{jk2\pi} \\
&= \frac{1}{jk\pi} \left[2 - e^{k\pi} - \cos\left(\frac{k\pi}{2}\right) \right]
\end{aligned}$$

$$X[k] = \begin{cases} \frac{1}{jk\pi} [2 - e^{k\pi} - \cos(\frac{k\pi}{2})] & k \neq 0 \\ -1, & k = 0 \end{cases}$$

3.93. The method for finding the FS coefficients described in the previous problem may be extended to signals that are piecewise linear by differentiating twice to obtain a sum of impulse trains and doublet trains. The time-shift property and FS of the impulse train and doublet train are then used to find the FS for the twice-differentiated signal and the differentiation property is used to obtain the FS coefficients of the original signal from the twice differentiated signal.

(a) Find the FS coefficients for the doublet train

$$d(t) = \sum_{l=-\infty}^{\infty} \delta^{(1)}(t - lT)$$

$$\begin{aligned}
a(t) &= \sum_{l=-\infty}^{\infty} \delta(t - lT) \xrightarrow{FT} A[k] = \frac{1}{T} \\
b(t) &= \frac{d}{dt} a(t) \xrightarrow{FT} B[k] = jk\omega_o A[k] \\
&\text{implies:} \\
\sum_{l=-\infty}^{\infty} \frac{d}{dt} \delta(t - lT) &\xrightarrow{FT} jk\omega_o \frac{1}{T} \\
\text{Thus:} \\
D[k] &= \frac{jk\omega_o}{T}
\end{aligned}$$

(b) Use this method to find the FS coefficients for the waveforms in Fig. P3.93.

(i)

Since two derivatives were taken to find the impulses, divide by $(jk\omega_o)^2$ to find $X[k]$.

$$\frac{d^2}{dt^2} x(t) = \sum_{l=-\infty}^{\infty} (-2\delta(t - l2) + 2\delta(t - 1 - l2)) \xrightarrow{FT} X[k] = \frac{1}{(jk\omega_o)^2} \left[\frac{2}{T} e^{-jk\pi(1)} - \frac{2}{T} \right]$$

$$X[k] = \begin{cases} \frac{1}{k^2\pi^2} [1 - (-1)^k] & k \neq 0 \\ \frac{1}{2} & k = 0 \end{cases}$$

(ii) $T = 4$, $\omega_o = \frac{\pi}{2}$. Take 2 derivatives to find:

$$\frac{d^2}{dt^2} x(t) = \sum_{l=-\infty}^{\infty} [\delta(t - lT) - 2\delta(t - 1 - lT) + \delta(t - 2 - lT)]$$

$$\begin{aligned}
& + \sum_{l=-\infty}^{\infty} \left[\delta^{(1)}(t+1-lT) - 2\delta^{(1)}(t-1-lT) + \delta^{(1)}(t-2-lT) \right] \\
x(t) & \xrightarrow{FT} X[k] = \frac{1}{jk\omega_o T} [e^{jk\omega_o} - 2e^{-jk\omega_o} + e^{-j2k\omega_o}] + \frac{1}{(jk\omega_o)^2 T} [1 - 2e^{-jk\omega_o} + e^{-j2k\omega_o}] \\
X[k] & = \frac{1}{jk2\pi} [e^{j\frac{\pi}{2}k} - 2e^{-j\frac{\pi}{2}k} + e^{-j\pi k}] + \frac{1}{2k^2\pi^2} [2e^{-j\frac{\pi}{2}k} + e^{-j\pi k} - 1] \\
\\
X[k] & = \begin{cases} \frac{1}{jk2\pi} [e^{j\frac{\pi}{2}k} - 2e^{-j\frac{\pi}{2}k} + e^{-j\pi k}] + \frac{1}{2k^2\pi^2} [2e^{-j\frac{\pi}{2}k} + e^{-j\pi k} - 1] & k \neq 0 \\ \frac{1}{4} \left(1 - \frac{2j}{\pi}\right) & k = 0 \end{cases}
\end{aligned}$$

3.94. The FT relates the electromagnetic field at a distant point to the electric field distribution at the antenna. This problem derives this result for a one-dimensional antenna with monochromatic (single frequency) excitation of ω_o . Let the electric field at a point z within the antenna aperture have amplitude $a(z)$ and phase $\phi(z)$ so the electric field as a function of z and t is $x(z, t) = a(z) \cos(\omega_o t + \phi(z))$. Define the complex amplitude of the field as $w(z) = a(z)e^{j\phi(z)}$ so that

$$x(z, t) = \text{Re} \{ w(z) e^{j\omega_o t} \}$$

Huygen's principle states that the electric field at a distant point is the superposition of the effects of each differential component of the aperture electric field. Suppose the point of interest is at a distance r . It takes time $t_o = r/c$ for the differential component between z and $z + dz$ to propagate a distance r where c is the propagation velocity. Thus the contribution to the field at r from this differential component of the aperture is given by

$$\begin{aligned}
y(z, t) dz &= x(z, t - t_o) dz \\
&= \text{Re} \{ w(z) e^{-j\omega_o t_o} dz e^{j\omega_o t} \}
\end{aligned}$$

Since the wavelength λ satisfies $\lambda = 2\pi c/\omega_o$, we have $\omega_o t_o = 2\pi r/\lambda$ and the complex amplitude associated with this differential volume is $w(z) e^{-j2\pi r/\lambda}$.

(a) Consider a point P at an angle θ with respect to the axis normal to the aperture and at a distance R from $z = 0$ as shown in Fig. P3.94. If R is much greater than the maximum extent of the aperture along the z -axis, then we may approximate r as $r = R + zs$ where $s = \sin \theta$. Use this approximation to determine the contribution of the differential component between z and $z + dz$ to P .

To an approximation

$$\begin{aligned}
r &= R + z \sin(\theta) \\
&= R + zs \\
s &= \sin(\theta)
\end{aligned}$$

then

$$\text{Re} \{ w(z) e^{-j2\pi r/\lambda} dz e^{j\omega_o t} \}$$

becomes

$$\text{Re} \{ w(z) e^{-j2\pi R/\lambda} e^{-j2\pi zs/\lambda} dz e^{j\omega_o t} \}$$

(b) Integrate all differential components of the antenna aperture to show that the field at P is given by

$$Y(s, R) = \text{Re} \left\{ G(s) e^{-j2\pi R/\lambda} e^{j\omega_o t} \right\}$$

where

$$G(s) = \int_{-\infty}^{\infty} w(z) e^{-j2\pi z s/\lambda} dz$$

represents the complex amplitude of the field as a function of $\sin \theta$. Comparison to the definition of the Fourier Transform indicates that $G(s)$ is the FT of $w(z)$ evaluated at $2\pi s/\lambda$.

Integrating out the differential component dz leads to:

$$\begin{aligned} Y(s, R) &= \int_{-\infty}^{\infty} \text{Re} \left\{ w(z) e^{-j2\pi R/\lambda} e^{-j2\pi z s/\lambda} dz e^{j\omega_o t} \right\} \\ &= \text{Re} \left\{ \int_{-\infty}^{\infty} w(z) e^{-j2\pi z s/\lambda} dz e^{-j2\pi R/\lambda} e^{j\omega_o t} \right\} \\ &= \text{Re} \left\{ G(s) e^{-j2\pi R/\lambda} e^{j\omega_o t} \right\} \end{aligned}$$

(c) Use the FT relationship developed in (b) to determine the far-field pattern, $|G(s)|$, for the following aperture distributions $w(z)$. Assume $\lambda = 1$ and sketch $|G(s)|$ for $-\pi/2 < \theta < \pi/2$.

$$(i) \ w(z) = \begin{cases} 1, & |z| < 5 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} |G(s)| &= \left| \int_{-\infty}^{\infty} w(z) e^{-j2\pi z s/\lambda} dz \right| \\ &= \left| \int_{-5}^5 e^{-j2\pi z s/\lambda} dz \right| \\ &= \left| \frac{\sin(10\pi \frac{s}{\lambda})}{\pi \frac{s}{\lambda}} \right| \end{aligned}$$

$$(ii) \ w(z) = \begin{cases} e^{j\pi z/4}, & |z| < 5 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} |G(s)| &= \left| \int_{-5}^5 e^{j\pi \frac{z}{4}} e^{-j2\pi z s/\lambda} dz \right| \\ &= \left| \frac{\sin(5\pi(0.25 - 2\frac{s}{\lambda}))}{\pi(0.25 - 2\frac{s}{\lambda})} \right| \end{aligned}$$

$$(iii) \ w(z) = \begin{cases} 1/2 + (1/2) \cos(\pi z/5), & |z| < 5 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} |G(s)| &= \left| \int_{-5}^5 \frac{1}{2} e^{-j2\pi z s/\lambda} dz + \int_{-5}^5 \cos\left(\frac{\pi z}{5}\right) e^{-j2\pi z s/\lambda} dz \right| \\ &= \left| \frac{1}{2} \frac{\sin(10\pi \frac{s}{\lambda})}{\pi \frac{s}{\lambda}} + \frac{\sin(5\pi(0.2 - 2\frac{s}{\lambda}))}{2\pi(0.2 - 2\frac{s}{\lambda})} + \frac{\sin(5\pi(0.2 + 2\frac{s}{\lambda}))}{2\pi(0.2 + 2\frac{s}{\lambda})} \right| \end{aligned}$$

(iv) $w(z) = e^{-z^2}$. Using Fourier Transform tables:

$$|G(s)| = \left| \int_{-\infty}^{\infty} e^{-z^2} e^{-j2\pi z \frac{s}{\lambda}} dz \right|$$

$$g(t) \xleftrightarrow{FT} G(s) = \sqrt{\pi} e^{-\frac{2\pi^2 s^2}{4}}$$

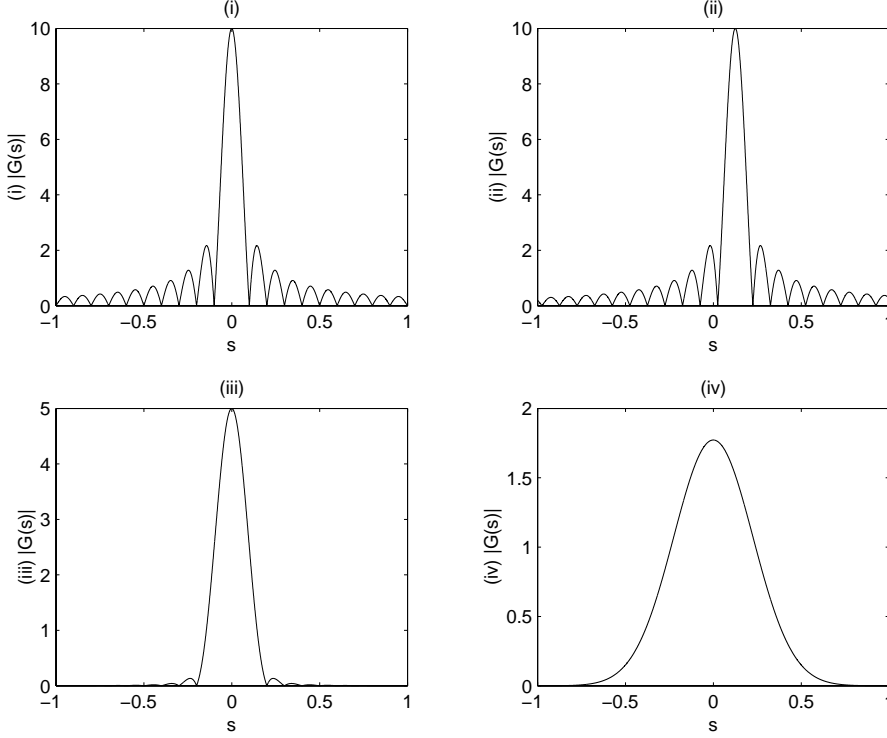


Figure P3.94. Plot 1 of 1

3.95. Figure P3.95 depicts a system known as a beamformer. The output of the beamformer is the weighted sum of signal measured at each antenna in the antenna array. We assume the antenna measures the complex amplitude of propagating plane waves of a single frequency, ω and that the antennas are equally spaced by a distance d along a vertical line. A plane wave $p(t) = e^{j\omega_0 t}$ is shown arriving at the array from direction θ . If the top antenna measures $p(t)$, then the second antenna measures $p(t - \tau(\theta))$ where $\tau(\theta) = (d \sin \theta)/c$ is the time delay required for the plane wavefront to propagate from the top to the second antenna. Since the antennas are equally spaced, the k^{th} antenna measures the signal $p(t - k\tau(\theta))$ and the output of the beamformer is

$$y(t) = \sum_{k=0}^{N-1} w_k p(t - k\tau(\theta))$$

$$= e^{j\omega_0 t} \sum_{k=0}^{N-1} w_k e^{-j\omega_0 k\tau(\theta)}$$

$$= e^{j\omega_0 t} \sum_{k=0}^{N-1} w_k e^{-j(\omega_0 k d \sin \theta)/c}$$

We may interpret this as a complex sinusoidal input from direction θ resulting in a complex sinusoidal output of the same frequency. The beamformer introduces a magnitude and phase change given by the complex number

$$b(\theta) = \sum_{k=0}^{N-1} w_k e^{-j(\omega_0 k d \sin \theta)/c}$$

The gain of the beamformer, $|b(\theta)|$ is termed the beampattern. Note that the gain is a function of the direction of arrival and thus the beamformer offers the potential for discriminating between signals arriving from different directions. For convenience we shall assume that operating frequency and spacing are chosen so that $\omega_0 d/c = \pi$. Assume θ is in the range $-\pi/2 < \theta < \pi/2$.

(a) Compare the expression for the beampattern to the frequency response of a discrete-time system having only N non-zero impulse response coefficients. That is, assume $h[k] = 0$ for $k < 0$ and $k \geq N$.

$$H(e^{j\Omega}) = \sum_{k=0}^{N-1} h[k] e^{-j\Omega k} \quad \text{compared to} \quad b(\theta) = \sum_{k=0}^{N-1} w_k e^{-j(\omega_0 k d \sin \theta)/c}$$

One can see that this is a 1-1 mapping where $\Omega \in (-\pi, \pi)$ is mapped into $\frac{\omega_0 d \sin(\theta)}{c}$, where $\theta \in (-\pi/2, \pi/2)$. For this case Ω is mapped into $\pi \sin(\theta)$ with $-\pi \leq \pi \sin(\theta) \leq \pi$.

(b) Evaluate and plot the beampattern for $N = 2$ with $w_0 = w_1 = 0.5$ and $w_1 = 0.5, w_2 = -0.5$.

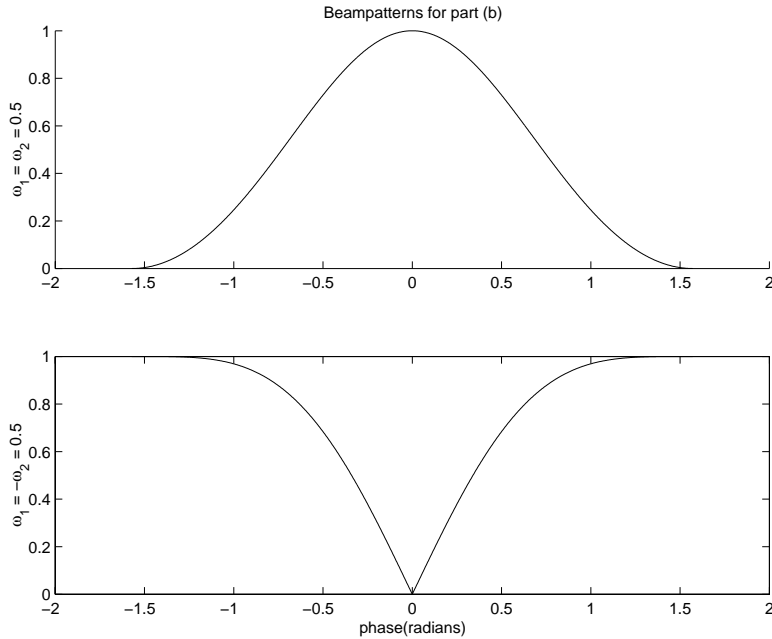


Figure P3.95. Plot 1 of 3

(c) Evaluate and plot the beampattern for $N = 4$ with $w_k = 0.25$, $k = 0, 1, 2, 3$.

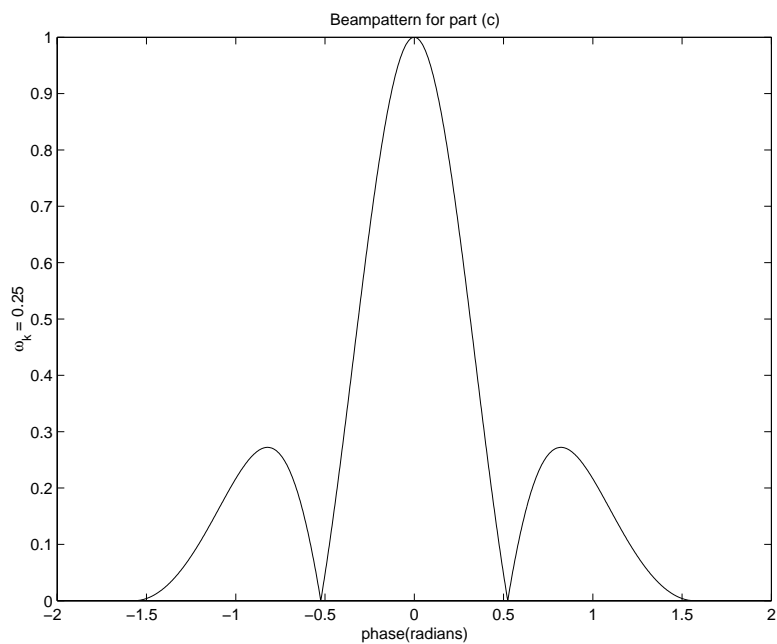


Figure P3.95. Plot 2 of 3

(d) Compare the beampatterns obtained for $N = 8$ with $w_k = 1/8$ $k = 0, 1, \dots, 7$ and $w_k = 1/8 e^{jk\pi/2}$ $k = 0, 1, \dots, 7$.

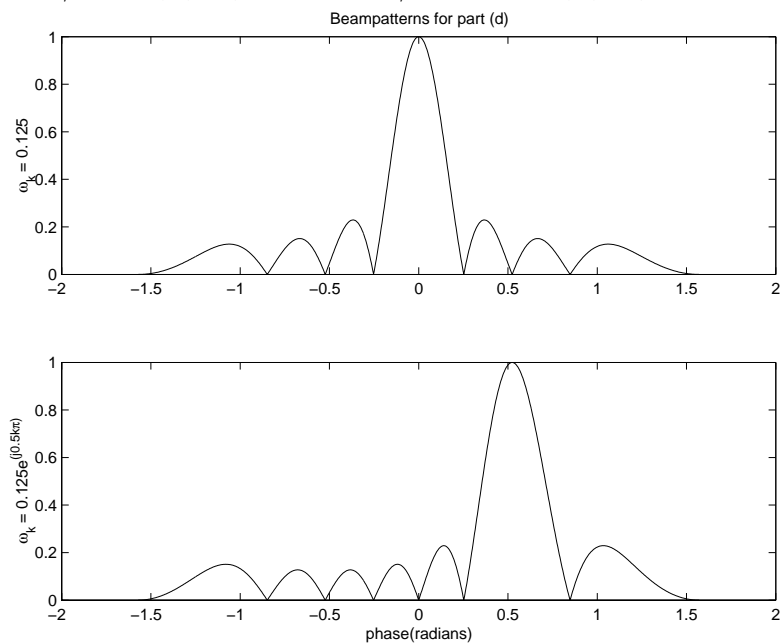


Figure P3.95. Plot 3 of 3

3.96. In Problem 2.97 we determined that the solution to the equations of motion for a vibrating string

have the form $y_k(l, t) = \phi_k(l)f_k(t)$, $0 \leq l \leq a$ where

$$f_k(t) = a_k \cos(\omega_k ct) + b_k \sin(\omega_k ct), \quad \phi_k(l) = \sin(\omega_k l)$$

and $\omega_k = k\pi/a$. Since $y_k(l, t)$ is a solution for any a_k and b_k , the most general solution has the form

$$y(l, t) = \sum_{k=1}^{\infty} \sin(\omega_k l) (a_k \cos(\omega_k ct) + b_k \sin(\omega_k ct))$$

We may use the initial conditions $y(l, 0) = x(l)$ to find the a_k and $\frac{\partial}{\partial t}y(l, t)|_{t=0}$ to find the b_k .

(a) Express the solution for a_k in terms of the FS coefficients of $x(l)$. Hint: consider Eqs.(3.25) and (3.26) with l replacing t .

Use the initial condition that:

$$y(l, 0) = x(l) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi l}{a}\right)$$

This solution is of the form of the Fourier sine series

By applying the principle of orthogonality:

$$a_k = \frac{2}{a} \int_0^a x(l) \sin\left(\frac{n\pi l}{a}\right) dl$$

(b) Express the solution for b_k in terms of the FS coefficients of $g(l)$.

Use the initial condition that:

$$\frac{d}{dt}y(l, t) = g(l) = \sum_{k=1}^{\infty} b_k \frac{k\pi}{a} c \sin\left(\frac{k\pi l}{a}\right)$$

This solution is of the form of the Fourier sine series

By applying the principle of orthogonality:

$$b_k = \frac{2}{k\pi c} \int_0^a g(l) \sin\left(\frac{k\pi l}{a}\right) dl$$

(c) Find $y(l, t)$ assuming $g(l) = 0$ and $x(l)$ is as shown in Fig. P3.96. Then $b_k = 0$ for $n = 1, 2, 3, \dots$ and

$$\begin{aligned} a_k &= \frac{2}{10} \left[\int_0^5 0.1 \frac{2l}{10} \sin\left(\frac{k\pi l}{10}\right) dl + \int_5^{10} 0.1 \left(2 - \frac{2l}{10}\right) \sin\left(\frac{k\pi l}{10}\right) dl \right] \\ &= \frac{8(0.1) \sin(0.5k\pi)}{\pi^2 k^2} \end{aligned}$$

Thus the complete solution is:

$$y(l, t) = \frac{8(0.1)}{\pi^2} \sum_{k=1}^{\infty} \frac{\sin(0.5k\pi)}{k^2} \sin\left(\frac{k\pi l}{10}\right) \cos\left(\frac{k\pi ct}{10}\right)$$

3.97. In this problem we explore a matrix representation for DTFS. The DTFS expresses the N time domain values of an N periodic signal $x[n]$ as a function of N frequency domain values, $X[k]$. Define

vectors

$$\mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

(a) Show that the DTFS representation

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o n}, \quad n = 0, 1, \dots, N-1$$

can be written in matrix vector form as $\mathbf{x} = \mathbf{V}\mathbf{X}$ where \mathbf{V} is an N by N matrix. Find the elements of \mathbf{V} .

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o n}, \quad n = 0, 1, \dots, N-1$$

is equivalent to:

$$x[0] = X[0]e^{j(0)\Omega_o(0)} + \dots + X[N-1]e^{j(N-1)\Omega_o(0)}$$

$$x[1] = X[0]e^{j(0)\Omega_o(1)} + \dots + X[N-1]e^{j(N-1)\Omega_o(1)}$$

.

.

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$$x[N-1] = X[0]e^{j(0)\Omega_o(N-1)} + \dots + X[N-1]e^{j(N-1)\Omega_o(N-1)}$$

can be written as:

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} e^{j(0)\Omega_o(0)} & \dots & e^{j(N-1)\Omega_o(0)} \\ e^{j(0)\Omega_o(1)} & \dots & e^{j(N-1)\Omega_o(1)} \\ \vdots & & \vdots \\ e^{j(0)\Omega_o(N-1)} & \dots & e^{j(N-1)\Omega_o(N-1)} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

which is: $\mathbf{x} = \mathbf{V}\mathbf{X}$, \mathbf{V} is an $N \times N$ matrix

The entries of \mathbf{V} are defined as:

$$[\mathbf{V}]_{r,c} = e^{j(c)\Omega_o(r)}$$

$$\text{where } r(= \text{row}) = 0, 1, \dots, N-1$$

$$\text{where } c(= \text{column}) = 0, 1, \dots, N-1$$

(b) Show that the expression for the DTFS coefficients

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_o n}, \quad k = 0, 1, \dots, N-1$$

can be written in matrix vector form as $\mathbf{X} = \mathbf{W}\mathbf{x}$ where \mathbf{W} is an N by N matrix. Find the elements of \mathbf{W} .

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_o n}, \quad k = 0, 1, \dots, N-1$$

$$\begin{aligned}
X[0] &= \frac{1}{N} \left[x[0]e^{-j(0)\Omega_o(0)} + \dots + x[N-1]e^{-j(N-1)\Omega_o(0)} \right] \\
X[1] &= \frac{1}{N} \left[x[0]e^{-j(0)\Omega_o(1)} + \dots + x[N-1]e^{-j(N-1)\Omega_o(1)} \right] \\
&\vdots \\
X[N-1] &= \frac{1}{N} \left[x[0]e^{-j(0)\Omega_o(N-1)} + \dots + x[N-1]e^{-j(N-1)\Omega_o(N-1)} \right]
\end{aligned}$$

can be written as:

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} e^{-j(0)\frac{\Omega_o}{N}(0)} & \dots & e^{j(N-1)\frac{\Omega_o}{N}(0)} \\ e^{j(0)\frac{\Omega_o}{N}(1)} & \dots & e^{j(N-1)\frac{\Omega_o}{N}(1)} \\ \vdots & & \vdots \\ e^{j(0)\frac{\Omega_o}{N}(N-1)} & \dots & e^{j(N-1)\frac{\Omega_o}{N}(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

which is: $\mathbf{X} = \mathbf{W}\mathbf{x}$, \mathbf{W} is an $N \times N$ matrix

the entries of \mathbf{W} are defined as:

$$\begin{aligned}
[\mathbf{W}]_{r,c} &= \frac{1}{N} e^{-j(c)\Omega_o(r)} \\
\text{where } r(= \text{row}) &= 0, 1, \dots, N-1 \\
\text{where } c(= \text{column}) &= 0, 1, \dots, N-1
\end{aligned}$$

(c) The expression $\mathbf{x} = \mathbf{V}\mathbf{X}$ implies that $\mathbf{X} = \mathbf{V}^{-1}\mathbf{x}$ provided \mathbf{V} is a nonsingular matrix. Comparing this equation to the results of (b) we conclude that $\mathbf{W} = \mathbf{V}^{-1}$. Show that this is true by establishing $\mathbf{W}\mathbf{V} = \mathbf{I}$. *Hint:* Use the definitions of \mathbf{V} and \mathbf{W} determined in (a) and (b) to obtain an expression for the element in the l^{th} row and m^{th} column of $\mathbf{W}\mathbf{V}$ and use the result of Problem 3.85.

$$\begin{aligned}
[\mathbf{W}\mathbf{V}]_{l,m} &= \sum_{k=0}^{N-1} e^{jk\Omega_o l} e^{-jk\Omega_o m} \frac{1}{N}, n = 0, 1, \dots, N-1 \\
&= \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o(l-m)} \\
\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o(r-c)} &= \begin{cases} 1 & l = m \\ \frac{1}{N} \left(\frac{1 - e^{jN\Omega_o(l-m)}}{1 - e^{j\Omega_o(l-m)}} \right) & l \neq m \end{cases}
\end{aligned}$$

Since $N\Omega_o = 2\pi$, we have:

$$[\mathbf{W}\mathbf{V}]_{l,m} = \begin{cases} 1 & l = m \\ 0 & l \neq m \end{cases}$$

which is an identity matrix, therefore:

$$\mathbf{W}\mathbf{V} = \mathbf{I}$$

3.98. We may find the FS coefficients by forming the inner product of the series expansion with the conjugate of the basis functions. Let

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_o t}$$

Derive the expression for $X[k]$ using the result of Problem 3.85 by multiplying both sides of this equation by $e^{-jk\omega_o t}$ and integrating over one period.

$$\begin{aligned}
\int_0^T x(t) e^{-jk\omega_o t} dt &= \int_0^T \left(\sum_{l=-\infty}^{\infty} X[l] e^{jl\omega_o t} \right) e^{-jk\omega_o t} dt \\
&= \sum_{l=-\infty}^{\infty} X[l] \int_0^T e^{j(l-k)\omega_o t} dt \\
&\quad \text{by the orthogonality of sinusoids (Problem 3.85)} \\
&= \begin{cases} TX[k] & l = k \\ 0 & l \neq k \end{cases} \\
&\quad \text{implies} \\
X[k] &= \frac{1}{T} \int_0^T x(t) e^{-jk\omega_o t} dt
\end{aligned}$$

3.99. In this problem we find the FS coefficients $X[k]$ by minimizing the mean squared error (MSE) between the signal $x(t)$ and its FS approximation. Define the J -term FS

$$\hat{x}_J(t) = \sum_{k=-J}^J A[k] e^{jk\omega_o t}$$

and the J term MSE as the average squared difference over one period

$$MSE_J = \frac{1}{T} \int_0^T |x(t) - \hat{x}_J(t)|^2 dt$$

(a) Substitute the series representation for $\hat{x}_J(t)$ and expand the magnitude squared using the identity $|a + b|^2 = (a + b)(a^* + b^*)$ to obtain

$$\begin{aligned}
MSE_J &= \frac{1}{T} \int_0^T |x(t)|^2 dt - \sum_{k=-J}^J A^*[k] \left(\frac{1}{T} \int_0^T x(t) e^{-jk\omega_o t} dt \right) - \\
&\quad \sum_{k=-J}^J A[k] \left(\frac{1}{T} \int_0^T x^*(t) e^{jk\omega_o t} dt \right) + \sum_{m=-J}^J \sum_{k=-J}^J A^*[k] A[m] \left(\frac{1}{T} \int_0^T e^{-jk\omega_o t} e^{jm\omega_o t} dt \right)
\end{aligned}$$

$$\begin{aligned}
|x(t) - \hat{x}_J(t)|^2 &= (x(t) - \hat{x}_J(t))(x^*(t) - \hat{x}_J^*(t)) \\
&= |x(t)|^2 - x(t)\hat{x}_J^*(t) - \hat{x}_J(t)x^*(t) + \hat{x}_J(t)\hat{x}_J^*(t) \\
\hat{x}_J^*(t) &= \sum_{k=-J}^J A^*[k] e^{-jk\omega_o t}
\end{aligned}$$

Thus:

$$\begin{aligned}
MSE &= \frac{1}{T} \int_0^T |x(t)|^2 dt - \sum_{k=-J}^J A^*[k] \left(\frac{1}{T} \int_0^T x(t) e^{-jk\omega_o t} dt \right) - \sum_{k=-J}^J A[k] \left(\frac{1}{T} \int_0^T x^*(t) e^{jk\omega_o t} dt \right) \\
&\quad + \sum_{m=-J}^J \sum_{k=-J}^J A^*[k] A[m] \left(\frac{1}{T} \int_0^T e^{-jk\omega_o t} e^{jm\omega_o t} dt \right)
\end{aligned}$$

(b) Define

$$X[k] = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_o t} dt$$

and use the orthogonality of $e^{jk\omega_o t}$ and $e^{jm\omega_o t}$ (See Problem 3.85) to show that

$$MSE_J = \frac{1}{T} \int_0^T |x(t)|^2 dt - \sum_{k=-J}^J A^*[k]X[k] - \sum_{k=-J}^J A[k]X^*[k] + \sum_{k=-J}^J |A[k]|^2$$

Substitute

$$A[k] = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_o t} dt$$

To obtain

$$\begin{aligned} MSE_J &= \frac{1}{T} \int_0^T |x(t)|^2 dt - \sum_{k=-J}^J A^*[k]X[k] - \sum_{k=-J}^J A[k]X^*[k] \\ &\quad + \sum_{k=-J}^J \sum_{m=-J}^J A^*[k]A[m] \left(\frac{1}{T} \int_0^T e^{-j(k-m)\omega_o t} dt \right) \end{aligned}$$

Now

$$\begin{aligned} \int_0^T e^{-j(k-m)\omega_o t} dt &= \begin{cases} T & k = m \\ 0 & k \neq m \end{cases} \\ &= T\delta_{m,k} \end{aligned}$$

And thus

$$\begin{aligned} \sum_{k=-J}^J \sum_{m=-J}^J A^*[k]A[m]\delta_{m,k} &= \sum_{k=-J}^J A^*[k]A[k] \\ &= \sum_{k=-J}^J |A[k]|^2 \end{aligned}$$

Thus

$$MSE_J = \frac{1}{T} \int_0^T |x(t)|^2 dt - \sum_{k=-J}^J A^*[k]X[k] - \sum_{k=-J}^J A[k]X^*[k] + \sum_{k=-J}^J |A[k]|^2$$

(c) Use the technique of completing the square to show that

$$MSE_J = \frac{1}{T} \int_0^T |x(t)|^2 dt - \sum_{k=-J}^J |A[k] - X[k]|^2 - \sum_{k=-J}^J |X[k]|^2$$

Note that

$$- \sum_{k=-J}^J A^*[k]X[k] - \sum_{k=-J}^J A[k]X^*[k] = \sum_{k=-J}^J |X[k] - A[k]|^2 - \sum_{k=-J}^J |X[k]|^2 - \sum_{k=-J}^J |A[k]|^2$$

Thus we can re-write the result in (b) as:

$$MSE_J = \frac{1}{T} \int_0^T |x(t)|^2 dt + \sum_{k=-J}^J |X[k] - A[k]|^2 - \sum_{k=-J}^J |A[k]|^2$$

(d) Find the value for $A[k]$ that minimizes MSE_J .

MSE_J is minimized when the sum at the middle vanishes, i.e., when $A[k] = X[k] = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_o t} dt$.

(e) Express the minimum MSE_J as a function of $x(t)$ and $X[k]$. What happens to MSE_J as J increases?

$$\min MSE_J = \frac{1}{T} \int_0^T |x(t)|^2 dt - \sum_{k=-J}^J |X[k]|^2$$

As J increases: $|X[k]|^2$ is always greater than or equal to zero, so the sum

$$\sum_{k=-J}^J |X[k]|^2 > 0$$

we also have $\frac{1}{T} \int_0^T |x(t)|^2 dt > 0$

As J increases: , $\sum_{k=-J}^J |X[k]|^2$ increases or remains constant and $\min MSE_J$ decreases.

3.100. Generalized Fourier Series. The concept of the Fourier Series may be generalized to sums of signals other than complex sinusoids. That is, given a signal $x(t)$ we may approximate $x(t)$ on an interval $[t_1, t_2]$ as a weighted sum of N functions $\phi_0(t), \phi_2(t), \dots, \phi_{N-1}(t)$

$$x(t) \approx \sum_{k=0}^{N-1} c_k \phi_k(t)$$

We shall assume that these N functions are mutually orthogonal on $[t_1, t_2]$. This means that

$$\int_{t_1}^{t_2} \phi_k(t) \phi_l^*(t) dt = \begin{cases} 0, & k \neq l, \\ f_k, & k = l \end{cases}$$

The MSE in using this approximation is

$$MSE = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left| x(t) - \sum_{k=1}^N c_k \phi_k(t) \right|^2 dt$$

(a) Show that the MSE is minimized by choosing $c_k = \frac{1}{f_k} \int_{t_1}^{t_2} x(t) \phi_k^*(t) dt$. *Hint:* Generalize steps outlined in Problem 3.99 (a) - (d) to this problem.

$$\begin{aligned} \left| x(t) - \sum_{k=1}^N c_k \phi_k(t) \right|^2 &= |x(t)|^2 - x(t) \sum_{k=0}^{N-1} c_k^* \phi_k^*(t) - x^*(t) \sum_{k=0}^N c_k \phi_k(t) \\ &\quad + \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} c_k c_m^* \phi_k(t) \phi_m^*(t) \end{aligned}$$

Thus:

$$\begin{aligned} MSE &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt - \sum_{k=0}^{N-1} c_k^* \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x(t) \phi_k^*(t) dt \right) \\ &\quad - \sum_{k=0}^{N-1} c_k \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x^*(t) \phi_k(t) dt \right) + \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} c_k c_m^* \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \phi_k(t) \phi_m^*(t) dt \right) \end{aligned}$$

$$\text{Let } A[k] = \frac{1}{f_k} \int_{t_1}^{t_2} x(t) \phi_k^*(t) dt \text{ and use orthogonality}$$

$$\begin{aligned}
MSE &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt - \frac{f_k}{t_2 - t_1} \sum_{k=0}^{N-1} c_k^* A[k] \\
&\quad - \frac{f_k}{t_2 - t_1} \sum_{k=0}^{N-1} c_k A^*[k] + \frac{f_k}{t_2 - t_1} \sum_{k=0}^{N-1} |c_k|^2 \\
&= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt + \frac{f_k}{t_2 - t_1} \sum_{k=0}^{N-1} |c_k - A[k]|^2 - \frac{f_k}{t_2 - t_1} \sum_{k=0}^{N-1} |A[k]|^2
\end{aligned}$$

Analogous to Problem 3.99(d), $c_k = A[k] = \frac{1}{f_k} \int_{t_1}^{t_2} x(t) \phi_k^*(t) dt$

(b) Show that the MSE is zero if

$$\int_{t_1}^{t_2} |x(t)|^2 dt = \sum_{k=0}^{N-1} f_k |c_k|^2$$

If this relationship holds for all $x(t)$ in a given class of functions, then the basis functions $\phi_0(t), \phi_1(t), \dots, \phi_{N-1}(t)$ is said to be “complete” for that class.

$$\begin{aligned}
\min MSE &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt - \frac{f_k}{t_2 - t_1} \sum_{k=0}^{N-1} |c_k|^2 \\
&= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} |x(t)|^2 dt - \sum_{k=0}^{N-1} f_k |c_k|^2 \right] \\
\min MSE &= 0 \text{ when:} \\
\int_{t_1}^{t_2} |x(t)|^2 dt &= \sum_{k=0}^{N-1} f_k |c_k|^2
\end{aligned}$$

(c) The Walsh functions are one set of orthogonal functions that are used for signal representation on $[0, 1]$. Determine the c_k and MSE obtained by approximating the following signals with the first six Walsh functions depicted in Fig. P3.100. Sketch the signal and the Walsh function approximation.

(i) $x(t) = \begin{cases} 2, & \frac{1}{2} < t < \frac{3}{4} \\ 0, & 0 < t < \frac{1}{2}, \frac{3}{4} < t < 1 \end{cases}$

We can see that the orthogonality relation for the Walsh function is as follows:

$$\begin{aligned}
\int_0^1 \phi_k(t) \phi_l^*(t) &= \begin{cases} 1 & k = l \\ 0 & k \neq l, \text{ so } f_k = 1 \end{cases} \\
c_0 &= \int_{0.5}^{0.75} 2 dt = \frac{1}{2} \\
c_1 &= - \int_{0.5}^{0.75} 2 dt = -\frac{1}{2} \\
c_2 &= - \int_{0.5}^{0.75} 2 dt = -\frac{1}{2} \\
c_3 &= 2 \frac{1}{4} = \frac{1}{2} \\
c_4 &= 2(0) = 0
\end{aligned}$$

$$c_5 = 2(0) = 0$$

Let

$$\hat{x}(t) = \sum_{k=0}^5 c_k \phi_k(t)$$

$$\hat{x}(t) = \frac{1}{2} (\phi_0(t) - \phi_1(t) - \phi_2(t) + \phi_3(t))$$

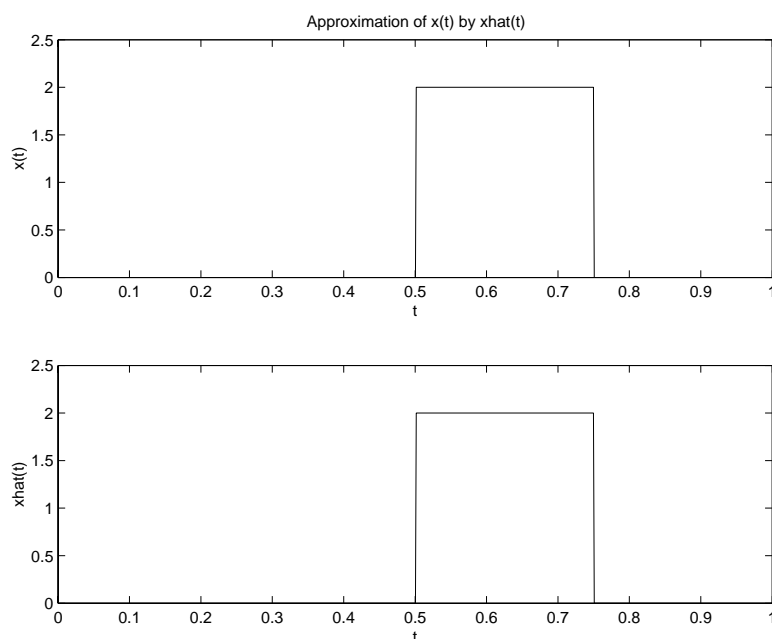


Figure P3.100. Plot 1 of 2

From the sketch, we can see that $x(t) = \hat{x}(t)$, so $MSE = 0$.

(ii) $x(t) = \sin(2\pi t)$

$$c_0 = \int_0^1 \sin(2\pi t) dt = 0$$

$$c_1 = \int_0^{\frac{1}{2}} \sin(2\pi t) dt - \int_{\frac{1}{2}}^1 \sin(2\pi t) dt = \frac{2}{\pi}$$

$$c_2 = \int_0^1 \phi_2(t) \sin(2\pi t) dt = \frac{1}{\pi}$$

$$c_3 = 0, \text{ by inspection.}$$

$$c_4 = 0, \text{ by inspection.}$$

$$\begin{aligned} c_5 &= 2 \left[\int_0^{\frac{1}{8}} \sin(2\pi t) dt - \int_{\frac{1}{8}}^{\frac{3}{8}} \sin(2\pi t) dt + \int_{\frac{3}{8}}^{\frac{1}{2}} \sin(2\pi t) dt \right] \\ &= \frac{2}{\pi} (1 - \sqrt{2}) \approx -\frac{1}{\pi} (0.83) \end{aligned}$$

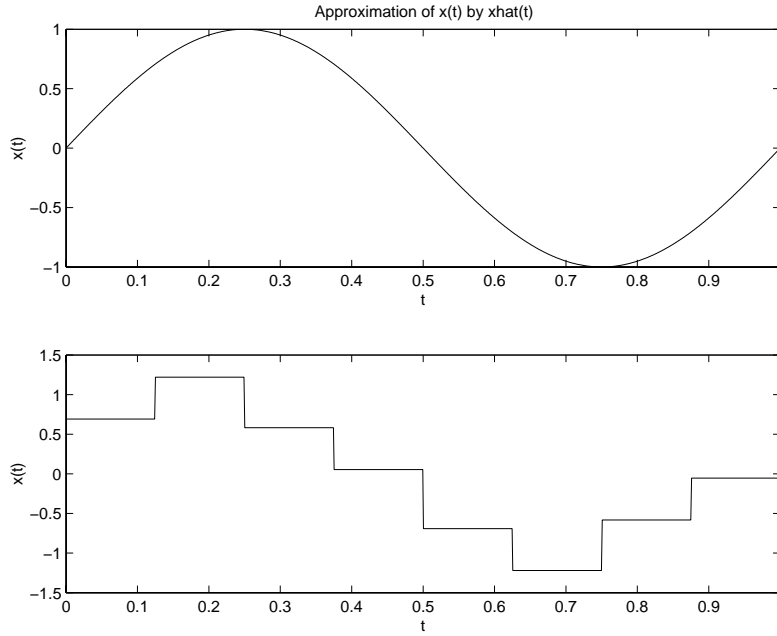


Figure P3.100. Plot 2 of 2

$$\begin{aligned}
 MSE &= \int_0^1 |\sin(2\pi t) - \hat{x}(t)|^2 dt \\
 &= 2 \int_0^{\frac{1}{8}} \left| \sin(2\pi t) - \frac{2.17}{\pi} \right|^2 dt + 2 \int_{\frac{1}{8}}^{\frac{2}{8}} \left| \sin(2\pi t) - \frac{3.83}{\pi} \right|^2 dt + 2 \int_{\frac{2}{8}}^{\frac{3}{8}} \left| \sin(2\pi t) - \frac{1.83}{\pi} \right|^2 dt \\
 &\quad + 2 \int_{\frac{3}{8}}^{\frac{1}{2}} \left| \sin(2\pi t) - \frac{0.17}{\pi} \right|^2 dt \\
 &\approx 0.1265
 \end{aligned}$$

(d) The Legendre polynomials are another set of orthogonal functions on the interval $[-1, 1]$. They are obtained from the difference equation

$$\phi_k(t) = \frac{2k-1}{k} t \phi_{k-1}(t) - \frac{k-1}{k} \phi_{k-2}(t)$$

using the initial functions $\phi_0(t) = 1$ and $\phi_1(t) = t$. Determine the c_k and MSE obtained by approximating the following signals with the first six Legendre polynomials.

$$\begin{aligned}
 \phi_0(t) &= 1 \\
 \phi_1(t) &= t \\
 \phi_2(t) &= \frac{3}{2}t(t) - \frac{1}{2}(1) = \frac{1}{2}(3t^2 - 1) \\
 \phi_3(t) &= \frac{5}{3}t\frac{1}{2}(3t^2 - 1) - \frac{2}{3}t = \frac{1}{2}(5t^3 - 3t) \\
 \phi_4(t) &= \frac{7}{4}t\frac{1}{2}(5t^3 - 3t) - \frac{3}{4}\frac{1}{2}(3t^2 - 1) = \frac{1}{8}(35t^4 - 30t^2 + 3) \\
 \phi_5(t) &= \frac{9}{5}t\frac{1}{8}(35t^4 - 30t^2 + 3) - \frac{4}{5}\frac{1}{2}(5t^3 - 3t) = \frac{1}{40}(315t^5 - 350t^3 + 75t)
 \end{aligned}$$

The orthogonality relation for Legendre polynomials is:

$$\int_{-1}^1 \phi_k(t) \phi_l^*(t) dt = \delta_{k,l} \frac{2}{2k+1}$$

so: $f_k = \frac{2}{2k+1}$

$$(i) \ x(t) = \begin{cases} 2, & 0 < t < \frac{1}{2} \\ 0, & -1 < t < 0, \frac{1}{2} < t < 1 \end{cases}$$

$$c_0 = \frac{1}{2} \int_0^{\frac{1}{2}} 2(1) dt = \frac{1}{2}$$

$$c_1 = \frac{3}{2} \int_0^{\frac{1}{2}} 2t dt = \frac{3}{8}$$

$$c_2 = \frac{5}{2} \int_0^{\frac{1}{2}} (3t^2 - 1) dt = -\frac{15}{16}$$

$$c_3 = \frac{7}{2} \int_0^{\frac{1}{2}} (5t^3 - 3t) dt = -1.039$$

$$c_4 = \frac{9}{2} \int_0^{\frac{1}{2}} \frac{1}{4} (35t^4 - 30t^2 + 3) dt = 0.527$$

$$c_5 = \frac{11}{2} \int_0^{\frac{1}{2}} \frac{1}{20} (315t^5 - 350t^3 + 75t) dt = 1.300$$

$$\hat{x}(t) = \sum_{k=0}^5 c_k \phi_k(t)$$

$$\begin{aligned} MSE &= \frac{1}{2} \left[\int_1^{\frac{1}{2}} 4 dt - \sum_{k=0}^5 \frac{2}{2k+1} |c_k|^2 \right] \\ &= 0.1886 \end{aligned}$$

(ii) $x(t) = \sin(\pi t)$

$$c_0 = \frac{1}{2} \int_{-1}^1 \sin(\pi t) dt = 0$$

$$c_1 = \frac{3}{2} \int_{-1}^1 t \sin(\pi t) dt = 0.955$$

$$c_2 = \frac{5}{2} \int_0^1 (3t^2 - 1) \sin(\pi t) dt = 0$$

$$c_3 = \frac{7}{2} \int_0^1 (5t^3 - 3t) \sin(\pi t) dt = -1.158$$

$$c_4 = \frac{9}{2} \int_0^1 \frac{1}{4} (35t^4 - 30t^2 + 3) \sin(\pi t) dt = 0$$

$$c_5 = \frac{11}{2} \int_0^1 \frac{1}{20} (315t^5 - 350t^3 + 75t) \sin(\pi t) dt = 0.213$$

$$\begin{aligned}
\hat{x}(t) &= \sum_{k=0}^5 c_k \phi_k(t) \\
MSE &= \frac{1}{2} \left[\int_{-1}^1 |\sin(\pi t)|^2 dt - \sum_{k=0}^5 \frac{2}{2k+1} |c_k|^2 \right] \\
&= 3 * 10^{-4}
\end{aligned}$$

3.101. We may derive the FT from the FS by describing an nonperiodic signal as the limiting form of a periodic signal whose period, T , approaches infinity. In order to take this approach, we assume that the FS of the periodic version of the signal exists, that the nonperiodic signal is zero for $|t| > \frac{T}{2}$, and that the limit as T approaches infinity is taken in a symmetric manner. Define the finite duration nonperiodic signal $x(t)$ as one period of the T periodic signal $\tilde{x}(t)$

$$x(t) = \begin{cases} \tilde{x}(t), & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & |t| > \frac{T}{2} \end{cases}$$

(a) Graph an example of $x(t)$ and $\tilde{x}(t)$ to demonstrate that as T increases, the periodic replicates of $x(t)$ in $\tilde{x}(t)$ are moved farther and farther away from the origin. Eventually, as T approaches infinity, these replicates are removed to infinity. Thus, we write

$$x(t) = \lim_{T \rightarrow \infty} \tilde{x}(t)$$

Notice how as $T \rightarrow \infty$, $x(t) \rightarrow \tilde{x}(t)$

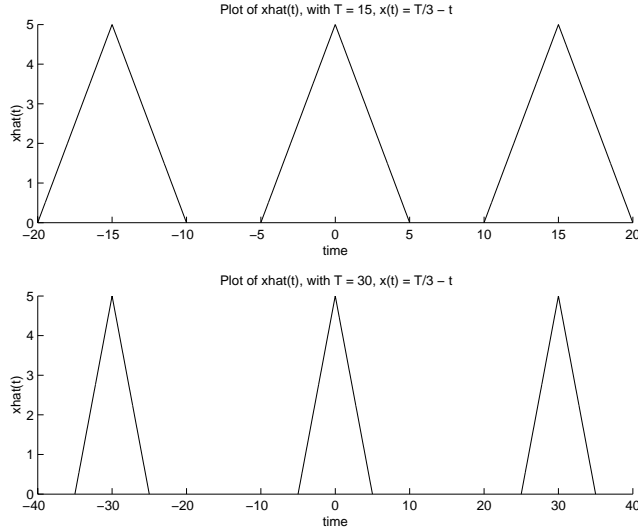


Figure P3.101. Plot 1 of 1

(b) The FS representation for the periodic signal $\tilde{x}(t)$ is

$$\begin{aligned}
\tilde{x}(t) &= \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_o t} \\
X[k] &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t) e^{-jk\omega_o t} dt
\end{aligned}$$

Show that $X[k] = \frac{1}{T}X(jk\omega_o)$ where

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t} dt$$

$$X[k] = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\omega_o t} dt$$

$$\text{Since } \tilde{x}(t) = x(t) \text{ for } -\frac{T}{2} < t < \frac{T}{2}$$

$$X[k] = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_o t} dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} x(t)e^{-jk\omega_o t} dt$$

$$\text{Since } x(t) = 0 \text{ for } |t| > \frac{T}{2}$$

Compare with the equation of the FT, and one can see,

$$X[k] = \frac{1}{T}X(jk\omega_o)$$

(c) Substitute this definition for $X[k]$ into the expression for $\tilde{x}(t)$ in (b) and show that

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_o)e^{jk\omega_o t}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_o)e^{jk\omega_o t}$$

$$\text{since } T = \frac{2\pi}{\omega_o}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_o)e^{jk\omega_o t}$$

(d) Use the limiting expression for $x(t)$ in (a) and define $\omega \approx k\omega_o$ to express the limiting form of the sum in (c) as the integral

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$$

$$x(t) = \lim_{T \rightarrow \infty} \tilde{x}(t)$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_o)e^{jk\omega_o t}$$

$$\text{let } \omega = k\omega_o$$

$$\text{as } T \rightarrow \infty, \omega_o \rightarrow 0, k \rightarrow \infty, \omega \rightarrow \infty$$

implies:

$$x(t) = \lim_{\omega_o \rightarrow 0} \frac{1}{2\pi} \sum_{\omega=-\infty}^{\infty} X(j\omega)e^{j\omega t}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$$

Solutions to Computer Experiments

3.102. Use MATLAB to repeat Example 3.7 for $N = 50$ and (a) $M = 12$. (b) $M = 5$. (c) $M = 20$.

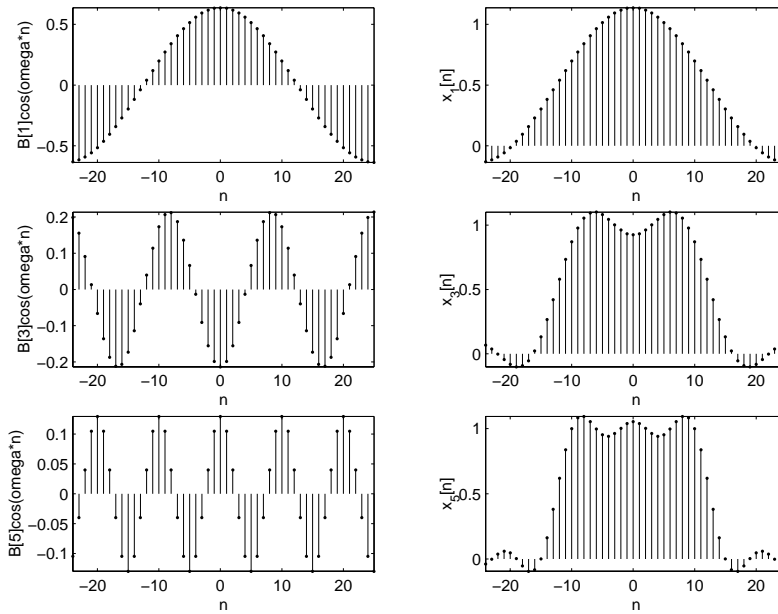


Figure P3.102. Plot 1 of 6

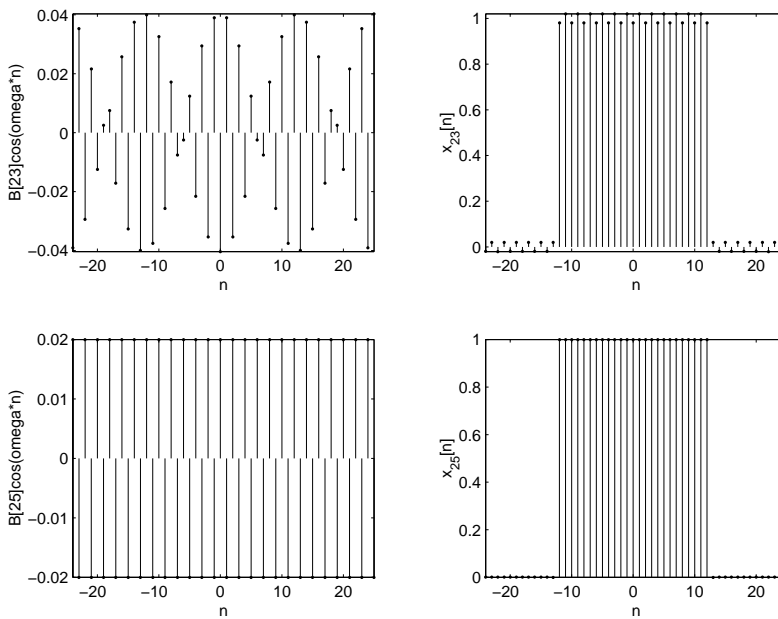


Figure P3.102. Plot 2 of 6

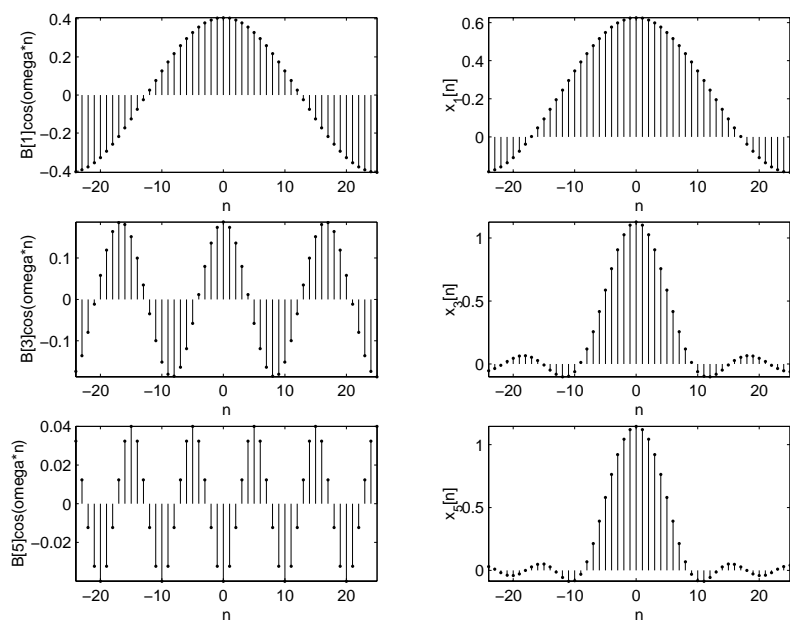


Figure P3.102. Plot 3 of 6

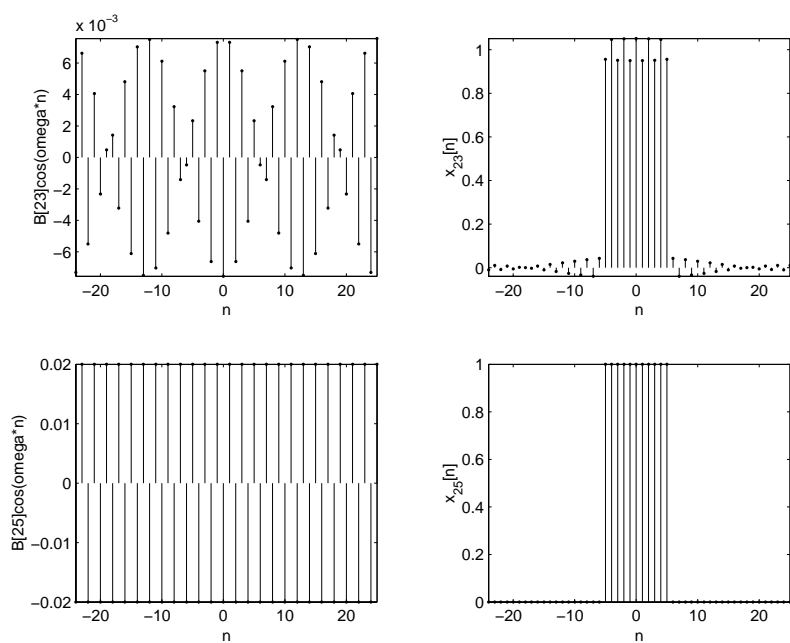


Figure P3.102. Plot 4 of 6

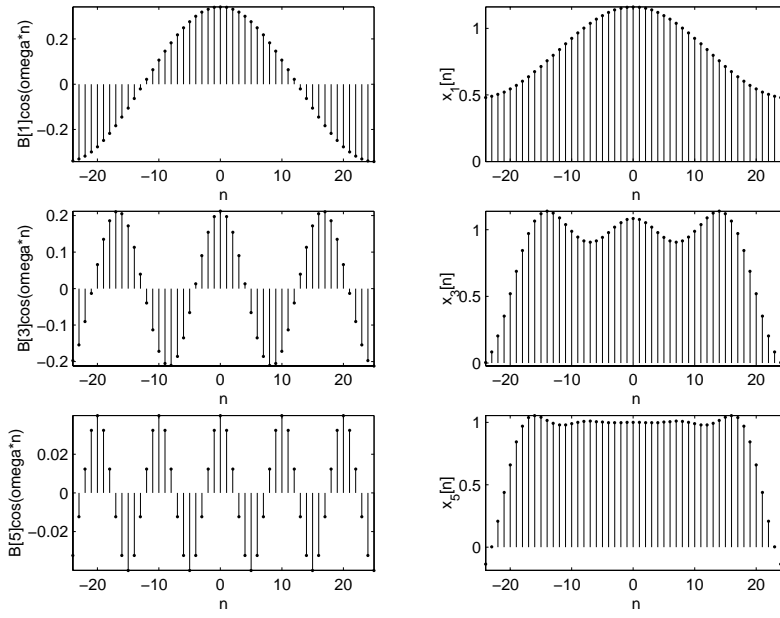


Figure P3.102. Plot 5 of 6

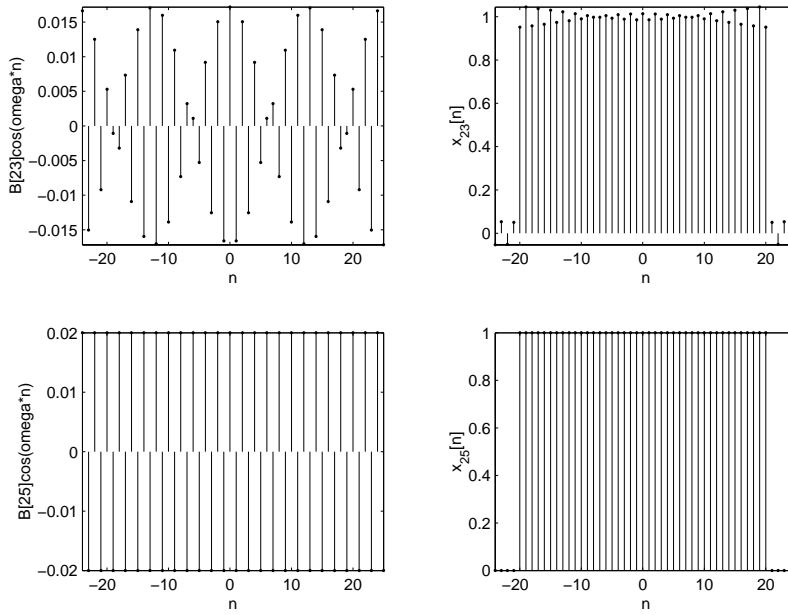


Figure P3.102. Plot 6 of 6

3.103. Use MATLAB's `fft` command to repeat Problem 3.48.

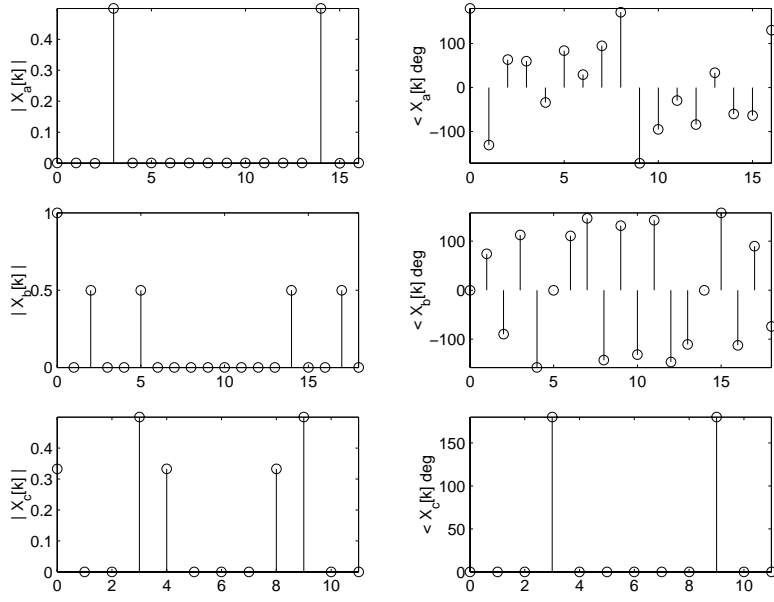


Figure P3.103. Plot 1 of 2

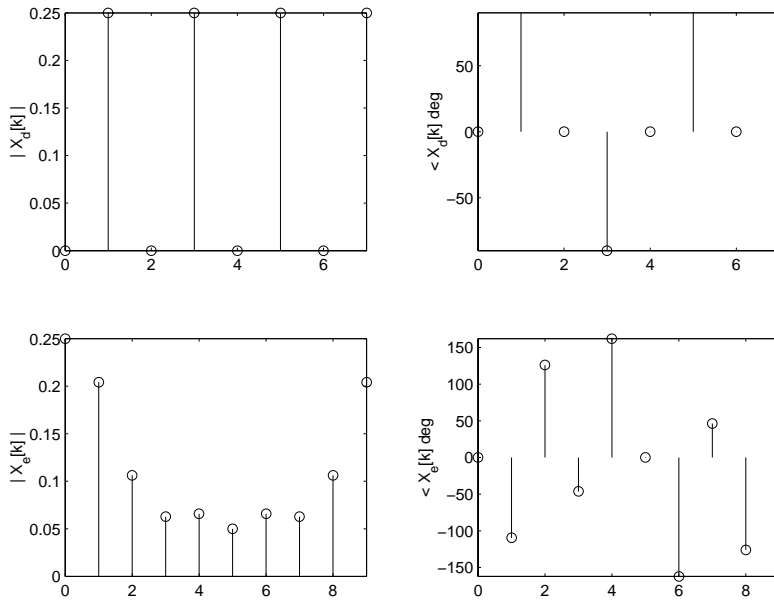


Figure P3.103. Plot 2 of 2

3.104. Use MATLAB's `ifft` command to repeat Problem 3.2.

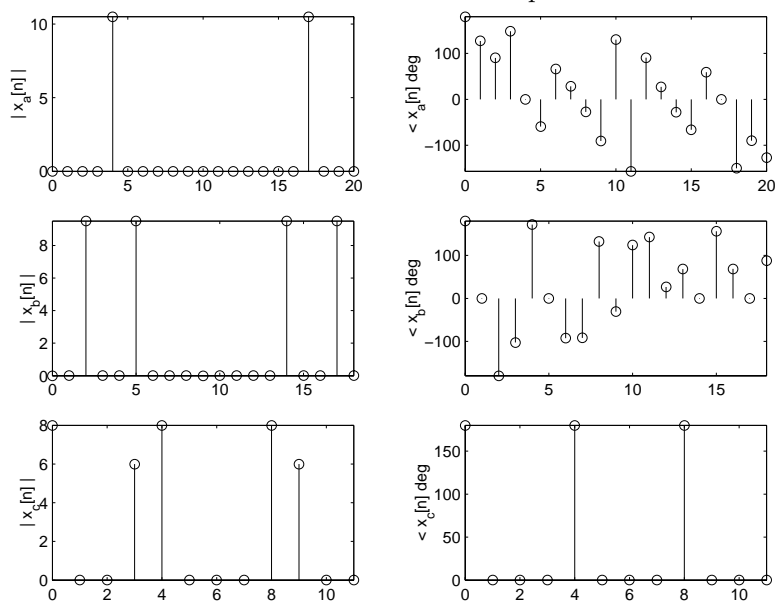


Figure P3.104. Plot 1 of 2

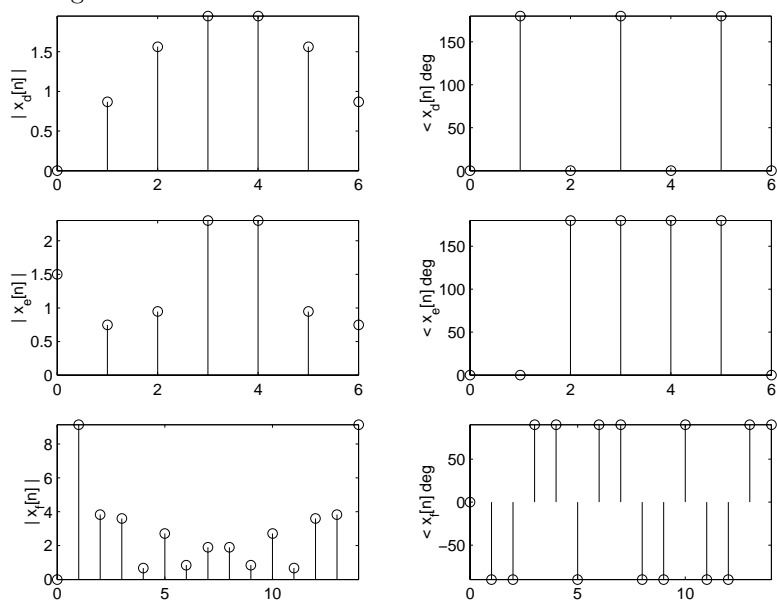


Figure P3.104. Plot 2 of 2

3.105. Use MATLAB's `fft` command to repeat Example 3.8.

Simply take the `fft` of the samples of one period of the time waveform and scale by the number of samples used.

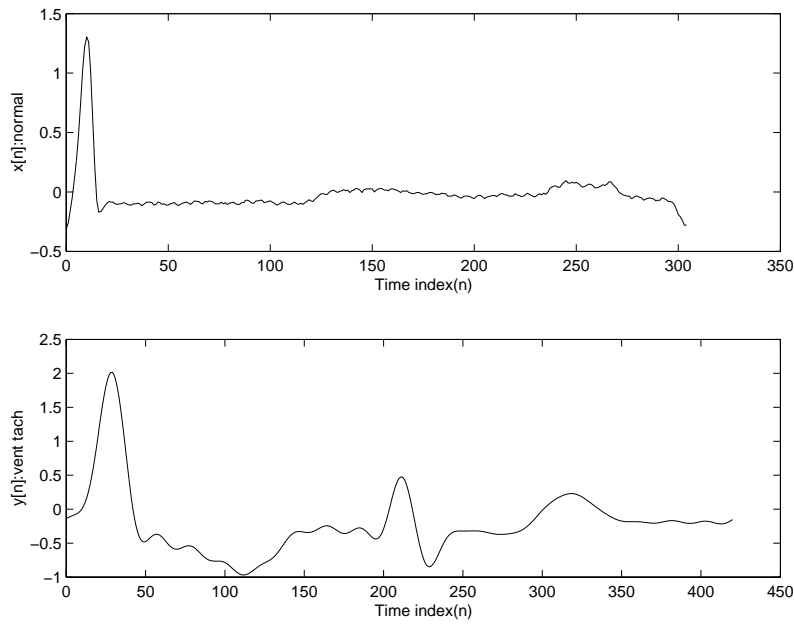


Figure P3.105. Plot 1 of 2

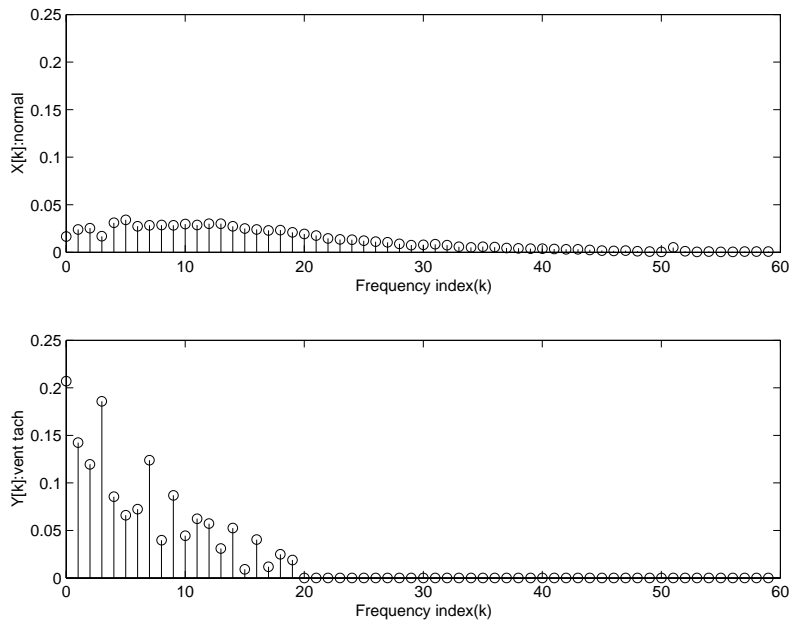


Figure P3.105. Plot 2 of 2

3.106. Use MATLAB to repeat Example 3.14. Evaluate the peak overshoot for $J = 29, 59$, and 99 . The peak overshoot for $J = 29$ is 0.0891 above 1.

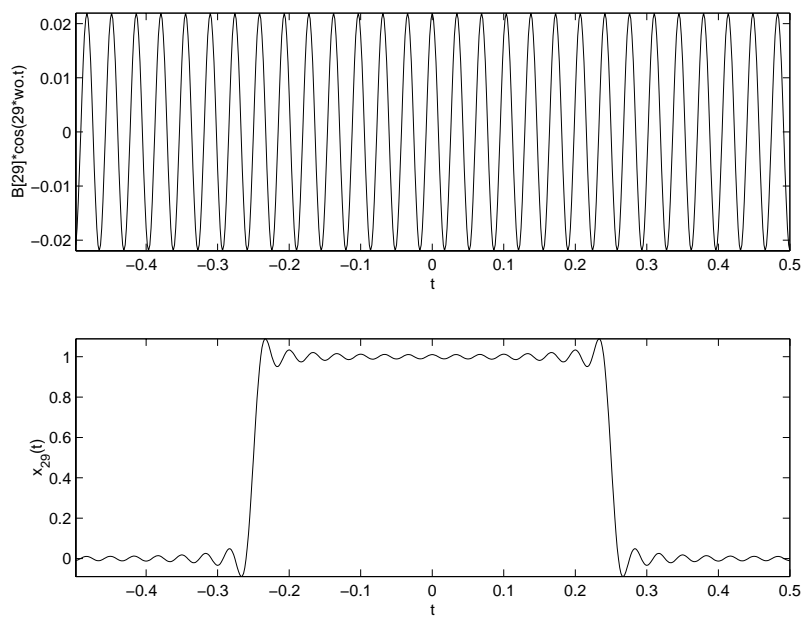


Figure P3.106. Plot 1 of 3

The peak overshoot for $J = 59$ is 0.0894 above 1.

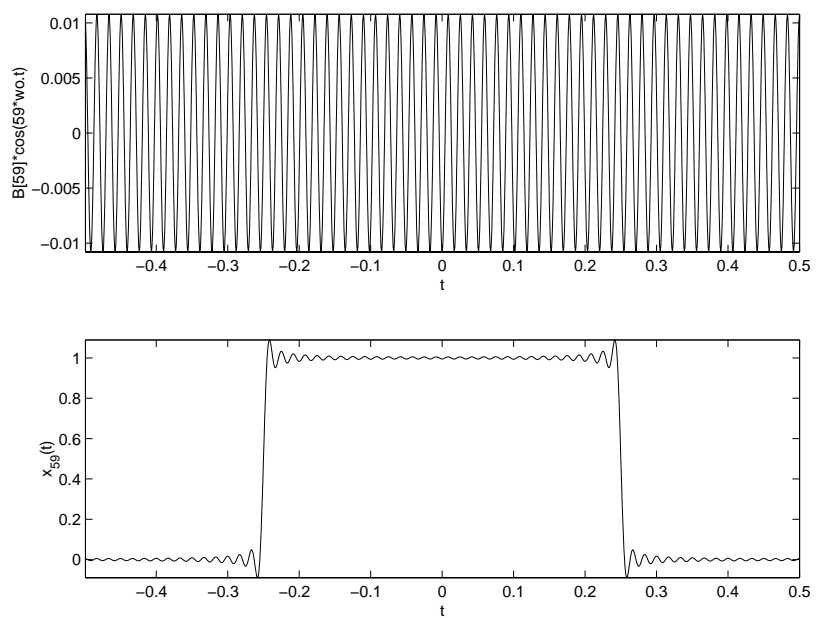


Figure P3.106. Plot 2 of 3

The peak overshoot for $J = 99$ is 0.0895 above 1.

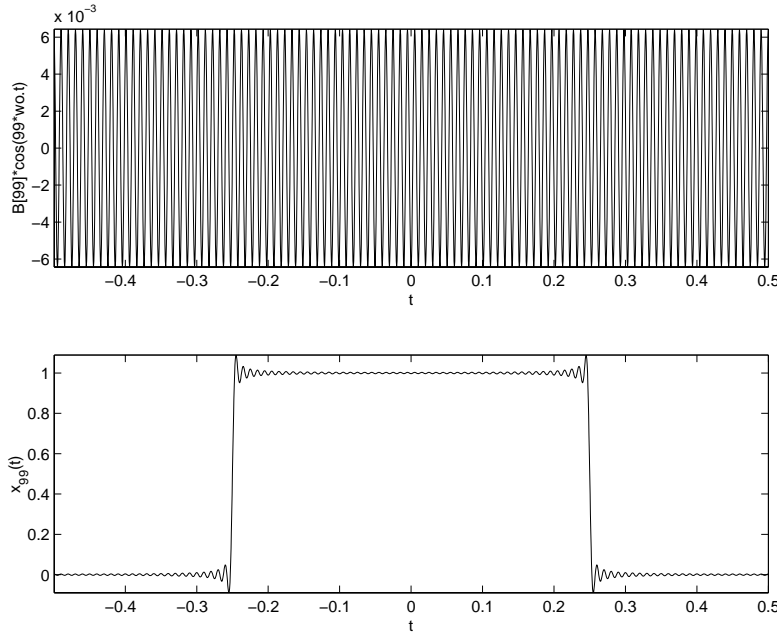


Figure P3.106. Plot 3 of 3

3.107. Let $x(t)$ be the triangular wave depicted in Fig. P3.107.

(a) Find the FS coefficients, $X[k]$.

$$X[k] = \frac{1}{T} \int_0^T x(t) e^{-j \frac{2\pi}{T} kt} dt, \quad T = 1$$

From Example 3.39 with $T = 1$, $x(t) = 2y(t + \frac{1}{4}) - 1$

$$\begin{aligned} Y[k] &= \begin{cases} \frac{1}{2}, & k = 0, \\ \frac{2 \sin(\frac{k\pi}{2})}{jk^2\pi^2}, & k \neq 0 \end{cases} \\ x(t - t_0) &\xleftrightarrow{FS; \omega_o} e^{-jk\omega_o t} X[k] \\ X[k] &= 2e^{jk2\pi\frac{1}{4}} Y[k] - \delta[k] \\ &\text{implies:} \\ X[k] &= \begin{cases} -\frac{1}{2}, & k = 0, \\ \frac{4 \sin(\frac{k\pi}{2})}{k^2\pi^2} e^{j\frac{\pi}{2}(k-1)}, & k \neq 0 \end{cases} \end{aligned}$$

(b) Show that the FS representation for $x(t)$ can be expressed in the form

$$x(t) = \sum_{k=0}^{\infty} B[k] \cos(k\omega_o t)$$

since $x(t)$ is real and even, $X[k]$ is also real and even with $X[k] = X[-k]$

$$\begin{aligned}
x(t) &= \sum_{k=1}^{\infty} X[k] e^{j\omega_o k t} + \sum_{k=-\infty}^{-1} X[k] e^{j\omega_o k t} + X[0] \\
&= \sum_{k=1}^{\infty} X[k] e^{j\omega_o k t} + \sum_{k=1}^{\infty} X[-k] e^{-j\omega_o k t} + X[0] \\
&\quad \text{since } X[k] = X[-k] \\
&= \sum_{k=1}^{\infty} X[k] (e^{j\omega_o k t} + e^{-j\omega_o k t}) + X[0] \\
&= \sum_{k=0}^{\infty} B[k] \cos(k\omega_o t) \\
&\quad \text{where} \\
B[k] &= \begin{cases} X[0], & k = 0, \\ 2X[k], & k \neq 0 \end{cases}
\end{aligned}$$

(c) Define the J term partial sum approximation to $x(t)$ as

$$\hat{x}_J(t) = \sum_{k=0}^J B[k] \cos(k\omega_o t)$$

Use MATLAB to evaluate and plot one period of the J^{th} term in this sum and $\hat{x}_J(t)$ for $J = 1, 3, 7, 29$, and 99.

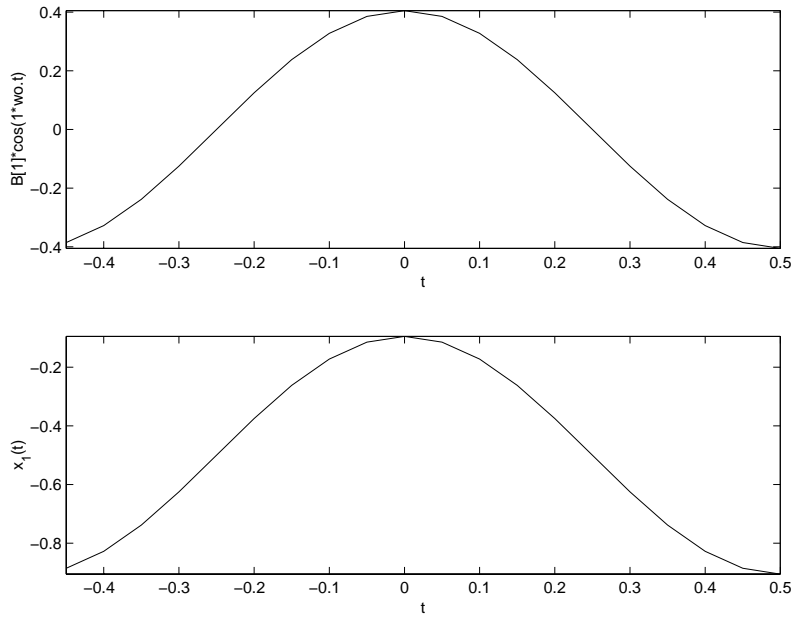


Figure P3.107. Plot 1 of 5

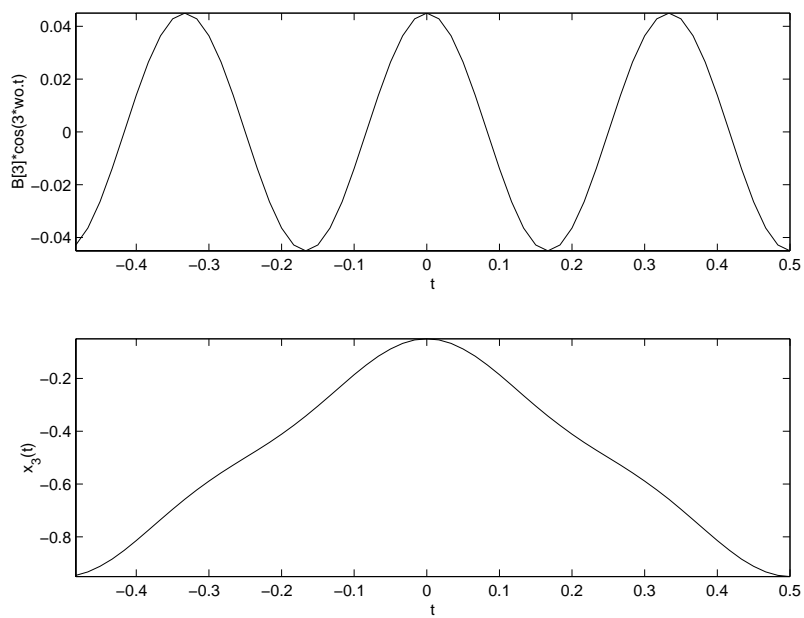


Figure P3.107. Plot 2 of 5

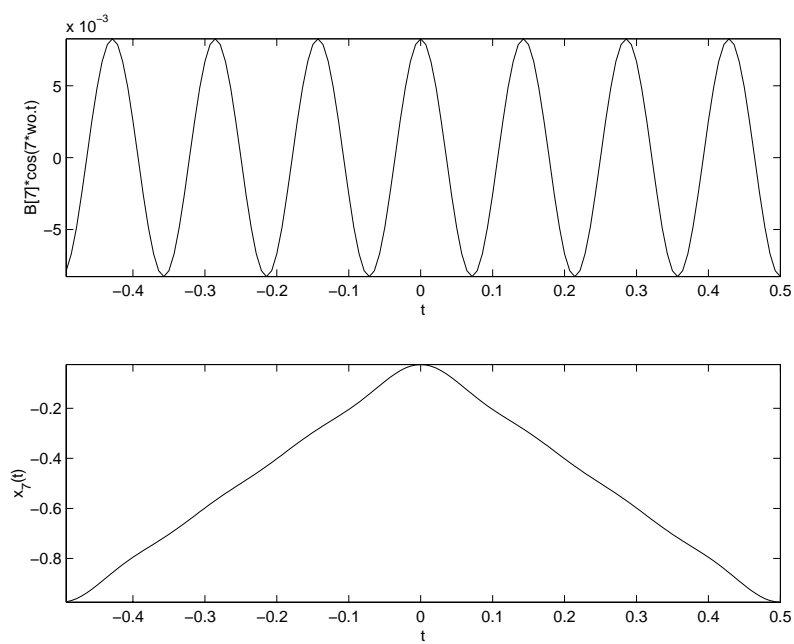


Figure P3.107. Plot 3 of 5

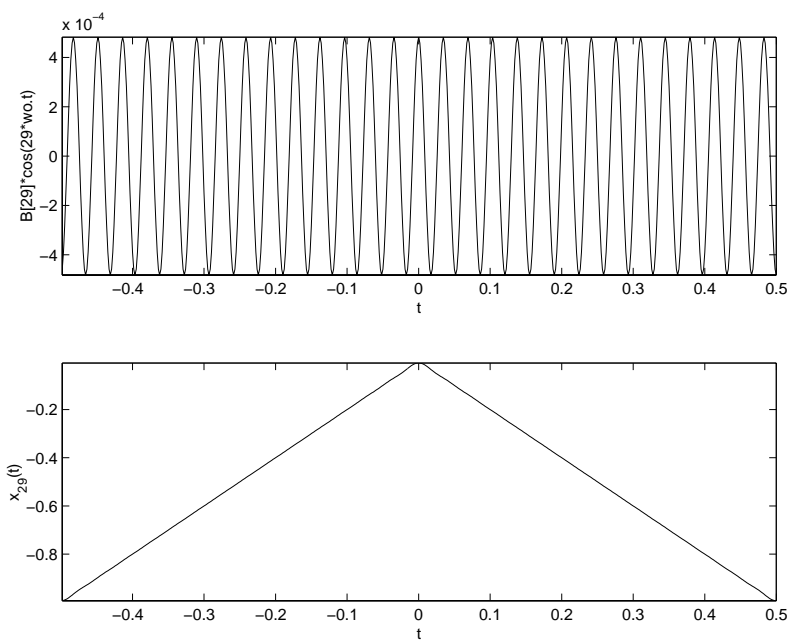


Figure P3.107. Plot 4 of 5

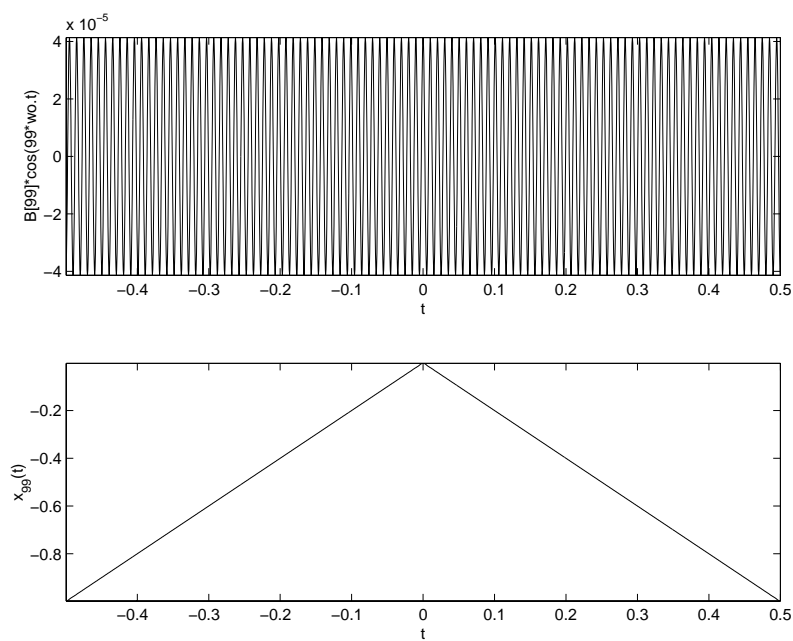


Figure P3.107. Plot 5 of 5

3.108. Repeat Problem 3.107 for the impulse train given by

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - n)$$

(a)

$$X[k] = \frac{1}{1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \delta(t) e^{-j2\pi kt} dt = 1 \text{ for all } k$$

(b)

$$B[k] = \begin{cases} 1, & k = 0, \\ 2, & k \neq 0 \end{cases}$$

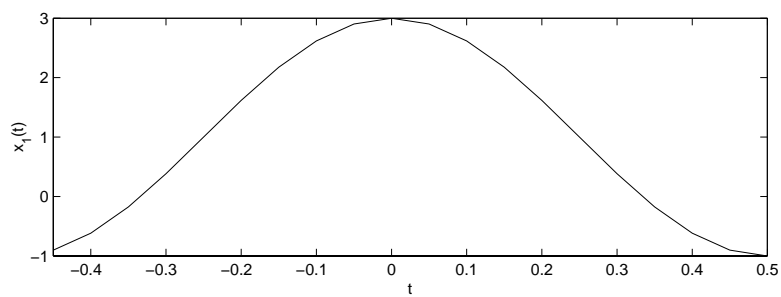
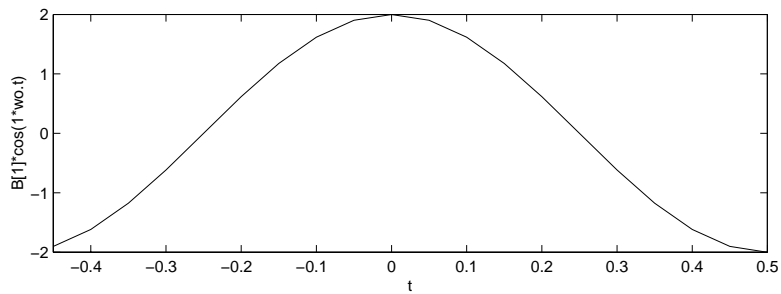


Figure P3.108. Plot 1 of 5

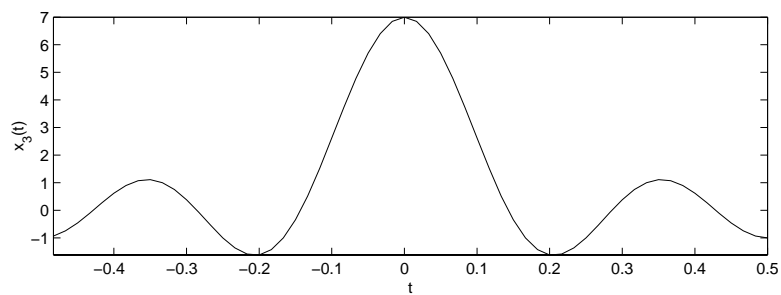
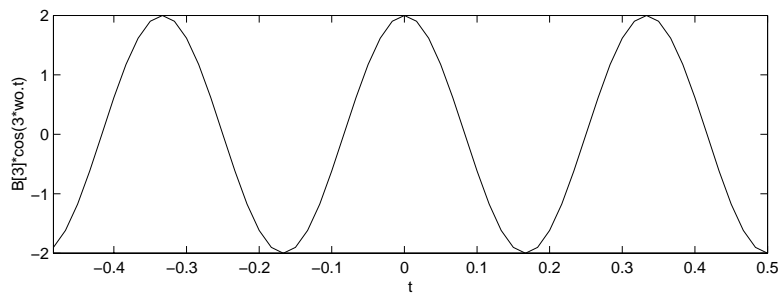


Figure P3.108. Plot 2 of 5

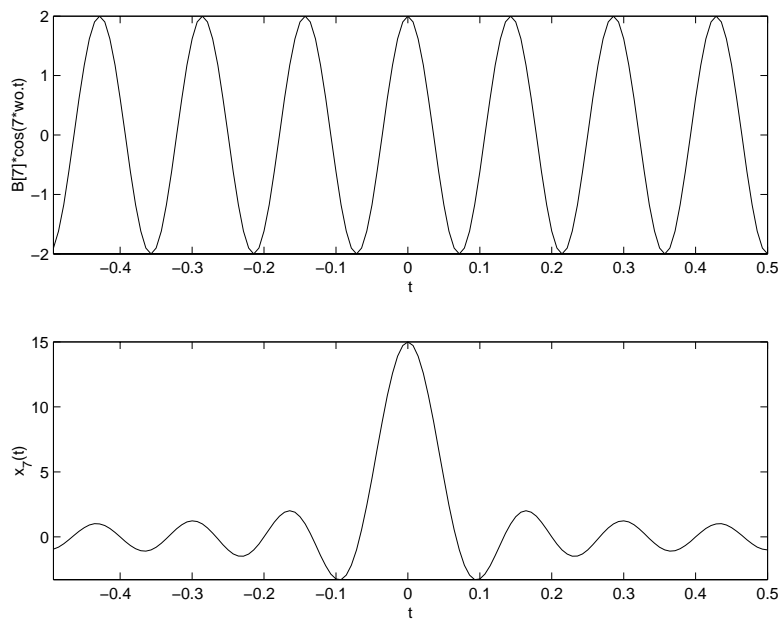


Figure P3.108. Plot 3 of 5

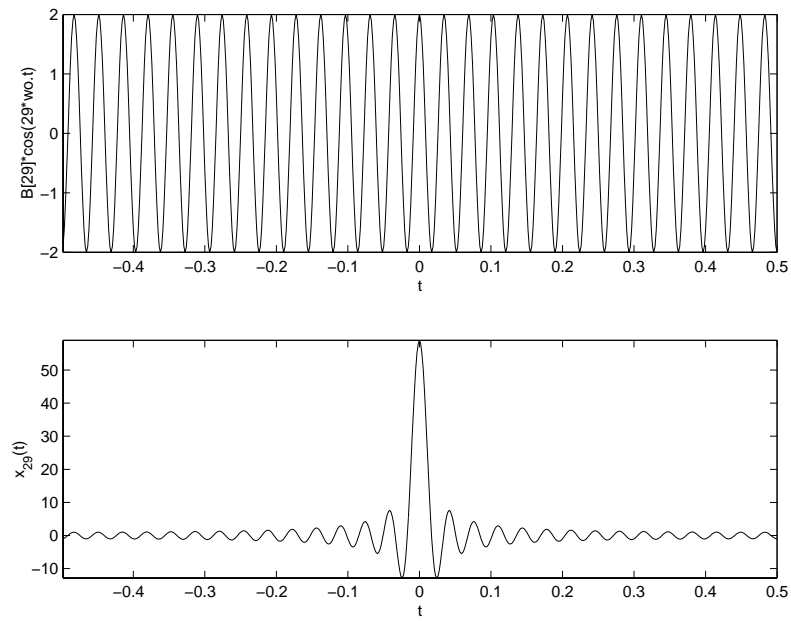


Figure P3.108. Plot 4 of 5

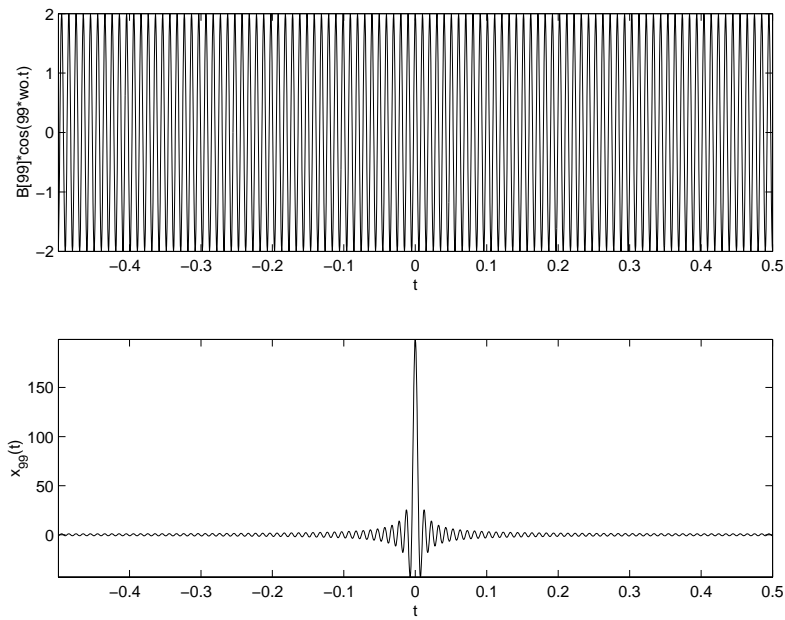


Figure P3.108. Plot 5 of 5

3.109. Use MATLAB to repeat Example 3.15 using the following values for the time constant.

- (a) $RC = .01$ s.
- (b) $RC = 0.1$ s.
- (c) $RC = 1$ s.

$$Y[k] = \frac{\frac{1}{RC}}{j2\pi k + \frac{1}{RC}} \frac{\sin(\frac{k\pi}{2})}{k\pi}$$

$$y(t) = \sum_{k=-100}^{100} Y[k] e^{j2\pi kt}$$

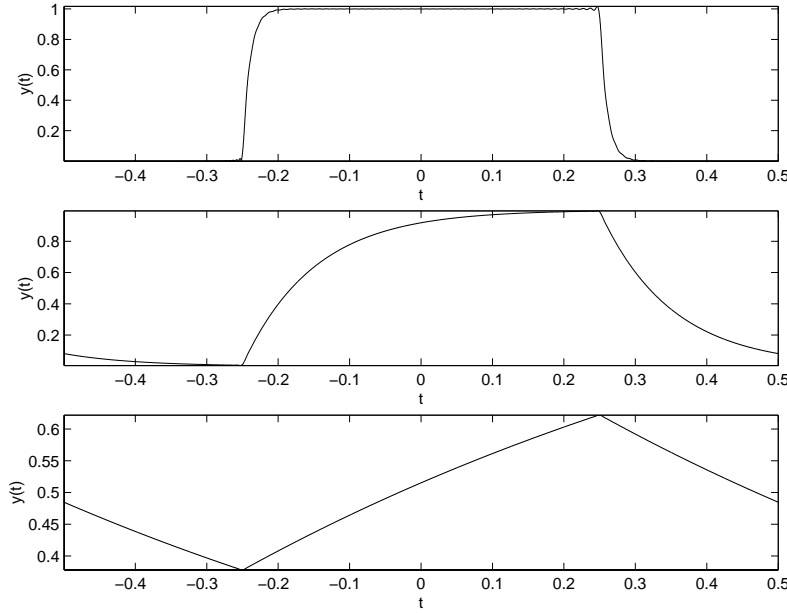


Figure P3.109. Plot 1 of 1

3.110. This experiment builds on Problem 3.71.

(a) Graph the magnitude response of the circuit depicted in Fig. P3.71 assuming the voltage across the inductor is the output. Use logarithmically spaced frequencies from 0.1 rads/S to 1000 rads/S. You may generate N logarithmically spaced values between 10^{d1} and 10^{d2} using the MATLAB command `logspace(d1,d2,N)`.

$$|H(j\omega)| = \frac{(\omega L)^2}{1 + (\omega L)^2}$$

(b) Determine and plot the voltage across the inductor if the input is the square wave depicted in Fig. 3.21 with $T = 1$ and $T_o = 1/4$. Use at least 99 harmonics in a truncated FS expansion for the output.

$$Y[k] = \frac{2j \sin(k\frac{\pi}{2})}{\frac{1}{L} + j2k\pi}$$

$$y(t) = \sum_{k=-99}^{99} Y[k] e^{jk2\pi t}$$

(c) Graph the magnitude response of the circuit depicted in Fig. P3.71 assuming the voltage across the resistor is the output. Use logarithmically spaced frequencies from 0.1 rad/s to 1000 rad/s.

$$|H(j\omega)| = \frac{1}{1 + (\omega L)^2}$$

(d) Determine and plot the voltage across the resistor if the input is the square wave depicted in Fig. 3.21 with $T = 1$ and $T_o = 1/4$. Use at least 99 harmonics in a truncated FS expansion for the output.

$$Y[k] = \frac{2j \sin(k\frac{\pi}{2})}{k\pi(1 + j2k\pi L)}$$

$$y(t) = \sum_{k=-99}^{99} Y[k]e^{jk2\pi t}$$

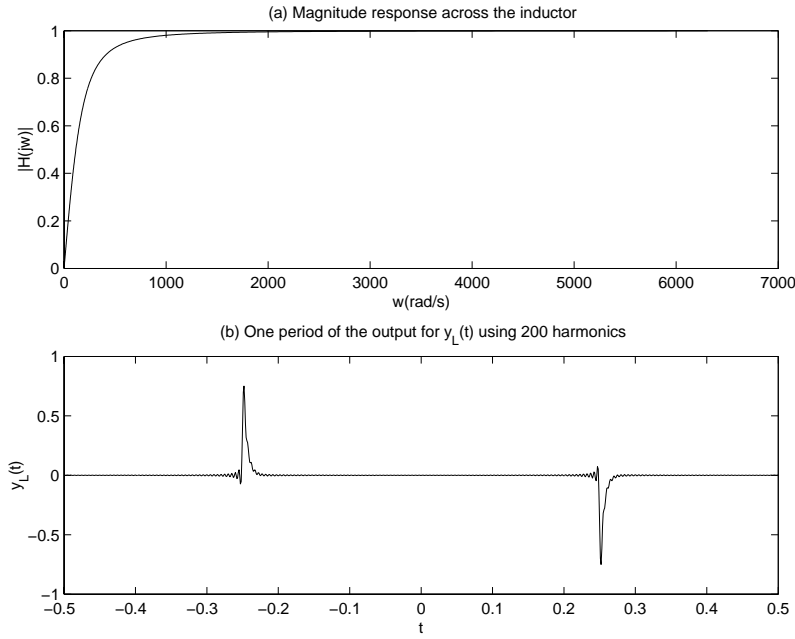


Figure P3.110. Plot 1 of 2

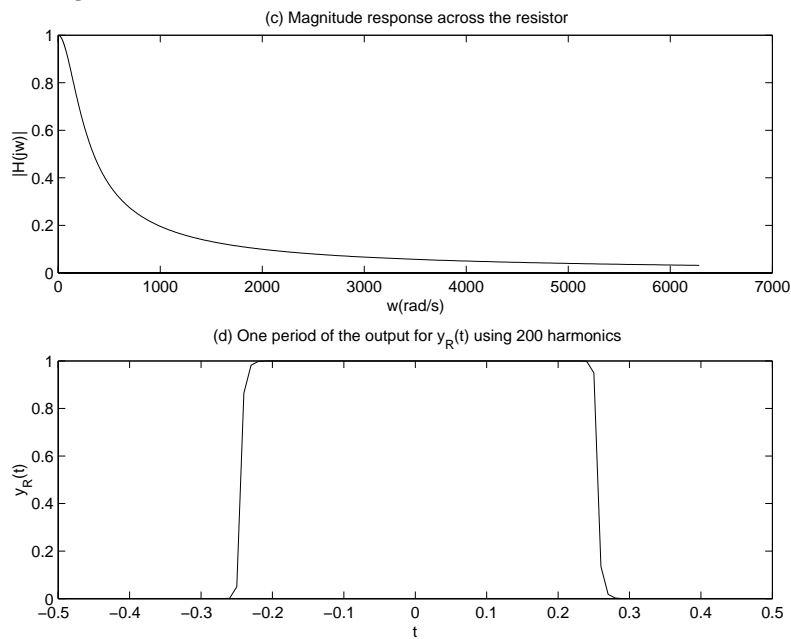


Figure P3.110. Plot 2 of 2

3.111. This experiment builds on Problem 3.72.

(a) Graph the magnitude response of the circuit depicted in Fig. P3.71 assuming the voltage across the inductor is the output. Use 401 logarithmically spaced frequencies from 10^0 rads/S to 10^4 rad/S. You may generate N logarithmically spaced values between 10^{d1} and 10^{d2} using the MATLAB command `logspace(d1,d2,N)`.

(i) Assume $L = 10\text{mH}$.

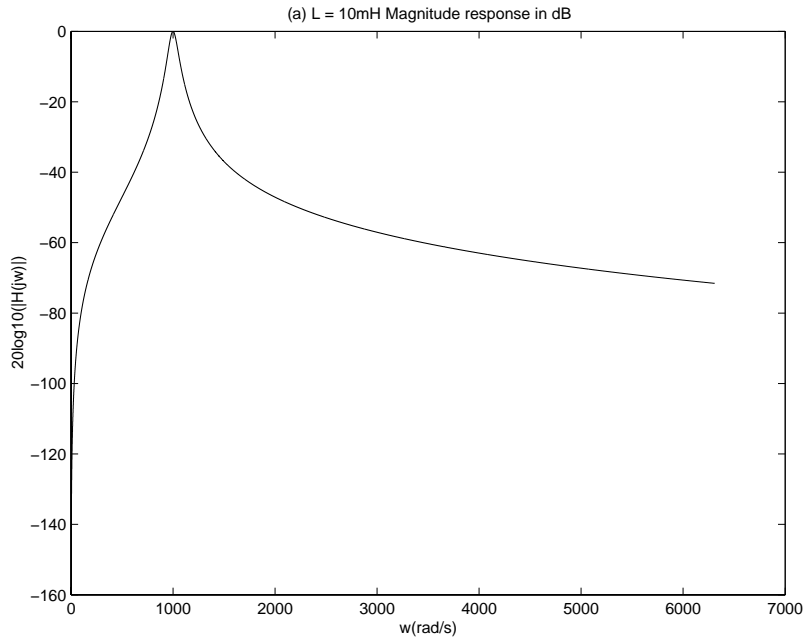


Figure P3.111. Plot 1 of 2

(b) Determine and plot the output if the input is the square wave depicted in Fig. 3.21 with $T = 2\pi * 10^{-3}$ and $T_o = \frac{\pi}{2} * 10^{-3}$. Use at least 99 harmonics in a truncated FS expansion for the output.

(i) Assume $L = 10\text{mH}$.

$$Y[k] = \frac{j1000 \sin(\frac{\pi k}{2})}{\pi(1/C + j1000k + L(j1000k)^2)}$$

$$y(t) = \sum_{k=-99}^{99} Y[k] e^{jk1000t}$$

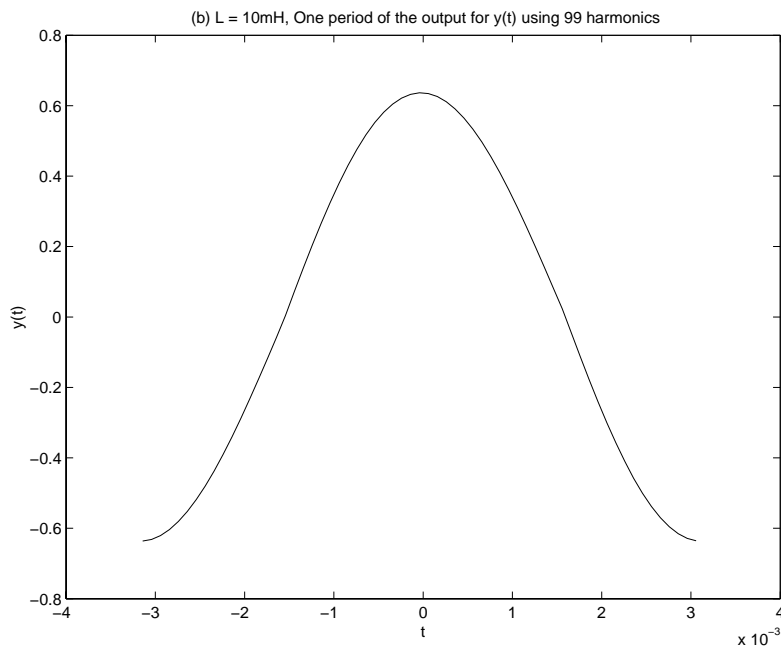


Figure P3.111. Plot 2 of 2

3.112. Evaluate the frequency response of the truncated filter in Example 3.46. You may do this in MATLAB by writing an m-file to evaluate

$$H_t(e^{j\Omega}) = \sum_{n=-M}^M h[n]e^{-j\Omega n}$$

for a large number (> 1000) values of Ω in the interval $-\pi < \Omega \leq \pi$. Plot the frequency response magnitude in dB ($20 \log_{10} |H_t(e^{j\Omega})|$) for the following values of M .

- (a) $M = 4$
- (b) $M = 10$
- (c) $M = 25$
- (d) $M = 50$

Discuss the effect of increasing M on the accuracy with which $H_t(e^{j\Omega})$ approximates $H(e^{j\Omega})$.

It can be seen that as M increases:

- (1) The ripple decreases
- (2) The transition rolls off faster

Observe that $h[n] = \frac{1}{\pi n} \sin(\frac{\pi n}{2})$ is symmetric, hence:

$$\begin{aligned} H_t(e^{j\Omega}) &= \sum_{n=1}^M h[n]e^{-j\Omega n} + \sum_{n=-M}^{-1} h[n]e^{-j\Omega n} + h[0] \\ &= \sum_{n=1}^M 2h[n] \cos(\Omega n) + h[0] \end{aligned}$$

This is the same basic form as a Fourier Series approximation using M terms. The approximation is

more accurate as M increases.

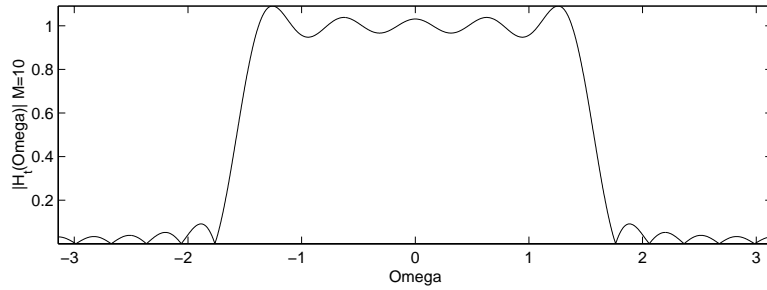
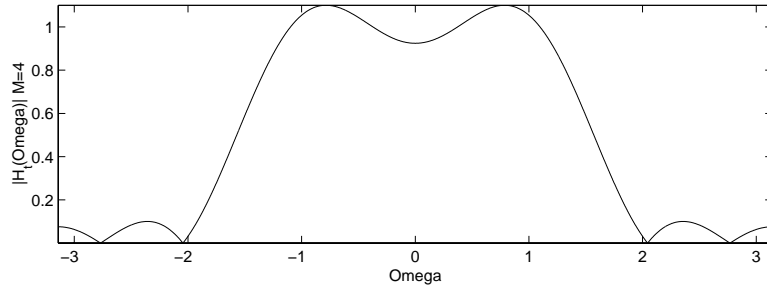


Figure P3.112. Plot 1 of 2

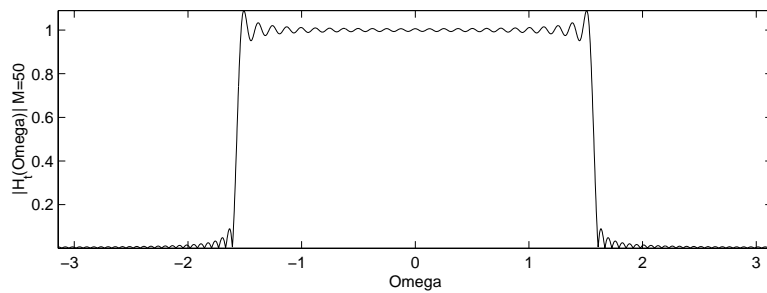
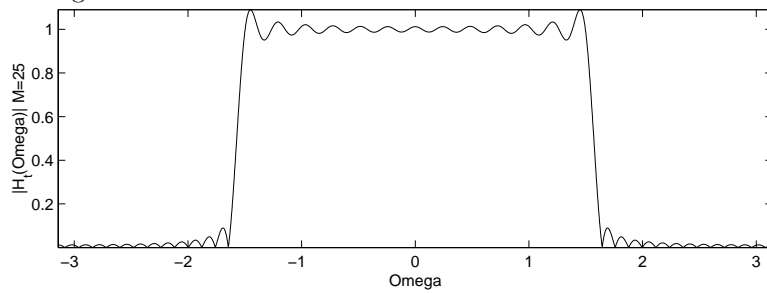


Figure P3.112. Plot 2 of 2

3.113. Use the MATLAB command `freqs` or `freqz` to plot the magnitude response of the following systems. Determine whether the system has a low-pass, high-pass, or band-pass characteristic.

(a) $H(j\omega) = \frac{8}{(j\omega)^3 + 4(j\omega)^2 + 8j\omega + 8}$
Low pass filter

(b) $H(j\omega) = \frac{(j\omega)^3}{(j\omega)^3 + 2(j\omega)^2 + 2j\omega + 1}$
High pass filter

$$(c) H(e^{j\Omega}) = \frac{1+3e^{-j\Omega}+3e^{-j2\Omega}+e^{-j3\Omega}}{6+2e^{-j2\Omega}}$$

Low pass filter

$$(d) H(e^{j\Omega}) = \frac{0.02426(1-e^{-j\Omega})^4}{(1+1.10416e^{-j\Omega}+0.4019e^{-j2\Omega})(1+0.56616e^{-j\Omega}+0.7657e^{-j2\Omega})}$$

High pass filter

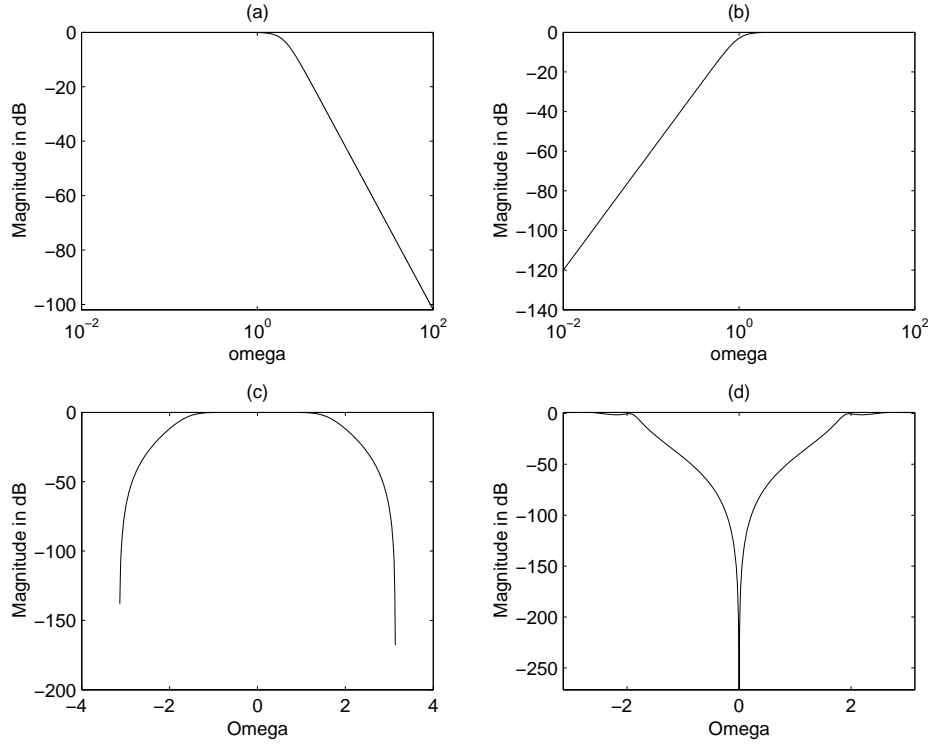
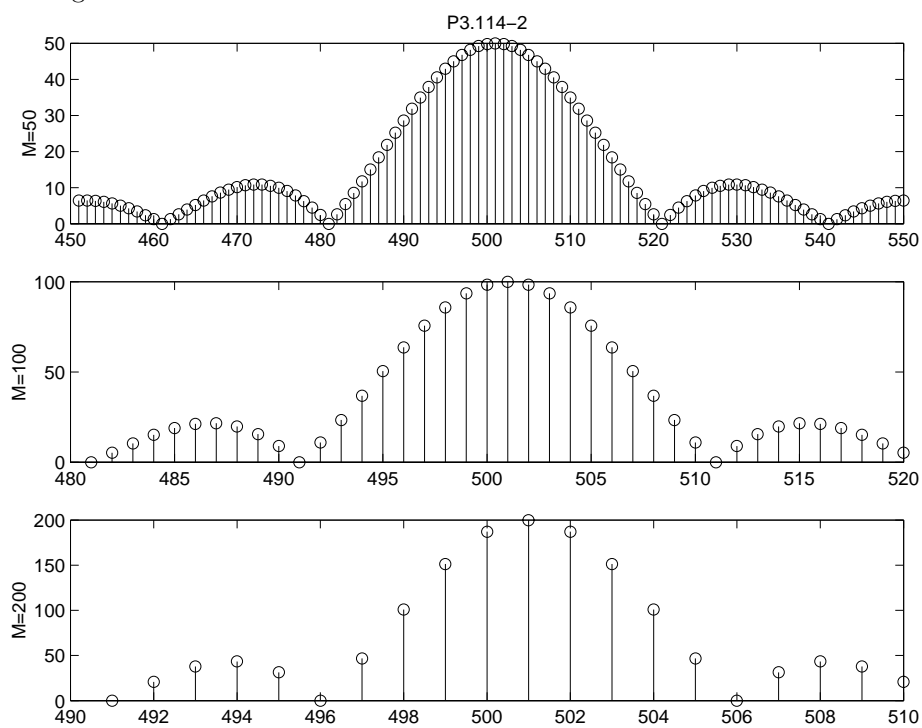
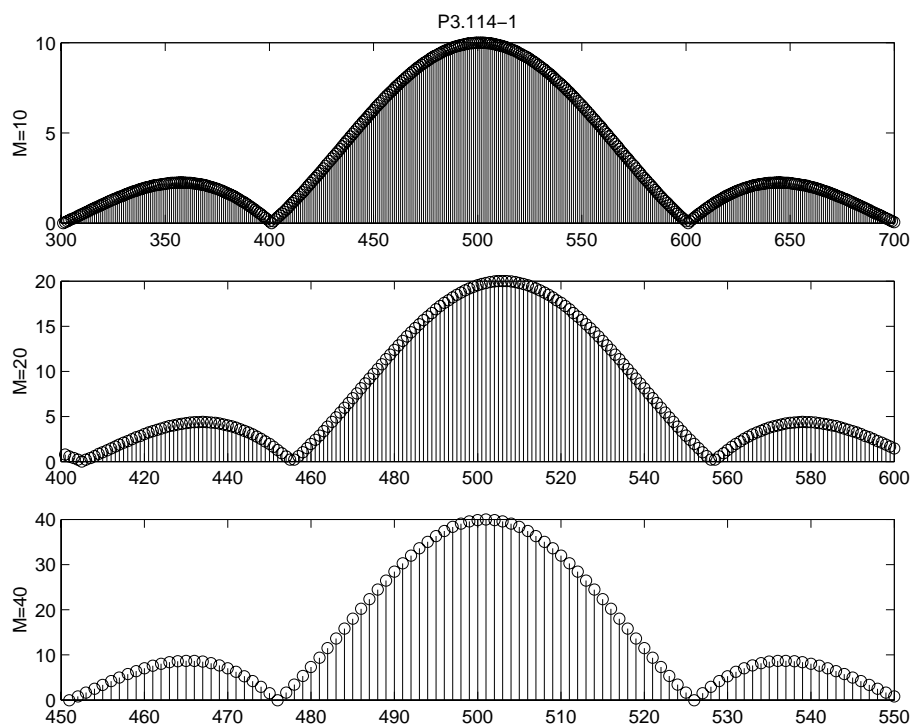


Figure P3.113. Plot 1 of 1

3.114. Use MATLAB to verify that the time-bandwidth product for a discrete-time square wave is approximately independent of the number of nonzero values in each period when duration is defined as the number of nonzero values in the square wave and bandwidth is defined as the mainlobe width. Define one period of the square wave as

$$x[n] = \begin{cases} 1, & 0 \leq n < M \\ 0, & M \leq n \leq 999 \end{cases}$$

Evaluate the bandwidth by first using `fft` and `abs` to obtain the magnitude spectrum and then count the number of DTFS coefficients in the mainlobe for $M = 10, 20, 40, 50, 100$, and 200 .



M	B_w
10	201
20	101
40	51
50	41
100	21
200	11

3.115. Use the MATLAB function `TdBw` introduced in Section 3.18 to evaluate and plot the time-bandwidth product as a function of duration for the following classes of signals:

- (a) Rectangular pulse trains. Let the pulse in a single period be of length M and vary M from 51 to 701 in steps of 50.
- (b) Raised cosine pulse trains. Let the pulse in a single period be of length M and vary M from 51 to 701 in steps of 50.
- (c) Gaussian pulse trains. Let $x[n] = e^{-an^2}$, $-500 \leq n \leq 500$ represent the Gaussian pulse in a single period. Vary the pulse duration by letting a take the following values: 0.00005, 0.0001, 0.0002, 0.0005, 0.001, 0.002, and 0.005.

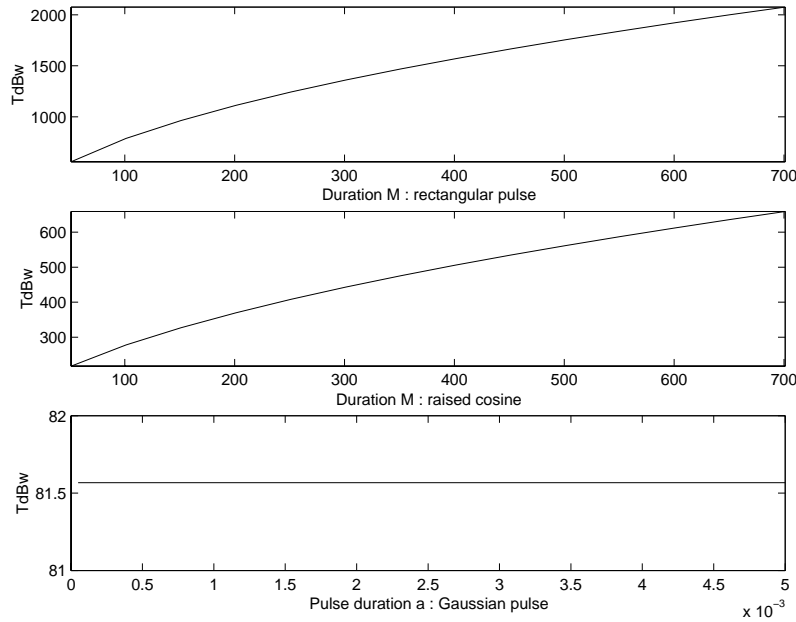


Figure P3.115. Plot 1 of 1

3.116. Use MATLAB to evaluate and plot the solution to Problem 3.96 on the interval $0 \leq l \leq 1$ at $t = 0, 1, 2, \dots, 20$ assuming $c = 1$. Use at least 99 harmonics in the sum.

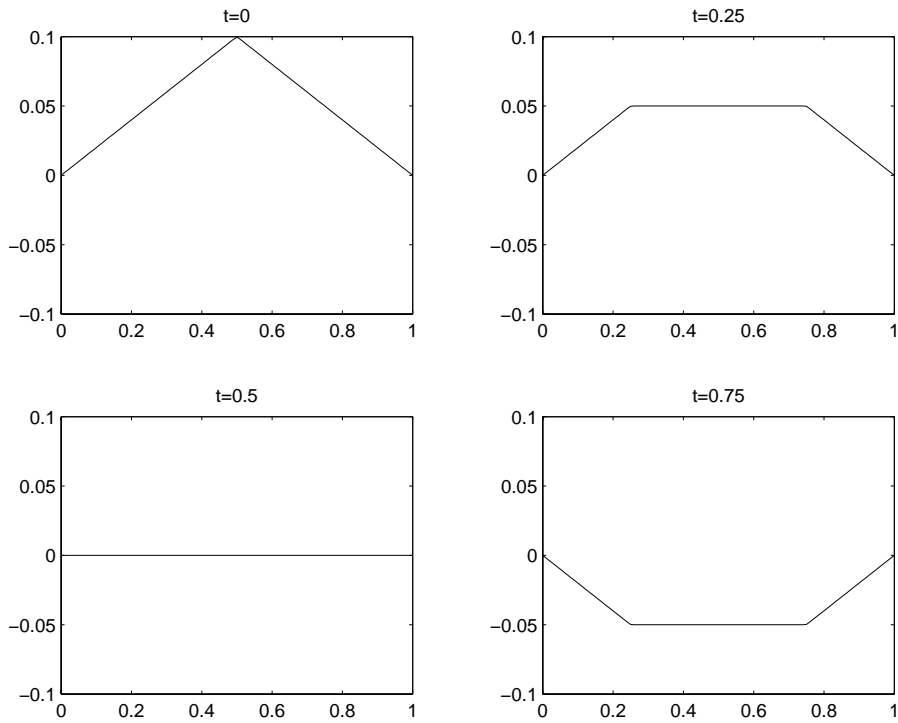


Figure P3.116. Plot 1 of 2

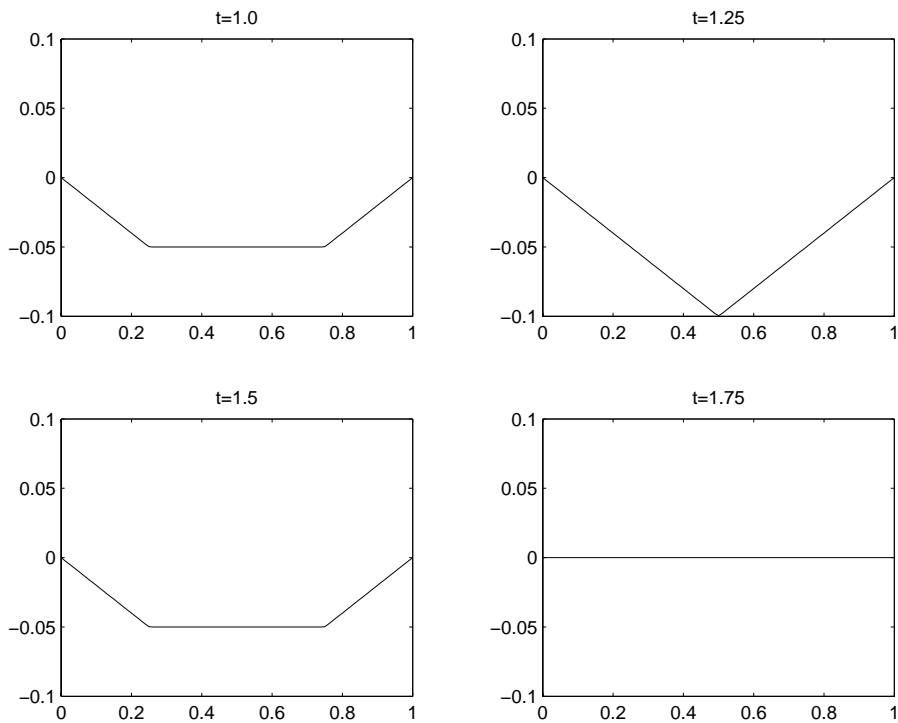


Figure P3.116. Plot 2 of 2

3.117. The frequency response of either a continuous- or discrete-time system described in state-variable form (see problem 3.87) may be computed using `freqresp`. The syntax is `h = freqresp(sys,w)`

where `sys` is the object containing the state-variable description (see section 2.13) and `w` is the vector containing the frequencies at which to evaluate the frequency response. `freqresp` applies in general to multiple-input, multiple-output systems so the output `h` is a multidimensional array. For the class of single-input, single-output systems considered in this text and `N` frequency points in `w`, `h` is a dimension of 1-by-1-by-`N`. The command `squeeze(h)` converts `h` to an `N`-dimensional vector that may be displayed with the `plot` command. Thus we obtain the frequency response using the commands:

```
>> h = freqresp(sys,w);
>> hmag = abs(squeeze(h));
>> plot(w,hmag);
>> title('System Magnitude Response');
>> xlabel('Frequency(rad/s)'); ylabel('Magnitude');
```

Use MATLAB to plot the magnitude and phase response for the systems with state-variable descriptions given in Problems 3.88 and 3.89.

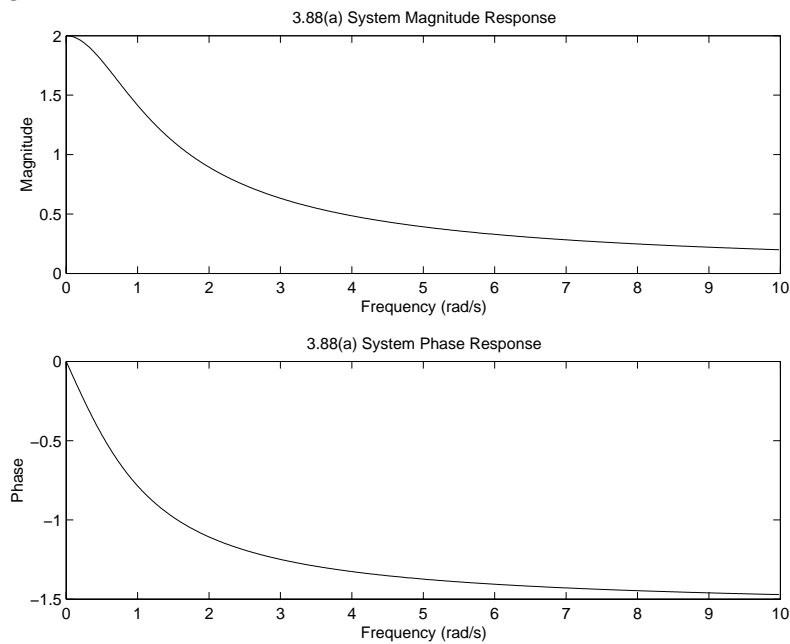


Figure P3.117. Plot 1 of 4

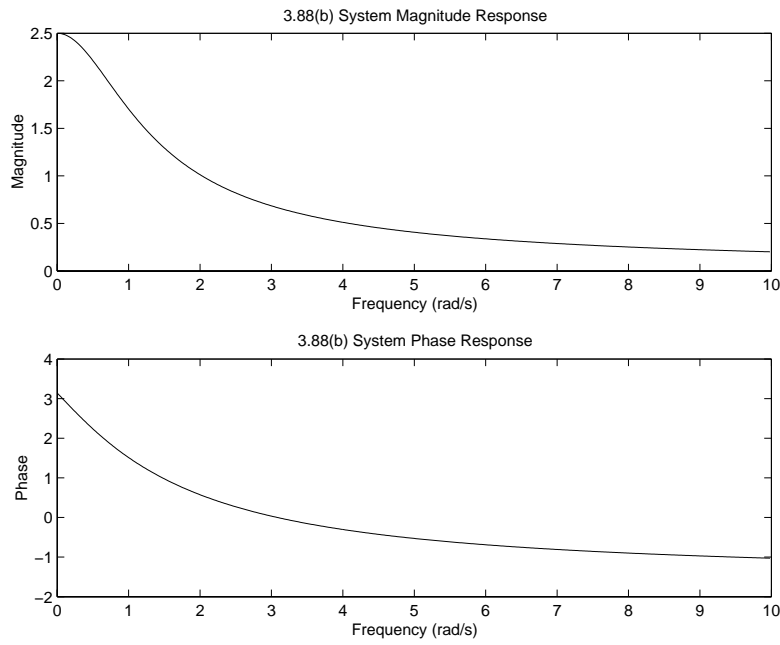


Figure P3.117. Plot 2 of 4

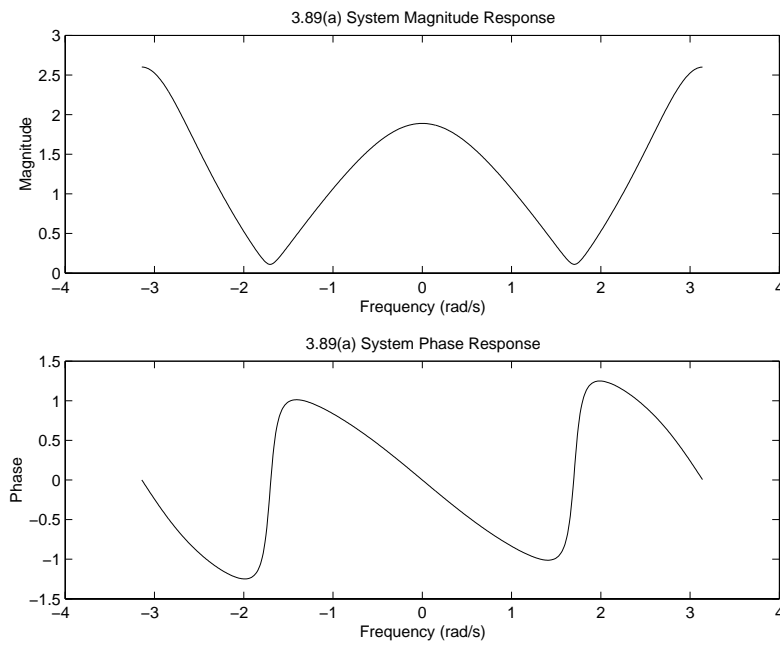


Figure P3.117. Plot 3 of 4

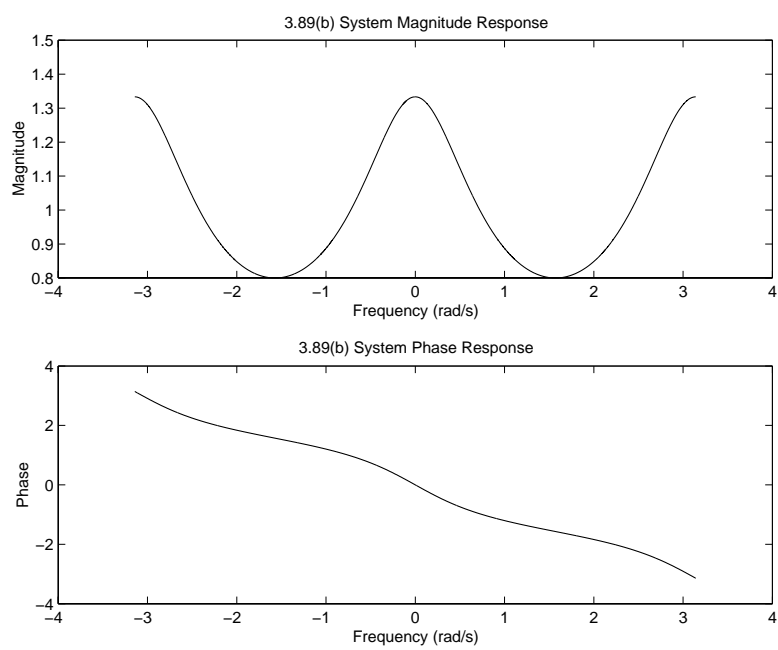


Figure P3.117. Plot 4 of 4

Solutions to Additional Problems

4.16. Find the FT representations for the following periodic signals: Sketch the magnitude and phase spectra.

(a) $x(t) = 2 \cos(\pi t) + \sin(2\pi t)$

$$\begin{aligned} x(t) &= e^{j\pi t} + e^{-j\pi t} + \frac{1}{2j} e^{j2\pi t} - \frac{1}{2j} e^{-j2\pi t} \\ \omega_o &= \text{lcm}(\pi, 2\pi) = \pi \\ x[1] &= x[-1] = 1 \\ x[2] &= -x[-2] = \frac{1}{2j} \\ X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\omega - k\omega_o) \\ X(j\omega) &= 2[\pi\delta(\omega - \pi) + \pi\delta(\omega + \pi)] + \frac{1}{j} [\pi\delta(\omega - 2\pi) - \pi\delta(\omega + 2\pi)] \end{aligned}$$

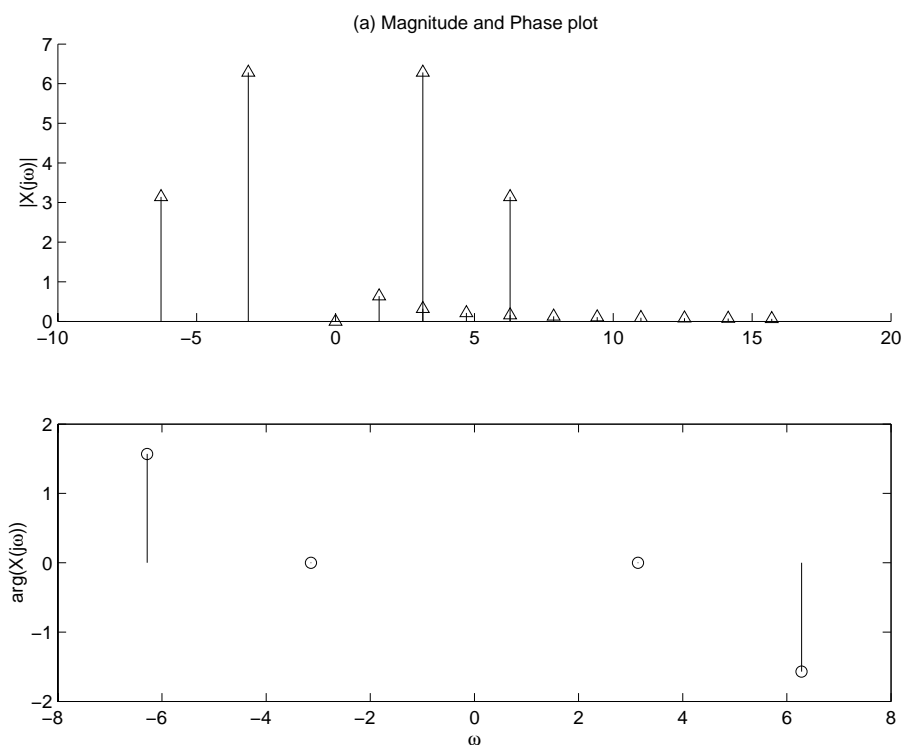


Figure P4.16. (a) Magnitude and Phase plot

(b) $x(t) = \sum_{k=0}^4 \frac{(-1)^k}{k+1} \cos((2k+1)\pi t)$

$$x(t) = \frac{1}{2} \sum_{k=0}^4 \frac{(-1)^k}{k+1} \left[e^{j(2k+1)\pi t} + e^{-j(2k+1)\pi t} \right]$$

$$X(j\omega) = \pi \sum_{k=0}^4 \frac{(-1)^k}{k+1} [\delta(\omega - (2k+1)\pi) + \delta(\omega + (2k+1)\pi)]$$

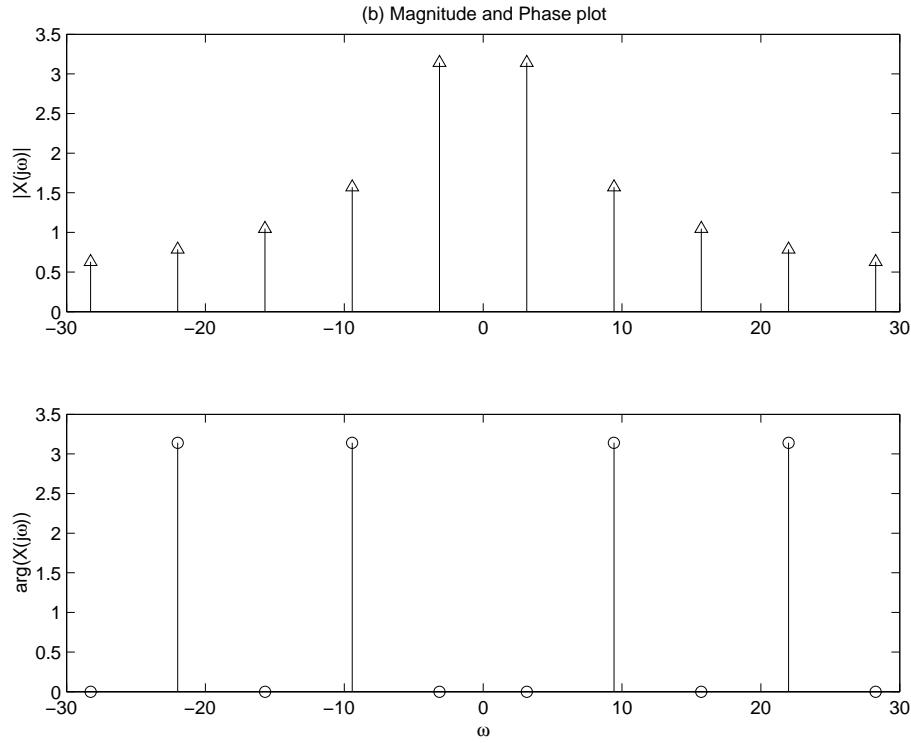


Figure P4.16. (b) Magnitude and Phase plot

(c) $x(t)$ as depicted in Fig. P4.16 (a).

$$x(t) = \begin{cases} 1 & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases} + \begin{cases} 2 & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \left[\frac{2 \sin(k \frac{2\pi}{3})}{k} + \frac{4 \sin(k \frac{\pi}{3})}{k} \right] \delta(\omega - k \frac{2\pi}{3})$$

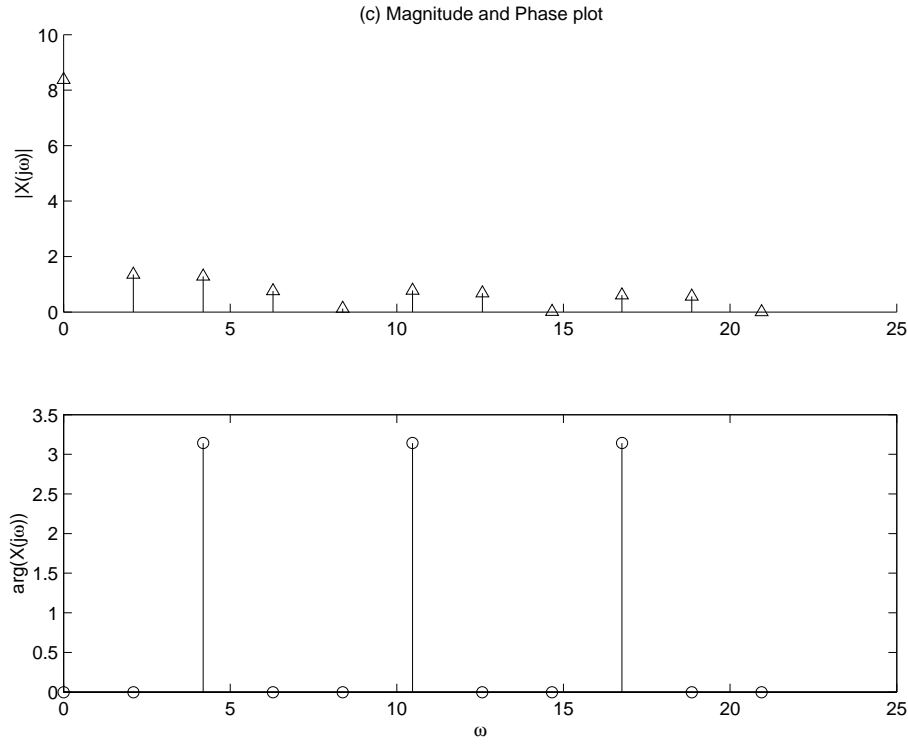


Figure P4.16. (c) Magnitude and Phase plot

(d) $x(t)$ as depicted in Fig. P4.16 (b).

$$\begin{aligned}
 T &= 4 & \omega_o &= \frac{\pi}{2} \\
 X[k] &= \frac{1}{4} \int_{-2}^2 2te^{-j\frac{\pi}{2}kt} dt \\
 &= \begin{cases} 0 & k = 0 \\ \frac{2j \cos(\pi k)}{\pi k} & k \neq 0 \end{cases} \\
 X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\omega - \frac{\pi}{2}k)
 \end{aligned}$$

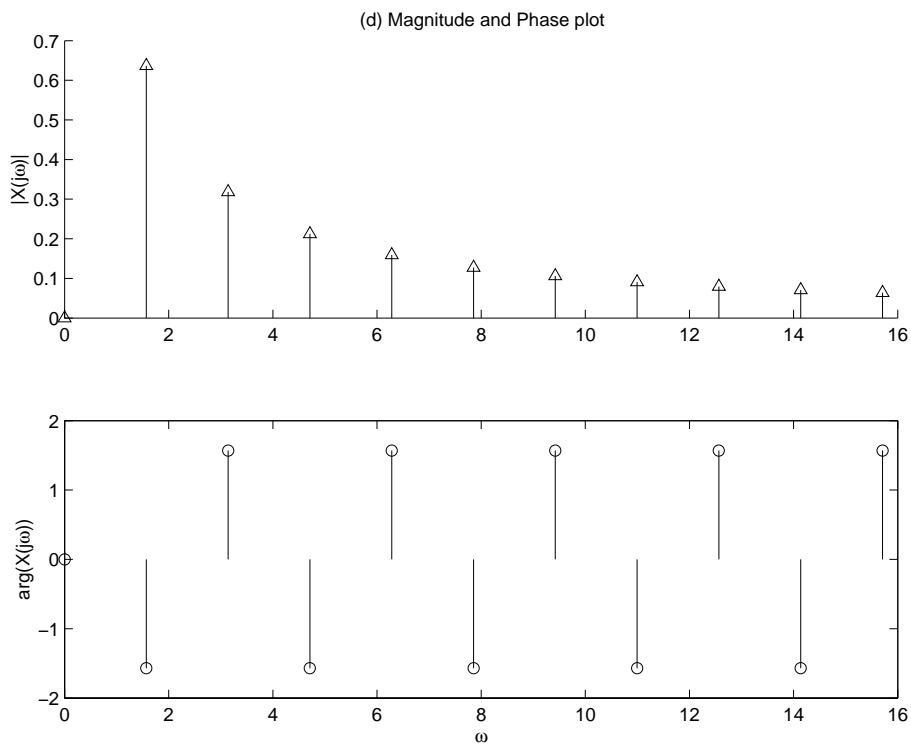


Figure P4.16. (d) Magnitude and Phase plot

4.17. Find the DTFT representations for the following periodic signals: Sketch the magnitude and phase spectra.

(a) $x[n] = \cos(\frac{\pi}{8}n) + \sin(\frac{\pi}{5}n)$

$$\begin{aligned}
 x[n] &= \frac{1}{2} [e^{j\frac{\pi}{8}n} + e^{-j\frac{\pi}{8}n}] + \frac{1}{2j} [e^{j\frac{\pi}{5}n} - e^{-j\frac{\pi}{5}n}] \\
 \Omega_o &= \text{lcm}(\frac{\pi}{8}, \frac{\pi}{5}) = \frac{\pi}{40} \\
 X[5] &= X[-5] = \frac{1}{2} \\
 X[8] &= -X[-8] = \frac{1}{2j} \\
 X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\Omega - k\Omega_o) \\
 X(e^{j\Omega}) &= \pi \left[\delta(\Omega - \frac{\pi}{8}) + \delta(\Omega + \frac{\pi}{8}) \right] + \frac{\pi}{j} \left[\delta(\Omega - \frac{\pi}{5}) - \delta(\Omega + \frac{\pi}{5}) \right]
 \end{aligned}$$

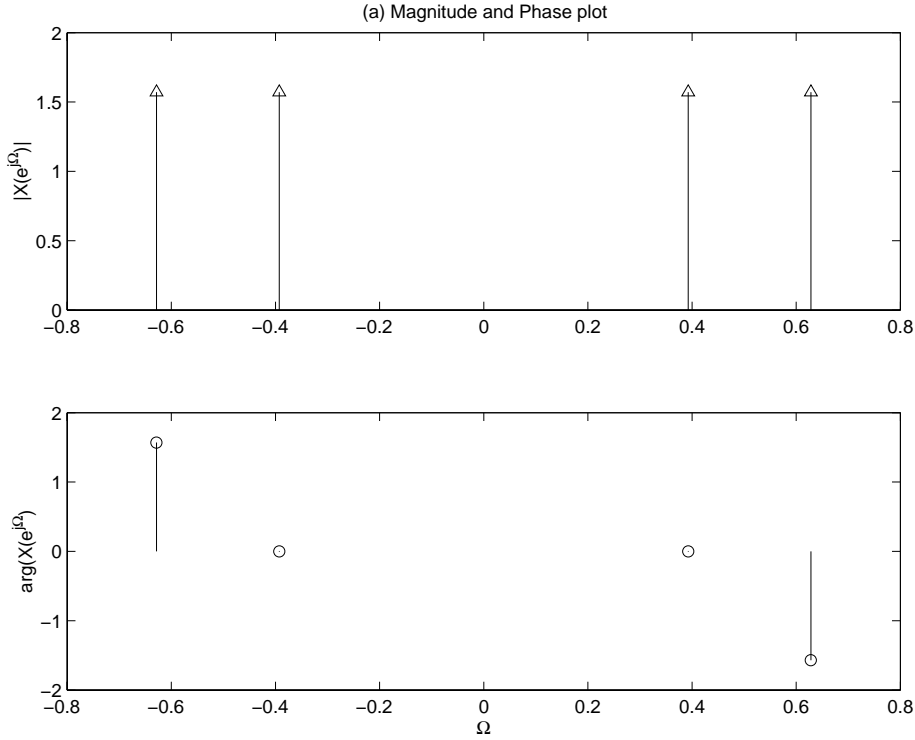


Figure P4.17. (a) Magnitude and Phase response

(b) $x[n] = 1 + \sum_{m=-\infty}^{\infty} \cos(\frac{\pi}{4}m)\delta[n - m]$

$$\begin{aligned}
 N &= 8 & \Omega_o &= \frac{\pi}{4} \\
 x[n] &= 1 + \sum_{m=-\infty}^{\infty} \cos(\frac{\pi}{4}m)\delta[n - m] \\
 &= 1 + \cos(\frac{\pi}{4}n) \\
 X[k] &= \frac{1}{8} \sum_{n=-4}^3 x[n]e^{-jk\frac{\pi}{4}n}
 \end{aligned}$$

For one period of $X[k]$, $k \in [-4, 3]$

$$\begin{aligned}
 X[-4] &= 0 \\
 X[-3] &= \frac{1 - 2^{-0.5}}{8} e^{jk\frac{3\pi}{4}} \\
 X[-2] &= \frac{1}{8} e^{jk\frac{2\pi}{4}} \\
 X[-1] &= \frac{1 + 2^{-0.5}}{8} e^{jk\frac{\pi}{4}} \\
 X[0] &= \frac{2}{8} \\
 X[1] &= \frac{1 + 2^{-0.5}}{8} e^{-jk\frac{\pi}{4}} \\
 X[2] &= \frac{1}{8} e^{-jk\frac{2\pi}{4}}
 \end{aligned}$$

$$\begin{aligned}
X[3] &= \frac{1 - 2^{-0.5}}{8} e^{-jk \frac{3\pi}{4}} \\
X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\Omega - k\Omega_o) \\
&= \pi \left[\frac{(1 - 2^{-0.5})}{4} \delta(\Omega + \frac{3\pi}{4}) + \frac{1}{4} \delta(\Omega + \frac{\pi}{2}) + \frac{(1 + 2^{-0.5})}{4} \delta(\Omega + \frac{\pi}{4}) + \frac{1}{4} \delta(2\Omega) \right] + \\
&\quad \pi \left[\frac{(1 + 2^{-0.5})}{4} \delta(\Omega - \frac{\pi}{4}) + \frac{1}{4} \delta(\Omega - \frac{\pi}{2}) + \frac{(1 - 2^{-0.5})}{4} \delta(\Omega - \frac{3\pi}{4}) \right]
\end{aligned}$$

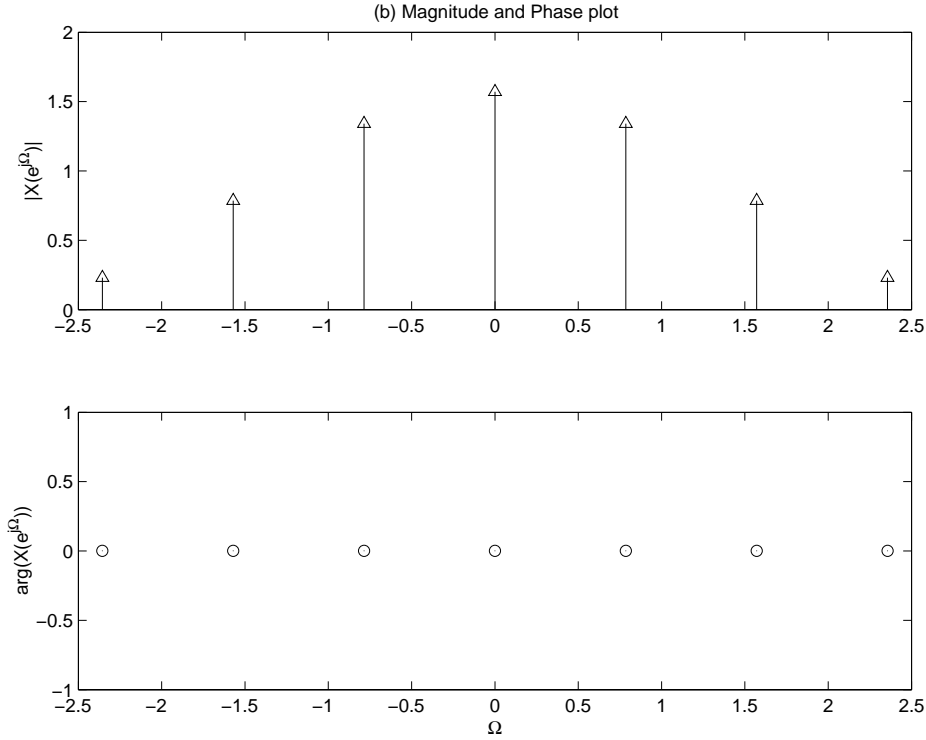


Figure P4.17. (b) Magnitude and Phase response

(c) $x[n]$ as depicted in Fig. P4.17 (a).

$$\begin{aligned}
N = 8 \quad \Omega_o &= \frac{\pi}{4} \\
X[k] &= \frac{\sin(k \frac{5\pi}{8})}{8 \sin(\frac{\pi}{8} k)} \\
X[0] &= \frac{5}{8} \\
X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\Omega - k \frac{\pi}{4})
\end{aligned}$$

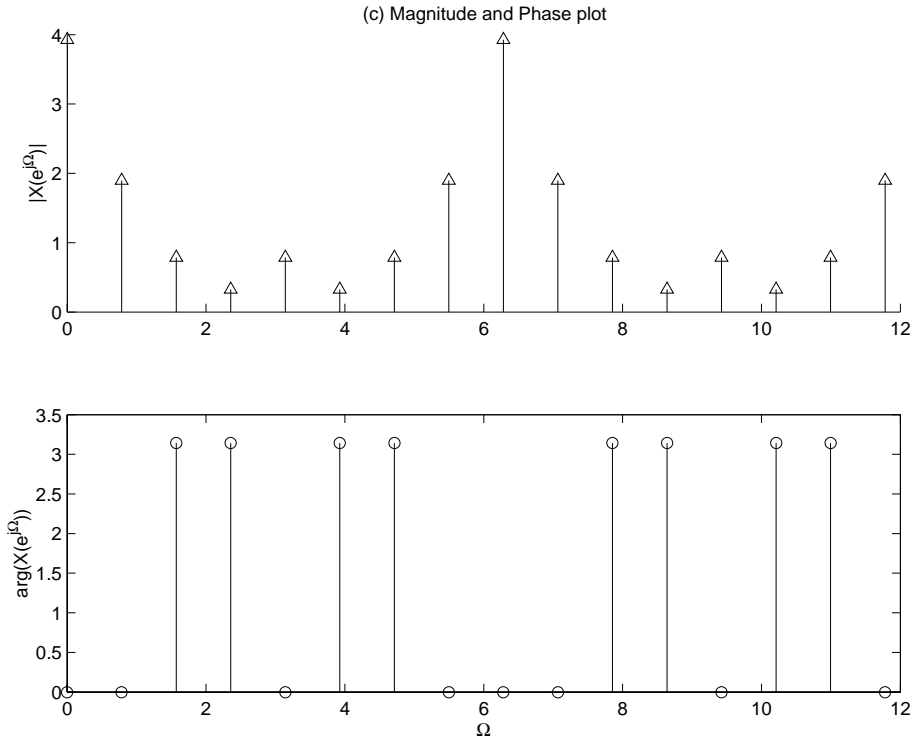


Figure P4.17. (c) Magnitude and Phase response

(d) $x[n]$ as depicted in Fig. P4.17 (b).

$$\begin{aligned}
 N = 7 \quad \Omega_o &= \frac{2\pi}{7} \\
 X[k] &= \frac{1}{7} \left(1 - e^{jk\frac{2\pi}{7}} - e^{-jk\frac{2\pi}{7}} \right) \\
 &= \frac{1}{7} \left(1 - 2 \cos\left(k\frac{2\pi}{7}\right) \right) \\
 X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta\left(\Omega - k\frac{2\pi}{7}\right)
 \end{aligned}$$

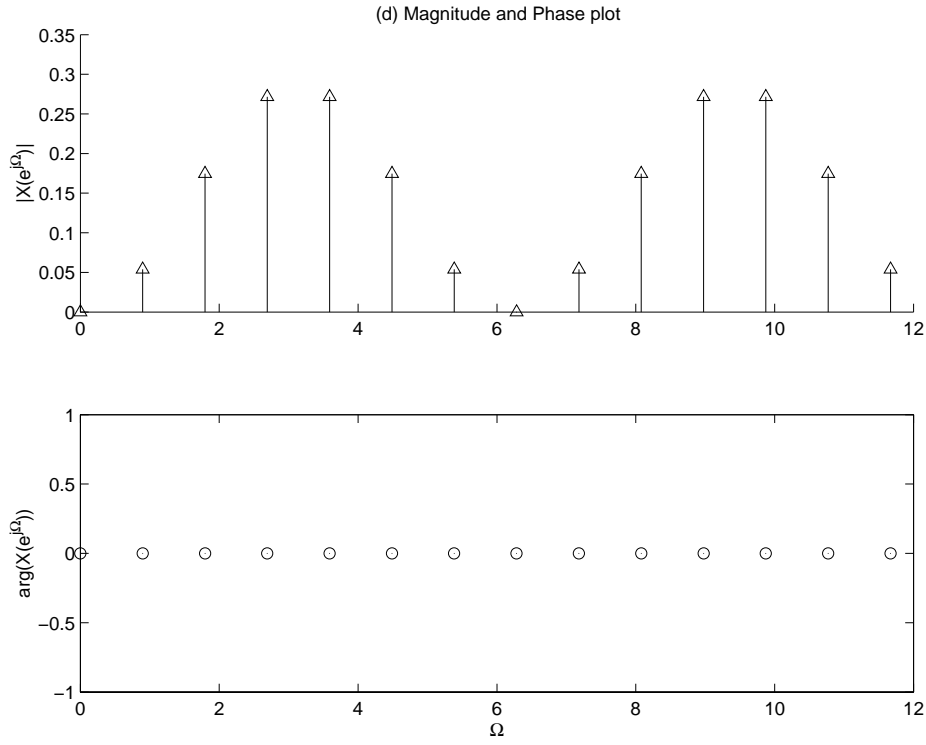


Figure P4.17. (d) Magnitude and Phase response

(e) $x[n]$ as depicted in Fig. P4.17 (c).

$$\begin{aligned}
 N &= 4 & \Omega_o &= \frac{\pi}{2} \\
 X[k] &= \frac{1}{4} \left(1 + e^{-jk\frac{\pi}{2}} - e^{-jk\pi} - e^{-jk\frac{3\pi}{2}} \right) \\
 &= \frac{1}{4} (1 - (-1)^k) - \frac{j}{2} \sin(k\frac{\pi}{2}) \\
 X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\Omega - k\frac{\pi}{2})
 \end{aligned}$$

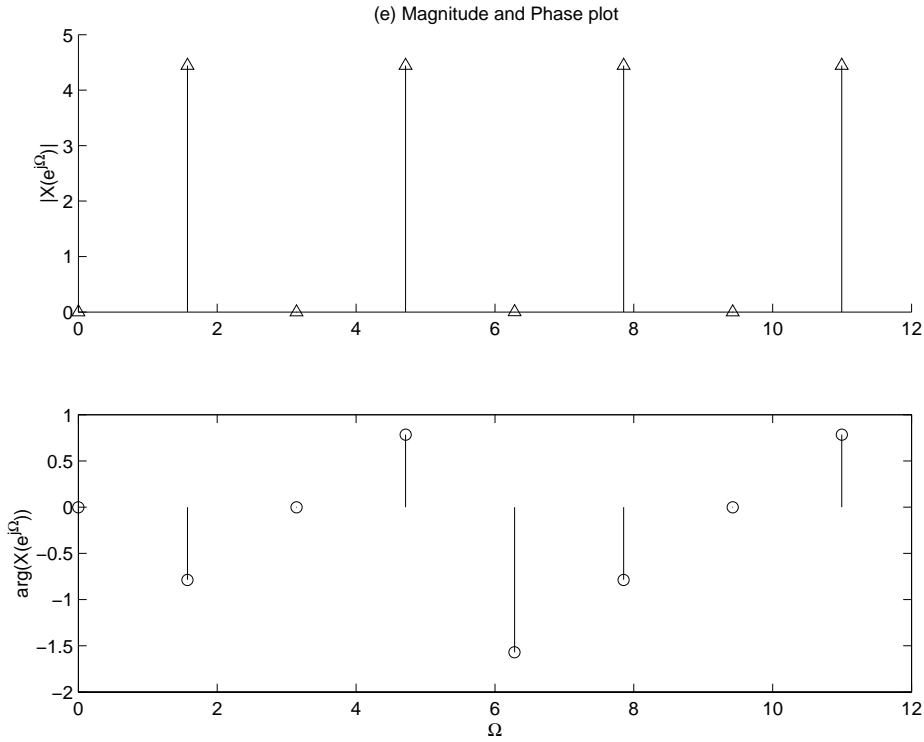


Figure P4.17. (e) Magnitude and Phase response

4.18. An LTI system has the impulse response

$$h(t) = 2 \frac{\sin(2\pi t)}{\pi t} \cos(7\pi t)$$

Use the FT to determine the system output for the following inputs, $x(t)$.

$$\begin{aligned} \text{Let } a(t) = \frac{\sin(2\pi t)}{\pi t} &\xleftrightarrow{FT} A(j\omega) = \begin{cases} 1 & |\omega| < 2\pi \\ 0 & \text{otherwise} \end{cases} \\ h(t) = 2a(t) \cos(7\pi t) &\xleftrightarrow{FT} H(j\omega) = A(j(\omega - 7\pi)) + A(j(\omega + 7\pi)) \end{aligned}$$

(a) $x(t) = \cos(2\pi t) + \sin(6\pi t)$

$$\begin{aligned} X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\omega - k\omega_o) \\ X(j\omega) &= \pi \delta(\omega - 2\pi) + \pi \delta(\omega + 2\pi) + \frac{\pi}{j} \pi \delta(\omega - 6\pi) - \frac{\pi}{j} \pi \delta(\omega + 6\pi) \\ Y(j\omega) &= X(j\omega) H(j\omega) \\ &= \frac{\pi}{j} \delta(\omega - 6\pi) - \frac{\pi}{j} \delta(\omega + 6\pi) \\ y(t) &= \sin(6\pi t) \end{aligned}$$

(b) $x(t) = \sum_{m=-\infty}^{\infty} (-1)^m \delta(t - m)$

$$\begin{aligned}
X[k] &= \frac{1}{2}(1 - e^{-jk\pi}) = \frac{1}{2}(1 - (-1)^k) \\
&= \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases} \\
X(j\omega) &= 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - l2\pi) \\
Y(j\omega) &= X(j\omega)H(j\omega) \\
&= 2\pi \sum_{l=3}^4 [\delta(\omega - l2\pi) + \delta(\omega + l2\pi)] \\
y(t) &= 2\cos(6\pi t) + 2\cos(8\pi t)
\end{aligned}$$

(c) $x(t)$ as depicted in Fig. P4.18 (a).

$$\begin{aligned}
T &= 1 \quad \omega_o = 2\pi \\
X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} \left(\frac{\sin(k\frac{\pi}{4})}{k\pi} (1 - e^{-jk\pi}) \right) \delta(\omega - k2\pi) \\
Y(j\omega) &= X(j\omega)H(j\omega) \\
&= 2\pi \left[\frac{\sin(3\frac{\pi}{4})}{3\pi} (1 - e^{-j3\pi}) \delta(\omega - 6\pi) + \frac{\sin(-3\frac{\pi}{4})}{-3\pi} (1 - e^{j3\pi}) \delta(\omega + 6\pi) \right] \\
&= \frac{4\sin(3\frac{\pi}{4})}{3} \delta(\omega - 6\pi) + \frac{4\sin(3\frac{\pi}{4})}{3} \delta(\omega + 6\pi) \\
y(t) &= \frac{4\sin(\frac{3\pi}{4})}{3\pi} \cos(6\pi t)
\end{aligned}$$

(d) $x(t)$ as depicted in Fig. P4.18 (b).

$$\begin{aligned}
X[k] &= 4 \int_{-\frac{1}{8}}^{\frac{1}{8}} 16te^{-jk8\pi t} dt \\
&= \frac{2j \cos(\pi k)}{\pi k} \\
X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\omega - k8\pi) \\
Y(j\omega) &= -4j\delta(\omega - 8\pi) + 4j\delta(\omega + 8\pi) \\
y[n] &= -\frac{4}{\pi} \sin(8\pi n)
\end{aligned}$$

(e) $x(t)$ as depicted in Fig. P4.18 (c).

$$T = 1 \quad \omega_o = 2\pi$$

$$\begin{aligned}
X[k] &= \int_0^1 e^{-t} e^{-jk2\pi t} dt \\
&= \frac{1 - e^{-(1+jk2\pi)}}{1 + jk2\pi} \\
&= \frac{1 - e^{-1}}{1 + jk2\pi} \\
X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\omega - k2\pi) \\
Y(j\omega) &= X(j\omega)H(j\omega) \\
&= 2\pi \left(\frac{1 - e^{-1}}{1 + j6\pi} \delta(\omega - 6\pi) + \frac{1 - e^{-1}}{1 + j8\pi} \delta(\omega - 8\pi) + \frac{1 - e^{-1}}{1 - j6\pi} \delta(\omega + 6\pi) + \frac{1 - e^{-1}}{1 - j8\pi} \delta(\omega + 8\pi) \right) \\
Y(j\omega) &= (1 - e^{-1}) \left[\frac{e^{j6\pi t}}{1 + j6\pi} + \frac{e^{j8\pi t}}{1 + j8\pi} + \frac{e^{-j6\pi t}}{1 - j6\pi} + \frac{e^{-j8\pi t}}{1 - j8\pi} \right] \\
y(t) &= 2(1 - e^{-1}) \left[\operatorname{Re} \left\{ \frac{e^{j6\pi t}}{1 + j6\pi} + \frac{e^{j8\pi t}}{1 + j8\pi} \right\} \right] \\
y(t) &= 2(1 - e^{-1}) [r_1 \cos(6\pi t + \phi_1) + r_2 \cos(8\pi t + \phi_2)]
\end{aligned}$$

Where

$$\begin{aligned}
r_1 &= \sqrt{1 + (6\pi)^2} \\
\phi_1 &= -\tan^{-1}(6\pi) \\
r_2 &= \sqrt{1 + (8\pi)^2} \\
\phi_2 &= -\tan^{-1}(8\pi)
\end{aligned}$$

4.19. We may design a dc power supply by cascading a full-wave rectifier and an RC circuit as depicted in Fig. P4.19. The full wave rectifier output is given by

$$z(t) = |x(t)|$$

Let $H(j\omega) = \frac{Y(j\omega)}{Z(j\omega)}$ be the frequency response of the RC circuit as shown by

$$H(j\omega) = \frac{1}{j\omega RC + 1}$$

Suppose the input is $x(t) = \cos(120\pi t)$.

(a) Find the FT representation for $z(t)$.

$$\begin{aligned}
\omega_o &= 240\pi & T &= \frac{1}{120} \\
Z[k] &= 120 \int_{-\frac{1}{240}}^{\frac{1}{240}} \frac{1}{2} (e^{j120\pi t} + e^{-j120\pi t}) e^{-jk240\pi t} dt \\
&= \frac{2(-1)^k}{\pi(1 - 4k^2)} \\
Z(j\omega) &= 4 \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\pi(1 - 4k^2)} \delta(\omega - k240\pi)
\end{aligned}$$

(b) Find the FT representation for $y(t)$.

$$H(j\omega) = \frac{Y(j\omega)}{Z(j\omega)}$$

In the time domain:

$$z(t) - y(t) = RC \frac{d}{dt} y(t) \xleftrightarrow{FT} Z(j\omega) = (1 + j\omega RC) Y(j\omega)$$

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

$$Y(j\omega) = Z(j\omega) H(j\omega)$$

$$= 4 \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\pi(1 - 4k^2)} \left(\frac{1}{1 + jk240\pi RC} \right) \delta(\omega - k240\pi)$$

(c) Find the range for the time constant RC such that the first harmonic of the ripple in $y(t)$ is less than 1% of the average value.

The ripple results from the exponential terms. Let $\tau = RC$.

Use first harmonic only:

$$Y(j\omega) \approx \frac{4}{\pi} \left[\delta(\omega) + \frac{1}{3} \left(\frac{\delta(\omega - 240\pi)}{1 + j240\pi RC} + \frac{\delta(\omega + 240\pi)}{1 + j240\pi RC} \right) \right]$$

$$y(t) = \frac{2}{\pi^2} + \frac{2}{3\pi^2} \left[\frac{e^{j240\pi t}}{1 + j240\pi RC} + \frac{e^{-j240\pi t}}{1 - j240\pi RC} \right]$$

$$|\text{ripple}| = \frac{2}{3\pi^2} \left[\frac{2}{\sqrt{1 + (240\pi\tau)^2}} \right] < 0.01 \left(\frac{2}{\pi^2} \right)$$

$$240\pi\tau > 66.659$$

$$\tau > 0.0884s$$

4.20. Consider the system depicted in Fig. P4.20 (a). The FT of the input signal is depicted in Fig. 4.20

(b). Let $z(t) \xleftrightarrow{FT} Z(j\omega)$ and $y(t) \xleftrightarrow{FT} Y(j\omega)$. Sketch $Z(j\omega)$ and $Y(j\omega)$ for the following cases.

(a) $w(t) = \cos(5\pi t)$ and $h(t) = \frac{\sin(6\pi t)}{\pi t}$

$$Z(j\omega) = \frac{1}{2\pi} X(j\omega) * W(j\omega)$$

$$W(j\omega) = \pi (\delta(\omega - 5\pi) + \delta(\omega + 5\pi))$$

$$Z(j\omega) = \frac{1}{2} (X(j(\omega - 5\pi)) + X(j(\omega + 5\pi)))$$

$$H(j\omega) = \begin{cases} 1 & |\omega| < 6\pi \\ 0 & \text{otherwise} \end{cases}$$

$$C(j\omega) = H(j\omega) Z(j\omega) = Z(j\omega)$$

$$Y(j\omega) = \frac{1}{2\pi} C(j\omega) * [\pi (\delta(\omega - 5\pi) + \delta(\omega + 5\pi))]$$

$$Y(j\omega) = \frac{1}{2} [Z(j(\omega - 5\pi)) + Z(j(\omega + 5\pi))]$$

$$= \frac{1}{4} [X(j(\omega - 10\pi)) + 2X(j\omega) + X(j(\omega + 10\pi))]$$

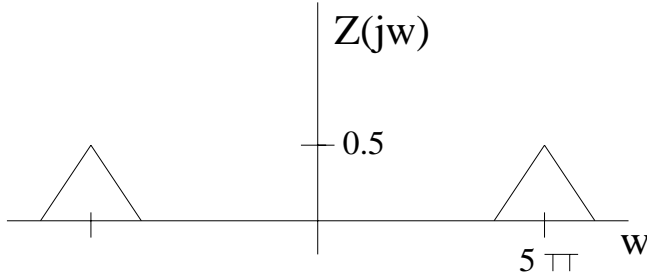


Figure P4.20. (a) Sketch of $Z(j\omega)$

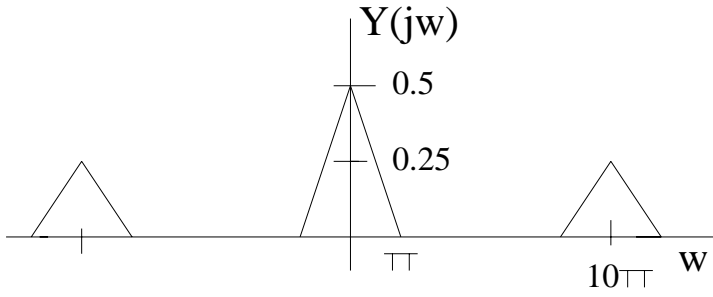


Figure P4.20. (a) Sketch of $Y(j\omega)$

(b) $w(t) = \cos(5\pi t)$ and $h(t) = \frac{\sin(5\pi t)}{\pi t}$

$$\begin{aligned} Z(j\omega) &= \frac{1}{2} [X(j(\omega - 5\pi)) + X(j(\omega + 5\pi))] \\ H(j\omega) &= \begin{cases} 1 & |\omega| < 5\pi \\ 0 & \text{otherwise} \end{cases} \\ C(j\omega) &= H(j\omega)Z(j\omega) \\ Y(j\omega) &= \frac{1}{2\pi} C(j\omega) * [\pi (\delta(\omega - 5\pi) + \delta(\omega + 5\pi))] \\ Y(j\omega) &= \frac{1}{2} [C(j(\omega - 5\pi)) + C(j(\omega + 5\pi))] \end{aligned}$$

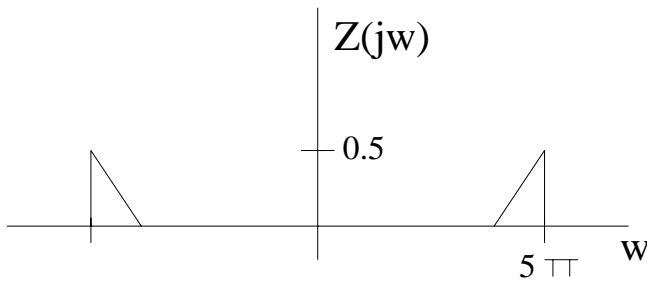


Figure P4.20. (b) Sketch of $Z(j\omega)$

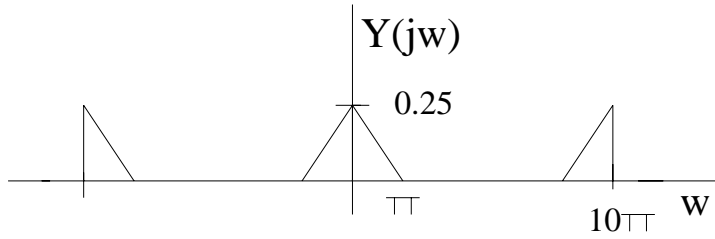


Figure P4.20. (b) Sketch of $Y(j\omega)$

(c) $w(t)$ depicted in Fig. P4.20 (c) and $h(t) = \frac{\sin(2\pi t)}{\pi t} \cos(5\pi t)$
 $T = 2$, $\omega_o = \pi$, $T_o = \frac{1}{2}$

$$\begin{aligned}
 W(j\omega) &= \sum_{k=-\infty}^{\infty} \frac{2 \sin(k\frac{\pi}{2})}{k} \delta(\omega - k\pi) \\
 Z(j\omega) &= \frac{1}{2\pi} X(j\omega) * W(j\omega) \\
 &= \sum_{k=-\infty}^{\infty} \frac{\sin(k\frac{\pi}{2})}{k\pi} X(j(\omega - k\pi)) \\
 C(j\omega) &= H(j\omega) Z(j\omega) \\
 &= \sum_{k=3}^7 \frac{\sin(k\frac{\pi}{2})}{k\pi} X(j(\omega - k\pi)) + \sum_{k=-3}^{-7} \frac{\sin(k\frac{\pi}{2})}{k\pi} X(j(\omega - k\pi)) \\
 Y(j\omega) &= \frac{1}{2} [C(j(\omega - 5\pi)) + C(j(\omega + 5\pi))]
 \end{aligned}$$

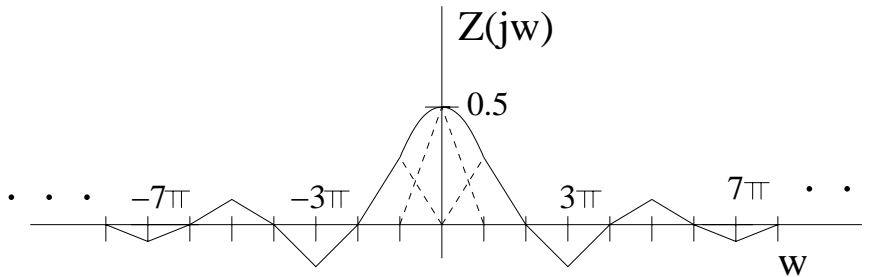


Figure P4.20. (c) Sketch of $Z(j\omega)$

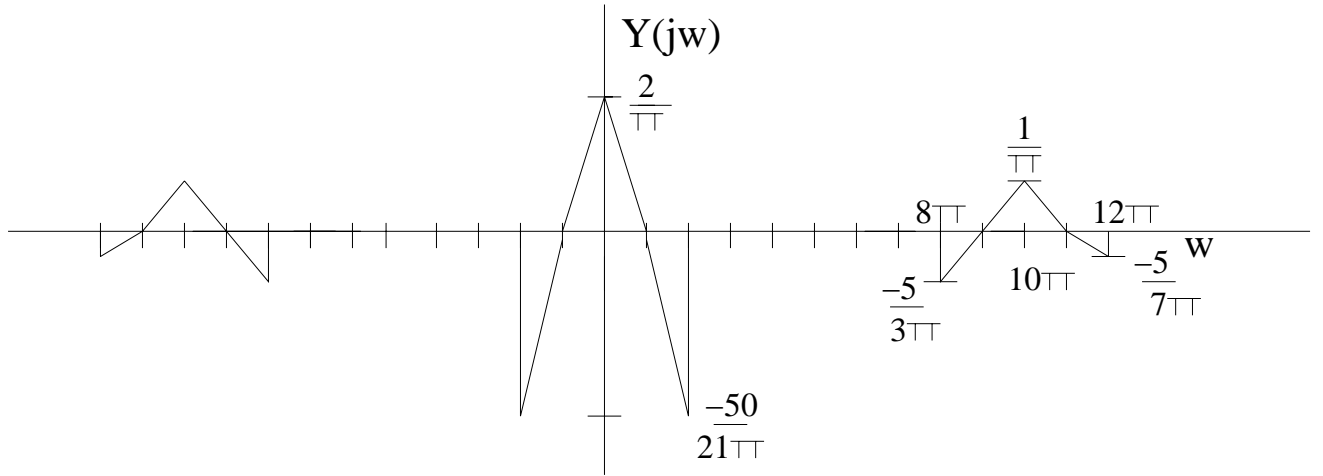


Figure P4.20. (c) Sketch of $Y(j\omega)$

4.21. Consider the system depicted in Fig. P4.21. The impulse response $h(t)$ is given by

$$h(t) = \frac{\sin(11\pi t)}{\pi t}$$

and we have

$$x(t) = \sum_{k=1}^{\infty} \frac{1}{k^2} \cos(k5\pi t)$$

$$g(t) = \sum_{k=1}^{10} \cos(k8\pi t)$$

Use the FT to determine $y(t)$.

$$\begin{aligned}
 y(t) &= [(x(t) * h(t))(g(t) * h(t))] * h(t) \\
 &= [x_h(t)g_h(t)] * h(t) \\
 &= m(t) * h(t) \\
 h(t) = \frac{\sin(11\pi t)}{\pi t} &\xleftrightarrow{FT} H(j\omega) = \begin{cases} 1 & |\omega| \leq 11\pi \\ 0 & \text{otherwise} \end{cases} \\
 x(t) = \sum_{k=1}^{\infty} \frac{1}{k^2} \cos(k5\pi t) &\xleftrightarrow{FT} X(j\omega) = \pi \sum_{k=1}^{\infty} \frac{1}{k^2} [\delta(\omega - 5k\pi) + \delta(\omega + 5k\pi)] \\
 g(t) = \sum_{k=1}^{10} \cos(k8\pi t) &= \pi \sum_{k=1}^{10} [\delta(\omega - 8k\pi) + \delta(\omega + 8k\pi)]
 \end{aligned}$$

$$\begin{aligned}
 X_h(j\omega) &= X(j\omega)H(j\omega) \\
 &= \pi \sum_{k=1}^2 \frac{1}{k^2} [\delta(\omega - 5k\pi) + \delta(\omega + 5k\pi)]
 \end{aligned}$$

$$\begin{aligned}
G_h(j\omega) &= G(j\omega)H(j\omega) \\
&= \pi\delta(\omega - 8\pi) + \pi\delta(\omega + 8\pi) \\
M(j\omega) &= \frac{1}{2\pi} X_h(j\omega) * G_h(j\omega) \\
&= \frac{1}{2} [X_h(j(\omega - 8\pi)) + X_h(j(\omega + 8\pi))] \\
&= \pi \sum_{k=1}^2 \frac{1}{k^2} [(\delta(\omega - 8\pi - 5k\pi) + \delta(\omega - 8\pi + 5k\pi)) + (\delta(\omega + 8\pi - 5k\pi) + \delta(\omega + 8\pi + 5k\pi))] \\
Y(j\omega) &= M(j\omega)H(j\omega) \\
&= \frac{\pi}{2} [\delta(\omega + 3\pi) + \delta(\omega - 3\pi)] + \frac{\pi}{8} [\delta(\omega - 2\pi) + \delta(\omega + 2\pi)] \\
y(t) &= \frac{1}{2} \cos(3\pi t) + \frac{1}{8} \cos(2\pi t)
\end{aligned}$$

4.22. The input to a discrete-time system is given by

$$x[n] = \cos\left(\frac{\pi}{8}n\right) + \sin\left(\frac{3\pi}{4}n\right)$$

Use the DTFT to find the output of the system, $y[n]$, for the following impulse responses $h[n]$, first note that

$$X(e^{j\Omega}) = \pi \left[\delta\left(\Omega - \frac{\pi}{8}\right) + \delta\left(\Omega + \frac{\pi}{8}\right) \right] + \frac{\pi}{j} \left[\delta\left(\Omega - \frac{3\pi}{4}\right) - \delta\left(\Omega + \frac{3\pi}{4}\right) \right]$$

(a) $h[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}$

$$\begin{aligned}
H(e^{j\Omega}) &= \begin{cases} 1 & |\Omega| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \leq |\Omega| < \pi, 2\pi \text{ periodic.} \end{cases} \\
Y(e^{j\Omega}) &= H(e^{j\Omega})X(e^{j\Omega}) \\
&= \pi \left[\delta\left(\Omega - \frac{\pi}{8}\right) + \delta\left(\Omega + \frac{\pi}{8}\right) \right] \\
y[n] &= \cos\left(\frac{\pi}{8}n\right)
\end{aligned}$$

(b) $h[n] = (-1)^n \frac{\sin(\frac{\pi}{2}n)}{\pi n}$

$$\begin{aligned}
h[n] &= e^{j\pi n} \frac{\sin(\frac{\pi}{2}n)}{\pi n} \\
H(e^{j\Omega}) &= \begin{cases} 1 & |\Omega - \pi| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \leq |\Omega - \pi| < \pi, 2\pi \text{ periodic.} \end{cases} \\
Y(e^{j\Omega}) &= H(e^{j\Omega})X(e^{j\Omega})
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{j} \left[\delta\left(\Omega - \frac{3\pi}{4}\right) + \delta\left(\Omega + \frac{3\pi}{4}\right) \right] \\
y[n] &= \sin\left(\frac{3\pi}{4}n\right)
\end{aligned}$$

(c) $h[n] = \cos\left(\frac{\pi}{2}n\right) \frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n}$

$$\begin{aligned}
h[n] &= \cos\left(\frac{\pi}{2}n\right) \frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n} \\
H(e^{j\Omega}) &= \begin{cases} \frac{1}{2} & |\Omega - \frac{\pi}{2}| \leq \frac{\pi}{5} \\ 0 & \frac{\pi}{5} \leq |\Omega - \frac{\pi}{2}| < \pi \end{cases} + \begin{cases} \frac{1}{2} & |\Omega + \frac{\pi}{2}| \leq \frac{\pi}{5} \\ 0 & \frac{\pi}{5} \leq |\Omega + \frac{\pi}{2}| < \pi \end{cases}, \quad 2\pi \text{ periodic.} \\
Y(e^{j\Omega}) &= H(e^{j\Omega})X(e^{j\Omega}) \\
&= 0 \\
y[n] &= 0
\end{aligned}$$

4.23. Consider the discrete-time system depicted in Fig. P4.23. Assume $h[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}$. Use the DTFT to determine the output, $y[n]$ for the following cases: Also sketch $G(e^{j\Omega})$, the DTFT of $g[n]$.

$$\begin{aligned}
y[n] &= (x[n]w[n]) * h[n] \\
&= g[n] * h[n] \\
h[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n} &\xleftrightarrow{FT} H(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq |\Omega| < \pi \end{cases} \\
&\quad H(e^{j\Omega}) \text{ is } 2\pi \text{ periodic.}
\end{aligned}$$

(a) $x[n] = \frac{\sin(\frac{\pi}{4}n)}{\pi n}$, $w[n] = (-1)^n$

$$\begin{aligned}
x[n] = \frac{\sin(\frac{\pi}{4}n)}{\pi n} &\xleftrightarrow{DTFT} X(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \leq |\Omega| < \pi \end{cases} \\
w[n] = e^{j\pi n} &\xleftrightarrow{DTFT} W(e^{j\Omega}) = 2\pi\delta(\Omega - \pi) \\
G(e^{j\Omega}) &= \frac{1}{2\pi} X(e^{j\Omega}) * W(e^{j\Omega}) \\
&= \begin{cases} 1 & |\Omega - \pi| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \leq |\Omega - \pi| < \pi \end{cases} \\
g[n] &= e^{j\pi n} \frac{\sin(\frac{\pi}{4}n)}{\pi n} \\
Y(e^{j\Omega}) &= G(e^{j\Omega})H(e^{j\Omega}) \\
&= 0 \\
y[n] &= 0
\end{aligned}$$

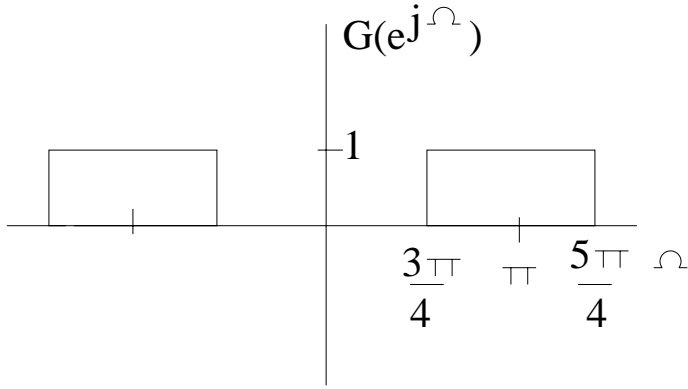


Figure P4.23. (a) The DTFT of $g[n]$

(b) $x[n] = \delta[n] - \frac{\sin(\frac{\pi}{4}n)}{\pi n}$, $w[n] = (-1)^n$

$$\begin{aligned}
 x[n] = \delta[n] - \frac{\sin(\frac{\pi}{4}n)}{\pi n} &\xleftrightarrow{DTFT} X(e^{j\Omega}) = \begin{cases} 0 & |\Omega| \leq \frac{\pi}{4} \\ 1 & \frac{\pi}{4} \leq |\Omega| < \pi \end{cases} \\
 w[n] = e^{j\pi n} &\xleftrightarrow{DTFT} W(e^{j\Omega}) = 2\pi\delta(\Omega - \pi) \\
 G(e^{j\Omega}) &= \frac{1}{2\pi} X(e^{j\Omega}) * W(e^{j\Omega}) \\
 &= \begin{cases} 0 & |\Omega - \pi| \leq \frac{3\pi}{4} \\ 1 & \frac{3\pi}{4} \leq |\Omega - \pi| < \pi \end{cases} \\
 g[n] &= \frac{\sin(\frac{3\pi}{4}n)}{\pi n} \\
 Y(e^{j\Omega}) &= G(e^{j\Omega})H(e^{j\Omega}) \\
 &= H(e^{j\Omega}) \\
 y[n] &= \frac{\sin(\frac{\pi}{2}n)}{\pi n}
 \end{aligned}$$

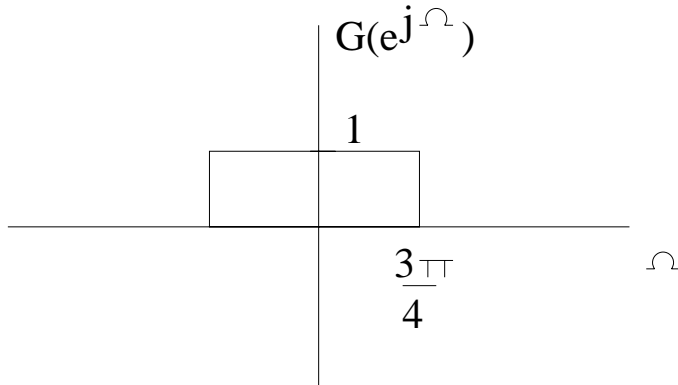


Figure P4.23. (b) The DTFT of $g[n]$

(c) $x[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}$, $w[n] = \cos(\frac{\pi}{2}n)$

$$\begin{aligned}
W(e^{j\Omega}) &= \pi \left[\delta(\Omega - \frac{\pi}{2}) + \delta(\Omega + \frac{\pi}{2}) \right], \text{ } 2\pi \text{ periodic} \\
G(e^{j\Omega}) &= \frac{1}{2\pi} X(e^{j\Omega}) * W(e^{j\Omega}) \\
&= \begin{cases} \frac{1}{2} & |\Omega - \frac{\pi}{2}| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq |\Omega - \frac{\pi}{2}| < \pi \end{cases} + \begin{cases} \frac{1}{2} & |\Omega + \frac{\pi}{2}| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq |\Omega + \frac{\pi}{2}| < \pi \end{cases} \\
g[n] &= \frac{1}{2} \frac{\sin(\frac{\pi}{2}n)}{\pi n} (e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}) \\
&= \frac{\sin(\frac{\pi}{2}n)}{\pi n} \cos(\frac{\pi}{2}n) \\
&= \frac{\sin(\pi n)}{2\pi n} \\
&= \frac{1}{2} \delta(n) \\
y[n] &= g[n] * h[n] \\
&= \frac{1}{2} h[n] \\
&= \frac{\sin(\pi n)}{2\pi n}
\end{aligned}$$

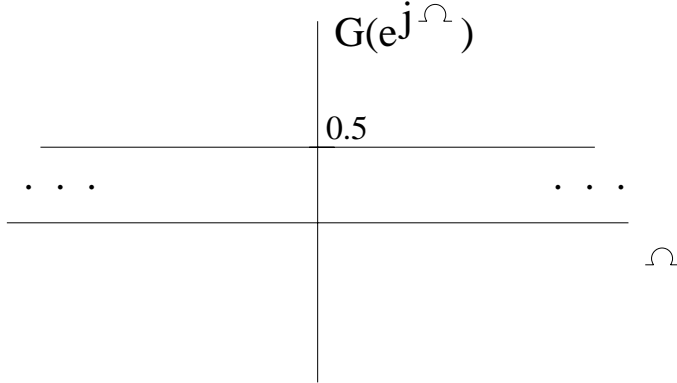


Figure P4.23. (c) The DTFT of $g[n]$

(d) $x[n] = 1 + \sin(\frac{\pi}{16}n) + 2\cos(\frac{3\pi}{4}n)$, $w[n] = \cos(\frac{3\pi}{8}n)$

$$\begin{aligned}
X(e^{j\Omega}) &= 2\pi\delta(\Omega) + \frac{\pi}{j} \left[\delta(\Omega - \frac{\pi}{16}) + \delta(\Omega + \frac{\pi}{16}) \right] + 2\pi \left[\delta(\Omega - \frac{3\pi}{4}) + \delta(\Omega + \frac{3\pi}{4}) \right], \text{ } 2\pi \text{ periodic.} \\
W(e^{j\Omega}) &= \pi \left[\delta(\Omega - \frac{3\pi}{8}) + \delta(\Omega + \frac{3\pi}{8}) \right], \text{ } 2\pi \text{ periodic} \\
G(e^{j\Omega}) &= \frac{1}{2\pi} X(e^{j\Omega}) * W(e^{j\Omega}) \\
&= \frac{1}{2\pi} \left[X(e^{j(\Omega - \frac{3\pi}{8})}) + X(e^{j(\Omega + \frac{3\pi}{8})}) \right] \\
&= \frac{1}{2\pi} \left[2\pi\delta(\Omega - \frac{3\pi}{8}) + \frac{\pi}{j} \left(\delta(\Omega - \frac{7\pi}{16}) - \delta(\Omega - \frac{5\pi}{16}) \right) + 2\pi \left(\delta(\Omega - \frac{9\pi}{8}) + \delta(\Omega + \frac{3\pi}{8}) \right) \right] \\
&\quad + \frac{1}{2\pi} \left[2\pi\delta(\Omega + \frac{3\pi}{8}) + \frac{\pi}{j} \left(\delta(\Omega + \frac{5\pi}{16}) - \delta(\Omega + \frac{7\pi}{16}) \right) + 2\pi \left(\delta(\Omega - \frac{3\pi}{8}) + \delta(\Omega + \frac{9\pi}{8}) \right) \right]
\end{aligned}$$

$$\begin{aligned}
g[n] &= \frac{2}{\pi} \cos\left(\frac{3\pi}{8}n\right) + \frac{1}{2\pi} \sin\left(\frac{7\pi}{16}n\right) - \frac{1}{2\pi} \sin\left(\frac{5\pi}{16}n\right) + \frac{1}{\pi} \cos\left(\frac{9\pi}{8}n\right) \\
Y(e^{j\Omega}) &= G(e^{j\Omega})H(e^{j\Omega}) \\
y[n] &= \frac{2}{\pi} \cos\left(\frac{3\pi}{8}n\right) + \frac{1}{2\pi} \sin\left(\frac{7\pi}{16}n\right) - \frac{1}{2\pi} \sin\left(\frac{5\pi}{16}n\right)
\end{aligned}$$

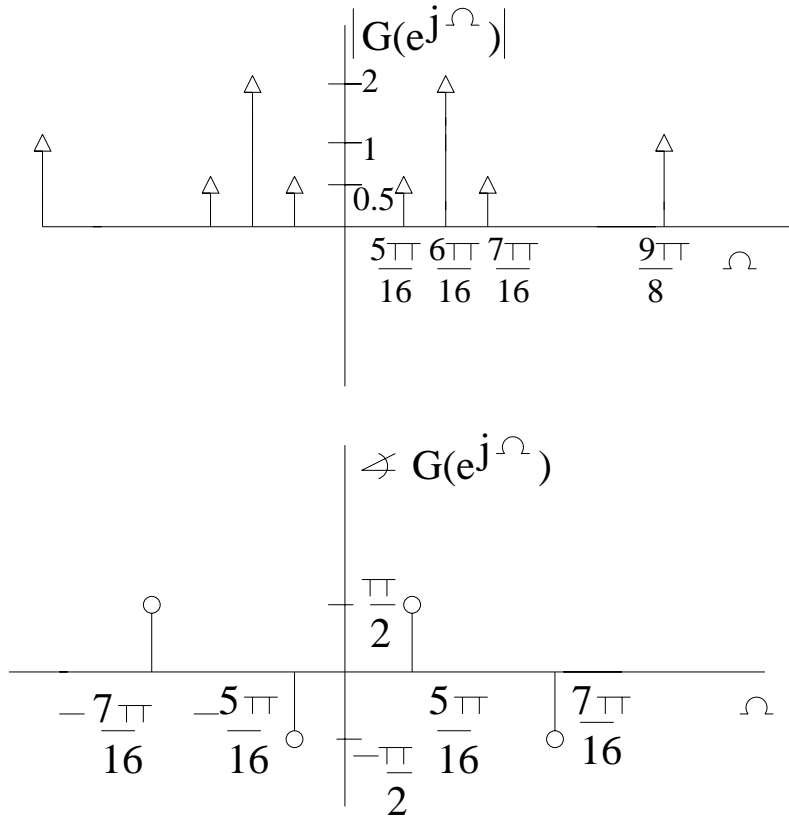


Figure P4.23. (d) The DTFT of $g[n]$

4.24. Determine and sketch the FT representation, $X_\delta(j\omega)$, for the following discrete-time signals with the sampling interval T_s as given:

$$\begin{aligned}
X_\delta(j\omega) &= \sum_{n=-\infty}^{\infty} e^{-j\omega T_s n} \\
&= X(e^{j\Omega})|_{\Omega=\omega T_s}
\end{aligned}$$

(a) $x[n] = \frac{\sin(\frac{\pi}{3}n)}{\pi n}$, $T_s = 2$

$$X(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \leq \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \leq |\Omega| < \pi \end{cases}$$

$$\begin{aligned}
X_\delta(j\omega) &= \begin{cases} 1 & |2\omega| \leq \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \leq |2\omega| < \pi \end{cases} \\
&= \begin{cases} 1 & |\omega| \leq \frac{\pi}{6} \\ 0 & \frac{\pi}{6} \leq |\omega| < \frac{\pi}{6} \end{cases} \\
&\quad X_\delta(j\omega) \text{ is } \pi \text{ periodic}
\end{aligned}$$

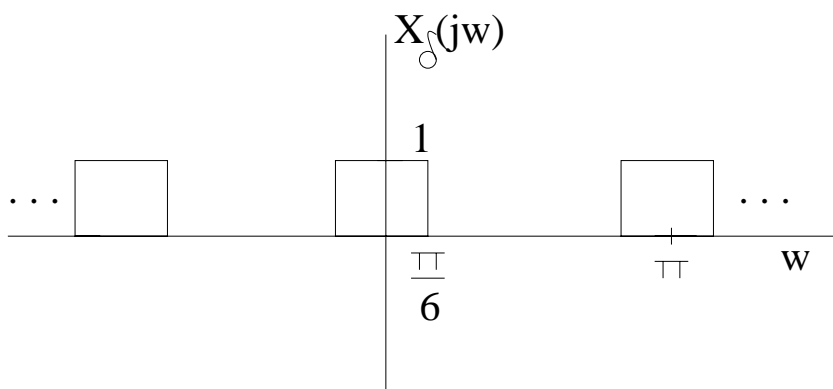


Figure P4.24. (a) FT of $X_\delta(j\omega)$

(b) $x[n] = \frac{\sin(\frac{\pi}{3}n)}{\pi n}$, $T_s = \frac{1}{4}$

$$\begin{aligned}
X(e^{j\Omega}) &= \begin{cases} 1 & |\Omega| \leq \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \leq |\Omega| < \pi \end{cases} \\
X_\delta(j\omega) &= \begin{cases} 1 & |\frac{1}{4}\omega| \leq \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \leq |\frac{1}{4}\omega| < \pi \end{cases} \\
&= \begin{cases} 1 & |\omega| \leq \frac{4\pi}{3} \\ 0 & \frac{4\pi}{3} \leq |\omega| < 4\pi \end{cases} \\
&\quad X_\delta(j\omega) \text{ is } 8\pi \text{ periodic}
\end{aligned}$$

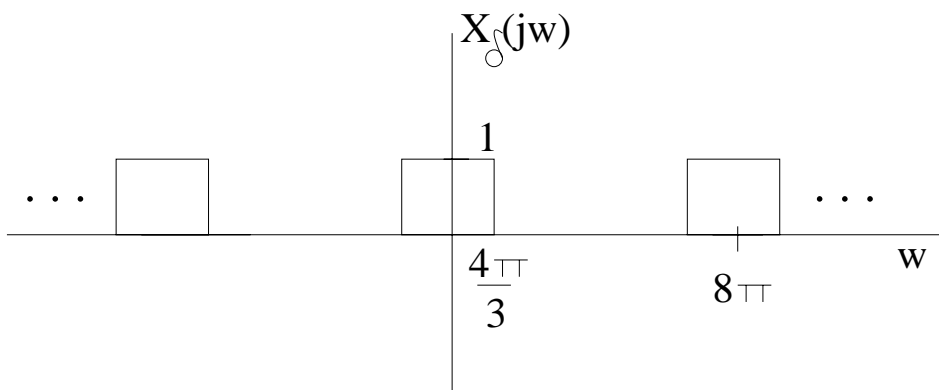


Figure P4.24. (b) FT of $X_\delta(j\omega)$

(c) $x[n] = \cos(\frac{\pi}{2}n) \frac{\sin(\frac{\pi}{4}n)}{\pi n}, \quad T_s = 2$

$$\begin{aligned}
 X(e^{j\Omega}) &= \begin{cases} \frac{1}{2} & |\Omega - \frac{\pi}{2}| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \leq |\Omega - \frac{\pi}{2}| < \pi \end{cases} + \begin{cases} \frac{1}{2} & |\Omega + \frac{\pi}{2}| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \leq |\Omega + \frac{\pi}{2}| < \pi \end{cases} \\
 X_\delta(j\omega) &= \begin{cases} \frac{1}{2} & |2\omega - \frac{\pi}{2}| \leq \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \leq |2\omega - \frac{\pi}{2}| < \pi \end{cases} + \begin{cases} \frac{1}{2} & |2\omega + \frac{\pi}{2}| \leq \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \leq |2\omega + \frac{\pi}{2}| < \pi \end{cases} \\
 &= \begin{cases} \frac{1}{2} & \frac{\pi}{8} < \omega < \frac{3\pi}{8} \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \frac{1}{2} & -\frac{3\pi}{8} < \omega < -\frac{\pi}{8} \\ 0 & \text{otherwise} \end{cases} \\
 &\quad X_\delta(j\omega) \text{ is } \pi \text{ periodic}
 \end{aligned}$$

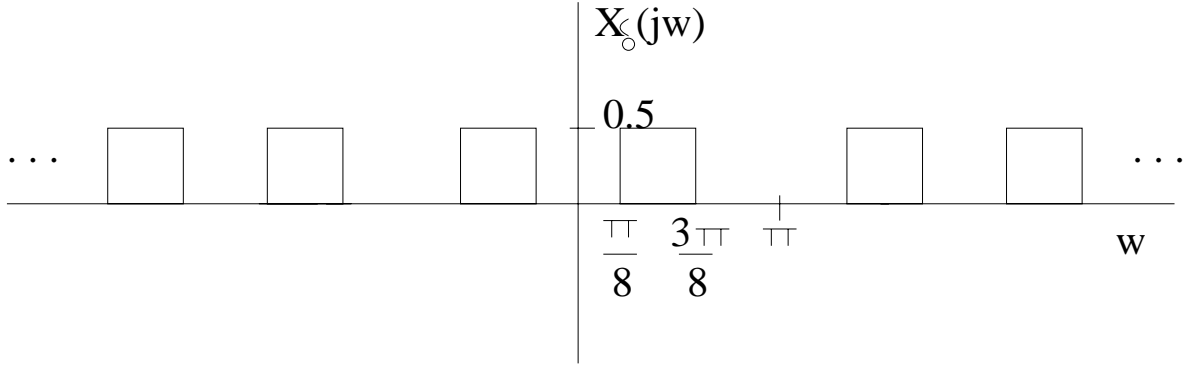


Figure P4.24. (c) FT of $X_\delta(j\omega)$

(d) $x[n]$ depicted in Fig. P4.17 (a) with $T_s = 4$.

$$\begin{aligned}
 \text{DTFS: } N = 8 \quad \Omega_o &= \frac{\pi}{4} \\
 X[k] &= \frac{\sin(k \frac{5\pi}{8})}{8 \sin(k \frac{\pi}{8})}, \quad k \in [-3, 4] \\
 \text{DTFT: } X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\Omega - k \frac{\pi}{4}) \\
 \text{FT: } X_\delta(j\omega) &= \frac{\pi}{4} \sum_{k=-\infty}^{\infty} \frac{\sin(k \frac{5\pi}{8})}{8 \sin(k \frac{\pi}{8})} \delta(4\omega - k \frac{\pi}{4}) \\
 &= \frac{\pi}{16} \sum_{k=-\infty}^{\infty} \frac{\sin(k \frac{5\pi}{8})}{8 \sin(k \frac{\pi}{8})} \delta(\omega - k \frac{\pi}{16}) \\
 &\quad X_\delta(j\omega) \text{ is } \frac{\pi}{2} \text{ periodic}
 \end{aligned}$$

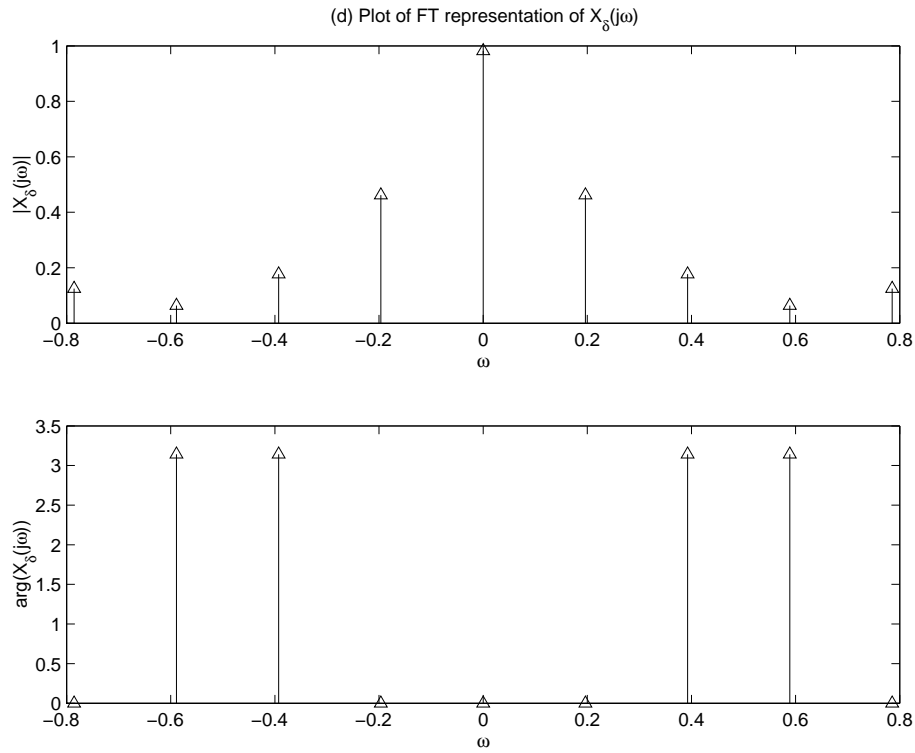


Figure P4.24. (d) FT of $X_\delta(j\omega)$

(e) $x[n] = \sum_{p=-\infty}^{\infty} \delta[n - 4p], \quad T_s = \frac{1}{8}$

$$\text{DTFS: } N = 4 \quad \Omega_o = \frac{\pi}{2}$$

$$X[k] = \frac{1}{4}, \quad k \in [0, 3]$$

$$\text{DTFT: } X(e^{j\Omega}) = \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\frac{\pi}{2})$$

$$\text{FT: } X_\delta(j\omega) = \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta(\frac{1}{8}\omega - k\frac{\pi}{2})$$

$$= 4\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k4\pi)$$

$X_\delta(j\omega)$ is 4π periodic

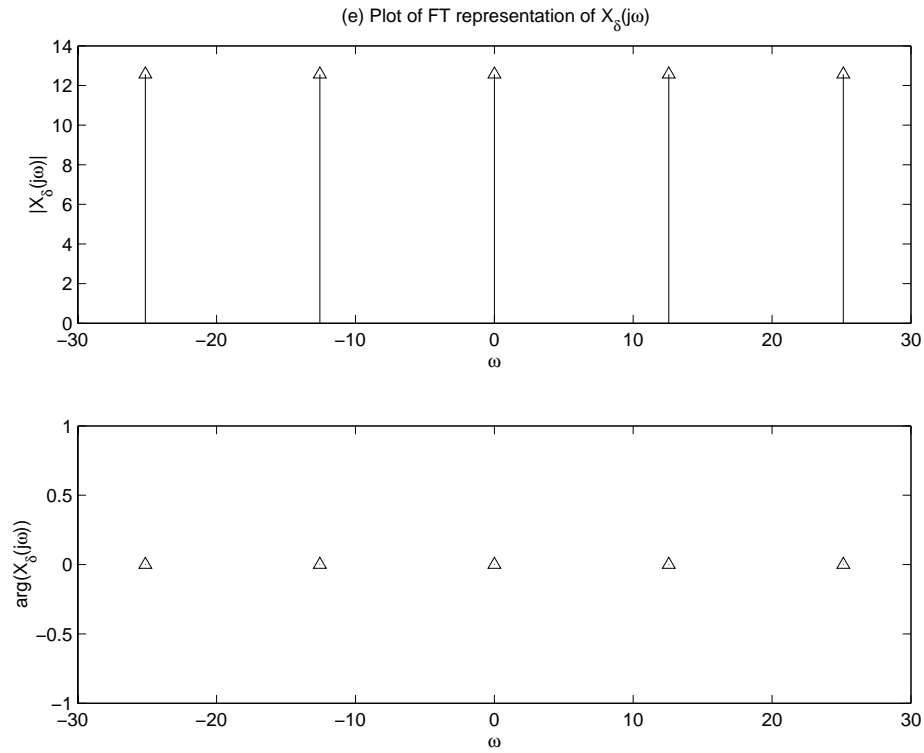


Figure P4.24. (e) FT of $X_\delta(j\omega)$

4.25. Consider sampling the signal $x(t) = \frac{1}{\pi t} \sin(2\pi t)$.

(a) Sketch the FT of the sampled signal for the following sampling intervals:

- (i) $T_s = \frac{1}{8}$
- (ii) $T_s = \frac{1}{3}$
- (iii) $T_s = \frac{1}{2}$
- (iv) $T_s = \frac{2}{3}$

In part (iv), aliasing occurs. The signals overlap and add, which can be seen in the following figure.

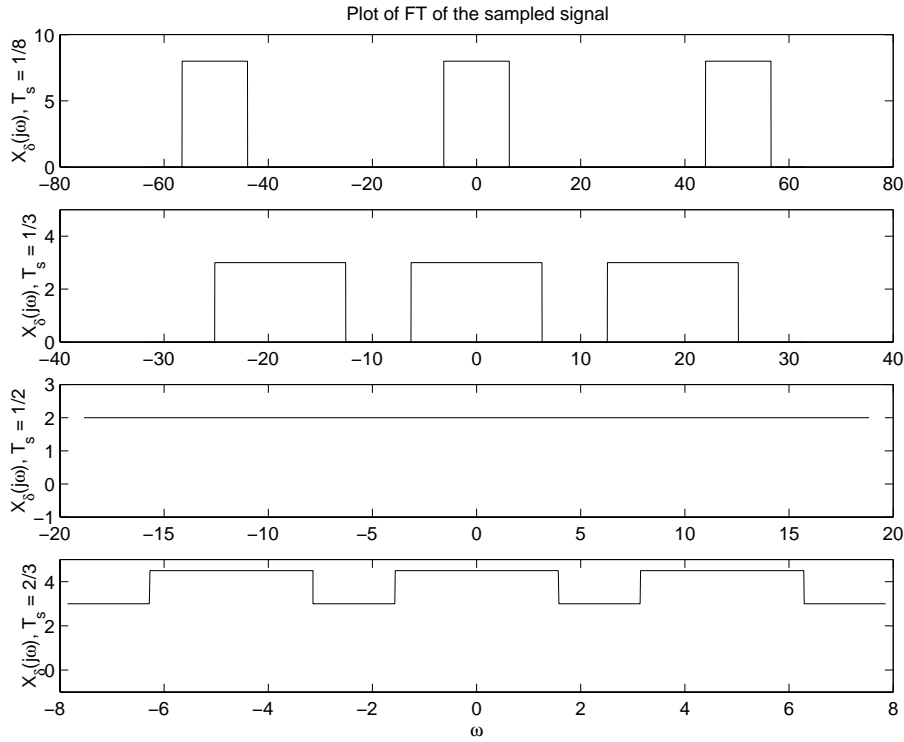


Figure P4.25. (a) Sketch of $X_\delta(j\omega)$

(b) Let $x[n] = x(nT_s)$. Sketch the DTFT of $x[n]$, $\tilde{X}(e^{j\Omega})$, for each of the sampling intervals given in (a).

$$x[n] = x(nT_s) = \frac{1}{\pi nT_s} \sin(2\pi nT_s) \xleftrightarrow{DTFT} \tilde{X}(e^{j\Omega})$$

$$\tilde{X}(e^{j\Omega}) = \begin{cases} \frac{1}{T_s} & |\Omega| \leq T_s\pi \\ 0 & T_s\pi \leq |\Omega| < \pi \end{cases} \quad 2\pi \text{ periodic}$$

Notice that the difference between the figure in (a) and (b) is that the ' ω ' axis has been scaled by the sampling rate.

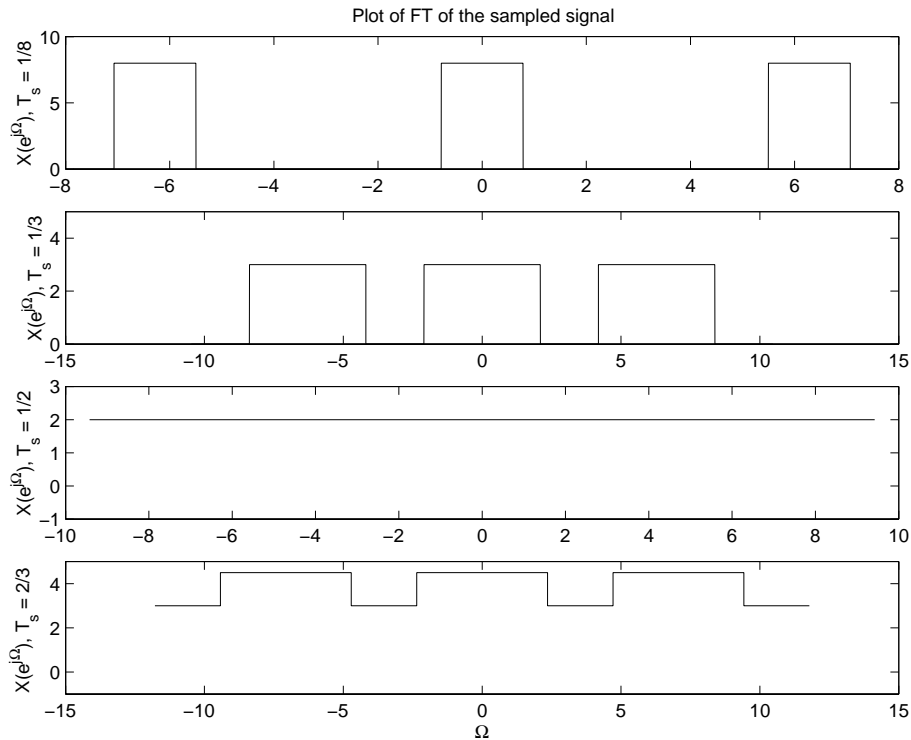


Figure P4.25. (b) Sketch of $X(e^{j\Omega})$

4.26. The continuous-time signal $x(t)$ with FT as depicted in Fig. P4.26 is sampled. Identify in each case if aliasing occurs.

(a) Sketch the FT of the sampled signal for the following sampling intervals:

(i) $T_s = \frac{1}{14}$

No aliasing occurs.

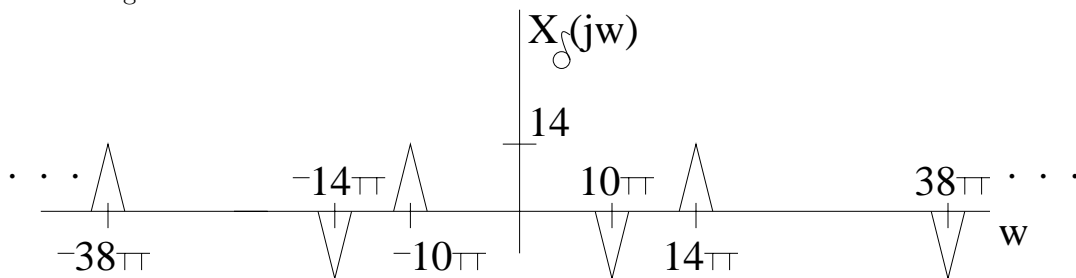


Figure P4.26. (i) FT of the sampled signal

(ii) $T_s = \frac{1}{7}$

Since $T_s > \frac{1}{11}$, aliasing occurs.

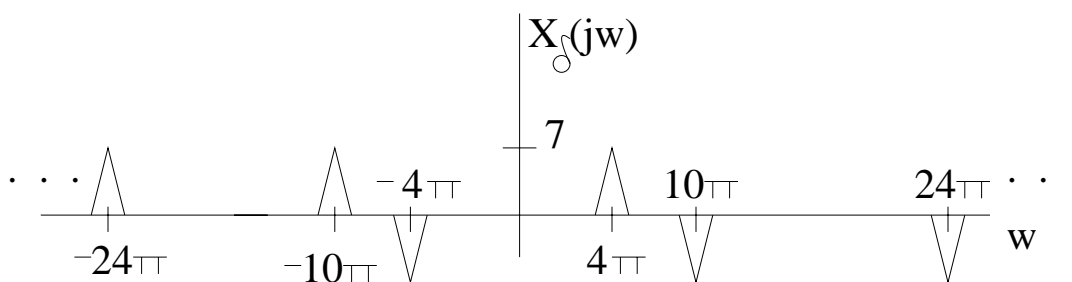


Figure P4.26. (ii) FT of the sampled signal

(iii) $T_s = \frac{1}{5}$
 Since $T_s > \frac{1}{11}$, aliasing occurs.

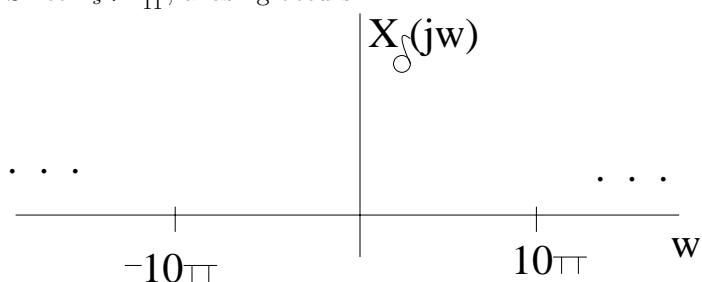


Figure P4.26. (iii) FT of the sampled signal

(b) Let $x[n] = x(nT_s)$. Sketch the DTFT of $x[n]$, $X(e^{j\Omega})$, for each of the sampling intervals given in (a).

The DTFT simple scales the 'x' axis by the sampling rate.

(i) $T_s = \frac{1}{14}$

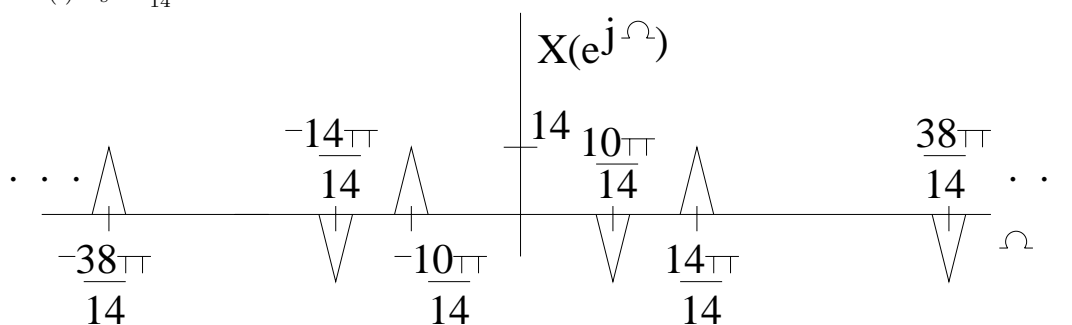


Figure P4.26. (i) DTFT of $x[n]$

(ii) $T_s = \frac{1}{7}$

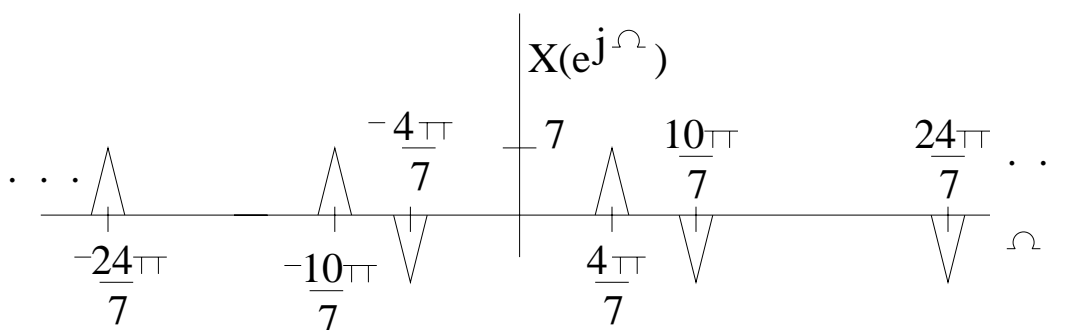


Figure P4.26. (ii) DTFT of $x[n]$

(iii) $T_s = \frac{1}{5}$

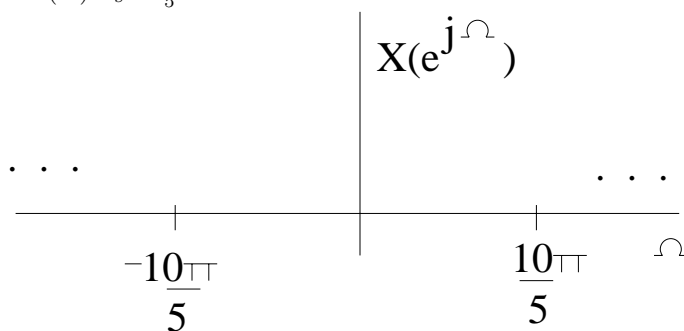


Figure P4.26. (iii) DTFT of $x[n]$

4.27. Consider subsampling the signal $x[n] = \frac{\sin(\frac{\pi}{6}n)}{\pi n}$ so that $y[n] = x[qn]$. Sketch $Y(e^{j\Omega})$ for the following choices of q :

$$\begin{aligned} X(e^{j\Omega}) &= \begin{cases} 1 & |\omega| \leq \frac{\pi}{6} \\ 0 & \frac{\pi}{6} \leq |\omega| < \pi \end{cases} \quad 2\pi\text{periodic} \\ q[n] &= x[qn] \\ Y(e^{j\Omega}) &= \frac{1}{q} \sum_{m=0}^{q-1} X\left(e^{j\frac{1}{q}(\Omega - m2\pi)}\right) \end{aligned}$$

(a) $q = 2$

$$Y(e^{j\Omega}) = \frac{1}{2} \sum_{m=0}^1 X\left(e^{j\frac{1}{2}(\Omega - m2\pi)}\right)$$

(b) $q = 4$

$$Y(e^{j\Omega}) = \frac{1}{4} \sum_{m=0}^3 X\left(e^{j\frac{1}{4}(\Omega - m2\pi)}\right)$$

(c) $q = 8$

$$Y(e^{j\Omega}) = \frac{1}{8} \sum_{m=0}^7 X\left(e^{j\frac{1}{8}(\Omega - m2\pi)}\right)$$

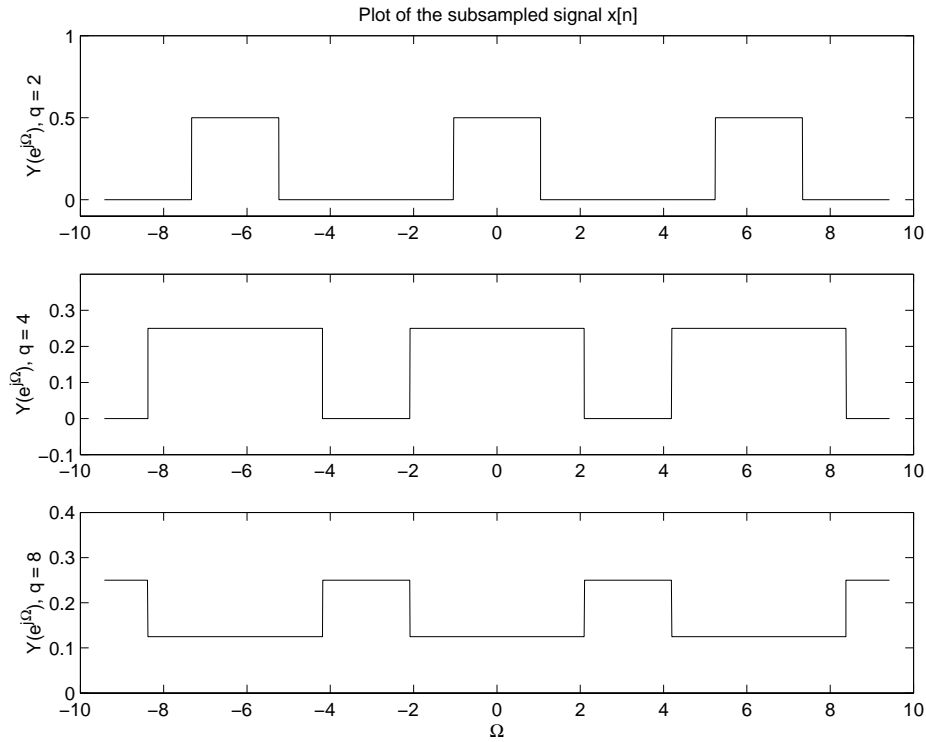


Figure P4.27. Sketch of $Y(e^{j\Omega})$

4.28. The discrete-time signal $x[n]$ with DTFT depicted in Fig. P4.28 is subsampled to obtain $y[n] = x[qn]$. Sketch $Y(e^{j\Omega})$ for the following choices of q :

- (a) $q = 3$
- (b) $q = 4$
- (c) $q = 8$

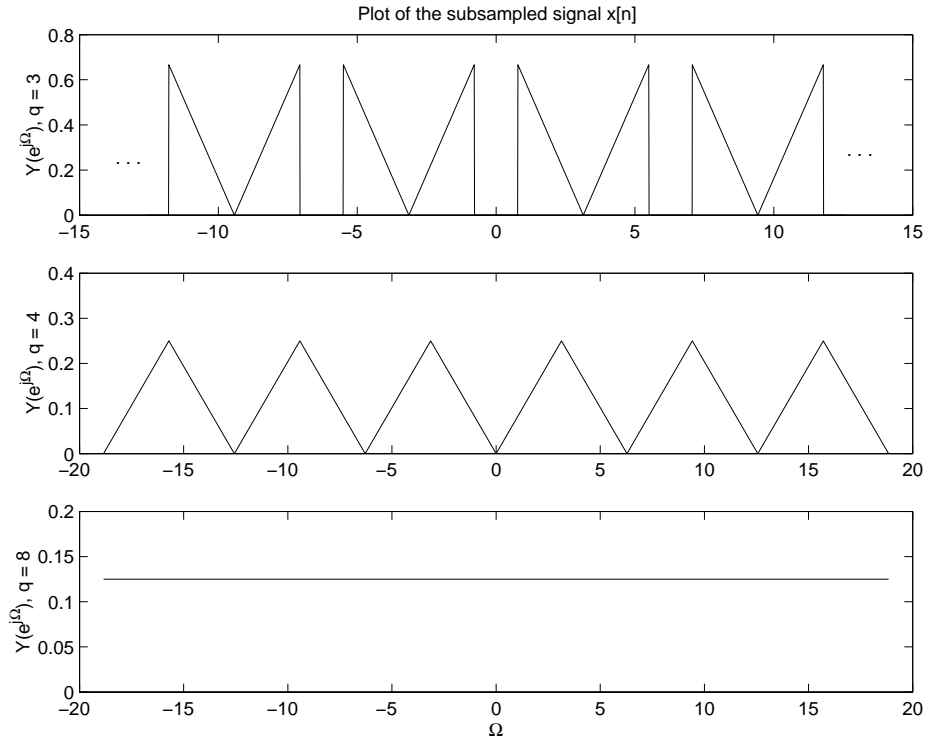


Figure P4.28. Sketch of $Y(e^{j\Omega})$

4.29. For each of the following signals sampled with sampling interval T_s , determine the bounds on T_s that guarantee there will be no aliasing.

(a) $x(t) = \frac{1}{t} \sin 3\pi t + \cos(2\pi t)$

$$\begin{aligned}
 \frac{1}{t} \sin(3\pi t) &\xleftrightarrow{FT} \begin{cases} \frac{1}{\pi} & |\omega| \leq 3\pi \\ 0 & \text{otherwise} \end{cases} \\
 \cos(2\pi t) &\xleftrightarrow{FT} \pi\delta(\Omega - 2\pi) + \pi\delta(\Omega + 2\pi) \\
 \omega_{max} &= 3\pi \\
 T &< \frac{\pi}{\omega_{max}} \\
 T &< \frac{1}{3}
 \end{aligned}$$

(b) $x(t) = \cos(12\pi t) \frac{\sin(\pi t)}{2t}$

$$\begin{aligned}
 X(j\omega) &= \begin{cases} \frac{1}{4\pi} & |\omega - 12\pi| \leq \pi \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \frac{1}{4\pi} & |\omega + 12\pi| \leq \pi \\ 0 & \text{otherwise} \end{cases} \\
 \omega_{max} &= 13\pi \\
 T &< \frac{\pi}{\omega_{max}}
 \end{aligned}$$

$$T < \frac{1}{13}$$

(c) $x(t) = e^{-6t}u(t) * \frac{\sin(Wt)}{\pi t}$

$$\begin{aligned} X(j\omega) &= \frac{1}{6+j\omega} [u(\omega+W) - u(\omega-W)] \\ \omega_{max} &= W \\ T &< \frac{\pi}{\omega_{max}} \\ T &< \frac{\pi}{W} \end{aligned}$$

(d) $x(t) = w(t)z(t)$, where the FTs $W(j\omega)$ and $Z(j\omega)$ are depicted in Fig. P4.29.

$$\begin{aligned} X(j\omega) &= \frac{1}{2\pi} W(j\omega) * G(j\omega) \\ \omega_{max} &= 4\pi + w_a \\ T &< \frac{\pi}{\omega_{max}} \\ T &< \frac{\pi}{4\pi + w_a} \end{aligned}$$

4.30. Consider the system depicted in Fig. P4.30. Assume $|X(j\omega)| = 0$ for $|\omega| > \omega_m$. Find the largest value of T such that $x(t)$ can be reconstructed from $y(t)$. Determine a system that will perform the reconstruction for this maximum value of T .

For reconstruction, we need to have $w_s > 2\omega_{max}$, or $T < \frac{\pi}{\omega_{max}}$. A finite duty cycle results in distortion.

$$\begin{aligned} W[k] &= \frac{\sin(\frac{\pi}{2}k)}{k\pi} e^{-j\frac{\pi}{2}k} \\ W(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} W[k] \delta(\omega - k\frac{2\pi}{T}) \end{aligned}$$

After multiplication:

$$Y(j\omega) = \sum_{k=-\infty}^{\infty} \frac{\sin(\frac{\pi}{2}k)}{k\pi} e^{-j\frac{\pi}{2}k} X(j(\omega - k\frac{2\pi}{T}))$$

To reconstruct:

$$H_r(j\omega)Y(j\omega) = X(j\omega), \quad |\omega| < \omega_{max}, \quad \frac{2\pi}{T} > 2\omega_{max}$$

$$k = 0$$

$$H_r(j\omega) \frac{1}{2} X(j\omega) = X(j\omega)$$

$$H_r(j\omega) = \begin{cases} 2 & |\omega| < \omega_{max} \\ \text{don't care} & \omega_{max} < |\omega| < \frac{2\pi}{T} - \omega_{max} \\ 0 & |\omega| > \frac{2\pi}{T} - \omega_{max} \end{cases}$$

4.31. Let $|X(j\omega)| = 0$ for $|\omega| > \omega_m$. Form the signal $y(t) = x(t)[\cos(3\pi t) + \sin(10\pi t)]$. Determine the maximum value of ω_m for which $x(t)$ can be reconstructed from $y(t)$ and specify a system that that will perform the reconstruction.

$$Y(j\omega) = \frac{1}{2} [X(j(\omega - 2\pi)) + X(j(\omega + 2\pi)) - jX(j(\omega - 10\pi)) + jX(j(\omega + 10\pi))]$$

$x(t)$ can be reconstructed from $y(t)$ if there is no overlap among the four shifted versions of $X(j\omega)$, yet $x(t)$ can still be reconstructed when overlap occurs, provided that there is at least one shifted $X(j\omega)$ that is not contaminated.

$$\omega_m \max = \frac{10\pi - 4\pi}{2} = 4\pi$$

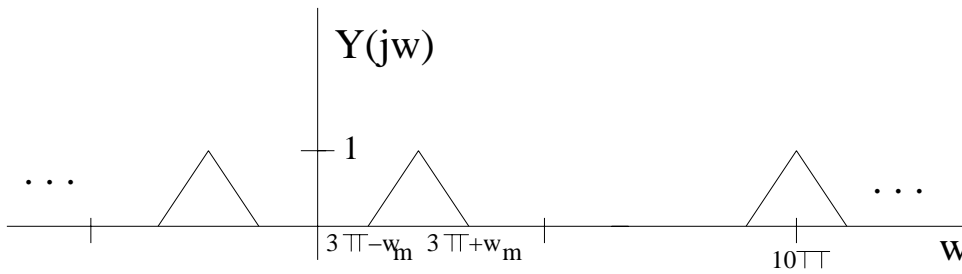


Figure P4.31. $Y(j\omega)$

We require $10\pi - \omega_m > 3\pi + \omega_m$, thus $\omega_m < \frac{7\pi}{2}$. To recover the signal, bandpassfilter with passband $6.5\pi \leq \omega \leq 13.5\pi$ and multiply with $2\sin(10\pi t)$ to retrieve $x(t)$.

4.32. A reconstruction system consists of a zero-order hold followed by a continuous-time anti-imaging filter with frequency response $H_c(j\omega)$. The original signal $x(t)$ is bandlimited to ω_m , that is, $X(j\omega) = 0$ for $|\omega| > \omega_m$ and is sampled with a sampling interval of T_s . Determine the constraints on the magnitude response of the anti-imaging filter so that the overall magnitude response of this reconstruction system is between 0.99 and 1.01 in the signal passband and less than 10^{-4} to the images of the signal spectrum for the following values:

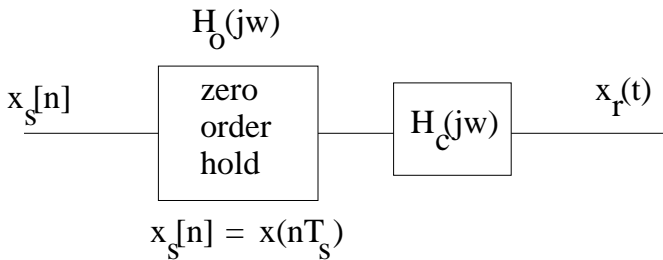


Figure P4.32. Reconstruction system.

(1) $0.99 < |H_o(j\omega)||H_c(j\omega)| < 1.01$, $-\omega_m \leq \omega \leq \omega_m$

Thus:

$$(i) \quad |H_c(j\omega)| > \frac{0.99\omega}{2\sin(\omega\frac{T_s}{2})}$$

$$(ii) |H_c(j\omega)| < \frac{1.01\omega}{2 \sin(\omega \frac{T_s}{2})}$$

The passband constraint for each case is:

$$\frac{0.99\omega}{2 \sin(\omega \frac{T_s}{2})} < |H_c(j\omega)| < \frac{1.01\omega}{2 \sin(\omega \frac{T_s}{2})}$$

The stopband constraint is $|H_o(j\omega)||H_c(j\omega)| < 10^{-4}$, at the worst case $\omega = \frac{2\pi}{T_s} - \omega_m$.

$$|H_c(j\omega)| < \left| \frac{10^{-4}\omega}{2 \sin(\omega \frac{T_s}{2})} \right|$$

(a) $\omega_m = 10\pi$, $T_s = 0.1$

$$\begin{aligned} \omega &= \frac{2\pi}{T_s} - \omega_m \\ &= \frac{2\pi}{0.1} - 10\pi = 10\pi \\ |H_c(j10\pi)| &< \left| \frac{10^{-4}10\pi}{2 \sin(10\pi(\frac{0.1}{2}))} \right| \\ &= |5\pi(10^{-4})| = 0.001571 \end{aligned}$$

(b) $\omega_m = 10\pi$, $T_s = 0.05$

$$\begin{aligned} \omega &= \frac{2\pi}{0.05} - 10\pi = 30\pi \\ |H_c(j30\pi)| &< \left| \frac{30\pi(10^{-4})}{2 \sin(30\pi(\frac{0.05}{2}))} \right| \\ &= \left| \frac{30\pi}{\sqrt{2}}(10^{-4}) \right| = 0.006664 \end{aligned}$$

(c) $\omega_m = 10\pi$, $T_s = 0.02$

$$\begin{aligned} \omega &= \frac{2\pi}{0.02} - 10\pi = 90\pi \\ |H_c(j90\pi)| &< \left| \frac{10^{-4}90\pi}{2 \sin(90\pi(\frac{0.02}{2}))} \right| \\ &= \left| \frac{90\pi(10^{-4})}{2 \sin(0.9\pi)} \right| = 0.04575 \end{aligned}$$

(d) $\omega_m = 2\pi$, $T_s = 0.05$

$$\omega = \frac{2\pi}{0.05} - 10\pi = 30\pi$$

$$\begin{aligned}
|H_c(j30\pi)| &< \left| \frac{30\pi(10^{-4})}{2\sin(30\pi(\frac{0.05}{2}))} \right| \\
&= \left| \frac{10^{-4}30\pi}{2\sin(\frac{3}{4}\pi)} \right| = 0.006664
\end{aligned}$$

4.33. The zero-order hold produces a staircase approximation to the sampled signal $x(t)$ from samples $x[n] = x(nT_s)$. A device termed a first-order hold linearly interpolates between the samples $x[n]$ and thus produces a smoother approximation to $x(t)$. The output of the first-order hold may be described as

$$x_1(t) = \sum_{n=-\infty}^{\infty} x[n]h_1(t - nT_s)$$

where $h_1(t)$ is the triangular pulse shown in Fig. P4.33 (a). The relationship between $x[n]$ and $x_1(t)$ is depicted in Fig. P4.33 (b).

(a) Identify the distortions introduced by the first-order hold and compare them to those introduced by the zero-order hold. *Hint:* $h_1(t) = h_o(t) * h_o(t)$.

$$\begin{aligned}
x_1(t) &= \sum_{n=-\infty}^{\infty} x[n]h_1(t - nT_s) \\
&= h_o(t) * h_o(t) * \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)
\end{aligned}$$

Thus:

$$X_1(j\omega) = H_o(j\omega)H_o(j\omega)X_{\Delta}(j\omega)$$

Which implies:

$$H_1(j\omega) = e^{-j\omega T_s} \frac{4\sin^2(\omega \frac{T_s}{2})}{\omega^2}$$

Distortions:

- (1) A linear phase shift corresponding to a time delay of T_s seconds (a unit of sampling time).
- (2) $\sin^2(\cdot)$ term introduces more distortion to the portion of $X_{\Delta}(j\omega)$, especially the higher frequency part is severely attenuated compared to the low frequency part which falls within the mainlobe, between $-\omega_m$ and ω_m .
- (3) Distorted and attenuated versions of $X(j\omega)$ still remain at the nonzero multiples of ω_m , yet it is lower than the case of the zero order hold.

(b) Consider a reconstruction system consisting of a first-order hold followed by an anti-imaging filter with frequency response $H_c(j\omega)$. Find $H_c(j\omega)$ so that perfect reconstruction is obtained.

$$\begin{aligned}
X_{\Delta}(j\omega)H_1(j\omega)H_c(j\omega) &= X(j\omega) \\
H_c(j\omega) &= \frac{e^{j\omega T_s}\omega^2}{4\sin^2(\omega \frac{T_s}{2})}T_s H_{LPF}(j\omega)
\end{aligned}$$

where $H_{LPF}(j\omega)$ is an ideal low pass filter.

$$H_c(j\omega) = \begin{cases} \frac{e^{j\omega T_s} \omega^2}{4 \sin^2(\omega \frac{T_s}{2})} T_s & |\omega| \leq \omega_m \\ \text{don't care} & \omega_m \leq |\omega| < \frac{2\pi}{T_s} - \omega_m \\ 0 & |\omega| > \frac{2\pi}{T_s} - \omega_m \end{cases}$$

Assuming $X(j\omega) = 0$ for $|\omega| > \omega_m$

(c) Determine the constraints on $|H_c(j\omega)|$ so that the overall magnitude response of this reconstruction system is between 0.99 and 1.01 in the signal passband and less than 10^{-4} to the images of the signal spectrum for the following values. Assume $x(t)$ is bandlimited to 12π , that is, $X(j\omega) = 0$ for $|\omega| > 12\pi$. Constraints:

(1) In the pass band:

$$0.99 < |H_1(j\omega)| |H_c(j\omega)| < 1.01$$

$$\frac{0.99\omega^2}{4 \sin^2(\omega \frac{T_s}{2})} < |H_c(j\omega)| < \frac{1.01\omega^2}{4 \sin^2(\omega \frac{T_s}{2})}$$

(2) In the image region: $\omega = \frac{2\pi}{T_s} - \omega_m$

$$\begin{aligned} |H_1(j\omega)| |H_c(j\omega)| &< 10^{-4} \\ |H_c(j\omega)| &< \frac{10^{-4}\omega^2}{4 \sin^2(\omega \frac{T_s}{2})} \end{aligned}$$

(i) $T_s = .05$

$$\begin{aligned} \omega &= \frac{2\pi}{T_s} - \omega_m \\ &= \frac{2\pi}{0.05} - 12\pi = 28\pi \\ |H_c(j\omega)| &< \left| \frac{10^{-4}(28\pi)^2}{4 \sin^2(28\pi(\frac{0.05}{2}))} \right| \\ &\approx 0.2956 \end{aligned}$$

$$\begin{aligned} \frac{0.99\omega^2}{4 \sin^2(\omega \frac{T_s}{2})} < |H_c(j\omega)| < \frac{1.01\omega^2}{4 \sin^2(\omega \frac{T_s}{2})} \\ 2926.01 < |H_c(j\omega)| < 2985.12 \end{aligned}$$

(ii) $T_s = .02$

$$\begin{aligned}\omega &= \frac{2\pi}{0.02} - 12\pi = 88\pi \\ |H_c(j\omega)| &< \left| \frac{10^{-4}(88\pi)^2}{4 \sin^2(88\pi(\frac{0.02}{2}))} \right| \\ &\approx 14.1\end{aligned}$$

$$\begin{aligned}\frac{0.99\omega^2}{4 \sin^2(\omega \frac{T_s}{2})} &< |H_c(j\omega)| < \frac{1.01\omega^2}{4 \sin^2(\omega \frac{T_s}{2})} \\ 139589 &< |H_c(j\omega)| < 142409\end{aligned}$$

4.34. Determine the maximum factor q by which $x[n]$ with DTFT $X(e^{j\Omega})$ depicted in Fig. P4.34 can be decimated without aliasing. Sketch the DTFT of the sequence that results when $x[n]$ is decimated by this amount.

Looking at the following equation:

$$Y(e^{j\Omega}) = \frac{1}{q} \sum_{m=0}^{q-1} X\left(e^{j\frac{1}{q}(\Omega - m2\pi)}\right)$$

For the bandlimited signal, overlap starts when:

$$2qW > 2\pi$$

Thus:

$$q_{max} = \frac{\pi}{W} = 3$$

After decimation:

$$Y(e^{j\Omega}) = \frac{1}{3} \sum_{m=0}^2 X\left(e^{j\frac{1}{3}(\Omega - m2\pi)}\right)$$

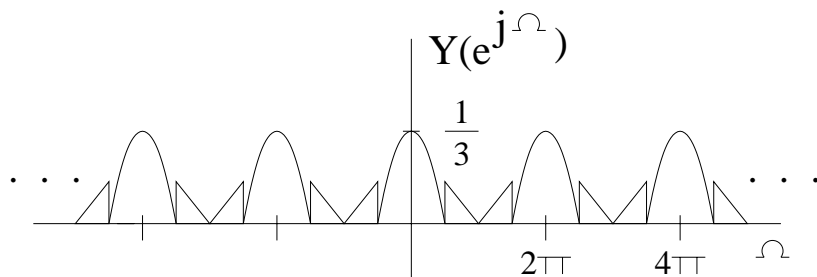


Figure P4.34. Sketch of the DTFT

4.35. A discrete-time system for processing continuous-time signals is shown in Fig. P4.35. Sketch the magnitude of the frequency response of an equivalent continuous-time system for the following cases:

$$|H_T(j\omega)| = |H_a(j\omega)| \frac{1}{T_s} |H(e^{j\omega T_s})| \left| \frac{2 \sin(\omega \frac{T_s}{2})}{\omega} \right| |H_c(j\omega)|$$

(a) $\Omega_1 = \frac{\pi}{4}$, $W_c = 20\pi$

$$\omega_{max} = \min(10\pi, \frac{\pi}{4}(20), 20\pi) = 5\pi$$

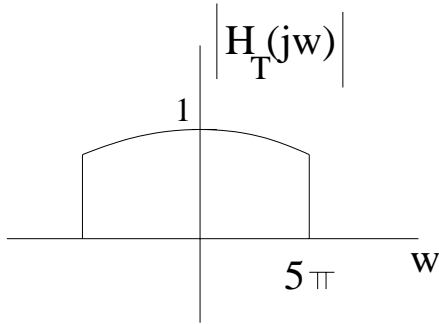


Figure P4.35. (a) Magnitude of the frequency response

(b) $\Omega_1 = \frac{3\pi}{4}$, $W_c = 20\pi$

$$\omega_{max} = \min(10\pi, \frac{3\pi}{4}(20), 20\pi) = 10\pi$$

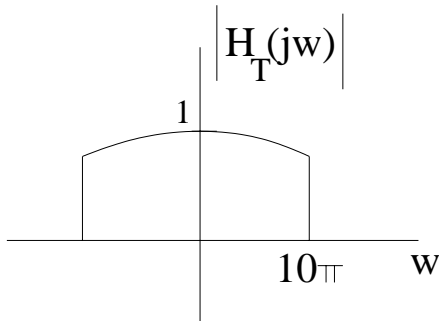


Figure P4.35. (b) Magnitude of the frequency response

(c) $\Omega_1 = \frac{\pi}{4}$, $W_c = 2\pi$

$$\omega_{max} = \min(10\pi, \frac{\pi}{4}(20), 2\pi) = 2\pi$$

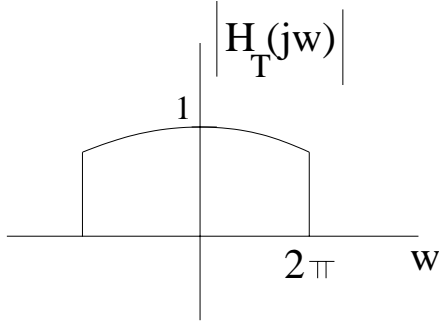


Figure P4.35. (c) Magnitude of the frequency response

4.36. Let $X(e^{j\Omega}) = \frac{\sin(\frac{11\Omega}{2})}{\sin(\frac{\Omega}{2})}$ and define $\tilde{X}[k] = X(e^{jk\Omega_o})$. Find and sketch $\tilde{x}[n]$ where $\tilde{x}[n] \xleftrightarrow{DTFS; \Omega_o} \tilde{X}[k]$ for the following values of Ω_o :

$$\tilde{X}[k] = \frac{\sin(\frac{11k\Omega_o}{2})}{\sin(\frac{k\Omega_o}{2})} \xleftrightarrow{DTFS; \Omega_o} \tilde{x}[n] = \begin{cases} N & |n| \leq 5 \\ 0 & 5 < |n| < \frac{N}{2}, \quad N \text{ periodic} \end{cases}$$

(a) $\Omega_o = \frac{2\pi}{15}$, $N = 15$

$$\tilde{x}[n] = \begin{cases} 15 & |n| \leq 5 \\ 0 & 5 < |n| < 7, \quad 15 \text{ periodic} \end{cases}$$

(b) $\Omega_o = \frac{\pi}{10}$, $N = 20$

$$\tilde{x}[n] = \begin{cases} 20 & |n| \leq 5 \\ 0 & 5 < |n| < 10, \quad 20 \text{ periodic} \end{cases}$$

(c) $\Omega_o = \frac{\pi}{3}$, $N = 6$

Overlap occurs

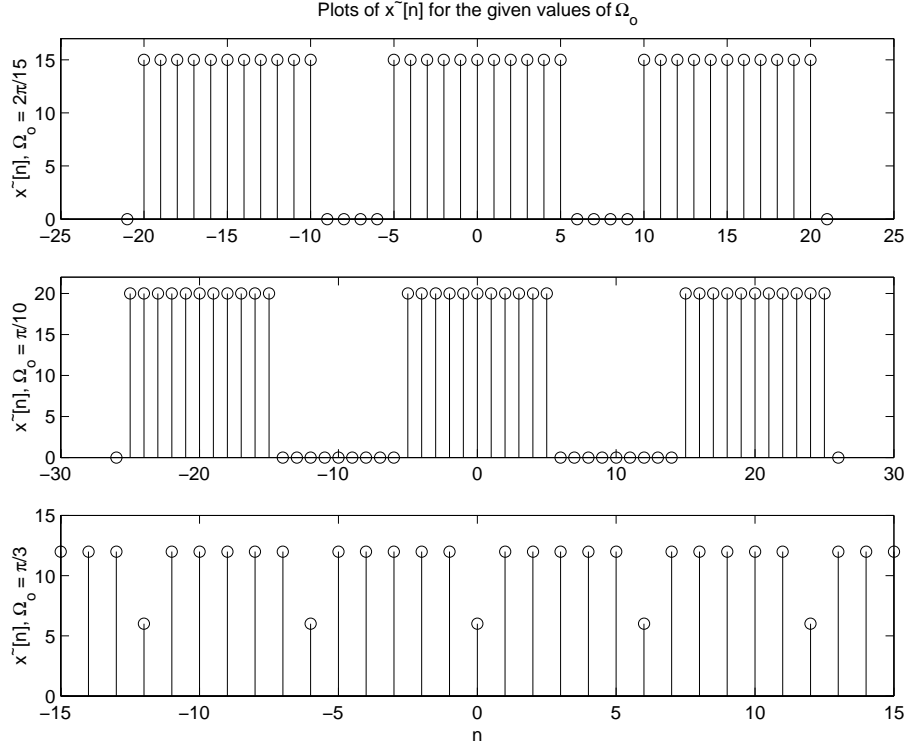


Figure P4.36. Sketch of $\tilde{x}[n]$

4.37. Let $X(j\omega) = \frac{\sin(2\omega)}{\omega}$ and define $\tilde{X}[k] = X(jk\omega_o)$. Find and sketch $\tilde{x}(t)$ where $\tilde{x}(t) \xleftrightarrow{FS; \omega_o} \tilde{X}[k]$ for the following values of ω_o :

$$\begin{aligned}
 x(t) &= \begin{cases} \frac{1}{2} & |t| \leq 2 \\ 0 & \text{otherwise} \end{cases} \\
 \tilde{X}[k] = X(jk\omega_o) &= \frac{\sin(2k\omega_o)}{k\omega_o} \\
 \tilde{x}(t) &= T \sum_{m=-\infty}^{\infty} x(t - mT)
 \end{aligned}$$

(a) $\omega_o = \frac{\pi}{8}$

$$\begin{aligned}
 \tilde{X}[k] &= \frac{\sin(\frac{\pi}{4}k)}{\frac{\pi}{8}k} \\
 T_s &= 2 \\
 T &= 16 \\
 \tilde{x}(t) &= \begin{cases} 8 & |t| < 2 \\ 0 & 2 < |t| < 8, \text{ 16 periodic} \end{cases}
 \end{aligned}$$

(b) $\omega_o = \frac{\pi}{4}$

$$\begin{aligned}\tilde{X}[k] &= \frac{\sin(\frac{\pi}{2}k)}{\frac{\pi}{4}k} \\ T_s &= 2 \\ T &= 8 \\ \tilde{x}(t) &= \begin{cases} 4 & |t| < 2 \\ 0 & 2 < |t| < 4, \text{ 8 periodic} \end{cases}\end{aligned}$$

(c) $\omega_o = \frac{\pi}{2}$

$$\begin{aligned}\tilde{X}[k] &= \frac{\sin(\pi k)}{\frac{\pi}{2}k} \\ &= 2\delta[k] \\ \tilde{x}(t) &= 2\end{aligned}$$

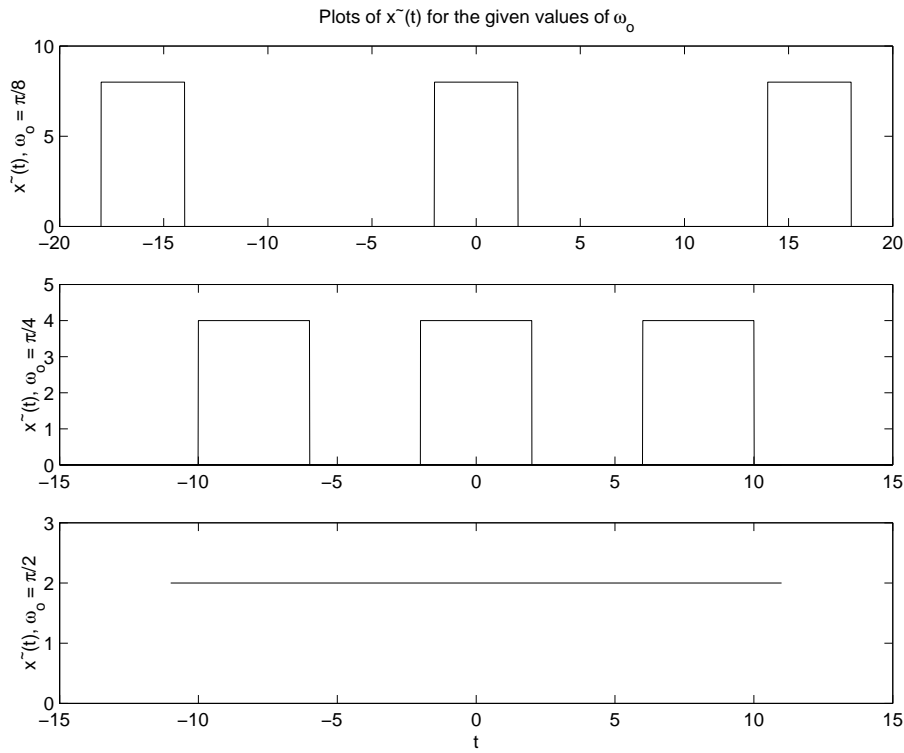


Figure P4.37. Sketch of $\tilde{x}(t)$

4.38. A signal $x(t)$ is sampled at intervals of $T_s = 0.01$ s. One hundred samples are collected and a 200-point DTFS is taken in an attempt to approximate $X(j\omega)$. Assume $|X(j\omega)| \approx 0$ for $|\omega| > 120\pi$ rad/s. Determine the frequency range $-\omega_a < \omega < \omega_a$ over which the DTFS offers a reasonable approximation to

$X(j\omega)$, the effective resolution of this approximation, ω_r , and the frequency interval between each DTFS coefficient, $\Delta\omega$.

$$\begin{array}{ll} x(t) & T_s = 0.01 \\ 100 \text{ samples} & M = 100 \end{array}$$

Use $N = 200$ DTFS to approximate $X(j\omega)$, $|X(j\omega)| \approx 0$, $|\omega| > 120\pi$, $\omega_m = 120\pi$

$$\begin{array}{ll} T_s & < \frac{2\pi}{\omega_m + \omega_a} \\ \omega_a & < \frac{2\pi}{T_s} - \omega_m \end{array}$$

Therefore:

$$\begin{array}{ll} \omega_a & < 80\pi \\ MT_s & > \frac{2\pi}{\omega_r} \end{array}$$

Therefore:

$$\begin{array}{ll} \omega_r & > 2\pi \\ N & > \frac{\omega_s}{\Delta\omega} \\ \Delta\omega & > \frac{\omega_s}{N} \\ \Delta\omega & = \frac{2\pi}{NT_s} \end{array}$$

Therefore:

$$\Delta\omega > \pi$$

4.39. A signal $x(t)$ is sampled at intervals of $T_s = 0.1$ s. Assume $|X(j\omega)| \approx 0$ for $|\omega| > 12\pi$ rad/s. Determine the frequency range $-\omega_a < \omega < \omega_a$ over which the DTFS offers a reasonable approximation to $X(j\omega)$, the minimum number of samples required to obtain an effective resolution $\omega_r = 0.01\pi$ rad/s, and the length of the DTFS required so the frequency interval between DTFS coefficients is $\Delta\omega = 0.001\pi$ rad/s.

$$\begin{array}{ll} T_s & < \frac{2\pi}{\omega_m + \omega_a} \\ \omega_m & = 12\pi \\ T_s & = 0.1 \\ \omega_a & < 8\pi \end{array}$$

The frequency range $|\omega| < 8\pi$ provides a reasonable approximation to the FT.

$$M \geq \frac{\omega_s}{\omega_r}$$

$$\begin{aligned}\omega_s &= \frac{2\pi}{T_s} = 20\pi \\ \omega_r &= 0.01\pi \\ M &\geq 2000\end{aligned}$$

$M = 2000$ samples is sufficient for the given resolution.

$$\begin{aligned}N &\geq \frac{\omega_s}{\Delta\omega} \\ \Delta\omega &= 0.001\pi \\ N &\geq 20,000\end{aligned}$$

The required length of the DTFS is $N = 20,000$.

4.40. Let $x(t) = a \sin(\omega_o t)$ be sampled at intervals of $T_s = 0.1$ s. Assume 100 samples of $x(t)$, $x[n] = x(nT_s)$, $n = 0, 1, \dots, 99$, are available. We use the DTFS of $x[n]$ to approximate the FT of $x(t)$ and wish to determine a from the DTFS coefficient of largest magnitude. The samples $x[n]$ are zero-padded to length N before taking the DTFS. Determine the minimum value of N for the following values of ω_o : Determine which DTFS coefficient has the largest magnitude in each case.

Choose $\Delta\omega$ so that ω_o is an integer multiple of $\Delta\omega$, ($\omega_o = p\Delta\omega$), where p is an integer, and set $N = M = 100$. Using these two conditions results in the DTFS sampling $W_\delta(j(\omega - \omega_o))$ at the peak of the mainlobe and at all of the zero crossings. Consequently,

$$Y[k] = \begin{cases} a & k = p \\ 0 & \text{otherwise on } 0 \leq k \leq N-1 \end{cases}$$

(a) $\omega_o = 3.2\pi$

$$\begin{aligned}N &= \frac{\omega_s}{\Delta\omega} \\ &= \frac{20\pi p}{\omega_o} \\ &= \frac{20\pi p}{3.2\pi} \\ &= 25 \\ p &= 4\end{aligned}$$

Since p and N have to be integers

$$Y[k] = \begin{cases} a & k = 4 \\ 0 & \text{otherwise on } 0 \leq k \leq 24 \end{cases}$$

(b) $\omega_o = 3.1\pi$

$$\begin{aligned}
 N &= \frac{\omega_s}{\Delta\omega} \\
 &= \frac{20\pi p}{\omega_o} \\
 &= \frac{20\pi p}{3.1\pi} \\
 &= 200 \\
 p &= 31
 \end{aligned}$$

Since p and N have to be integers

$$Y[k] = \begin{cases} a & k = 31 \\ 0 & \text{otherwise on } 0 \leq k \leq 199 \end{cases}$$

(c) $\omega_o = 3.15\pi$

$$\begin{aligned}
 N &= \frac{\omega_s}{\Delta\omega} \\
 &= \frac{20\pi p}{\omega_o} \\
 &= \frac{20\pi p}{3.15\pi} \\
 &= 400 \\
 p &= 63
 \end{aligned}$$

Since p and N have to be integers

$$Y[k] = \begin{cases} a & k = 63 \\ 0 & \text{otherwise on } 0 \leq k \leq 399 \end{cases}$$

Solutions to Advanced Problems

4.41. A continuous-time signal lies in the frequency band $|\omega| < 5\pi$. This signal is contaminated by a large sinusoidal signal of frequency 120π . The contaminated signal is sampled at a sampling rate of $\omega_s = 13\pi$.

(a) After sampling, at what frequency does the sinusoidal interfering signal appear?

$X(j\omega)$ is bandlimited to 5π

$$\begin{aligned}
 s(t) &= x(t) + A \sin(120\pi t) \\
 s[n] = s(nT_s) &= x[n] + A \sin\left(\frac{240\pi}{13}n\right)
 \end{aligned}$$

$$\begin{aligned}
&= x[n] + A \sin(9(2\pi)n + \frac{6\pi}{13}n) \\
&= x[n] + A \sin(\frac{6\pi}{13}n) \\
\Omega \sin &= \frac{6\pi}{13} \\
\omega &= \frac{\Omega}{T_s} = \left(\frac{6\pi}{13}\right) \left(\frac{13}{2}\right) = 3\pi
\end{aligned}$$

The sinusoid appears at $\omega = 3\pi$ rads/sec in $S_\delta(j\omega)$.

(b) The contaminated signal is passed through an anti-aliasing filter consisting of the RC circuit depicted in Fig. P4.41. Find the value of the time constant RC required so that the contaminating sinusoid is attenuated by a factor of 1000 prior to sampling.

Before the sampling, $s(t)$ is passed through a LPF.

$$\begin{aligned}
T(j\omega) &= \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \\
&= \frac{1}{1 + j\omega RC} \\
&= \frac{1}{1 + j\omega\tau} \\
|T(j\omega)| &= \frac{1}{\sqrt{1 + \omega^2\tau^2}} \Big|_{\omega=120\pi} \\
&= \frac{1}{1000} \\
\tau &= 2.65 \text{ s}
\end{aligned}$$

(c) Sketch the magnitude response in dB that the anti-aliasing filter presents to the signal of interest for the value of RC identified in (b).

$$\begin{aligned}
T(j\omega) &= \frac{1}{1 + j\omega 2.65} \\
|T(j\omega)| &= \frac{1}{\sqrt{1 + \omega^2 (2.65)^2}}
\end{aligned}$$

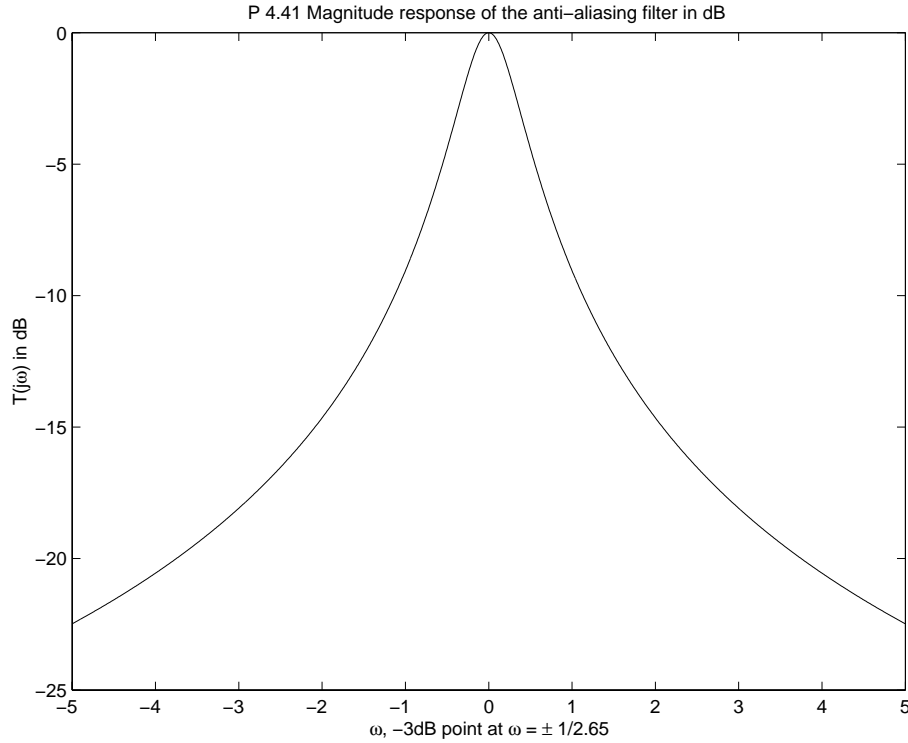


Figure P4.41. Sketch of the magnitude response

4.42. This problem derives the frequency-domain relationship for subsampling given in Eq.(4.27). Use Eq. (4.17) to represent $x[n]$ as the impulse-sampled continuous-time signal with sampling interval T_s , and thus write

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

Suppose $x[n]$ are the samples of a continuous-time signal $x(t)$, obtained at integer multiples of T_s . That is, $x[n] = x(nT_s)$. Let $x(t) \xrightarrow{FT} X(j\omega)$. Define the subsampled signal $y[n] = x[qn]$ so that $y[n] = x(nqT_s)$ is also expressed as samples of $x(t)$.

(a) Apply Eq. (4.23) to express $X_\delta(j\omega)$ as a function of $X(j\omega)$. Show that

$$Y_\delta(j\omega) = \frac{1}{qT_s} \sum_{k=-\infty}^{\infty} X(j(\omega - \frac{k}{q}\omega_s))$$

Since $y[n]$ is formed by using every q th sample of $x[n]$, the effective sampling rate is $T'_s = qT_s$

$$y_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT'_s) \xrightarrow{FT} Y_\delta(j\omega) = \frac{1}{T'_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega'_s))$$

Substituting $T'_s = qT_s$, and $\omega'_s = \frac{\omega_s}{q}$ yields:

$$Y_\delta(j\omega) = \frac{1}{qT_s} \sum_{k=-\infty}^{\infty} X(j(\omega - \frac{k}{q}\omega_s))$$

(b) The goal is to express $Y_\delta(j\omega)$ as a function of $X_\delta(j\omega)$ so that $Y(e^{j\Omega})$ can be expressed in terms of $X(e^{j\Omega})$. To this end, write $\frac{k}{q}$ in $Y_\delta(j\omega)$ as the proper fraction

$$\frac{k}{q} = l + \frac{m}{q}$$

where l is the integer portion of $\frac{k}{q}$ and m is the remainder. Show that we may thus rewrite $Y_\delta(j\omega)$ as

$$Y_\delta(j\omega) = \frac{1}{q} \sum_{m=0}^{q-1} \left\{ \frac{1}{T_s} \sum_{l=-\infty}^{\infty} X(j(\omega - l\omega_s - \frac{m}{q}\omega_s)) \right\}$$

Letting k to range from $-\infty$ to ∞ corresponds to having l range from $-\infty$ to ∞ and m from 0 to $q-1$, which permits us to rewrite $Y_\delta(j\omega)$ as :

$$Y_\delta(j\omega) = \frac{1}{q} \sum_{m=0}^{q-1} \left\{ \frac{1}{T_s} \sum_{l=-\infty}^{\infty} X(j(\omega - l\omega_s - \frac{m}{q}\omega_s)) \right\}$$

Next show that

$$Y_\delta(j\omega) = \frac{1}{q} \sum_{m=0}^{q-1} X_\delta(j(\omega - \frac{m}{q}\omega_s))$$

Recognizing that the term in braces corresponds to $X_\delta(j(\omega - \frac{m}{q}\omega_s))$, allows us to rewrite the equation as the following double sum:

$$Y_\delta(j\omega) = \frac{1}{q} \sum_{m=0}^{q-1} X_\delta(j(\omega - \frac{m}{q}\omega_s))$$

(c) Now we convert from the FT representation back to the DTFT in order to express $Y(e^{j\Omega})$ as a function of $X(e^{j\Omega})$. The sampling interval associated with $Y_\delta(j\omega)$ is qT_s . Using the relationship $\Omega = \omega qT_s$ in

$$Y(e^{j\Omega}) = Y_\delta(j\omega)|_{\omega=\frac{\Omega}{qT_s}}$$

show that

$$Y(e^{j\Omega}) = \frac{1}{q} \sum_{m=0}^{q-1} X_\delta\left(\frac{j}{T_s} \left(\frac{\Omega}{q} - \frac{m}{q}2\pi\right)\right)$$

Substituting $\omega = \frac{\Omega}{qT_s}$ yields

$$Y(e^{j\Omega}) = \frac{1}{q} \sum_{m=0}^{q-1} X_\delta\left(\frac{j}{T_s} \left(\frac{\Omega}{q} - \frac{m}{q}2\pi\right)\right)$$

(d) Lastly, use $X(e^{j\Omega}) = X_\delta(j\frac{\Omega}{T_s})$ to obtain

$$Y(e^{j\Omega}) = \frac{1}{q} \sum_{m=0}^{q-1} X\left(e^{j\frac{1}{q}(\Omega - m2\pi)}\right)$$

The sampling interval associated with $X_\delta(j\omega)$ is T_s , so $X(e^{j\Omega}) = X_\delta(j\frac{\Omega}{T_s})$. Hence we may substitute $X\left(e^{j(\frac{\Omega}{q} - m\frac{2\pi}{q})}\right)$ for $X_\delta\left(\frac{j}{T_s} \left(\frac{\Omega}{q} - \frac{m}{q}2\pi\right)\right)$ and obtain

$$Y(e^{j\Omega}) = \frac{1}{q} \sum_{m=0}^{q-1} X\left(e^{j\frac{1}{q}(\Omega - m2\pi)}\right)$$

4.43. A bandlimited signal $x(t)$ satisfies $|X(j\omega)| = 0$ for $|\omega| < \omega_1$ and $|\omega| > \omega_2$. Assume $\omega_1 > \omega_2 - \omega_1$. In this case we can sample $x(t)$ at a rate less than that indicated by the sampling interval and still perform perfect reconstruction by using a bandpass reconstruction filter $H_r(j\omega)$. Let $x[n] = x(nT_s)$. Determine the maximum sampling interval T_s such that $x(t)$ can be perfectly reconstructed from $x[n]$. Sketch the frequency response of the reconstruction filter required for this case.

We can tolerate aliasing as long as there is no overlap on $\omega_1 \leq |\omega| \leq \omega_2$

We require:

$$\begin{aligned}\omega_s - \omega_2 &\geq -\omega_1 \\ \omega_s &\geq \omega_2 - \omega_1\end{aligned}$$

Implies:

$$\begin{aligned}T_s &\leq \frac{2\pi}{\omega_2 - \omega_1} \\ H_r(j\omega) &= \begin{cases} T_s & \omega_1 \leq |\omega| \leq \omega_2 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

4.44. Suppose a periodic signal $x(t)$ has FS coefficients

$$X[k] = \begin{cases} (\frac{3}{4})^k, & |k| \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

The period of this signal is $T = 1$.

(a) Determine the minimum sampling interval for this signal that will prevent aliasing.

$$\begin{aligned}X(j\omega) &= 2\pi \sum_{k=-4}^4 \left(\frac{3}{4}\right)^k \delta(\omega - k2\pi) \\ \omega_m &= 8\pi \\ \frac{2\pi}{T_s} &> 2(8\pi) \\ \min T_s &= \frac{1}{8}\end{aligned}$$

(b) The constraints of the sampling theorem can be relaxed somewhat in the case of periodic signals if we allow the reconstructed signal to be a time-scaled version of the original. Suppose we choose a sampling interval $T_s = \frac{20}{19}$ and use a reconstruction filter

$$H_r(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}$$

Show that the reconstructed signal is a time-scaled version of $x(t)$ and identify the scaling factor.

$$\begin{aligned}T_s &= \frac{20}{19} \\ X_\delta(j\omega) &= \frac{19}{20} \sum_{l=-\infty}^{\infty} X(j(\omega - l1.9\pi))\end{aligned}$$

Aliasing produces a “frequency scaled” replica of $X(j\omega)$ centered at zero. The scaling is by a factor of 20 from $\omega_o = 2\pi$ to $\omega'_o = 0.1\pi$. Applying the LPF, $|\omega| < \pi$ gives $x(\frac{t}{20})$, and $x_{reconstructed}(t) = \frac{19}{20}x(\frac{t}{20})$

(c) Find the constraints on the sampling interval T_s so that use of $H_r(j\omega)$ in (b) results in the reconstruction filter being a time-scaled version of $x(t)$ and determine the relationship between the scaling factor and T_s .

The choice of T_s is so that no aliasing occurs.

$$\begin{aligned}
 (1) \quad & \frac{2\pi}{T_s} < 2\pi \\
 & T_s > 1 \text{ period of the original signal.} \\
 (2) \quad & \left(2\pi - \frac{2\pi}{T_s}\right) 4 < \frac{1}{2} \frac{2\pi}{T_s} \\
 & T_s < \frac{9}{8} \\
 & 1 < T_s < \frac{9}{8}
 \end{aligned}$$

4.45. In this problem we reconstruct a signal $x(t)$ from its samples $x[n] = x(nT_s)$ using pulses of width less than T_s followed by an anti-imaging filter with frequency response $H_c(j\omega)$. Specifically, we apply

$$x_p(t) = \sum_{n=-\infty}^{\infty} x[n]h_p(t - nT_s)$$

to the anti-imaging filter, where $h_p(t)$ is a pulse of width T_o as depicted in Fig. P4.45 (a). An example of $x_p(t)$ is depicted in Fig. P4.45 (b). Determine the constraints on $|H_c(j\omega)|$ so that the overall magnitude response of this reconstruction system is between 0.99 and 1.01 in the signal passband and less than 10^{-4} to the images of the signal spectrum for the following values with $x(t)$ bandlimited to 10π , that is, $X(j\omega) = 0$ for $|\omega| > 10\pi$:

$$\begin{aligned}
 X_p(j\omega) &= H_p(j\omega)X_\Delta(j\omega) \\
 H_p(j\omega) &= \frac{2 \sin(\omega \frac{T_o}{2})}{T_o \omega} e^{-j\omega \frac{T_o}{2}}
 \end{aligned}$$

Constraints:

(1) Passband

$$\begin{aligned}
 0.99 < |H_p(j\omega)||H_c(j\omega)| &< 1.01, \text{ using } \omega_{max} = 10\pi \\
 \frac{0.99T_o\omega}{2 \sin(\omega \frac{T_o}{2})} < |H_c(j\omega)| &< \frac{1.01T_o\omega}{2 \sin(\omega \frac{T_o}{2})}
 \end{aligned}$$

(2) In the image location

$$|H_p(j\omega)| < \frac{10^{-4}T_o\omega}{2 \sin(\omega \frac{T_o}{2})}$$

$$\text{where} \quad \omega = \frac{2\pi}{T_s} - 10\pi$$

$$(a) \quad T_s = 0.08, \quad T_o = 0.04$$

(1) Passband

$$1.1533 < |H_c(j\omega)| < 1.1766$$

(2) In the image location

$$\begin{array}{rcl} \omega & = & 15\pi \\ |H_c(j15\pi)| & < & 1.165 \times 10^{-4} \end{array}$$

$$(b) \quad T_s = 0.08, \quad T_o = 0.02$$

(1) Passband

$$1.0276 < |H_c(j\omega)| < 1.0484$$

(2) In the image location

$$\begin{array}{rcl} \omega & = & 15\pi \\ |H_c(j15\pi)| & < & 1.038 \times 10^{-4} \end{array}$$

$$(c) \quad T_s = 0.04, \quad T_o = 0.02$$

(1) Passband

$$1.3081 < |H_c(j\omega)| < 1.3345$$

(2) In the image location

$$\begin{array}{rcl} \omega & = & 40\pi \\ |H_c(j40\pi)| & < & 1.321 \times 10^{-4} \end{array}$$

$$(d) \quad T_s = 0.04, \quad T_o = 0.01$$

(1) Passband

$$1.0583 < |H_c(j\omega)| < 1.0796$$

(2) In the image location

$$\begin{array}{rcl} \omega & = & 40\pi \\ |H_c(j40\pi)| & < & 1.0690 \times 10^{-4} \end{array}$$

4.46. A non-ideal sampling operation obtains $x[n]$ from $x(t)$ as

$$x[n] = \int_{(n-1)T_s}^{nT_s} x(t) dt$$

(a) Show that this can be written as ideal sampling of a filtered signal $y(t) = x(t) * h(t)$, that is, $x[n] = y(nT_s)$, and find $h(t)$.

$$\begin{aligned}
 y(t) &= \int_{t-T_s}^{T_s} x(\tau) d\tau \text{ by inspection} \\
 &= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \\
 &= x(t) * h(t) \\
 \text{choose } h(t) &= \begin{cases} 1 & 0 \leq t \leq T_s \\ 0 & \text{otherwise} \end{cases} \\
 h(t-\tau) &= \begin{cases} 1 & t-T_s \leq \tau \leq T_s \\ 0 & \text{otherwise} \end{cases} \\
 h(t) &= u(t) - u(t-T_s)
 \end{aligned}$$

(b) Express the FT of $x[n]$ in terms of $X(j\omega)$, $H(j\omega)$, and T_s .

$$\begin{aligned}
 Y(j\omega) &= X(j\omega)H(j\omega) \\
 y(nT_s) &\xleftrightarrow{FT} \frac{1}{T_s} \sum_{k=-\infty}^{\infty} Y(j(\omega - k\frac{2\pi}{T_s})) \\
 FT\{x[n]\} &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\frac{2\pi}{T_s}))H(j(\omega - k\frac{2\pi}{T_s}))
 \end{aligned}$$

(c) Assume that $x(t)$ is bandlimited to the frequency range $|\omega| < \frac{3\pi}{4T_s}$. Determine the frequency response of a discrete-time system that will correct the distortion in $x[n]$ introduced by non-ideal sampling.

$x(t)$ is bandlimited to:

$$|\omega| < \frac{3\pi}{4T_s} < \frac{2\pi}{T_s}$$

We can use:

$$\begin{aligned}
 H_r(j\omega) &= \begin{cases} \frac{T_s}{H(j\omega)} & |\omega| \leq \frac{3\pi}{4T_s} \\ 0 & \text{otherwise} \end{cases} \\
 h(t) &= \begin{cases} 1 & 0 \leq t \leq T_s \\ 0 & \text{otherwise} \end{cases} \\
 h(t + \frac{T_s}{2}) &\xleftrightarrow{FT} \frac{2 \sin(\omega \frac{T_s}{2})}{\omega} \\
 H(j\omega) &= \frac{2 \sin(\omega \frac{T_s}{2})}{\omega} e^{-j\omega \frac{T_s}{2}} \\
 H_r(j\omega) &= \begin{cases} \frac{\omega T_s e^{j\omega \frac{T_s}{2}}}{2 \sin(\omega \frac{T_s}{2})} & |\omega| \leq \frac{3\pi}{4T_s} \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

4.47. The system depicted in Fig. P4.47 (a) converts a continuous-time signal $x(t)$ to a discrete-time signal $y[n]$. We have

$$H(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < \frac{\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

Find the sampling frequency $\omega_s = \frac{2\pi}{T_s}$ and the constraints on the anti-aliasing filter frequency response $H_a(j\omega)$ so that an input signal with FT $X(j\omega)$ shown in Fig. P4.47 (b) results in the output signal with DTFT $Y(e^{j\Omega})$.

$$\begin{aligned} X_a(j\omega) &= X(j\omega)H_a(j\omega) \\ X_a(j\omega) &= \frac{1}{T_s} \sum_k X(j(\omega - k\frac{2\pi}{T_s})) \end{aligned}$$

To discard the high frequency of $X(j\omega)$, and anticipate $\frac{1}{T_s}$, use:

$$H_a(j\omega) = \begin{cases} T_s & |\omega| \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

Given $Y(e^{j\Omega})$, we can conclude that $T_s = \frac{1}{4}$, since $Y(j\omega)$ is

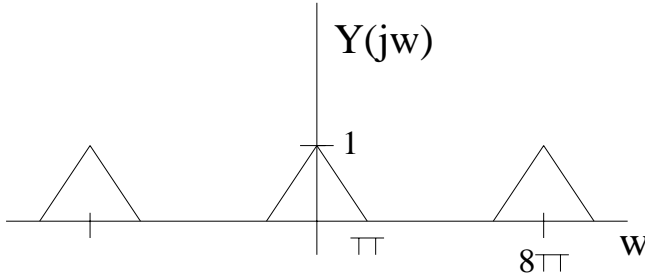


Figure P4.47. Graph of $Y(j\omega)$

Also, the bandwidth of $x(t)$ should not change, therefore:

$$\begin{aligned} \omega_s &= 8\pi \\ H_a(j\omega) &= \begin{cases} \frac{1}{4} & |\omega| \leq \pi \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

4.48. The discrete-time signal $x[n]$ with DTFT $X(e^{j\Omega})$ shown in Fig. P4.48 (a) is decimated by first passing $x[n]$ through the filter with frequency response $H(e^{j\Omega})$ shown in Fig. P4.48 (b) and then sub-sampling by the factor q . For the following values of q and W , determine the minimum value of Ω_p and maximum value of Ω_s so that the subsampling operation does not change the shape of the portion of $X(e^{j\Omega})$ on $|\Omega| < W$.

$$Y(e^{j\Omega}) = \frac{1}{q} \sum_{m=0}^{q-1} X\left(e^{j\frac{1}{q}(\Omega - m2\pi)}\right)$$

From the following figure, one can see to preserve the shape within $|\Omega| < W$, we need:

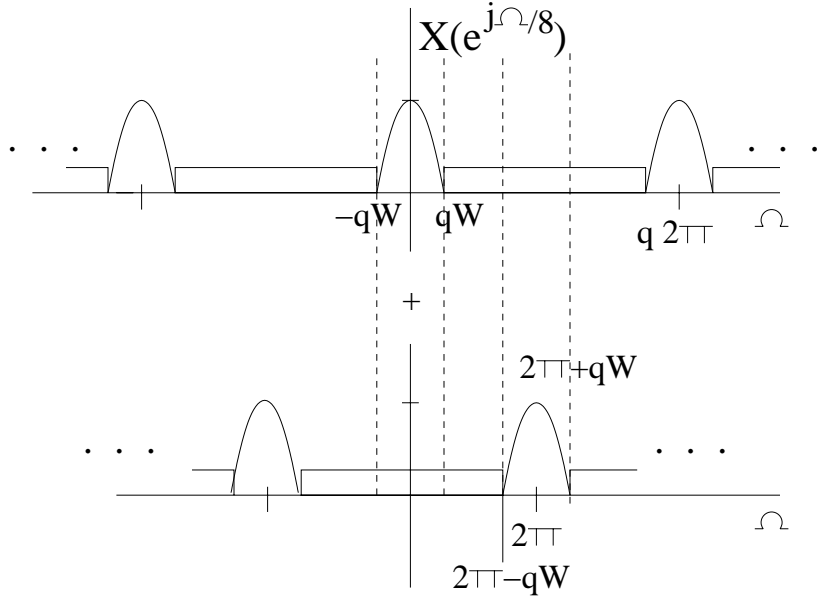


Figure P4.48. Figure showing the necessary constraints to preserve the signal

$$\begin{aligned} \min \Omega_p &= \frac{qW}{q} = W \\ \max \Omega_s &= \frac{2\pi - qW}{q} = \frac{2\pi}{q} - W \end{aligned}$$

(a) $q = 2$, $W = \frac{\pi}{3}$

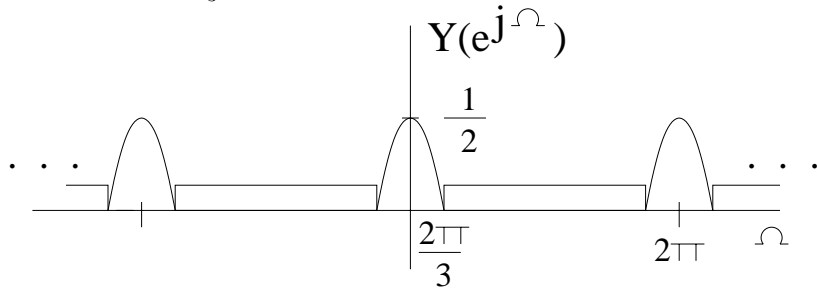


Figure P4.48. (a) Sketch of the DTFT

(b) $q = 2$, $W = \frac{\pi}{4}$

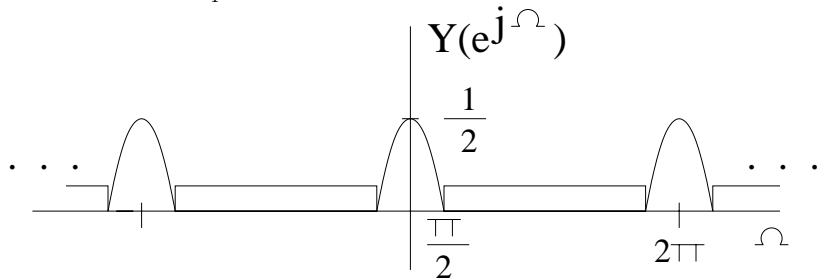


Figure P4.48. (b) Sketch of the DTFT

(c) $q = 3$, $W = \frac{\pi}{4}$

In each case sketch the DTFT of the subsampled signal.

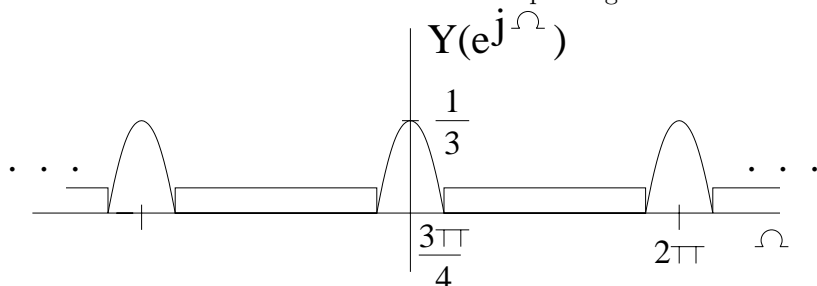


Figure P4.48. (c) Sketch of the DTFT

4.49. A signal $x[n]$ is interpolated by the factor q by first inserting $q - 1$ zeros between each sample and next passing the zero-stuffed sequence through a filter with frequency response $H(e^{j\Omega})$ depicted in Fig. P4.48 (b). The DTFT of $x[n]$ is depicted in Fig. P4.49. Determine the minimum value of Ω_p and maximum value of Ω_s so that ideal interpolation is obtained for the following cases. In each case sketch the DTFT of the interpolated signal.

$$X_z(e^{j\Omega}) = X(e^{j\Omega q})$$

For ideal interpolation,

$$\begin{aligned} \min \Omega_p &= \frac{W}{q} \\ \max \Omega_s &= 2\pi - \frac{W}{q} \end{aligned}$$

(a) $q = 2$, $W = \frac{\pi}{2}$

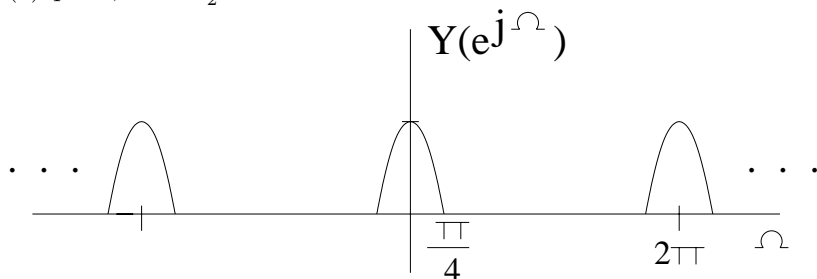


Figure P4.49. (a) Sketch of the DTFT of the interpolated signal

(b) $q = 2$, $W = \frac{3\pi}{4}$

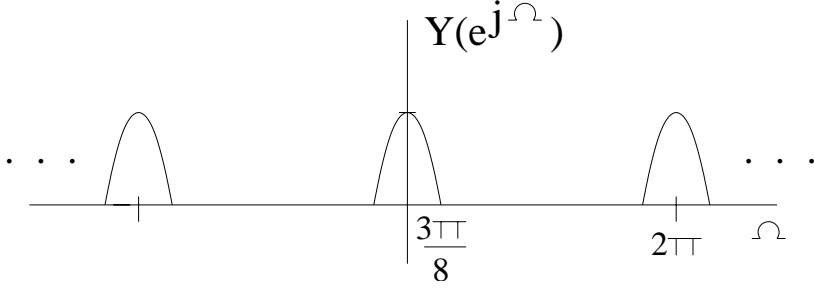


Figure P4.49. (b) Sketch of the DTFT of the interpolated signal

(c) $q = 3$, $W = \frac{3\pi}{4}$

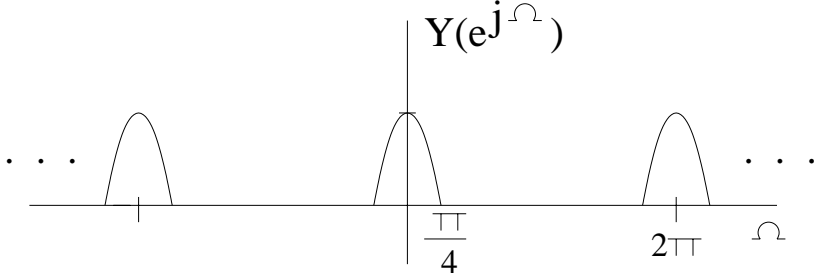


Figure P4.49. (c) Sketch of the DTFT of the interpolated signal

4.50. Consider interpolating a signal $x[n]$ by repeating each value q times as depicted in Fig. P4.50. That is, we define $x_o[n] = x[\text{floor}(\frac{n}{q})]$ where $\text{floor}(z)$ is the integer less than or equal to z . Letting $x_z[n]$ be derived from $x[n]$ by inserting $q - 1$ zeros between each value of $x[n]$, that is,

$$x_z[n] = \begin{cases} x[\frac{n}{q}], & \frac{n}{q} \text{ integer} \\ 0, & \text{otherwise} \end{cases}$$

We may then write $x_o[n] = x_z[n] * h_o[n]$, where $h_o[n]$ is:

$$h_o[n] = \begin{cases} 1, & 0 \leq n \leq q - 1 \\ 0, & \text{otherwise} \end{cases}$$

Note that this is the discrete-time analog of the zero-order hold. The interpolation process is completed by passing $x_o[n]$ through a filter with frequency response $H(e^{j\Omega})$.

$$X_z(e^{j\Omega}) = X(e^{j\Omega q})$$

(a) Express $X_o(e^{j\Omega})$ in terms of $X(e^{j\Omega})$ and $H_o(e^{j\Omega})$. Sketch $|X_o(e^{j\Omega})|$ if $x[n] = \frac{\sin(\frac{3\pi}{4}n)}{\pi n}$.

$$\begin{aligned} X_o(e^{j\Omega}) &= X(e^{j\Omega q})H_o(e^{j\Omega}) \\ x[n] = \frac{\sin(\frac{3\pi}{4}n)}{\pi n} &\xleftrightarrow{DTFT} X(e^{j\Omega}) = \begin{cases} 1 & |\Omega| < \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} \leq |\Omega| < \pi, \text{ } 2\pi \text{ periodic} \end{cases} \end{aligned}$$

$$|X_o(e^{j\Omega})| = |X(e^{j\Omega q})| \left| \frac{\sin(\Omega \frac{q}{2})}{\sin(\frac{\Omega}{2})} \right|$$

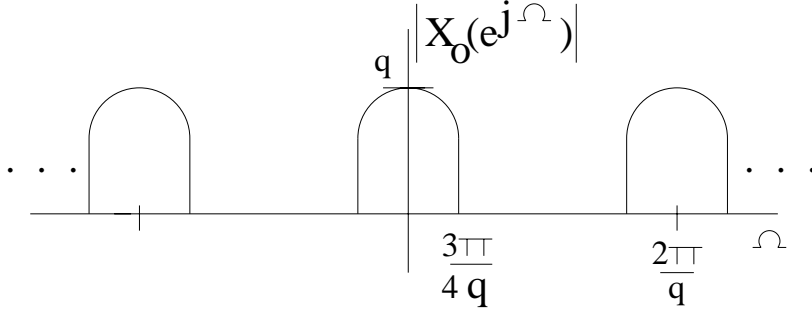


Figure P4.50. (a) Sketch of $|X_o(e^{j\Omega})|$

(b) Assume $X(e^{j\Omega})$ is as shown in Fig. P4.49. Specify the constraints on $H(e^{j\Omega})$ so that ideal interpolation is obtained for the following cases.

For ideal interpolation, discard components other than those centered at multiples of 2π . Also, some correction is needed to correct for magnitude and phase distortion.

$$H(e^{j\Omega}) = \begin{cases} \frac{\sin(\frac{\Omega}{2})}{\sin(\Omega \frac{q}{2})} e^{j\Omega \frac{q}{2}} & |\Omega| < \frac{W}{q} \\ 0 & \frac{W}{q} \leq |\Omega| < 2\pi - \frac{W}{q}, \quad 2\pi \text{ periodic} \end{cases}$$

(i) $q = 2, \quad W = \frac{3\pi}{4}$

$$H(e^{j\Omega}) = \begin{cases} \frac{\sin(\frac{\Omega}{2}) e^{j\Omega}}{\sin(\Omega)} & |\Omega| < \frac{3\pi}{8} \\ 0 & \frac{3\pi}{8} \leq |\Omega| < \frac{13\pi}{8}, \quad 2\pi \text{ periodic} \end{cases}$$

(ii) $q = 4, \quad W = \frac{3\pi}{4}$

$$H(e^{j\Omega}) = \begin{cases} \frac{\sin(\frac{\Omega}{2})}{\sin(2\Omega)} e^{j2\Omega} & |\Omega| < \frac{3\pi}{16} \\ 0 & \frac{3\pi}{16} \leq |\Omega| < \frac{29\pi}{16}, \quad 2\pi \text{ periodic} \end{cases}$$

4.51. The system shown in Fig. P4.51 is used to implement a bandpass filter. The discrete-time filter $H(e^{j\Omega})$ has frequency response on $-\pi < \Omega \leq \pi$

$$H(e^{j\Omega}) = \begin{cases} 1, & \Omega_a \leq |\Omega| \leq \Omega_b \\ 0, & \text{otherwise} \end{cases}$$

Find the sampling interval T_s , Ω_a , Ω_b , W_1 , W_2 , W_3 , and W_4 , so that the equivalent continuous-time frequency response $G(j\omega)$ satisfies

$$0.9 < |G(j\omega)| < 1.1, \quad \text{for } 100\pi < \omega < 200\pi$$

$$G(j\omega) = 0 \quad \text{elsewhere}$$

In solving this problem, choose W_1 and W_3 as small as possible and choose T_s , W_2 and W_4 as large as possible.

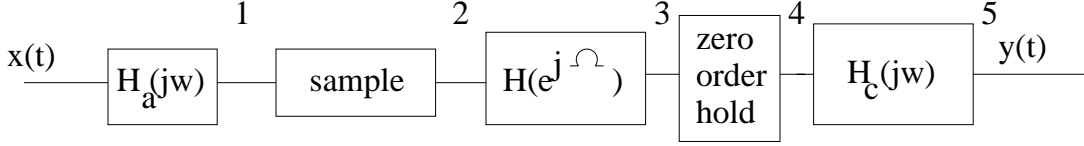


Figure P4.51. Graph of the system

(3) Passband:

$$100\pi < \omega < 200\pi$$

Thus

$$\Omega_a = 100\pi T_s$$

$$\Omega_b = 200\pi T_s$$

$$(4) |H_o(j\omega)| = \left| \frac{2 \sin(\omega \frac{T_s}{2})}{\omega} \right|$$

$$\text{at } \omega = 100\pi \quad \frac{2 \sin(50\pi T_s)}{100\pi T_s} < 1.1$$

$$\text{at } \omega = 200\pi \quad \frac{2 \sin(100\pi T_s)}{200\pi T_s} > 0.9$$

implies:

$$T_s(100\pi) < 0.785$$

$$\max T_s = 0.0025$$

$$(5) \min W_3 = 200\pi$$

$$\max W_4 = \frac{2\pi}{T_s} - 200\pi = 600\pi$$

$$(3) \Omega_a = 0.25\pi$$

$$\Omega_b = 0.5\pi$$

(1) and (2)

$$\min W_1 = 200\pi$$

$$\max W_2 = \frac{1}{2} \frac{2\pi}{T_s} = 400\pi, \text{ No overlap.}$$

4.52. A time-domain interpretation of the interpolation procedure described in Fig. 4.50 (a) is derived in this problem. Let $h_i[n] \xrightarrow{DTFT} H_i(e^{j\Omega})$ be an ideal low-pass filter with transition band of zero width.

That is

$$H_i(e^{j\Omega}) = \begin{cases} q, & |\Omega| < \frac{\pi}{q} \\ 0, & \frac{\pi}{q} < |\Omega| < \pi \end{cases}$$

(a) Substitute for $h_i[n]$ in the convolution sum

$$\begin{aligned} x_i[n] &= \sum_{k=-\infty}^{\infty} x_z[k] * h_i[n-k] \\ h_i[n] &= q \frac{\sin\left(\frac{\pi}{q}n\right)}{\pi n} \\ x_i[n] &= \sum_{k=-\infty}^{\infty} x_z[k] \frac{q \sin\left(\frac{\pi}{q}(n-k)\right)}{\pi(n-k)} \end{aligned}$$

(b) The zero-insertion procedure implies $x_z[k] = 0$ unless $k = qm$ where m is integer. Rewrite $x_i[n]$ using only the non-zero terms in the sum as a sum over m and substitute $x[m] = x_z[qm]$ to obtain the following expression for ideal discrete-time interpolation:

$$x_i[n] = \sum_{m=-\infty}^{\infty} x[m] \frac{q \sin\left(\frac{\pi}{q}(n - qm)\right)}{\pi(n - qm)}$$

Substituting $k = qm$ yields:

$$\begin{aligned} x_i[n] &= \sum_{m=-\infty}^{\infty} x_z[qm] \frac{q \sin\left(\frac{\pi}{q}(n - qm)\right)}{\pi(n - qm)} \\ \text{Now use } x_z[qm] &= x[m]. \end{aligned}$$

4.53. The continuous-time representation for a periodic discrete-time signal $x[n] \xleftrightarrow{DTFS; \frac{2\pi}{N}} X[k]$ is periodic and thus has a FS representation. This FS representation is a function of the DTFS coefficients $X[k]$, as we show in this problem. The result establishes the relationship between the FS and DTFS representations. Let $x[n]$ have period N and let $x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$.

$$\begin{aligned} x[n] &= x[n + N] \\ x_\delta(t) &= \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \end{aligned}$$

(a) Show $x_\delta(t)$ is periodic and find the period, T .

$$x_\delta(t + T) = \sum_{n=-\infty}^{\infty} x[n]\delta(t + T - nT_s)$$

Now use $x[n] = x[n - N]$ to rewrite

$$x_\delta(t + T) = \sum_{n=-\infty}^{\infty} x[n - N] \delta(t + T - nT_s)$$

let $k = n - N$

$$x_\delta(t + T) = \sum_{k=-\infty}^{\infty} x[k] \delta(t - kT_s + (T - NT_s))$$

It is clear that if $T = NT_s$, then $x_\delta(t + T) = x_\delta(t)$.
Therefore, $x_\delta(t)$ is periodic with $T = NT_s$.

(b) Begin with the definition of the FS coefficients

$$X_\delta[k] = \frac{1}{T} \int_0^T x_\delta(t) e^{-jk\omega_o t} dt.$$

Substitute for T, ω_o , and one period of $x_\delta(t)$ to show

$$\begin{aligned} X_\delta[k] &= \frac{1}{T} \int_0^T x_\delta(t) e^{-jk\omega_o t} dt \\ &= \frac{1}{T} \int_0^T \sum_{n=0}^{N-1} x[n] \delta(t - nT_s) e^{-jk\omega_o t} dt \\ &= \frac{1}{NT_s} \int_0^T \sum_{n=0}^{N-1} x[n] \delta(t - nT_s) e^{-jk\omega_o t} dt \\ &= \frac{1}{NT_s} \sum_{n=0}^{N-1} x[n] \int_0^T \delta(t - nT_s) e^{-jk\omega_o t} dt \\ &= \frac{1}{NT_s} \sum_{n=0}^{N-1} x[n] e^{-jk\omega_o nT_s} \\ &= \frac{1}{T_s} \left(\frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk \frac{2\pi}{N} n} \right) \\ &= \frac{1}{T_s} X[k] \end{aligned}$$

4.54. The fast algorithm for evaluating the DTFS (FFT) may be used to develop a computationally efficient algorithm for determining the output of a discrete-time system with a finite-length impulse response. Instead of directly computing the convolution sum, the DTFS is used to compute the output by performing multiplication in the frequency domain. This requires that we develop a correspondence between the periodic convolution implemented by the DTFS and the linear convolution associated with the system output, the goal of this problem. Let $h[n]$ be an impulse response of length M so that $h[n] = 0$ for $n < 0, n \geq M$. The system output $y[n]$ is related to the input via the convolution sum

$$y[n] = \sum_{k=0}^{M-1} h[k] x[n - k]$$

(a) Consider the N -point periodic convolution of $h[n]$ with N consecutive values of the input sequence $x[n]$ and assume $N > M$. Let $\tilde{x}[n]$ and $\tilde{h}[n]$ be N periodic versions of $x[n]$ and $h[n]$, respectively

$$\begin{aligned}\tilde{x}[n] &= x[n], & \text{for } 0 \leq n \leq N-1 \\ \tilde{x}[n+mN] &= \tilde{x}[n], & \text{for all integer } m, 0 \leq n \leq N-1\end{aligned}$$

$$\begin{aligned}\tilde{h}[n] &= h[n], & \text{for } 0 \leq n \leq N-1 \\ \tilde{h}[n+mN] &= \tilde{h}[n], & \text{for all integer } m, 0 \leq n \leq N-1\end{aligned}$$

The periodic convolution between $\tilde{h}[n]$ and $\tilde{x}[n]$ is

$$\tilde{y}[n] = \sum_{k=0}^{N-1} \tilde{h}[k] \tilde{x}[n-k]$$

Use the relationship between $h[n], x[n]$ and $\tilde{h}[n], \tilde{x}[n]$ to prove that $\tilde{y}[n] = y[n]$, $M-1 \leq n \leq N-1$. That is, the periodic convolution is equal to the linear convolution at $L = N - M + 1$ values of n .

$$\tilde{y}[n] = \sum_{k=0}^{N-1} h[k] \tilde{x}[n-k] \quad (1)$$

Now since $\tilde{x}[n] = x[n]$, for $0 \leq n \leq N-1$, we know that

$$\tilde{x}[n-k] = x[n-k], \quad \text{for } 0 \leq n-k \leq N-1$$

In (1), the sum over k varies from 0 to $M-1$, and so the condition $0 \leq n-k \leq N-1$ is always satisfied provided $M-1 \leq n \leq N-1$. Substituting $x[n-k] = \tilde{x}[n-k]$, $M-1 \leq n \leq N-1$ into (1) yields

$$\begin{aligned}\tilde{y}[n] &= \sum_{k=0}^{M-1} h[k] x[n-k] & M-1 \leq n \leq N-1 \\ &= y[n]\end{aligned}$$

(b) Show that we may obtain values of $y[n]$ other than those on the interval $M-1 \leq n \leq N-1$ by shifting $x[n]$ prior to defining $\tilde{x}[n]$. That is, if

$$\begin{aligned}\tilde{x}_p[n] &= x[n+pL], & 0 \leq n \leq N-1 \\ \tilde{x}_p[n+mN] &= \tilde{x}_p[n], & \text{for all integer } m, 0 \leq n \leq N-1\end{aligned}$$

and

$$\tilde{y}_p[n] = \tilde{h}[n] \circledast \tilde{x}_p[n]$$

then show

$$\tilde{y}_p[n] = y[n+pL], \quad M-1 \leq n \leq N-1$$

This implies that the last L values in one period of $\tilde{y}_p[n]$ correspond to $y[n]$ for $M-1+pL \leq n \leq N-1+pL$. Each time we increment p the N point periodic convolution gives us L new values of the linear convolution. This result is the basis for the so-called *overlap and save method* for evaluating a linear convolution

with the DTFS.

Overlap and Save Method of Implementing Convolution

1. Compute the N DTFS coefficients $H[k] : h[n] \xleftrightarrow{DTFS; 2\pi/N} H[k]$
2. Set $p = 0$ and $L = N - M + 1$
3. Define $\tilde{x}_p[n] = x[n - (M - 1) + pL]$, $0 \leq n \leq N - 1$
4. Compute the N DTFS coefficients $\tilde{X}_p[k] : \tilde{x}_p[n] \xleftrightarrow{DTFS; 2\pi/N} \tilde{X}_p[k]$
5. Compute the product $\tilde{Y}_p[k] = NH[k]\tilde{X}_p[k]$
6. Compute the time signal $\tilde{y}_p[n]$ from the DTFS coefficients, $\tilde{Y}_p[k] : \tilde{y}_p[n] \xleftrightarrow{DTFS; 2\pi/N} \tilde{Y}_p[k]$.
7. Save the L output points: $y[n + pL] = \tilde{y}_p[n + M - 1]$, $0 \leq n \leq L - 1$
8. Set $p = p + 1$ and return to step 3.

Solutions to Computer Experiments

4.55. Repeat Example 4.7 using zero-padding and the MATLAB commands `fft` and `fftshift` to sample and plot $Y(e^{j\Omega})$ at 512 points on $-\pi < \Omega \leq \pi$ for each case.

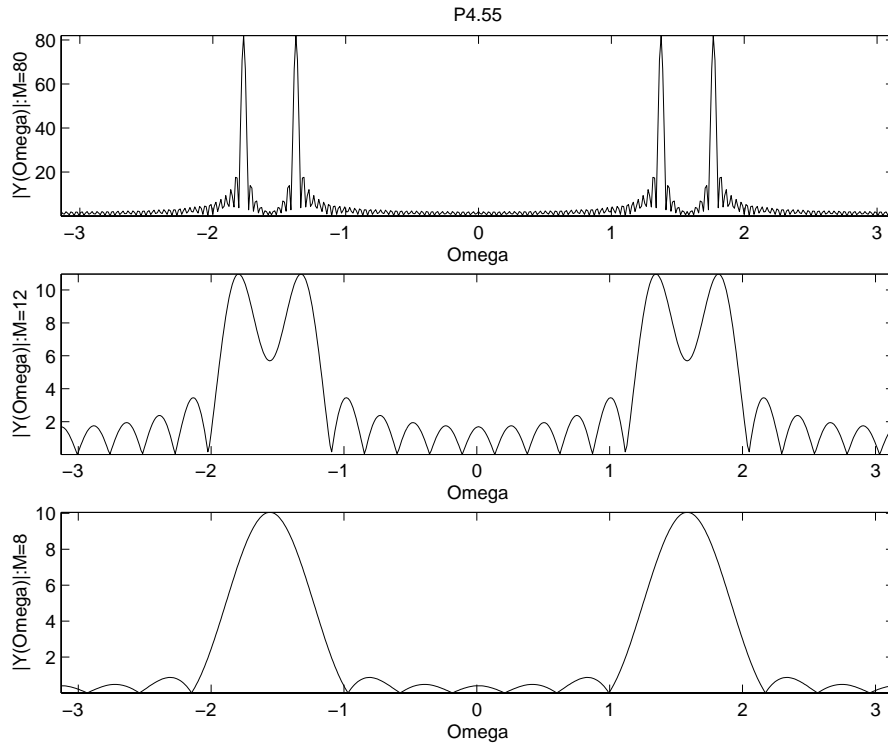


Figure P4.55. Plot of $Y(e^{j\Omega})$

4.56. The rectangular window is defined as

$$w_r[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

We may truncate a signal to the interval $0 \leq n \leq M$ by multiplying the signal with $w[n]$. In the frequency domain we convolve the DTFT of the signal with

$$W_r(e^{j\Omega}) = e^{-j\frac{M}{2}\Omega} \frac{\sin\left(\frac{\Omega(M+1)}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

The effect of this convolution is to smear detail and introduce ripples in the vicinity of discontinuities. The smearing is proportional to the mainlobe width, while the ripple is proportional to the size of the sidelobes. A variety of alternative windows are used in practice to reduce sidelobe height in return for increased main lobe width. In this problem we evaluate the effect windowing time-domain signals on their DTFT. The role of windowing in filter design is explored in Chapter 8. The Hanning window is defined as

$$w_h[n] = \begin{cases} 0.5 - 0.5 \cos\left(\frac{2\pi n}{M}\right), & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

- (a) Assume $M = 50$ and use the MATLAB command `fft` to evaluate magnitude spectrum of the rectangular window in dB at intervals of $\frac{\pi}{50}$, $\frac{\pi}{100}$, and $\frac{\pi}{200}$.
- (b) Assume $M = 50$ and use the MATLAB command `fft` to evaluate the magnitude spectrum of the Hanning window in dB at intervals of $\frac{\pi}{50}$, $\frac{\pi}{100}$, and $\frac{\pi}{200}$.
- (c) Use the results from (a) and (b) to evaluate the mainlobe width and peak sidelobe height in dB for each window.

Using an interval of $\frac{\pi}{200}$, the mainlobe width and peak sidelobe for each window can be estimated from the figure, or finding the local minima and local nulls in the vicinity of the mainlobe.

	Ω :rad	(dB)
	Mainlobe width	Sidelobe height
Rectangular	0.25	-13.48
Hanning	0.5	-31.48

Note: sidelobe height is relative to the mainlobe.

The Hanning window has lower sidelobes, but at the cost of a wider mainlobe when compared to the rectangular window

- (d) Let $y_r[n] = x[n]w_r[n]$ and $y_h[n] = x[n]w_h[n]$ where $x[n] = \cos(\frac{26\pi}{100}n) + \cos(\frac{29\pi}{100}n)$ and $M = 50$. Use the the MATLAB command `fft` to evaluate $|Y_r(e^{j\Omega})|$ in dB and $|Y_h(e^{j\Omega})|$ in dB at intervals of $\frac{\pi}{200}$. Does the window choice affect whether you can identify the presence of two sinusoids? Why?

Yes, since the two sinusoids are very close to one another in frequency, $(\frac{26\pi}{100}$ and $\frac{29\pi}{100})$.

Since the Hanning window has a wider mainlobe, its capability to resolve these two sinusoid is inferior to the rectangular window. Notice from the plot that the existence of two sinusoids are visible for the rectangular window, but not for the Hanning.

- (e) Let $y_r[n] = x[n]w_r[n]$ and $y_h[n] = x[n]w_h[n]$ where $x[n] = \cos(\frac{26\pi}{100}n) + 0.02 \cos(\frac{51\pi}{100}n)$ and $M = 50$. Use the the MATLAB command `fft` to evaluate $|Y_r(e^{j\Omega})|$ in dB and $|Y_h(e^{j\Omega})|$ in dB at intervals of $\frac{\pi}{200}$. Does the window choice affect whether you can identify the presence of two sinusoids? Why?

Yes, here the two sinusoid frequencies are significantly different from one another. The separation of $\frac{\pi}{4}$ from each other is significantly larger than the mainlobe width of either window. Hence, resolution is not a problem for the Hanning window. Since the sidelobe magnitude is greater than 0.02 of the mainlobe in the rectangular window, the sinusoid at $\frac{51\pi}{100}$ is not distinguishable. In contrast, the sidelobes of the Hanning window are much lower, which allows the two sinusoids to be resolved.

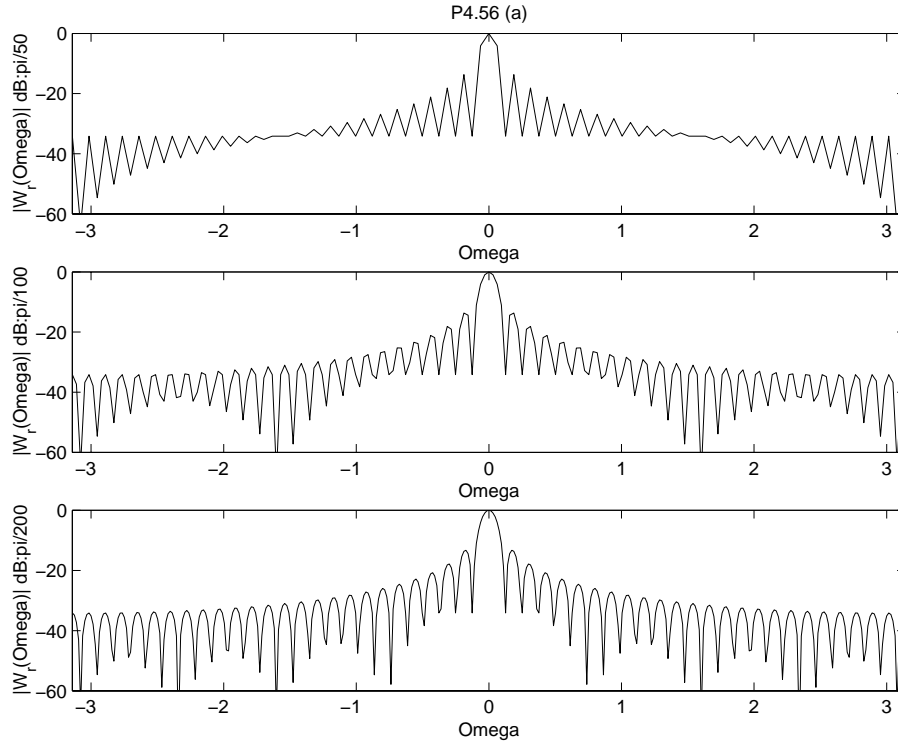


Figure P4.56. Magnitude spectrum of the rectangular window

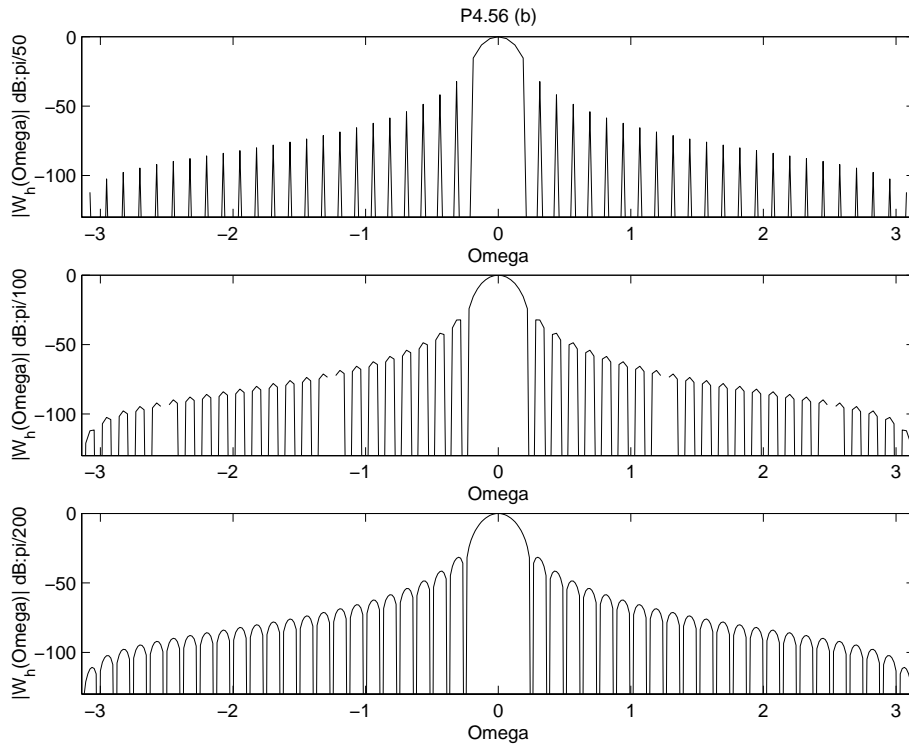


Figure P4.56. Magnitude spectrum of the Hanning window

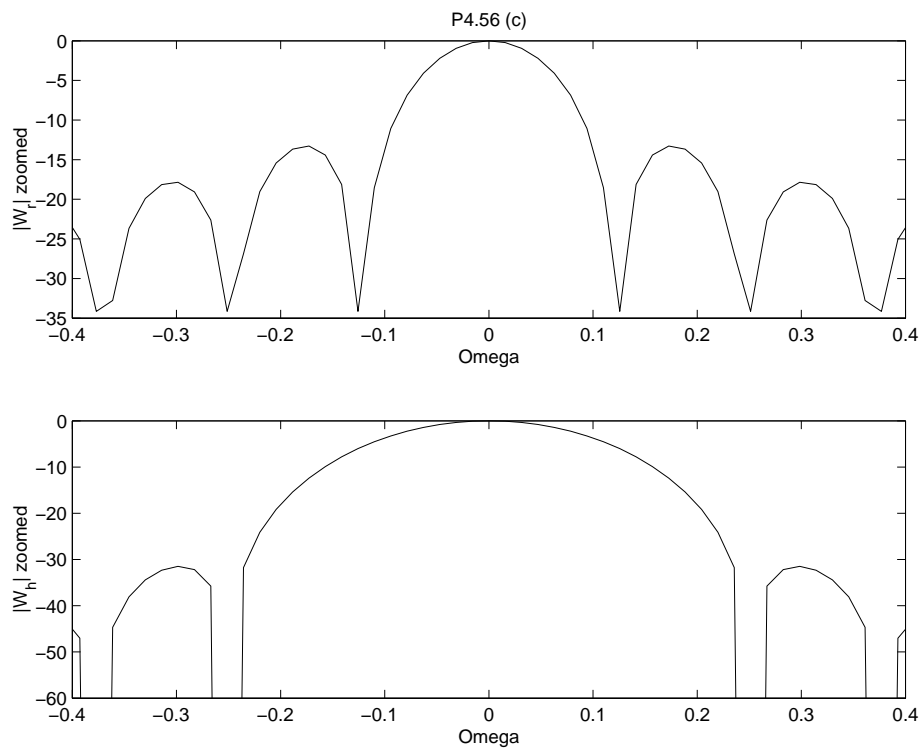


Figure P4.56. Zoomed in plots of $W_r(e^{j\Omega})$ and $W_h(e^{j\Omega})$

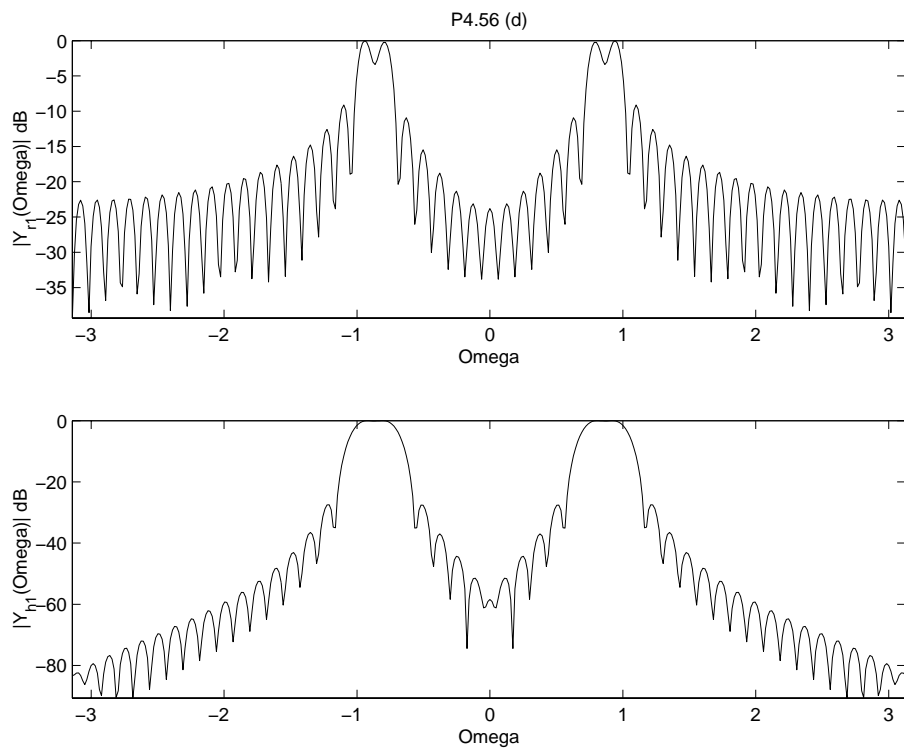


Figure P4.56. (d) Plots of $|Y_r(e^{j\Omega})|$ and $|Y_h(e^{j\Omega})|$

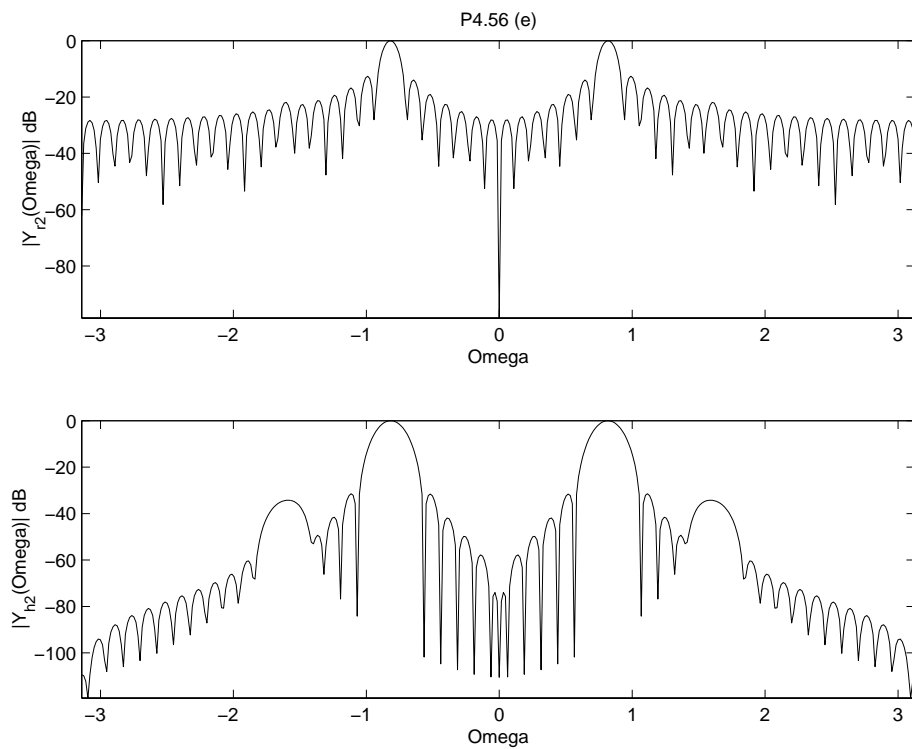


Figure P4.56. (e) Plots of $|Y_r(e^{j\Omega})|$ and $|Y_h(e^{j\Omega})|$

4.57. Let a discrete-time signal $x[n]$ be defined as

$$x[n] = \begin{cases} e^{-\frac{(0.1n)^2}{2}}, & |n| \leq 50 \\ 0, & \text{otherwise} \end{cases}$$

Use the MATLAB commands `fft` and `fftshift` to numerically evaluate and plot the DTFT of $x[n]$ and the following subsampled signals at 500 values of Ω on the interval $-\pi < \Omega \leq \pi$:

(a) $y[n] = x[2n]$

(b) $z[n] = x[4n]$

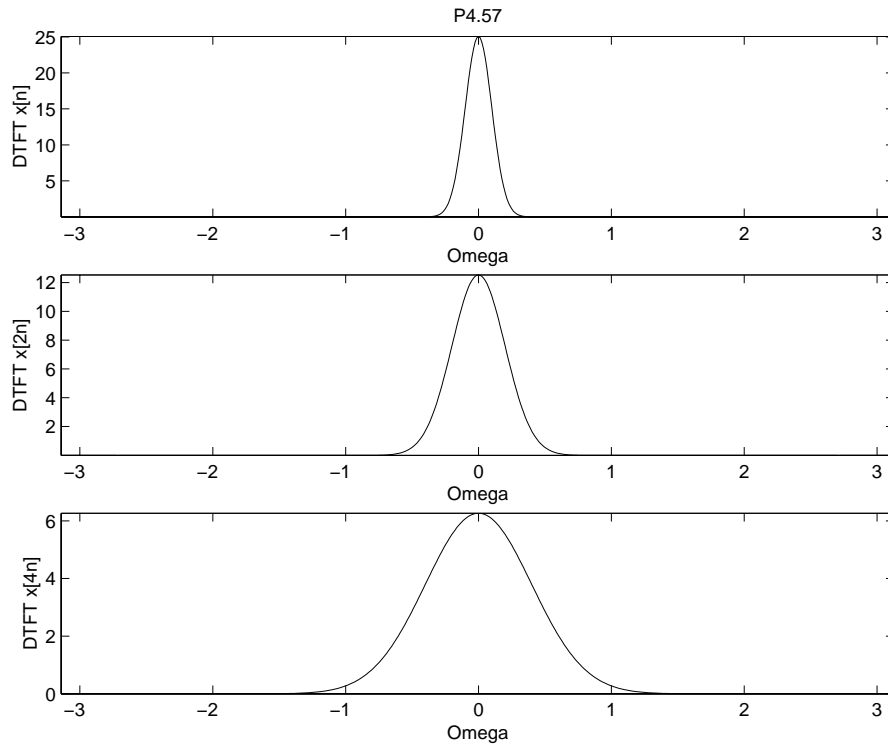


Figure P4.57. Plot of the DTFT of $x[n]$

4.58. Repeat Problem 4.57 assuming

$$x[n] = \begin{cases} \cos(\frac{\pi}{2}n)e^{-\frac{(0.1n)^2}{2}}, & |n| \leq 50 \\ 0, & \text{otherwise} \end{cases}$$

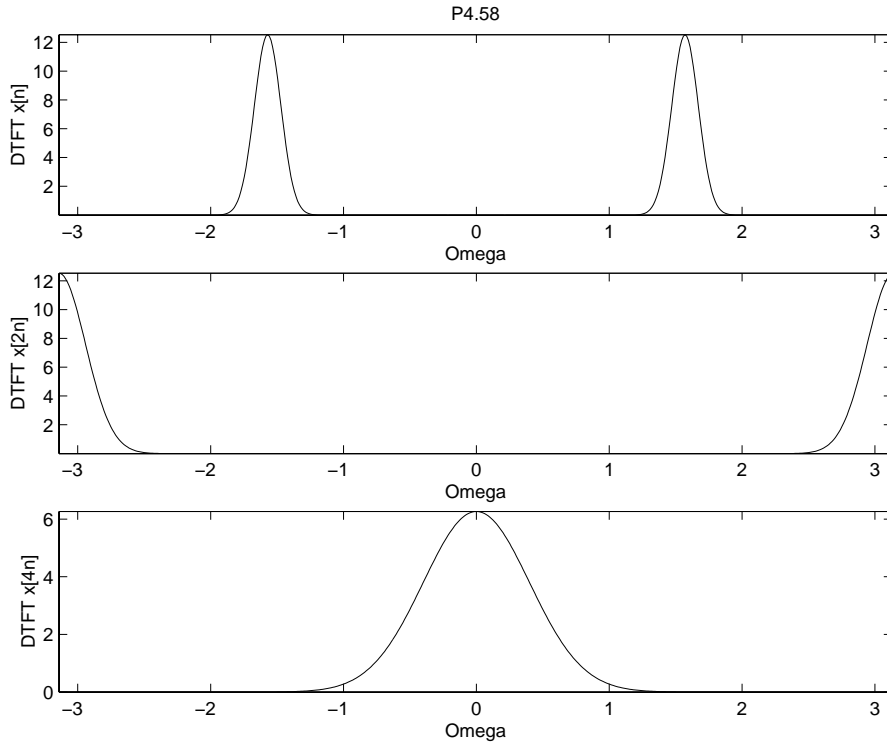


Figure P4.58. Plot of the DTFT of $x[n]$

4.59. A signal $x(t)$ is defined as

$$x(t) = \cos\left(\frac{3\pi}{2}t\right)e^{-\frac{t^2}{2}}$$

(a) Evaluate the FT $X(j\omega)$ and show that $|X(j\omega)| \approx 0$ for $|\omega| > 3\pi$.

$$\begin{aligned} X(j\omega) &= \pi \left(\delta\left(\omega - \frac{3\pi}{2}\right) + \delta\left(\omega + \frac{3\pi}{2}\right) \right) * \frac{e^{-\frac{\omega^2}{2}}}{\sqrt{2\pi}} \\ &= \sqrt{2\pi^3} \left(e^{-\frac{(\omega - \frac{3\pi}{2})^2}{2}} + e^{-\frac{(\omega + \frac{3\pi}{2})^2}{2}} \right) \end{aligned}$$

for $|\omega| > 3\pi$

$$\begin{aligned} |X(j\omega)| &= \sqrt{2\pi^3} \left| e^{-\frac{(\omega - \frac{3\pi}{2})^2}{2}} + e^{-\frac{(\omega + \frac{3\pi}{2})^2}{2}} \right| \\ |X(j\omega)| &\leq \sqrt{2\pi^3} \left| e^{-\frac{(\frac{3\pi}{2})^2}{2}} + e^{-\frac{(\frac{9\pi}{2})^2}{2}} \right| \\ &\approx 1.2 \times 10^{-4} \end{aligned}$$

Which is small enough to be approximated as zero.

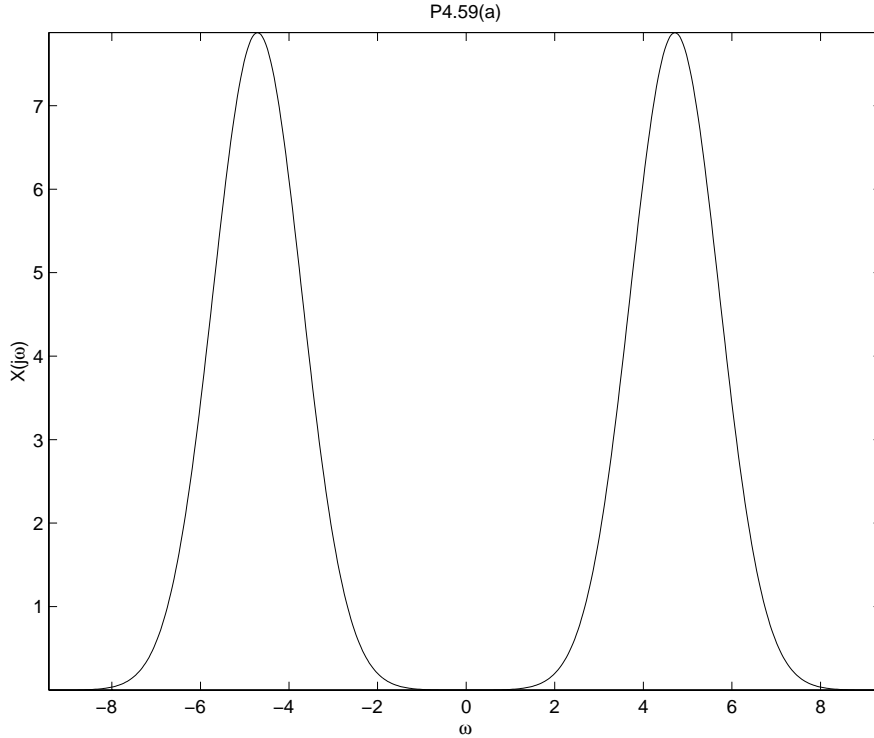


Figure P4.59. The Fourier Transform of $x(t)$

In parts (b)-(d), we compare $X(j\omega)$ to the FT of the sampled signal, $x[n] = x(nT_s)$, for several sampling intervals. Let $x[n] \xleftrightarrow{FT} X_\delta(j\omega)$ be the FT of the sampled version of $x(t)$. Use MATLAB to numerically determine $X_\delta(j\omega)$ by evaluating

$$X_\delta(j\omega) = \sum_{n=-25}^{25} x[n] e^{-j\omega T_s}$$

at 500 values of ω on the interval $-3\pi < \omega < 3\pi$. In each case, compare $X(j\omega)$ and $X_\delta(j\omega)$ and explain any differences.

Notice that $x[n]$ is symmetric with respect to n , so

$$\begin{aligned} X_\delta(j\omega) &= x[0] + 2 \sum_{n=1}^{25} x[n] \cos(\omega n T_s) \\ &= 1 + 2 \sum_{n=1}^{25} x[n] \cos(\omega n T_s) \end{aligned}$$

Aside from the magnitude difference of $\frac{1}{T_s}$, for each T_s , $X(j\omega)$ and $X_\delta(j\omega)$ are different within $[-3\pi, 3\pi]$ only when the sampling period is large enough to cause noticeable aliasing. It is obvious that the worst aliasing occurs when $T_s = \frac{1}{2}$.

(b) $T_s = \frac{1}{3}$

(c) $T_s = \frac{2}{5}$

(d) $T_s = \frac{1}{2}$

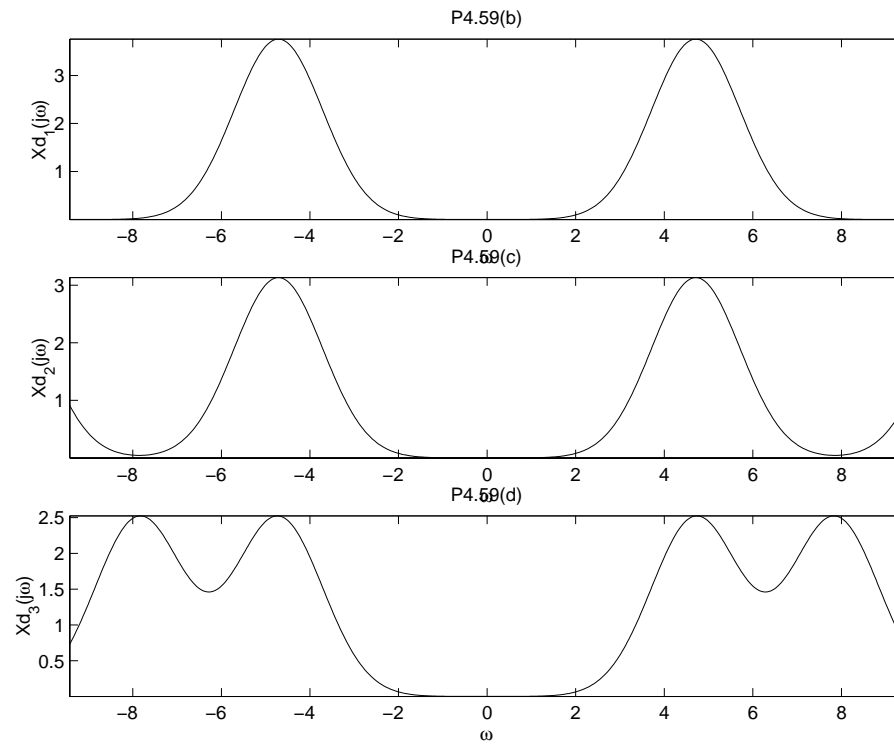


Figure P4.59. Comparison of the FT to the sampled FT

4.60. Use the MATLAB command `fft` to repeat Example 4.14.

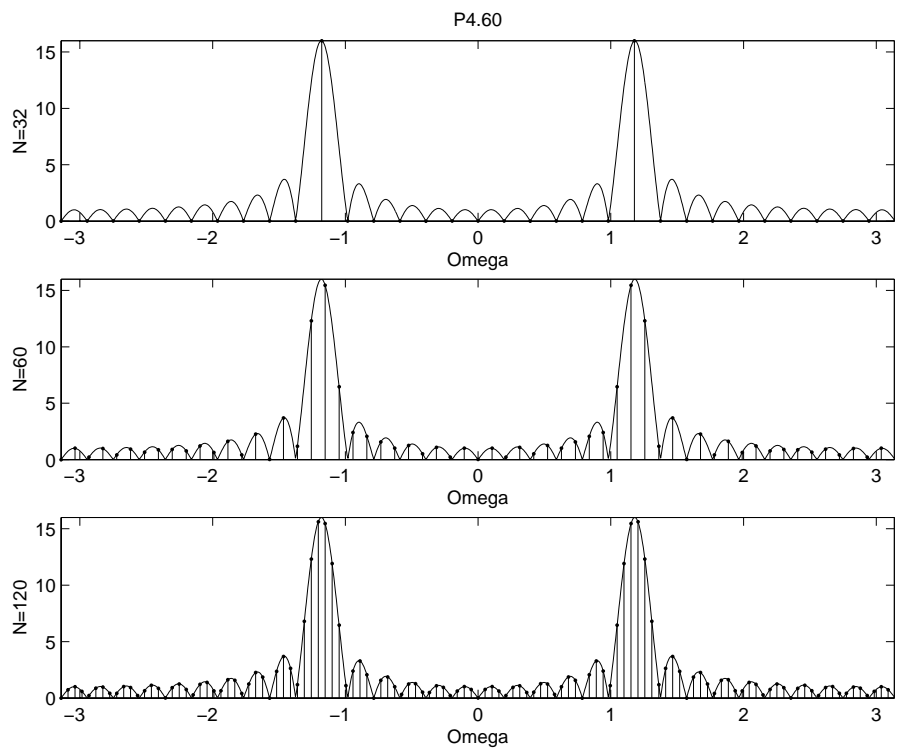


Figure P4.60. Plot of $|X(e^{j\Omega})|$ and $N|X[k]|$

4.61. Use the MATLAB command `fft` to repeat Example 4.15.

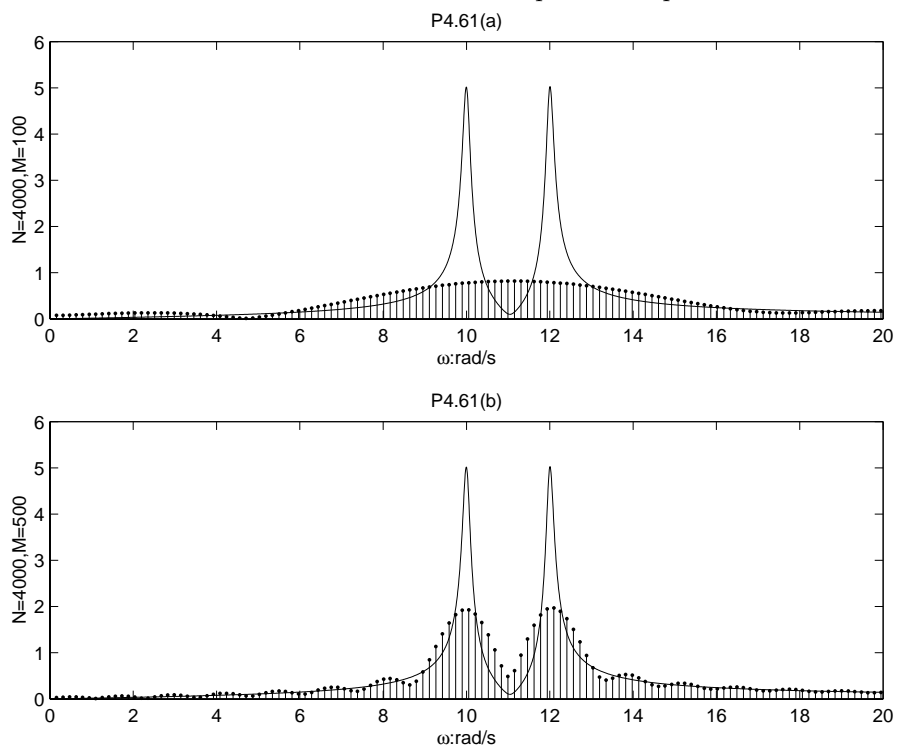


Figure P4.61. DTFT approximation to the FT, graphs of $|X(e^{j\Omega})|$ and $NT_s|Y[k]|$

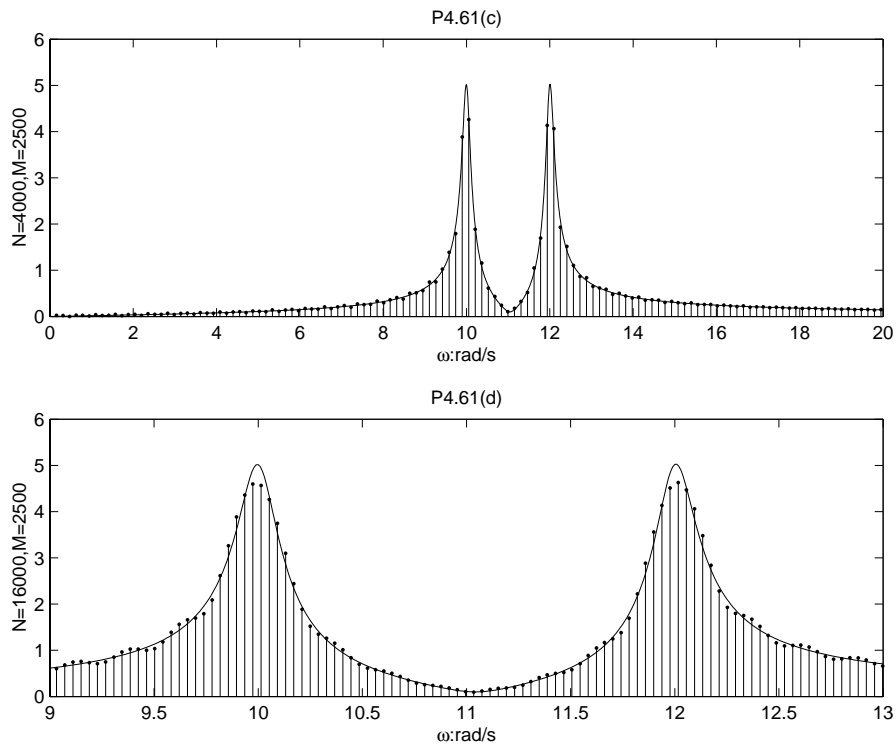


Figure P4.61. DTFT approximation to the FT, graphs of $|X(e^{j\Omega})|$ and $NT_s|Y[k]|$

4.62. Use the MATLAB command `fft` to repeat Example 4.16. Also depict the DTFS approximation and the underlying DTFT for $M = 2001$ and $M=2005$.

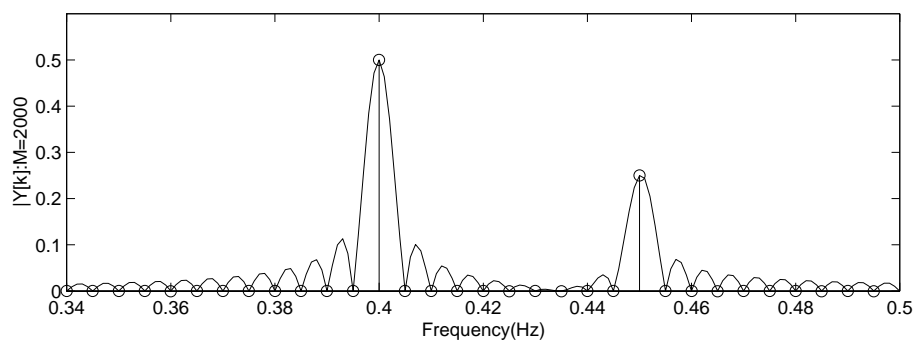
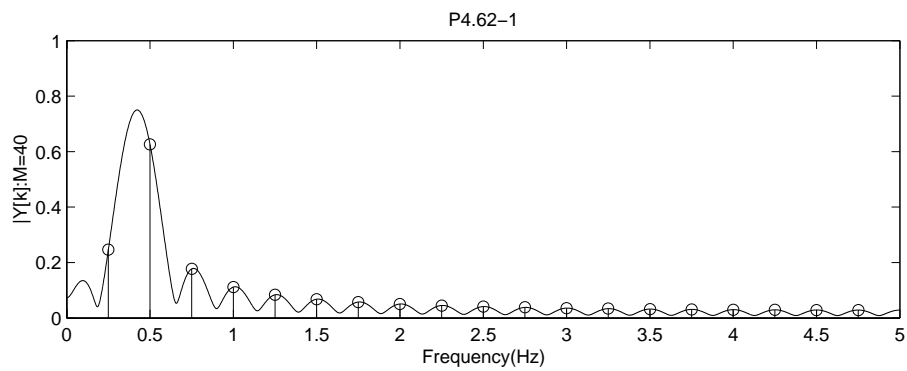


Figure P4.62. DTFS Approximation

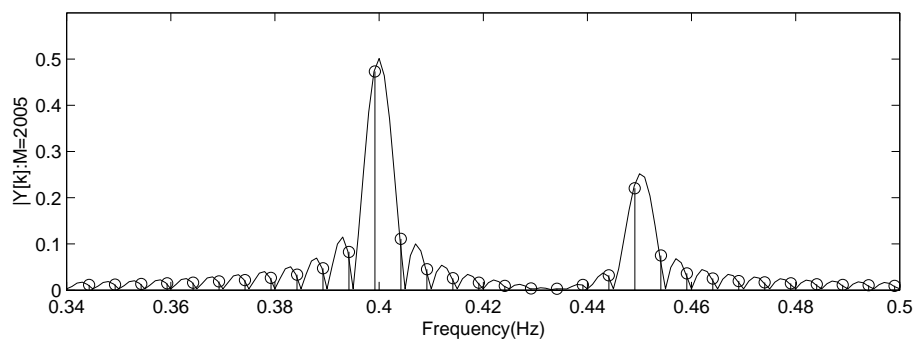
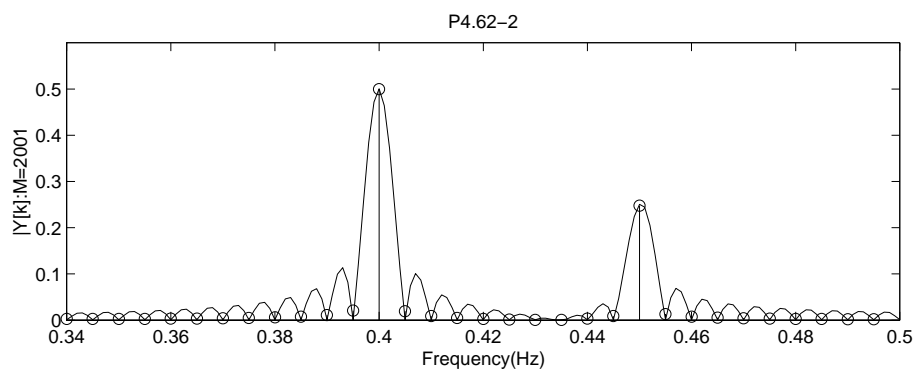


Figure P4.62. DTFS Approximation

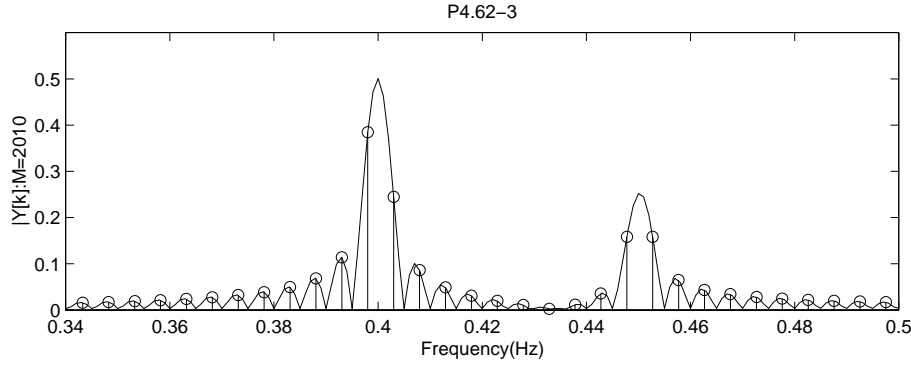


Figure P4.62. DTFS Approximation

4.63. Consider the sum of sinusoids

$$x(t) = \cos(2\pi t) + 2 \cos(2\pi(0.8)t) + \frac{1}{2} \cos(2\pi(1.1)t)$$

Assume the frequency band of interest is $-5\pi < \omega < 5\pi$.

(a) Determine the sampling interval T_s so that the DTFS approximation to the FT of $x(t)$ spans the desired frequency band.

$$\begin{aligned} \omega_a &= 5\pi \\ T_s &< \frac{2\pi}{3\omega_a} = 0.133 \end{aligned}$$

choose:

$$T_s = 0.1$$

(b) Determine the minimum number of samples M_o so that the DTFS approximation consists of discrete-valued impulses located at the frequency corresponding to each sinusoid.

For a given M_o , the frequency interval for sampling the DTFS is $\frac{2\pi}{M_o}$

$$\begin{aligned} \frac{2\pi}{M_o} k_1 &= 2\pi T_s \\ M_o &= \frac{k_1}{T_s} \\ \frac{2\pi}{M_o} k_2 &= 2\pi(0.8)T_s \\ M_o &= \frac{k_2}{0.8T_s} \\ \frac{2\pi}{M_o} k_3 &= 2\pi(1.1)T_s \\ M_o &= \frac{k_3}{1.1T_s} \end{aligned}$$

where k_1, k_2, k_3 are integers.

By choosing $T_s = 0.1$, the minimum $M_o = 100$ with $k_1 = 10$, $k_2 = 8$, $k_3 = 11$.

(c) Use MATLAB to plot $\frac{1}{M}|Y_\delta(j\omega)|$ and $|Y[k]|$ for the value of T_s chosen in parts (a) and $M = M_o$.

(d) Repeat part (c) using $M = M_o + 5$ and $M = M_o + 8$.

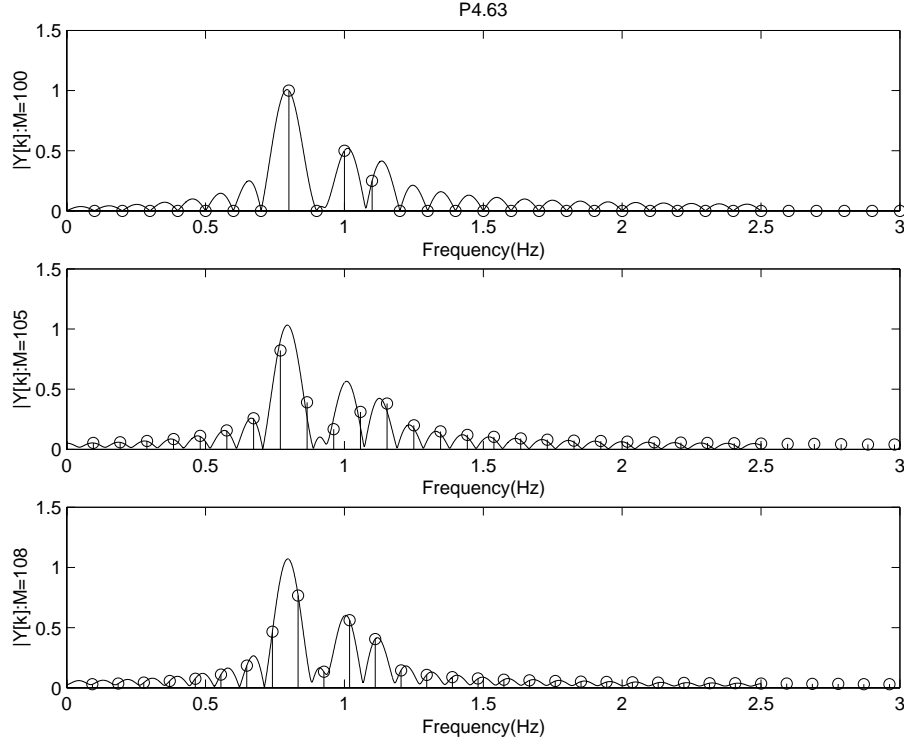


Figure P4.63. Plots of $(1/M)|Y_\delta(j\omega)|$ and $|Y[k]|$ for the appropriate values of T_s and M

4.64. We desire to use the DTFS to approximate the FT of a continuous time signal $x(t)$ on the band $-\omega_a < \omega < \omega_a$ with resolution ω_r and a maximum sampling interval in frequency of $\Delta\omega$. Find the sampling interval T_s , number of samples M , and DTFS length N . You may assume that the signal is effectively bandlimited to a frequency ω_m for which $|X(j\omega_a)| \geq 10|X(j\omega)|$, $\omega > \omega_m$. Plot the FT and the DTFS approximation for each of the following cases using the MATLAB command `fft`.

We use $T_s < \frac{2\pi}{\omega_m + \omega_a}$; $M \geq \frac{\omega_s}{\omega_r}$; $N \geq \frac{\omega_s}{\Delta\omega}$ to find the required T_s , M , N for part (a) and (b).

$$(a) \ x(t) = \begin{cases} 1, & |t| < 1 \\ 0, & \text{otherwise} \end{cases}, \ \omega_a = \frac{3\pi}{2}, \ \omega_r = \frac{3\pi}{4}, \text{ and } \Delta\omega = \frac{\pi}{8}$$

$$X(j\omega) = \frac{2 \sin(\omega)}{\omega}$$

We want :

$$\left| \frac{\sin(\omega)}{\omega} \right| \leq \frac{1}{15\pi} \text{ for } |\omega| \geq \omega_m$$

which gives

$$\omega_m = 15\pi$$

Hence:

$$T_s < \frac{2\pi}{15\pi + \frac{3\pi}{2}} \approx 0.12$$

$$\text{choose } T_s = 0.1$$

$$M \geq 26.67$$

$$\text{choose } M = 28$$

$$N \geq 160$$

$$\text{choose } N = 160$$

(b) $x(t) = \frac{1}{2\pi} e^{-\frac{t^2}{2}}$, $\omega_a = 3$, $\omega_r = \frac{1}{2}$, and $\Delta\omega = \frac{1}{8}$

$$X(j\omega) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}}$$

We want :

$$e^{-\frac{\omega^2}{2}} \leq \frac{1}{10} e^{-\frac{9}{2}}$$

which gives

$$\omega_m = 3.69$$

Hence:

$$T_s < 0.939$$

$$\text{choose } T_s = 0.9$$

$$M \geq 3.49$$

$$\text{choose } M = 4$$

$$N \geq 55.85$$

$$\text{choose } N = 56$$

(c) $x(t) = \cos(20\pi t) + \cos(21\pi t)$, $\omega_a = 40\pi$, $\omega_r = \frac{\pi}{3}$, and $\Delta\omega = \frac{\pi}{10}$

$$\omega_s = 2\omega_a = 80\pi$$

$$X(j\omega) = \pi (\delta(\omega - 20\pi) + \delta(\omega + 20\pi))$$

Hence:

$$T_s < \frac{2\pi}{3\omega_a} = 0.0167$$

$$\text{choose } T_s = 0.01$$

$$M \geq 600$$

$$\text{choose } M = 600$$

$$\text{choose } N = M = 600$$

(d) Repeat case (c) using $\omega_r = \frac{\pi}{10}$.

Hint: Be sure to sample the pulses in (a) and (b) symmetrically about $t = 0$.

choose $T_s = 0.01$

$M \geq 2000$

choose $M = 2000$

choose $N = M = 2000$

For (c) and (d), one needs to consider $X(j\omega) * W_\delta(j\omega)$, where

$$W_\delta(j\omega) = e^{-j\omega T_s \left(\frac{M-1}{2}\right)} \left(\frac{\sin\left(M \frac{\omega T_s}{2}\right)}{\sin\left(\frac{\omega T_s}{2}\right)} \right)$$

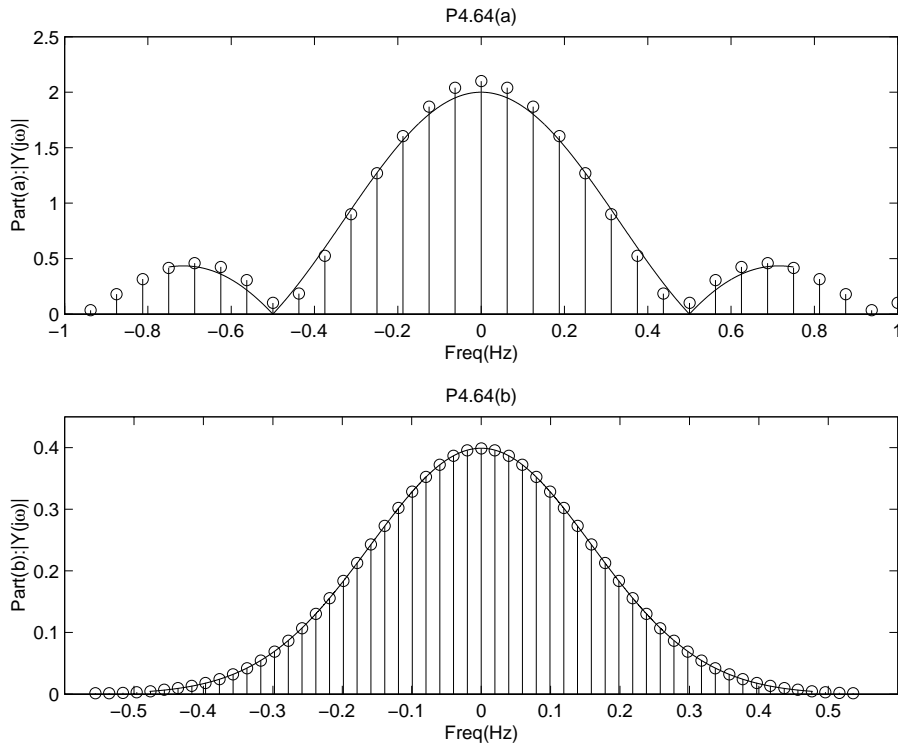


Figure P4.64. FT and DTFS approximation

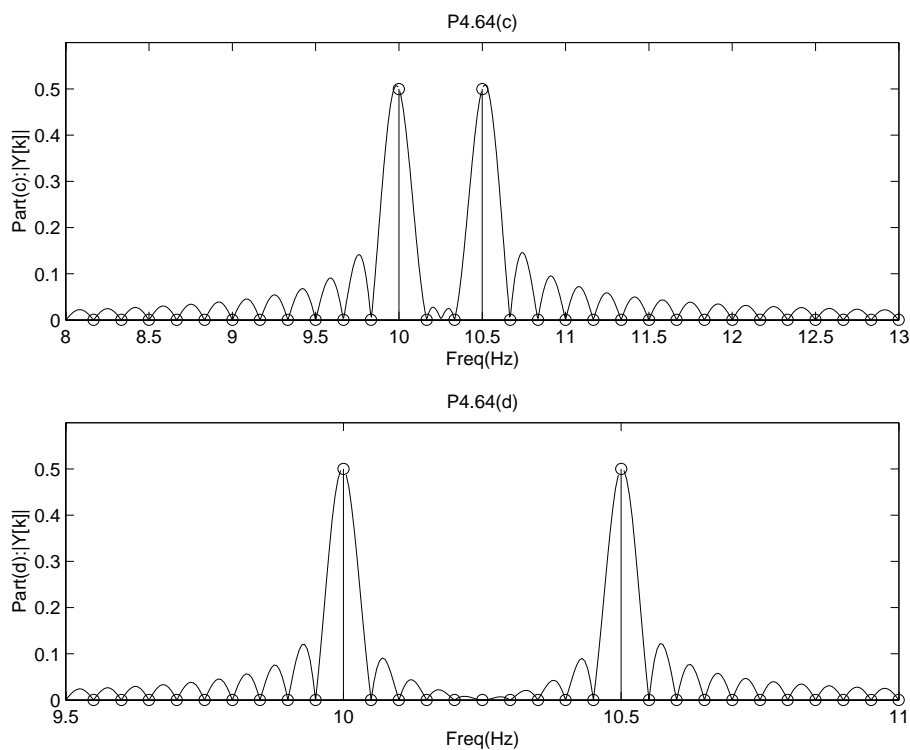


Figure P4.64. FT and DTFS approximation

4.65. The overlap and save method for linear filtering is discussed in Problem 4.54. Write a MATLAB m-file that implements the overlap and save method using `fft` to evaluate the convolution $y[n] = h[n] * x[n]$ on $0 \leq n < L$ for the following signals.

(a) $h[n] = \frac{1}{5}(u[n] - u[n - 5])$, $x[n] = \cos(\frac{\pi}{6}n)$, $L = 30$

$$M = 5$$

$$L = 30$$

$$N = 34$$

(b) $h[n] = \frac{1}{5}(u[n] - u[n - 5])$, $x[n] = (\frac{1}{2})^n u[n]$, $L = 20$

$$M = 5$$

$$L = 20$$

$$N = 24$$

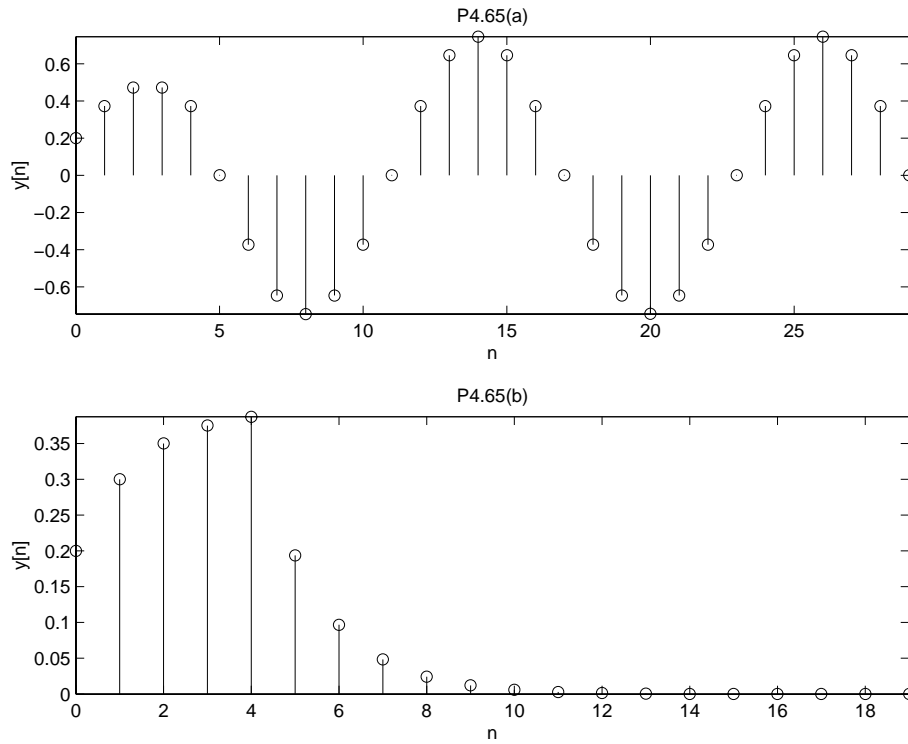


Figure P4.65. Overlap-and-save method

4.66. Plot the ratio of the number of multiplies in the direct method for computing the DTFS coefficients to that of the FFT approach when $N = 2^p$ for $p = 2, 3, 4, \dots, 16$.

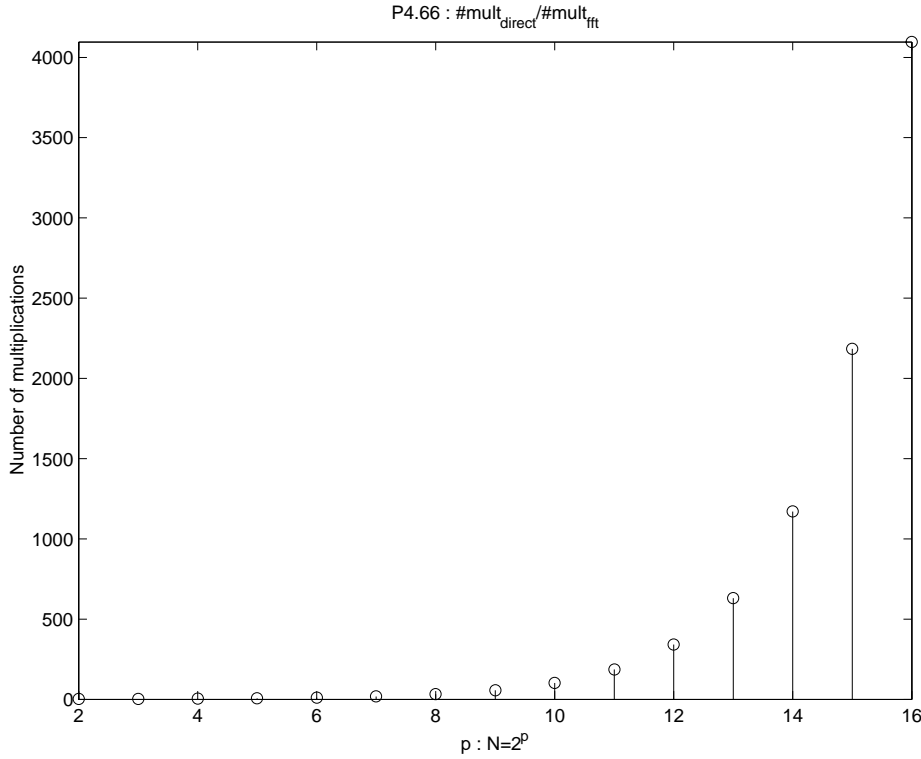


Figure P4.66. (a)

4.67. In this experiment we investigate evaluation of the time-bandwidth product with the DTFS. Let $x(t) \xleftrightarrow{FT} X(j\omega)$.

(a) Use the Riemann sum approximation to an integral

$$\int_a^b f(u)du \approx \sum_{m=m_a}^{m_b} f(m\Delta u)\Delta u$$

to show that

$$\begin{aligned} T_d &= \left[\frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \right]^{\frac{1}{2}} \\ &\approx T_s \left[\frac{\sum_{n=-M}^M n^2 |x[n]|^2}{\sum_{n=-M}^M |x[n]|^2} \right]^{\frac{1}{2}} \end{aligned}$$

provided $x[n] = x(nT_s)$ represents the samples of $x(t)$ and $x(nT_s) \approx 0$ for $|n| > M$.

By setting $t = nT_s$, $dt \approx \Delta t = T_s$, so

$$\begin{aligned} \int_{-\infty}^{\infty} t^2 |x(t)|^2 dt &\approx \sum_{n=-M}^M (nT_s)^2 |x(nT_s)|^2 T_s \\ &= T_s^3 \sum_{n=-M}^M n^2 |x[n]|^2 \end{aligned}$$

similarly

$$\int_{-\infty}^{\infty} |x(t)|^2 dt \approx T_s \sum_{n=-M}^M |x[n]|^2$$

Therefore

$$T_d \approx T_s \left[\frac{\sum_{n=-M}^M n^2 |x[n]|^2}{\sum_{n=-M}^M |x[n]|^2} \right]^{\frac{1}{2}}$$

(b) Use the DTFS approximation to the FT and the Riemann sum approximation to an integral to show that

$$\begin{aligned} B_w &= \left[\frac{\int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega} \right]^{\frac{1}{2}} \\ &\approx \frac{\omega_s}{2M+1} \left[\frac{\sum_{k=-M}^M |k|^2 |X[k]|^2}{\sum_{k=-M}^M |X[k]|^2} \right]^{\frac{1}{2}} \end{aligned}$$

where $x[n] \xleftrightarrow{DTFS; \frac{2\pi}{2M+1}} X[k]$, $\omega_s = \frac{2\pi}{T_s}$ is the sampling frequency, and $X(jk\frac{\omega_s}{2M+1}) \approx 0$ for $|k| > M$.

Using the $(2M+1)$ -point DTFS approximation, we have:

$$\begin{aligned} \omega_k &= k \frac{\omega_s}{2M+1} \\ \text{hence} \\ d\omega &\approx \Delta\omega = \frac{\omega_s}{2M+1} \\ X[k] &= \frac{1}{(2M+1)T_s} X(jk\frac{\omega_s}{2M+1}) \\ \int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2 d\omega &= \left(\frac{\omega_s}{2M+1} \right)^3 \sum_{k=-M}^M k^2 |X[k]|^2 \\ \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega &= \left(\frac{\omega_s}{2M+1} \right) \sum_{k=-M}^M |X[k]|^2 \end{aligned}$$

therefore

$$B_w \approx \frac{\omega_s}{2M+1} \left[\frac{\sum_{k=-M}^M |k|^2 |X[k]|^2}{\sum_{k=-M}^M |X[k]|^2} \right]^{\frac{1}{2}}$$

(c) Use the result from (a) and (b) and Eq. (3.65) to show that the time-bandwidth product computed using the DTFS approximation satisfies

$$\left[\frac{\sum_{n=-M}^M n^2 |x[n]|^2}{\sum_{n=-M}^M |x[n]|^2} \right]^{\frac{1}{2}} \left[\frac{\sum_{k=-M}^M |k|^2 |X[k]|^2}{\sum_{k=-M}^M |X[k]|^2} \right]^{\frac{1}{2}} \geq \frac{2M+1}{4\pi}$$

$$\text{Note } T_d B_w \approx \frac{T_s \omega_s}{2M+1} \left[\frac{\sum_{n=-M}^M n^2 |x[n]|^2}{\sum_{n=-M}^M |x[n]|^2} \right]^{\frac{1}{2}} \left[\frac{\sum_{k=-M}^M |k|^2 |X[k]|^2}{\sum_{k=-M}^M |X[k]|^2} \right]^{\frac{1}{2}}$$

Since $T_d B_w \geq \frac{1}{2}$, and $\frac{T_s \omega_s}{2M+1} = \frac{2\pi}{2M+1}$, we have

$$\left[\frac{\sum_{n=-M}^M n^2 |x[n]|^2}{\sum_{n=-M}^M |x[n]|^2} \right]^{\frac{1}{2}} \left[\frac{\sum_{k=-M}^M |k|^2 |X[k]|^2}{\sum_{k=-M}^M |X[k]|^2} \right]^{\frac{1}{2}} \geq \frac{2M+1}{4\pi}$$

(d) Repeat Computer Experiment 3.115 to demonstrate that the bound in (c) is satisfied and that Gaussian pulses satisfy the bound with equality.

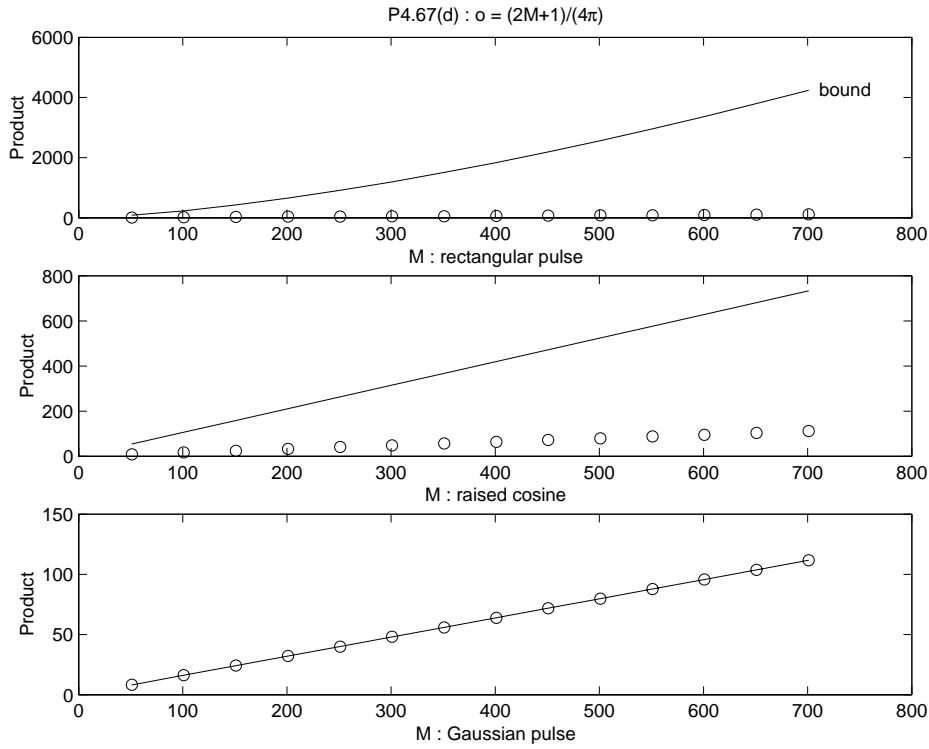


Figure P4.67. Plot of Computer Experiment 3.115

CHAPTER 5

Additional Problems

5.17 (a) The AM signal is defined by

$$\begin{aligned} s(t) &= A_c(1 + k_a m(t)) \cos(\omega_c t) \\ &= A_c \left(1 + \frac{k_a}{1 + t^2} \right) \cos(\omega_c t) \end{aligned}$$

To obtain 50% modulation, we choose $k_a = 1$, which results in the modulated waveform of Fig. 1 for $s(t)$ with $A_c = 1$ volt:

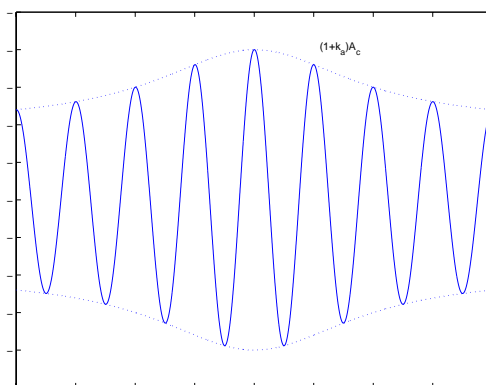


Figure 1

(b) The DSB-SC modulated signal is defined by

$$\begin{aligned} s(t) &= A_c m(t) \cos(\omega_c t) \\ &= \frac{A_c}{1 + t^2} \cos(\omega_c t) \end{aligned}$$

which has the modulated waveform of Fig. 2 for $A_c = 1$ volt:

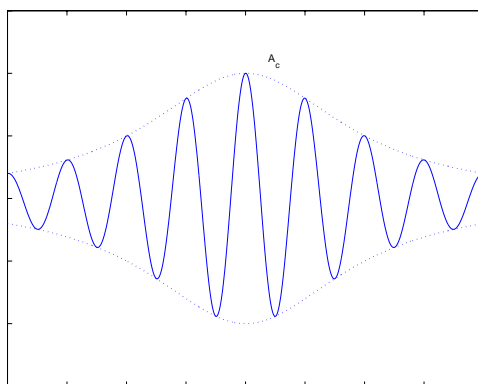


Figure 2

Typically, a DSB-SC signal exhibits phase reversals. No such reversals are exhibited in Fig. 2 because the modulating signal $m(t)$ is nonnegative for all time t .

5.18 The AM signal is defined by

$$s(t) = A_c(1 + k_a m(t)) \cos(\omega_c t)$$

where $A_c \cos(\omega_c t)$ is the carrier and k_a is a constant. We are given $\omega_c = 2\pi \times 10^5$ rad/sec; that is, $f_c = 100$ kHz.

(a) $m(t) = A_0 \cos(2\pi \times 10^3 t)$

$$\omega_0 = 2\pi \times 10^3 \text{ rad/sec}$$

$$f_c = 1 \text{ kHz}$$

The frequency components of $s(t)$ for positive frequencies are:

$$f_0 = 100 \text{ kHz}$$

$$f_c + f_0 = 100 + 1 = 101 \text{ kHz}$$

$$f_c - f_0 = 100 - 1 = 99 \text{ kHz}$$

(b) $m(t) = A_0 \cos(2\pi \times 10^3 t) + A_0 \cos(4\pi \times 10^3 t)$

which consists of two sinusoidal components with frequencies $f_0 = 1$ kHz and $f_1 = 2$ kHz. Hence, the frequency components of $s(t)$ for positive frequencies are:

$$f_c = 100 \text{ kHz}$$

$$f_c + f_0 = 100 + 1 = 101 \text{ kHz}$$

$$f_c - f_0 = 100 - 1 = 99 \text{ kHz}$$

$$f_c + f_1 = 100 + 2 = 102 \text{ kHz}$$

$$f_c - f_1 = 100 - 2 = 98 \text{ kHz}$$

(c) First, we note that

$$\cos(X) \sin(Y) = \frac{1}{2} \sin(Y + X) + \frac{1}{2} \sin(Y - X)$$

Hence,

$$\begin{aligned} m(t) &= A_0 \cos(2\pi \times 10^3 t) \sin(4\pi \times 10^3 t) \\ &= \frac{A_0}{2} \sin(6\pi \times 10^3 t) + \frac{A_0}{2} \sin(2\pi \times 10^3 t) \end{aligned}$$

which consists of two sinusoidal components with frequencies $f_0 = 3$ kHz and $f_1 = 1$ kHz. Correspondingly, the frequency components of $s(t)$ for positive frequencies are:

$$f_c = 100 \text{ kHz}$$

$$f_c + f_0 = 100 + 3 = 103 \text{ kHz}$$

$$f_c - f_0 = 100 - 3 = 97 \text{ kHz}$$

$$f_c + f_1 = 100 + 1 = 101 \text{ kHz}$$

$$f_c - f_1 = 100 - 1 = 99 \text{ kHz}$$

(d) Here we use the formula:

$$\cos^2 \theta = \frac{1}{2}[1 + \cos(2\theta)]$$

Hence,

$$\begin{aligned} m(t) &= A_0 \cos^2(2\pi \times 10^3 t) \\ &= \frac{A_0}{2}[1 + \cos(4\pi \times 10^3 t)] \end{aligned}$$

which consists of dc component and sinusoidal component of frequency $f_0 = 2 \text{ kHz}$.

The frequency components of $s(t)$ for positive frequencies are therefore:

$$f_c = 100 \text{ kHz}$$

$$f_c + f_0 = 100 + 2 = 102 \text{ kHz}$$

$$f_c - f_0 = 100 - 2 = 98 \text{ kHz}$$

(e) For this part of the problem, we find use of the formulas:

$$\cos^2 \theta = \frac{1}{2}[1 + \cos(2\theta)]$$

$$\sin^2 \theta = \frac{1}{2}[1 - \cos(2\theta)]$$

Hence,

$$\begin{aligned} m(t) &= \frac{1}{2}[1 + \cos(4\pi \times 10^3 t)] + \frac{1}{2}[1 - \cos(8\pi \times 10^3 t)] \\ &= 1 + \frac{1}{2}\cos(4\pi \times 10^3 t) - \frac{1}{2}\cos(8\pi \times 10^3 t) \end{aligned}$$

which consists of dc component, and two sinusoidal components with $f_0 = 2 \text{ kHz}$ and $f_1 = 4 \text{ kHz}$. Correspondingly, the frequency components of $s(t)$ for positive frequencies are:

$$f_c = 100 \text{ kHz}$$

$$f_c + f_0 = 100 + 2 = 102 \text{ kHz}$$

$$f_c - f_0 = 100 - 2 = 98 \text{ kHz}$$

$$f_c + f_1 = 100 + 4 = 104 \text{ kHz}$$

$$f_c - f_1 = 100 - 4 = 96 \text{ kHz}$$

(f) We first use the formula

$$\cos^3 \theta = \cos \theta \cdot \frac{1}{2}[1 + \cos(2\theta)]$$

$$\begin{aligned}
&= \frac{1}{2} \cos \theta + \frac{1}{2} \cos \theta \cos(2\theta) \\
&= \frac{1}{2} \cos \theta + \frac{1}{2} \left[\frac{1}{2} \cos(\theta + 2\theta) + \frac{1}{2} \cos(\theta - 2\theta) \right] \\
&= \frac{1}{4} \cos(3\theta) + \frac{3}{4} \cos \theta
\end{aligned}$$

Hence,

$$\begin{aligned}
m(t) &= A_0 \cos^2(2\pi \times 10^3 t) \\
&= \frac{A_0}{4} \cos(6\pi \times 10^3 t) + \frac{3A_0}{4} \cos(2\pi \times 10^3 t)
\end{aligned}$$

which consists of two sinusoidal components with frequencies $f_0 = 1$ kHz and $f_1 = 3$ kHz. The frequency components of $s(t)$ are therefore:

$$\begin{aligned}
f_c &= 100 \text{ kHz} \\
f_c + f_0 &= 100 + 1 = 101 \text{ kHz} \\
f_c - f_0 &= 100 - 1 = 99 \text{ kHz} \\
f_c + f_1 &= 100 + 3 = 103 \text{ kHz} \\
f_c - f_1 &= 100 - 3 = 97 \text{ kHz}
\end{aligned}$$

Note: For negative frequencies, the frequency components of $s(t)$ are the negative of those for positive frequencies.

- 5.19 (a) For a square wave with equal mark-to-space ratio and frequency $f_0 = 500$ Hz, alternating between 0 and 1, has the frequency expansion (see Example 3.13 of the text):

$$m(t) = \sum_{k=-\infty}^{\infty} m[k] e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

$$\begin{aligned}
\text{where } m[k] &= \frac{2 \sin(k\omega_0 T_0)}{Tk\omega_0}, \quad \omega_0 T_0 = \frac{2\pi}{T} \cdot T_0 = \frac{2\pi}{4} = \frac{\pi}{2} \\
&= \frac{1}{k\pi} \sin\left(\frac{k\pi}{2}\right)
\end{aligned}$$

That is,

$$\begin{aligned}
m(t) &= \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{k} \sin\left(\frac{k\pi}{2}\right) e^{jk\omega_0 t} \\
&= \frac{1}{2} + \frac{2}{\pi} \cos(\omega_0 t) + \frac{2}{3\pi} \cos 3(\omega_0 t) + \frac{2}{5\pi} \cos 5(\omega_0 t) + \dots
\end{aligned}$$

With $\omega_0 = 2\pi f_0 = 2\pi \times 500$ rad/sec, $m(t)$ consists of a dc component, and sinusoidal components of frequencies 0.5 kHz, 1.5 kHz, 2.5 kHz (i.e., odd harmonics). Hence, the following components of the AM signal $s(t)$ for positive frequencies (with progressively decreasing amplitude) are as follows:

$$\begin{aligned}
f_0 &= 100 \text{ kHz} \\
f_0 + 0.5 &= 100.5 \text{ kHz} \\
f_0 - 0.5 &= 99.5 \text{ kHz} \\
f_0 + 1.5 &= 101.5 \text{ kHz} \\
f_0 - 1.5 &= 98.5 \text{ kHz} \\
f_0 + 2.5 &= 102.5 \text{ kHz} \\
f_0 - 2.5 &= 97.5 \text{ kHz}
\end{aligned}$$

and so on.

- (b) When the square modulating wave $m(t)$ alternates between -1 and +1, the dc component is zero. Nevertheless, the frequency components of the AM signal $m(t)$ remain the same as in part (a); the only difference is in the carrier amplitude.

Note: The frequency components for negative frequencies are the negative of those for positive frequencies.

- 5.20 (a) For positive frequencies, the spectral content of the AM signal consists of the following:

Carrier: $f_c = 100$ kHz

Upper sideband, occupying the band from 100.3 to 103.1 kHz

Lower sideband, occupying the band from 99.7 to 96.9 kHz

- (b) For positive frequencies, the spectral content of the AM signal consists of the following:

Carrier: $f_c = 100$ kHz

Upper sideband, occupying the band from 100.05 to 115 kHz

Lower sideband, occupying the band from 99.95 to 85 kHz

Note: For negative frequencies, the spectral contents of the AM signal $s(t)$ are the negative of those for positive frequencies.

5.21 The percentage modulation is defined by

$$\begin{aligned}\frac{A_{\max} - A_{\min}}{A_{\max} + A_{\min}} &= \frac{9.75 - 0.25}{9.75 + 0.25} \\ &= \frac{9.5}{10} = 95\%\end{aligned}$$

5.22 Building on the solution to Problem 5.17, the frequency components of the DSB-Sc signal for positive frequencies are as follows:

$$\begin{aligned}\text{(a)} \quad f_c + f_0 &= 101 \text{ kHz} \\ f_c - f_0 &= 99 \text{ kHz}\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad f_c + f_0 &= 101 \text{ kHz} \\ f_c - f_0 &= 99 \text{ kHz} \\ f_c + f_1 &= 102 \text{ kHz} \\ f_c - f_1 &= 98 \text{ kHz}\end{aligned}$$

$$\begin{aligned}\text{(c)} \quad f_c + f_0 &= 103 \text{ kHz} \\ f_c - f_0 &= 97 \text{ kHz} \\ f_c + f_1 &= 101 \text{ kHz} \\ f_c - f_1 &= 99 \text{ kHz}\end{aligned}$$

$$\begin{aligned}\text{(d)} \quad f_c + f_0 &= 102 \text{ kHz} \\ f_c - f_0 &= 98 \text{ kHz}\end{aligned}$$

$$\begin{aligned}\text{(e)} \quad f_c + f_0 &= 102 \text{ kHz} \\ f_c - f_0 &= 98 \text{ kHz} \\ f_c + f_1 &= 104 \text{ kHz} \\ f_c - f_1 &= 96 \text{ kHz}\end{aligned}$$

$$\text{(f)} \quad f_c + f_0 = 101 \text{ kHz}$$

$$f_c - f_0 = 99 \text{ kHz}$$

$$f_c + f_1 = 103 \text{ kHz}$$

$$f_c - f_1 = 97 \text{ kHz}$$

Note: For negative frequencies, the frequency components of the DSB-SC signal are the negative of those for positive frequencies.

5.23 Building on the solution for Problem 5.19, we find that the DSB-SC signal has the same frequency content for both forms of the square wave described in parts (a) and (b), as shown by:

- For positive frequencies, we have the frequency components 100.5, 99.5, 101.5, 98.5, 102.5, 97.5 kHz, and so on, with progressively decreasing amplitude.
- For negative frequencies, the frequency components are the negative of those for positive frequencies.

Note: By definition, the carrier is suppressed from the modulated signal.

5.24 (a) For positive frequencies, we have
 Upper sideband, extending from 100.3 to 103.1 kHz
 Lower sideband, extending from 99.7 to 96.9 kHz

(b) For positive frequencies, we have
 Upper sideband, extending from 100.05 to 115 kHz
 Lower sideband, extending from 99.95 to 85 kHz

Note: For negative frequencies, the spectral contents of the DSB-SC signal are the negatives of those for positive frequencies.

5.25 Building on the solution to Problem 5.18, we may describe the frequency components of SSB modulator for positive frequencies as follows:

(i) Upper sideband transmission:

$$(a) f_c + f_0 = 101 \text{ kHz}$$

$$(b) f_c + f_0 = 101 \text{ kHz}$$

$$f_c - f_1 = 102 \text{ kHz}$$

$$\begin{aligned}\text{(c)} \quad f_c + f_0 &= 103 \text{ kHz} \\ f_c - f_1 &= 101 \text{ kHz}\end{aligned}$$

$$\text{(d)} \quad f_c + f_0 = 102 \text{ kHz}$$

$$\begin{aligned}\text{(e)} \quad f_c + f_0 &= 102 \text{ kHz} \\ f_c - f_1 &= 104 \text{ kHz}\end{aligned}$$

$$\begin{aligned}\text{(f)} \quad f_c + f_0 &= 101 \text{ kHz} \\ f_c + f_1 &= 103 \text{ kHz}\end{aligned}$$

(ii) Lower sideband transmission:

$$\text{(a)} \quad f_c - f_0 = 99 \text{ kHz}$$

$$\begin{aligned}\text{(b)} \quad f_c - f_0 &= 99 \text{ kHz} \\ f_c - f_1 &= 98 \text{ kHz}\end{aligned}$$

$$\begin{aligned}\text{(c)} \quad f_c - f_0 &= 97 \text{ kHz} \\ f_c - f_1 &= 99 \text{ kHz}\end{aligned}$$

$$\text{(d)} \quad f_c - f_0 = 98 \text{ kHz}$$

$$\begin{aligned}\text{(e)} \quad f_c - f_0 &= 98 \text{ kHz} \\ f_c - f_1 &= 96 \text{ kHz}\end{aligned}$$

$$\begin{aligned}\text{(f)} \quad f_c - f_0 &= 99 \text{ kHz} \\ f_c - f_1 &= 97 \text{ kHz}\end{aligned}$$

Note: For both (i) and (ii), the frequency contents of SSB modulated signal for negative frequencies are the negative of the frequency contents for positive frequencies.

5.26 With the carrier suppressed, the frequency contents of the SSB signal are the same for both square waves described under (a) and (b), as shown by (for positive frequencies):

(i) Upper sideband transmission: 100.5, 101.5, 102.5 kHz, and so on.

(ii) Lower sideband transmission: 99.5, 98.5, 97.5 kHz, and so on.

Note: For negative frequencies, the frequency components of the SSB signal are the negative of those for positive frequencies.

5.27 (i) Upper sideband transmission:

(a) Upper sideband, occupying the band from 100.3 to 103.1 kHz

(b) Upper sideband, occupying the band from 100.05 to 115 kHz

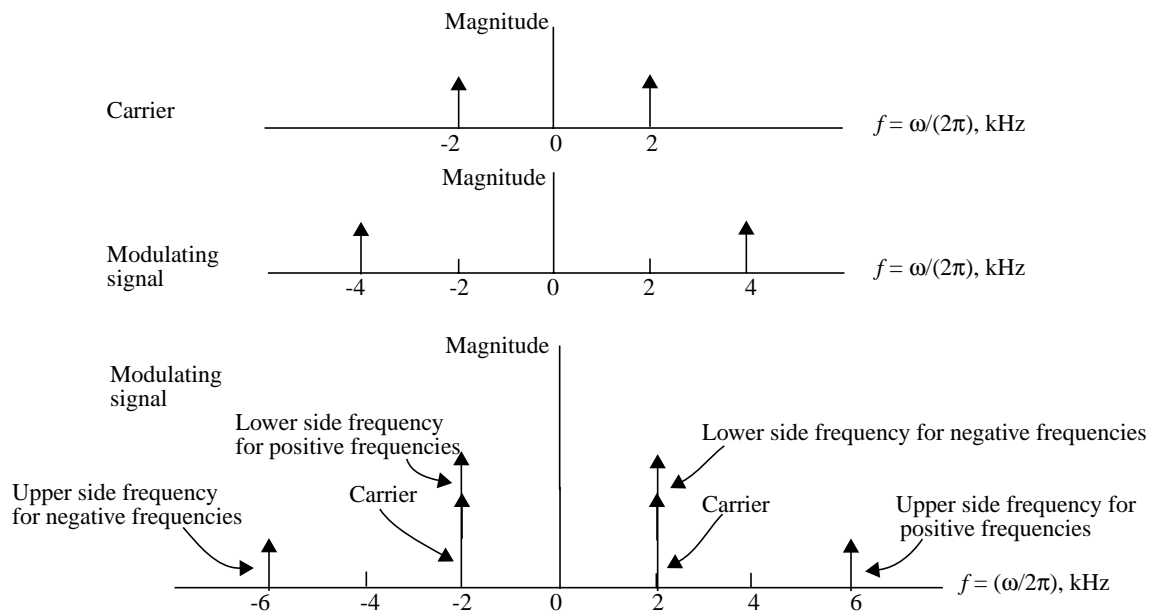
(ii) Lower sideband transmission:

(a) Lower sideband, occupying the band from 99.7 to 96.9 kHz

(b) Lower sideband, occupying the band from 99.5 to 85 kHz

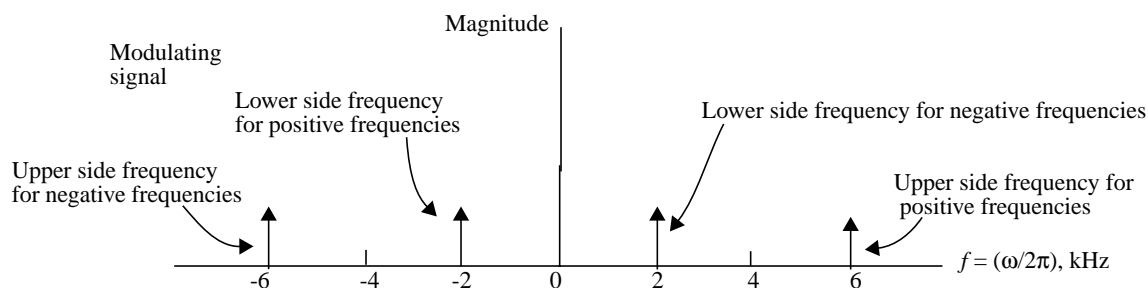
Note: For negative frequencies, the spectrum contents of the SSB signal are the negative of those for positive frequencies.

5.28 The spectra of the pertinent signals are as follows:



We clearly see that this spectrum suffers from frequency overlap, with the lower side frequency for negative frequencies coinciding with the carrier for positive frequencies; similarly, for negative frequencies. This spectrum is therefore radically different from the spectrum of a regular AM signal; hence, it is not possible to recover the modulating signal using an envelope detector.

5.29 (a)



The modulated signal therefore consists of two sinusoidal components, one at frequency 2 kHz and the other at frequency 6 kHz.

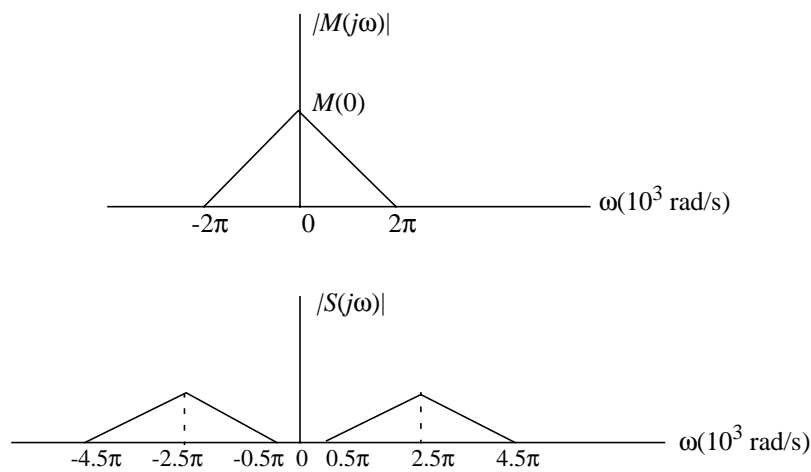
- (b) A conventional coherent detector consists of a product modulator followed by a low-pass filter. The product modulator is supplied with a carrier of frequency 2 kHz. The spectrum of this modulator consists of the following components:

dc component (at zero frequency)
 sinusoidal component at 4 kHz
 another sinusoidal component at 8 kHz

To extract the sinusoidal modulating signal of 4 kHz, the low-pass filter has a cutoff frequency slightly in excess of 4 kHz. The resulting output therefore consists of the desired sinusoidal modulating signal, plus a dc component that is undesired.

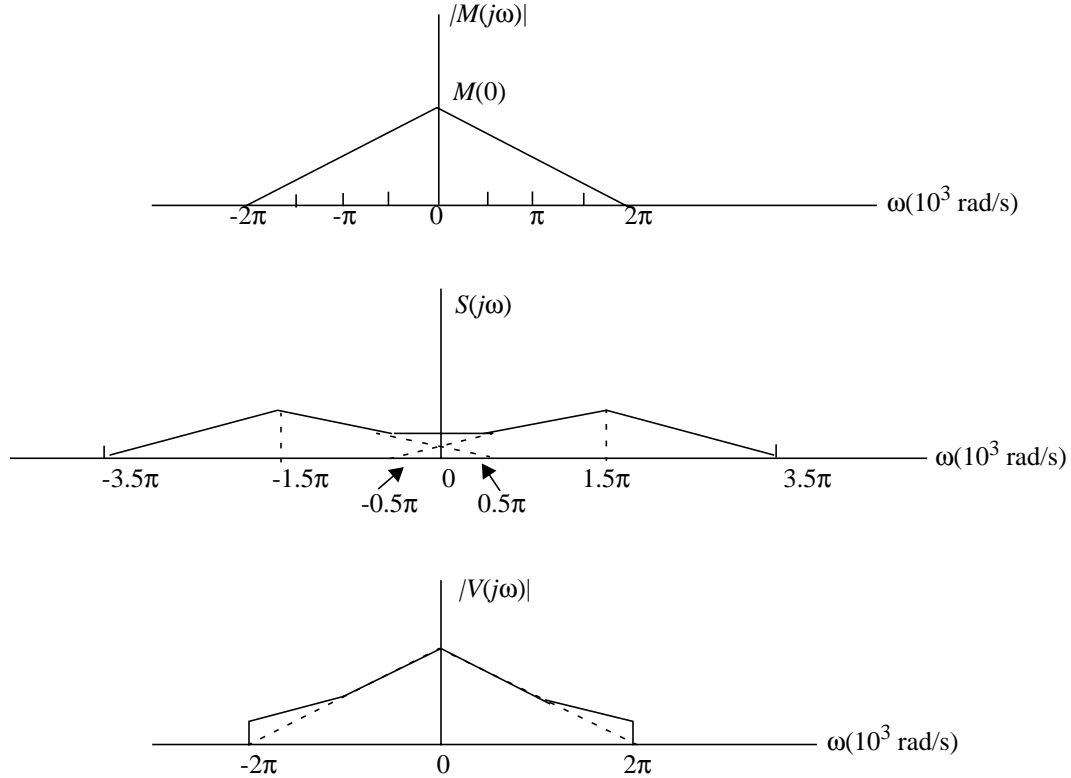
To suppress the dc component, we have to modify the coherent detector by passing the detector output through a capacitor.

- 5.30 For a message bandwidth $\omega_m = 2.5\pi \times 10^3$ rad/s and carrier frequency $\omega_c = 2\pi \times 10^3$ rad/s, the spectra of the message signal and double sideband-suppressed carrier (DSB-SC) modulated signal may be depicted as follows:



The important point to note from the picture depicted here is that there is a clear separation between the sidebands lying in the negative frequency region and those in the positive frequency region. Consequently, when the DSB-SC modulated signal is applied to a coherent detector, the resulting demodulated signal is a replica of the original message signal except for a scale change.

When, however, the carrier frequency is reduced to $\omega_c = 1.5\pi \times 10^3$ rad/s, the lower sideband for negative frequencies overlaps the lower sideband for positive frequencies, and the situation changes dramatically as depicted below:



The spectrum labeled $V(j\omega)$ refers to the demodulated signal appearing at the output of the coherent detector. Comparing this spectrum with the original message spectrum $M(j\omega)$, we now see that the modulation-demodulation process results in a message distortion.

The conclusions to be drawn from the results presented above are:

1. To avoid message distortion on demodulation due to sideband overlap, we must choose the carrier frequency in accordance with the condition $\omega_c \geq \omega_m$. The minimum acceptable value of ω_c is therefore ω_m .
2. For $\omega_c < \omega_m$, we have sideband overlap and therefore message distortion.

5.31 The two AM modulator outputs are

$$s_1(t) = A_c(1 + k_a m(t))\cos(\omega_c t)$$

$$s_2(t) = A_c(1 - k_a m(t)) \cos(\omega_c t)$$

Here it is assumed that the two modulators are identical, that is, they have the same amplitude modulation sensitivity k_a and the same carrier $A_c \cos(\omega_c t)$. Subtracting $s_2(t)$ from $s_1(t)$:

$$\begin{aligned} s(t) &= s_1(t) - s_2(t) \\ &= A_c(1 + k_a m(t)) \cos(\omega_c t) - A_c(1 - k_a m(t)) \cos(\omega_c t) \\ &= 2k_a A_c m(t) \cos(\omega_c t) \end{aligned}$$

which is the formula for a DSB-SC modulated signal. For this method of generating a DSB-SC modulated signal to work satisfactorily, the two AM modulators must be carefully matched.

5.32 From Eq. (5.30) of the text, we have

$$S(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} M(j(\omega - k\omega_s)) H(j\omega) \quad (1)$$

where

$$H(j\omega) = T_0 \text{sinc}(\omega T_0 / (2\pi)) e^{-j\omega T_0 / 2}$$

We are given

$$T_0 = 10 \mu s$$

$$\omega_0 = \frac{2\pi}{10} = 2\pi \times 10^5 \text{ rad/s}$$

$$M(j\omega) = \frac{A_m}{2} [\delta(j\omega - j\omega_m) + \delta(j\omega + j\omega_m)]$$

$$A_m = \text{amplitude}$$

$$\omega_m = 2\pi \times 10^3 \text{ rad/s}$$

$$\omega_s = \frac{2\pi}{T_s}$$

With $\omega_0 = 100\omega_m$, it follows that the effect of flat-top sampling is small enough to approximate Eq. (1) as follows:

$$S(j\omega) \approx \frac{1}{T_s} \sum_{k=-\infty}^{\infty} M(j(\omega - k\omega_s))$$

With $\omega_s > 2\omega_m$, the side frequencies of the modulated signal are as follows (for positive frequencies)

$$\begin{aligned} k=0: & \quad \omega_m \\ k=1: & \quad \omega_s - \omega_m, \omega_s + \omega_m \\ k=2: & \quad 2\omega_s - \omega_m, 2\omega_s + \omega_m \\ k=3: & \quad 3\omega_s - \omega_m, 3\omega_s + \omega_m \end{aligned}$$

and so on.

For negative frequencies, the side frequencies are the negative of those for positive frequencies.

5.33 (a) The radio frequency (RF) pulse is defined by

$$s(t) = \begin{cases} A_c \cos(\omega_c t), & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

The modulated signal $s(t)$ is obtained by multiplying the carrier $A_c \cos(\omega_c t)$ by a rectangular pulse of unit amplitude and duration T (centered about the origin). The Fourier transform of $A_c \cos(\omega_c t)$ is

$$\frac{A_c}{2} \delta(\omega - \omega_c) + \frac{A_c}{2} \delta(\omega + \omega_c).$$

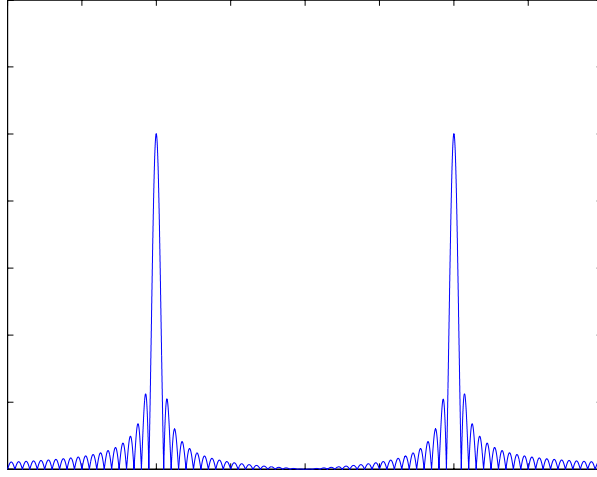
The Fourier transform of the rectangular pulse is equal to the sinc function $T \text{sinc}(\omega T)$. Since multiplication in the time domain is transformed into convolution in the frequency domain, we may express the Fourier transform of $s(t)$ as

$$S(j\omega) = \frac{A_c T}{2} [\text{sinc}(T(\omega - \omega_c)) + \text{sinc}(T(\omega + \omega_c))] \quad (1)$$

When $\omega_c T$ is much greater than 2π , the overlap between the two sinc functions $\text{sinc}(T(\omega - \omega_c))$ and $\text{sinc}(T(\omega + \omega_c))$ is correspondingly small. We may then approximate Eq. (1) as follows:

$$S(j\omega) \approx \begin{cases} \frac{A_c T}{2} \text{sinc}(T(\omega - \omega_c)), & \omega > 0 \\ 0, & \omega = 0 \\ \frac{A_c T}{2} \text{sinc}(T(\omega + \omega_c)), & \omega < 0 \end{cases} \quad (2)$$

(b) For $\omega_c T = 20\pi$, use of the approximate formula of Eq. (2) is justified. The width of the main lobe of the sinc function is $4\pi/T = \omega_c/5$. The width of each sidelobe is $\omega_c/10$. We may thus plot the magnitude spectrum of $S(j\omega)$ as shown on the next page.



- 5.34 (a) The spectrum of each transmitted radar pulse is closely defined by Eq. (2) given in the solution to Problem 5.33. The periodic transmission of each such pulse has the equivalent effect of sampling this spectrum at a rate equal to the pulse repetition frequency of the radar. Accordingly, we may express the spectrum of the transmitted radar signal $s(t)$ as follows:

$$S(j\omega) \approx \begin{cases} \sum_n a_n \delta(\omega - \omega_c - n\omega_0) & \text{for } \omega > 0 \\ 0 & \text{for } \omega = 0 \\ \sum_n a_n \delta(\omega + \omega_c + n\omega_0) & \text{for } \omega < 0 \end{cases} \quad (1)$$

where

ω_0 = fundamental frequency

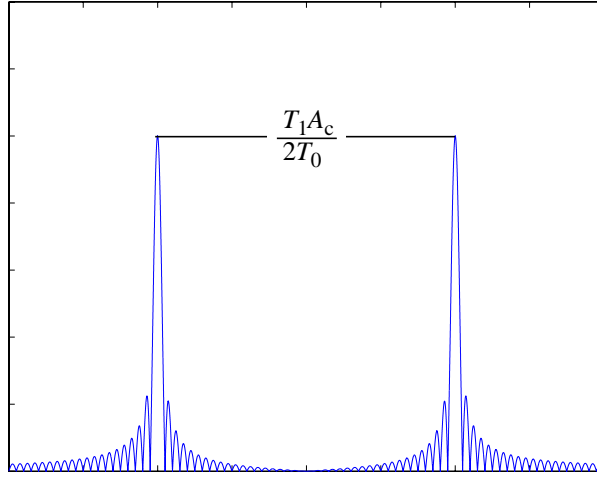
$$= \frac{2\pi}{T_0}$$

and the coefficient a_n is defined by

$$a_n = \frac{T_1 A_c}{2T_0} \text{sinc}\left(\frac{nT_1}{T_0}\right)$$

where T_1 is the pulse duration.

- (b) The spectrum $S(j\omega)$ defined in Eq. (1) is discrete in nature, consisting of a set of impulse functions located at $\omega = \pm(\omega_c + n\omega_0)$, where $n = 0, \pm 1, \pm 2, \dots$. The envelope of the magnitude spectrum $|S(j\omega)|$ is therefore as shown on the next page.



The mainlobe of the spectrum has a width of

$$\frac{4\pi}{T_1} = 4\pi \times 10^6 \text{ rad/s}$$

and the sidelobes have a width of $2\pi \times 10^6 \text{ rad/s}$.

The impulse functions are separated by

$$\omega_0 = 2\pi 10^3 \text{ rad/s}$$

and the carrier frequency $\omega_c = 2\pi \times 10^9 \text{ rad/s}$. Thus, there are 2000 impulse functions enveloped by the mainlobe and 1000 impulse functions enveloped by each sidelobe.

5.35 The DSB-SC modulated signal is defined by

$$s(t) = A_c m(t) \cos(\omega_c t)$$

Let the local oscillator output in the coherent detector be denoted by $\cos(\omega_c t + \Delta\omega t)$, where $\Delta\omega$ is the frequency error. Multiplying $s(t)$ by the local carrier yields

$$\begin{aligned} s_1(t) &= s(t) \cos(\omega_c t + \Delta\omega t) \\ &= A_c m(t) \cos(\omega_c t) \cos(\omega_c t + \Delta\omega t) \\ &= \frac{A_c}{2} m(t) [\cos(\Delta\omega t) + \cos(2\omega_c t + \Delta\omega t)] \end{aligned}$$

Low-pass filtering $s_1(t)$ results in the output

$$s_2(t) = \frac{A_c}{2} m(t) \cos(\Delta\omega t)$$

In words, the effect of frequency error $\Delta\omega$ in the local oscillator is to produce a new DSB-SC modulated signal with an effective carrier frequency of $\Delta\omega$. It is only when $\Delta\omega = 0$ that the coherent detector works properly.

5.36 The mixer produces an output signal of frequency equal to the sum or the difference between its input signal frequencies. The range of sum- or difference-frequencies is from 100 kHz (representing the difference between input frequencies 1 MHz and 900 kHz) to 9.9 MHz (representing the sum of input frequencies 9 MHz and 900 kHz). The frequency resolution is 100 kHz.

5.37 The basic similarity between full AM and PAM is that in both cases the envelope of the modulated signal faithfully follows the original message (modulating) signal.

They differ from each other in the following respects:

1. In AM, the carrier is a sinusoidal signal; whereas in PAM, the carrier is a periodic sequence of rectangular pulses.
2. The spectrum of an AM signal consists of a carrier plus an upper sideband and a lower sideband. The spectrum of a PAM signal consists of a carrier plus an upper sideband and a lower sideband, which repeat periodically at a rate equal to the sampling rate.

5.38 (a) $g(t) = \text{sinc}(200t)$

This sinc pulse corresponds to a bandwidth of 100 Hz. Hence the Nyquist rate is 200 Hz, and the Nyquist interval is $(1/200)$ seconds.

(b) $g(t) = \text{sinc}^2(200t)$

This signal may be viewed as the product of the sinc signal $\text{sinc}(200t)$ with itself. Since multiplication in the time domain corresponds to convolution in the frequency domain, we find that the signal $g(t)$ has a bandwidth twice that of the sinc pulse $\text{sinc}(200t)$, that is, 200 Hz and the Nyquist interval is $1/400$ seconds.

(c) $g(t) = \text{sinc}(200t) + \text{sinc}^2(200t)$

The bandwidth of $g(t)$ is determined by its highest frequency component. With $\text{sinc}(200t)$ having a bandwidth of 100 Hz and $\text{sinc}^2(200t)$ having a bandwidth of 200 Hz, it follows that the bandwidth of $g(t)$ is 200 Hz. Correspondingly, the Nyquist rate of $g(t)$ is 400 Hz and its Nyquist interval is $1/400$ seconds.

5.39 (a) With a sampling rate of 8 kHz, the sampling interval is

$$\begin{aligned} T_s &= \frac{1}{8 \times 10^3} \\ &= 125 \mu s \end{aligned}$$

There are 24 voice channels and 1 synchronizing pulse, so the time allotted to each channel is

$$T_{\text{channel}} = \frac{\tau}{25} = 5 \mu s$$

- (b) If each voice signal is sampled at the Nyquist rate, the sampling rate would be twice the highest frequency component 3.4 kHz, that is, 6.8 kHz. The sampling interval is then

$$T_s = \frac{1}{6.8 \times 10^3}$$

$$= 147 \mu s$$

Hence,

$$T_{\text{channel}} = \frac{147}{25} = 6.68 \mu s$$

- 5.40 (a) The bandwidth required for each single sideband modulated channel is 10 kHz. The total bandwidth for 12 such channels is $12 \times 10 = 120$ kHz.

- (b) The Nyquist rate for each channel is $2 \times 10 = 20$ kHz. For 12 TDM signals, the total data rate is $12 \times 20 = 240$ kHz.

By using a sinc pulse whose amplitude varies in accordance with the modulating signal, and with zero crossings at multiples of $(1/240)$ ms, we would need a minimum bandwidth of 120 kHz.

- 5.41 (a) The Nyquist rate for $s_1(t)$ and $s_2(t)$ is 160 Hz. Therefore, $\frac{2400}{2^R}$ must be greater than 160 Hz. Hence, the maximum value of R is 3.

- (b) With $R = 3$, we may use the signal format shown in Fig. 1 to multiplex the signals $s_1(t)$ and $s_2(t)$ into a new signal, and then multiplex $s_3(t)$, $s_4(t)$ and $s_5(t)$ markers for synchronization:

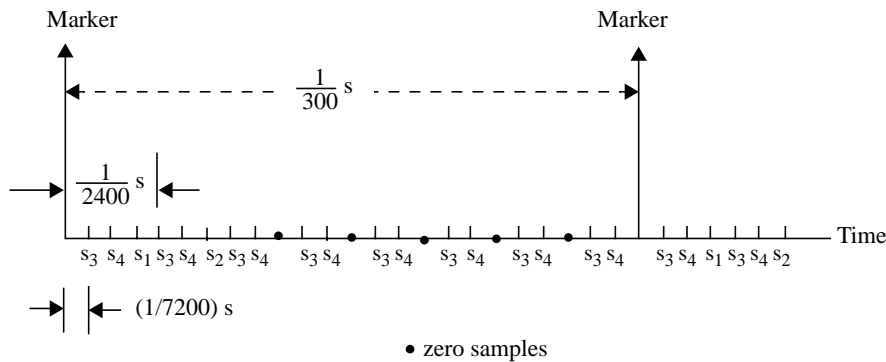
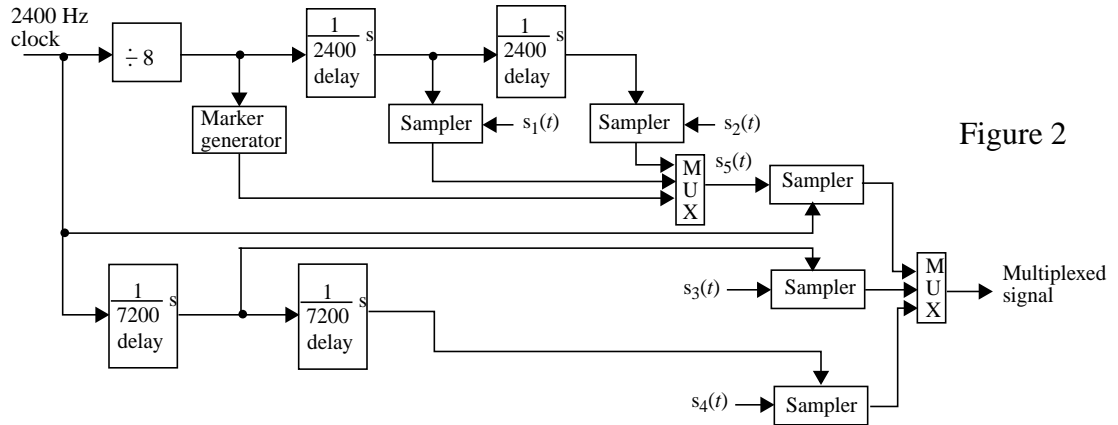


Figure 1

Based on this signal format, we may develop the multiplexing system shown in Fig. 2.



5.42 The Fourier transform provides a tool for displaying the spectral content of a continuous-time signal. A continuous-wave (CW) modulated signal $s(t)$ involves the multiplication of a message signal $m(t)$ by a sinusoidal carrier $c(t) = A_c \cos(\omega_c t)$ in one form or another. The CW modulated signal $s(t)$ may be viewed as a mixture signal that involves, in one form or another, the multiplication of $m(t)$ by $c(t)$. Multiplication in the time-domain is transformed into the convolution of two spectra, namely, the Fourier transform $M(j\omega)$ and the Fourier transform $C(j\omega)$. The Fourier transform $C(j\omega)$ consists of a pair of impulses at $\pm\omega_c$. Hence, the Fourier transform of the modulated signal, denoted by $S(j\omega)$, contains a component $M(j\omega - j\omega_c)$ for positive frequencies. For negative frequencies, we have the image of this spectrum with respect to the vertical axis. The picture so portrayed teaches us that the carrier frequency ω_c must be greater than the highest frequency component of the message spectrum $M(j\omega)$. Moreover, given that this condition is satisfied, Fourier analysis teaches us that recovery of the original message signal $m(t)$ from the modulated signal $s(t)$ is indeed a practical reality. For example, we may use a product modulator consisting of a product modulator followed by a low-pass filter. The product modulator, supplied by a local carrier of frequency ω_c , produces a replica of the original message signal $m(t)$ plus a new modulated signal with carrier frequency $2\omega_c$. By designing the low-pass filter to have a cutoff frequency slightly higher than the highest frequency of $m(t)$, recovery of $m(t)$ except for a scaling factor is realized.

Consider next the case of pulse-amplitude modulation (PAM). The simplest form of PAM consists of multiplying the message signal $m(t)$ by a periodic train of uniformly spaced impulse functions, with adjacent impulses being spaced T_s seconds apart. What we have just described is the instantaneous form of sampling. Thus, the sampling process represents another form of a mixture signal. The Fourier transform of the periodic pulse train just described consists of a new uniformly spaced periodic train of impulses in the frequency domain, with each pair of impulses spaced apart by $1/T_s$ hertz. As already mentioned, multiplication of two time functions is transformed into the convolution of

their spectra in the frequency domain. Accordingly, Fourier analysis teaches us that there will be no spectral overlap provided that the sampling frequency $1/T_s$ is not less than twice the highest frequency component of the original message signal $m(t)$. Hence, provided that this condition is satisfied, the recovery of a replica of $m(t)$ is indeed possible at the receiver. This recovery may, for example, take the form of a low-pass interpolation filter with a cutoff frequency just slightly greater than the highest frequency component of $m(t)$.

Advanced Problems

5.43 The nonlinear device is defined by

$$i_0(t) = a_1 v_i(t) + a_2 v_i^2(t) \quad (1)$$

The input $v_i(t)$ is given by

$$v_i(t) = A_c \cos(\omega_c t) + A_m \cos(\omega_m t) \quad (2)$$

where $A_c \cos(\omega_c t)$ is the carrier wave and $A_m \cos(\omega_m t)$ is the modulating wave. Substituting Eq. (2) into (1):

$$\begin{aligned} i_0(t) &= a_1 (A_c \cos(\omega_c t) + A_m \cos(\omega_m t)) \\ &\quad + a_2 (A_c \cos(\omega_c t) + A_m \cos(\omega_m t))^2 \\ &= a_1 A_c \cos(\omega_c t) + a_1 A_m \cos(\omega_m t) \\ &\quad + a_2 A_c^2 \cos^2(\omega_c t) + 2a_2 A_c A_m \cos(\omega_c t) \cos(\omega_m t) \\ &\quad + a_2 A_m^2 \cos^2(\omega_m t) \end{aligned} \quad (3)$$

Using the trigonometric identity

$$\cos^2 \theta = \frac{1}{2} (\cos(2\theta) + 1)$$

we may rewrite Eq. (3) in the equivalent form (after a rearrangement of terms):

$$\begin{aligned} i_0(t) &= a_1 A_c \cos(\omega_c t) + 2a_2 A_c A_m \cos(\omega_c t) \cos(\omega_m t) \\ &\quad + a_1 A_m \cos(\omega_m t) + \frac{1}{2} a_2 A_m^2 \cos(2\omega_m t) \\ &\quad + \frac{1}{2} a_2 A_c^2 \cos(2\omega_c t) + \frac{1}{2} a_2 A_c^2 + \frac{1}{2} a_2 A_m^2 \end{aligned} \quad (4)$$

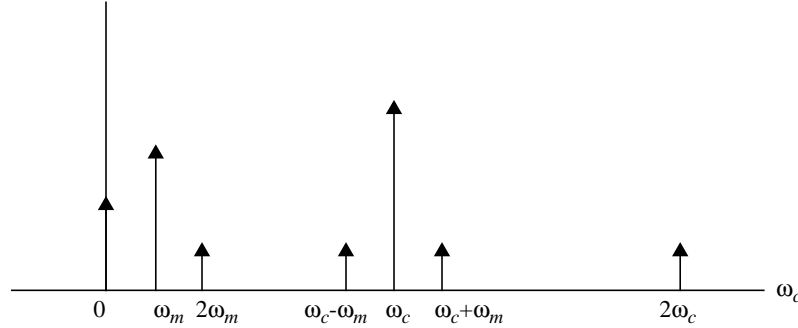
We may now recognize the following components in the output $i_0(t)$:

1. An amplitude modulated component (AM) represented by

$$s(t) = a_1 A_c \left(1 + \frac{2a_2 A_m}{a_1} \cos(\omega_m t) \right) \cos(\omega_c t)$$

2. A set of undesirable components represented by the remaining components of Eq. (4).

- (a) In amplitude modulation, the carrier frequency ω_c is typically much larger than the modulation frequency ω_m . Hence, the frequency context of $i_0(t)$ may be depicted as follows (showing only the positive frequency components so as to simplify the presentation):



- (b) From this figure, we see that the desired AM component occupies a frequency band extending from the lower side-frequency $\omega_c - \omega_m$ to the upper side frequency $\omega_c + \omega_m$. To extract this component, we need to pass the output of the nonlinear device, $i_0(t)$, through a band-pass filter. The frequency specifications of this filter are as follows:

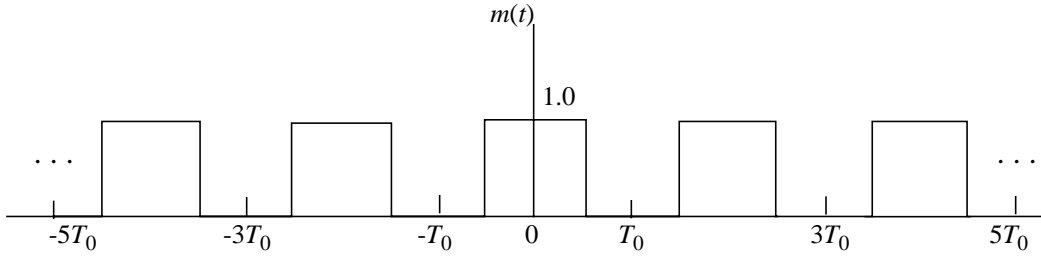
- Passband of a width slightly larger than $2\omega_m$, centered on the carrier frequency ω_c .
- Lower stopband, lying below the lower side-frequency $\omega_c - \omega_m$ and thereby suppressing the dc component as well as the frequency components ω_m and $2\omega_m$.
- Upper stopband, lying above the upper side-frequency $\omega_c + \omega_m$ and thereby suppressing the frequency component $2\omega_c$.

- 5.44 (a) For the special case of an infinite unipolar sequence (binary sequence consisting of a square wave of equal mark-to-space ratio), we have

$$\begin{aligned} m(t) &= \frac{1}{2} + \frac{2}{\pi} \cos(\omega_0 t) + \frac{2}{3\pi} \cos(3\omega_0 t) + \frac{2}{5\pi} \cos(5\omega_0 t) + \dots \\ &= \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \cos((2k+1)\omega_0 t) \end{aligned}$$

where $\omega_0 = 2\pi f_0 = 2\pi(1/2T_0) = \pi/T_0$. (See the solution to Problem 5.19). The resulting 00K signal is given by (assuming a sinusoidal carrier of unit amplitude and frequency ω_c)

$$\begin{aligned} s(t) &= m(t) \cos(\omega_c t) \\ &= \frac{1}{2} \cos(\omega_c t) + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \cos((2k+1)\omega_0 t) \cos(\omega_c t) \end{aligned} \quad (1)$$



Using the formula

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

we may rewrite Eq. (1) in the equivalent form

$$s(t) = \frac{1}{2}\cos(\omega_c t) + \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{1}{2k+1}[\cos(((2k+1)\omega_0 + \omega_c)t) + \cos((2k+1)\omega_0 - \omega_c)t)] \quad (2)$$

The spectrum of $s(t)$ defined in Eq. (2) is depicted in Fig. 1 (for positive frequencies):

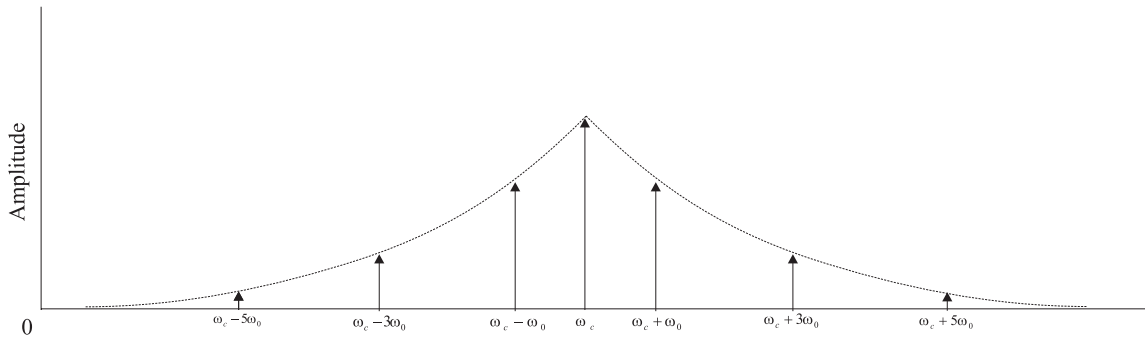
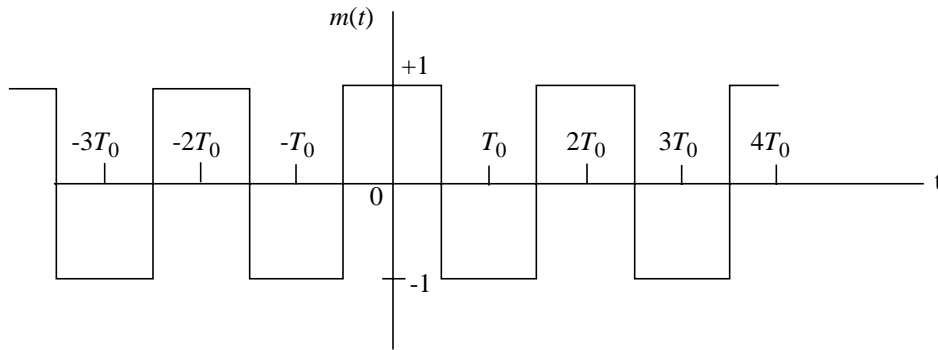


Figure 1

- (b) For the BPSK signal, the binary sequence is represented in its polar nonreturn-to-zero sequence with the following waveform:



Correspondingly, $m(t)$ has the Fourier series representation:

$$m(t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k+1} \cos((2k+1)\omega_0 t) \quad (3)$$

where $\omega_0 = \pi/T_0$. The binary sequence of Eq. (3) differs from that of Eq. (1) in two respects:

- It has no dc component
- It is scaled by a factor of 2.

The resulting BPSK signal is defined by (assuming a carrier of unit amplitude and frequency ω_c)

$$\begin{aligned} s(t) &= m(t) \cos(\omega_c t) \\ &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k+1} \cos((2k+1)\omega_0 t) \cos(\omega_c t) \\ &= \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k+1} [\cos(((2k+1)\omega_0 + \omega_c)t) + \cos(((2k+1)\omega_0 - \omega_c)t)] \end{aligned} \quad (4)$$

The spectrum of $s(t)$ defined in Eq. (4) is depicted in Fig. 2:

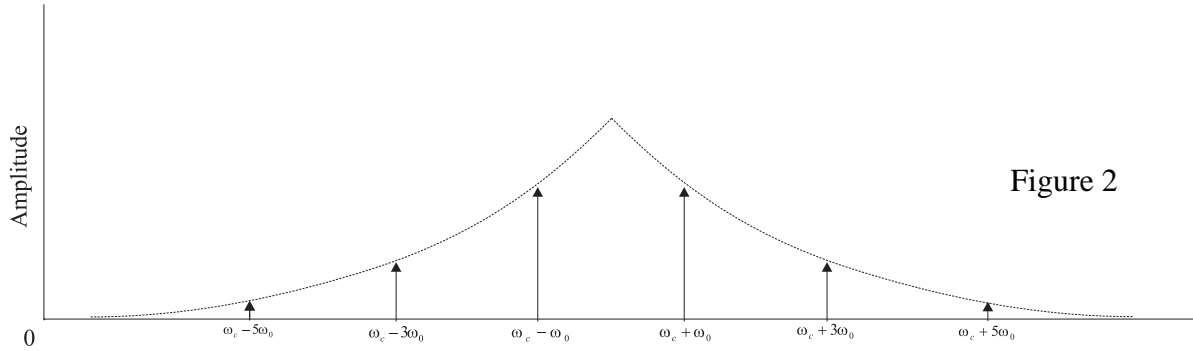


Figure 2

(c) The energy of the unipolar signal defined in Eq. (2) is given by

$$\frac{1}{T} \int_0^T |s(t)|^2 dt = \sum_{k=-\infty}^{\infty} |S(k)|^2 \quad (5)$$

The left-hand side of Eq. (5) yields

$$\begin{aligned} E &= \frac{1}{T} \int_0^T |s(t)|^2 dt = \frac{1}{2T_0} \left[\int_{-T_0}^{-T_0/2} |s(t)|^2 dt + \int_{-T_0/2}^0 |s(t)|^2 dt + \int_0^{T_0/2} |s(t)|^2 dt + \int_{T_0/2}^{T_0} |s(t)|^2 dt \right] \\ &= \frac{1}{2T_0} \left[\int_{-T_0/2}^{T_0/2} (1)^2 dt \right] \\ &= \frac{1}{2} \end{aligned} \quad (6)$$

The right-hand side of Eq. (5) yields, in light of Eq. (2), the energy

$$\begin{aligned}
E &= \left(\frac{1}{2}\right)^2 + \frac{(2/\pi)^2}{(\sqrt{2})^2} \left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right) \\
&= \frac{1}{4} + \frac{2}{\pi^2} \left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right) \\
&= \frac{1}{4} + \frac{2}{\pi^2} \cdot \frac{\pi^2}{8} \\
&= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\end{aligned}$$

which checks the result of Eq. (2).

For the polar nonreturn-to-zero signal of Eq. (3), the energy is

$$\begin{aligned}
E &= \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 dt \\
&= \frac{1}{2T_0} \left[\int_{-T_0}^{T_0/2} |s(t)|^2 dt + \int_{-T_0/2}^{T_0/2} |s(t)|^2 dt + \int_{-T_0/2}^{T_0} |s(t)|^2 dt \right] \\
&= \frac{1}{2T_0} \left[\int_{-T_0}^{T_0/2} (-1)^2 dt + \int_{-T_0/2}^{T_0/2} (1)^2 dt + \int_{-T_0/2}^{T_0} (-1)^2 dt \right] \\
&= \frac{1}{2T_0} \left[\frac{T_0}{2} + T_0 + \frac{T_0}{2} \right] = 1
\end{aligned}$$

Application of Parseval's theorem yields

$$\begin{aligned}
E &= \frac{(4/\pi)^2}{(\sqrt{2})^2} \left[(1)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{7}\right)^2 + \dots \right] \\
&= \frac{8}{\pi^2} \left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots\right) \\
&= \frac{8}{\pi^2} \cdot \frac{\pi^2}{8} = 1
\end{aligned}$$

- (d) To raise the energy of the unipolar signal to equal that of the polar nonreturn-to-zero signal, the amplitude of the former signal has to be increased by the scale factor $\sqrt{2}$.
- (e) Examining the OOK signal of Eq. (2), we see that it contains a carrier component in addition to side frequencies, hence the similarity to an AM signal. On the other hand, examining Eq. (4) for the BPSK signal, we see that it lacks a carrier component, hence the similarity to a DSB-SC signal.

5.45 The multiplexed signal is

$$s(t) = A_c m_1(t) \cos(\omega_c t) + A_c m_2(t) \sin(\omega_c t)$$

where $m_1(t)$ and $m_2(t)$ are the incoming message signals. Taking the Fourier transform of $s(t)$:

$$\begin{aligned} S(j\omega) &= \frac{A_c}{2} [M_1(j\omega - j\omega_c) + M_1(j\omega + j\omega_c)] \\ &\quad + \frac{A_c}{2j} [M_2(j\omega - j\omega_c) + M_2(j\omega + j\omega_c)] \end{aligned}$$

where $m_1(t) \xleftrightarrow{\text{FT}} M_1(j\omega)$ and $m_2(t) \xleftrightarrow{\text{FT}} M_2(j\omega)$. With $H(j\omega)$ denoting the transfer function of the channel, the Fourier transform of the received signal is

$$\begin{aligned} R(j\omega) &= H(j\omega) S(j\omega) \\ &= \frac{A_c}{2} H(j\omega) [M_1(j\omega - j\omega_c) + M_1(j\omega + j\omega_c)] \\ &\quad + \frac{A_c}{2j} H(j\omega) [M_2(j\omega - j\omega_c) + M_2(j\omega + j\omega_c)] \end{aligned}$$

where $r(t) \xleftrightarrow{\text{FT}} R(j\omega)$. To recover $m_1(t)$, we multiply the received signal $r(t)$ by $\cos(\omega_c t)$ and then pass the resulting output through a low-pass filter with cutoff frequency equal to the message bandwidth. The result of this processing is a signal with the spectrum

$$\begin{aligned} S_1(j\omega) &= \frac{1}{2} (R(j\omega - j\omega_c) + R(j\omega + j\omega_c)) \\ &= \frac{A_c}{4} H(j\omega - j\omega_c) [M_1(j\omega - j2\omega_c) + M_1(j\omega)] \\ &\quad + \frac{1}{j} M_2(j\omega - j2\omega_c) - \frac{1}{j} M_2(j\omega)] \\ &\quad + \frac{A_c}{4} H(j\omega + j\omega_c) [M_1(j\omega) + M_1(j\omega + j2\omega_c) \\ &\quad + \frac{1}{j} M_2(j\omega) - \frac{1}{j} M_2(j\omega + j2\omega_c)] \end{aligned} \tag{1}$$

For a real channel, the condition

$$H(j\omega_c + j\omega) = H^*(j\omega_c - j\omega)$$

is equivalent to

$$H(j\omega + j\omega_c) = H(j\omega - j\omega_c)$$

This equivalence follows from the fact that for a channel with real-value impulse response $h(t)$, we have $H(-j\omega) = H^*(j\omega)$. Hence, substituting this condition into Eq. (1):

$$\begin{aligned} S_1(j\omega) &= \frac{A_c}{2} H(j\omega - j\omega_c) M_1(j\omega) \\ &\quad + \frac{A_c}{4} H(j\omega - j\omega_c) [M_1(j\omega - j2\omega_c) \end{aligned}$$

$$+ M_1(j\omega + j2\omega_c) + \frac{1}{j}M_2(j\omega - j2\omega_c) - \frac{1}{j}M_2(j\omega + j2\omega_c) \Big]$$

Passing the signal defined by $S_1(j\omega)$ through a low-pass filter of cutoff frequency equal to the message bandwidth, ω_m , we get an output whose Fourier transform is equal to

$$\frac{A_c}{2}H(j\omega - j\omega_c)M_1(j\omega) \quad \text{for } -\omega_m \leq \omega \leq \omega_m$$

Recognizing the band-pass nature of $H(j\omega)$, we immediately see that the output so obtained is indeed a replica of the original message signal $m_1(t)$.

Similarly, to recover $m_2(t)$, we multiply $r(t)$ by $\sin(\omega_c t)$ and then pass the resulting signal through a low-pass filter. In this case, we get an output with a spectrum equal to

$$\frac{A_c}{2}H(j\omega - j\omega_c)M_2(j\omega)$$

5.46 The block diagram of the scrambler is as follows:

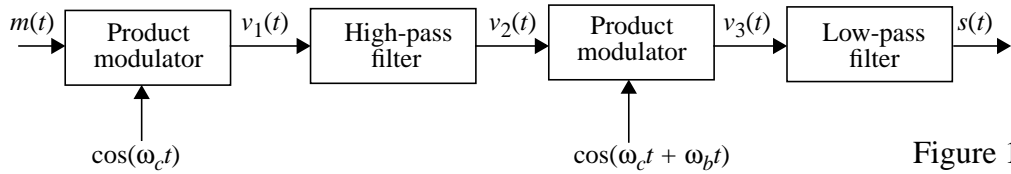


Figure 1

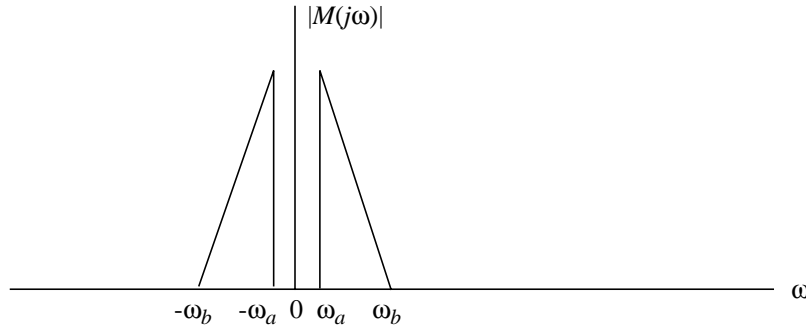
(a) The first product modulator output is

$$v_1(t) = m(t)\cos(\omega_c t)$$

When $v_1(t)$ is processed by the high-pass filter, we get $v_2(t)$. The second product modulator output is

$$v_3(t) = v_2(t)\cos(\omega_c t + \omega_b t)$$

The magnitude spectra of $m(t)$, $v_1(t)$, $v_2(t)$, $v_3(t)$ and $s(t)$ are illustrated in Fig. 2. From this figure, we observe that $v_1(t)$ is a DSB-SC modulated signal, $v_2(t)$ is a SSB-modulated signal, $v_3(t)$ is a DSB-SC modulated signal whose lowest frequency is zero, and $s(t)$ is a low-pass signal whose spectrum is uniquely defined by $m(t)$.



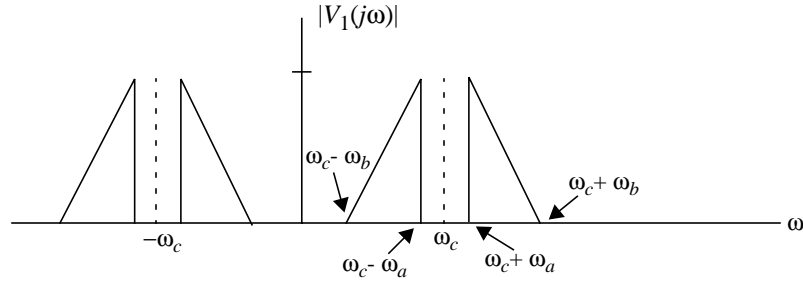
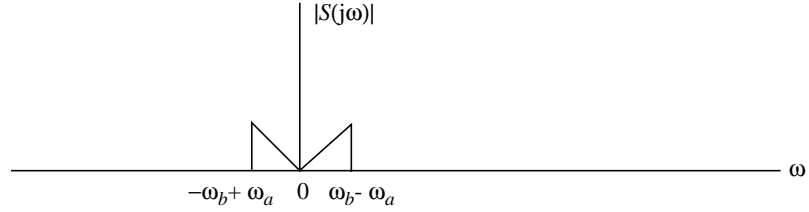
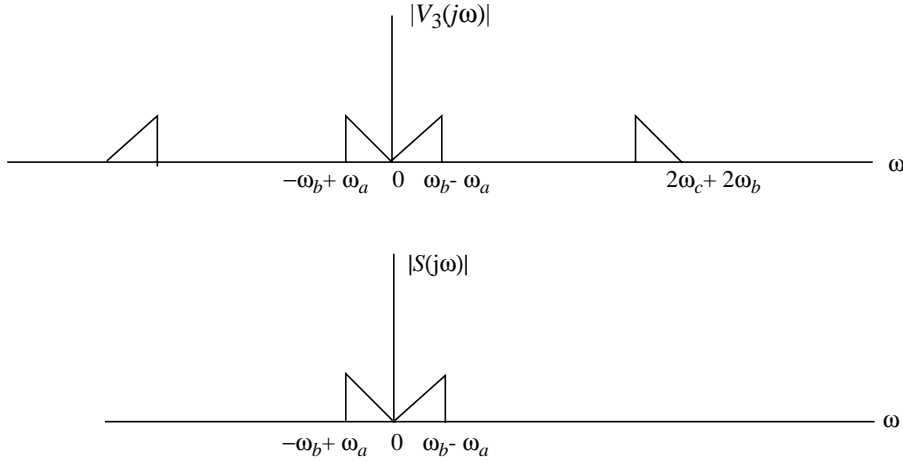
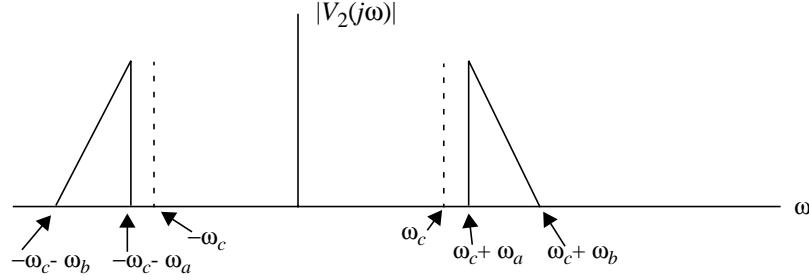


Figure 2



To find an expression for the scrambled voice signal $s(t)$, we need to invoke the Hilbert transform. Specifically, the Hilbert transform of $m(t)$ is defined by

$$\hat{m}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{m(\tau)}{t - \tau} d\tau$$

The $\hat{m}(t)$ may be viewed as the convolution of $m(t)$ with $1/(\pi t)$. The Fourier transform of $1/(\pi t)$ is equal to $-j \text{sgn}(\omega)$, where

$$\text{sgn}(\omega) = \begin{cases} 1 & \text{for } \omega > 0 \\ 0 & \text{for } \omega = 0 \\ -1 & \text{for } \omega < 0 \end{cases}$$

Hence,

$$\hat{M}(j\omega) = -j \text{sgn}(\omega) M(j\omega)$$

Using this relation, it is a straightforward matter to show that the spectrum of $s(t)$ defined previously is the same as that of the following expression:

$$s(t) = \frac{1}{4}m(t)\cos(\omega_b t) + \frac{1}{4}\hat{m}(t)\sin(\omega_b t)$$

(b) With $s(t)$ as the input to the scrambler, the output of the first product modulator is

$$\begin{aligned} v_1(t) &= s(t)\cos(\omega_c t) \\ &= \frac{1}{4}m(t)\cos(\omega_b t)\cos(\omega_c t) + \frac{1}{4}\hat{m}(t)\sin(\omega_b t)\cos(\omega_c t) \\ &= \frac{1}{8}m(t)[\cos(\omega_c t + \omega_b t) + \cos(\omega_c t - \omega_b t)] \\ &\quad + \frac{1}{8}\hat{m}(t)[\sin(\omega_c t + \omega_b t) - \sin(\omega_c t - \omega_b t)] \end{aligned}$$

The high-pass filter output is therefore

$$v_2(t) = \frac{1}{8}m(t)\cos((\omega_c + \omega_b)t) + \frac{1}{8}\hat{m}(t)\sin((\omega_c + \omega_b)t)$$

Correspondingly, the output of the second product modulator is

$$\begin{aligned} v_3(t) &= v_2(t)\cos((\omega_c + \omega_b)t) \\ &= \frac{1}{8}m(t)\cos^2((\omega_c + \omega_b)t) + \frac{1}{8}\hat{m}(t)\sin((\omega_c + \omega_b)t)\cos((\omega_c + \omega_b)t) \\ &= \frac{1}{16}m(t) \\ &\quad + \frac{1}{16}m(t)\cos(2((\omega_c + \omega_b)t)) + \frac{1}{16}\hat{m}(t)\sin((\omega_c + \omega_b)t) \end{aligned}$$

The scrambler output is therefore

$$v_0(t) = \frac{1}{16}m(t)$$

which is a scaled version of the original message signal.

- 5.47 Consider a message signal $m(t)$ whose frequency content lies inside the band $\omega_l \leq |\omega| \leq \omega_m$. Assuming that the carrier frequency $\omega_c > \omega_m$, we may describe the spectrum of the SSB modulated signal $s(t)$ (with only its upper sideband retained) as follows:

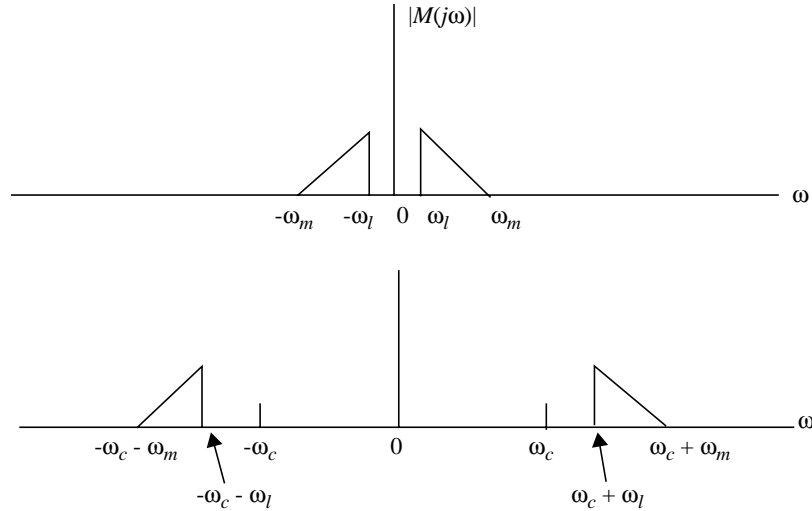


Figure 1

To demodulate $s(t)$, we apply it to the coherent detector of Fig. P5.47. Let $v(t)$ denote the output of the product modulator in this figure, and let $v_0(t)$ denote the output of the low-pass filter. We may then describe the corresponding spectra of $v(t)$ and $v_0(t)$ in graphical forms as shown in Fig. 2.

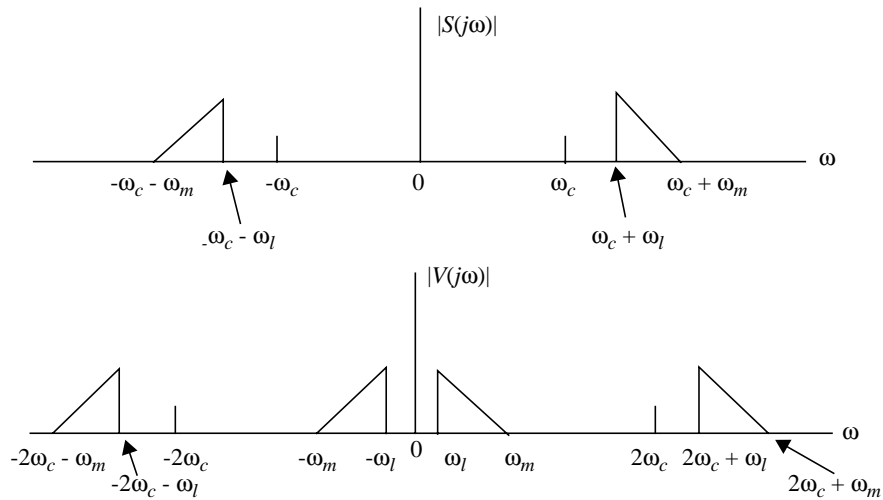


Figure 2

From this figure we see that the product modulator output $v(t)$ consists of two components:

- A scaled version of the original message signal $m(t)$
- A new SSB modulated signal with carrier frequency $2\omega_c$

The latter component is suppressed by the low-pass filter, leaving the original message signal as the detector output as shown by the spectrum of Fig. 3.

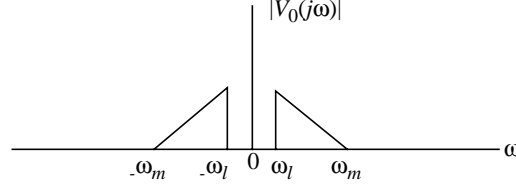


Figure 3

5.48 The multiplexed signal is

$$s(t) = \sum_{k=1}^4 [\cos(\omega_a t + \alpha_{k-1}) + \cos(\omega_b t + \beta_{k-1})] m_k(t) \quad (1)$$

where $\alpha_0 = \beta_0 = 0$. The corresponding output of the product modulator in the coherent detector of the receiver is

$$v_i(t) = s(t) [\cos(\omega_a t + \alpha_{i-1}) + \cos(\omega_b t + \beta_{i-1})] \quad (2)$$

where $i = 1, 2, 3, 4$. There using Eq. (1) in (2):

$$v_i(t) = \sum_{k=1}^4 m_k(t) [\cos(\omega_a t + \alpha_{k-1}) + \cos(\omega_b t + \beta_{k-1})] \\ \times [\cos(\omega_a t + \alpha_{i-1}) + \cos(\omega_b t + \beta_{i-1})]$$

Expanding terms:

$$v_i(t) = \sum_{k=1}^4 m_k(t) [\cos(\omega_a t + \alpha_{k-1}) \cos(\omega_a t + \alpha_{i-1}) \\ + \cos(\omega_a t + \alpha_{k-1}) \cos(\omega_b t + \beta_{i-1}) \\ + \cos(\omega_b t + \beta_{k-1}) \cos(\omega_a t + \alpha_{i-1}) \\ + \cos(\omega_b t + \beta_{k-1}) \cos(\omega_b t + \beta_{i-1})] \\ = \frac{1}{2} \sum_{k=1}^4 m_k(t) [\cos(\alpha_{k-1} - \alpha_{i-1}) + \cos(\beta_{k-1} - \beta_{i-1})] \\ + \cos(2\omega_a t + \alpha_{k-1} + \alpha_{i-1}) + \cos(2\omega_b t + \beta_{k-1} + \beta_{i-1}) \\ + \cos((\omega_a + \omega_b)t + \alpha_{k-1} + \beta_{i-1}) \\ + \cos((\omega_a - \omega_b)t + \alpha_{k-1} - \beta_{i-1}) \\ + \cos((\omega_a + \omega_b)t + \alpha_{i-1} + \beta_{k-1}) \\ + \cos((\omega_a - \omega_b)t + \alpha_{i-1} - \beta_{k-1})]$$

The low-pass filter in the coherent detector removes the six high-frequency components of $v_i(t)$, leaving the output

$$v'_i(t) = \frac{1}{2} \sum_{k=1}^4 m_k(t) [\cos(\alpha_{k-1} - \alpha_{i-1}) + \cos(\beta_{k-1} - \beta_{i-1})]$$

The requirement on α_k and β_k is therefore

$$\cos(\alpha_{k-1} - \alpha_{i-1}) + \cos(\beta_{k-1} - \beta_{i-1}) = \begin{cases} 2, & i = k \\ 0, & i \neq k \end{cases}$$

where $(i, k) = 1, 2, 3, 4$.

- 5.49 Consider an incoming AM signal of bandwidth 10 kHz and carrier frequency ω_c that lies in the range (0.535 - 1.605) MHz. It is required to frequency translate this modulated signal to a fixed band centered at 0.455 MHz. The problem is to determine the range of tuning that must be provided in the local oscillator.

Let ω_{local} denote the local oscillator frequency, which is required to satisfy the condition

$$\omega_c - \omega_{\text{local}} = 2\pi \times 10^6 \times 0.455 \text{ rad/s}$$

or

$$f_c - f_{\text{local}} = 0.455 \text{ MHz}$$

where both f_c and f_{local} are expressed in MHz. That is,

$$f_{\text{local}} = f_c - 0.455$$

When $f_c = 0.535$ MHz, $f_{\text{local}} = 0.535 - 0.455 = 0.08$ MHz.

When $f_c = 1.605$ MHz, $f_{\text{local}} = 1.605 - 0.455 = 1.15$ MHz.

Thus the required range of tuning of the local oscillator is 0.08 - 1.15 MHz, independent of the AM signal's bandwidth.

- 5.50 Consider a periodic train of rectangular pulses, each of duration T_0 . Assuming that a pulse of the train is centered on the origin, we may expand it as a Fourier series:

$$c(t) = \sum_{n=-\infty}^{\infty} f_s \text{sinc}(n f_s T_0) e^{j2\pi n f_s t}$$

where f_s is the pulse repetition frequency and the amplitude of each rectangular pulse is $1/T_0$ (i.e., each pulse has unit area). The assumption that $f_s T_0 \gg 1$ means that the impulse functions in the spectrum of the periodic pulse train $c(t)$ are well separated from each other.

Multiplying a message signal $g(t)$ by $c(t)$ yields the PAM signal

$$s(t) = c(t)g(t)$$

$$= \sum_{n=-\infty}^{\infty} f_s \text{sinc}(n f_s T_0) g(t) e^{j2\pi n f_s t}$$

Hence the Fourier transform of $s(t)$ is given by

$$S(j\omega) = \sum_{n=-\infty}^{\infty} f_s \text{sinc}(n f_s T_0) G(j\omega - jn\omega_s)$$

where $\omega_s = 2\pi f_s$. Thus the spectrum of a naturally sampled signal consists of frequency-shifted replicas of the message spectrum $G(j\omega)$, with the n th replica being scaled in amplitude by the factor $f_s \text{sinc}(n f_s T_0)$, which decreases with increasing n .

5.51 (a) The spectrum of the carrier

$$c[n] = \cos(\Omega_c n), \quad n = \pm 1, \pm 2, \dots$$

is defined by

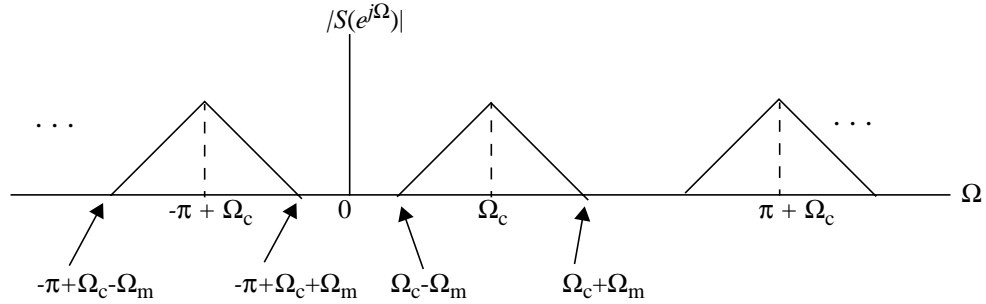
$$C(e^{j\Omega}) = \frac{1}{2} [\delta(j\Omega - j\Omega_c) + \delta(j\Omega + j\Omega_c)], \quad n = \pm 1, \pm 2, \dots$$

where $\Omega_c > \Omega_m$, with Ω_m denoting the highest frequency component of $m[n]$ inside the range $0 \leq |\Omega| \leq \pi$. The spectrum of the modulated signal

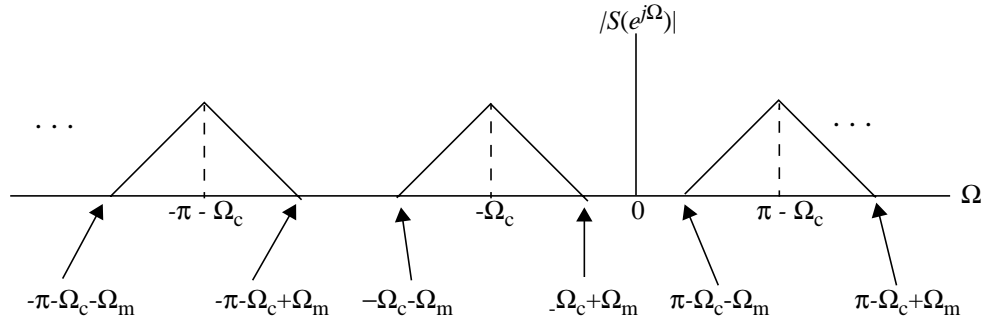
$$s[n] = c[n]m[n],$$

namely, $S(e^{j\Omega})$, is obtained by convolving $C(e^{j\Omega})$ with $M(e^{j\Omega})$.

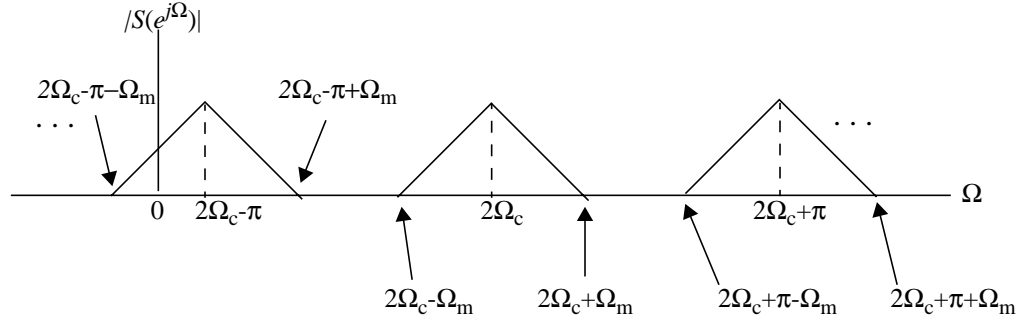
(i) $n = 1$:



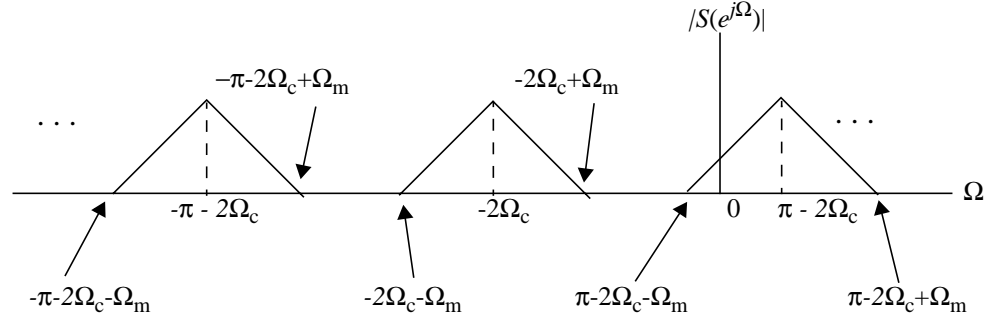
(ii) $n = -1$:



(iii) $n = 2$



(iv) $n = -2$



and so on for $n = \pm 3, \dots$

The transmitter includes a band-pass digital filter designed to pass the spectral bands centered on $\pm\Omega_c$ and reject all other bands.

- (b) Following the coherent detector described in Section 5.5 for demodulation of a DSB-SC modulated signal of the continuous-time variety, we may postulate the scheme of Fig. 1 for demodulation of the discrete-time DSB-SC modulated signal:

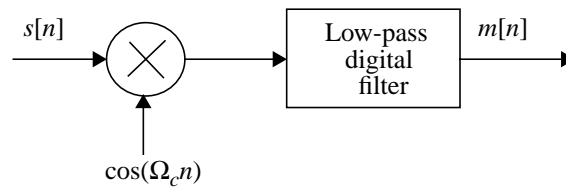


Figure 1

Here it is assumed that (1) the local oscillator supplying $\cos(\Omega_c n)$ is synchronous with the carrier generator used in the transmitter, and (2) the low-pass digital filter is designed with a cutoff frequency slightly greater than Ω_m .

Computer Experiments

%Problem 5.52

clear

fm = 1e3;

fc = 2e4;

mu = 0.75;

t = 1e-6:1e-6:0.1;

y = (1+mu*cos(2*pi*fm*t)).*cos(2*pi*fc*t);

figure(1)

plot(t,y);

xlim([0 0.002])

xlabel('Time (s)')

ylabel('Amplitude')

title('AM Wave: fc = 2e4 Hz, fm = 1e3 Hz, \mu = 0.75')

L = length(y);

Y = fft(y,L);

Y = fftshift(Y);

Ys = Y.*conj(Y);

f = 1e6/L*(-(L/2):L/2-1);

figure(2)

plot(f,Ys)

xlim([-0.5e5 0.5e5])

xlabel('Frequency (Hz)')

ylabel('Power')

title('Spectrum AM Wave: fc = 2e4 Hz, fm = 1e3 Hz, \mu = 0.75')

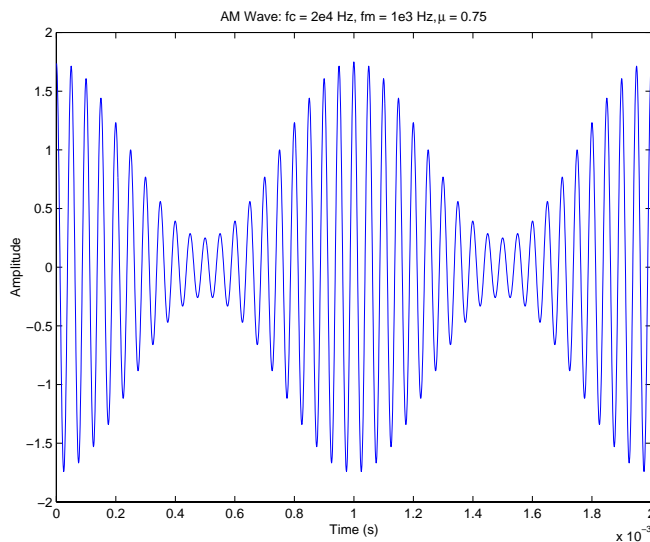


Figure 1: Modulated Signal

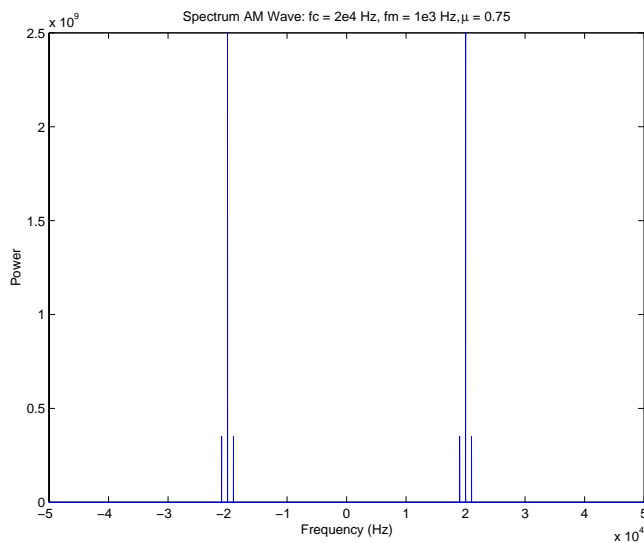


Figure 2: Spectrum of modulated signal

%Problem 5.53

```
fo = 1;
t = 0:0.0001:5;
m = sawtooth(2*pi*fo*t,0.5);
figure(1)
plot(t,m)
xlabel('Time (s)')
ylabel('Amplitude')
title('Triangular Wave 1 Hz')

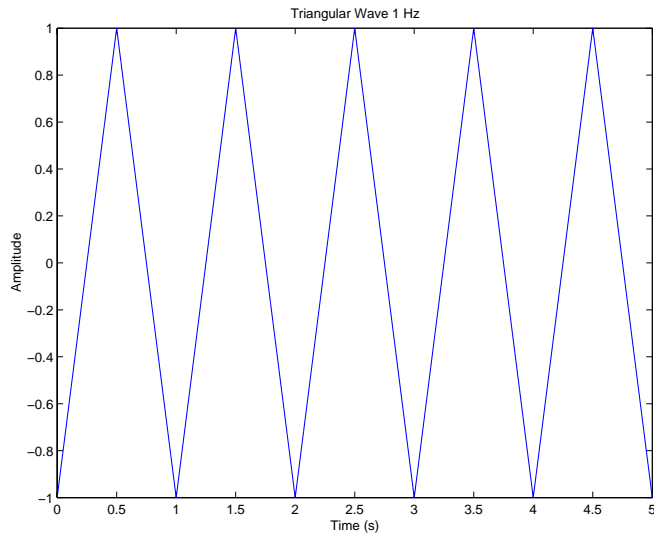
mu = 0.8;
y = (1+mu*m).*cos(2*pi*25*t);
figure(2);
plot(t,y);
xlim([0 2])
xlabel('Time (s)')
ylabel('Amplitude')
title('Modulated Wave')

figure(3)
L = length(y);
Y = fft(y,L);
Y = fftshift(Y);
Ys = abs(Y);
f = 1e4/L*(-(L/2):L/2-1);
plot(f,Ys)
```

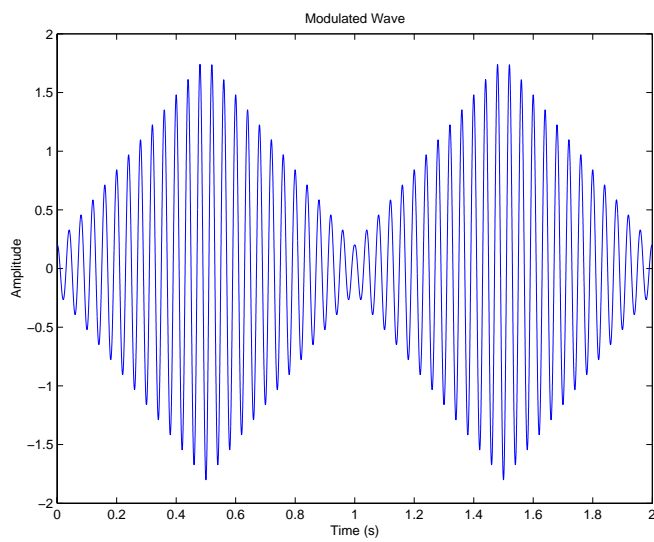
```

xlim([-50 50])
xlabel('Frequency (Hz)')
ylabel('Magnitude')
title('Spectrum AM Wave: fc = 25 Hz, fm = 1 Hz, \mu = 0.8')

```



(a) Figure 1: Waveform of modulating signal



(b) Figure 2: Modulated signal

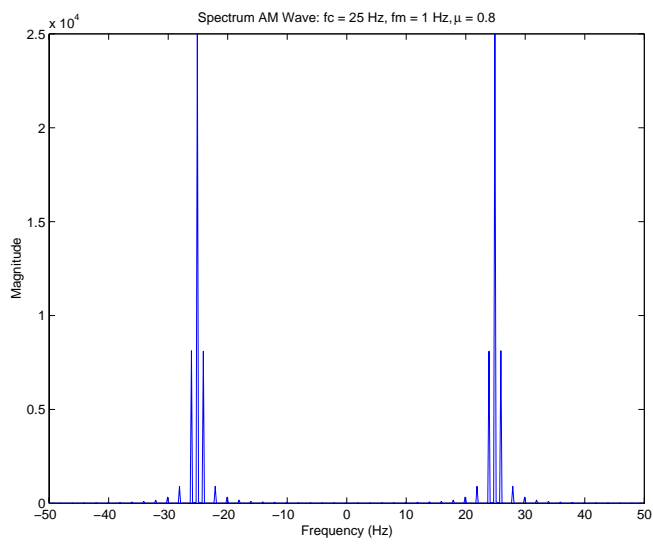


Figure 3

5.54

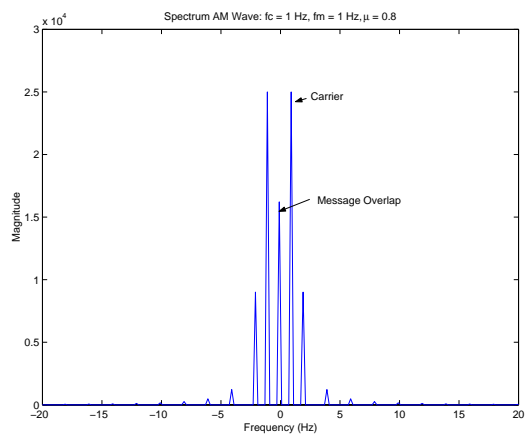


Figure 1

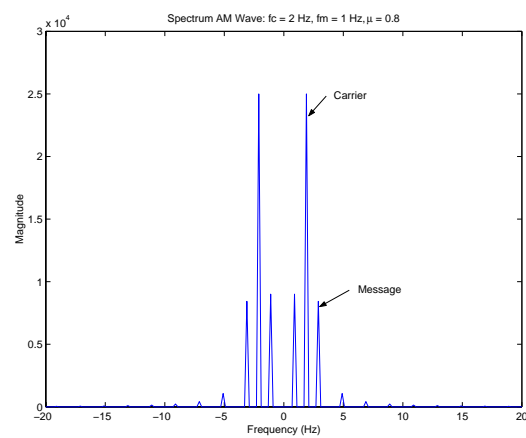


Figure 2

```

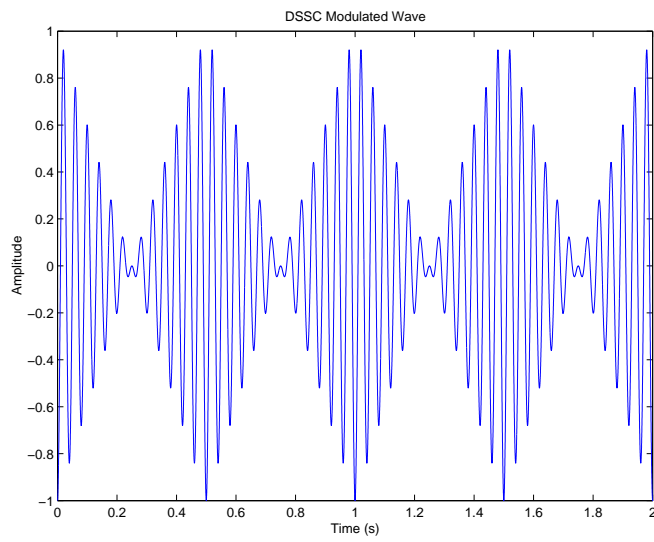
%Problem 5.55
fo = 1;
t = 0:0.0001:5;
m = sawtooth(2*pi*fo*t,0.5);

figure(1)
plot(t,m)
xlabel('Time (s)')
ylabel('Amplitude')
title('Triangular Wave 1 Hz')
mu = 0.8;
y = m.*cos(2*pi*25*t);

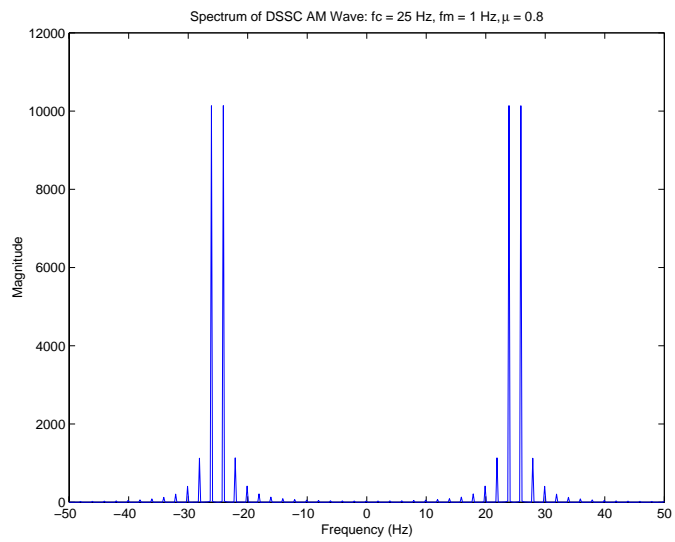
figure(2);
plot(t,y);
xlim([0 2])
xlabel('Time (s)')
ylabel('Amplitude')
title('DSSC Modulated Wave')

figure(3)
L = length(y);
Y = fft(y,L);
Y = fftshift(Y);
Ys = abs(Y);
f = 1e4/L*(-(L/2):L/2-1);
plot(f,Ys)
xlim([-50 50])
xlabel('Frequency (Hz)')
ylabel('Magnitude')
title('Spectrum of DSSC AM Wave: fc = 25 Hz, fm = 1 Hz, \mu = 0.8')

```

(a) Figure 1: Modulated signal



(b) Figure 2: Spectrum of modulated signal

%Problem 5.56

```
clear;clc;
wc = 0.5*pi;
res = 0.001;
Fs = 1;
sam = floor(1/(Fs*res));
t = 0:res:10-res;
m = sin(wc*t);
f = 1/res * ([0:1/(length(t)-1):1] - 0.5);

T = 0.5; %Pick pulse duration value (0.05, 0.1, 0.2, 0.3, 0.4, 0.5)
durInSam = floor(T/res); %number of samples in pulse
sm = zeros(size(t)); %Reserve vector for output waveform

for i=0:floor(length(t)/sam)-1; %loop over number of samples
    sm((i*sam)+1:(i*sam)+durInSam) = m(i*sam+1);
end

y = fftshift(abs(fft(sm)));

figure(1)
subplot(2,1,1)
plot(t,sm);
xlabel('Time (s)')
ylabel('Amplitude')
title('PAM Wave: T_0 = 0.5 s')
subplot(2,1,2)
plot(f,y)
xlabel('Frequency (Hz)')
ylabel('Amplitude')
xlim([-10 10])
```

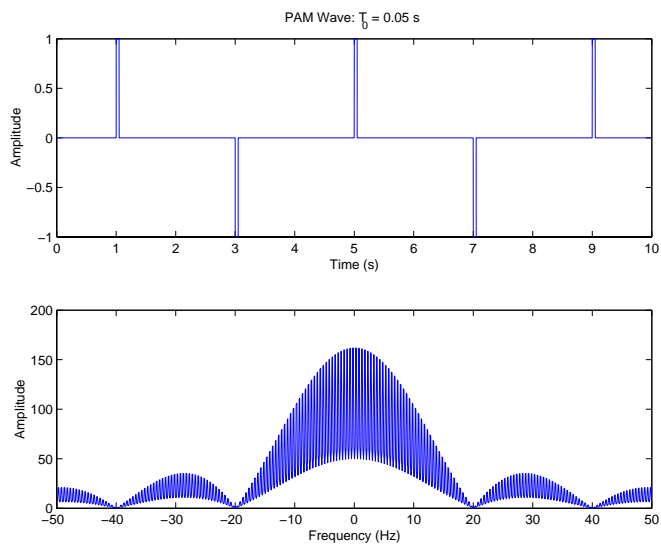


Figure 1: $T_0 = 0.05$ s

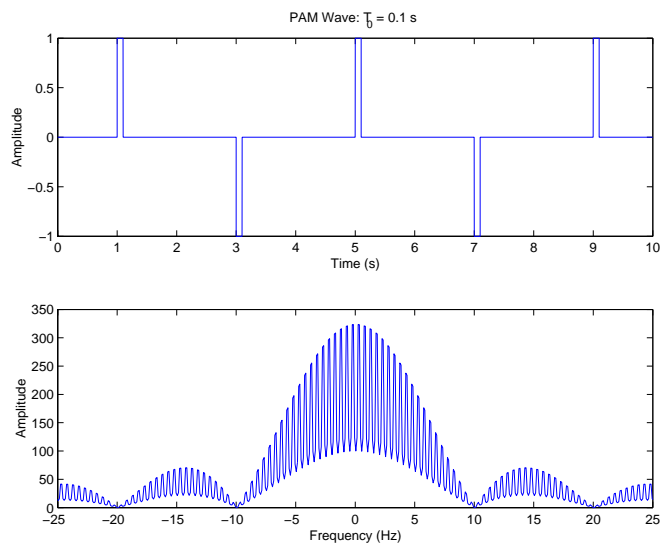


Figure 2: $T_0 = 0.1$ s

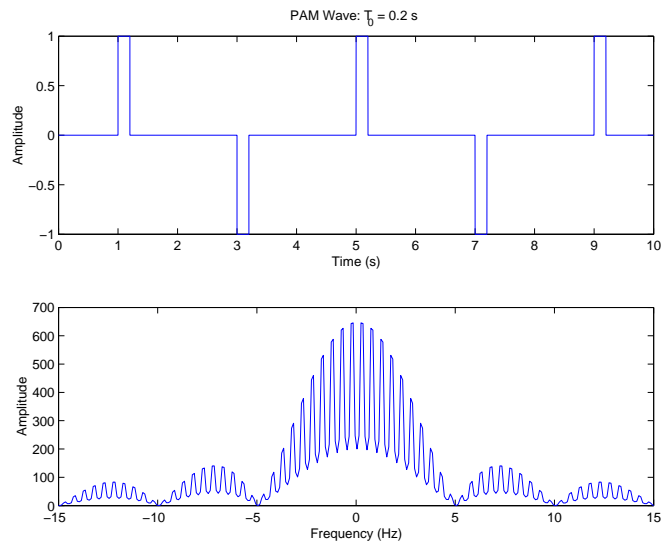


Figure 3: $T_0 = 0.2$ s

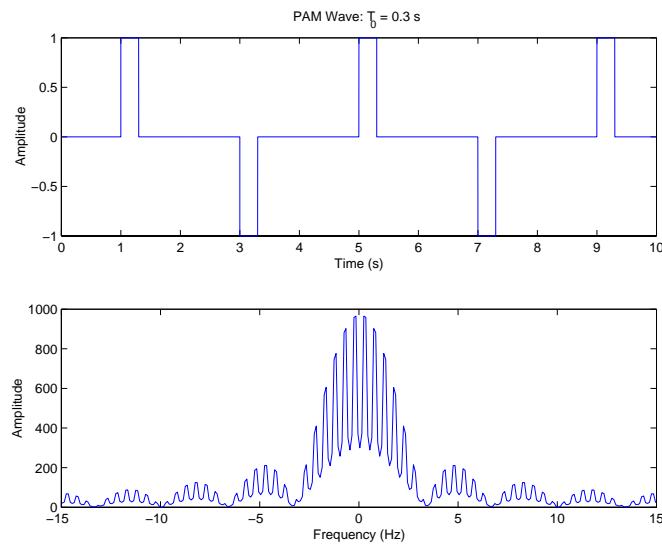


Figure 4: $T_0 = 0.3$ s

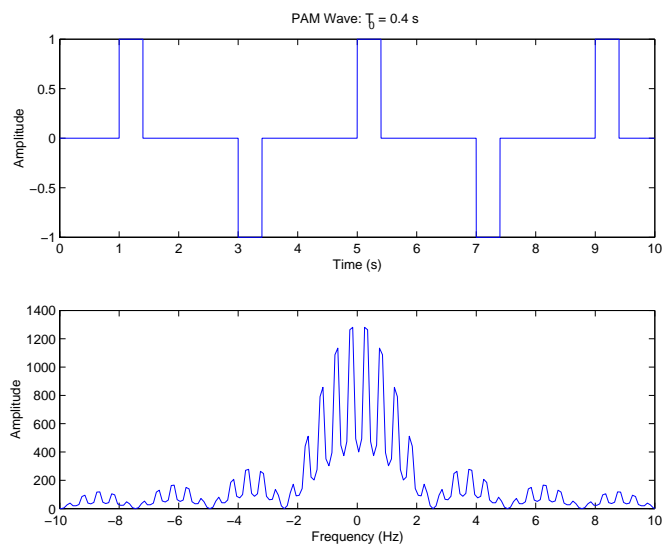


Figure 5: $T_0 = 0.4$ s

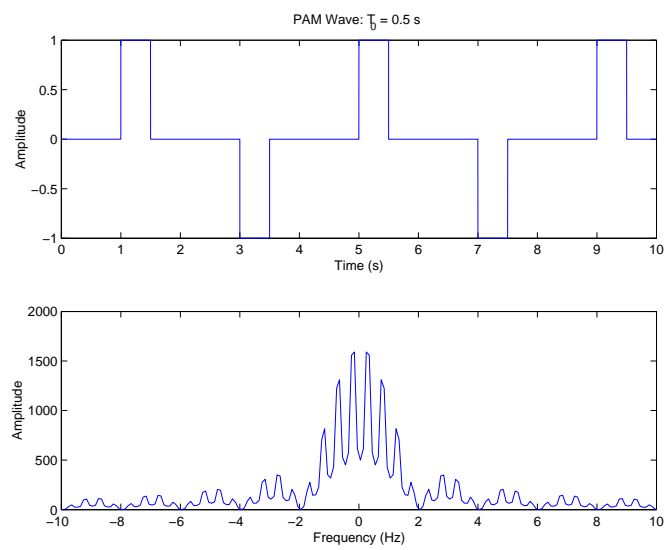


Figure 6: $T_0 = 0.5$ s

%Problem 5.57

```
clear;clc;
res = 1e-6;
t = 0:res:0.01-res;
modf=1000;
m = sin(modf*2*pi*t);

%Samples per sampling period
freqT = 1e4;
periodT = 1/freqT;
pulseDur = 1e-5;
samplesPerPulse = floor(pulseDur/res);
samplesPerT = floor(periodT/res);

%Number of complete sampling cycles we can get i on the signal m
num = floor(length(m) / samplesPerT);

r=[];
for i=1:num,
    r = [r ones(1,samplesPerPulse) zeros(1,samplesPerT-samplesPerPulse)];
end

figure(1)
subplot(2,1,1)
y=r.*m;
plot(t,y)
title('Naturally Sampled Waveform')
xlabel('Time (s)')
ylabel('Amplitude')

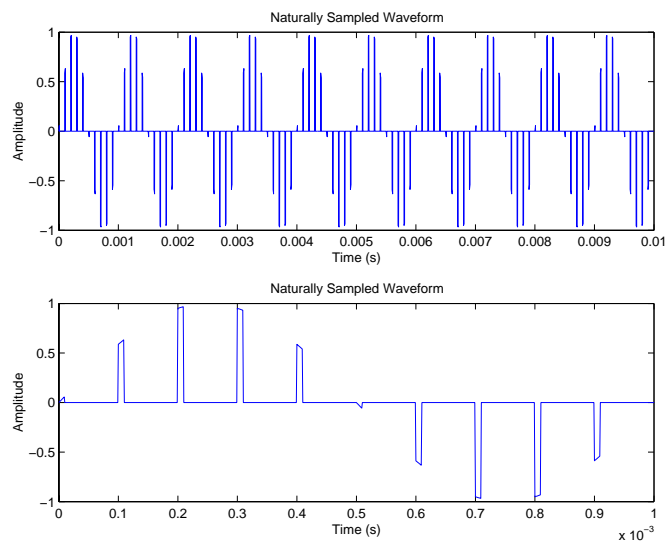
subplot(2,1,2)
plot(t,y)
title('Naturally Sampled Waveform')
xlabel('Time (s)')
ylabel('Amplitude')
set(gca, 'xlim', [0 0.001])

figure(2)
Fy = fftshift(abs(fft(y)));
f = 1/res * ([0:1/(length(t)-1):1]-0.5);
subplot(2,1,1)
plot(f,Fy)
title('Spectrum of the Modulated Waveform')
xlabel('Frequency (Hz)')
ylabel('Magnitude')
```

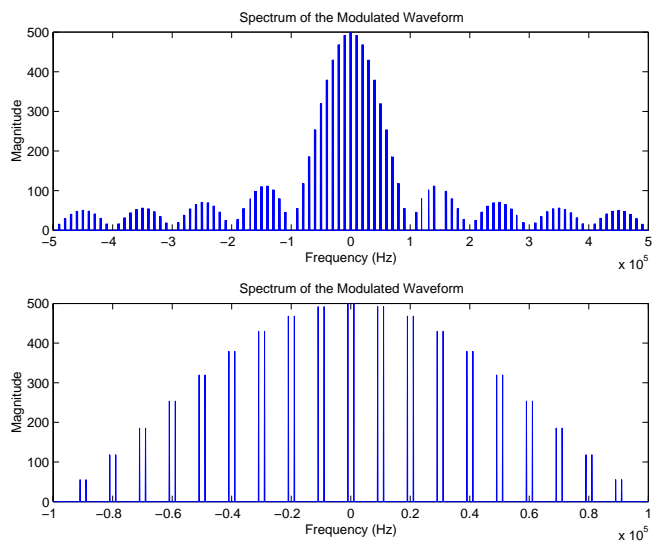
```

subplot(2,1,2)
plot(f,Fy)
title('Spectrum of the Modulated Waveform')
xlabel('Frequency (Hz)')
ylabel('Magnitude')
set(gca, 'xlim', [-1e5 1e5])

```



(a) Figure 1



(b) Figure 2

Solutions to Additional Problems

6.26. A signal $x(t)$ has Laplace transform $X(s)$ as given below. Plot the poles and zeros in the s -plane and determine the Fourier transform of $x(t)$ without inverting $X(s)$.

(a) $X(s) = \frac{s^2+1}{s^2+5s+6}$

$$X(s) = \frac{(s+j)(s-j)}{(s+3)(s+2)}$$

zeros at: $\pm j$

poles at: $-3, -2$

$$X(j\omega) = X(s)|_{s=j\omega}$$

$$= \frac{-\omega^2 + 1}{-\omega^2 + 5j\omega + 6}$$

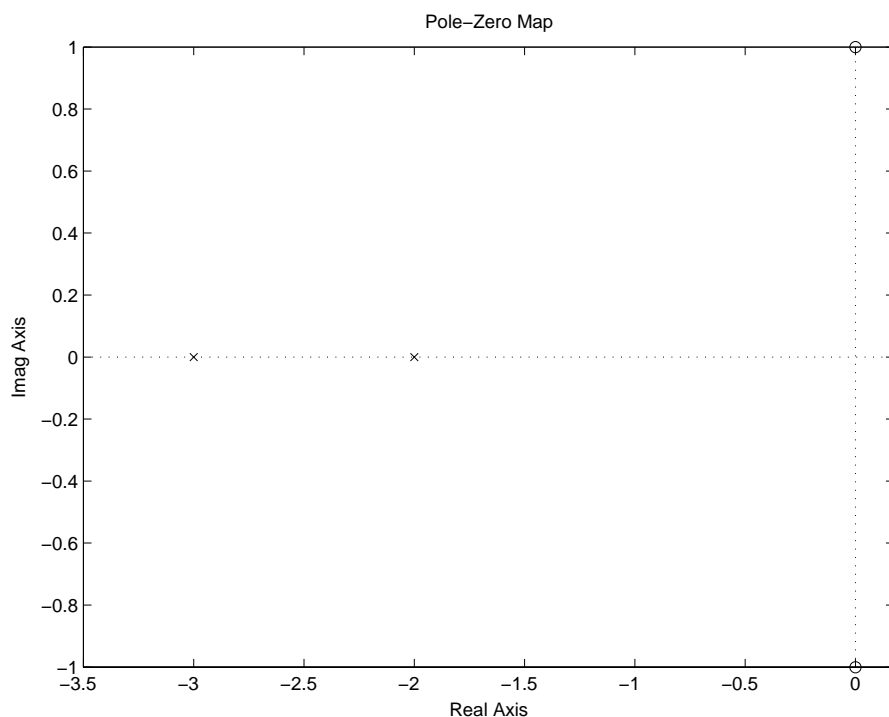


Figure P6.26. (a) Pole-Zero Plot of $X(s)$

(b) $X(s) = \frac{s^2-1}{s^2+s+1}$

$$X(s) = \frac{(s+1)(s-1)}{(s+0.5-j\sqrt{\frac{3}{4}})(s+0.5+j\sqrt{\frac{3}{4}})}$$

zeros at: ± 1

poles at: $\frac{-1 \pm j\sqrt{3}}{2}$

$$\begin{aligned} X(j\omega) &= X(s)|_{s=j\omega} \\ &= \frac{-\omega^2 - 1}{-\omega^2 + j\omega + 1} \end{aligned}$$

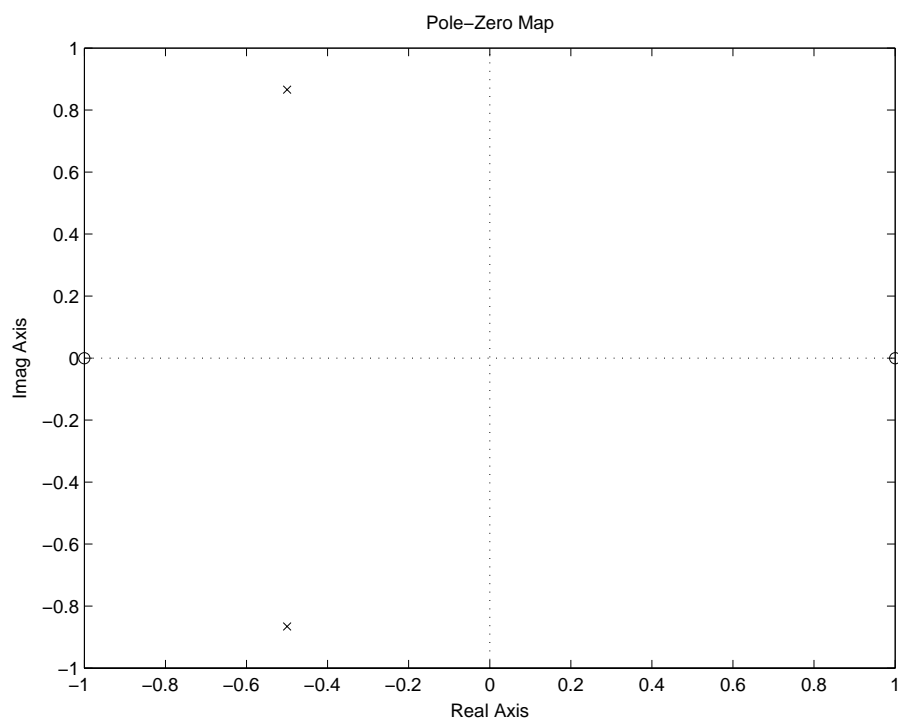


Figure P6.26. (b) Pole-Zero Plot of $X(s)$

(c) $X(s) = \frac{1}{s-4} + \frac{2}{s-2}$

$$\begin{aligned} X(s) &= \frac{3(s - \frac{10}{3})}{(s-4)(s-2)} \\ \text{zero at: } &\frac{10}{3} \\ \text{poles at: } &4, 2 \end{aligned}$$

$$\begin{aligned} X(j\omega) &= X(s)|_{s=j\omega} \\ &= \frac{1}{j\omega - 4} + \frac{2}{j\omega - 2} \end{aligned}$$

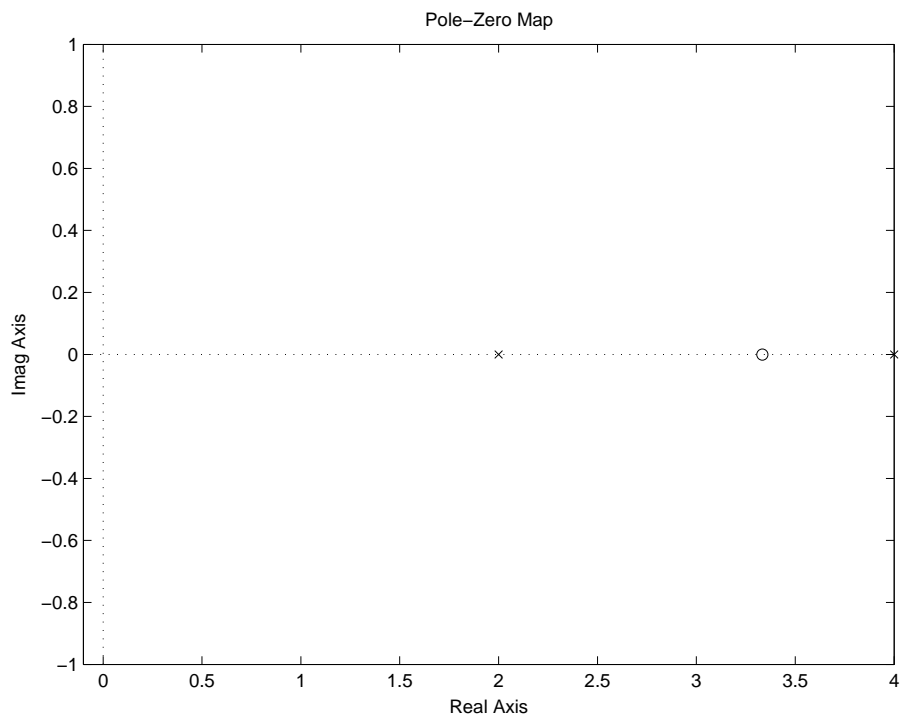


Figure P6.26. (b) Pole-Zero Plot of $X(s)$

6.27. Determine the bilateral Laplace transform and ROC for the following signals:

(a) $x(t) = e^{-t}u(t+2)$

$$\begin{aligned}
 X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st} dt \\
 &= \int_{-\infty}^{\infty} e^{-t}u(t+2)e^{-st} dt \\
 &= \int_{-2}^{\infty} e^{-t(1+s)} dt \\
 &= \frac{e^{2(1+s)}}{1+s} \\
 \text{ROC: } \text{Re}(s) &> -1
 \end{aligned}$$

(b) $x(t) = u(-t+3)$

$$\begin{aligned}
 X(s) &= \int_{-\infty}^3 e^{-st} dt \\
 &= \frac{-e^{-3s}}{s} \\
 \text{ROC: } \text{Re}(s) &< 0
 \end{aligned}$$

(c) $x(t) = \delta(t + 1)$

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} \delta(t + 1) e^{-st} dt \\ &= e^s \\ \text{ROC: all } s \end{aligned}$$

(d) $x(t) = \sin(t)u(t)$

$$\begin{aligned} X(s) &= \int_0^{\infty} \frac{1}{2j} (e^{jt} - e^{-jt}) e^{-st} dt \\ &= \int_0^{\infty} \frac{1}{2j} e^{t(j-s)} dt - \int_0^{\infty} \frac{1}{2j} e^{-t(j+s)} dt \\ &= \frac{1}{2j} \left(\frac{-1}{j-s} - \frac{1}{j+s} \right) \\ &= \frac{1}{(1+s^2)} \\ \text{ROC: } \text{Re}(s) > 0 \end{aligned}$$

6.28. Determine the unilateral Laplace transform of the following signals using the defining equation:

(a) $x(t) = u(t - 2)$

$$\begin{aligned} X(s) &= \int_{0^-}^{\infty} x(t) e^{-st} dt \\ &= \int_{0^-}^{\infty} u(t - 2) e^{-st} dt \\ &= \int_2^{\infty} e^{-st} dt \\ &= \frac{e^{-2s}}{s} \end{aligned}$$

(b) $x(t) = u(t + 2)$

$$\begin{aligned} X(s) &= \int_{0^-}^{\infty} u(t + 2) e^{-st} dt \\ &= \int_{0^-}^{\infty} e^{-st} dt \\ &= \frac{1}{s} \end{aligned}$$

(c) $x(t) = e^{-2t}u(t + 1)$

$$\begin{aligned}
X(s) &= \int_{0^-}^{\infty} e^{-2t} u(t+1) e^{-st} dt \\
&= \int_{0^-}^{\infty} e^{-t(s+2)} dt \\
&= \frac{1}{s+2}
\end{aligned}$$

(d) $x(t) = e^{2t} u(-t+2)$

$$\begin{aligned}
X(s) &= \int_{0^-}^{\infty} e^{2t} u(-t+2) e^{-st} dt \\
&= \int_{0^-}^2 e^{t(2-s)} dt \\
&= \frac{e^{2(2-s)} - 1}{2-s}
\end{aligned}$$

(e) $x(t) = \sin(\omega_o t)$

$$\begin{aligned}
X(s) &= \int_{0^-}^{\infty} \frac{1}{2j} (e^{j\omega_o t} - e^{-j\omega_o t}) e^{-st} dt \\
&= \frac{1}{2j} \left[\int_{0^-}^{\infty} e^{t(j\omega_o - s)} dt - \int_{0^-}^{\infty} e^{-t(j\omega_o + s)} dt \right] \\
&= \frac{1}{2j} \left[\frac{-1}{j\omega_o - s} - \frac{1}{j\omega_o + s} \right] \\
&= \frac{\omega_o}{s^2 + \omega_o^2}
\end{aligned}$$

(f) $x(t) = u(t) - u(t-2)$

$$\begin{aligned}
X(s) &= \int_{0^-}^2 e^{-st} dt \\
&= \frac{1 - e^{-2s}}{s}
\end{aligned}$$

(g) $x(t) = \begin{cases} \sin(\pi t), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

$$\begin{aligned}
X(s) &= \int_{0^-}^1 \frac{1}{2j} (e^{j\pi t} - e^{-j\pi t}) e^{-st} dt \\
&= \frac{\pi(1 + e^{-s})}{s^2 + \pi^2}
\end{aligned}$$

6.29. Use the basic Laplace transforms and the Laplace transform properties given in Tables D.1 and D.2 to determine the unilateral Laplace transform of the following signals:

(a) $x(t) = \frac{d}{dt} \{te^{-t}u(t)\}$

$$\begin{aligned} a(t) = te^{-t}u(t) &\xleftrightarrow{\mathcal{L}_u} A(s) = \frac{1}{(s+1)^2} \\ x(t) = \frac{d}{dt}a(t) &\xleftrightarrow{\mathcal{L}_u} X(s) = \frac{s}{(s+1)^2} \end{aligned}$$

(b) $x(t) = tu(t) * \cos(2\pi t)u(t)$

$$\begin{aligned} a(t) = tu(t) &\xleftrightarrow{\mathcal{L}_u} A(s) = \frac{1}{s^2} \\ b(t) = \cos(2\pi t)u(t) &\xleftrightarrow{\mathcal{L}_u} \frac{s}{s^2 + 4\pi^2} \\ x(t) = a(t) * b(t) &\xleftrightarrow{\mathcal{L}_u} X(s) = A(s)B(s) \\ X(s) &= \frac{1}{s^2(s^2 + 4\pi^2)} \end{aligned}$$

(c) $x(t) = t^3u(t)$

$$\begin{aligned} a(t) = tu(t) &\xleftrightarrow{\mathcal{L}_u} A(s) = \frac{1}{s^2} \\ b(t) = -ta(t) &\xleftrightarrow{\mathcal{L}_u} B(s) = \frac{d}{ds}A(s) = \frac{-2}{s^3} \\ x(t) = -tb(t) &\xleftrightarrow{\mathcal{L}_u} X(s) = \frac{d}{ds}B(s) = \frac{6}{s^4} \end{aligned}$$

(d) $x(t) = u(t-1) * e^{-2t}u(t-1)$

$$\begin{aligned} a(t) = u(t) &\xleftrightarrow{\mathcal{L}_u} A(s) = \frac{1}{s} \\ b(t) = a(t-1) &\xleftrightarrow{\mathcal{L}_u} B(s) = \frac{e^{-s}}{s} \\ c(t) = e^{-2t}u(t) &\xleftrightarrow{\mathcal{L}_u} C(s) = \frac{1}{s+2} \\ d(t) = e^{-2}c(t-1) &\xleftrightarrow{\mathcal{L}_u} D(s) = \frac{e^{-(s+2)}}{s+2} \\ x(t) = b(t) * d(t) &\xleftrightarrow{\mathcal{L}_u} X(s) = B(s)D(s) \\ X(s) &= \frac{e^{-2(s+1)}}{s(s+2)} \end{aligned}$$

(e) $x(t) = \int_0^t e^{-3\tau} \cos(2\tau) d\tau$

$$\begin{aligned} a(t) = e^{-3t} \cos(2t)u(t) & \xleftrightarrow{\mathcal{L}_u} A(s) = \frac{s+3}{(s+3)^2+4} \\ \int_{-\infty}^t a(\tau) d\tau & \xleftrightarrow{\mathcal{L}_u} \frac{1}{s} \int_{-\infty}^{0^-} a(\tau) d\tau + \frac{A(s)}{s} \\ X(s) & = \frac{s+3}{s((s+3)^2+4)} \end{aligned}$$

(f) $x(t) = t \frac{d}{dt} (e^{-t} \cos(t)u(t))$

$$\begin{aligned} a(t) = e^{-t} \cos(t)u(t) & \xleftrightarrow{\mathcal{L}_u} A(s) = \frac{s+1}{(s+1)^2+1} \\ b(t) = \frac{d}{dt} a(t) & \xleftrightarrow{\mathcal{L}_u} B(s) = \frac{s(s+1)}{(s+1)^2+1} \\ x(t) = tb(t) & \xleftrightarrow{\mathcal{L}_u} X(s) = -\frac{d}{ds} B(s) \\ X(s) & = \frac{-s^2-4s-2}{(s^2+2s+2)^2} \end{aligned}$$

6.30. Use the basic Laplace transforms and the Laplace transform properties given in Tables D.1 and D.2 to determine the time signals corresponding to the following unilateral Laplace transforms:

(a) $X(s) = \left(\frac{1}{s+2}\right) \left(\frac{1}{s+3}\right)$

$$\begin{aligned} X(s) & = \frac{1}{s+2} + \frac{-1}{s+3} \\ x(t) & = (e^{-2t} - e^{-3t})u(t) \end{aligned}$$

(b) $X(s) = e^{-2s} \frac{d}{ds} \left(\frac{1}{(s+1)^2} \right)$

$$\begin{aligned} A(s) = \frac{1}{(s+1)^2} & \xleftrightarrow{\mathcal{L}_u} a(t) = te^{-t}u(t) \\ B(s) = \frac{d}{ds} A(s) & \xleftrightarrow{\mathcal{L}_u} b(t) = -ta(t) = -t^2e^{-t}u(t) \\ X(s) = e^{-2s} B(s) & \xleftrightarrow{\mathcal{L}_u} x(t) = b(t-2) = -(t-2)^2e^{-(t-2)}u(t-2) \end{aligned}$$

(c) $X(s) = \frac{1}{(2s+1)^2+4}$

$$\begin{aligned} B\left(\frac{s}{a}\right) & \xleftrightarrow{\mathcal{L}_u} ab(at) \\ \frac{1}{(s+1)^2+4} & \xleftrightarrow{\mathcal{L}_u} \frac{1}{2}e^{-t} \sin(2t)u(t) \\ x(t) & = \frac{1}{4}e^{-0.5t} \sin(t)u(t) \end{aligned}$$

(d) $X(s) = s \frac{d^2}{ds^2} \left(\frac{1}{s^2+9} \right) + \frac{1}{s+3}$

$$\begin{aligned}
 A(s) &= \frac{1}{s^2+9} \xleftarrow{\mathcal{L}_u} a(t) = \frac{1}{3} \sin(3t)u(t) \\
 B(s) &= \frac{d}{ds} A(s) \xleftarrow{\mathcal{L}_u} b(t) = -ta(t) = -\frac{t}{3} \sin(3t)u(t) \\
 C(s) &= \frac{d}{ds} B(s) \xleftarrow{\mathcal{L}_u} c(t) = -tb(t) = \frac{t^2}{3} \sin(3t)u(t) \\
 D(s) &= sC(s) \xleftarrow{\mathcal{L}_u} d(t) = \frac{d}{dt} c(t) - c(0^-) = \frac{2t}{3} \sin(3t)u(t) + t^2 \cos(3t)u(t) \\
 E(s) &= \frac{1}{s+3} \xleftarrow{\mathcal{L}_u} e(t) = e^{-3t}u(t) \\
 x(t) &= e(t) + d(t) = \left[e^{-3t} + \frac{2t}{3} \sin(3t) + t^2 \cos(3t) \right] u(t)
 \end{aligned}$$

6.31. Given the transform pair $\cos(2t)u(t) \xleftarrow{\mathcal{L}_u} X(s)$, determine the time signals corresponding to the following Laplace transforms:

(a) $(s+1)X(s)$

$$\begin{aligned}
 sX(s) + X(s) &\xleftarrow{\mathcal{L}_u} \frac{d}{dt} x(t) + x(t) \\
 &= [-2 \sin(2t) + \cos(2t)] u(t)
 \end{aligned}$$

(b) $X(3s)$

$$\begin{aligned}
 X\left(\frac{s}{a}\right) &\xleftarrow{\mathcal{L}_u} ax(at) \\
 x(t) &= \frac{1}{3} \cos\left(\frac{2}{3}t\right)u(t)
 \end{aligned}$$

(c) $X(s+2)$

$$\begin{aligned}
 X(s+2) &\xleftarrow{\mathcal{L}_u} e^{-2t}x(t) \\
 x(t) &= e^{-2t} \cos(2t)u(t)
 \end{aligned}$$

(d) $s^{-2}X(s)$

$$\begin{aligned}
 B(s) = \frac{1}{s}X(s) &\xleftarrow{\mathcal{L}_u} \int_{-\infty}^t x(\tau) d\tau \\
 &\xleftarrow{\mathcal{L}_u} \int_{-\infty}^t \cos(2\tau)u(\tau) d\tau \\
 &\xleftarrow{\mathcal{L}_u} \int_0^t \cos(2\tau) d\tau
 \end{aligned}$$

$$\begin{aligned}
B(s) &\xleftrightarrow{\mathcal{L}_u} \frac{1}{2} \sin(2t) \\
\frac{1}{s} B(s) &\xleftrightarrow{\mathcal{L}_u} \int_0^t \frac{1}{2} \sin(2\tau) d\tau \\
&\xleftrightarrow{\mathcal{L}_u} \frac{1 - \cos(2t)}{4} u(t)
\end{aligned}$$

(e) $\frac{d}{ds} (e^{-3s} X(s))$

$$\begin{aligned}
A(s) = e^{-3s} X(s) &\xleftrightarrow{\mathcal{L}_u} a(t) = x(t-3) = \cos(2(t-3))u(t-3) \\
B(s) = \frac{d}{ds} A(s) &\xleftrightarrow{\mathcal{L}_u} b(t) = -ta(t) = -t \cos(2(t-3))u(t-3)
\end{aligned}$$

6.32. Given the transform pair $x(t) \xleftrightarrow{\mathcal{L}_u} \frac{2s}{s^2+2}$, where $x(t) = 0$ for $t < 0$, determine the Laplace transform of the following time signals:

(a) $x(3t)$

$$\begin{aligned}
x(3t) &\xleftrightarrow{\mathcal{L}_u} \frac{1}{3} X\left(\frac{s}{3}\right) \\
X(s) &= \frac{2\frac{s}{3}}{\left(\frac{s}{3}\right)^2 + 2} \\
&= \frac{6s}{s^2 + 18}
\end{aligned}$$

(b) $x(t-2)$

$$x(t-2) \xleftrightarrow{\mathcal{L}_u} e^{-2s} X(s) = e^{-2s} \frac{2s}{s^2+2}$$

(c) $x(t) * \frac{d}{dt} x(t)$

$$\begin{aligned}
b(t) = \frac{d}{dt} x(t) &\xleftrightarrow{\mathcal{L}_u} B(s) = sX(s) \\
y(t) = x(t) * b(t) &\xleftrightarrow{\mathcal{L}_u} Y(s) = B(s)X(s) = s[X(s)]^2 \\
Y(s) &= s \left(\frac{2s}{s^2+2} \right)^2
\end{aligned}$$

(d) $e^{-t} x(t)$

$$e^{-t} x(t) \xleftrightarrow{\mathcal{L}_u} X(s+1) = \frac{2(s+1)}{(s+1)^2+2}$$

(e) $2tx(t)$

$$2tx(t) \xrightarrow{\mathcal{L}_u} -2\frac{d}{ds}X(s) = \frac{4s^2 - 8}{(s^2 + 2)^2}$$

(f) $\int_0^t x(3\tau)d\tau$

$$\begin{aligned} \int_0^t x(3\tau)d\tau &\xrightarrow{\mathcal{L}_u} Y(s) = \frac{X(\frac{s}{3})}{3s} \\ Y(s) &= \frac{2}{s^2 + 18} \end{aligned}$$

6.33. Use the s -domain shift property and the transform pair $e^{-at}u(t) \xrightarrow{\mathcal{L}_u} \frac{1}{s+a}$ to derive the unilateral Laplace transform of $x(t) = e^{-at} \cos(\omega_1 t)u(t)$.

$$\begin{aligned} e^{-at}u(t) &\xrightarrow{\mathcal{L}_u} \frac{1}{s+a} \\ x(t) &= e^{-at} \cos(\omega_1 t)u(t) \\ &= \frac{1}{2}e^{-at} (e^{j\omega_1 t} + e^{-j\omega_1 t}) u(t) \\ &\text{Using the s-domain shift property:} \\ X(s) &= \frac{1}{2} \left(\frac{1}{(s - j\omega_1) + a} + \frac{1}{(s + j\omega_1) + a} \right) \\ &= \frac{1}{2} \frac{2(s+a)}{(s+a)^2 + \omega_1^2} \\ &= \frac{(s+a)}{(s+a)^2 + \omega_1^2} \end{aligned}$$

6.34. Prove the following properties of the unilateral Laplace transform:

(a) Linearity

$$\begin{aligned} z(t) &= ax(t) + by(t) \\ Z(s) &= \int_0^\infty z(t)e^{-st} dt \\ &= \int_0^\infty (ax(t) + by(t)) e^{-st} dt \\ &= \int_0^\infty ax(t)e^{-st} dt + \int_0^\infty by(t)e^{-st} dt \\ &= a \int_0^\infty x(t)e^{-st} dt + b \int_0^\infty y(t)e^{-st} dt \\ &= aX(s) + bY(s) \end{aligned}$$

(b) Scaling

$$z(t) = x(at)$$

$$\begin{aligned}
Z(s) &= \int_0^\infty x(at)e^{-st} dt \\
&= \frac{1}{a} \int_0^\infty x(\tau)e^{-\frac{s}{a}\tau} d\tau \\
&= \frac{1}{a} X\left(\frac{s}{a}\right)
\end{aligned}$$

(c) Time shift

$$\begin{aligned}
z(t) &= x(t - \tau) \\
Z(s) &= \int_0^\infty x(t - \tau)e^{-st} dt \\
&\quad \text{Let } m = t - \tau \\
Z(s) &= \int_{-\tau}^\infty x(m)e^{-s(m+\tau)} dm \\
&\quad \text{If } x(t - \tau)u(t) = x(t - \tau)u(t - \tau) \\
Z(s) &= \int_0^\infty x(m)e^{-sm}e^{-s\tau} dm \\
&= e^{-s\tau} X(s)
\end{aligned}$$

(d) s -domain shift

$$\begin{aligned}
z(t) &= e^{s_o t} x(t) \\
Z(s) &= \int_0^\infty e^{s_o t} x(t)e^{-st} dt \\
&= \int_0^\infty x(t)e^{-(s_o - s)t} dt \\
&= X(s - s_o)
\end{aligned}$$

(e) Convolution

$$\begin{aligned}
z(t) &= x(t) * y(t) \\
&= \int_0^\infty x(\tau)y(t - \tau) d\tau; x(t), y(t) \text{ causal} \\
Z(s) &= \int_0^\infty \left(\int_0^\infty x(\tau)y(t - \tau) d\tau \right) e^{-st} dt \\
&= \int_0^\infty \left(\int_0^\infty x(\tau)y(m) d\tau \right) e^{-sm}e^{-s\tau} dm \\
&= \left(\int_0^\infty x(\tau)e^{-s\tau} d\tau \right) \left(\int_0^\infty y(m)e^{-sm} dm \right) \\
&= X(s)Y(s)
\end{aligned}$$

(f) Differentiation in the s -domain

$$\begin{aligned}
 z(t) &= -tx(t) \\
 Z(s) &= \int_0^\infty -tx(t)e^{-st} dt \\
 &= \int_0^\infty x(t) \frac{d}{ds}(e^{-st}) dt \\
 &= \int_0^\infty \frac{d}{ds}(x(t)e^{-st}) dt \\
 &\quad \text{Assume: } \int_0^\infty (\cdot) dt \text{ and } \frac{d}{ds}(\cdot) \text{ are interchangeable.} \\
 Z(s) &= \frac{d}{ds} \int_0^\infty x(t)e^{-st} dt \\
 Z(s) &= \frac{d}{ds} X(s)
 \end{aligned}$$

6.35. Determine the initial value $x(0^+)$ given the following Laplace transforms $X(s)$:

(a) $X(s) = \frac{1}{s^2+5s-2}$

$$x(0^+) = \lim_{s \rightarrow \infty} sX(s) = \frac{s}{s^2+5s-2} = 0$$

(b) $X(s) = \frac{s+2}{s^2+2s-3}$

$$x(0^+) = \lim_{s \rightarrow \infty} sX(s) = \frac{s^2+2s}{s^2+2s-3} = 1$$

(c) $X(s) = e^{-2s} \frac{6s^2+s}{s^2+2s-2}$

$$x(0^+) = \lim_{s \rightarrow \infty} sX(s) = e^{-2s} \frac{6s^3+s^2}{s^2+2s-2} = 0$$

6.36. Determine the final value $x(\infty)$ given the following Laplace transforms $X(s)$:

(a) $X(s) = \frac{2s^2+3}{s^2+5s+1}$

$$x(\infty) = \lim_{s \rightarrow 0} sX(s) = \frac{2s^3+3s}{s^2+5s+1} = 0$$

(b) $X(s) = \frac{s+2}{s^3+2s^2+s}$

$$x(\infty) = \lim_{s \rightarrow 0} sX(s) = \frac{s+2}{s^2+2s+1} = 2$$

$$(c) X(s) = e^{-3s} \frac{2s^2+1}{s(s+2)^2}$$

$$x(\infty) = \lim_{s \rightarrow 0} sX(s) = e^{-3s} \frac{2s^2+1}{(s+2)^2} = \frac{1}{4}$$

6.37. Use the method of partial fractions to find the time signals corresponding to the following unilateral Laplace transforms:

$$(a) X(s) = \frac{s+3}{s^2+3s+2}$$

$$\begin{aligned} X(s) &= \frac{s+3}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2} \\ 1 &= A+B \\ 3 &= 2A+B \\ X(s) &= \frac{2}{s+1} + \frac{-1}{s+2} \\ x(t) &= [2e^{-t} - e^{-2t}] u(t) \end{aligned}$$

$$(b) X(s) = \frac{2s^2+10s+11}{s^2+5s+6}$$

$$\begin{aligned} X(s) &= \frac{2s^2+10s+11}{s^2+5s+6} = 2 - \frac{1}{(s+2)(s+3)} \\ \frac{1}{(s+2)(s+3)} &= \frac{A}{s+2} + \frac{B}{s+3} \\ 0 &= A+B \\ 1 &= 3A+2B \\ X(s) &= 2 - \frac{1}{s+2} + \frac{1}{s+3} \\ x(t) &= 2\delta(t) + [e^{-3t} - e^{-2t}] u(t) \end{aligned}$$

$$(c) X(s) = \frac{2s-1}{s^2+2s+1}$$

$$\begin{aligned} X(s) &= \frac{2s-1}{s^2+2s+1} = \frac{A}{s+1} + \frac{B}{(s+1)^2} \\ 2 &= A \\ -1 &= A+B \\ x(t) &= [2e^{-t} - 3te^{-t}] u(t) \end{aligned}$$

$$(d) X(s) = \frac{5s+4}{s^3+3s^2+2s}$$

$$X(s) = \frac{5s+4}{s^3+3s^2+2s} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$\begin{aligned}
0 &= A + B + C \\
5 &= 3A + B + 2C \\
4 &= 2A \\
X(s) &= \frac{2}{s} + \frac{-3}{s+2} + \frac{1}{s+1} \\
x(t) &= [2 - 3e^{-2t} + e^{-t}] u(t)
\end{aligned}$$

$$(e) X(s) = \frac{s^2-3}{(s+2)(s^2+2s+1)}$$

$$\begin{aligned}
X(s) &= \frac{s^2-3}{(s+2)(s^2+2s+1)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2} \\
1 &= A + B \\
0 &= 2A + 3B + C \\
-3 &= A + 2B + 2C \\
X(s) &= \frac{1}{s+2} + \frac{-2}{(s+1)^2} \\
x(t) &= [e^{-2t} - 2te^{-t}] u(t)
\end{aligned}$$

$$(f) X(s) = \frac{3s+2}{s^2+2s+10}$$

$$\begin{aligned}
X(s) &= \frac{3s+2}{s^2+2s+10} = \frac{3(s+1)-1}{(s+1)^2+3^2} \\
x(t) &= \left[3e^{-t} \cos(3t) - \frac{1}{3}e^{-t} \sin(3t) \right] u(t)
\end{aligned}$$

$$(g) X(s) = \frac{4s^2+8s+10}{(s+2)(s^2+2s+5)}$$

$$\begin{aligned}
X(s) &= \frac{2}{s+2} + \frac{2(s+1)}{(s+1)^2+2^2} + \frac{-2}{(s+1)^2+2^2} \\
x(t) &= [2e^{-2t} + 2e^{-t} \cos(2t) - e^{-t} \sin(2t)] u(t)
\end{aligned}$$

$$(h) X(s) = \frac{3s^2+10s+10}{(s+2)(s^2+6s+10)}$$

$$\begin{aligned}
X(s) &= \frac{3s^2+10s+10}{(s+2)(s^2+6s+10)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+6s+10} \\
3 &= A + B \\
10 &= 6A + 2B + C \\
10 &= 10A + 2C \\
X(s) &= \frac{1}{s+2} + \frac{2(s+3)}{(s+3)^2+1} - \frac{6}{(s+3)^2+1} \\
x(t) &= [e^{-2t} + 2e^{-3t} \cos(t) - 6e^{-3t} \sin(t)] u(t)
\end{aligned}$$

$$(i) X(s) = \frac{2s^2 + 11s + 16 + e^{-2s}}{s^2 + 5s + 6}$$

$$\begin{aligned} X(s) &= \frac{2s^2 + 11s + 16 + e^{-2s}}{s^2 + 5s + 6} = 2 + \frac{s + 4}{s^2 + 5s + 6} + \frac{e^{-2s}}{s^2 + 5s + 6} \\ X(s) &= 2 + \frac{-1}{s + 3} + \frac{2}{s + 2} - e^{-2s} \frac{1}{s + 3} + e^{-2s} \frac{1}{s + 2} \\ x(t) &= 2\delta(t) + [2e^{-2t} - e^{-t}] u(t) + [e^{-2(t-2)} - e^{-3(t-2)}] u(t - 2) \end{aligned}$$

6.38. Determine the forced and natural responses for the LTI systems described by the following differential equations with the specified input and initial conditions:

$$(a) \frac{d}{dt}y(t) + 10y(t) = 10x(t), \quad y(0^-) = 1, \quad x(t) = u(t)$$

$$\begin{aligned} X(s) &= \frac{1}{s} \\ Y(s)(s + 10) &= 10X(s) + y(0^-) \\ Y^f(s) &= \frac{10X(s)}{s + 10} \\ &= \frac{10}{s(s + 10)} \\ &= \frac{1}{s} + \frac{-1}{s + 10} \\ y^f(t) &= [1 - e^{-10t}] u(t) \\ Y^n(s) &= \frac{y(0^-)}{s + 10} \\ y^n(t) &= e^{-10t} u(t) \end{aligned}$$

$$(b) \frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = -4x(t) - 3\frac{d}{dt}x(t), \quad y(0^-) = -1, \quad \frac{d}{dt}y(t)|_{t=0^-} = 5, \quad x(t) = e^{-t}u(t)$$

$$\begin{aligned} Y(s)(s^2 + 5s + 6) - 5 + s + 5 &= (-4 - 3s) \frac{1}{s + 1} \\ Y(s) &= \frac{-1}{(s + 1)(s + 2)(s + 3)} + \frac{s}{(s + 2)(s + 3)} \\ &= Y^f(s) + Y^n(s) \\ Y^f(s) &= \frac{-0.5}{s + 1} + \frac{-2}{s + 2} + \frac{2.5}{s + 3} \\ y^f(t) &= (-0.5e^{-t} - 2e^{-2t} + 2.5e^{-3t}) u(t) \\ Y^n(s) &= \frac{-2}{s + 2} + \frac{3}{s + 3} \\ y^n(t) &= (-2e^{-2t} + 3e^{-3t}) u(t) \end{aligned}$$

$$(c) \frac{d^2}{dt^2}y(t) + y(t) = 8x(t), \quad y(0^-) = 0, \quad \frac{d}{dt}y(t)|_{t=0^-} = 2, \quad x(t) = e^{-t}u(t)$$

$$Y(s)(s^2 + 1) = 8X(s) + 2$$

$$\begin{aligned}
Y^f(s) &= \frac{4}{s+1} + \frac{4}{s^2+1} - \frac{4s}{s^2+1} \\
y^f(t) &= 4(e^{-t} + \sin(t) - \cos(t))u(t) \\
Y^n(s) &= \frac{2}{s^2+1} \\
y^n(t) &= 2\sin(t)u(t)
\end{aligned}$$

(d) $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 5y(t) = \frac{d}{dt}x(t)$, $y(0^-) = 2$, $\frac{d}{dt}y(t)|_{t=0^-} = 0$, $x(t) = u(t)$

$$\begin{aligned}
Y(s)(s^2 + 2s + 5) &= sX(s) + sy(0^-) + 2y(0^-) \\
Y^f(s) &= \frac{1}{(s+1)^2 + 2^2} \\
y^f(t) &= \frac{1}{2}e^{-t}\sin(2t)u(t) \\
Y^n(s) &= \frac{2(s+2)}{(s+2)^2 + 1} \\
y^n(t) &= 2e^{-t}\cos(t)u(t)
\end{aligned}$$

6.39. Use Laplace transform circuit models to determine the current $y(t)$ in the circuit of Fig. P6.39 assuming normalized values $R = 1\Omega$ and $L = \frac{1}{2}$ H for the specified inputs. The current through the inductor at time $t = 0^-$ is 2 A.

$$\begin{aligned}
X(s) + Li_L(0^-) &= (R + Ls)Y(s) \\
Y(s) &= \frac{1}{L} \frac{X(s)}{s + R/L} + \frac{i_L(0^-)}{s + R/L} \\
&= \frac{2X(s)}{s+2} + \frac{2}{s+2}
\end{aligned}$$

(a) $x(t) = e^{-t}u(t)$

$$\begin{aligned}
Y(s) &= \frac{2}{(s+1)(s+2)} + \frac{2}{s+2} \\
&= \frac{2}{s+1} - \frac{2}{s+2} + \frac{2}{s+2} \\
&= \frac{2}{s+1} \\
y(t) &= 2e^{-t}u(t)
\end{aligned}$$

(b) $x(t) = \cos(t)u(t)$

$$Y(s) = \frac{2s}{(s^2+1)(s+2)} + \frac{2}{s+2}$$

$$\begin{aligned}
&= \frac{2}{5} \left[\frac{2s+1}{s^2+1} - \frac{2}{s+2} \right] + \frac{2}{s+2} \\
&= \frac{4}{5} \left(\frac{s}{s^2+1} \right) + \frac{2}{5} \left(\frac{1}{s^2+1} \right) + \frac{1}{5} \left(\frac{1}{s+2} \right) \\
y(t) &= \frac{1}{5} (4 \cos(t) + 2 \sin(t) + e^{-2t}) u(t)
\end{aligned}$$

6.40. The circuit in Fig. P6.40 represents a system with input $x(t)$ and output $y(t)$. Determine the forced response and the natural response of this system under the specified conditions.

Forced Response:

$$\begin{aligned}
X(s) &= RI(s) + Y^{(f)}(s) + \frac{1}{SC}I(s) \\
\text{but} \\
I(s) &= \frac{Y^{(f)}(s)}{sL} \\
X(s) &= R \frac{Y^{(f)}(s)}{sL} + Y^{(f)}(s) + \frac{1}{SC} \frac{Y^{(f)}(s)}{sL} \\
s^2CLX(s) &= sRCY^{(f)}(s) + s^2CLY^{(f)}(s) + Y^{(f)}(s) \\
Y^{(f)}(s) &= \frac{X(s)s^2CL}{s^2CL + sRC + 1}
\end{aligned}$$

Natural Response:

$$\begin{aligned}
0 &= RI(s) + Y^{(n)}(s) + \frac{1}{SC}I_c(s) \\
I(s) = I_L(s) + \frac{i_L(0^-)}{s} &= \frac{Y^{(n)}(s)}{sL} + \frac{i_L(0^-)}{s} \\
I_c(s) = I(s) + Cv_c(0^-) &= \frac{Y^{(n)}(s)}{sL} + \frac{i_L(0^-)}{s} + Cv_c(0^-)
\end{aligned}$$

So:

$$\begin{aligned}
R \frac{Y^{(n)}(s)}{sL} + \frac{Ri_L(0^-)}{s} + Y^{(n)}(s) + \frac{1}{SC} \left(\frac{Y^{(n)}(s)}{sL} + \frac{i_L(0^-)}{s} + Cv_c(0^-) \right) &= 0 \\
sCRY^{(n)}(s) + sRCLi_L(0^-) + s^2CLY^{(n)}(s) + Y^{(n)}(s) + Li_L(0^-) + sLCv_c(0^-) &= 0
\end{aligned}$$

$$Y^{(n)}(s) = \frac{-L(sRC + 1)i_L(0^-) - sLCv_c(0^-)}{1 + sCR + s^2CL}$$

(a) $R = 3\Omega$, $L = 1$ H, $C = \frac{1}{2}$ F, $x(t) = u(t)$, current through the inductor at $t = 0^-$ is 2 A, and the voltage across the capacitor at $t = 0^-$ is 1 V.

$$Y^{(f)}(s) = \frac{\frac{1}{2}s}{\frac{1}{2}s^2 + \frac{3}{2}s + 2}$$

$$\begin{aligned}
&= \frac{-1}{s+1} + \frac{2}{s+2} \\
y^{(f)}(t) &= (2e^{-2t} - e^{-t}) u(t) \\
Y^{(n)}(s) &= \frac{-(\frac{3}{2}s+1)2 - \frac{1}{2}s}{1 + \frac{3}{2}s + \frac{1}{2}s^2} \\
&= \frac{-(3s+2)2 - s}{s^2 + 3s + 2} \\
&= \frac{3}{s+1} - \frac{10}{s+2} \\
y^{(n)}(t) &= (3e^{-t} - 10e^{-2t}) u(t)
\end{aligned}$$

(b) $R = 2\Omega$, $L = 1$ H, $C = \frac{1}{5}$ F, $x(t) = u(t)$, current through the inductor at $t = 0^-$ is 2 A, and the voltage across the capacitor at $t = 0^-$ is 1 V.

$$\begin{aligned}
Y^{(f)}(s) &= \frac{\frac{1}{5}s}{\frac{1}{5}s^2 + \frac{2}{5}s + 1} \\
&= \frac{s+1}{(s+1)^2 + 2^2} + \frac{-1}{2} \frac{2}{(s+1)^2 + 2^2} \\
y^{(f)}(t) &= \left(e^{-t} \cos(2t) - \frac{1}{2} e^{-t} \sin(2t) \right) u(t) \\
Y^{(n)}(s) &= \frac{-(\frac{2}{5}s+1)2 - \frac{1}{5}s}{1 + \frac{2}{5}s + \frac{1}{5}s^2} \\
&= -5 \frac{s+1}{(s+1)^2 + 2^2} - \frac{5}{2} \frac{2}{(s+1)^2 + 2^2} \\
y^{(n)}(t) &= \left(-5e^{-t} \cos(2t) - \frac{5}{2} e^{-t} \sin(2t) \right) u(t)
\end{aligned}$$

6.41. Determine the bilateral Laplace transform and the corresponding ROC for the following signals:

(a) $x(t) = e^{-t/2}u(t) + e^{-t}u(t) + e^t u(-t)$

$$\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt \\
&= \int_0^{\infty} e^{-t/2} dt + \int_0^{\infty} e^{-t} dt + \int_{-\infty}^0 e^t dt \\
&= \frac{1}{s+0.5} + \frac{1}{s+1} - \frac{1}{s-1} \\
\text{ROC: } &-0.5 < \text{Re}(s) < 1
\end{aligned}$$

(b) $x(t) = e^t \cos(2t)u(-t) + e^{-t}u(t) + e^{t/2}u(t)$

$$X(s) = -\frac{s-1}{(s-1)^2 + 4} + \frac{1}{s+1} + \frac{1}{s-\frac{1}{2}}$$

$$\text{ROC: } 0.5 < \text{Re}(s) < 1$$

$$(c) \ x(t) = e^{3t+6}u(t+3)$$

$$\begin{aligned} x(t) &= e^{-3}e^{3(t+3)}u(t+3) \\ a(t) = e^{3t}u(t) &\xleftrightarrow{\mathcal{L}} A(s) = \frac{1}{s-3} \\ b(t) = a(t+3) &\xleftrightarrow{\mathcal{L}} B(s) = e^{3s}A(s) = e^{3s}\frac{1}{s-3} \\ X(s) &= \frac{e^{3(s-1)}}{s-3} \\ \text{ROC: } \text{Re}(s) &> 3 \end{aligned}$$

$$(d) \ x(t) = \cos(3t)u(-t) * e^{-t}u(t)$$

$$\begin{aligned} a(t) * b(t) &\xleftrightarrow{\mathcal{L}} A(s)B(s) \\ X(s) &= -\frac{s}{s^2+9} \left(\frac{1}{s+1} \right) \\ \text{ROC: } -1 &< \text{Re}(s) < 0 \end{aligned}$$

$$(e) \ x(t) = e^t \sin(2t+4)u(t+2)$$

$$\begin{aligned} x(t) &= e^{-2}e^{t+2}\sin(2(t+2))u(t+2) \\ a(t+2) &\xleftrightarrow{\mathcal{L}} e^{2s}A(s) \\ X(s) &= e^{2s}\frac{2e^{-2}}{(s-1)^2+4} \\ \text{ROC: } \text{Re}(s) &> 1 \end{aligned}$$

$$(f) \ x(t) = e^t \frac{d}{dt} (e^{-2t}u(-t))$$

$$\begin{aligned} a(t) = e^{-2t}u(-t) &\xleftrightarrow{\mathcal{L}} A(s) = \frac{-1}{s+2} \\ b(t) = \frac{d}{dt}a(t) &\xleftrightarrow{\mathcal{L}} B(s) = sA(s) \\ x(t) = e^tb(t) &\xleftrightarrow{\mathcal{L}} X(s) = B(s-1) = \frac{1-s}{s+1} \\ \text{ROC: } \text{Re}(s) &< -1 \end{aligned}$$

6.42. Use the tables of transforms and properties to determine the time signals that correspond to the following bilateral Laplace transforms:

(a) $X(s) = e^{5s} \frac{1}{s+2}$ with ROC $\text{Re}(s) < -2$

(left-sided)

$$A(s) = \frac{1}{s+2} \xleftarrow{\mathcal{L}} a(t) = -e^{-2t}u(-t)$$

$$X(s) = e^{5s}A(s) \xleftarrow{\mathcal{L}} x(t) = a(t+5) = -e^{-2(t+5)}u(-(t+5))$$

(b) $X(s) = \frac{d^2}{ds^2} \left(\frac{1}{s-3} \right)$ with ROC $\text{Re}(s) > 3$

(right-sided)

$$A(s) = \frac{1}{s-3} \xleftarrow{\mathcal{L}} a(t) = e^{3t}u(t)$$

$$X(s) = \frac{d^2}{ds^2}A(s) \xleftarrow{\mathcal{L}} x(t) = t^2 e^{3t}u(t)$$

(c) $X(s) = s \left(\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s} \right)$ with ROC $\text{Re}(s) < 0$

(left-sided)

$$x(t) = \frac{d}{dt} (-tu(-t) + tu(-t-1) + u(-t-2))$$

$$x(t) = -u(-t) + u(-t-1) - \delta(t+2)$$

(d) $X(s) = s^{-2} \frac{d}{ds} \left(\frac{e^{-3s}}{s} \right)$ with ROC $\text{Re}(s) > 0$

(right-sided)

$$A(s) = \frac{1}{s} \xleftarrow{\mathcal{L}} a(t) = u(t)$$

$$B(s) = e^{-3s}A(s) \xleftarrow{\mathcal{L}} b(t) = a(t-3) = u(t-3)$$

$$C(s) = \frac{d}{ds}B(s) \xleftarrow{\mathcal{L}} c(t) = -tb(t) = -tu(t-3)$$

$$D(s) = \frac{1}{s} \xleftarrow{\mathcal{L}} d(t) = \int_{-\infty}^t c(\tau) d\tau$$

$$\xleftarrow{\mathcal{L}} d(t) = -\int_3^t \tau d\tau = -\frac{1}{2}(t^2 - 9)$$

$$X(s) = \frac{1}{s}D(s) \xleftarrow{\mathcal{L}} x(t) = \int_{-\infty}^t d(\tau) d\tau$$

$$\xleftarrow{\mathcal{L}} x(t) = -\frac{1}{2} \int_3^t (\tau^2 - 9) d\tau$$

$$\xleftarrow{\mathcal{L}} x(t) = \left[-\frac{1}{6}(t^3 - 27) + \frac{9}{2}(t-3) \right] u(t-3)$$

6.43. Use the method of partial fractions to determine the time signals corresponding to the following bilateral Laplace transforms:

(a) $X(s) = \frac{-s-4}{s^2+3s+2}$

$$X(s) = \frac{-3}{s+1} + \frac{2}{s+2}$$

(i) with ROC $\text{Re}(s) < -2$

(left-sided)

$$x(t) = (3e^{-t} - 2e^{-2t}) u(-t)$$

(ii) with ROC $\text{Re}(s) > -1$

(right-sided)

$$x(t) = (-3e^{-t} + 2e^{-2t}) u(t)$$

(iii) with ROC $-2 < \text{Re}(s) < -1$

(two sided)

$$x(t) = 3e^{-t}u(-t) + 2e^{-2t}u(t)$$

(b) $X(s) = \frac{4s^2+8s+10}{(s+2)(s^2+2s+5)}$

$$X(s) = \frac{2}{s+2} + \frac{2(s+1)}{(s+1)^2+2^2} + \frac{-2}{(s+1)^2+2^2}$$

(i) with ROC $\text{Re}(s) < -2$

(left-sided)

$$x(t) = (-2e^{-2t} - 2e^{-t}\cos(2t) + e^{-t}\sin(2t)) u(-t)$$

(ii) with ROC $\text{Re}(s) > -1$

(right-sided)

$$x(t) = (2e^{-2t} + 2e^{-t}\cos(2t) - e^{-t}\sin(2t)) u(t)$$

(iii) with ROC $-2 < \text{Re}(s) < -1$

(two sided)

$$x(t) = 2e^{-2t}u(t) + (-2e^{-t}\cos(2t) + e^{-t}\sin(2t))u(-t)$$

(c) $X(s) = \frac{5s+4}{s^2+2s+1}$

$$X(s) = \frac{5}{s+1} - \frac{1}{(s+1)^2}$$

(i) with ROC $\text{Re}(s) < -1$

(left-sided)

$$x(t) = (-5e^{-t} + te^{-t})u(-t)$$

(ii) with ROC $\text{Re}(s) > -1$

(right-sided)

$$x(t) = (5e^{-t} - te^{-t})u(t)$$

(d) $X(s) = \frac{2s^2+2s-2}{s^2-1}$

$$X(s) = 2 + \frac{1}{s+1} + \frac{1}{s-1}$$

(i) with ROC $\text{Re}(s) < -1$

(left-sided)

$$x(t) = 2\delta(t) - (e^{-t} + e^t)u(t)$$

(ii) with ROC $\text{Re}(s) > 1$

(right-sided)

$$x(t) = 2\delta(t) + (e^{-t} + e^t)u(t)$$

(iii) with ROC $-1 < \text{Re}(s) < 1$

(two sided)

$$x(t) = 2\delta(t) + e^{-t}u(t) - e^t u(-t)$$

6.44. Consider the RC -circuit depicted in Fig. P6.44.

(a) Find the transfer function assuming $y_1(t)$ is the output. Plot the poles and zeros and characterize the system as lowpass, highpass, or bandpass.

$$\begin{aligned}
 x(t) &= i(t)R + \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau \\
 y_1(t) &= \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau \\
 \frac{d}{dt} y_1(t) &= \frac{1}{C} i(t) \\
 sY_1(s) &= \frac{1}{C} I(s) \\
 \frac{d}{dt} x(t) &= R \frac{d}{dt} i(t) + \frac{1}{C} i(t) \\
 sX(s) &= I(s) \left(Rs + \frac{1}{C} \right) \\
 H(s) = \frac{Y_1(s)}{X(s)} &= \frac{1}{RC} \frac{1}{s + \frac{1}{RC}}
 \end{aligned}$$

Low pass filter with a pole at $s = -\frac{1}{RC}$
 $RC = 10^{-3}$:

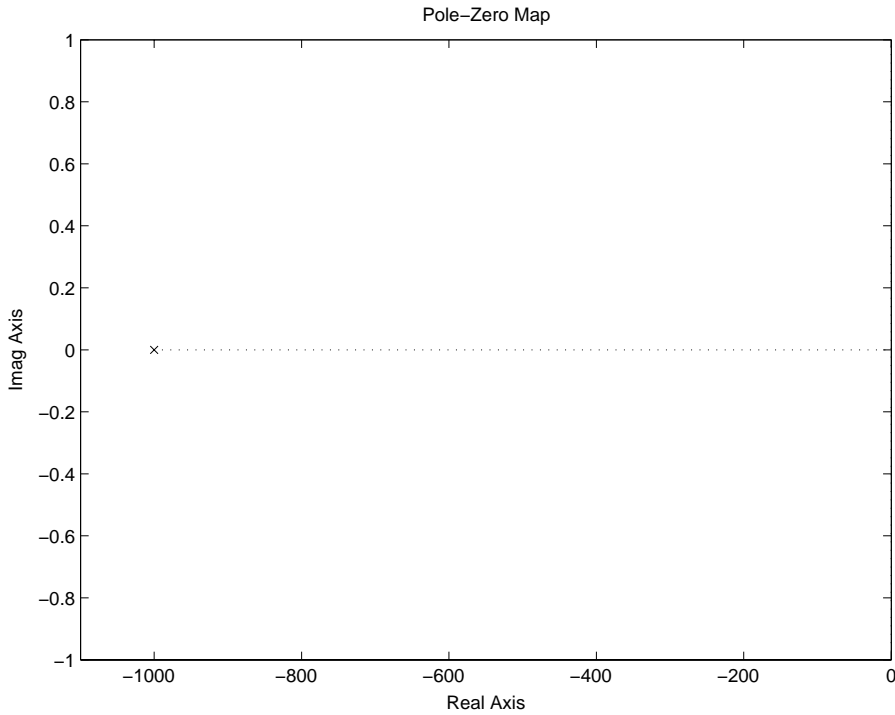


Figure P6.44. (a) Pole-Zero Plot of $H(s)$

$$\begin{aligned}
 H(j\omega) &= \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}} \\
 H(j0) &= 1
 \end{aligned}$$

$$H(j\infty) = 0$$

The filter is low pass.

(b) Repeat part (a) assuming $y_2(t)$ is the system output.

$$\begin{aligned} y_2(t) &= i(t)R \\ Y_2(s) &= I(s)R \\ x(t) &= y_2(t) + \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau \\ x(t) &= i(t)R + \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau \\ \frac{d}{dt}x(t) &= R \frac{d}{dt}i(t) + \frac{1}{C}i(t) \\ sX(s) &= I(s) \left(sR + \frac{1}{C} \right) \\ H(s) &= \frac{s}{s + \frac{1}{RC}} \end{aligned}$$

High pass filter with a zero at $s = 0$, and a pole at $s = -\frac{1}{RC}$

$RC = 10^{-3}$:

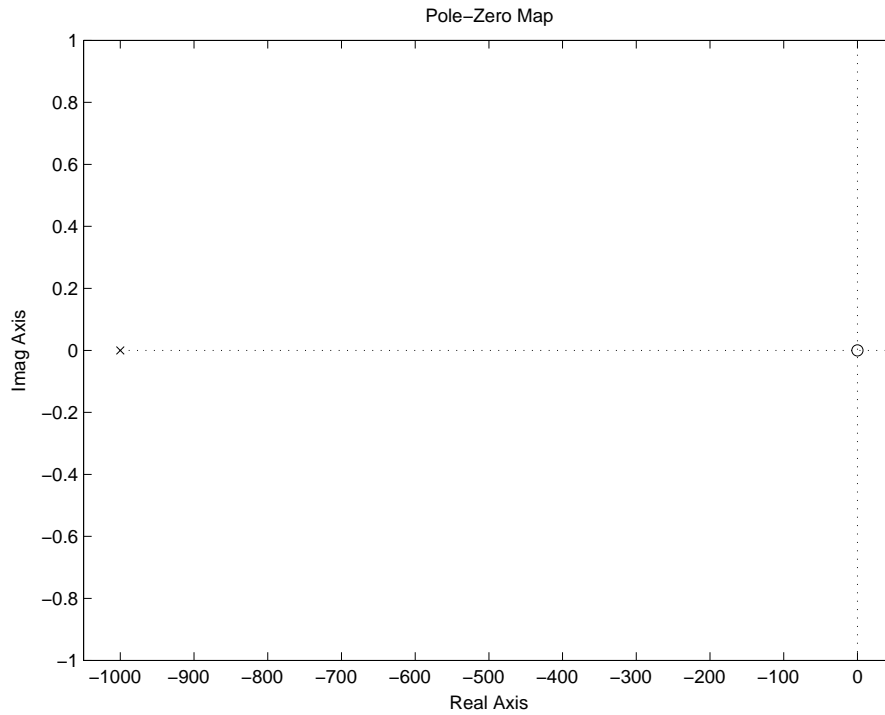


Figure P6.44. (b) Pole-Zero Plot of $H(s)$

$$\begin{aligned}
H(j\omega) &= \frac{j\omega}{j\omega + \frac{1}{RC}} \\
H(j0) &\rightarrow 0 \\
H(j\infty) &= 1
\end{aligned}$$

The filter is high pass.

(c) Find the impulse responses for the systems in parts (a) and (b).

For part (a), $h(t) = \frac{1}{RC}e^{-\frac{1}{RC}t}u(t)$.

For part (b):

$$\begin{aligned}
A(s) = \frac{1}{s + \frac{1}{RC}} &\xleftrightarrow{\mathcal{L}} a(t) = e^{-\frac{1}{RC}t}u(t) \\
H(s) = sA(s) &\xleftrightarrow{\mathcal{L}} h(t) = \frac{d}{dt}a(t) = -\frac{1}{RC}e^{-\frac{1}{RC}t}u(t) + \delta(t)
\end{aligned}$$

6.45. A system has transfer function $H(s)$ as given below. Determine the impulse response assuming

(i) that the system is causal, and (ii) that the system is stable.

(a) $H(s) = \frac{2s^2+2s-2}{s^2-1}$

$$H(s) = 2 + \frac{1}{s+1} + \frac{1}{s-1}$$

(i) system is causal

$$h(t) = 2\delta(t) + (e^{-t} + e^t)u(t)$$

(ii) system is stable

$$h(t) = 2\delta(t) + e^{-t}u(t) + -e^t u(-t)$$

(b) $H(s) = \frac{2s-1}{s^2+2s+1}$

$$H(s) = \frac{2}{s+1} + \frac{-3}{(s+1)^2}$$

(i) system is causal

$$h(t) = (2e^{-t} - 3te^{-t})u(t)$$

(ii) system is stable

$$h(t) = (2e^{-t} - 3te^{-t}) u(t)$$

(c) $H(s) = \frac{s^2+5s-9}{(s+1)(s^2-2s+10)}$

$$H(s) = \frac{-1}{s+1} + \frac{2(s-1)}{(s-1)^2+3^2} + \frac{3}{(s-1)^2+3^2}$$

(i) system is causal

$$h(t) = (-e^{-t} + 2e^t \cos(3t) + e^t \sin(3t)) u(t)$$

(ii) system is stable

$$h(t) = -e^{-t}u(t) - (2e^t \cos(3t) + e^t \sin(3t)) u(-t)$$

(d) $H(s) = e^{-5s} + \frac{2}{s-2}$

(i) system is causal

$$h(t) = \delta(t-5) + 2e^{-2t}u(t)$$

(ii) system is stable

$$h(t) = \delta(t-5) - 2e^{2t}u(-t)$$

6.46. A stable system has input $x(t)$ and output $y(t)$ as given below. Use Laplace transforms to determine the transfer function and impulse response of the system.

(a) $x(t) = e^{-t}u(t)$, $y(t) = e^{-2t} \cos(t)u(t)$

$$\begin{aligned} X(s) &= \frac{1}{s+1} \\ Y(s) &= \frac{s+2}{(s+1)^2+1} \\ H(s) &= \frac{Y(s)}{X(s)} = \frac{s^2+3s+2}{s^2+4s+5} \\ &= 1 + \frac{-(s+2)}{(s+2)^2+1} + \frac{-1}{(s+2)^2+1} \\ h(t) &= \delta(t) - (e^{-2t} \cos(t) + e^{-2t} \sin(t)) u(t) \end{aligned}$$

(b) $x(t) = e^{-2t}u(t)$, $y(t) = -2e^{-t}u(t) + 2e^{-3t}u(t)$

$$\begin{aligned} X(s) &= \frac{1}{s+2} \\ Y(s) &= \frac{-2}{s+1} + \frac{2}{s+3} \\ H(s) &= \frac{Y(s)}{X(s)} = \frac{-4(s+2)}{(s+1)(s+3)} \\ &= \frac{-2}{s+1} + \frac{-2}{s+3} \\ h(t) &= (-2e^{-t} - 2e^{-3t})u(t) \end{aligned}$$

6.47. The relationship between the input $x(t)$ and output $y(t)$ of a causal system is described by the differential equation given below. Use Laplace transforms to determine the transfer function and impulse response of the system.

(a) $\frac{d}{dt}y(t) + 10y(t) = 10x(t)$

$$\begin{aligned} sY(s) + 10Y(s) &= 10X(s) \\ H(s) = \frac{Y(s)}{X(s)} &= \frac{10}{s+10} \\ h(t) &= 10e^{-10t}u(t) \end{aligned}$$

(b) $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = x(t) + \frac{d}{dt}x(t)$

$$\begin{aligned} Y(s)(s^2 + 5s + 6) &= X(s)(1 + s) \\ H(s) &= \frac{s+1}{(s+3)(s+2)} \\ &= \frac{-1}{s+2} + \frac{2}{s+3} \\ h(t) &= (2e^{-3t} - e^{-2t})u(t) \end{aligned}$$

(c) $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 2y(t) = -4x(t) + 5\frac{d}{dt}x(t)$

$$\begin{aligned} Y(s)(s^2 - s - 2) &= X(s)(5s - 4) \\ H(s) &= \frac{5s - 4}{(s-2)(s+1)} \\ &= \frac{3}{s+1} + \frac{2}{s-2} \\ h(t) &= (3e^{-t} + 2e^{2t})u(t) \end{aligned}$$

6.48. Determine a differential equation description for a system with the following transfer function.

(a) $H(s) = \frac{1}{s(s+3)}$

$$\begin{aligned}
H(s) = \frac{Y(s)}{X(s)} &= \frac{1}{s(s+3)} \\
Y(s)(s^2 + 3s) &= X(s) \\
\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) &= x(t)
\end{aligned}$$

(b) $H(s) = \frac{6s}{s^2 - 2s + 8}$

$$\begin{aligned}
Y(s)(s^2 - 2s + 8) &= X(s)(6s) \\
\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + 8y(t) &= 6\frac{d}{dt}x(t)
\end{aligned}$$

(c) $H(s) = \frac{2(s-2)}{(s+1)^2(s+3)}$

$$\begin{aligned}
Y(s)(s^3 + 5s^2 + 7s + 3) &= X(s)(3s^2 + 6s + 3) \\
\frac{d^3}{dt^3}y(t) + 5\frac{d^2}{dt^2}y(t) + 7\frac{d}{dt}y(t) + 3y(t) &= 2\frac{d}{dt}x(t) - 4x(t)
\end{aligned}$$

6.49. (a) Use the time-differentiation property to show that the transfer function of a LTI system is expressed in terms of the state-variable description as shown by

$$H(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D$$

$$\begin{aligned}
\frac{d}{dt}\mathbf{q}(t) &= \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t) \\
s\mathbf{Q}(s) &= \mathbf{A}\mathbf{Q}(s) + \mathbf{b}X(s) \\
\mathbf{Q}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}X(s) \\
y(t) &= \mathbf{c}\mathbf{q}(t) + Dx(t) \\
Y(s) &= \mathbf{c}\mathbf{Q}(s) + DX(s) \\
H(s) &= \frac{Y(s)}{X(s)} = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D
\end{aligned}$$

(b) Determine the transfer function, impulse response, and differential equation descriptions for a stable LTI system represented by the following state variable descriptions:

(i) $\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $D = [0]$

$$\begin{aligned}
H(s) &= \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D \\
&= \frac{s+3}{s^2 + 3s + 1} \\
&= \frac{2}{s+1} + \frac{-1}{s+2}
\end{aligned}$$

$$\begin{aligned}
h(t) &= (2e^{-t} - e^{-2t})u(t) \\
\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + y(t) &= \frac{d}{dt}x(t) + 3x(t)
\end{aligned}$$

$$(ii) \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & -6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = [0]$$

$$\begin{aligned}
H(s) &= \frac{2s-1}{s^2+5s-8} \\
&= \frac{2(s+2.5)-6}{(s+2.5)^2-14.25} \\
&= \frac{2(s+2.5)}{(s+2.5)^2-14.25} + \frac{-6}{(s+2.5)^2-14.25} \\
h(t) &= \left[2e^{-2.5t} \cos(t\sqrt{14.25}) - \frac{6}{\sqrt{14.25}} e^{-2.5t} \sin(t\sqrt{14.25}) \right] u(t) \\
\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) - 8y(t) &= 2\frac{d}{dt}x(t) - x(t)
\end{aligned}$$

6.50. Determine whether the systems described by the following transfer functions are (i) both stable and causal, and (ii) whether a stable and causal inverse system exists:

$$(a) H(s) = \frac{(s+1)(s+2)}{(s+1)(s^2+2s+10)}$$

$$\begin{aligned}
H(s) &= \frac{s+2}{s^2+2s+10} \\
\text{zero at:} & -2 \\
\text{poles at:} & -1 \pm 3j
\end{aligned}$$

- (i) All poles are in the LHP, and with ROC: $\text{Re}(s) > -1$, the system is both stable and causal.
- (ii) All zeros are in the LHP, so a stable and causal inverse system exists.

$$(b) H(s) = \frac{s^2+2s-3}{(s+3)(s^2+2s+5)}$$

$$\begin{aligned}
H(s) &= \frac{s-1}{s^2+2s+5} \\
\text{zero at:} & 1 \\
\text{poles at:} & -1 \pm 2j
\end{aligned}$$

- (i) All poles are in the LHP, and with ROC: $\text{Re}(s) > -1$, the system is both stable and causal.
- (ii) Not all zeros are in the LHP, so no stable and causal inverse system exists.

$$(c) H(s) = \frac{s^2-3s+2}{(s+2)(s^2-2s+8)}$$

$$H(s) = \frac{s^2-3s+2}{s^3+4s+16}$$

$$\begin{aligned}\text{zeros at:} & \quad 1, 2 \\ \text{poles at:} & \quad -2, 1 \pm j\sqrt{7}\end{aligned}$$

- (i) Not all poles are in the LHP, so the system is not stable and causal.
(ii) No zeros are in the LHP, so no stable and causal inverse system exists.

$$(d) H(s) = \frac{s^2 + 2s}{(s^2 + 3s - 2)(s^2 + s + 2)}$$

$$\begin{aligned}H(s) &= \frac{s^2 + 2s}{s^4 + 4s^3 + 3s^2 + 4s - 4} \\ \text{zeros at:} & \quad 0, -2 \\ \text{poles at:} & \quad \frac{-3 \pm \sqrt{17}}{2}, -0.5 \pm j\sqrt{\frac{7}{4}}\end{aligned}$$

- (i) Not all poles are in the LHP, so the system is not stable and causal.
(ii) There is a zero at $s = 0$, so no stable and causal inverse system exists.

6.51. The relationship between the input $x(t)$ and output $y(t)$ of a system is described by the differential equation

$$\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + 5y(t) = \frac{d^2}{dt^2}x(t) - 2\frac{d}{dt}x(t) + x(t)$$

- (a) Does this system have a stable and causal inverse? why?

$$\begin{aligned}Y(s)(s^2 + s + 5) &= X(s)(s^2 - 2s + 1) \\ H(s) &= \frac{(s - 1)^2}{s^2 + s + 5}\end{aligned}$$

Since $H(s)$ has zeros in the RHP, this system does not have a causal and stable inverse.

- (b) Find a differential equation description for the inverse system.

$$\begin{aligned}H^{inv}(s) &= \frac{1}{H(s)} \\ &= \frac{s^2 + s + 5}{s^2 - 2s + 1}\end{aligned}$$

Inverse system:

$$\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + y(t) = \frac{d^2}{dt^2}x(t) + \frac{d}{dt}x(t) + 5x(t)$$

6.52. A stable, causal system has a rational transfer function $H(s)$. The system satisfies the following conditions: (i) The impulse response $h(t)$ is real valued; (ii) $H(s)$ has exactly two zeros, one of which is at $s = 1 + j$; (iii) The signal $\frac{d^2}{dt^2}h(t) + 3\frac{d}{dt}h(t) + 2h(t)$ contains an impulse and doublet of unknown

strengths and a unit amplitude step. Find $H(s)$.

$$\begin{aligned}\frac{d^2}{dt^2}h(t) + 3\frac{d}{dt}h(t) + 2h(t) &= b\delta^{(1)}(t) + a\delta(t) + u(t) \\ H(s)(s^2 + 3s + 2) &= bs + a + \frac{1}{s} \\ H(s) &= \frac{bs + a + \frac{1}{s}}{s^2 + 3s + 2} \\ &= \frac{bs^2 + as + 1}{s(s+2)(s+1)}\end{aligned}$$

One zero is $1 + j$ and $h(t)$ is real valued, which implies zeros occur in conjugate pairs, so the other zero is $1 - j$.

$$H(s) = \frac{\frac{1}{2}s^2 - s + 1}{s(s+2)(s+1)}$$

6.53. Sketch the magnitude response for the systems described by the following transfer functions using the relationship between the pole and zero locations and the $j\omega$ axis in the s -plane.

(a) $H(s) = \frac{s}{s^2 + 2s + 101}$

$$|H(j\omega)| = \frac{|\omega|}{|j(\omega - 10) + 1||j(\omega + 10) + 1|}$$

(b) $H(s) = \frac{s^2 + 16}{s + 1}$

$$|H(j\omega)| = \frac{|j(\omega + 4)||j(\omega - 4)|}{|j\omega - 1|}$$

(c) $H(s) = \frac{s-1}{s+1}$

$$|H(j\omega)| = \frac{|j\omega - 1|}{|j\omega + 1|}$$

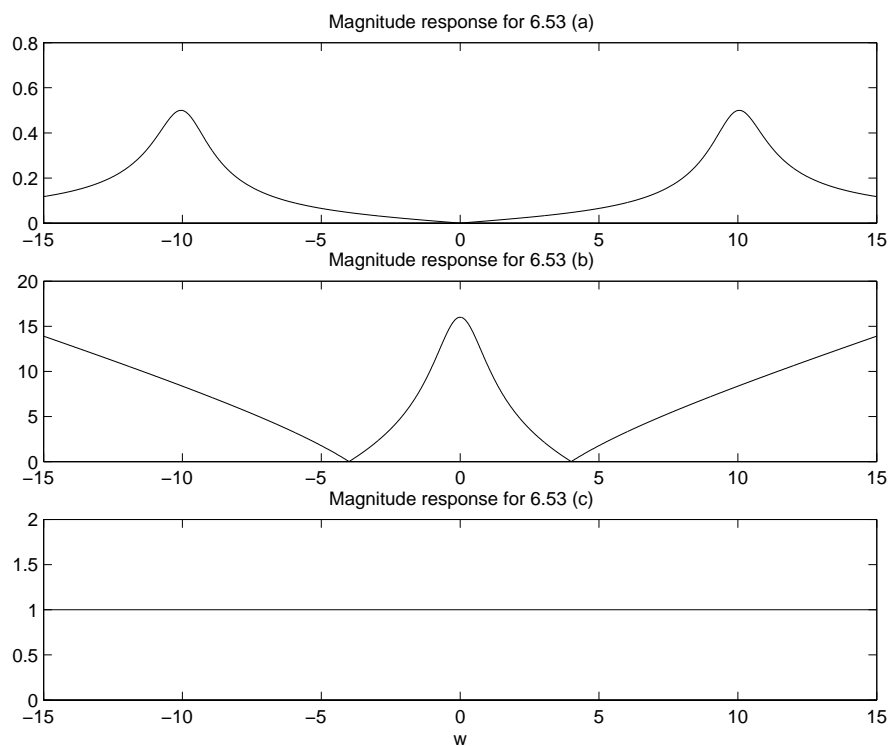


Figure P6.53. Magnitude response for the systems

6.54. Sketch the phase response for the systems described by the following transfer functions using the relationship between the pole and zero locations and the $j\omega$ axis in the s -plane.

(a) $H(s) = \frac{s-1}{s+2}$

$$\angle H(j\omega) = \pi - \arctan(\omega) - \arctan\left(\frac{\omega}{2}\right)$$

(b) $H(s) = \frac{s+1}{s+2}$

$$\angle H(j\omega) = \arctan(\omega) - \arctan\left(\frac{\omega}{2}\right)$$

(c) $H(s) = \frac{1}{s^2+2s+17}$

$$\begin{aligned} H(s) &= \frac{1}{(s+1)^2+4^2} \\ H(j\omega) &= \frac{1}{(j\omega+1+j4)(j\omega+1-j4)} \\ \angle H(j\omega) &= -[\arctan(\omega+4) + \arctan(\omega-4)] \end{aligned}$$

(d) $H(s) = s^2$

$$H(j\omega) = -\omega^2$$

$$\angle H(j\omega) = \pi$$

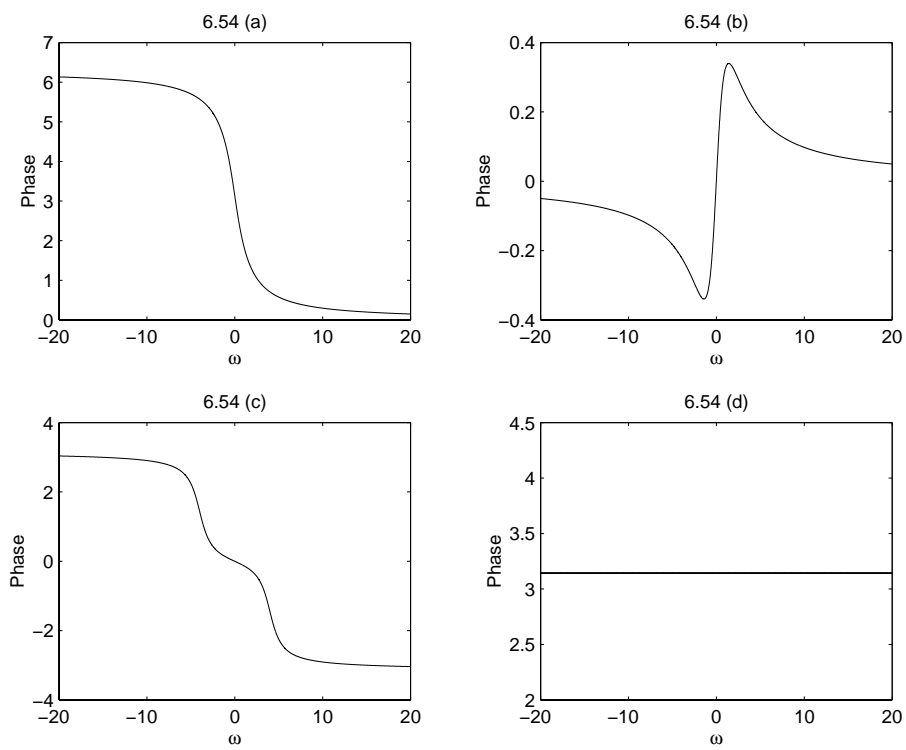


Figure P6.54. Phase Plot of $H(s)$

6.55. Sketch the Bode diagrams for the systems described by the following transfer functions.

- (a) $H(s) = \frac{50}{(s+1)(s+10)}$
- (b) $H(s) = \frac{20(s+1)}{s^2(s+10)}$
- (c) $H(s) = \frac{5}{(s+1)^3}$
- (d) $H(s) = \frac{s+2}{s^2+s+100}$
- (e) $H(s) = \frac{s+2}{s^2+10s+100}$

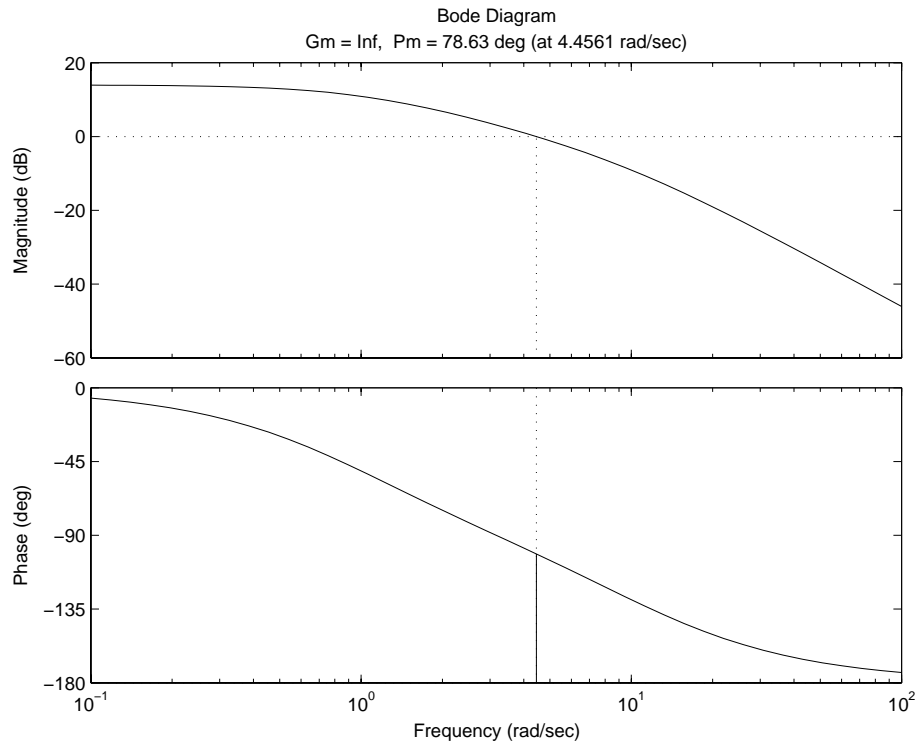


Figure P6.55. (a) Bode Plot of $H(s)$

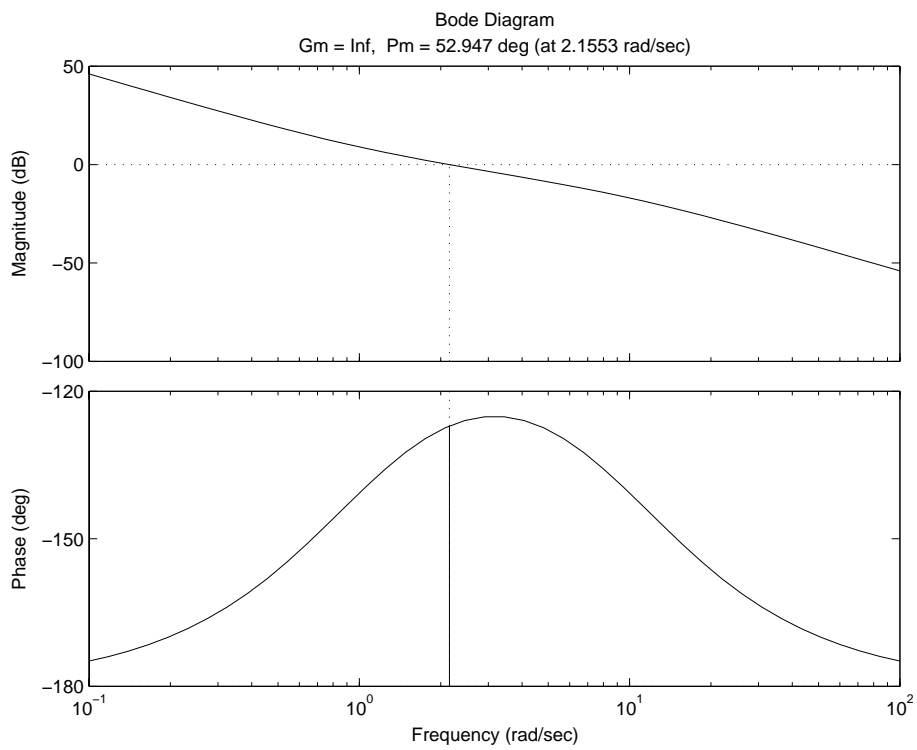


Figure P6.55. (b) Bode Plot of $H(s)$

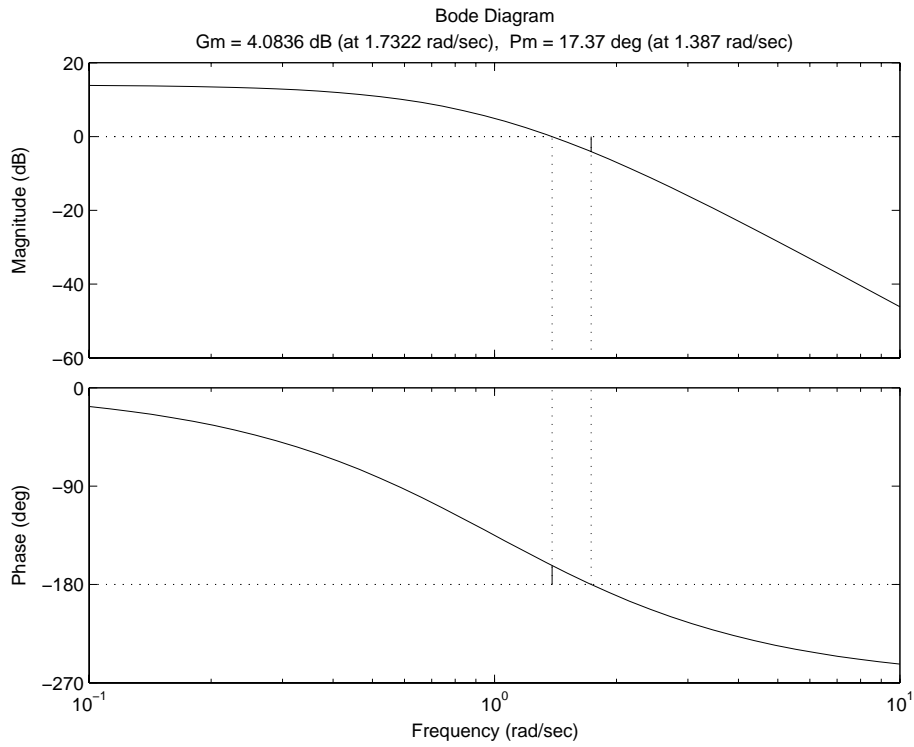


Figure P6.55. (c) Bode Plot of $H(s)$

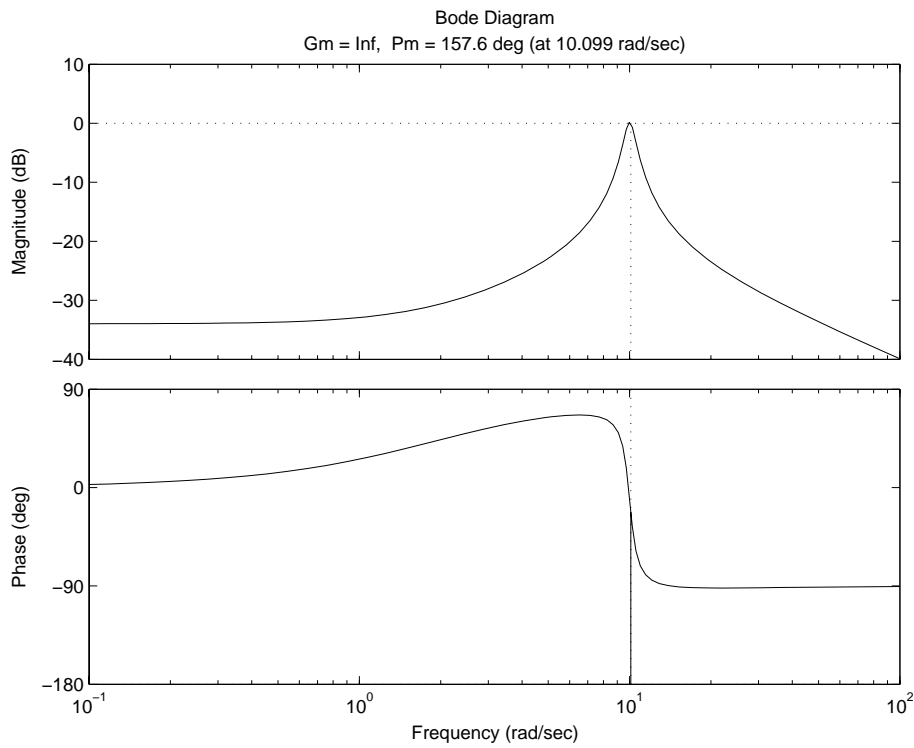


Figure P6.55. (d) Bode Plot of $H(s)$

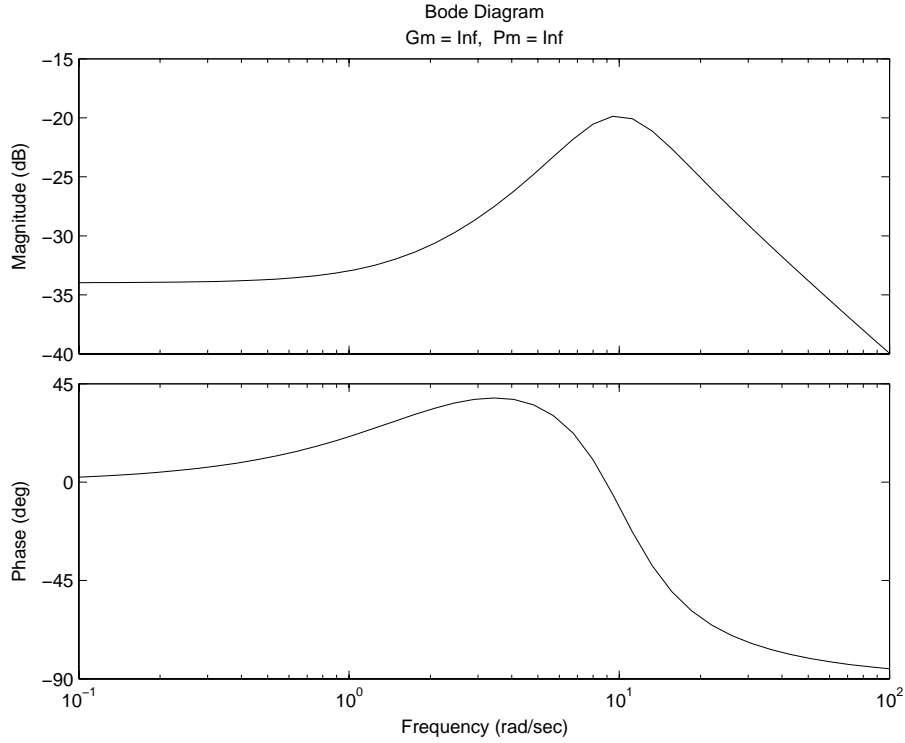


Figure P6.55. (e) Bode Plot of $H(s)$

6.56. The output of a multipath system $y(t)$ may be expressed in terms of the input $x(t)$ as

$$y(t) = x(t) + ax(t - T_{diff})$$

where a and T_{diff} respectively represent the relative strength and time delay of the second path.

(a) Find the transfer function of the multipath system.

$$\begin{aligned} Y(s) &= X(s) + ae^{-sT_{diff}}X(s) \\ H(s) &= 1 + ae^{-sT_{diff}} \end{aligned}$$

(b) Express the transfer function of the inverse system as an infinite sum using the formula for summing a geometric series.

$$\begin{aligned} H^{-1}(s) = \frac{1}{H(s)} &= \frac{1}{1 + ae^{-sT_{diff}}} \\ &= \sum_{n=0}^{\infty} (-ae^{-sT_{diff}})^n \end{aligned}$$

(c) Determine the impulse response of the inverse system. What condition must be satisfied for the inverse system to be both stable and causal?

$$h^{inv}(t) = \sum_{n=0}^{\infty} (-a)^n \delta(t - nT_{diff})$$

$|a| < 1$ for the system to be both stable and causal.

(d) Find a stable inverse system assuming the condition determined in part (c) is violated.

The following system is stable, but not causal.

$$H^{inv}(s) = \frac{1}{1 + ae^{-sT_{diff}}}$$

Use long division in the positive powers of $e^{sT_{diff}}$, i.e., divide 1 by $ae^{-sT_{diff}} + 1$. This yields:

$$\begin{aligned} H^{inv}(s) &= -\sum_{n=1}^{\infty} \left(-\frac{1}{a}\right)^n e^{nsT_{diff}} \\ h^{inv}(t) &= -\sum_{n=1}^{\infty} \left(-\frac{1}{a}\right)^n \delta(t + nT_{diff}) \end{aligned}$$

6.57. In Section 2.12 we derived block-diagram descriptions for systems described by linear constant-coefficient differential equations by rewriting the differential equation as an integral equation. Consider the second-order system with the integral equation description

$$y(t) = -a_1 y^{(1)}(t) - a_0 y^{(2)}(t) + b_2 x(t) + b_1 x^{(1)}(t) + b_0 x^{(2)}(t)$$

Recall that $v^{(n)}(t)$ is the n -fold integral of $v(t)$ with respect to time. Use the integration property to take the Laplace transform of the integral equation and derive the direct form I and II block diagrams for the transfer function of this system.

$$Y(s) = -a_1 \frac{Y(s)}{s} - a_0 \frac{Y(s)}{s^2} + b_2 X(s) + b_1 \frac{X(s)}{s} + b_0 \frac{X(s)}{s^2}$$

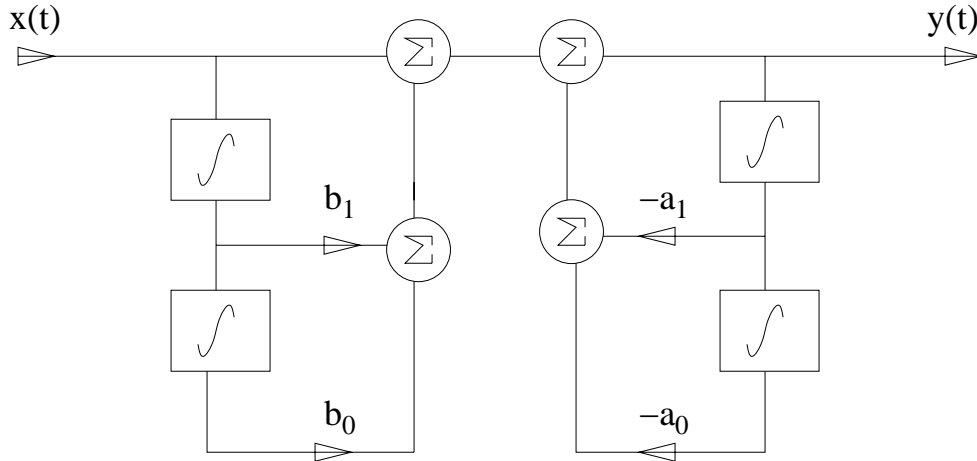


Figure P6.57. Direct Form I

By the properties of linear systems, $H_1(s)$ can be interchanged with $H_2(s)$, which leads to direct form II after combining the integrators.

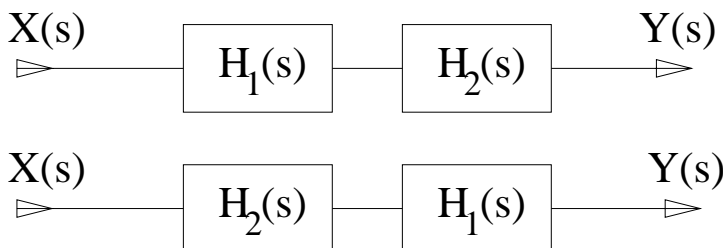


Figure P6.57. Properties of Linear Systems

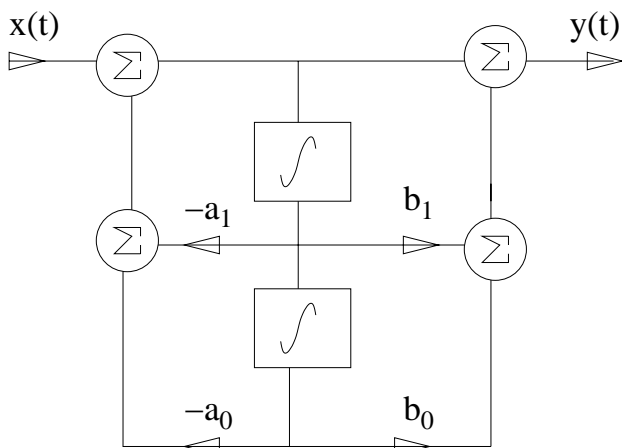


Figure P6.57. Direct Form II

Solutions to Advanced Problems

6.58. Prove the initial value theorem by assuming $x(t) = 0$ for $t < 0$ and taking the Laplace transform of the Taylor series expansion of $x(t)$ about $t = 0^+$.

Assume that there exists function $f(t)$ such that $x(t) = f(t)u(t)$. The Taylor series expansion of $f(t)$ is

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0^+)}{n!} (t - 0^+)^n \\ &= f(0^+) + \frac{f'(0^+)}{1!} (t - 0^+) + \frac{f''(0^+)}{2!} (t - 0^+)^2 + \dots \end{aligned}$$

Assuming the expansion of t is around $a = 0^+$. The Laplace transform for $x(t)$ is thus:

$$x(t) = f(0^+)u(t) + \frac{f'(0^+)}{1!} (t - 0^+)u(t) + \frac{f''(0^+)}{2!} (t - 0^+)^2 u(t) + \dots$$

$$\begin{aligned}
X(s) &= \frac{f(0^+)}{s} + \frac{f'(0^+)}{s^2} - \frac{2f''(0^+)}{s^3} + \dots \\
sX(s) &= f(0^+) + \frac{f'(0^+)}{s} - \frac{2f''(0^+)}{s^2} + \dots
\end{aligned}$$

Thus:

$$\lim_{s \rightarrow \infty} sX(s) = f(0^+)$$

6.59. The system with impulse response $h(t)$ is causal and stable and has a rational transfer function. Identify the conditions on the transfer function so that the system with impulse response $g(t)$ is stable and causal, where

(a) $g(t) = \frac{d}{dt}h(t)$

$$g(t) = \frac{d}{dt}h(t) \xrightarrow{\mathcal{L}} G(s) = sH(s) - h(0^-) = sH(s)$$

All poles of $H(s)$ are in the left half plane, so no conditions are needed.

(b) $g(t) = \int_{-\infty}^t h(\tau) d\tau$

$$g(t) = \int_{-\infty}^t h(\tau) d\tau \xrightarrow{\mathcal{L}} G(s) = \frac{1}{s} \int_{-\infty}^{0^-} h(\tau) d\tau + \frac{H(s)}{s} = \frac{H(s)}{s}$$

$H(s)$ must have at least one zero at $s = 0$ for the transfer function to be stable.

6.60. Use the continuous-time representation $x_\delta(t)$ for the discrete-time signal $x[n]$ introduced in Section 4.4 to determine the Laplace transforms of the following discrete-time signals.

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \xrightarrow{\mathcal{L}} X_\delta(s) = \sum_{n=-\infty}^{\infty} x[n]e^{-snT_s}$$

(a) $x[n] = \begin{cases} 1, & -2 \leq n \leq 2 \\ 0, & \text{otherwise} \end{cases}$

$$X_\delta(s) = e^{2sT_s} + e^{sT_s} + 1 + e^{-sT_s} + e^{-2sT_s}$$

(b) $x[n] = (1/2)^n u[n]$

$$\begin{aligned}
X_\delta(s) &= \sum_{n=-\infty}^{\infty} x[n]e^{-snT_s} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-snT_s}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-sT_s} \right)^n \\
&= \frac{1}{1 - \frac{1}{2} e^{-sT_s}}
\end{aligned}$$

(c) $x[n] = e^{-2t}u(t)|_{t=nT}$

$$\begin{aligned}
X_{\delta}(s) &= \sum_{n=0}^{\infty} e^{-2nT_s} e^{-snT_s} \\
&= \sum_{n=0}^{\infty} \left(e^{-T_s(2+s)} \right)^n \\
&= \frac{1}{1 - e^{-T_s(2+s)}}
\end{aligned}$$

6.61. The autocorrelation function for a signal $x(t)$ is defined as

$$r(t) = \int_{-\infty}^{\infty} x(\tau)x(t+\tau)d\tau$$

(a) Write $r(t) = x(t) * h(t)$. Express $h(t)$ in terms of $x(t)$. The system with impulse response $h(t)$ is called a matched filter for $x(t)$.

$$\begin{aligned}
&\text{let:} \\
h(t) &= x(-t) \\
h(t) * x(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(-\tau)x(t-\tau)d\tau \\
&\quad \text{let } \gamma = -\tau \\
&= - \int_{\infty}^{-\infty} x(\gamma)x(t+\gamma)d\gamma \\
&= \int_{-\infty}^{\infty} x(\gamma)x(t+\gamma)d\gamma
\end{aligned}$$

(b) Use the result from part (a) to find the Laplace transform of $r(t)$.

$$\begin{aligned}
x(-t) &\xleftrightarrow{\mathcal{L}} X(-s) \\
r(t) = x(t) * x(-t) &\xleftrightarrow{\mathcal{L}} R(s) = X(s)X(-s)
\end{aligned}$$

(c) If $x(t)$ is real and $X(s)$ has two poles, one of which is located at $s = \sigma_p + j\omega_p$, determine the location of all the poles of $R(s)$.

Since $x(t)$ is real, then the poles of $X(s)$ are conjugate symmetric, thus $s = \sigma_p \pm j\omega_p$. Therefore the poles of $X(-s)$ are $s = -\sigma_p \pm j\omega_p$, which implies that the poles of $R(s)$ are at $s = \sigma_p \pm j\omega_p, -\sigma_p \pm j\omega_p$.

6.62. Suppose a system has M poles at $d_k = \alpha_k + j\beta_k$ and M zeros at $c_k = -\alpha_k + j\beta_k$. That is, the pole and zero locations are symmetric about the $j\omega$ -axis.

(a) Show that the magnitude response of any system that satisfies this condition is unity. Such a system is termed an **all-pass** system since it passes all frequencies with unit gain.

$$\begin{aligned}
 H(s) &= \frac{\prod_{k=1}^M (s - c_k)}{\prod_{k=1}^M (s - d_k)} \\
 |H(j\omega)| &= \frac{\prod_{k=1}^M |j\omega - \alpha_k - j\beta_k|}{\prod_{k=1}^M |j\omega + \alpha_k - j\beta_k|} \\
 |H(j\omega)| &= \frac{\prod_{k=1}^M |j(\omega - \beta_k) - \alpha_k|}{\prod_{k=1}^M |j(\omega - \beta_k) + \alpha_k|} \\
 |H(j\omega)| &= \frac{\prod_{k=1}^M \sqrt{(\omega - \beta_k)^2 - \alpha_k^2}}{\prod_{k=1}^M \sqrt{(\omega - \beta_k)^2 + \alpha_k^2}} \\
 H(s) &= 1
 \end{aligned}$$

(b) Evaluate the phase response of a single real pole-zero pair, that is, sketch the phase response of $\frac{s-\alpha}{s+\alpha}$ where $\alpha > 0$.

$$\begin{aligned}
 H(s) &= \frac{s - \alpha}{s + \alpha} \\
 H(j\omega) &= \frac{j\omega - \alpha}{j\omega + \alpha} \\
 \angle H(j\omega) &= \pi - \arctan \frac{\omega}{\alpha} - \arctan \frac{\omega}{\alpha} \\
 \angle H(j\omega) &= \pi - 2 \arctan \frac{\omega}{\alpha}
 \end{aligned}$$

For $\alpha = 1$, then $H(j\alpha) = \frac{\pi}{2}$, and $H(-j\alpha) = \frac{3\pi}{2}$.

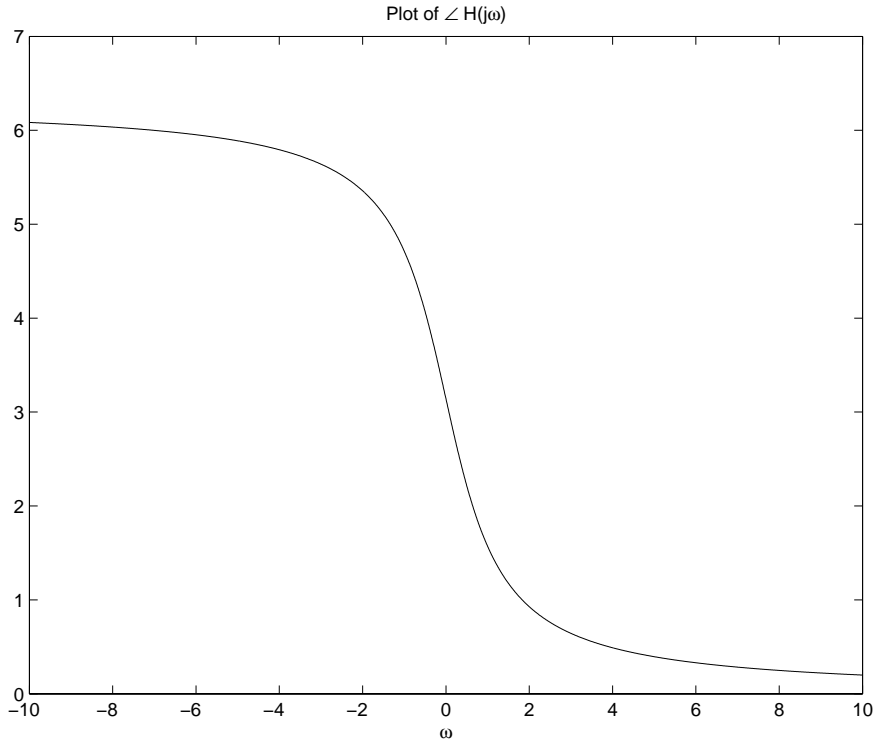


Figure P6.62. Phase Plot of $H(j\omega)$

6.63. Consider the nonminimum phase system described by the transfer function

$$H(s) = \frac{(s+2)(s-1)}{(s+4)(s+3)(s+5)}$$

(a) Does this system have a stable and causal inverse system?

The zeros of $H(s)$: $s = -2, 1$. Since one of the zeros is in the right half plane, the inverse system can not be stable and causal.

(b) Express $H(s)$ as the product of a minimum phase system, $H_{min}(s)$, and an all-pass system, $H_{ap}(s)$ containing a single pole and zero. (See Problem 6.62 for the definition of an all-pass system.)

$$\begin{aligned} H_{min}(s) &= \frac{(s+2)(s-1)}{(s+4)(s+3)(s+5)} \\ H_{ap}(s) &= \frac{s-1}{s+1} \\ H(s) &= H_{min}(s)H_{ap}(s) \end{aligned}$$

(c) Let $H_{min}^{inv}(s)$ be the inverse system for $H_{min}(s)$. Find $H_{min}^{inv}(s)$. Can it be both stable and causal?

$$H_{min}^{inv}(s) = \frac{(s+4)(s+3)(s+5)}{(s+2)(s-1)}$$

The poles of $H_{min}^{inv}(s)$ are: $s = -1, -2$. All are in the left half plane, so $h_{min}^{inv}(t)$ can be both causal and stable.

(d) Sketch the magnitude response and phase response of the system $H(s)H_{min}^{inv}(s)$.

$$\begin{aligned}
 H(s)H_{min}^{inv}(s) &= \frac{s-1}{s+1} \\
 &= H_{ap}(s) \\
 H_{ap}(j\omega) &= \frac{j\omega-1}{j\omega+1} \\
 |H_{ap}(j\omega)| &= 1 \\
 \angle H_{ap}(j\omega) &= \pi - \arctan(\omega) - \arctan(\omega) \\
 &= \pi - 2\arctan(\omega)
 \end{aligned}$$

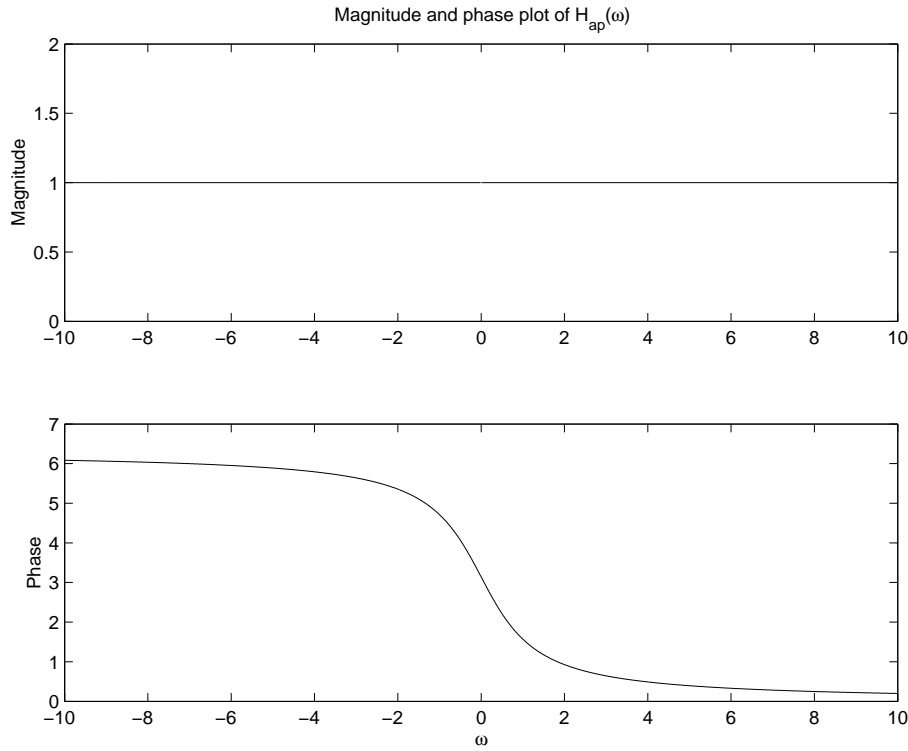


Figure P6.63. Magnitude and Phase Plot of $H_{ap}(j\omega)$

(e) Generalize your results from parts (b) and (c) to an arbitrary nonminimum phase system $H(s)$ and determine the magnitude response of the system $H(s)H_{min}^{inv}(s)$.

Generalization for $H(s) = H'(s)(s-c)$ where $H'(s)$ is the minimum phase part, assume $\text{Re}(c) > 0$

$$\begin{aligned}
 H_{min}(s) &= H'(s)(s+c) \\
 H_{ap}(s) &= \frac{s-c}{s+c} \\
 H_{min}^{inv}(s) &= \frac{1}{H'(s)(s+c)} \\
 H(s)H_{min}^{inv}(s) &= H_{ap}(s) \\
 H_{ap}(j\omega) &= \frac{j\omega-c}{j\omega+c} \\
 |H_{ap}(j\omega)| &= 1
 \end{aligned}$$

$$\angle H_{ap}(j\omega) = \pi - \arctan\left(\frac{\omega}{c}\right)$$

6.64. An N -th order lowpass **Butterworth** filter has squared magnitude response

$$|H(j\omega)|^2 = \frac{1}{1 + (j\omega/j\omega_c)^{2N}}$$

The Butterworth filter is said to be maximally flat because the first $2N$ derivatives of $|H(j\omega)|^2$ are zero at $\omega = 0$. The cutoff frequency, defined as the value for which $|H(j\omega)|^2 = 1/2$, is $\omega = \omega_c$. Assuming the impulse response is real, then the conjugate symmetry property of the Fourier transform may be used to write $|H(j\omega)|^2 = H(j\omega)H^*(j\omega) = H(j\omega)H(-j\omega)$. Noting that $H(s)|_{s=j\omega} = H(j\omega)$, we conclude that the Laplace transform of the Butterworth filter is characterized by the equation

$$H(s)H(-s) = \frac{1}{1 + (s/j\omega_c)^{2N}}$$

(a) Find the poles and zeros of $H(s)H(-s)$ and sketch them in the s -plane.

The roots of the denominator polynomial are located at the following points in the s -plane:

$$\begin{aligned} s &= j\omega_c(-1)^{\frac{1}{2N}} \\ &= \omega_c e^{j\pi \frac{(2k+N-1)}{2N}} \text{ for } k = 0, 1, \dots, 2N-1 \end{aligned}$$

Assuming $N = 3$ and $\omega_c = 1$:

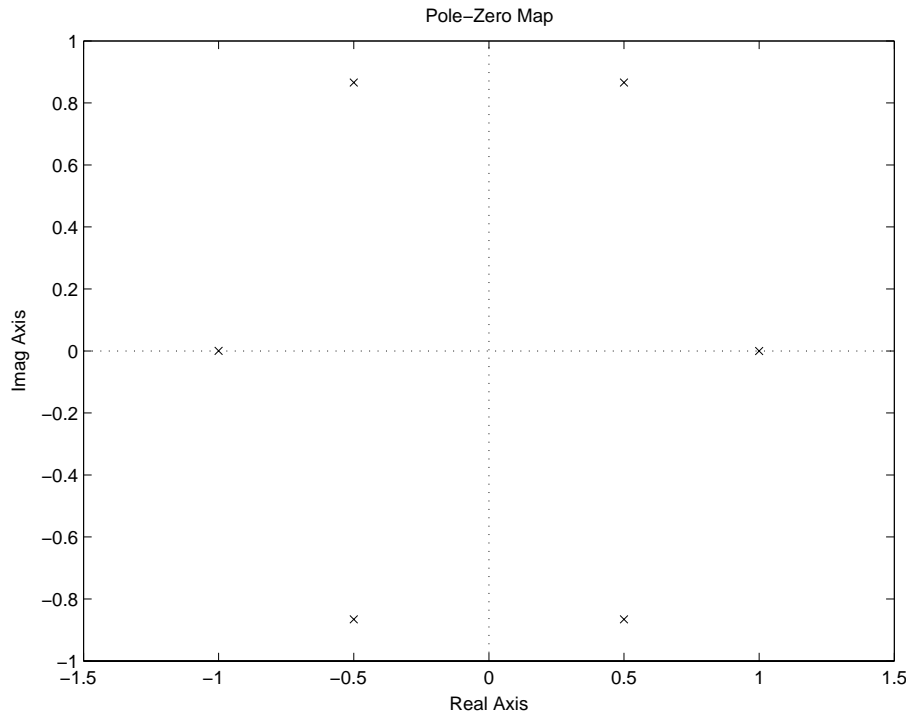


Figure P6.64. (a) Pole-Zero Plot of $H(s)H(-s)$

(b) Choose the poles and zeros of $H(s)$ so that the impulse response is both stable and causal. Note that if s_p is a pole or zero of $H(s)$, then $-s_p$ is a pole or zero of $H(-s)$.

From the graph in part (a), picking the poles in the left half plane makes the impulse response stable

and causal. This implies selecting the poles for $k = 0, 1, \dots, N - 1$.

(c) Note that $H(s)H(-s)|_{s=0} = 1$. Find $H(s)$ for $N = 1$ and $N = 2$.

For $N = 1$, the poles are at:

$$s_0 = \omega_c$$

$$s_1 = -\omega_c$$

which implies:

$$H(s) = \frac{1}{s + \omega_c}$$

For $N = 2$, the poles are at:

$$s_0 = \omega_c e^{j\pi\frac{1}{4}}$$

$$s_1 = \omega_c e^{j\pi\frac{3}{4}}$$

$$s_2 = \omega_c e^{j\pi\frac{5}{4}}$$

$$s_3 = \omega_c e^{j\pi\frac{7}{4}}$$

which implies:

$$H(s) = \frac{1}{(s - \omega_c e^{j\pi\frac{3}{4}})(s - \omega_c e^{j\pi\frac{5}{4}})}$$

(d) Find the third-order differential equation that describes a Butterworth filter with cutoff frequency $\omega_c = 1$. For $N = 3$, the $2N = 6$ poles of $H(s)H(-s)$ are located on a circle of unit radius with angular spacing of 60 degrees. Hence the left half plane poles of $H(s)$ are:

$$s_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$s_2 = -1$$

$$s_3 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

The transfer function is therefore:

$$\begin{aligned} H(s) &= \frac{1}{(s + 1)(s + \frac{1}{2} + j\frac{\sqrt{3}}{2})(s - \frac{1}{2} - j\frac{\sqrt{3}}{2})} \\ &= \frac{1}{s^3 + 2s^2 + 2s + 1} \end{aligned}$$

Which implies

$$\frac{d^3}{dt^3}y(t) + 2\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = x(t)$$

6.65. It is often convenient to change the cutoff frequency of a filter, or change a lowpass filter to a highpass filter. Consider a system described by the transfer function

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

(a) Find the poles and zeros and sketch the magnitude response of this system. Determine whether this system is lowpass or highpass, and find the cutoff frequency (the value of ω for which $|H(j\omega)| = 1/\sqrt{2}$).

$$\text{poles at } s : = -1, \frac{-1 \pm \sqrt{3}}{2}$$

This system is lowpass with cutoff frequency at $\omega_c = 1$.

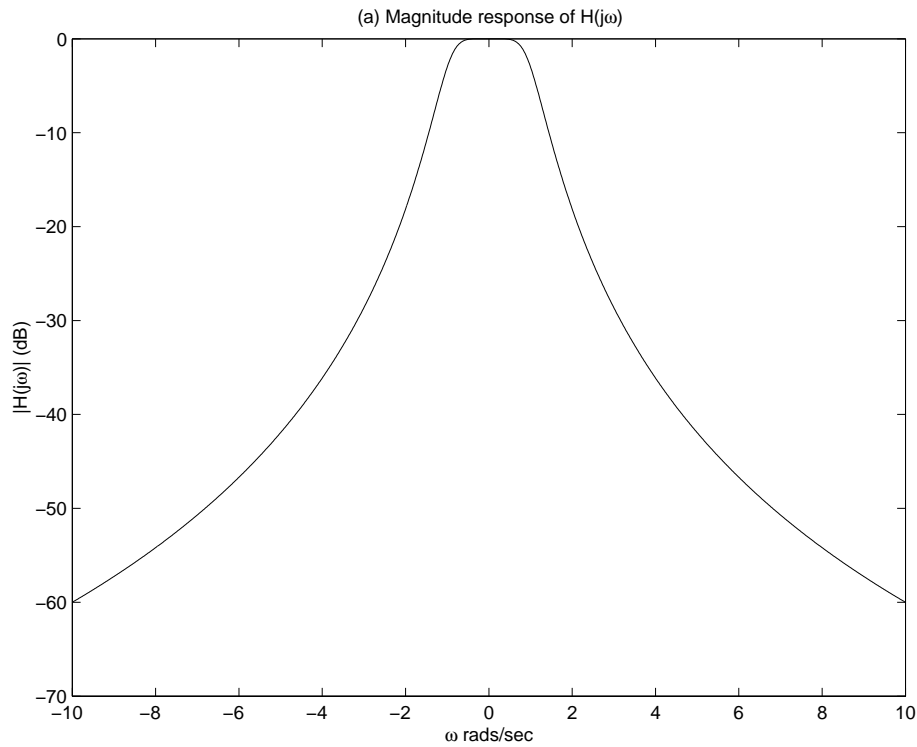


Figure P6.65. (a) Magnitude response of $H(s)$, $20\log_{10}|H(j\omega)|$.

(b) Perform the transformation of variables in which s is replaced by $s/10$ in $H(s)$. Repeat part (a) for the transformed system.

$$H(s) = \frac{1}{(\frac{s}{10}+1)(\frac{s^2}{10}+\frac{s}{10}+1)}$$

$$\text{poles at } s : = -10, -5 \pm \sqrt{5}$$

This system is lowpass with cutoff frequency at $\omega_c = 10$.

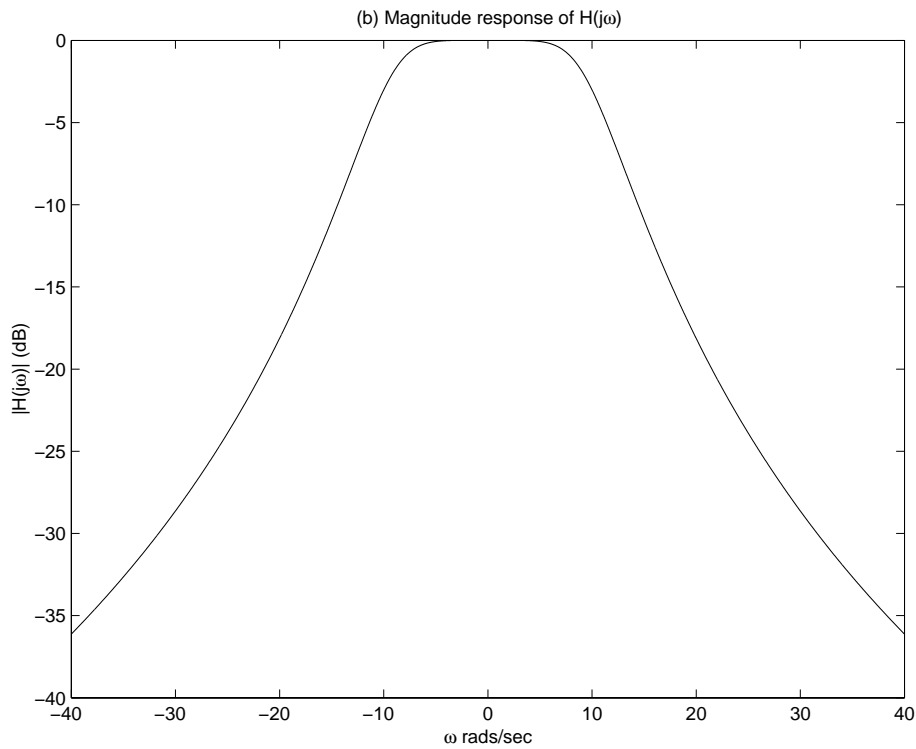


Figure P6.65. (b) Magnitude response of $H(s)$, $20\log_{10}|H(j\omega)|$.

(c) Perform the transformation of variables in which s is replaced by $1/s$ in $H(s)$. Repeat part (a) for the transformed system.

$$H(s) = \frac{1}{\left(\frac{1}{s} + 1\right)\left(\frac{1}{s^2} + \frac{1}{s} + 1\right)}$$

$$\text{poles at } s : = -1, \frac{-1 \pm \sqrt{3}}{2}$$

$$\text{Three zeros at } s : = 0$$

This system is a high pass filter with cutoff frequency $\omega_c = 1$.

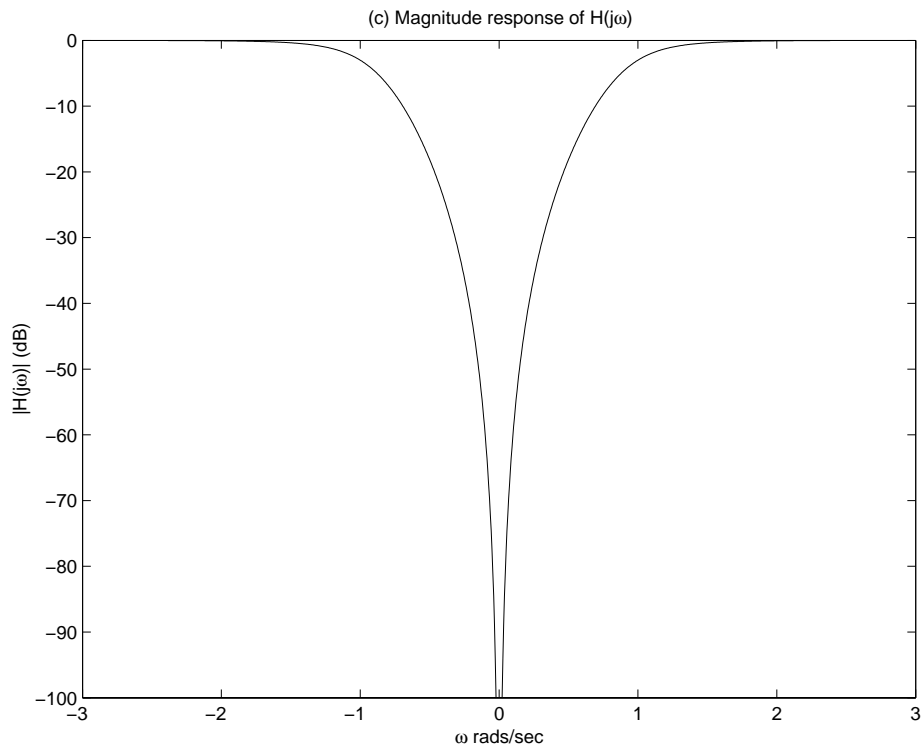


Figure P6.65. (c) Magnitude response of $H(s)$, $20\log_{10}|H(j\omega)|$.

(d) Find the transformation that converts $H(s)$ to a highpass system with cutoff frequency $\omega = 100$. Replace s by $\frac{100}{s}$:

$$H(s) = \frac{1}{\left(\frac{100}{s} + 1\right)\left(\frac{100}{s^2} + \frac{100}{s} + 1\right)}$$

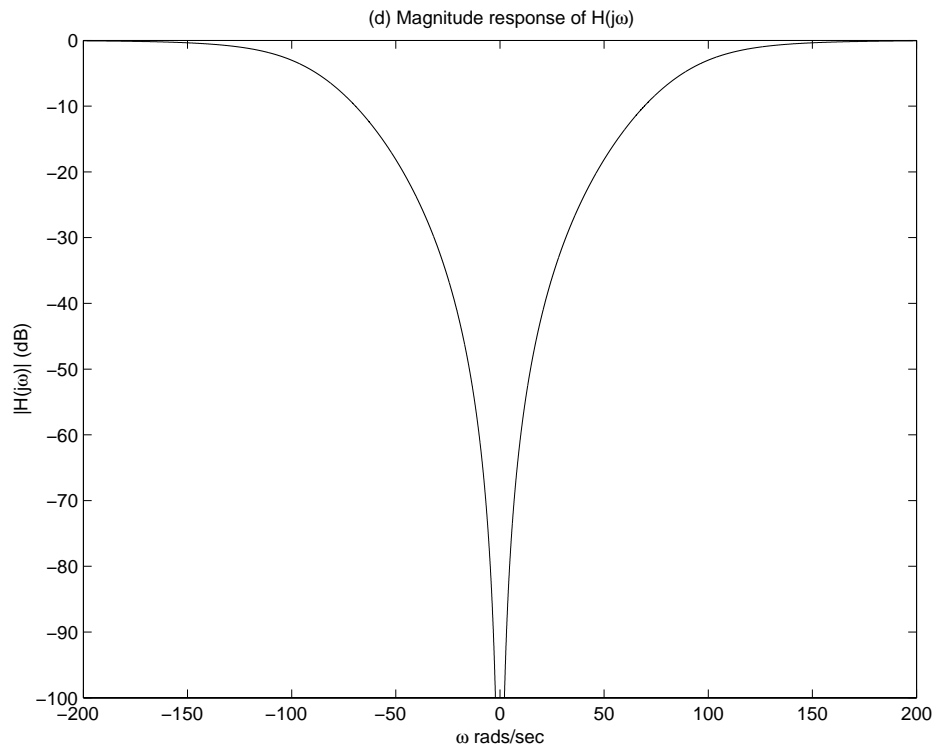


Figure P6.65. (d) Magnitude response of $H(s)$, $20\log_{10}|H(j\omega)|$.

Solutions to Computer Experiments

6.66. Use the MATLAB command `roots` to determine the poles and zeros of the following systems:

(a) $H(s) = \frac{s^2+2}{s^3+2s^2-s+1}$

(b) $H(s) = \frac{s^3+1}{s^4+2s^2+1}$

(c) $H(s) = \frac{4s^2+8s+10}{2s^3+8s^2+18s+20}$

P6.66 :

===== Part (a) :

ans =

0 + 1.4142i

0 - 1.4142i

ans =

-2.5468

0.2734 + 0.5638i

0.2734 - 0.5638i

Part (b) :

```

ans =
-1.0000
0.5000 + 0.8660i
0.5000 - 0.8660i
ans =
0.0000 + 1.0000i
0.0000 - 1.0000i
-0.0000 + 1.0000i
-0.0000 - 1.0000i

```

Part (c) :

```

ans =
-1.0000 + 1.2247i
-1.0000 - 1.2247i
ans =
-1.0000 + 2.0000i
-1.0000 - 2.0000i
-2.0000

```

6.67. Use the MATLAB command `pzmap` to plot the poles and zeros for the following systems:

(a) $H(s) = \frac{s^3+1}{s^4+2s^2+1}$

(b) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & -6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $D = [0]$

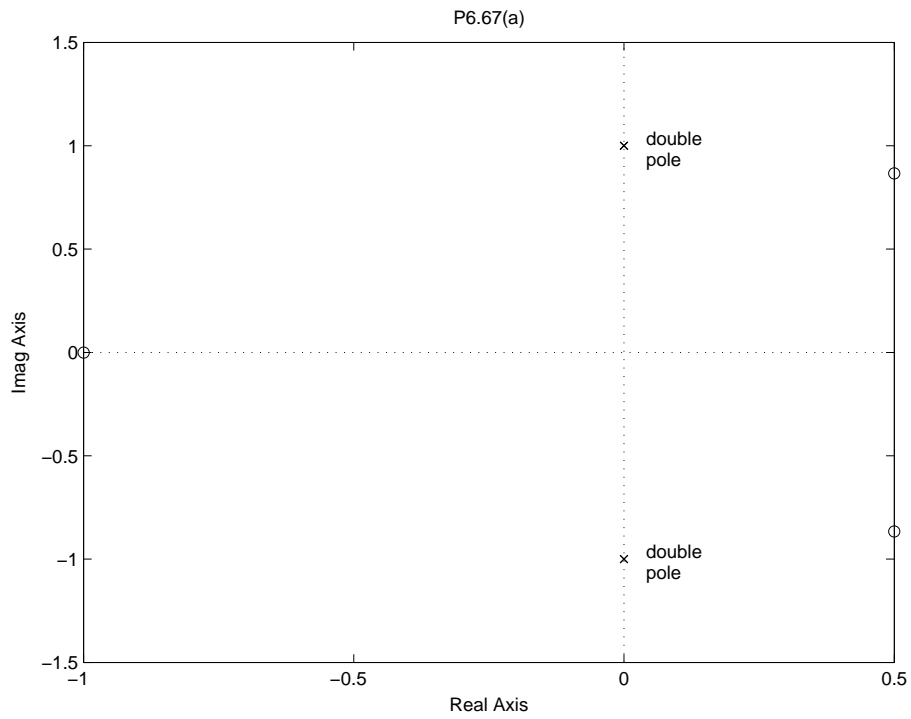


Figure P6.67. (a) Pole-Zero Plot of $H(s)$

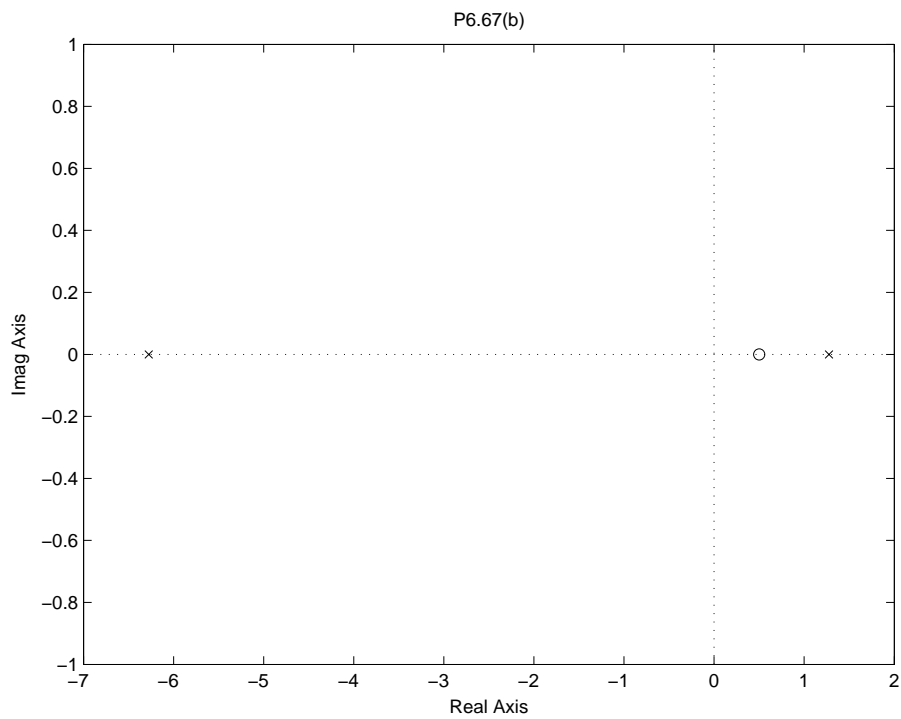


Figure P6.67. (b) Pole-Zero Plot

6.68. Use the MATLAB command `freqresp` evaluate and plot the magnitude and phaseresponses for

Examples 6.23 and 6.24.

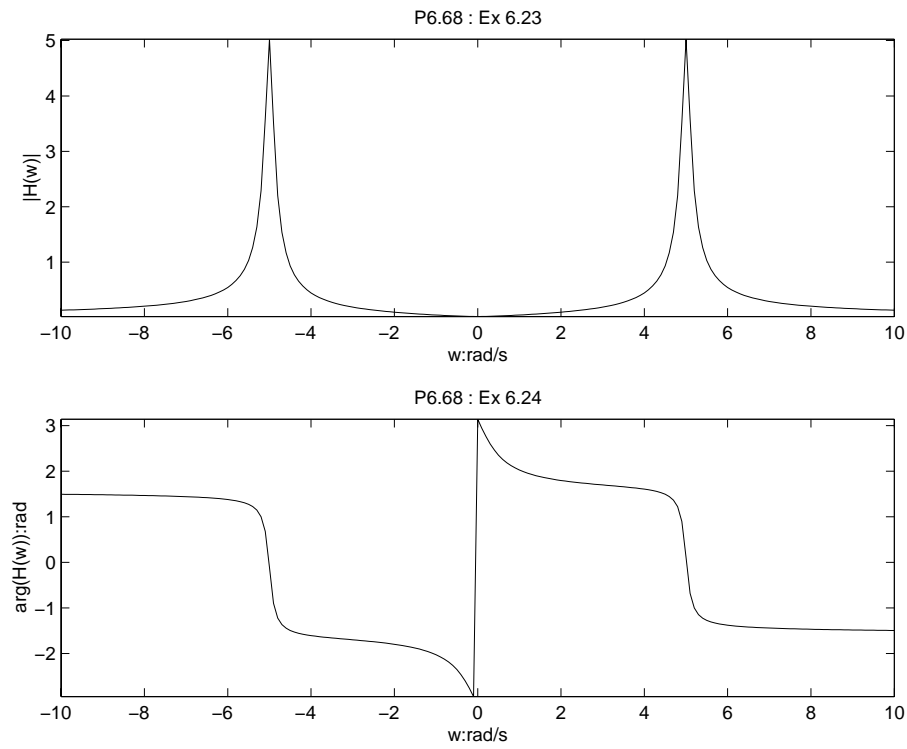


Figure P6.68. Magnitude and phase plot for Ex 6.23 & 6.24

6.69. Use the MATLAB command `freqresp` evaluate and plot the magnitude and phase responses for Problem 6.53.

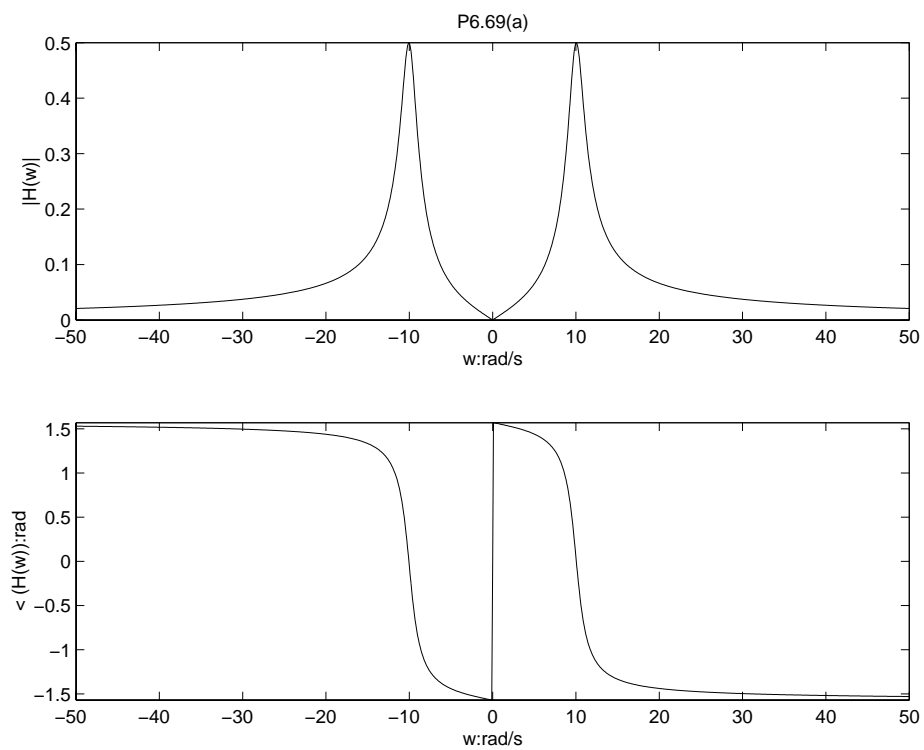


Figure P6.69. Magnitude and phase plot for Prob 6.53 (a)

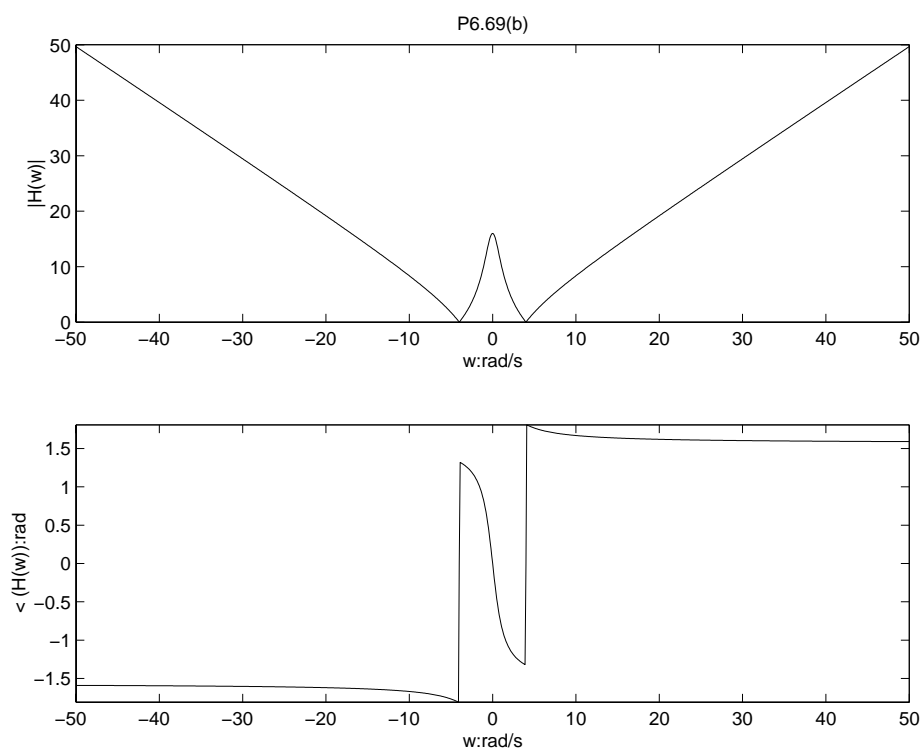


Figure P6.69. Magnitude and phase plot for Prob 6.53 (b)

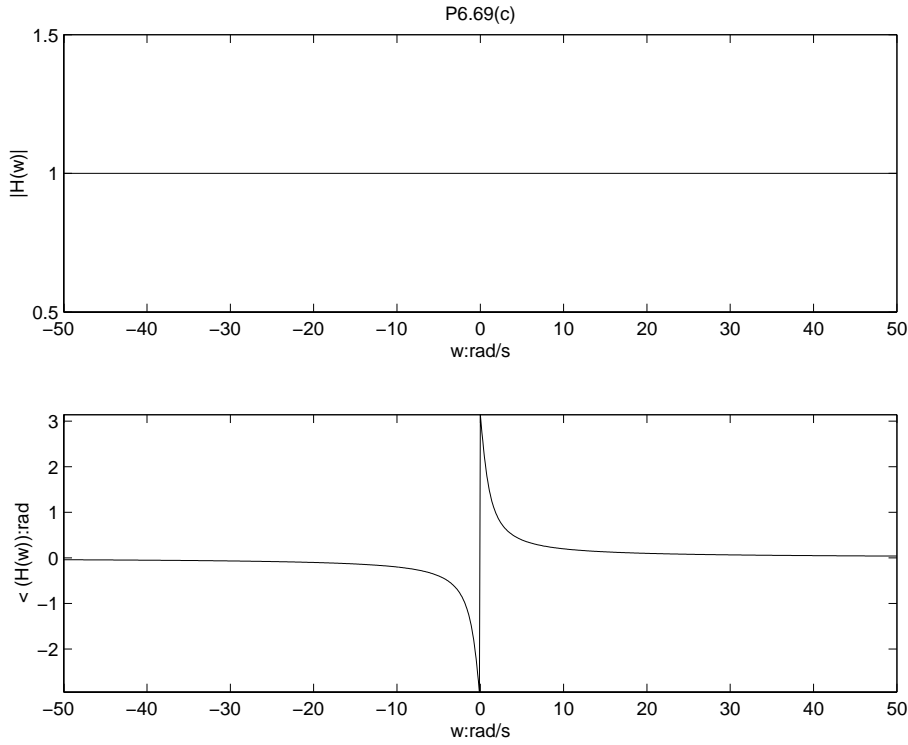


Figure P6.69. Magnitude and phase plot for Prob 6.53 (c)

6.70. Use your knowledge of the effect of poles and zeros on the magnitude response to design systems having the specified magnitude response. Place poles and zeros in the s -plane, and evaluate the corresponding magnitude response using the MATLAB command `freqresp`. Repeat this process until you find pole and zero locations that satisfy the specifications.

(a) Design a high-pass filter with two poles and two zeros that satisfies $|H(j0)| = 0$, $0.8 \leq |H(j\omega)| \leq 1.2$ for $|\omega| > 100\pi$, and has real valued coefficients.

Two conjugate poles are needed around the transition, which implies one possible solution is :

$$H(s) = \frac{s^2}{(s + 25 + j10\pi)(s + 25 - j10\pi)}$$

(b) Design a low-pass filter with real valued coefficients that satisfies $0.8 \leq |H(j\omega)| \leq 1.2$ for $|\omega| < \pi$, and $|H(j\omega)| < 0.1$ for $|\omega| > 10\pi$.

One possible solution is:

$$H(s) = \frac{(s - j50)(s + j50)}{(s + 2 + j\pi)(s + 2 - j\pi)}$$

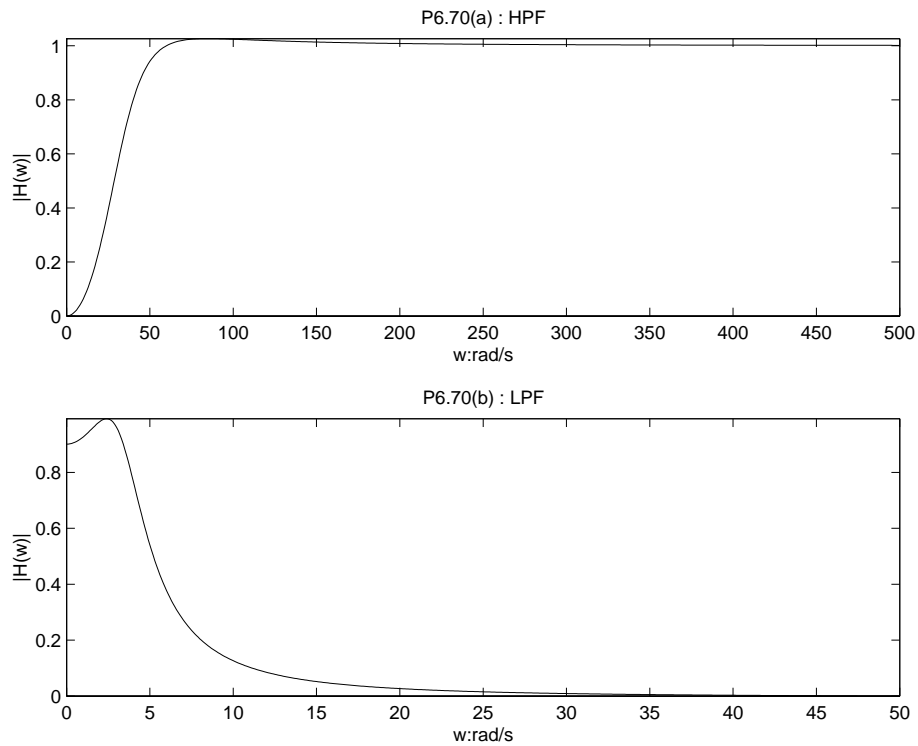


Figure P6.70. Magnitude response

6.71. Use the MATLAB command `bode` to find the bode diagrams for the systems in Problem 6.55.

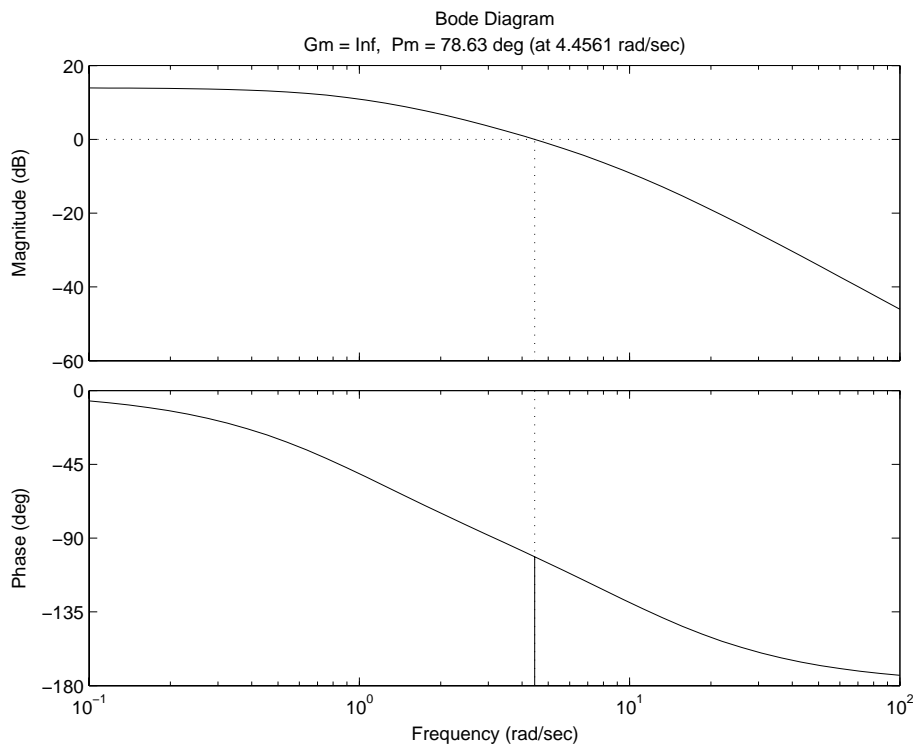


Figure P6.71. (a) Bode Plot of $H(s)$

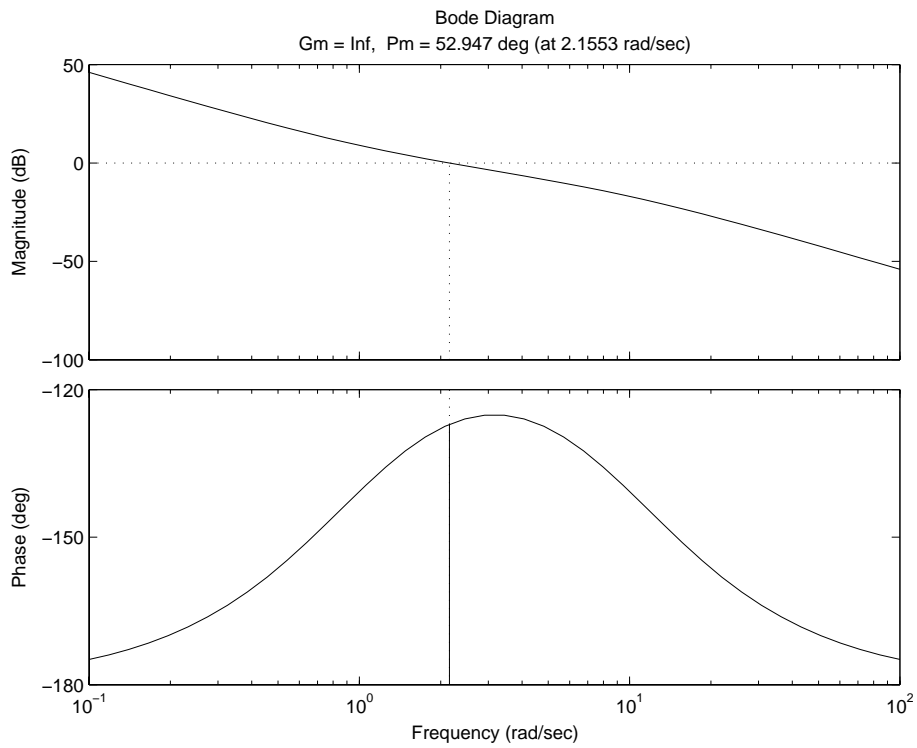


Figure P6.71. (b) Bode Plot of $H(s)$

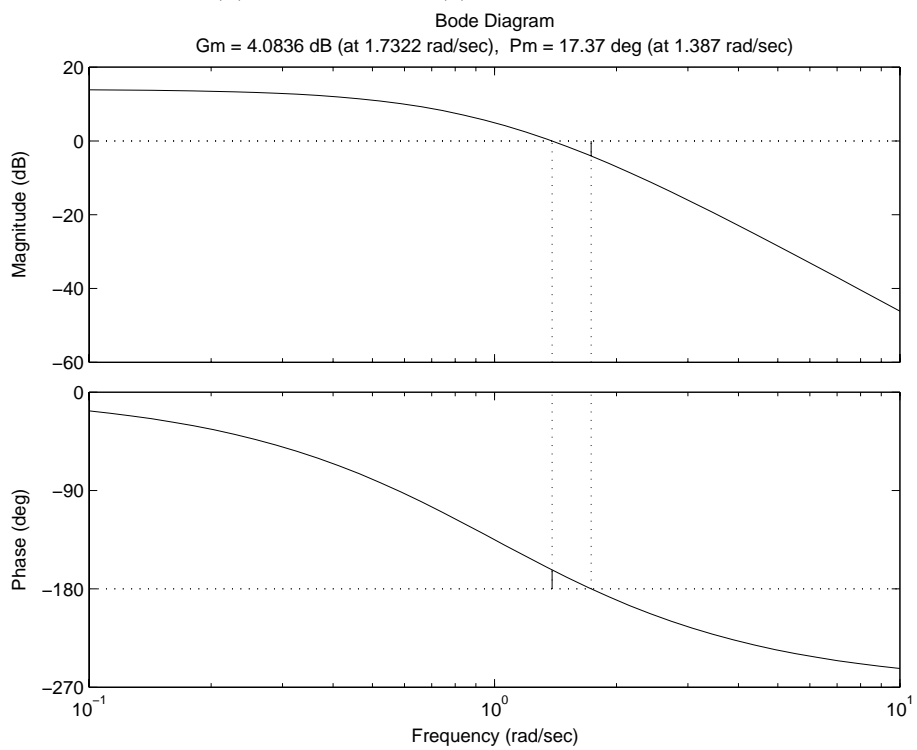


Figure P6.71. (c) Bode Plot of $H(s)$

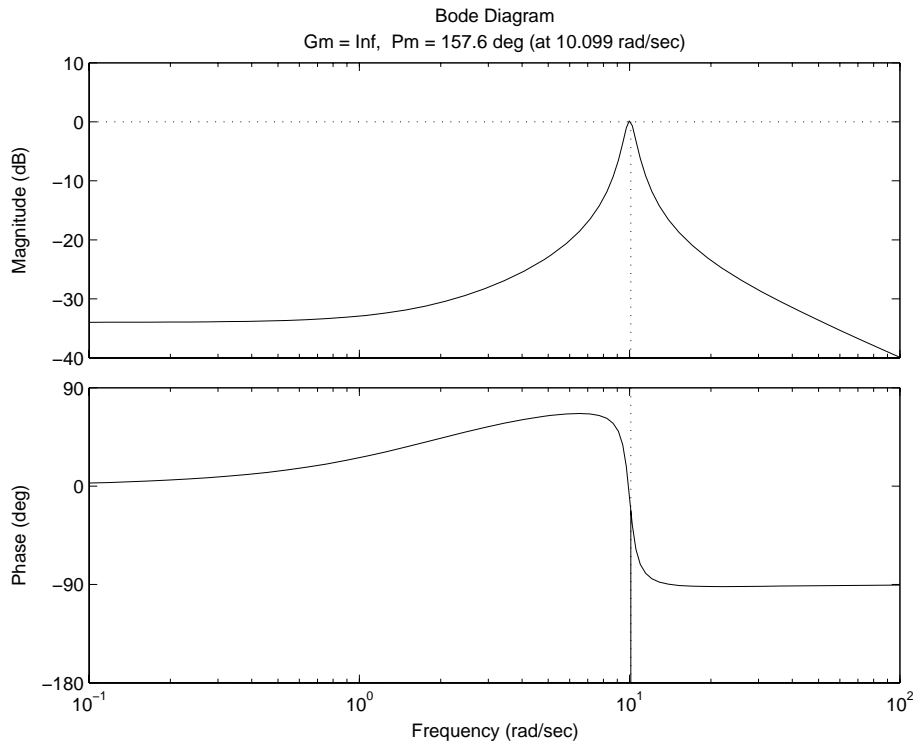


Figure P6.71. (d) Bode Plot of $H(s)$

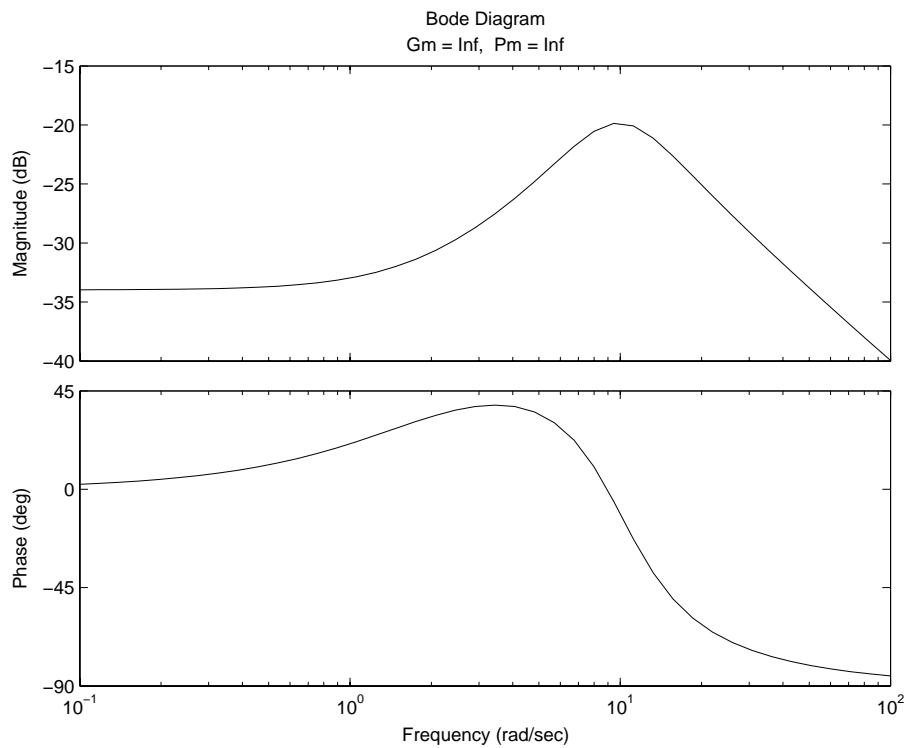


Figure P6.71. (e) Bode Plot of $H(s)$

6.72. Use the MATLAB command `ss` to find state-variable descriptions for the systems in Problem 6.48.

P6.72 :

=====

Part (a) :

=====

a =

x1 x2

x1 -3 -0

x2 1 0

b =

u1

x1 1

x2 0

c =

x1 x2

y1 0 1

d =

u1

y1 0

Continuous-time model.

Part (b) :

=====

a =

x1 x2

x1 2 -2

x2 4 0

b =

u1

x1 2

x2 0

c =

x1 x2

y1 3 0

d =

u1
y1 0

Continuous-time model.

Part (c) :

=====

a =

x1 x2 x3

x1 -5 -0.875 -0.09375

x2 8 0 0

x3 0 4 0

b =

u1

x1 0.5

x2 0

x3 0

c =

x1 x2 x3

y1 0 0.5 -0.25

d =

u1

y1 0

Continuous-time model.

6.73. Use the MATLAB command `tf` to find transfer function descriptions for the systems in Problem 6.49.

P6.73 :

=====

Part (a) :

=====

Transfer function:

$$\frac{s+3}{s^2+3s+2}$$

Part (b) :

=====

Transfer function:

$$\frac{2s-1}{s^2+5s-8}$$

Solutions to Additional Problems

7.17. Determine the z -transform and ROC for the following time signals: Sketch the ROC, poles, and zeros in the z -plane.

(a) $x[n] = \delta[n - k]$, $k > 0$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= z^{-k}, \quad z \neq 0 \end{aligned}$$

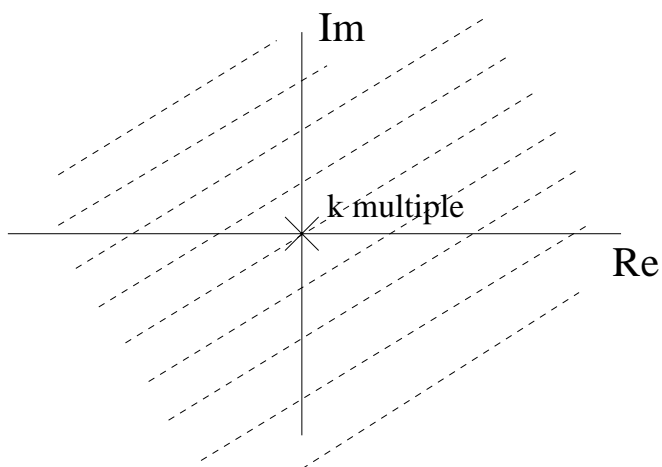


Figure P7.17. (a) ROC

(b) $x[n] = \delta[n + k]$, $k > 0$

$$X(z) = z^k, \quad \text{all } z$$

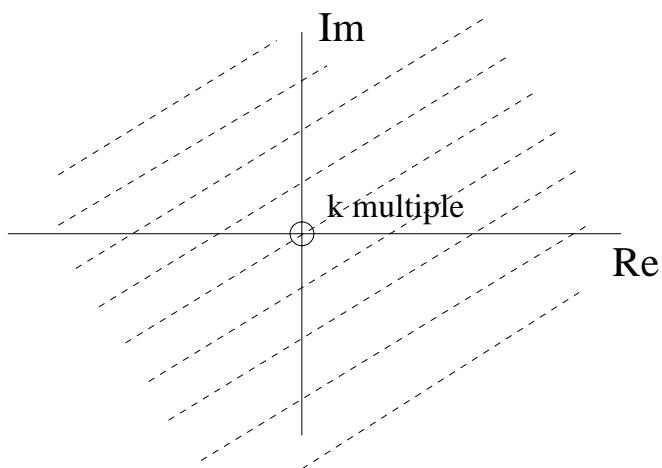


Figure P7.17. (b) ROC

(c) $x[n] = u[n]$

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} z^{-n} \\ &= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \end{aligned}$$

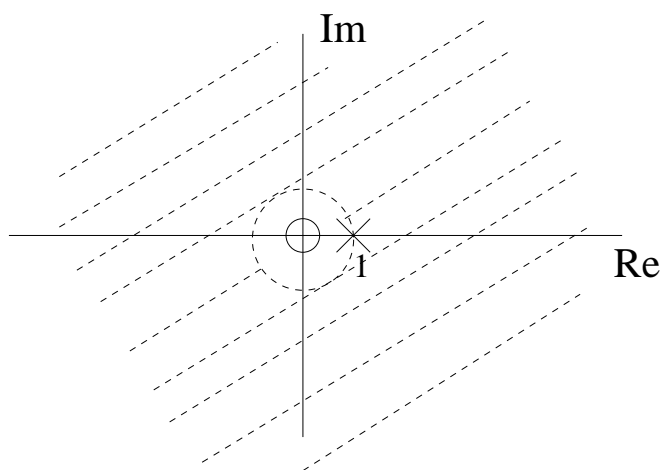


Figure P7.17. (c) ROC

(d) $x[n] = \left(\frac{1}{4}\right)^n (u[n] - u[n - 5])$

$$\begin{aligned} X(z) &= \sum_{n=0}^4 \left(\frac{1}{4}z^{-1}\right)^n \\ &= \frac{1 - \left(\frac{1}{4}z^{-1}\right)^5}{1 - \frac{1}{4}z^{-1}} \\ &= \frac{\left[z^5 - \left(\frac{1}{4}\right)^5\right]}{z^4\left(z - \frac{1}{4}\right)}, \quad \text{all } z \end{aligned}$$

4 poles at $z = 0$, 1 pole at $z = \frac{1}{4}$

5 zeros at $z = \frac{1}{4}e^{jk\frac{2\pi}{5}}$ $k = 0, 1, 2, 3, 4$

Note zero for $k = 0$ cancels pole at $z = \frac{1}{4}$

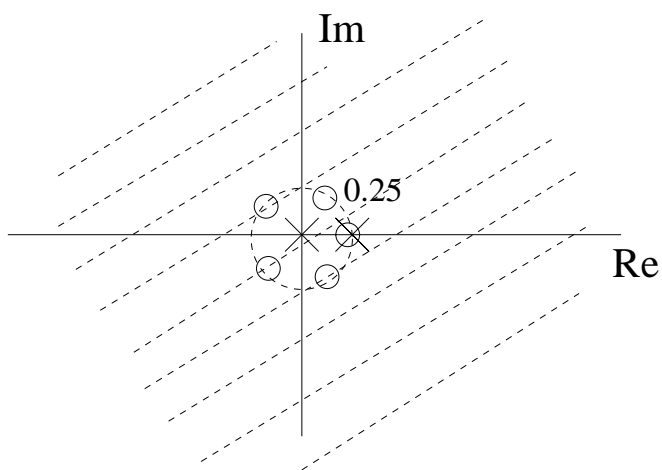


Figure P7.17. (d) ROC

(e) $x[n] = \left(\frac{1}{4}\right)^n u[-n]$

$$\begin{aligned}
 X(z) &= \sum_{n=-\infty}^0 \left(\frac{1}{4}z^{-1}\right)^n \\
 &= \sum_{n=0}^{\infty} (4z)^n \\
 &= \frac{1}{1-4z}, \quad |z| < \frac{1}{4}
 \end{aligned}$$

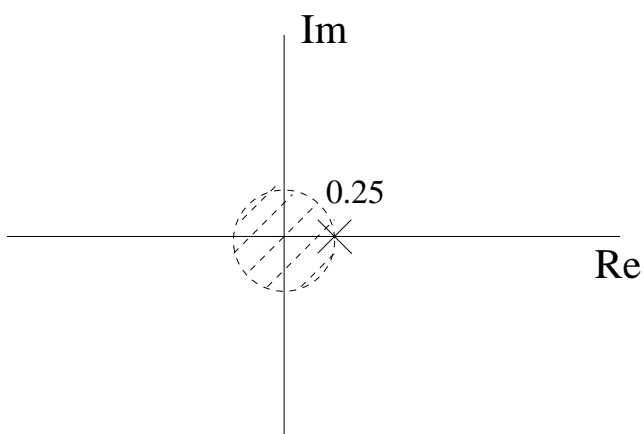


Figure P7.17. (e) ROC

(f) $x[n] = 3^n u[-n-1]$

$$\begin{aligned}
 X(z) &= \sum_{n=-\infty}^{-1} (3z^{-1})^n \\
 &= \sum_{n=1}^{\infty} \left(\frac{1}{3}z\right)^n
 \end{aligned}$$

$$\begin{aligned}
&= \frac{\frac{1}{3}z}{1 - \frac{1}{3}z} \\
&= \frac{-1}{1 - 3z^{-1}}, \quad |z| < 3 \\
&\text{Pole at } z = 3 \\
&\text{Zero at } z = 0
\end{aligned}$$

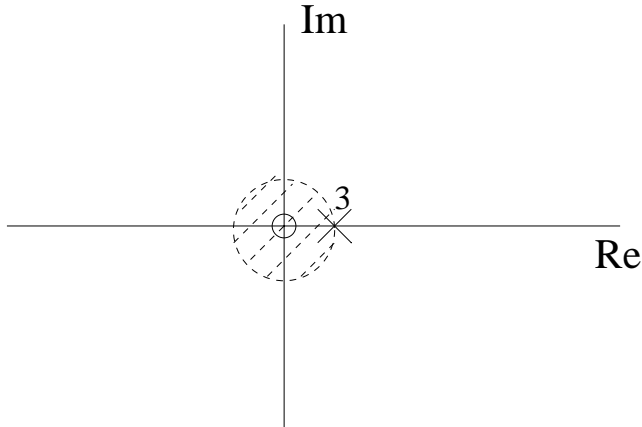


Figure P7.17. (f) ROC

(g) $x[n] = \left(\frac{2}{3}\right)^{|n|}$

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{-1} \left(\frac{3}{2}z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(\frac{2}{3}z^{-1}\right)^n \\
&= \frac{-1}{1 - \frac{3}{2}z^{-1}} + \frac{1}{1 - \frac{2}{3}z^{-1}} \\
&= \frac{-\frac{5}{6}z}{(z - \frac{3}{2})(z - \frac{2}{3})}, \quad \frac{2}{3} < |z| < \frac{3}{2}
\end{aligned}$$

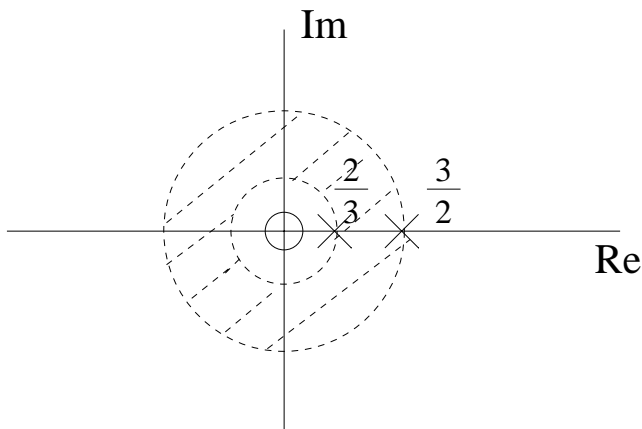


Figure P7.17. (g) ROC

(h) $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[-n-1]$

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n + \sum_{n=-\infty}^{-1} \left(\frac{1}{4}z^{-1}\right)^n \\
&= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - \frac{2}{3}z^{-1}}, \quad |z| > \frac{1}{2} \text{ and } |z| < \frac{1}{4}
\end{aligned}$$

No region of convergence exists.

7.18. Given the following z -transforms, determine whether the DTFT of the corresponding time signals exists without determining the time signal, and identify the DTFT in those cases where it exists:

(a) $X(z) = \frac{5}{1 + \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$

ROC includes $|z| = 1$, DTFT exists.

$$X(e^{j\Omega}) = \frac{5}{1 + \frac{1}{3}e^{-j\Omega}}$$

(b) $X(z) = \frac{5}{1 + \frac{1}{3}z^{-1}}, \quad |z| < \frac{1}{3}$

ROC does not include, $|z| = 1$, DTFT does not exist.

(c) $X(z) = \frac{z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + 3z^{-1})}, \quad |z| < \frac{1}{2}$

ROC does not include, $|z| = 1$, DTFT does not exist.

(d) $X(z) = \frac{z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + 3z^{-1})}, \quad \frac{1}{2} < |z| < 3$

ROC includes $|z| = 1$, DTFT exists.

$$X(e^{j\Omega}) = \frac{e^{-j\Omega}}{(1 - \frac{1}{2}e^{-j\Omega})(1 + 3e^{-j\Omega})}$$

7.19. The pole and zero locations of $X(z)$ are depicted in the z -plane on the following figures. In each case, identify all valid ROCs for $X(z)$ and specify the characteristics of the time signal corresponding to each ROC.

(a) Fig. P7.19 (a)

$$X(z) = \frac{Cz(z - \frac{3}{2})}{(z + \frac{3}{4})(z - \frac{1}{3})}$$

There are 4 possible ROCs

(1) $|z| > \frac{3}{4}$

$x[n]$ is right-sided.

(2) $\frac{1}{3} < |z| < \frac{3}{4}$
 $x[n]$ is two-sided.

(3) $|z| < \frac{1}{3}$
 $x[n]$ is left-sided.

(b) Fig. P7.19 (b)

$$X(z) = \frac{C(z^4 - 1)}{z(z - \sqrt{2}e^{j\frac{\pi}{4}})(z - \sqrt{2}e^{-j\frac{\pi}{4}})}$$

There are 2 possible ROCs

(1) $|z| > \sqrt{2}$
 $x[n]$ is right-sided.

(2) $|z| < \sqrt{2}$
 $x[n]$ is two-sided.

(c) Fig. P7.19 (c)

$$X(z) = (z - \frac{1}{2})(z + 1)(z^2 + \frac{9}{16})C, \quad |z| < \infty$$

$x[n]$ is stable and left-sided.

7.20. Use the tables of z -transforms and z -transform properties given in Appendix E to determine the z -transforms of the following signals:

(a) $x[n] = (\frac{1}{2})^n u[n] * 2^n u[-n - 1]$

$$\begin{aligned} a[n] &= \left(\frac{1}{2}\right)^n u[n] \xrightarrow{z} A(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2} \\ b[n] &= 2^n u[-n - 1] \xrightarrow{z} B(z) = \frac{1}{1 - 2z^{-1}}, \quad |z| < 2 \\ x[n] &= a[n] * b[n] \xrightarrow{z} X(z) = A(z)B(z) \\ X(z) &= \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right) \left(\frac{1}{1 - 2z^{-1}}\right), \quad \frac{1}{2} < |z| < 2 \end{aligned}$$

(b) $x[n] = n \left((\frac{1}{2})^n u[n] * (\frac{1}{4})^n u[n - 2] \right)$

$$a[n] = \left(\frac{1}{2}\right)^n u[n] \xrightarrow{z} A(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$\begin{aligned}
b[n] &= \left(\frac{1}{4}\right)^n u[n] \xleftrightarrow{z} B(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4} \\
c[n] &= b[n-2] \xleftrightarrow{z} C(z) = \frac{z^{-2}}{1 - \frac{1}{4}z^{-1}} \\
x[n] &= n[a[n] * b[n]] \xleftrightarrow{z} X(z) = -z \frac{d}{dz} A(z) B(z) \\
X(z) &= \frac{2z^{-2} - \frac{3}{4}z^{-3}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}, \quad |z| > \frac{1}{2}
\end{aligned}$$

(c) $x[n] = u[-n]$

$$\begin{aligned}
X(z) &= \frac{1}{1 - \left(\frac{1}{z}\right)^{-1}} \\
&= \frac{1}{1 - z}, \quad |z| < 1
\end{aligned}$$

(d) $x[n] = n \sin\left(\frac{\pi}{2}n\right)u[-n]$

$$\begin{aligned}
x[n] &= -n \sin\left(-\frac{\pi}{2}n\right)u[-n] \\
X(z) &= -z \frac{d}{dz} \left(\frac{z^{-1}}{1 + z^{-2}} \right) \Big|_{z=\frac{1}{z}} \\
&= -z \left[\frac{-z^{-2}}{1 + z^{-2}} - \frac{z^{-1}(-2z^{-3})}{(1 + z^{-2})^2} \right] \Big|_{z=\frac{1}{z}} \\
&= \frac{z}{1 + z^2} + \frac{2z^3}{(1 + z^2)^2}
\end{aligned}$$

(e) $x[n] = 3^{n-2}u[n] * \cos\left(\frac{\pi}{6}n + \pi/3\right)u[n]$

$$\begin{aligned}
a[n] &= \frac{1}{9}3^n u[n] \xleftrightarrow{z} A(z) = \frac{\frac{1}{9}}{1 - 3z^{-3}} \\
b[n] &= \cos\left(\frac{\pi}{6}n + \frac{\pi}{3}\right)u[n] = \left[\cos\left(\frac{\pi}{6}n\right)\cos(\pi/3) - \sin\left(\frac{\pi}{6}n\right)\sin(\pi/3) \right] u[n] \\
b[n] &\xleftrightarrow{z} B(z) = \frac{\cos(\frac{\pi}{3})(1 + z^{-1}\cos(\frac{\pi}{6}))}{1 - 2z^{-1}\cos(\frac{\pi}{6}) + z^{-2}} - \frac{\sin(\frac{\pi}{3})(z^{-1}\sin(\frac{\pi}{6}))}{1 - 2z^{-1}\cos(\frac{\pi}{6}) + z^{-2}} \\
X(z) = A(z)B(z) &= \left(\frac{\frac{1}{9}}{1 - 3z^{-3}} \right) \frac{\cos(\frac{\pi}{3})(1 + z^{-1}\cos(\frac{\pi}{6})) - \sin(\frac{\pi}{3})\sin(\frac{\pi}{6})z^{-1}}{1 - 2z^{-1}\cos(\frac{\pi}{6}) + z^{-2}}
\end{aligned}$$

7.21. Given the z -transform pair $x[n] \xleftrightarrow{z} \frac{z^2}{z^2 - 16}$ with ROC $|z| < 4$, use the z -transform properties to determine the z -transform of the following signals:

(a) $y[n] = x[n-2]$

$$y[n] = x[n-2] \xleftarrow{z} Y(z) = z^{-2}X(z) = \frac{1}{z^2 - 16}$$

(b) $y[n] = (1/2)^n x[n]$

$$y[n] = \left(\frac{1}{2}\right)^n x[n] \xleftarrow{z} Y(z) = X(2z) = \frac{z^2}{z^2 - 4}$$

(c) $y[n] = x[-n] * x[n]$

$$y[n] = x[-n] * x[n] \xleftarrow{z} Y(z) = X\left(\frac{1}{z}\right)X(z) = \frac{z^2}{257z^2 - 16z^4 - 16}$$

(d) $y[n] = nx[n]$

$$y[n] = nx[n] \xleftarrow{z} Y(z) = -z \frac{d}{dz} X(z) = \frac{32z^2}{(z^2 - 16)^2}$$

(e) $y[n] = x[n+1] + x[n-1]$

$$y[n] = x[n+1] + x[n-1] \xleftarrow{z} Y(z) = (z^1 + z^{-1})X(z) = \frac{z^3 + z}{z^2 - 16}$$

(f) $y[n] = x[n] * x[n-3]$

$$y[n] = x[n] * x[n-3] \xleftarrow{z} Y(z) = X(z)z^{-3}X(z) = \frac{z}{(z^2 - 16)^2}$$

7.22. Given the z -transform pair $n^2 3^n u[n] \xleftarrow{z} X(z)$, use the z -transform properties to determine the time-domain signals corresponding to the following z transforms:

(a) $Y(z) = X(2z)$

$$Y(z) = X(2z) \xleftarrow{z} y[n] = \left(\frac{1}{2}\right)^n x[n] = \left(\frac{1}{2}\right)^n n^2 3^n u[n]$$

(b) $Y(z) = X(z^{-1})$

$$Y(z) = X\left(\frac{1}{z}\right) \xleftarrow{z} y[n] = x[-n] = n^2 3^{-n} u[-n]$$

(c) $Y(z) = \frac{d}{dz}X(z)$

$$Y(z) = \frac{d}{dz}X(z) = -z^{-1} \left(-z \frac{d}{dz}X(z) \right) \xleftrightarrow{z} y[n] = -(n-1)x[n-1] = -(n-1)^3 3^{n-1} u[n-1]$$

(d) $Y(z) = \frac{z^2 - z^{-2}}{2} X(z)$

$$\begin{aligned} Y(z) = \frac{z^2 - z^{-2}}{2} X(z) &\xleftrightarrow{z} y[n] = \frac{1}{2}(x[n+2] - x[n-2]) \\ y[n] &= \frac{1}{2}[(n+2)^2 3^{n+2} u[n+2] - (n-2)^2 3^{n-2} u[n-2]] \end{aligned}$$

(e) $Y(z) = [X(z)]^2$

$$\begin{aligned} Y(z) = X(z)X(z) &\xleftrightarrow{z} y[n] = x[n] * x[n] \\ y[n] &= u[n] \sum_{k=0}^n k^2 3^k (n-k)^2 3^{n-k} \\ &= 3^n u[n] \sum_{k=0}^n [k^2 n^2 - 2nk^3 + k^4] \end{aligned}$$

7.23. Prove the following z -transform properties:

(a) Time reversal

$$\begin{aligned} x[n] &\xleftrightarrow{z} X\left(\frac{1}{z}\right) \\ y[n] &= x[-n] \\ Y(z) &= \sum_{n=-\infty}^{\infty} x[-n] z^{-n} \\ \text{let } l &= -n \\ &= \sum_{l=-\infty}^{\infty} x[l] \left(\frac{1}{z}\right)^{-l} \\ &= X\left(\frac{1}{z}\right) \end{aligned}$$

(b) Time shift

$$\begin{aligned} x[n - n_o] &\xleftrightarrow{z} z^{-n_o} X(z) \\ y[n] &= x[n - n_o] \\ Y(z) &= \sum_{n=-\infty}^{\infty} x[n - n_o] z^{-n} \\ \text{let } l &= n - n_o \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=-\infty}^{\infty} x[l]z^{-(l+n_o)} \\
&= \left(\sum_{l=-\infty}^{\infty} x[l]z^{-l} \right) z^{-n_o} \\
&= z^{-n_o} X(z)
\end{aligned}$$

(c) Multiplication by exponential sequence

$$\begin{aligned}
\alpha^n x[n] &\xleftrightarrow{z} X\left(\frac{z}{\alpha}\right) \\
y[n] &= \alpha^n x[n] \\
Y(z) &= \sum_{n=-\infty}^{\infty} \alpha^n x[n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{z}{\alpha}\right)^{-n} \\
&= X\left(\frac{z}{\alpha}\right)
\end{aligned}$$

(d) Convolution

Let $c[n] = x[n] * y[n]$

$$\begin{aligned}
x[n] * y[n] &\xleftrightarrow{z} X(z)Y(z) \\
C(z) &= \sum_{n=-\infty}^{\infty} (x[n] * y[n]) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{p=-\infty}^{\infty} x[p]y[n-p] \right) z^{-n} \\
&= \sum_{p=-\infty}^{\infty} x[p] \underbrace{\left(\sum_{n=-\infty}^{\infty} y[n-p] z^{-(n-p)} \right)}_{Y(z)} z^{-p} \\
&= \underbrace{\left(\sum_{p=-\infty}^{\infty} x[p] z^{-p} \right)}_{X(z)} Y(z) \\
&= X(z)Y(z)
\end{aligned}$$

(e) Differentiation in the z -domain.

$$\begin{aligned}
nx[n] &\xleftrightarrow{z} -z \frac{d}{dz} X(z) \\
X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n}
\end{aligned}$$

Differentiate with respect to z and multiply by $-z$.

$$-z \frac{d}{dz} X(z) \xleftrightarrow{z} \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

Therefore

$$nx[n] \xleftrightarrow{z} -z \frac{d}{dz} X(z)$$

7.24. Use the method of partial fractions to obtain the time-domain signals corresponding to the following z -transforms:

(a) $X(z) = \frac{1 + \frac{7}{6}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{2}$

$x[n]$ is right-sided

$$\begin{aligned} X(z) &= \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 + \frac{1}{3}z^{-1}} \\ 1 &= A + B \\ \frac{7}{6} &= \frac{1}{3}A - \frac{1}{2}B \\ X(z) &= \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{-1}{1 + \frac{1}{3}z^{-1}} \\ x[n] &= \left[2\left(\frac{1}{2}\right)^n - \left(-\frac{1}{3}\right)^n \right] u[n] \end{aligned}$$

(b) $X(z) = \frac{1 + \frac{7}{6}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}, \quad |z| < \frac{1}{3}$

same as (a), but $x[n]$ is left-sided

$$x[n] = \left[-2\left(\frac{1}{2}\right)^n + \left(-\frac{1}{3}\right)^n \right] u[-n - 1]$$

(c) $X(z) = \frac{1 + \frac{7}{6}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}, \quad \frac{1}{3} < |z| < \frac{1}{2}$

same as (a), but $x[n]$ is two-sided

$$x[n] = -2\left(\frac{1}{2}\right)^n u[-n - 1] - \left(-\frac{1}{3}\right)^n u[n]$$

(d) $X(z) = \frac{z^2 - 3z}{z^2 + \frac{3}{2}z - 1}, \quad \frac{1}{2} < |z| < 2$

$x[n]$ is two-sided

$$\begin{aligned} X(z) &= \frac{1 - 3z^{-1}}{1 + \frac{3}{2}z^{-1} - z^{-2}} \\ &= \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 + 2z^{-1}} \end{aligned}$$

$$\begin{aligned}
1 &= A + B \\
-3 &= 2A - \frac{1}{2}B \\
X(z) &= \frac{-1}{1 + \frac{1}{2}z^{-1}} + \frac{2}{1 - 2z^{-1}} \\
x[n] &= -\left(-\frac{1}{2}\right)^n u[n] - 2(2)^n u[-n - 1]
\end{aligned}$$

(e) $X(z) = \frac{3z^2 - \frac{1}{4}z}{z^2 - 16}, \quad |z| > 4$

$x[n]$ is right-sided

$$\begin{aligned}
X(z) &= \frac{A}{1 + 4z^{-1}} + \frac{B}{1 - 4z^{-1}} \\
3 &= A + B \\
-\frac{1}{4} &= -4A + 4B \\
X(z) &= \frac{\frac{49}{32}}{1 + 4z^{-1}} + \frac{\frac{47}{32}}{1 - 4z^{-1}} \\
x[n] &= \left[\frac{49}{32}(-4)^n + \frac{47}{32}4^n \right] u[n]
\end{aligned}$$

(f) $X(z) = \frac{z^3 + z^2 + \frac{3}{2}z + \frac{1}{2}}{z^3 + \frac{3}{2}z^2 + \frac{1}{2}z}, \quad |z| < \frac{1}{2}$

$x[n]$ is left-sided

$$\begin{aligned}
X(z) &= z^{-1} + \frac{2}{1 + z^{-1}} + \frac{-1}{1 + \frac{1}{2}z^{-1}} \\
x[n] &= \delta[n - 1] + \left[\left(-\frac{1}{2}\right)^n - 2(-1)^n \right] u[-n - 1]
\end{aligned}$$

(g) $X(z) = \frac{2z^4 - 2z^3 - 2z^2}{z^2 - 1}, \quad |z| > 1$

$x[n]$ is right-sided

$$\begin{aligned}
X(z) &= \left(2 + \frac{1}{1 + z^{-1}} + \frac{-1}{1 - z^{-1}} \right) z^2 \\
x[n] &= 2\delta[n + 2] + [(-1)^n - 1] u[n + 2]
\end{aligned}$$

7.25. Determine the time-domain signals corresponding to the following z -transforms:

(a) $X(z) = 1 + 2z^{-6} + 4z^{-8}, \quad |z| > 0$

$$x[n] = \delta[n] + 2\delta[n-6] + 4\delta[n-8]$$

(b) $X(z) = \sum_{k=5}^{10} \frac{1}{k} z^{-k}, \quad |z| > 0$

$$x[n] = \sum_{k=5}^{10} \frac{1}{k} \delta[n-k]$$

(c) $X(z) = (1 + z^{-1})^3, \quad |z| > 0$

$$\begin{aligned} X(z) &= (\delta[n] + \delta[n-1]) * (\delta[n] + \delta[n-1]) * (\delta[n] + \delta[n-1]) \\ x[n] &= \delta[n] + 3\delta[n-1] + 3\delta[n-2] + \delta[n-3] \end{aligned}$$

(d) $X(z) = z^6 + z^2 + 3 + 2z^{-3} + z^{-4}, \quad |z| > 0$

$$x[n] = \delta[n+6] + \delta[n+2] + 3\delta[n] + 2\delta[n-3] + \delta[n-4]$$

7.26. Use the following clues to determine the signal $x[n]$ and rational z -transform $X(z)$.

(a) $X(z)$ has poles at $z = 1/2$ and $z = -1$, $x[1] = 1$, $x[-1] = 1$, and the ROC includes the point $z = 3/4$.

Since the ROC includes the point $z = 3/4$, the ROC is $\frac{1}{2} < |z| < 1$.

$$\begin{aligned} X(z) &= \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 + z^{-1}} \\ x[n] &= A \left(\frac{1}{2}\right)^n u[n] - B(-1)^n u[-n-1] \\ x[1] = 1 &= A \left(\frac{1}{2}\right) \\ A &= 2 \\ x[-1] = 1 &= -1B(-1) \\ B &= 1 \\ x[n] &= 2 \left(\frac{1}{2}\right)^n u[n] - (-1)^n u[-n-1] \end{aligned}$$

(b) $x[n]$ is right-sided, $X(z)$ has a single pole, and $x[0] = 2$, $x[2] = 1/2$.

$$\begin{aligned} x[n] &= c(p)^n u[n] \quad \text{where } c \text{ and } p \text{ are unknown constants.} \\ x[0] = 2 &= c(p)^0 \end{aligned}$$

$$\begin{aligned}
c &= 2 \\
x[2] = \frac{1}{2} &= 2(p)^2 \\
p &= \frac{1}{2} \\
x[n] &= 2\left(\frac{1}{2}\right)^n u[n]
\end{aligned}$$

(c) $x[n]$ is two-sided, $X(z)$ has one pole at $z = 1/4$, $x[-1] = 1$, $x[-3] = 1/4$, and $X(1) = 11/3$.

$$\begin{aligned}
X(z) &= \frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B}{1 - cz^{-1}} \\
x[n] &= A\left(\frac{1}{4}\right)^n u[n] - B(c)^n u[-n-1] \\
x[-1] = 1 &= -Bc^{-1} \\
x[-3] = \frac{1}{4} &= -Bc^{-3} \\
c &= 2 \\
B &= -2 \\
X(1) = \frac{11}{3} &= \frac{A}{1 - \frac{1}{4}} + \frac{-2}{1 - 2} \\
A &= \frac{5}{4} \\
x[n] &= \frac{5}{4}\left(\frac{1}{4}\right)^n u[n] + 2(2)^n u[-n-1]
\end{aligned}$$

7.27. Determine the impulse response corresponding to the following transfer functions if (i) the system is stable, or (ii) the system is causal:

(a) $H(z) = \frac{2 - \frac{3}{2}z^{-1}}{(1 - 2z^{-1})(1 + \frac{1}{2}z^{-1})}$

$$H(z) = \frac{1}{1 - 2z^{-1}} + \frac{1}{1 + \frac{1}{2}z^{-1}}$$

(i) $h[n]$ is stable, ROC $\frac{1}{2} < |z| < 2$, ROC includes $|z| = 1$.

$$h[n] = -(2)^n u[-n-1] + \left(-\frac{1}{2}\right)^n u[n]$$

(ii) $h[n]$ is causal, ROC $|z| > 2$

$$h[n] = \left[2^n + \left(-\frac{1}{2}\right)^n\right] u[n]$$

(b) $H(z) = \frac{5z^2}{z^2 - z - 6}$

$$H(z) = \frac{3}{1 - 3z^{-1}} + \frac{2}{1 + 2z^{-1}}$$

(i) $h[n]$ is stable, ROC $|z| < 2$, ROC includes $|z| = 1$.

$$h[n] = [-(3)^n - (-2)^n] u[-n - 1]$$

(ii) $h[n]$ is causal, ROC $|z| > 3$

$$h[n] = [3^n + (-2)^n] u[n]$$

(c) $H(z) = \frac{4z}{z^2 - \frac{1}{4}z + \frac{1}{16}}$

$$H(z) = \frac{4z^{-1}}{(1 - \frac{1}{4}z^{-1})^2}$$

(i) $h[n]$ is stable, ROC $|z| > \frac{1}{4}$, ROC includes $|z| = 1$.

$$h[n] = 16n(\frac{1}{4})^n u[n]$$

(ii) $h[n]$ is causal, ROC $|z| > \frac{1}{4}$

$$h[n] = 16n(\frac{1}{4})^n u[n]$$

7.28. Use a power series expansion to determine the time-domain signal corresponding to the following z -transforms:

(a) $X(z) = \frac{1}{1 - \frac{1}{4}z^{-2}}, \quad |z| > \frac{1}{4}$

$$X(z) = \sum_{k=0}^{\infty} (\frac{1}{4}z^{-2})^k$$

$$\begin{aligned}
x[n] &= \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \delta[n-2k] \\
&= \begin{cases} \left(\frac{1}{4}\right)^{\frac{n}{2}} & n \text{ even and } n \geq 0 \\ 0 & n \text{ odd} \end{cases}
\end{aligned}$$

(b) $X(z) = \frac{1}{1-\frac{1}{4}z^{-2}}, \quad |z| < \frac{1}{4}$

$$\begin{aligned}
X(z) &= -4z^2 \sum_{k=0}^{\infty} (2z)^{2k} \\
&= -\sum_{k=0}^{\infty} 2^{2(k+1)} z^{2(k+1)} \\
x[n] &= -\sum_{k=0}^{\infty} 2^{2(k+1)} \delta[n+2(k+1)]
\end{aligned}$$

(c) $X(z) = \cos(z^{-3}), \quad |z| > 0$

Note:

$$\cos(\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (\alpha)^{2k}$$

$$\begin{aligned}
X(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (z^{-3})^{2k} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{-6k} \\
x[n] &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \delta[n-6k]
\end{aligned}$$

(d) $X(z) = \ln(1+z^{-1}), \quad |z| > 0$

Note:

$$\ln(1+\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k} (\alpha)^k$$

$$\begin{aligned}
X(z) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (z^{-1})^k \\
x[n] &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \delta[n-k]
\end{aligned}$$

7.29. A causal system has input $x[n]$ and output $y[n]$. Use the transfer function to determine the impulse response of this system.

(a) $x[n] = \delta[n] + \frac{1}{4}\delta[n-1] - \frac{1}{8}\delta[n-2]$, $y[n] = \delta[n] - \frac{3}{4}\delta[n-1]$

$$\begin{aligned} X(z) &= 1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2} \\ Y(z) &= 1 - \frac{3}{4}z^{-1} \\ H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{-\frac{2}{3}}{1 - \frac{1}{4}z^{-1}} + \frac{\frac{5}{3}}{1 + \frac{1}{2}z^{-1}} \\ h[n] &= \frac{1}{3} \left[5\left(-\frac{1}{2}\right)^n - 2\left(\frac{1}{4}\right)^n \right] u[n] \end{aligned}$$

(b) $x[n] = (-3)^n u[n]$, $y[n] = 4(2)^n u[n] - \left(\frac{1}{2}\right)^n u[n]$

$$\begin{aligned} X(z) &= \frac{1}{1 + 3z^{-1}} \\ Y(z) &= \frac{3}{(1 - 2z^{-1})(1 - \frac{1}{2}z^{-1})} \\ H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{10}{1 - 2z^{-1}} + \frac{-7}{1 - \frac{1}{2}z^{-1}} \\ h[n] &= \left[10(2)^n - 7\left(\frac{1}{2}\right)^n \right] u[n] \end{aligned}$$

7.30. A system has impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$. Determine the input to the system if the output is given by

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

(a) $y[n] = 2\delta[n-4]$

$$\begin{aligned} Y(z) &= 2z^{-4} \\ X(z) &= \frac{Y(z)}{H(z)} \\ &= 2z^{-4} - z^{-5} \\ x[n] &= 2\delta[n-4] - \delta[n-5] \end{aligned}$$

(b) $y[n] = \frac{1}{3}u[n] + \frac{2}{3}\left(\frac{-1}{2}\right)^n u[n]$

$$\begin{aligned} Y(z) &= \frac{\frac{1}{3}}{1 - z^{-1}} + \frac{\frac{2}{3}}{1 + \frac{1}{2}z^{-1}} \\ X(z) &= \frac{Y(z)}{H(z)} \\ &= -\frac{1}{2} + \frac{\frac{1}{6}}{1 - z^{-1}} + \frac{\frac{4}{3}}{1 + \frac{1}{2}z^{-1}} \\ x[n] &= -\frac{1}{2}\delta[n] + \frac{1}{6}u[n] + \frac{4}{3}\left(-\frac{1}{2}\right)^n u[n] \end{aligned}$$

7.31. Determine (i) transfer function and (ii) impulse response representations for the systems described by the following difference equations:

(a) $y[n] - \frac{1}{2}y[n-1] = 2x[n-1]$

$$\begin{aligned} Y(z)\left(1 - \frac{1}{2}z^{-1}\right) &= 2z^{-1}X(z) \\ H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{2z^{-1}}{1 - \frac{1}{2}z^{-1}} \\ h[n] &= 2\left(\frac{1}{2}\right)^{n-1}u[n-1] \end{aligned}$$

(b) $y[n] = x[n] - x[n-2] + x[n-4] - x[n-6]$

$$\begin{aligned} Y(z) &= (1 - z^{-2} + z^{-4} - z^{-6})X(z) \\ H(z) &= \frac{Y(z)}{X(z)} \\ &= 1 - z^{-2} + z^{-4} - z^{-6} \\ h[n] &= \delta[n] - \delta[n-2] + \delta[n-4] - \delta[n-6] \end{aligned}$$

(c) $y[n] - \frac{4}{5}y[n-1] - \frac{16}{25}y[n-2] = 2x[n] + x[n-1]$

$$\begin{aligned} Y(z)\left(1 - \frac{4}{5}z^{-1} - \frac{16}{25}z^{-2}\right) &= (2 + z^{-1})X(z) \\ H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{2}{1 - \frac{4}{5}z^{-1}} + \frac{\frac{13}{5}z^{-1}}{\left(1 - \frac{4}{5}z^{-1}\right)^2} \\ h[n] &= \left[2\left(\frac{4}{5}\right)^n + \frac{13}{4}n\left(\frac{4}{5}\right)^n\right]u[n] \end{aligned}$$

7.32. Determine (i) transfer function and (ii) difference-equation representations for the systems with the following impulse responses:

(a) $h[n] = 3 \left(\frac{1}{4}\right)^n u[n-1]$

$$\begin{aligned} h[n] &= 3 \left(\frac{1}{4}\right) \left(\frac{1}{4}\right)^{n-1} u[n-1] \\ H(z) &= \frac{\frac{3}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}} \\ &= \frac{Y(z)}{X(z)} \end{aligned}$$

Taking the inverse z -transform yields:

$$y[n] - \frac{1}{4}y[n-1] = \frac{3}{4}x[n-1]$$

(b) $h[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^{n-2} u[n-1]$

$$\begin{aligned} h[n] &= \left(\frac{1}{3}\right)^n u[n] + 2 \left(\frac{1}{2}\right)^{n-1} u[n-1] \\ H(z) &= \frac{1 + \frac{3}{2}z^{-1} - \frac{2}{3}z^{-2}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \\ &= \frac{Y(z)}{X(z)} \end{aligned}$$

Taking the inverse z -transform yields:

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] + \frac{3}{2}x[n-1] - \frac{2}{3}x[n-2]$$

(c) $h[n] = 2 \left(\frac{2}{3}\right)^n u[n-1] + \left(\frac{1}{4}\right)^n [\cos(\frac{\pi}{6}n) - 2\sin(\frac{\pi}{6}n)]u[n]$

$$\begin{aligned} H(z) &= \frac{\frac{4}{3}z^{-1}}{1 - \frac{2}{3}z^{-1}} + \frac{1 - \frac{1}{4}z^{-1}\cos(\frac{\pi}{6}) - \frac{1}{2}z^{-1}\sin(\frac{\pi}{6})}{1 - z^{-1}\frac{1}{2}\cos(\frac{\pi}{6}) + \frac{1}{16}z^{-2}} \\ &= \frac{1 + (\frac{5}{12} - \frac{1}{8}\sqrt{3})z^{-1} + (\frac{1}{6} - \frac{1}{4}\sqrt{3})z^{-2} + \frac{1}{12}z^{-3}}{1 - (\frac{2}{3} + \frac{1}{4}\sqrt{3})z^{-1} + (\frac{1}{16} + \frac{1}{6}\sqrt{3})z^{-2} - \frac{1}{24}z^{-3}} \end{aligned}$$

Taking the z -transform yields:

$$\begin{aligned} y[n] &- \left(\frac{2}{3} + \frac{1}{4}\sqrt{3}\right)y[n-1] + \left(\frac{1}{16} + \frac{1}{6}\sqrt{3}\right)y[n-2] - \frac{1}{24}y[n-3] \\ &= x[n] + \left(\frac{5}{12} - \frac{1}{8}\sqrt{3}\right)x[n-1] + \left(\frac{1}{6} - \frac{1}{4}\sqrt{3}\right)x[n-2] + \frac{1}{12}x[n-3] \end{aligned}$$

(d) $h[n] = \delta[n] - \delta[n - 5]$

$$H(z) = 1 - z^{-5}$$

Taking the z -transform yields:

$$y[n] = x[n] - x[n - 5]$$

7.33. (a) Take the z -transform of the state-update equation Eq. (2.62) using the time-shift property Eq. (7.13) to obtain

$$\tilde{\mathbf{q}}(z) = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}X(z)$$

where

$$\tilde{\mathbf{q}}(z) = \begin{bmatrix} Q_1(z) \\ Q_2(z) \\ \vdots \\ Q_N(z) \end{bmatrix}$$

is the z -transform of $\mathbf{q}[n]$. Use this result to show that the transfer function of a LTI system is expressed in terms of the state-variable description as

$$H(z) = \mathbf{c}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D$$

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

$$\tilde{\mathbf{q}}(z) = \mathbf{A}\tilde{\mathbf{q}}(z) + \mathbf{b}X(z)$$

$$\tilde{\mathbf{q}}(z)(z\mathbf{I} - \mathbf{A}) = \mathbf{b}X(z)$$

$$\tilde{\mathbf{q}}(z) = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}X(z)$$

$$y[n] = \mathbf{c}\mathbf{q}[n] + Dx[n]$$

$$Y(z) = \mathbf{c}\tilde{\mathbf{q}}(z) + DX(z)$$

$$Y(z) = \mathbf{c}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}X(z) + DX(z)$$

$$H(z) = \mathbf{c}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D$$

(b) Determine transfer function and difference-equation representations for the systems described by the following state-variable descriptions. Plot the pole and zero locations in the z -plane.

(i) $\mathbf{A} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $D = [1]$

$$\begin{aligned} H(z) &= \mathbf{c}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + D \\ &= \frac{z^2 - 2z - \frac{5}{4}}{z^2 - \frac{1}{4}} \end{aligned}$$

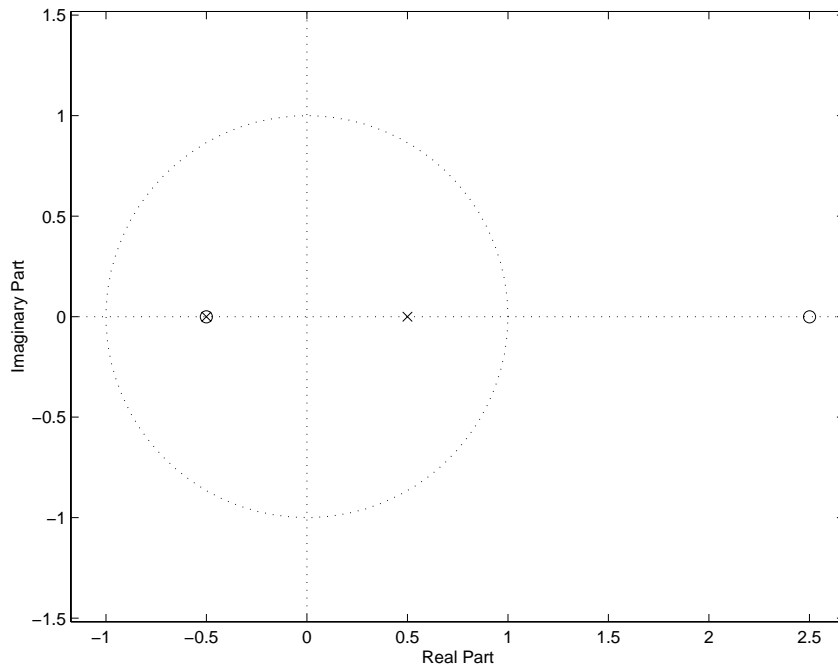


Figure P7.33. (b)-(i) Pole-Zero Plot

(ii) $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{4} \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 2 & 1 \end{bmatrix}$, $D = [0]$

$$H(z) = \frac{2z}{z^2 - \frac{1}{4}z - \frac{3}{8}}$$

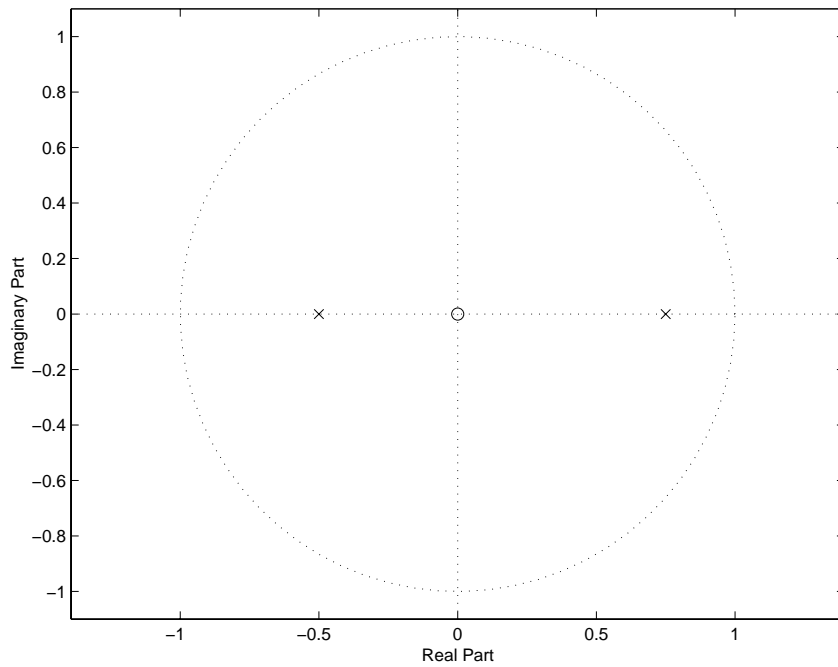


Figure P7.33. (b)-(ii) Pole-Zero Plot

$$(iii) \mathbf{A} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{8} \\ -\frac{7}{2} & \frac{3}{4} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = [0]$$

$$H(z) = \frac{2z - \frac{13}{2}}{z^2 - \frac{1}{2}z + \frac{1}{4}}$$

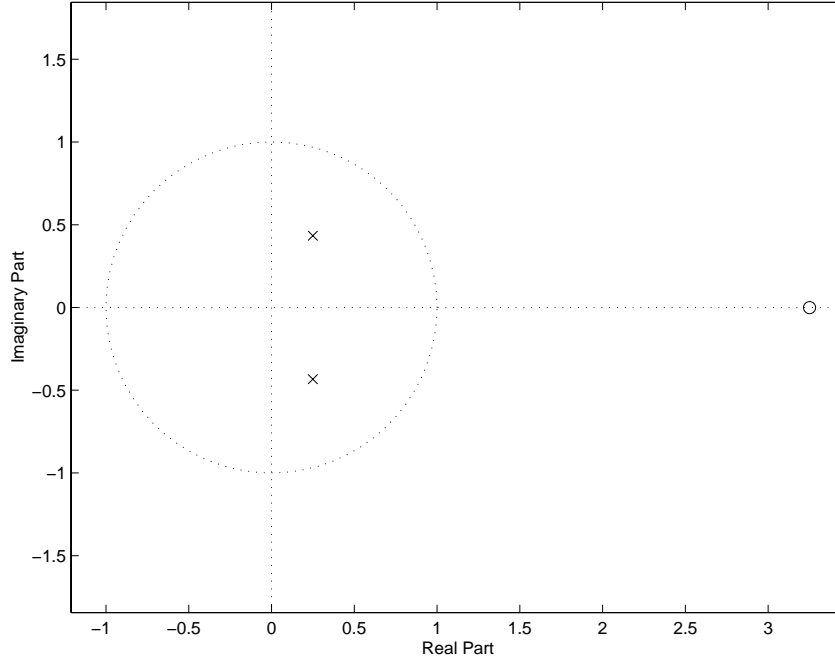


Figure P7.33. (b)-(iii) Pole-Zero Plot

7.34. Determine whether each of the systems described below are (i) causal and stable and (ii) minimum phase.

(a) $H(z) = \frac{2z+3}{z^2+z-\frac{5}{16}}$

$$\begin{aligned} \text{zero at:} \quad z &= -\frac{3}{2} \\ \text{poles at:} \quad z &= -\frac{5}{4}, \frac{1}{4} \end{aligned}$$

(i) Not all poles are inside $|z| = 1$, the system is not causal and stable.

(ii) Not all poles and zeros are inside $|z| = 1$, the system is not minimum phase.

(b) $y[n] - y[n-1] - \frac{1}{4}y[n-2] = 3x[n] - 2x[n-1]$

$$\begin{aligned} \text{zeros at:} \quad z &= 0, \frac{2}{3} \\ \text{poles at:} \quad z &= \frac{1 \pm \sqrt{2}}{2} \end{aligned}$$

- (i) Not all poles are inside $|z| = 1$, the system is not causal and stable.
(ii) Not all poles and zeros are inside $|z| = 1$, the system is not minimum phase.

(c) $y[n] - 2y[n-2] = x[n] - \frac{1}{2}x[n-1]$

$$\begin{aligned} H(z) &= \frac{z(z - \frac{1}{2})}{z^2 - 2} \\ \text{zeros at:} & \quad z = 0, \frac{1}{2} \\ \text{poles at:} & \quad z = \pm\sqrt{2} \end{aligned}$$

- (i) Not all poles are inside $|z| = 1$, the system is not causal and stable.
(ii) Not all poles and zeros are inside $|z| = 1$, the system is not minimum phase.

7.35. For each system described below, identify the transfer function of the inverse system, and determine whether it can be both causal and stable.

(a) $H(z) = \frac{1-8z^{-1}+16z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$

$$\begin{aligned} H(z) &= \frac{(z-4)^2}{(z-\frac{1}{2})^2} \\ H^{inv}(z) &= \frac{(z-\frac{1}{2})^2}{(z-4)^2} \\ \text{poles at:} & \quad z = 4 \text{ (double)} \end{aligned}$$

For the inverse system, not all poles are inside $|z| = 1$, so the system is not causal and stable.

(b) $H(z) = \frac{z^2 - \frac{81}{100}}{z^2 - 1}$

$$\begin{aligned} H^{inv}(z) &= \frac{z^2 - 1}{z^2 - \frac{81}{100}} \\ \text{poles at:} & \quad z = \frac{9}{10} \text{ (double)} \end{aligned}$$

For the inverse system, all poles are inside $|z| = 1$, so the system can be causal and stable.

(c) $h[n] = 10\left(\frac{-1}{2}\right)^n u[n] - 9\left(\frac{-1}{4}\right)^n u[n]$

$$\begin{aligned} H(z) &= \frac{z(z-2)}{(z+\frac{1}{2})(z+\frac{1}{4})} \\ H^{inv}(z) &= \frac{(z+\frac{1}{2})(z+\frac{1}{4})}{z(z-2)} \end{aligned}$$

poles at: $z = 0, 2$

For the inverse system, not all poles are inside $|z| = 1$, so the system cannot be both causal and stable.

(d) $h[n] = 24 \left(\frac{1}{2}\right)^n u[n-1] - 30 \left(\frac{1}{3}\right)^n u[n-1]$

$$\begin{aligned} H(z) &= \frac{2(z + \frac{1}{2})}{(z - \frac{1}{2})(z - \frac{1}{3})} \\ H^{inv}(z) &= \frac{(z - \frac{1}{2})(z - \frac{1}{3})}{2(z + \frac{1}{2})} \\ \text{pole at: } &z = -\frac{1}{2} \end{aligned}$$

For the inverse system, all poles are inside $|z| = 1$, so the system can be both causal and stable.

(e) $y[n] - \frac{1}{4}y[n-2] = 6x[n] - 7x[n-1] + 3x[n-2]$

$$\begin{aligned} H^{inv}(z) &= \frac{X(z)}{Y(z)} \\ &= \frac{(z - \frac{1}{2})(z + \frac{1}{2})}{6z^2 + 7z + 3} \\ \text{poles at: } &z = \frac{7 \pm j\sqrt{23}}{12} \end{aligned}$$

For the inverse system, all poles are inside $|z| = 1$, so the system can be both causal and stable.

(f) $y[n] - \frac{1}{2}y[n-1] = x[n]$

$$\begin{aligned} H^{inv}(z) &= \frac{z - \frac{1}{2}}{z} \\ \text{pole at: } &z = 0 \end{aligned}$$

For the inverse system, all poles are inside $|z| = 1$, so the system can be both causal and stable.

7.36. A system described by a rational transfer function $H(z)$ has the following properties: 1) the system is causal; 2) $h[n]$ is real; 3) $H(z)$ has a pole at $z = j/2$ and exactly one zero; 4) the inverse system has two zeros; 5) $\sum_{n=0}^{\infty} h[n]2^{-n} = 0$; 6) $h[0] = 1$.

By 2) and 3)

poles at $z = \pm \frac{j}{2}$

By 4)

$$H^{inv} = \frac{1}{H}$$

H has two poles. $z = \frac{1}{2}e^{\pm j\frac{\pi}{2}}$

$$\begin{aligned} H(z) &= \frac{A(1 - Cz^{-1})}{1 - z^{-1} \cos(\frac{\pi}{2}) + \frac{1}{4}z^{-2}} \\ &= \frac{A(1 - Cz^{-1})}{1 + \frac{1}{4}z^{-2}} \end{aligned}$$

By 5)

$$\sum_{n=0}^{\infty} h[n]2^{-n} = \sum_{n=0}^{\infty} h[n]z^{-n} \Big|_{z=2} = H(z)$$

Since $H(z) = \frac{A(1 - Cz^{-1})}{1 + \frac{1}{4}z^{-2}},$

$H(z) = 0$ implies

$$\begin{aligned} C &= 2 \\ h[n] &= \left[A \left(\frac{1}{2} \right)^n \cos\left(\frac{\pi}{2}n\right) - 2A \left(\frac{1}{2} \right)^n \sin\left(\frac{\pi}{2}n\right) \right] u[n] \\ h[0] = 1 &= A \\ H(z) &= \frac{1 - 2z^{-1}}{1 + \frac{1}{4}z^{-2}} \\ h[n] &= \left[\left(\frac{1}{2} \right)^n \cos\left(\frac{\pi}{2}n\right) - 2 \left(\frac{1}{2} \right)^n \sin\left(\frac{\pi}{2}n\right) \right] u[n] \end{aligned}$$

(a) Is this system stable?

The poles are inside $|z| = 1$, so the system is stable.

(b) Is the inverse system both stable and causal?

No, the inverse system has a pole at $z = 2$, which is not inside $|z| = 1$.

(c) Find $h[n]$.

$$h[n] = \left[\left(\frac{1}{2} \right)^n \cos\left(\frac{\pi}{2}n\right) - 2 \left(\frac{1}{2} \right)^n \sin\left(\frac{\pi}{2}n\right) \right] u[n]$$

(d) Find the transfer function of the inverse system.

$$\begin{aligned} H^{inv}(z) &= \frac{1}{H(z)} \\ &= \frac{1 + \frac{1}{4}z^{-2}}{1 - 2z^{-1}} \end{aligned}$$

7.37. Use the graphical method to sketch the magnitude response of the systems having the following transfer functions:

(a) $H(z) = \frac{z^{-2}}{1 + \frac{49}{64}z^{-2}}$

$$H(z) = \frac{1}{(z + j\frac{7}{8})(z - j\frac{7}{8})}$$

$$H(e^{j\Omega}) = \frac{1}{(e^{j\Omega} + j\frac{7}{8})(e^{j\Omega} - j\frac{7}{8})}$$

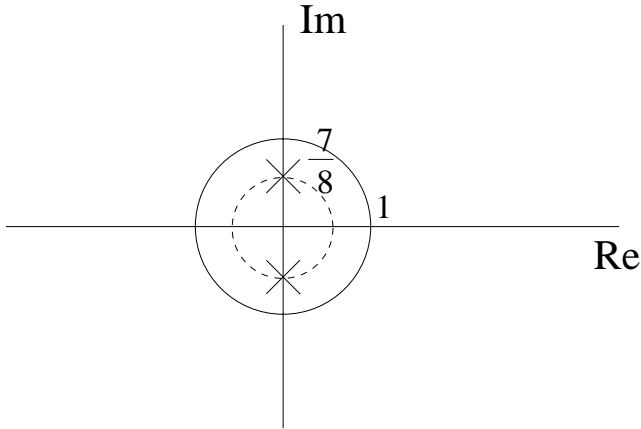


Figure P7.37. (a) Graphical method.

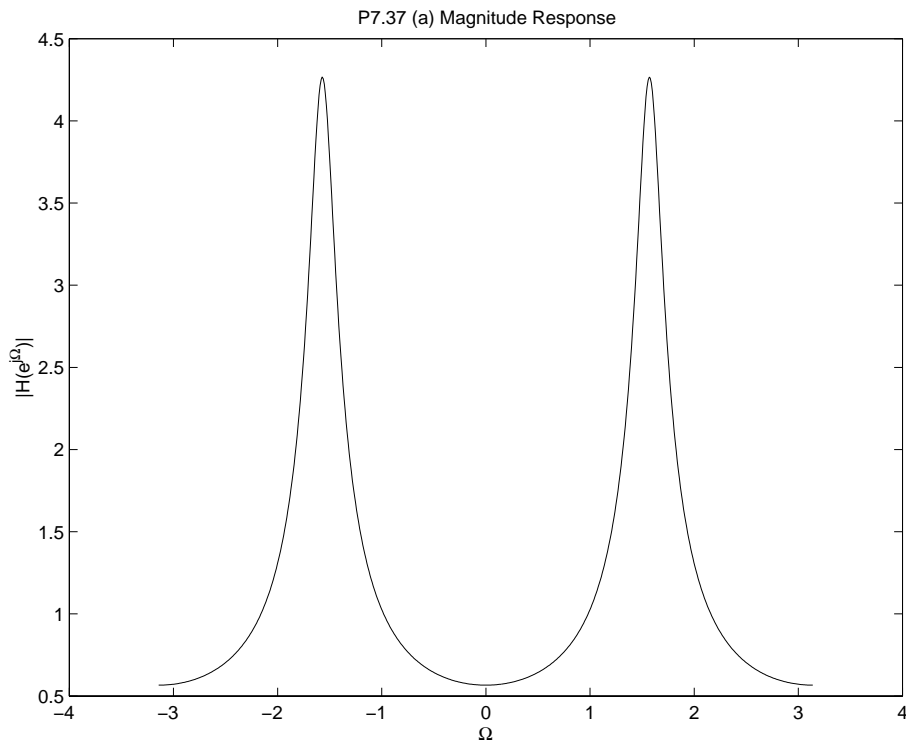


Figure P7.37. (a) Magnitude Response

(b) $H(z) = \frac{1+z^{-1}+z^{-2}}{3}$

$$H(z) = \frac{z^2 + z^1 + 1}{3z^2}$$

$$H(e^{j\Omega}) = \frac{e^{j2\Omega} + e^{j\Omega} + 1}{3e^{j2\Omega}}$$

poles at: $z = 0$, (double)

zeros at: $z = e^{\pm j\frac{2\pi}{3}}$

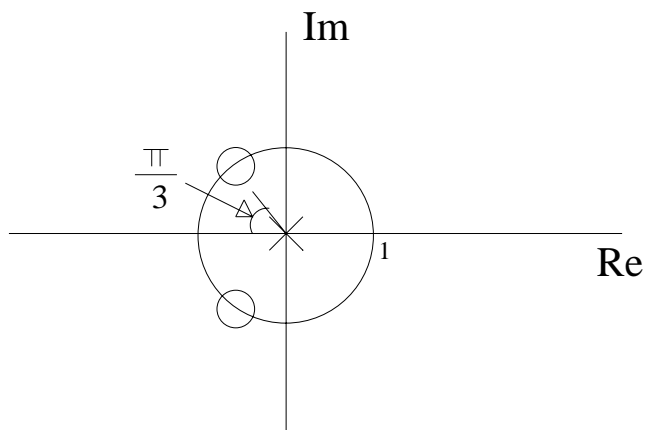


Figure P7.37. (b) Graphical method.

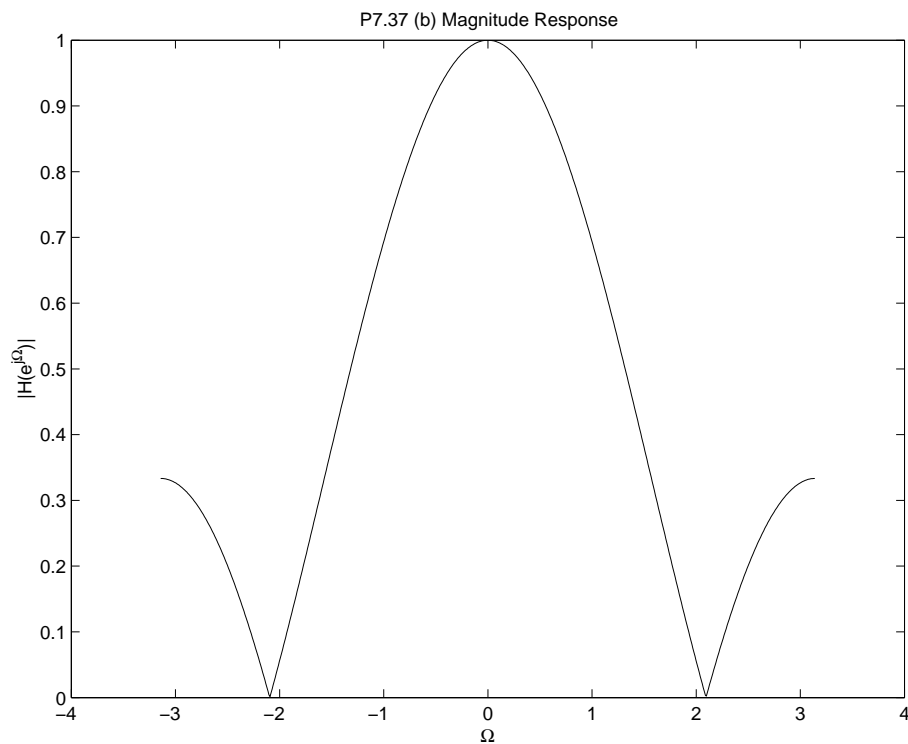


Figure P7.37. (b) Magnitude Response

(c) $H(z) = \frac{1+z^{-1}}{1+(18/10)\cos(\frac{\pi}{4})z^{-1}+(81/100)z^{-2}}$

$$H(z) = \frac{1 + z^{-1}}{(1 - \frac{9}{10}e^{j\frac{3}{4}\pi}z^{-1})(1 - \frac{9}{10}e^{-j\frac{3}{4}\pi}z^{-1})}$$

$$H(e^{j\Omega}) = \frac{e^{j2\Omega} + e^{j\Omega}}{e^{j\Omega} + (18/10)\cos(\frac{\pi}{4})e^{j\Omega} + (81/100)}$$

zeros: $z = -1$

poles: $z = \frac{9}{10}e^{\pm j\frac{3}{4}\pi}$

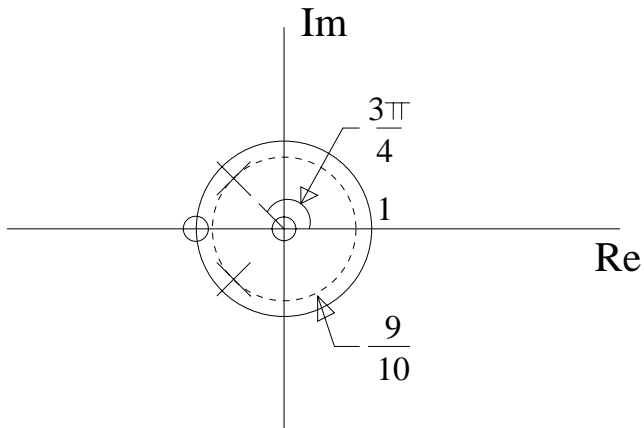


Figure P7.37. (c) Graphical method.

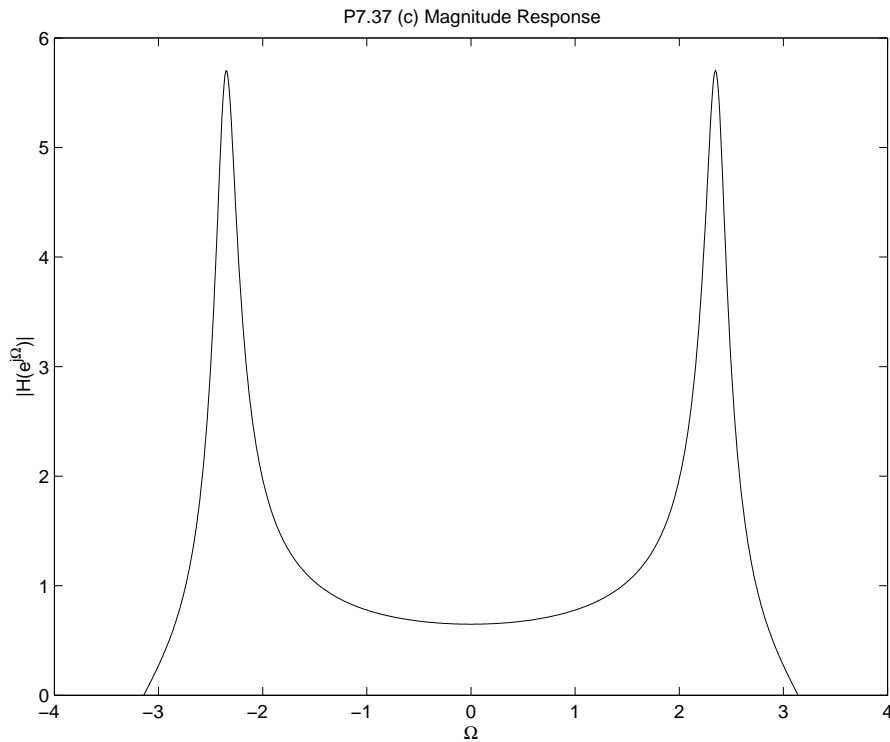


Figure P7.37. (c) Magnitude Response

7.38. Draw block-diagram implementations of the following systems as a cascade of second-order sec-

tions with real-valued coefficients:

$$(a) \quad H(z) = \frac{(1 - \frac{1}{4}e^{j\frac{\pi}{4}}z^{-1})(1 - \frac{1}{4}e^{-j\frac{\pi}{4}}z^{-1})(1 + \frac{1}{4}e^{j\frac{\pi}{8}}z^{-1})(1 + \frac{1}{4}e^{-j\frac{\pi}{8}}z^{-1})}{(1 - \frac{1}{2}e^{j\frac{\pi}{3}}z^{-1})(1 - \frac{1}{2}e^{-j\frac{\pi}{3}}z^{-1})(1 - \frac{3}{4}e^{j\frac{7\pi}{8}}z^{-1})(1 - \frac{3}{4}e^{-j\frac{7\pi}{8}}z^{-1})}$$

$$\begin{aligned} H(z) &= H_1(z)H_2(z) \\ H_1(z) &= \frac{1 - \frac{1}{2}\cos(\frac{\pi}{4})z^{-1} + \frac{1}{16}z^{-2}}{1 - \cos(\frac{\pi}{3})z^{-1} + \frac{1}{4}z^{-2}} \\ H_2(z) &= \frac{1 + \frac{1}{2}\cos(\frac{\pi}{8})z^{-1} + \frac{1}{16}z^{-2}}{1 - \frac{3}{2}\cos(\frac{7\pi}{8})z^{-1} + \frac{9}{16}z^{-2}} \end{aligned}$$

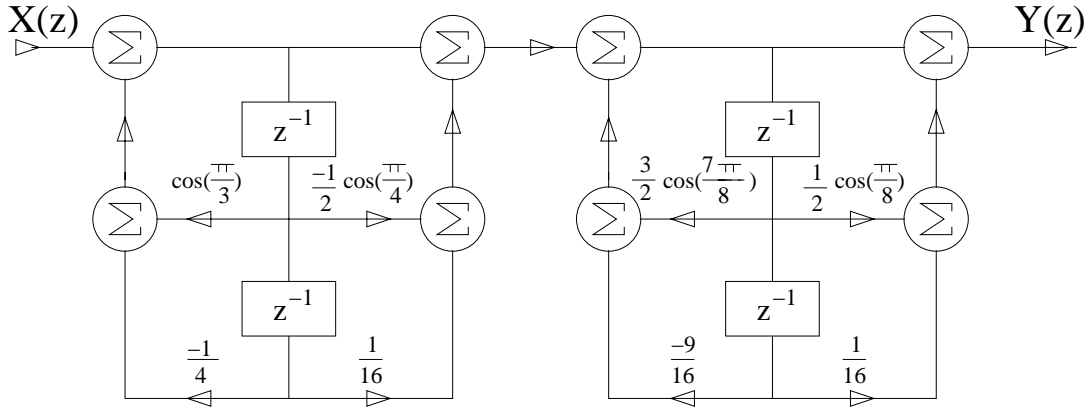


Figure P7.38. (a) Block diagram.

$$(b) \quad H(z) = \frac{(1+2z^{-1})^2(1-\frac{1}{2}e^{j\frac{\pi}{2}}z^{-1})(1-\frac{1}{2}e^{-j\frac{\pi}{2}}z^{-1})}{(1-\frac{3}{8}z^{-1})(1-\frac{3}{8}e^{j\frac{\pi}{3}}z^{-1})(1-\frac{3}{8}e^{-j\frac{\pi}{3}}z^{-1})(1+\frac{3}{4}z^{-1})}$$

$$\begin{aligned} H(z) &= H_1(z)H_2(z) \\ H_1(z) &= \frac{1 + 4z^{-1} + 4z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{9}{32}z^{-2}} \\ H_2(z) &= \frac{1 + \frac{1}{4}z^{-2}}{1 - \frac{3}{4}\cos(\frac{\pi}{3})z^{-1} + \frac{9}{64}z^{-2}} \end{aligned}$$

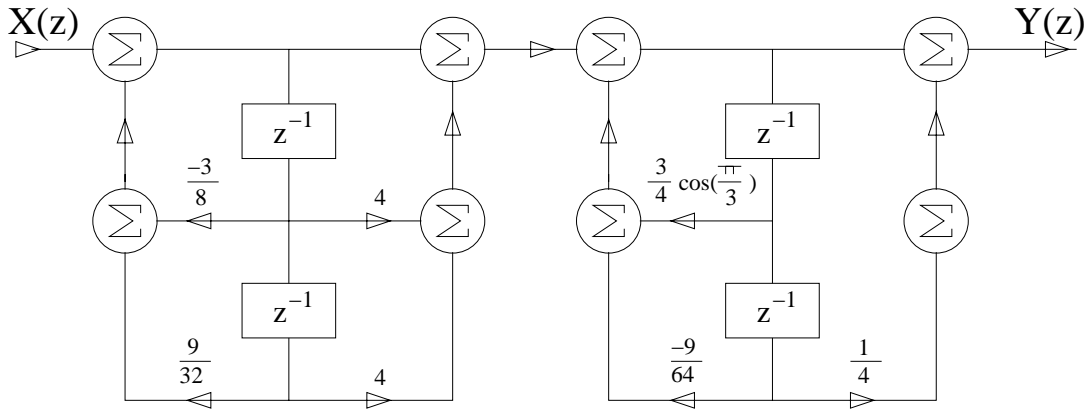


Figure P7.38. (b) Block diagram.

7.39. Draw block diagram implementations of the following systems as a parallel combination of second-order sections with real-valued coefficients:

(a) $h[n] = 2 \left(\frac{1}{2}\right)^n u[n] + \left(\frac{j}{2}\right)^n u[n] + \left(\frac{-j}{2}\right)^n u[n] + \left(\frac{-1}{2}\right)^n u[n]$

$$\begin{aligned} H(z) &= \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - \frac{j}{2}z^{-1}} + \frac{1}{1 + \frac{j}{2}z^{-1}} + \frac{1}{1 + \frac{1}{2}z^{-1}} \\ &= \frac{3 + \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} + \frac{2}{1 + \frac{1}{4}z^{-2}} \\ H(z) &= H_1(z) + H_2(z) \end{aligned}$$

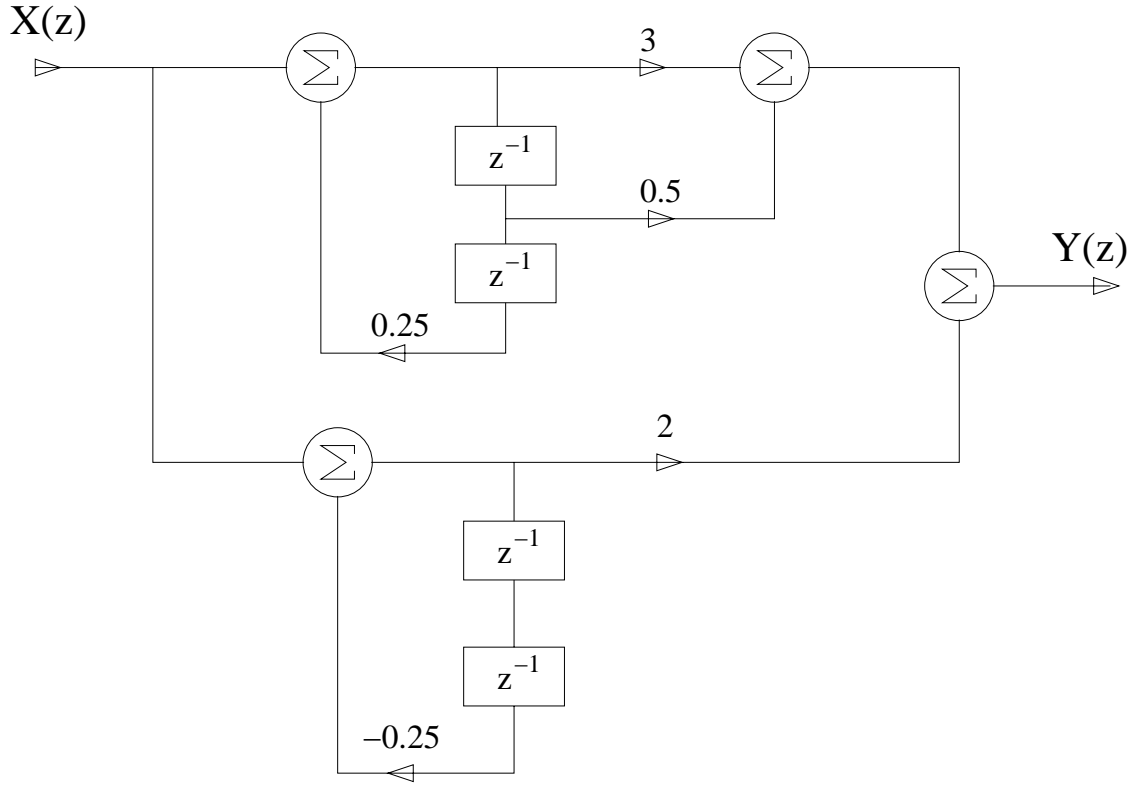


Figure P7.39. (a) Block diagram.

(b) $h[n] = 2 \left(\frac{1}{2}e^{j\frac{\pi}{4}}\right)^n u[n] + \left(\frac{1}{4}e^{j\frac{\pi}{3}}\right)^n u[n] + \left(\frac{1}{4}e^{-j\frac{\pi}{3}}\right)^n u[n] + 2 \left(\frac{1}{2}e^{-j\frac{\pi}{4}}\right)^n u[n]$

$$\begin{aligned} H(z) &= \frac{2}{1 - \frac{1}{2}e^{j\frac{\pi}{4}}z^{-1}} + \frac{1}{1 - \frac{1}{4}e^{j\frac{\pi}{3}}z^{-1}} + \frac{1}{1 - \frac{1}{4}e^{-j\frac{\pi}{3}}z^{-1}} + \frac{2}{1 - \frac{1}{2}e^{-j\frac{\pi}{4}}z^{-1}} \\ &= \frac{4 - \sqrt{2}z^{-1}}{1 - \frac{1}{\sqrt{2}}z^{-1} + \frac{1}{4}z^{-2}} + \frac{2 - \frac{1}{4}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{1}{16}z^{-2}} \\ H(z) &= H_1(z) + H_2(z) \end{aligned}$$

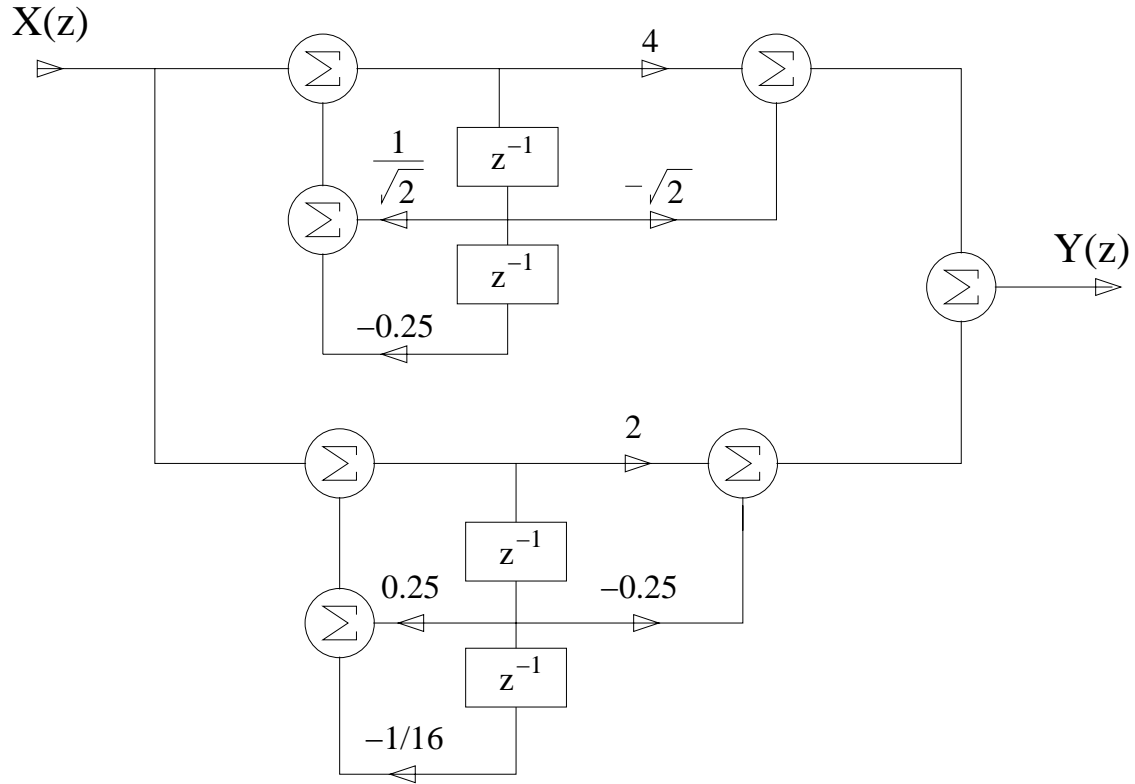


Figure P7.39. (b) Block diagram.

7.40. Determine the transfer function of the system depicted in Fig. P7.40.

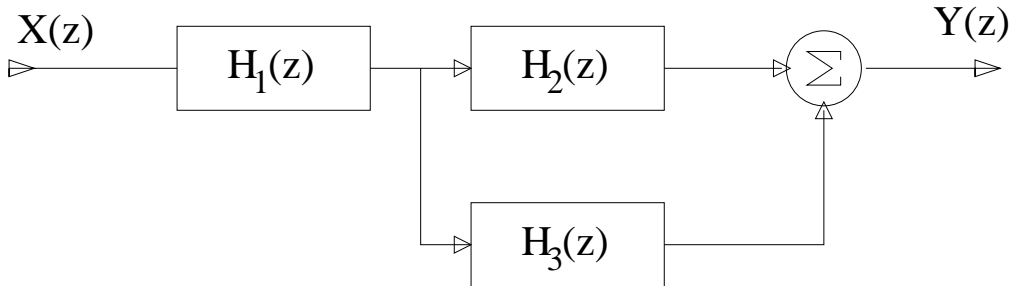


Figure P7.40. System diagram

$$\begin{aligned}
 H_1(z) &= \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-2}} \\
 H_2(z) &= \frac{z^{-2}}{1 + \frac{3}{4}z^{-1} - \frac{1}{8}z^{-2}} \\
 H_3(z) &= \frac{1}{1 + \frac{2}{3}z^{-1}} \\
 H(z) &= H_1(z)H_2(z) + H_1(z)H_3(z)
 \end{aligned}$$

7.41. Let $x[n] = u[n + 4]$.

(a) Determine the unilateral z -transform of $x[n]$.

$$\begin{aligned} x[n] = u[n + 4] &\xleftrightarrow{z_u} X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \\ X(z) &= \sum_{n=0}^{\infty} z^{-n} \\ &= \frac{1}{1 - z^{-1}} \end{aligned}$$

(b) Use the unilateral z -transform time-shift property and the result of (a) to determine the unilateral z -transform of $w[n] = x[n - 2]$.

$$\begin{aligned} w[n] = x[n - 2] &\xleftrightarrow{z_u} W(z) = x[-2] + x[-1]z^{-1} + z^{-2}X(z) \\ W(z) &= 1 + z^{-1} + \frac{z^{-2}}{1 - z^{-1}} \end{aligned}$$

7.42. Use the unilateral z -transform to determine the forced response, the natural response, and the complete response of the systems described by the following difference equations with the given inputs and initial conditions.

(a) $y[n] - \frac{1}{3}y[n - 1] = 2x[n]$, $y[-1] = 1$, $x[n] = (\frac{-1}{2})^n u[n]$

$$\begin{aligned} X(z) &= \frac{1}{1 + \frac{1}{2}z^{-1}} \\ Y(z) - \frac{1}{3}(z^{-1}Y(z) + 1) &= 2X(z) \\ Y(z) \left(1 - \frac{1}{3}z^{-1}\right) &= \frac{1}{3} + 2X(z) \\ Y(z) &= \underbrace{\frac{1}{3} \frac{1}{1 - \frac{1}{3}z^{-1}}}_{Y^{(n)}(z)} + 2 \underbrace{\frac{1}{1 - \frac{1}{3}z^{-1}} X(z)}_{Y^{(f)}(z)} \end{aligned}$$

Natural Response

$$\begin{aligned} Y^{(n)}(z) &= \frac{1}{3} \frac{1}{1 - \frac{1}{3}z^{-1}} \\ y^{(n)}[n] &= \frac{1}{3} \left(\frac{1}{3}\right)^n u[n] \end{aligned}$$

Forced Response

$$\begin{aligned} Y^{(f)}(z) &= \frac{\frac{6}{5}}{1 + \frac{1}{2}z^{-1}} + \frac{\frac{4}{5}}{1 - \frac{1}{3}z^{-1}} \\ y^{(f)}[n] &= \left[\frac{6}{5} \left(-\frac{1}{2}\right)^n + \frac{4}{5} \left(\frac{1}{3}\right)^n \right] u[n] \end{aligned}$$

Complete Response

$$y[n] = \left[\frac{6}{5} \left(-\frac{1}{2} \right)^n + \frac{17}{15} \left(\frac{1}{3} \right)^n \right] u[n]$$

(b) $y[n] - \frac{1}{9}y[n-2] = x[n-1], \quad y[-1] = 1, y[-2] = 0, \quad x[n] = 2u[n]$

$$\begin{aligned} X(z) &= \frac{2}{1-z^{-1}} \\ Y(z) - \frac{1}{9}(z^{-2}Y(z) + z^{-1}) &= z^{-1}X(z) \\ Y(z) \left(1 - \frac{1}{9}z^{-2} \right) &= \frac{1}{9}z^{-1} + z^{-1}X(z) \\ Y(z) &= \underbrace{\frac{1}{9} \frac{z^{-1}}{1 - \frac{1}{9}z^{-2}}}_{Y^{(n)}(z)} + \underbrace{\frac{z^{-1}X(z)}{1 - \frac{1}{9}z^{-2}}}_{Y^{(f)}(z)} \end{aligned}$$

Natural Response

$$\begin{aligned} Y^{(n)}(z) &= \frac{1}{9} \frac{z^{-1}}{1 - \frac{1}{9}z^{-2}} \\ &= \frac{1}{6} \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{6} \frac{1}{1 + \frac{1}{3}z^{-1}} \\ y^{(n)}[n] &= \frac{1}{6} \left[\left(\frac{1}{3} \right)^n - \left(-\frac{1}{3} \right)^n \right] u[n] \end{aligned}$$

Forced Response

$$\begin{aligned} Y^{(f)}(z) &= \frac{\frac{9}{4}}{1-z^{-1}} - \frac{\frac{3}{4}}{1+\frac{1}{3}z^{-1}} - \frac{\frac{3}{2}}{1-\frac{1}{3}z^{-1}} \\ y^{(f)}[n] &= \left[\frac{9}{4} - \frac{3}{4} \left(-\frac{1}{3} \right)^n - \frac{3}{2} \left(\frac{1}{3} \right)^n \right] u[n] \end{aligned}$$

Complete Response

$$y[n] = \left[\frac{9}{4} - \frac{3}{4} \left(-\frac{1}{3} \right)^n - \frac{3}{2} \left(\frac{1}{3} \right)^n \right] u[n] + \frac{1}{6} \left[\left(\frac{1}{3} \right)^n - \left(-\frac{1}{3} \right)^n \right] u[n]$$

(c) $y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + x[n-1], \quad y[-1] = 1, y[-2] = -1, \quad x[n] = 3^n u[n]$

$$\begin{aligned} X(z) &= \frac{1}{1-3z^{-1}} \\ Y(z) - \frac{1}{4}(z^{-1}Y(z) + 1) - \frac{1}{8}(z^{-2}Y(z) + z^{-1} - 1) &= X(z) + z^{-1}X(z) \\ Y(z) \left(1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2} \right) &= \frac{1}{8} + \frac{1}{8}z^{-1} + (1+z^{-1})X(z) \\ Y(z) &= \underbrace{\frac{1}{8} \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}}_{Y^{(n)}(z)} + \underbrace{\frac{(1+z^{-1})X(z)}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}}_{Y^{(f)}(z)} \end{aligned}$$

Natural Response

$$Y^{(n)}(z) = \frac{1}{4} \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{8} \frac{1}{1+\frac{1}{4}z^{-1}}$$

$$y^{(n)}[n] = \frac{1}{8} \left[2 \left(\frac{1}{2} \right)^n - \left(-\frac{1}{4} \right)^n \right] u[n]$$

Forced Response

$$Y^{(f)}(z) = \frac{\frac{96}{65}}{1 - 3z^{-1}} + \frac{-\frac{2}{5}}{1 - \frac{1}{2}z^{-1}} + \frac{-\frac{1}{13}}{1 + \frac{1}{4}z^{-1}}$$

$$y^{(f)}[n] = \left[\frac{96}{65} (3)^n - \frac{2}{5} \left(\frac{1}{2} \right)^n - \frac{1}{13} \left(-\frac{1}{4} \right)^n \right] u[n]$$

Complete Response

$$y[n] = \left[\frac{96}{65} (3)^n - \frac{3}{20} \left(\frac{1}{2} \right)^n - \frac{21}{104} \left(-\frac{1}{4} \right)^n \right] u[n]$$

Solutions to Advanced Problems

7.43. Use the z -transform of $u[n]$ and the differentiation in the z -domain property to derive the formula for evaluating the sum

$$\sum_{n=0}^{\infty} n^2 a^n$$

assuming $|a| < 1$.

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

let

$$x[n] = a^n u[n]$$

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} -n x[n] z^{-(n-1)}$$

$$\frac{d^2}{dz^2} X(z) = \sum_{n=-\infty}^{\infty} n(n-1) x[n] z^{-(n-2)}$$

$$= \sum_{n=-\infty}^{\infty} n^2 x[n] z^{-(n-2)} - \sum_{n=-\infty}^{\infty} n x[n] z^{-(n-2)}$$

$$\frac{d^2}{dz^2} \left(\frac{1}{1 - az^{-1}} \right) = \sum_{n=0}^{\infty} n^2 a^n z^{-(n-2)} - \sum_{n=0}^{\infty} n a^n z^{-(n-2)}$$

Evaluate at $z = 1$

$$\left. \frac{d^2}{dz^2} \left(\frac{1}{1 - az^{-1}} \right) \right|_{z=1} = \sum_{n=0}^{\infty} n^2 a^n - \underbrace{\sum_{n=0}^{\infty} n a^n}_{\left. \frac{d}{dz} X(z) \right|_{z=1}}$$

$$\sum_{n=0}^{\infty} n^2 a^n = \left. \frac{d^2}{dz^2} \left(\frac{1}{1 - az^{-1}} \right) \right|_{z=1} - \left. \frac{d}{dz} \left(\frac{1}{1 - az^{-1}} \right) \right|_{z=1}$$

$$= \frac{2a}{(1-a)^3} + \frac{a}{(1-a)^2}$$

$$= \frac{3a - a^2}{(1 - a)^3}$$

7.44. A continuous-time signal $y(t)$ satisfies the first-order differential equation

$$\frac{d}{dt}y(t) + 2y(t) = x(t)$$

Use the approximation $\frac{d}{dt}y(t) \approx [y(nT_s) - y((n-1)T_s)]/T_s$ to show that the sampled signal $y[n] = y(nT_s)$ satisfies the first-order difference equation

$$y[n] + \alpha y[n-1] = v[n]$$

Express α and $v[n]$ in terms of T_s and $x[n] = x(nT_s)$.

$$\begin{aligned} \frac{y[n] - y[n-1]}{T_s} + 2y[n] &= x[n] \\ y[n] \left(\frac{1}{T_s} + 2 \right) - \frac{1}{T_s}y[n-1] &= x[n] \\ y[n] - \frac{1}{1+2T_s}y[n-1] &= \frac{T_s}{1+2T_s}x[n] \\ \alpha &= -\frac{1}{1+2T_s} \\ v[n] &= \frac{T_s}{1+2T_s}x[n] \end{aligned}$$

7.45. The autocorrelation signal for a real-valued causal signal $x[n]$ is defined as

$$r_x[n] = \sum_{l=0}^{\infty} x[l]x[n+l]$$

Assume the z -transform of $r_x[n]$ converges for some values of z .

Find $x[n]$ if

$$R_x(z) = \frac{1}{(1 - \alpha z^{-1})(1 - \alpha z)}$$

where $|\alpha| < 1$.

$$\begin{aligned} r_x[n] &= x[n] * x[-n] \\ \text{Let } y[n] &= x[-n] \\ r_x[n] &= \sum_{k=-\infty}^{\infty} y[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[-k]x[n-k] \\ \text{let } p &= -k \end{aligned}$$

$$\begin{aligned}
r_x[n] &= \sum_{p=-\infty}^{\infty} x[p]x[n+p] \\
&= \sum_{l=-\infty}^{\infty} x[l]x[n+l] \\
&= \sum_{l=0}^{\infty} x[l]x[n+l] \\
&\quad \text{since } x[l] = 0, \text{ for } l < 0 \\
R_x(z) &= X\left(\frac{1}{z}\right)X(z) \\
&= \left(\frac{1}{1-\alpha z}\right)\frac{1}{1-\alpha z^{-1}} \\
\text{Implies} \\
X(z) &= \frac{1}{1-\alpha z^{-1}} \\
x[n] &= \alpha^n u[n]
\end{aligned}$$

7.46. The cross-correlation of two real-valued signals $x[n]$ and $y[n]$ is expressed as

$$r_{xy}(n) = \sum_{l=-\infty}^{\infty} x[l]y[n+l]$$

(a) Express $r_{xy}[n]$ as a convolution of two sequences.

$$r_{xy}[n] = x[n] * y[-n], \text{ see previous problem}$$

(b) Find the z -transform of $r_{xy}[n]$ as a function of the z -transforms of $x[n]$ and $y[n]$.

$$R_{xy}(z) = X(z)Y\left(\frac{1}{z}\right)$$

7.47. A signal with rational z -transform has even symmetry, that is, $x[n] = x[-n]$.

(a) What constraints must the poles of such a signal satisfy?

$$\begin{aligned}
x[n] &\xleftrightarrow{z} X(z) \\
x[-n] &\xleftrightarrow{z} X\left(\frac{1}{z}\right) \\
\text{Implies} \\
X(z) &= X\left(\frac{1}{z}\right)
\end{aligned}$$

This means if there is a pole at z_o , there must also be a pole at $\frac{1}{z_o}$. Hence poles occur in reciprocal pairs.

(b) Show that the z -transform corresponds to a stable system if and only if $|\sum_{n=-\infty}^{\infty} x[n]| < \infty$.

The reciprocal poles are $(z_o, \frac{1}{z_o})$ assume they take the following form:

$$\begin{aligned} z_o &= r_o e^{j\theta_o} \\ \frac{1}{z_o} &= \frac{1}{r_o} e^{-j\theta_o} \end{aligned}$$

If z_o is inside $|z| = 1$, its z -transform is right-sided and stable. For the pole at $\frac{1}{z_o}$, its corresponding z -transform is either right-sided unstable, or left-sided stable. For convergence, the ROC must include the unit circle, $|z| = 1$, which means the z -transforms are exponentially decaying as they approach ∞ , $-\infty$ respectively.

(c) Suppose

$$X(z) = \frac{2 - (17/4)z^{-1}}{(1 - (1/4)z^{-1})(1 - 4z^{-1})}$$

Determine the ROC and find $x[n]$.

For the system to be stable, the pole at $z = \frac{1}{4}$ must be right-sided, and the pole at $z = 4$ must be left sided so their z -transforms are exponentially decaying as they approach ∞ , $-\infty$ respectively. This implies the ROC is $\frac{1}{4} < |z| < 4$

7.48. Consider a LTI system with transfer function

$$H(z) = \frac{1 - a^*z}{z - a}, \quad |a| < 1$$

Here the pole and zero are a conjugate reciprocal pair.

(a) Sketch a pole-zero plot for this system in the z -plane.

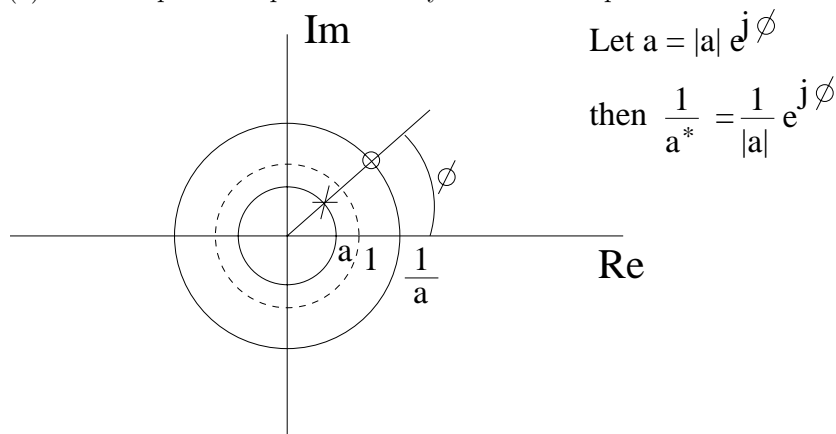


Figure P7.48. (a) Pole-Zero plot.

(b) Use the graphical method to show that the magnitude response of this system is unity for all frequency. A system with this characteristic is termed an **all-pass** system.

$$\begin{aligned} |H(e^{j\Omega})| &= \left| \frac{1 - a^* e^{j\Omega}}{e^{j\Omega} - a} \right| \\ &= |1 - a^* e^{j\Omega}| \frac{1}{|e^{j\Omega} - a|} \end{aligned}$$

As shown below, $|H(e^{j\Omega})| = 1$ for all Ω .

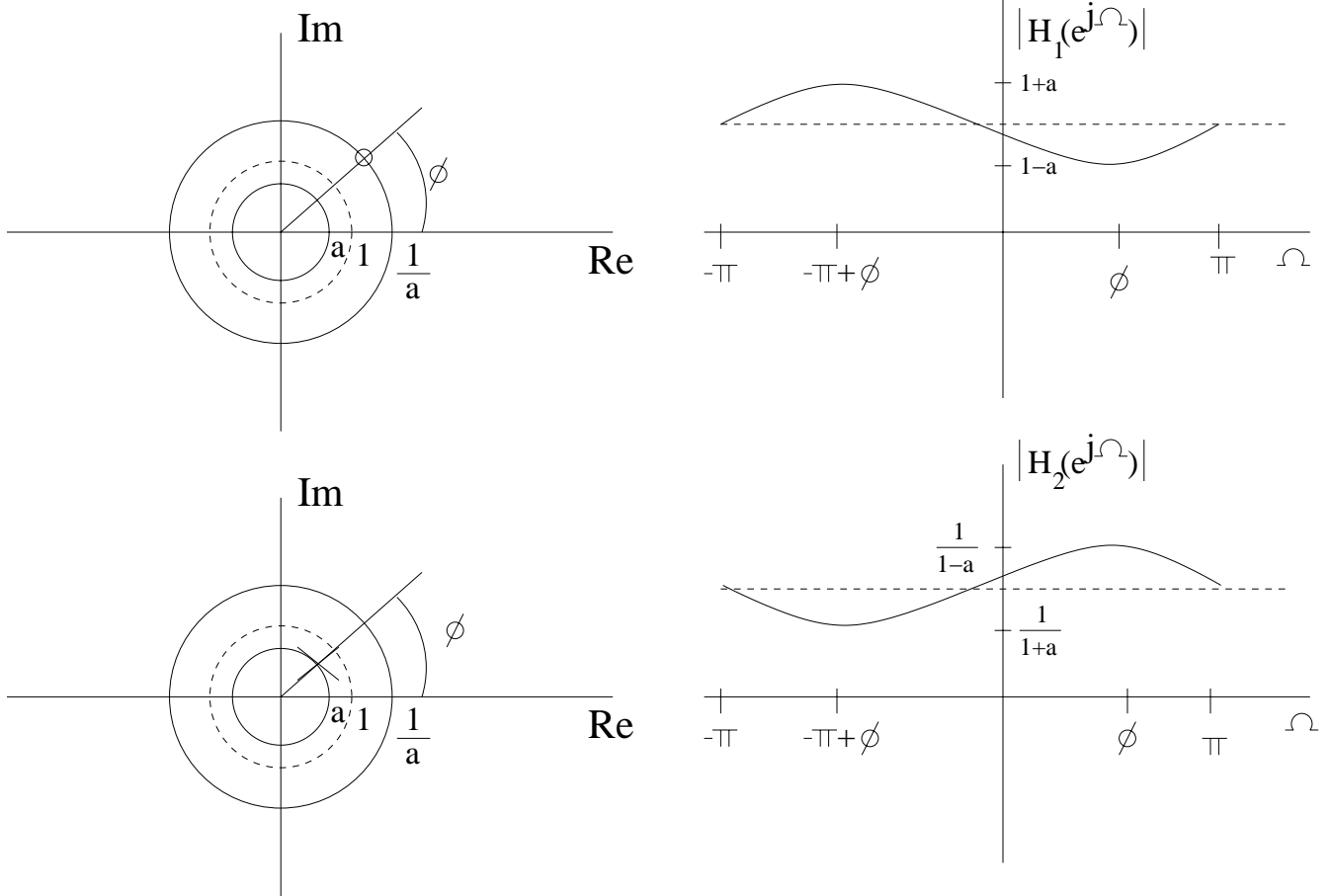


Figure P7.48. (b) Magnitude Response.

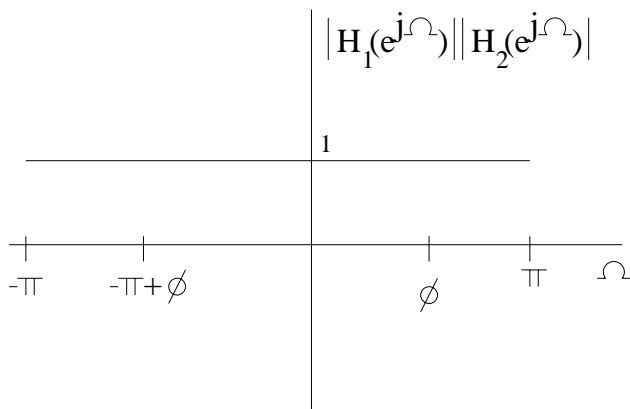


Figure P7.48. (b) Magnitude Response.

(c) Use the graphical method to sketch the phase response of this system for $a = \frac{1}{2}$.

$$\begin{aligned}
 H(e^{j\Omega}) &= \frac{1 - \frac{1}{2}e^{j\Omega}}{e^{j\Omega} - \frac{1}{2}} \\
 \arg\{H(e^{j\Omega})\} &= \arg\left\{1 - \frac{1}{2}e^{j\Omega}\right\} - \arg\left\{e^{j\Omega} - \frac{1}{2}\right\} \\
 &= \pi + \arg\{e^{j\Omega} - 2\} - \arg\left\{e^{j\Omega} - \frac{1}{2}\right\}
 \end{aligned}$$

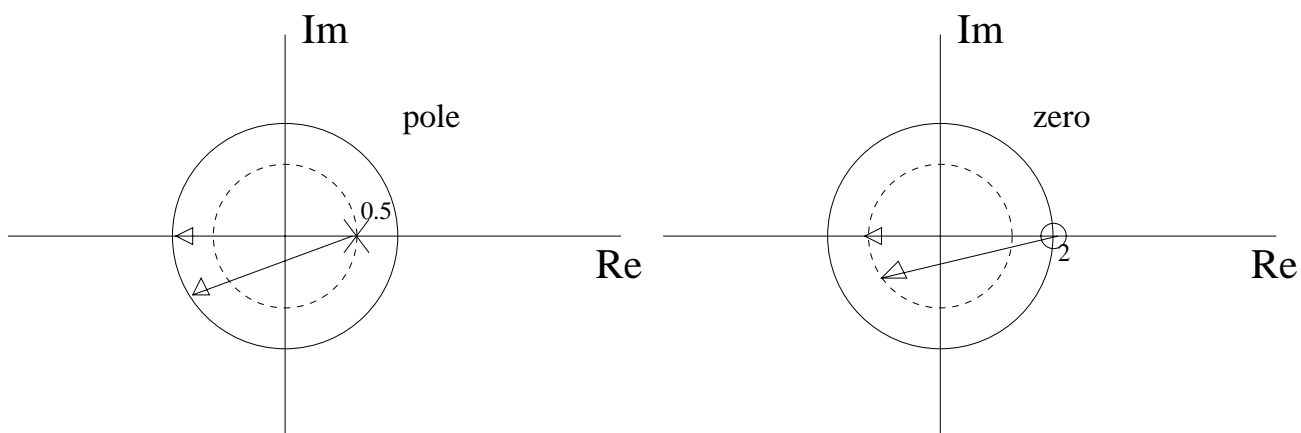


Figure P7.48. (c) Pole/Zero graphical method.

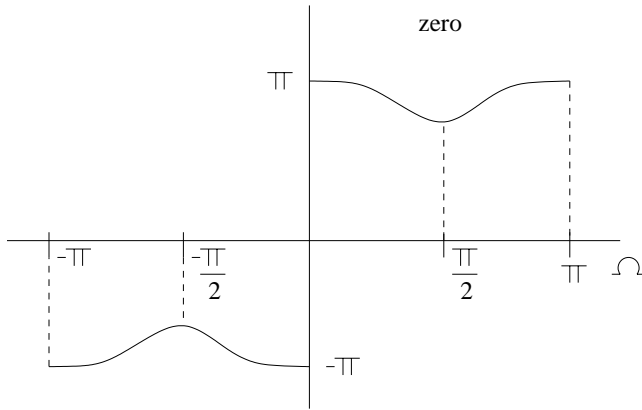


Figure P7.48. (c) Zero phase response.

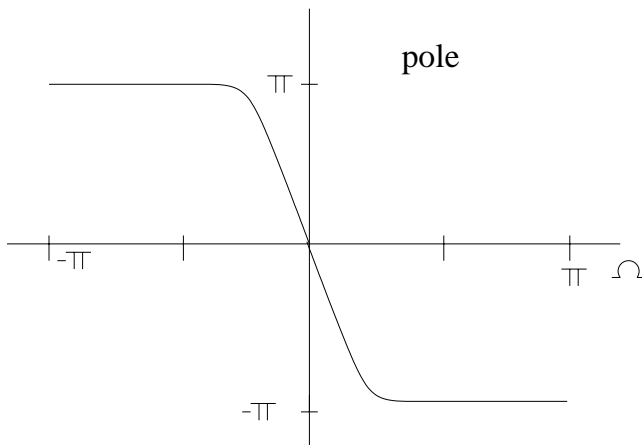


Figure P7.48. (c) Pole phase response.

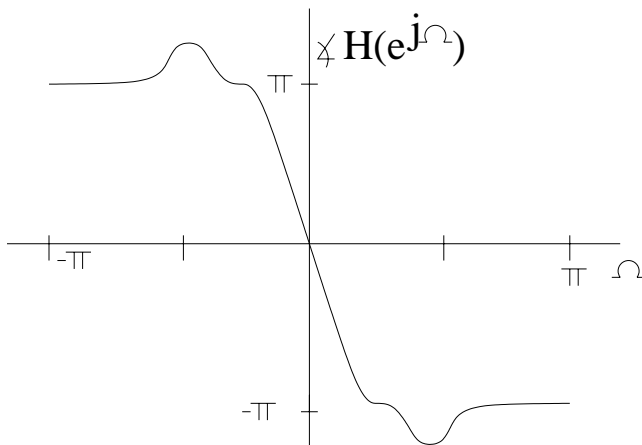


Figure P7.48. (c) Phase response.

(d) Use the result from (b) to prove that any system with a transfer function of the form

$$H(z) = \prod_{k=1}^P \frac{1 - a_k^* z}{z - a_k} \quad |a_k| < 1$$

corresponds to a stable and causal all-pass system.

Since $|a_k| < 1$, if the system is causal, then the system is stable.

$$\begin{aligned}
 H(e^{j\Omega}) &= \prod_{k=1}^P \frac{1 - a_k^* e^{j\Omega}}{e^{j\Omega} - a_k} \\
 |H(e^{j\Omega})| &= \left| \frac{1 - a_1^* e^{j\Omega}}{e^{j\Omega} - a_1} \frac{1 - a_2^* e^{j\Omega}}{e^{j\Omega} - a_2} \cdots \frac{1 - a_p^* e^{j\Omega}}{e^{j\Omega} - a_p} \right| \\
 H(e^{j\Omega}) &= \prod_{k=1}^P \frac{1 - a_k^* e^{j\Omega}}{e^{j\Omega} - a_k} \\
 |H(e^{j\Omega})| &= \left| \frac{1 - a_1^* e^{j\Omega}}{e^{j\Omega} - a_1} \right| \left| \frac{1 - a_2^* e^{j\Omega}}{e^{j\Omega} - a_2} \right| \cdots \left| \frac{1 - a_p^* e^{j\Omega}}{e^{j\Omega} - a_p} \right| \\
 &= 1
 \end{aligned}$$

The system is all-pass.

(e) Can a stable and causal all-pass system also be minimum phase? Explain.

For a stable and causal all-pass system, $|a_k| < 1$ for all k . Using $P=1$:

$$\begin{aligned}
 H(z) &= \frac{1 - a_1^* z}{z - a_1} \\
 \text{The zero is } z_z &= \frac{1}{a_1^*} \\
 \text{Which implies} \\
 |z_z| &= \frac{1}{|a_1^*|} > 1
 \end{aligned}$$

This system cannot also be minimum phase.

7.49. Let

$$H(z) = F(z)(z - a) \text{ and } G(z) = F(z)(1 - az)$$

where $0 < a < 1$ is real.

(a) Show that $|G(e^{j\Omega})| = |H(e^{j\Omega})|$.

$$\begin{aligned}
 G(z) &= H(z) \underbrace{\frac{1 - az}{z - a}}_{\text{all-pass term}} \\
 |G(e^{j\Omega})| &= |H(e^{j\Omega})| \left| \frac{1 - az}{z - a} \right| \\
 &= |H(e^{j\Omega})|
 \end{aligned}$$

Since $\left| \frac{1-az}{z-a} \right| = 1$, see prob 7.48

(b) Show that $g[n] = h[n] * v[n]$ where

$$V(z) = \frac{z^{-1} - a}{1 - az^{-1}}$$

$V(z)$ is thus the transfer function of an all-pass system (see Problem P7.48).

$$\begin{aligned} V(z) &= \frac{1-az}{z-a} \\ &= \frac{z^{-1}-a}{1-az^{-1}} \\ G(z) = H(z)V(z) &\xleftrightarrow{z} g[n] = h[n] * v[n] \end{aligned}$$

(c) One definition of the average delay introduced by a causal system is the normalized first moment

$$d = \frac{\sum_{k=0}^{\infty} kv^2[k]}{\sum_{k=0}^{\infty} v^2[k]}$$

Calculate the average delay introduced by the all-pass system $V(z)$.

$$\begin{aligned} V(z) &= \frac{z^{-1}}{1-az^{-1}} - \frac{a}{1-az^{-1}} \\ v[k] &= a^{k-1}u[k-1] - aa^k u[k] \\ \text{for } k=0 & \\ v[0] &= -a \\ v^2[0] &= a^2 \\ \text{for } k \geq 1 & \\ v[k] &= a^{k-1} - a^{k+1} \\ v^2[k] &= a^{2k-2} + a^{2k+2} - 2a^{(k-1)+(k+1)} \\ &= a^{2k}(a^{-2} + a^2 - 2) \\ \sum_{k=0}^{\infty} kv^2[k] &= (a^{-2} + a^2 - 2) \sum_{k=0}^{\infty} k(a^2)^k \\ &= \frac{1 + a^4 - 2a^2}{(1 - a^2)^2} \\ \sum_{k=0}^{\infty} v^2[k] &= a^2 + (a^{-2} + a^2 - 2) \sum_{k=1}^{\infty} (a^2)^k \\ &= a^2 + \frac{1 + a^4 - 2a^2}{1 - a^2} \\ &= \frac{1 - a^2}{1 - a^2} \\ &= 1, \text{ which also follows from Parseval's Theorem.} \\ d &= \frac{\sum_{k=0}^{\infty} kv^2[k]}{\sum_{k=0}^{\infty} v^2[k]} \end{aligned}$$

$$= \frac{1 + a^4 - 2a^2}{(1 - a^2)^2}$$

7.50. The transfer function of a LTI system is expressed as

$$H(z) = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{\prod_{k=1}^N (1 - d_k z^{-1})}$$

where $|d_k| < 1, k = 1, 2, \dots, N$, $|c_k| < 1, k = 1, 2, \dots, M - 1$, and $|c_M| > 1$.

(a) Show that $H(z)$ can be factored in the form $H(z) = H_{min}(z)H_{ap}(z)$ where $H_{min}(z)$ is minimum phase and $H_{ap}(z)$ is all-pass (see Problem P7.48).

$$\begin{aligned} H(z) &= (1 - c_M z^{-1}) \frac{b_0 \prod_{k=1}^{M-1} (1 - c_k z^{-1})}{\prod_{k=1}^N (1 - d_k z^{-1})} \\ &= \underbrace{\frac{b_0 \prod_{k=1}^{M-1} (1 - c_k z^{-1}) (1 - \frac{1}{c_M^*} z^{-1})}{\prod_{k=1}^N (1 - d_k z^{-1})}}_{H_{min}(z)} \left(\underbrace{\frac{1 - c_M z^{-1}}{1 - \frac{1}{c_M^*} z^{-1}}}_{H_{ap}(z)} \right) \\ &= H_{min}(z) H_{ap}(z) \end{aligned}$$

(b) Find a minimum phase equalizer with transfer function $H_{eq}(z)$ chosen so that $|H(e^{j\Omega})H_{eq}(e^{j\Omega})| = 1$ and determine the transfer function of the cascade $H(z)H_{eq}(z)$.

$$\begin{aligned} H_{eq}(z) &= \frac{1}{H_{min}(z)} \\ \text{so } H(z)H_{eq}(z) &= H_{ap}(z) \\ |H(z)H_{eq}(z)| &= |H_{ap}(z)| = 1 \\ H_{ap}(z) &= \frac{1 - c_M z^{-1}}{1 - \frac{1}{c_M^*} z^{-1}} \end{aligned}$$

7.51. A very useful structure for implementing nonrecursive systems is the so-called lattice structure. The lattice is constructed as a cascade of two input, two output sections of the form depicted in Fig. P7.51

(a). An M^{th} order lattice structure is depicted in Fig. P7.51 (b).

(a) Find the transfer function of a second-order ($M = 2$) lattice having $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{4}$.

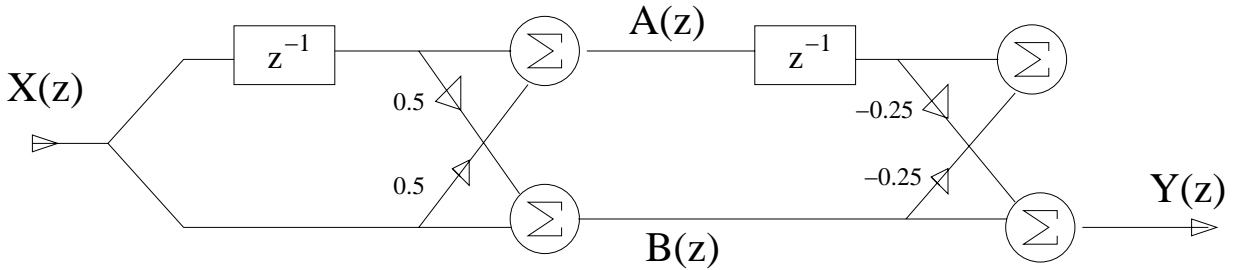


Figure P7.51. (a) Lattice Diagram.

$$\begin{aligned}
Y(z) &= -\frac{1}{4}A(z)z^{-1} + B(z) \\
A(z) &= X(z)(z^{-1} + \frac{1}{2}) \\
B(z) &= X(z)(\frac{1}{2}z^{-1} + 1) \\
Y(z) &= \left(-\frac{1}{4}z^{-2} + \frac{3}{8}z^{-1} + 1\right)X(z) \\
H(z) &= \frac{Y(z)}{X(z)} \\
&= 1 + \frac{3}{8}z^{-1} - \frac{1}{4}z^{-2}
\end{aligned}$$

(b) We may determine the relationship between the transfer function and lattice structure by examining the effect of adding a section on the transfer function, as depicted in Fig. P7.51(c).

Here we have defined $H_i(z)$ as the transfer function between the input and the output of the lower branch in the i^{th} section and $\tilde{H}_i(z)$ as the transfer function between the input and the output of the upper branch in the i^{th} section. Write the relationship between the transfer functions to the $(i-1)^{st}$ and i^{th} stages as

$$\begin{bmatrix} \tilde{H}_i(z) \\ H_i(z) \end{bmatrix} = \mathbf{T}(z) \begin{bmatrix} \tilde{H}_{i-1}(z) \\ H_{i-1}(z) \end{bmatrix}$$

where $\mathbf{T}(z)$ is a two-by-two matrix. Express $\mathbf{T}(z)$ in terms of c_i and z^{-1} .

$$\begin{aligned}
\tilde{H}_i(z) &= \tilde{H}_{i-1}(z)z^{-1} + c_i H_{i-1}(z) \\
H_i(z) &= \tilde{H}_{i-1}(z)z^{-1}c_i + H_{i-1}(z) \\
\begin{bmatrix} \tilde{H}_i(z) \\ H_i(z) \end{bmatrix} &= \begin{bmatrix} z^{-1} & c_i \\ z^{-1}c_i & 1 \end{bmatrix} \begin{bmatrix} \tilde{H}_{i-1}(z) \\ H_{i-1}(z) \end{bmatrix} \\
\mathbf{T}(z) &= \begin{bmatrix} z^{-1} & c_i \\ z^{-1}c_i & 1 \end{bmatrix}
\end{aligned}$$

(c) Use induction to prove that $\tilde{H}_i(z) = z^{-i}H_i(z^{-1})$.

$$\begin{aligned}
i = 1 \quad & \tilde{H}_{i-1}(z) = H_{i-1}(z) = 1 \\
& \tilde{H}_1(z) = z^{-1} + c_1 \\
& H_1(z) = z^{-1}c_1 + 1 \\
& H_1(z^{-1}) = 1 + zc_1 \\
& \tilde{H}_1(z) = z^{-1}H_1(z^{-1}) \\
i = k \quad & \text{assume } \tilde{H}_k(z) = z^{-k}H_k(z^{-1}) \\
i = k + 1 \quad & \tilde{H}_{k+1}(z) = z^{-1}\tilde{H}_k(z) + c_{k+1}H_k(z)
\end{aligned}$$

$$\begin{aligned}
\tilde{H}_{k+1}(z) &= z^{-1}c_{k+1}\tilde{H}_k(z) + H_k(z) \\
\text{substitute } \tilde{H}_k(z) &= z^{-k}H_k(z^{-1}) \\
\tilde{H}_{k+1}(z) &= z^{-(k+1)}H_k(z^{-1}) + c_{k+1}H_k(z) \\
H_{k+1}(z) &= z^{-(k+1)}c_{k+1}H_k(z^{-1}) + H_k(z) \\
H_{k+1}(z^{-1}) &= z^{(k+1)}c_{k+1}H_k(z) + H_k(z^{-1}) \\
\tilde{H}_{k+1}(z) &= z^{-(k+1)}\left(z^{(k+1)}c_{k+1}H_k(z) + H_k(z^{-1})\right) \\
\tilde{H}_{k+1}(z) &= z^{-(k+1)}H_{k+1}(z^{-1}) \\
\text{Therefore} \\
\tilde{H}_i(z) &= z^{-i}H_i(z^{-1})
\end{aligned}$$

(d) Show that the coefficient of z^{-i} in $H_i(z)$ is given by c_i .

$$\begin{aligned}
\begin{bmatrix} \tilde{H}_i \\ H_i \end{bmatrix} &= \begin{bmatrix} z^{-1} & c_i \\ z^{-1}c_i & 1 \end{bmatrix} \begin{bmatrix} \tilde{H}_{i-1} \\ H_{i-1} \end{bmatrix} \\
H_i &= z^{-1}c_i\tilde{H}_{i-1} + H_{i-1} \\
\tilde{H}_{i-1}(z) &= z^{-(i-1)} + \dots + c_{i-1}
\end{aligned}$$

The highest order of (z^{-1}) in $H_i(z)$ is (i) , and the coefficients of z^{-i} is (c_i) , since $H_{i-1}(z)$ does not contribute to z^{-i} , therefore the coefficient of z^{-i} is $H_i(z)$ is given by (c_i) .

(e) By combining the results of (b) - (d) we may derive an algorithm for finding the c_i required by the lattice structure to implement an arbitrary order M nonrecursive transfer function $H(z)$. Start with $i = M$ so that $H_M(z) = H(z)$. The result of (d) implies that c_M is the coefficient of z^{-M} in $H(z)$. By decreasing i , continue this algorithm to find the remaining c_i . *Hint:* Use the result of (b) to find a two-by-two matrix $\mathbf{A}(z)$ such that

$$\begin{aligned}
\begin{bmatrix} \tilde{H}_{i-1}(z) \\ H_{i-1}(z) \end{bmatrix} &= \mathbf{A}(z) \begin{bmatrix} \tilde{H}_i(z) \\ H_i(z) \end{bmatrix} \\
\begin{bmatrix} \tilde{H}_i(z) \\ H_i(z) \end{bmatrix} &= \mathbf{T} \begin{bmatrix} \tilde{H}_{i-1}(z) \\ H_{i-1}(z) \end{bmatrix} \\
\begin{bmatrix} \tilde{H}_{i-1}(z) \\ H_{i-1}(z) \end{bmatrix} &= \mathbf{A} \begin{bmatrix} \tilde{H}_{i-1}(z) \\ H_{i-1}(z) \end{bmatrix} \tag{1} \\
\text{where } \mathbf{A} &= \mathbf{T}^{-1} \\
\mathbf{A} &= \frac{1}{z^{-1}(1 - c_i^2)} \begin{bmatrix} 1 & -c_i \\ -z^{-1}c_i & \end{bmatrix} \\
\text{let} \\
H(z) &= \sum_{k=0}^M b_k z^{-k}
\end{aligned}$$

$$= H_M(z)$$

Where $H(z)$ is given. From this we have $c_M = b_M$. The following is an algorithm to obtain all c_i 's.

- (1) $c_M = b_M$
- (2) for $i = M$ to 2, descending
 - compute $\tilde{H}_{i-1}(z)$ and $H_{i-1}(z)$ from (1)
 - get c_{i-1} from $H_{i-1}(z)$
- end

Thus we will have c_1, c_2, \dots, c_M .

7.52. Causal filters always have a nonzero phase response. One technique for attaining zero phase response from a causal filter involves filtering the signal twice, once in the forward direction and the second time in the reverse direction. We may describe this operation in terms of the input $x[n]$ and filter impulse response $h[n]$ as follows. Let $y_1[n] = x[n] * h[n]$ represent filtering the signal in the forward direction. Now filter $y_1[n]$ backwards to obtain $y_2[n] = y_1[-n] * h[n]$. The output is then given by reversing $y_2[n]$ to obtain $y[n] = y_2[-n]$.

(a) Show that this set of operations is equivalently represented by a filter with impulse response $h_o[n]$ as $y[n] = x[n] * h_o[n]$ and express $h_o[n]$ in terms of $h[n]$.

$$\begin{aligned}
 y[n] &= y_1[n] * h[n] \\
 &= (x[n] * h[n]) * h[-n] \\
 &= x[n] * (h[n] * h[-n]) \\
 &= x[n] * h_o[n] \\
 h_o[n] &= h[n] * h[-n]
 \end{aligned}$$

(b) Show that $h_o[n]$ is an even signal and that the phase response of any system with an even impulse response is zero.

$$\begin{aligned}
 h_o[-n] &= h[-n] * h[n] \\
 &= h[n] * h[-n] \\
 h_o[-n] &= h_o[n]
 \end{aligned}$$

Which shows that $h_o[n]$ is an even signal. Since $h_o[n]$ is even, the phase response can be found from the following:

$$\begin{aligned}
 H_o(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} h_o[n] e^{-j\Omega n} \\
 H_o^*(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} h_o[n] e^{j\Omega n}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=-\infty}^{\infty} h_o[-n]e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} h_o[n]e^{-j\Omega n} \\
H_o^*(e^{j\Omega}) &= H_o(e^{j\Omega}) \\
\arg\{H_o(e^{j\Omega})\} &= 0
\end{aligned}$$

(c) For every pole or zero at $z = \beta$ in $h[n]$, show that $h_o[n]$ has a pair of poles or zeros at $z = \beta$ and $z = \frac{1}{\beta}$.

$$\begin{aligned}
h_o[n] &= h[n] * h[-n] \\
\text{so} \\
H_o(z) &= H(z)H(z^{-1}) \\
\text{let} \\
H(z) &= \frac{z - c}{z - \beta} \\
H_o(z) &= \frac{z - c}{z - \beta} \frac{z^{-1} - c}{z^{-1} - \beta} \\
&= \frac{(z - c)(1 - cz)}{(z - \beta)(\frac{1}{\beta} - z)\beta}
\end{aligned}$$

Therefore, $H_o(z)$ has a pair of poles at $z = \beta, \frac{1}{\beta}$

$$\begin{aligned}
\text{let} \\
H(z) &= \frac{z - \beta}{z - p} \\
H_o(z) &= \frac{z - \beta}{z - p} \frac{z^{-1} - \beta}{z^{-1} - p} \\
&= \frac{\beta}{p} \frac{(z - \beta)(z - \frac{1}{\beta})}{(z - p)(z - \frac{1}{p})\beta}
\end{aligned}$$

Therefore, $H_o(z)$ has a pair of zeros at $z = \beta, \frac{1}{\beta}$

7.53. The present value of a loan with interest compounded monthly may be described in terms of the first-order difference equation

$$y[n] = \rho y[n - 1] - x[n]$$

where $\rho = \left(1 + \frac{r/12}{100}\right)$, r is the annual interest rate expressed as a percent, $x[n]$ is the payment credited at the end of the n^{th} month, and $y[n]$ is the loan balance at the beginning of the $n + 1^{st}$ month. The beginning loan balance is the initial condition $y[-1]$. If uniform payments of \$ c are made for L consecutive months, then $x[n] = c\{u[n] - u[n - L]\}$.

(a) Use the unilateral z -transform to show that

$$Y(z) = \frac{y[-1]\rho - c \sum_{n=0}^{L-1} z^{-n}}{1 - \rho z^{-1}}$$

Hint: Use long division to show that

$$\frac{1 - z^{-L}}{1 - z^{-1}} = \sum_{n=0}^{L-1} z^{-n}$$

$$\begin{aligned} Y(z) &= \frac{y[-1]\rho - X(z)}{1 - \rho z^{-1}} \\ X(z) &= c \frac{1 - z^{-L}}{1 - z^{-1}} \\ &= c (1 + z^{-1} + z^{-2} + \dots + z^{-L+1}) \\ &= c \sum_{n=0}^{L-1} z^{-n} \end{aligned}$$

which implies

$$Y(z) = \frac{y[-1]\rho - c \sum_{n=0}^{L-1} z^{-n}}{1 - \rho z^{-1}}$$

(b) Show that $z = \rho$ must be a zero of $Y(z)$ if the loan is to have zero balance after L payments.

$$Y(z) = \frac{y[-1]p + c \sum_{n=0}^{L-1} z^{-n}}{1 - \rho z^{-1}}$$

The pole at $z = p$ results in an infinite length $y[n]$ in general. If the loan reaches zero after the L^{th} payment, we have:

$$y[n] = 0, \quad n \geq L - 1$$

$$\text{So, } Y(z) = \sum_{n=0}^{L-2} y[n] z^{-n}$$

Thus we must have:

$$y[-1]\rho - c \sum_{n=0}^{L-1} z^{-n} = (1 - \rho z^{-1}) \sum_{n=0}^{L-2} y[n] z^{-n}$$

to cancel the pole at $z = p$

From polynomial theory, the first term is zero if $f(z = \rho) = 0$, or $\rho = 1 + \frac{r}{12}$ is a zero of $Y(z)$

(c) Find the monthly payment $\$c$ as a function of the initial loan value $y[-1]$ and the interest rate r assuming the loan has zero balance after L payments.

$$y[-1]\rho - c \sum_{n=0}^{L-1} \rho^{-n} = 0$$

$$c \frac{1 - \rho^{-L}}{1 - \rho^{-1}} = y[-1]\rho$$

$$c = y[-1] \frac{\rho - 1}{1 - \rho^{-L}}$$

Solutions to Computer Experiments

7.54. Use the MATLAB command `zplane` to obtain a pole-zero plot for the following systems:

(a) $H(z) = \frac{1+z^{-2}}{2+z^{-1}-\frac{1}{2}z^{-2}+\frac{1}{4}z^{-3}}$

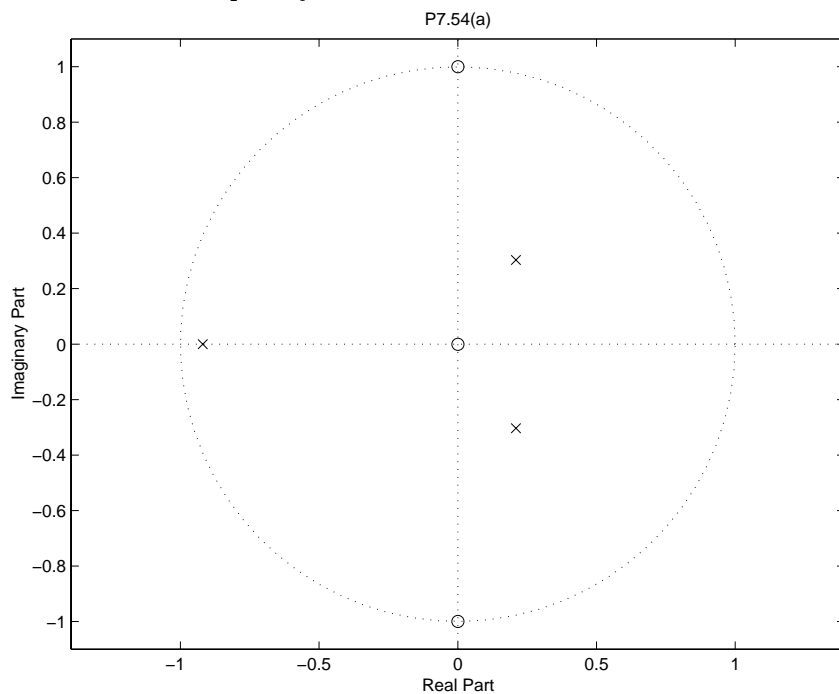


Figure P7.54. (a) Pole-Zero plot of $H(z)$

(b) $H(z) = \frac{1+z^{-1}+\frac{3}{2}z^{-2}+\frac{1}{2}z^{-3}}{1+\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$

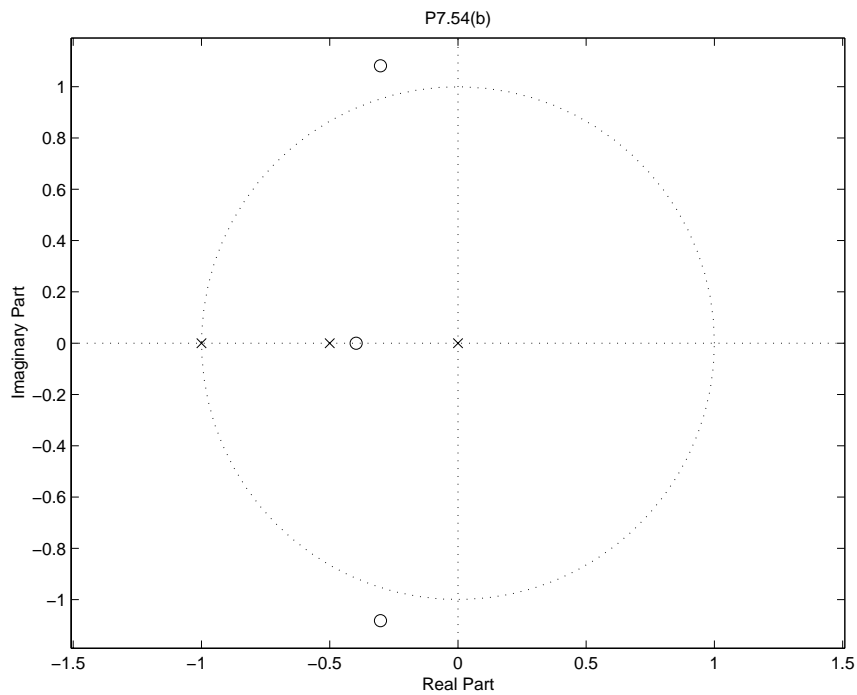


Figure P7.54. (b) Pole-Zero plot of $H(z)$

7.55. Use the MATLAB command `residuez` to obtain the partial fraction expansions required to solve Problem 7.24 (d) - (g).

P7.55 :

=====

Part (d) :

=====

r =

2
-1

p =

-2.0000
0.5000

k =

0

Part (e) :

=====

r =

1.4688
1.5312

p =

4
-4

k =

0

Part (f) :

=====

r =

2
-1

p =

-1.0000
-0.5000

k =

0 1

Part (g) :

=====

r =

1
-1

p =

-1

1

k =

2 0 0

7.56. Use the MATLAB command `tf2ss` to find state-variable descriptions for the systems in Problem 7.27.

P7.56 :

=====

Part (a) :

=====

A =

1.5000 1.0000
1.0000 0

B =

1
0

C =

1.5000 2.0000

D =

2

Part (b) :

=====

A =

1 6
1 0

B =

1
0

C =

5 30

D =

5

Part (c) :

=====

A =

0.2500 -0.0625
1.0000 0

B =

1
0

C =

4 0

D =

0

7.57. Use the MATLAB command `ss2tf` to find the transfer functions in Problem 7.33.

P7.57 :

=====

Part (b)-(i) :

=====

Num =

1.0000 -2.0000 -1.2500

Den =

1.0000 0 -0.2500

Part (b)-(ii) :

=====

Num =

0 2 0

Den =

1.0000 -0.2500 -0.3750

Part (b)-(iii) :

=====

Num =

0 2.0000 -6.5000

Den =

1.0000 -0.5000 0.2500

7.58. Use the MATLAB command `zplane` to solve Problem 7.35 (a) and (b).

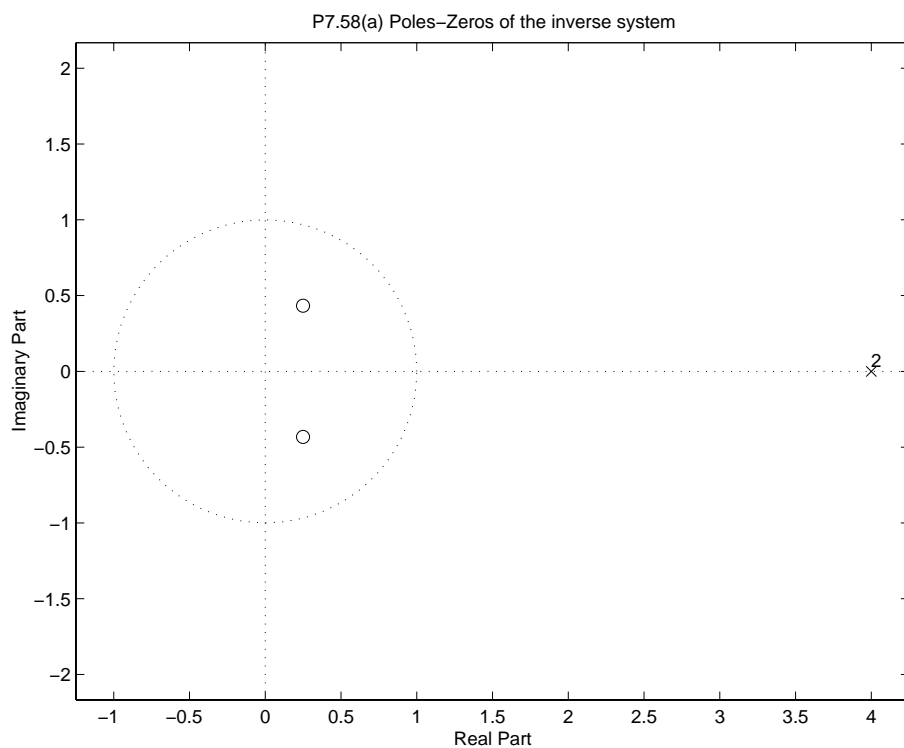


Figure P7.58. (a) Pole-Zero plots of the inverse system.

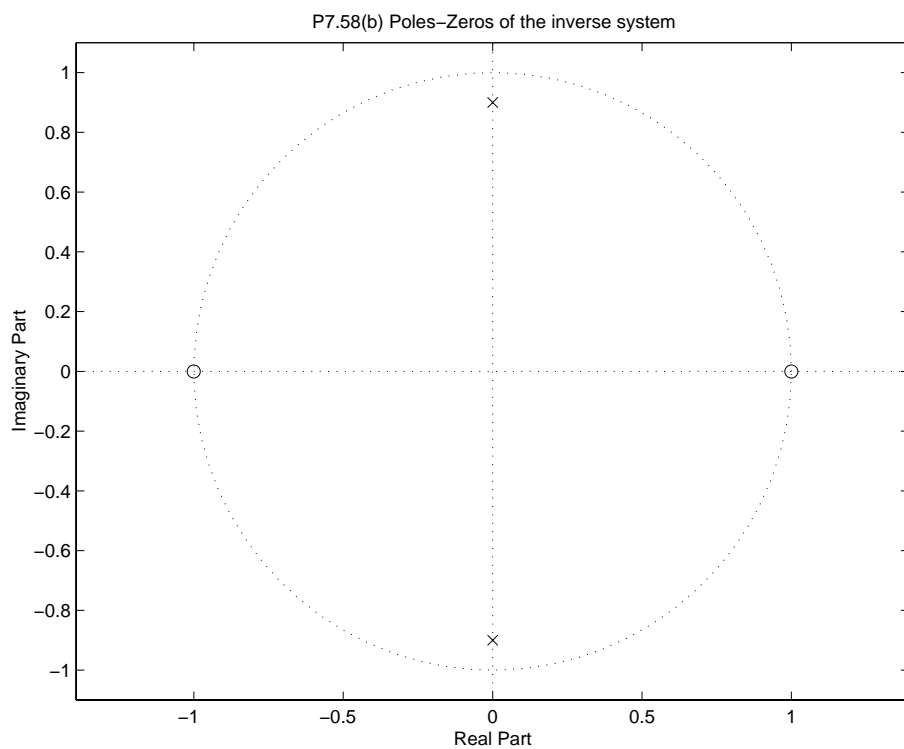


Figure P7.58. (b) Pole-Zero plots of the inverse system.

7.59. Use the MATLAB command `freqz` to evaluate and plot the magnitude and phase response of the system given in Example 7.21.

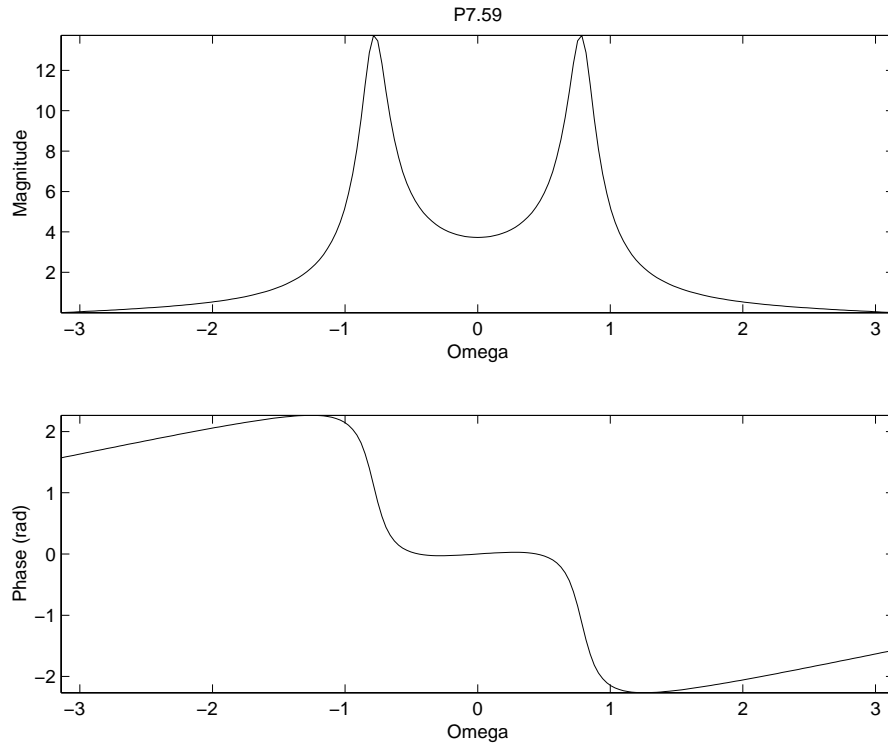


Figure P7.59. Magnitude Response for Example 7.21

7.60. Use the MATLAB command `freqz` to evaluate and plot the magnitude and phase response of the systems given in Problem 7.37.

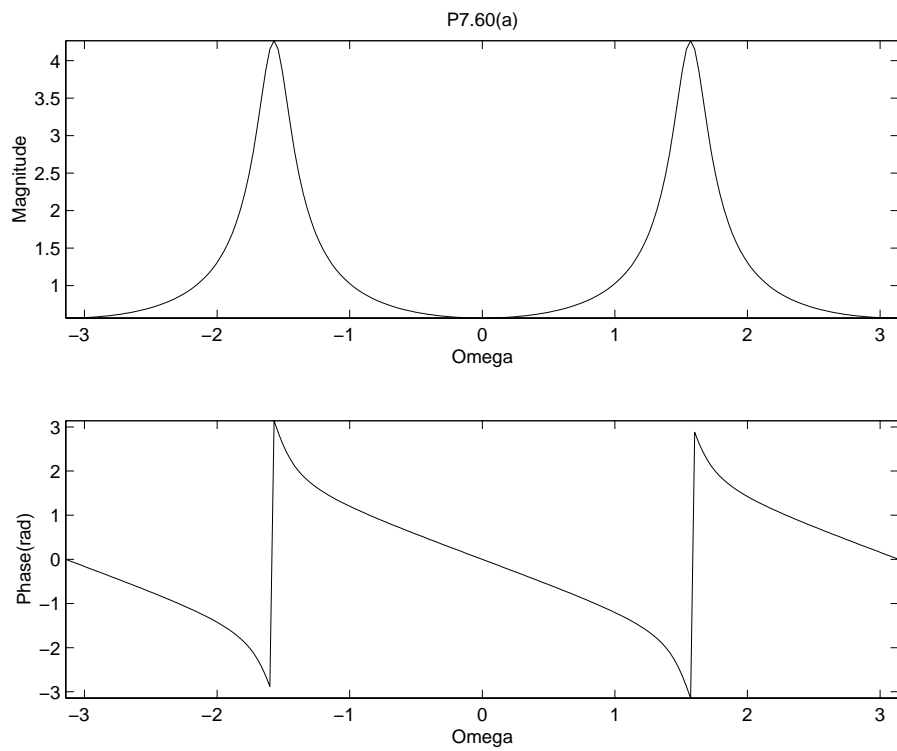


Figure P7.60. (a) Magnitude and phase response

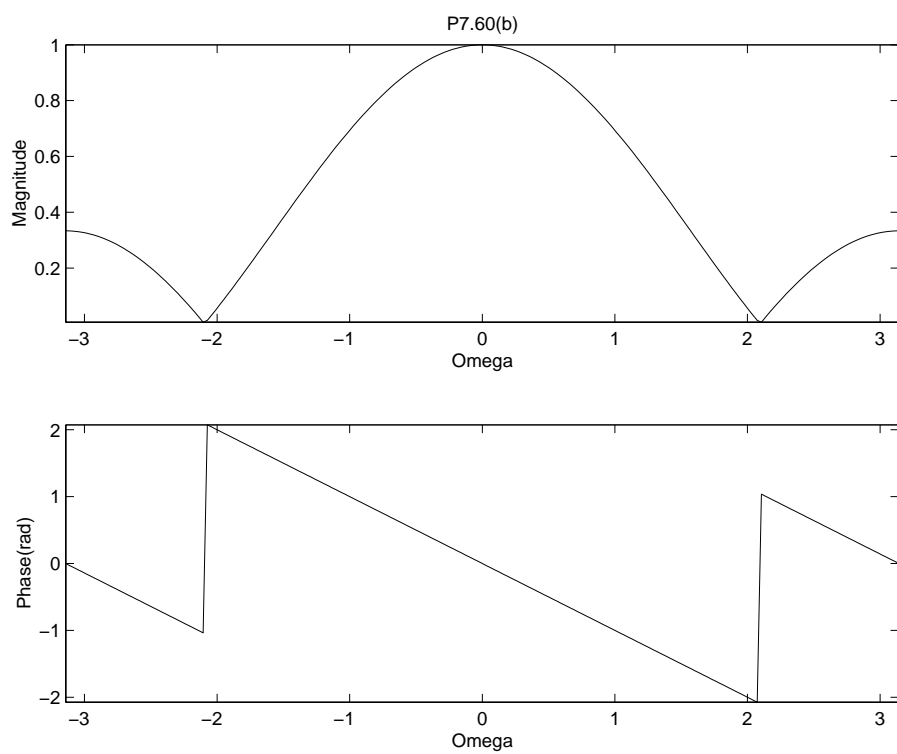


Figure P7.60. (b) Magnitude and phase response

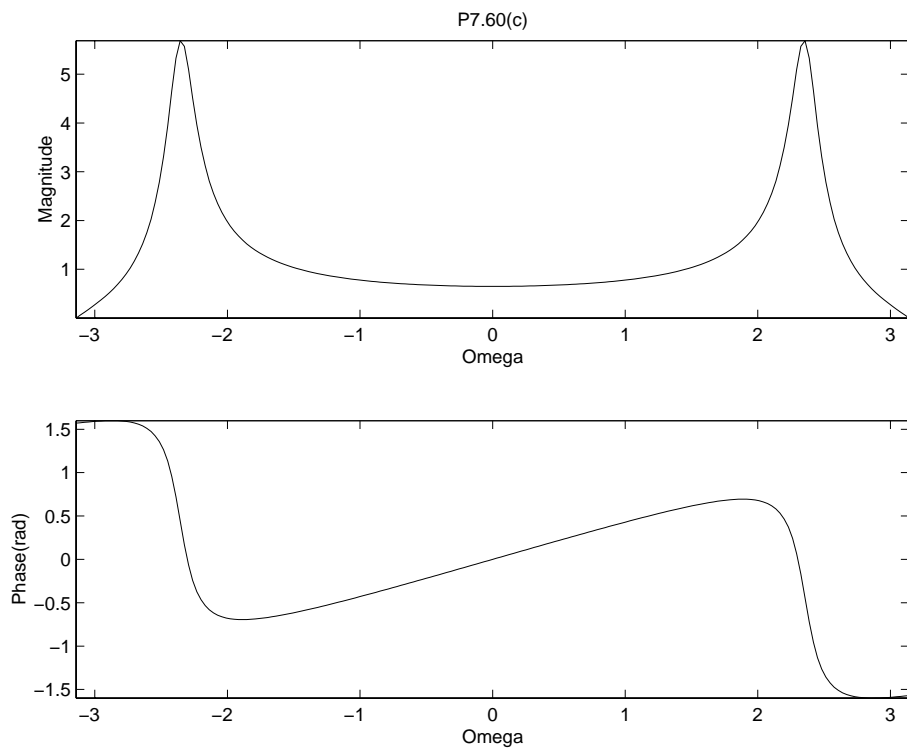


Figure P7.60. (c) Magnitude and phase response

7.61. Use the MATLAB commands `filter` and `filtic` to plot the loan balance at the start of each month $n = 0, 1, \dots, L + 1$ for Problem 7.53. Assume that $y[-1] = \$10,000$, $L = 60$, $r = 0.1$ and the monthly payment is chosen to bring the loan balance to zero after 60 payments.

From Problem 7.53:

$$\begin{aligned}
 c &= y[-1] \frac{\rho - 1}{1 - \rho^{-L}} \\
 y[-1] &= 10,000 \\
 L &= 60 \\
 \rho &= 1 + \frac{r}{12} = 1.00833 \\
 c &= 0.02125 \\
 \underline{b} &= [y[-1]\rho - c, \quad -c(\text{ones}(1, 59))] \\
 \underline{a} &= [1, \quad \rho]
 \end{aligned}$$

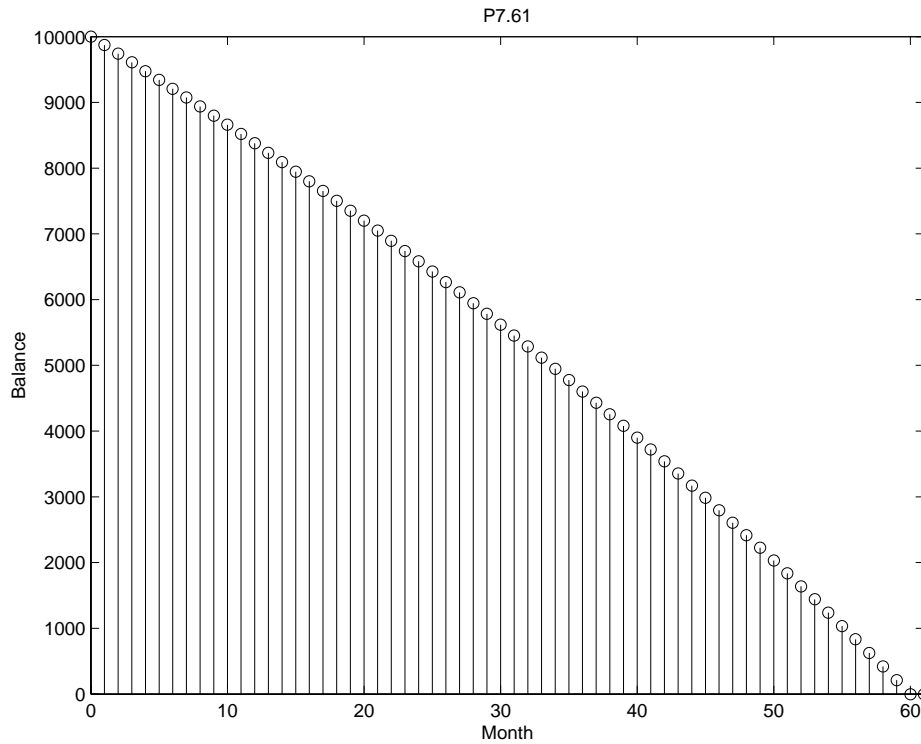


Figure P7.61. Monthly loan balance.

7.62. Use the MATLAB command `zp2sos` to determine a cascade connection of second-order sections for implementing the systems in Problem 7.38.

=====

Part (a) :

=====

sos =

1.0000	-0.3536	0.0625	1.0000	-0.5000	0.2500
1.0000	0.4619	0.0625	1.0000	1.3858	0.5625

Part (b) :

=====

sos =

1.0000	4.0000	4.0000	1.0000	0.3750	-0.2813
1.0000	-0.0000	0.2500	1.0000	-0.3750	0.1406

7.63. A causal discrete-time LTI system has the transfer function

$$H(z) = \frac{0.0976(z-1)^2(z+1)^2}{(z-0.3575-j0.5889)(z-0.3575+j0.5889)(z-0.7686-j0.3338)(z-0.7686+j0.3338)}$$

(a) Use the pole and zero locations to sketch the magnitude response.

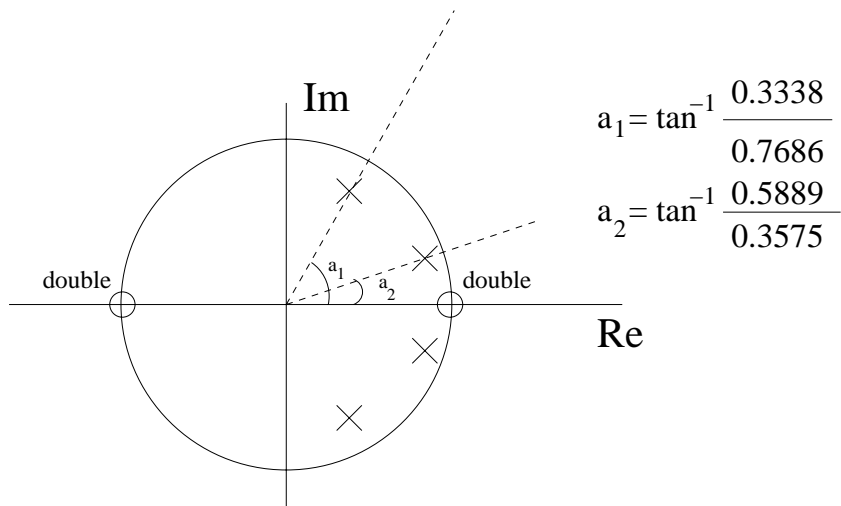


Figure P7.63. (a) Pole Zero plot

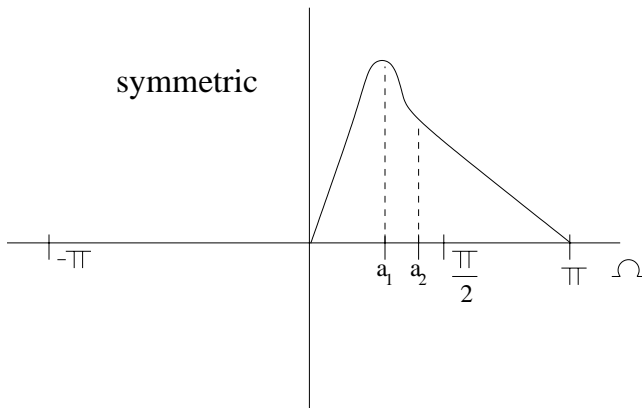


Figure P7.63. (a) Sketch of the Magnitude Response.

(b) Use the MATLAB commands `zp2tf` and `freqz` to evaluate and plot the magnitude and phase response.

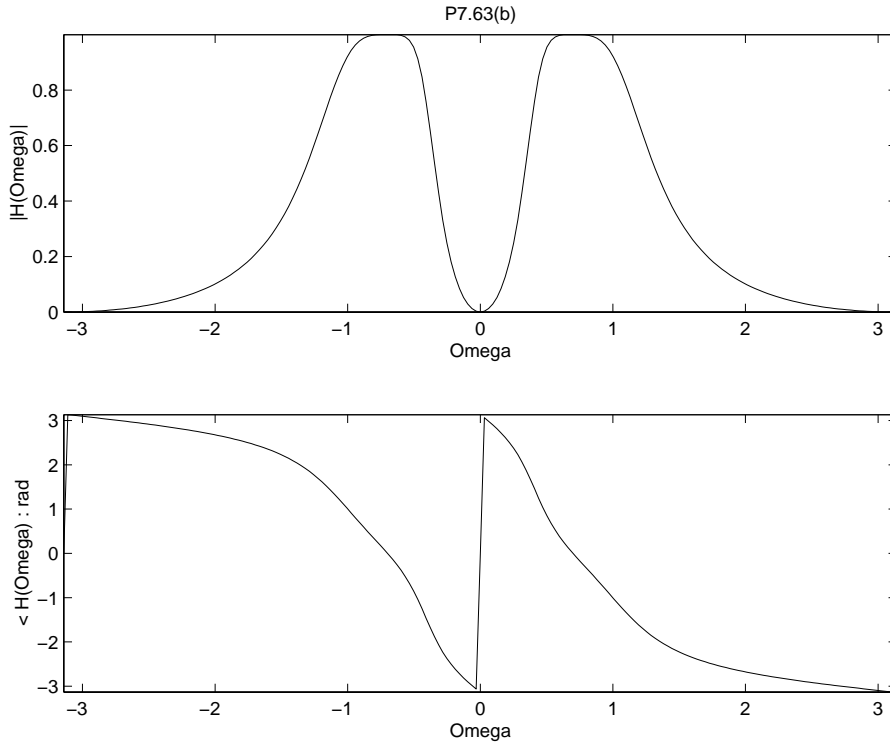


Figure P7.63. (b) Plot of the Magnitude Response.

(c) Use the MATLAB command `zp2sos` to obtain a representation for this filter as a cascade of two second-order sections with real-valued coefficients.

$$H(z) = \left(\frac{0.1413(1 + 2z^{-1} + z^{-2})}{1 - 0.715z^{-1} + 0.4746z^{-2}} \right) \left(\frac{0.6907(1 - 2z^{-1} + z^{-2})}{1 - 1.5372z^{-1} + 0.7022z^{-2}} \right)$$

(d) Use the MATLAB command `freqz` to evaluate and plot the magnitude response of each section in (c).

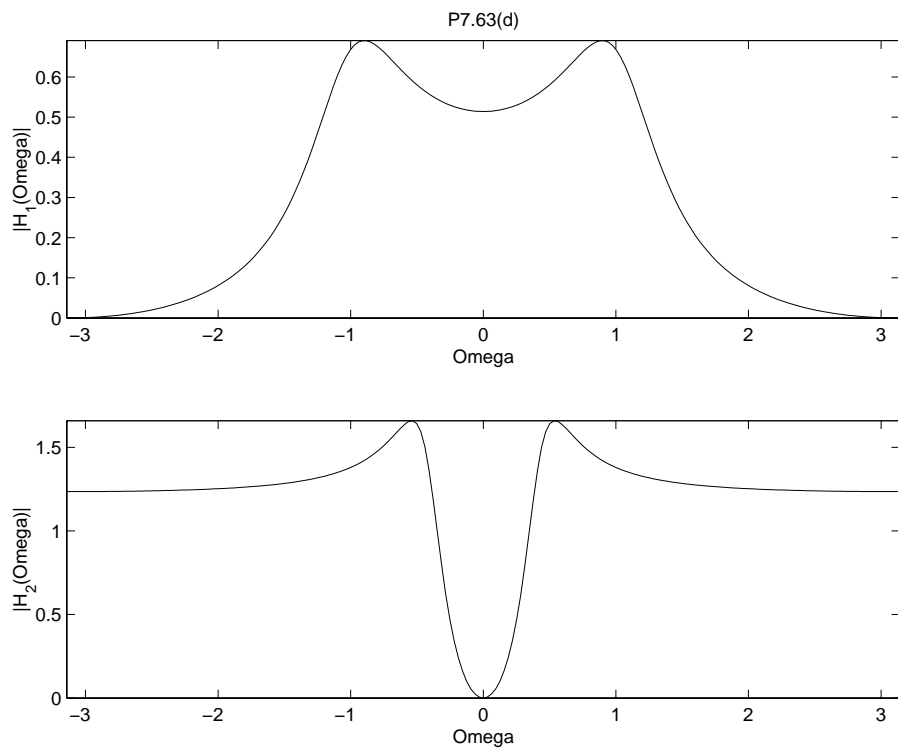


Figure P7.63. (d) Plot of the Magnitude Response for each section.
 (e) Use the MATLAB command `filter` to determine the impulse response of this system by obtaining the output for an input $x[n] = \delta[n]$.

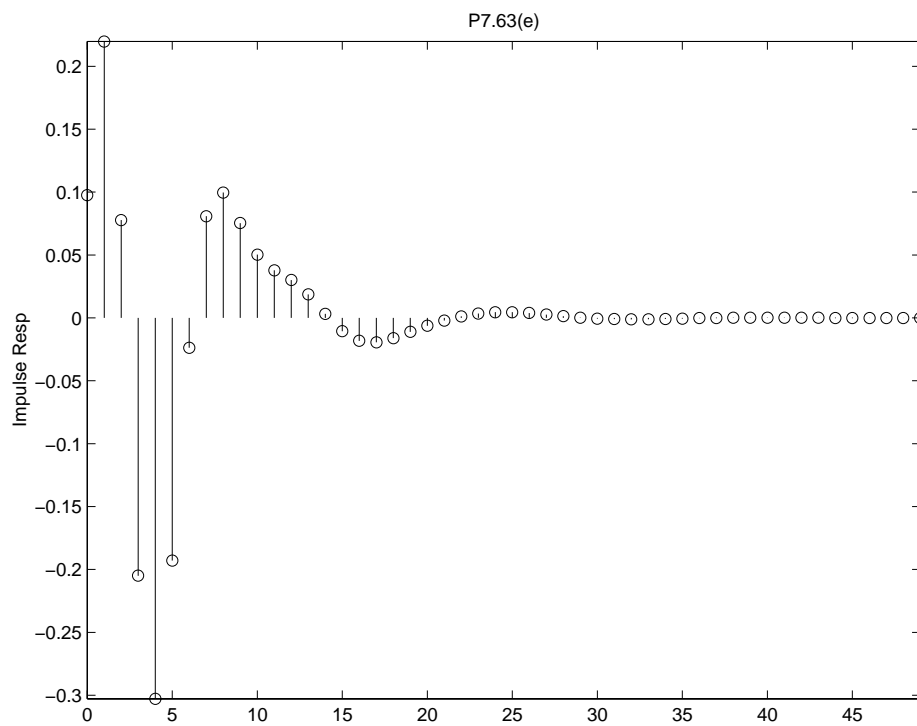


Figure P7.63. (e) Plot of the Impulse Response.

(f) Use the MATLAB command `filter` to determine the system output for the input

$$x[n] = \left(1 + \cos\left(\frac{\pi}{4}n\right) + \cos\left(\frac{\pi}{2}n\right) + \cos\left(\frac{3\pi}{4}n\right) + \cos(\pi n) \right) u[n]$$

Plot the first 250 points of the input and output.

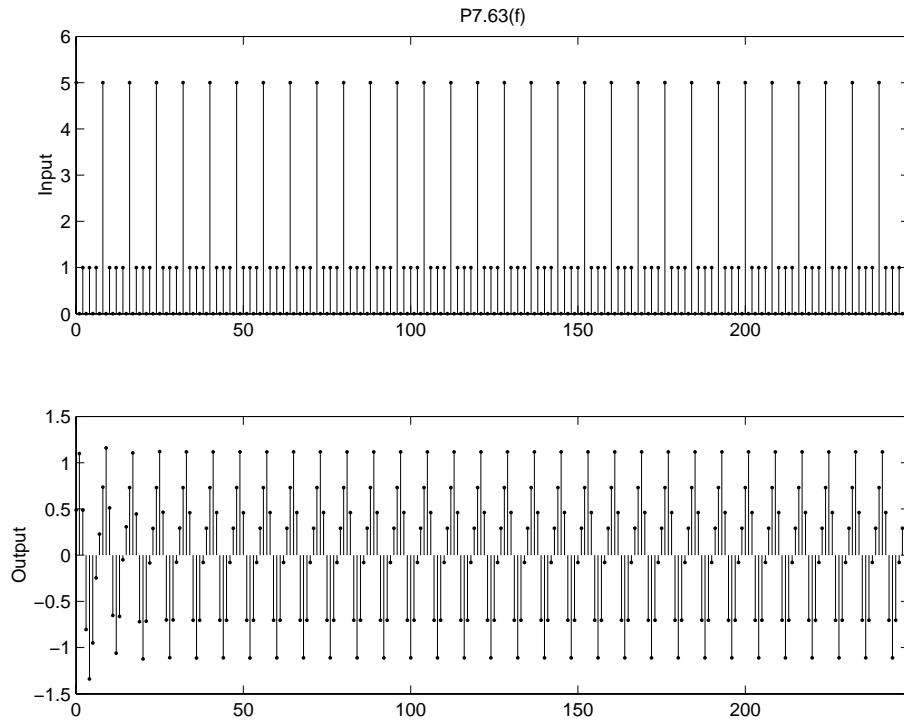


Figure P7.63. (f) System output for given input.

The system eliminates “1” and “ $\cos(\pi n)$ ” terms. It also greatly attenuates the “ $\cos(\frac{3\pi}{4}n)$ ” term. The “ $\cos(\frac{\pi}{2}n)$ ” term is also attenuated, so the output is dominated by “ $\cos(\frac{\pi}{4}n)$ ”.

CHAPTER 8

Additional Problems

8.16 Provided that the channel bandwidth is not smaller than the reciprocal of the transmitted pulse duration T , the received pulse is recognizable at the channel output. With $T = 1\mu\text{s}$, a small enough value for the channel bandwidth is $(1/T) = 10^6 \text{ Hz} = 1 \text{ MHz}$.

8.17 For a low-pass filter of the Butterworth type, the squared magnitude response is defined by

$$|H(j\omega)|^2 = \frac{1}{1 + (\omega/\omega_c)^{2N}} \quad (1)$$

At the edge of the passband, $\omega = \omega_p$, we have (by definition)

$$|H(j\omega_p)| = 1 - \epsilon$$

We may therefore write

$$(1 - \epsilon)^2 = \frac{1}{1 + (\omega_p/\omega_c)^{2N}} \quad (2)$$

Define

$$\epsilon_0 = 1 - (1 - \epsilon)^2 = 2\epsilon - \epsilon^2$$

Then solving Eq. (2) for ω_p :

$$\omega_p = \left(\frac{\epsilon_0}{1 - \epsilon_0} \right) \omega_c$$

Next, by definition, at the edge of the stopband, $\omega = \omega_s$, we have

$$|H(j\omega_s)| = \delta$$

Hence

$$\delta^2 = \frac{1}{1 + (\omega_s/\omega_c)^{2N}} \quad (3)$$

Define

$$\delta_0 = \delta^2$$

Hence, solving Eq. (3) for ω_s :

$$\omega_s = \left(\frac{1 - \delta_0}{\delta_0} \right) \omega_c$$

8.18 We start with the relation

$$|H(j\omega)|_{j\omega=s}^2 = H(s)H(-s)$$

For a Butterworth low-pass filter of order 5, the 10 poles of $H(s)H(-s)$ are uniformly distributed around the unit circle in the s -plane as shown in Fig. 1.

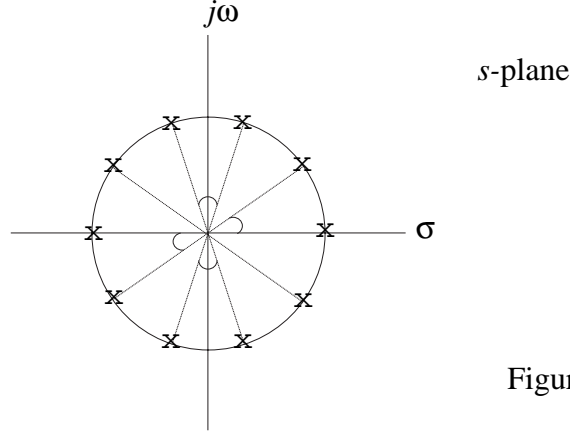


Figure 1

Let $D(s)D(-s)$ denote the denominator polynomial of $H(s)H(-s)$. Hence

$$\begin{aligned} D(s)D(-s) &= (s+1)(s+\cos 144^\circ + j\sin 144^\circ)(s+\cos 144^\circ - j\sin 144^\circ) \\ &\quad \times (s+\cos 108^\circ + j\sin 108^\circ)(s+\cos 108^\circ - j\sin 108^\circ) \\ &\quad \times (s-1)(s-\cos 36^\circ + j\sin 36^\circ)(s-\cos 36^\circ - j\sin 36^\circ) \\ &\quad \times (s-\cos 72^\circ + j\sin 72^\circ)(s-\cos 72^\circ - j\sin 72^\circ) \end{aligned}$$

Identifying the zeros of $D(s)D(-s)$ in the left-half plane with $D(s)$ and those in the right-half plane with $D(-s)$, we may express $D(s)$ as

$$\begin{aligned} D(s) &= (s+1)(s+\cos 144^\circ + j\sin 144^\circ)(s+\cos 144^\circ - j\sin 144^\circ) \\ &\quad \times (s+\cos 108^\circ + j\sin 108^\circ)(s+\cos 108^\circ - j\sin 108^\circ) \\ &= s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1 \end{aligned}$$

Hence,

$$\begin{aligned} H(s) &= \frac{1}{D(s)} \\ &= \frac{1}{s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1} \end{aligned}$$

8.19 (a) For filter order N that is odd, the transfer function $H(s)$ of the filter must have a real pole in the left-half plane. Let this pole be $s = -a$ where $a > 0$. We may then write

$$H(s) = \frac{1}{(s+a)D'(s)}$$

where $D'(s)$ is the remainder of the denominator polynomial. For a Butterworth low-pass filter of cutoff frequency ω_c , all the poles of $H(s)$ lie on a circle of radius ω_c in the left-half plane. Hence, we must have $a = -\omega_c$.

- (b) For a Butterworth low-pass filter of even order N , all the poles of the transfer function $H(s)$ are complex. They all lie on a circle of radius ω_c in the left-half plane. Let $s = -a - jb$, with $a > 0$ and $b > 0$, denote a complex pole of $H(s)$. All the coefficients of $H(s)$ are real. This condition can only be satisfied if we have a complex conjugate pole at $s = -a + jb$. We may then express the contribution of this pair of poles as

$$\frac{1}{(s + a + jb)(s + a - jb)} = \frac{1}{(s + a)^2 + b^2}$$

whose coefficients are all real. We therefore conclude that for even filter order N , all the poles of $H(s)$ occur in complex-conjugate pairs.

8.20 The transfer function of a Butterworth low-pass filter of order 5 is

$$H(s) = \frac{1}{(s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)} \quad (1)$$

The low-pass to high-pass transformation is defined by

$$s \rightarrow \frac{1}{s}$$

where it is assumed that the cutoff frequency of the high-pass filter is unity. Hence replacing s with $1/s$ in Eq. (1), we find that the transfer function of a Butterworth high-pass filter of order 5 is

$$\begin{aligned} H(s) &= \frac{1}{\left(\frac{1}{s} + 1\right)\left(\frac{1}{s^2} + \frac{0.618}{s} + 1\right)\left(\frac{1}{s^2} + \frac{0.618}{s} + 1\right)} \\ &= \frac{s^5}{(s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)} \end{aligned}$$

The magnitude response of this high-pass filter is plotted in Fig. 1.

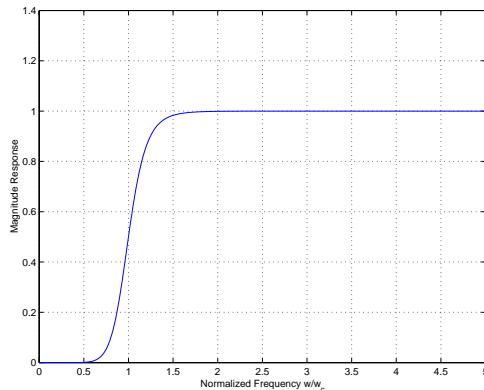


Figure 1

8.21 We are given the transfer function

$$H(s) = \frac{1}{(s+1)(s^2+0.618s+1)(s+1.618s+1)} \quad (1)$$

To modify this low-pass filter so as to assume a cutoff frequency ω_c , we use the transformation

$$s \rightarrow \frac{s}{\omega_c}$$

Hence, replacing s with s/ω_c in Eq. (1), we obtain:

$$\begin{aligned} H(s) &= \frac{1}{\left(\frac{s}{\omega_c} + 1\right) \left(\frac{s^2}{\omega_c^2} + 0.618 \frac{s}{\omega_c} + 1\right) \left(\frac{s^2}{\omega_c^2} + 1.618 \frac{s}{\omega_c} + 1\right)} \\ &= \frac{\omega_c^5}{(s + \omega_c)(s^2 + 0.618\omega_c s + \omega_c^2)(s^2 + 1.618\omega_c s + \omega_c^2)} \end{aligned}$$

8.22 We are given the transfer function

$$H(s) = \frac{1}{(s+1)(s^2+s+1)} \quad (1)$$

To transform this low-pass filter into a band-pass filter with bandwidth B centered on ω_0 , we use the following transformation:

$$s \rightarrow \frac{s^2 + \omega_0^2}{B s}$$

With $\omega_0 = 1$ and $B = 0.1$, the transformation takes the value

$$s \rightarrow \frac{s^2 + 1}{0.1s} \quad (2)$$

Substituting Eq. (2) into (1):

$$\begin{aligned} H(s) &= \frac{1}{\left(\frac{s^2+1}{0.1s} + 1\right) \left(\left(\frac{s^2+1}{0.1s}\right)^2 + \left(\frac{s^2+1}{0.1s}\right) + 1\right)} \\ &= \frac{0.001s^3}{(s^2 + 0.1s + 1)(s^4 + 2s^2 + 1 + 0.1s^3 + 0.1s + 0.01s^2)} \end{aligned}$$

$$= \frac{0.001s^3}{(s^2 + 0.1s + 1)(s^4 + 0.1s^3 + 2.01s^2 + 0.1s + 1)} \quad (3)$$

The magnitude response of this band-pass filter of the Butterworth type, obtained by putting $s = j\omega$ in Eq. 3, is plotted in Fig. 1.

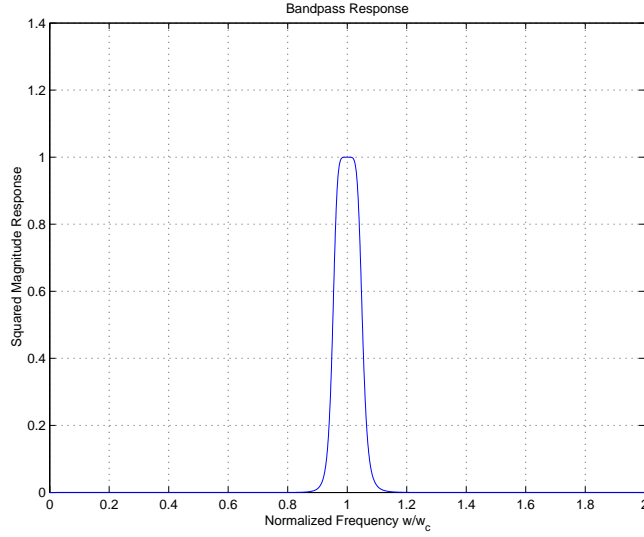
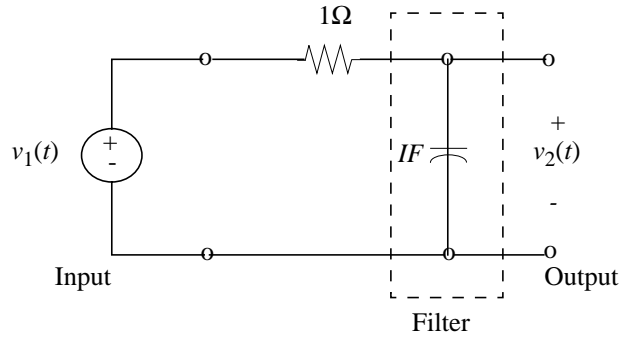
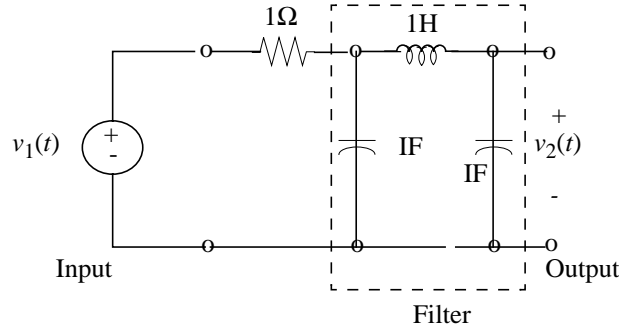


Figure 1

8.23 For the counterpart to the low-pass filter of order one in Fig. 8.14(a), we have



For the counterpart to the low-pass filter of order three in Fig. 8.14(b), we have



Note that in both of these figures, the 1Ω resistance may be used to account for the source resistance. Note also that the load resistance is infinitely large.

- 8.24 (a) For Fig. 8.14(a), the capacitor has the value

$$\begin{aligned} C &= \frac{1}{(10^4)(2\pi \times 10^5)} \text{F} \\ &= \frac{10^3}{2\pi} \text{pF} \\ &= 159 \text{ pF} \end{aligned}$$

- (b) For Fig. 8.14(a), the two capacitors are 159 PF each. The inductor has the value

$$\begin{aligned} L &= \frac{(10^4)}{(2\pi \times 10^5)} \text{H} \\ &= \frac{100}{2\pi} \text{mH} \\ &= 15.9 \text{ mH} \end{aligned}$$

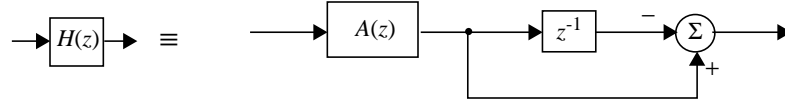
- 8.25 Let the transfer function of the FIR filter be defined by

$$H(z) = (1 - z^{-1})A(z) \quad (1)$$

where $A(z)$ is an arbitrary polynomial in z^{-1} . The $H(z)$ of Eq. (1) has a zero at $z = 1$ as prescribed. Let the sequence $a[n]$ denote the inverse z -transform of $A(z)$. Expanding Eq. (1):

$$H(z) = A(z) - z^{-1}A(z) \quad (2)$$

which may be represented by the block diagram:



From Eq. (2) we readily find that the impulse response of the filter must satisfy the condition

$$h[n] = a[n] - a[n-1]$$

where $a[n]$ is the inverse z -transform of an arbitrary polynomial $A(z)$.

8.26 Let the transfer function of the filter be defined by

$$H'(e^{j\Omega}) = \sum_{n=-M/2}^{M/2} h_d[n] e^{-jn\Omega}$$

Replacing n with $n - M/2$ so as to make the filter causal, we may thus write

$$\begin{aligned} H'(e^{j\Omega}) &= \sum_{n=0}^M h_d\left[n - \frac{M}{2}\right] e^{-j\left(n - \frac{M}{2}\right)\Omega} \\ &= e^{jM\Omega/2} \sum_{n=0}^M h_d\left[n - \frac{M}{2}\right] e^{-jn\Omega} \end{aligned} \quad (1)$$

We are given that

$$h[n] = h_d[n] \quad \text{for } -\frac{M}{2} \leq n \leq \frac{M}{2}$$

This condition is equivalent to

$$h_d\left[n - \frac{M}{2}\right] = h[n] \quad \text{for } 0 \leq n \leq M$$

Hence, we may rewrite Eq. (1) in the form

$$\begin{aligned} H'(e^{j\Omega}) &= e^{jM\Omega/2} \sum_{n=0}^M h[n] e^{-jn\Omega} \\ &= e^{jM\Omega/2} H(e^{j\Omega}) \end{aligned}$$

Equivalently we have

$$H(e^{j\Omega}) = e^{-jM\Omega/2} H'(e^{j\Omega})$$

which is the desired result.

8.27 According to Eqs. (8.64) and (8.65), the magnitude $r = |z|$ and phase $\theta = \arg\{z\}$ are defined by

$$r = \left(\frac{(1 + \sigma)^2 + \omega^2}{(1 - \sigma)^2 + \omega^2} \right)^{1/2} \quad (1)$$

$$\theta = \tan^{-1} \left(\frac{\omega}{1 + \sigma} \right) - \tan^{-1} \left(\frac{\omega}{1 - \sigma} \right) \quad (2)$$

These two relations are based on the transformation

$$z = re^{j\theta} = \frac{1 + s}{1 - s}, \quad s = \sigma + j\omega$$

For the more general case of a sampling rate $1/T_s$ for which we have

$$s = \frac{1}{T_s} \frac{z - 1}{z + 1}$$

or

$$z = \frac{1 + \frac{T_s}{2}s}{1 - \frac{T_s}{2}s}$$

we may rewrite Eqs. (1) and (2) by replacing ω with $\frac{T_s}{2}\omega$ and σ with $\frac{T_s}{2}\sigma$, obtaining

$$r = \left(\frac{\left(\frac{2}{T_s} + \sigma \right)^2 + \omega^2}{\left(\frac{2}{T_s} - \sigma \right)^2 + \omega^2} \right)^{1/2}$$

$$\theta = \tan^{-1} \left(\frac{\omega}{\frac{2}{T_s} + \sigma} \right) - \tan^{-1} \left(\frac{\omega}{\frac{2}{T_s} - \sigma} \right)$$

8.28 (a) From Section 1.10, we recall the input-output relation

$$y[n] = x[n] + \rho y[n - 1]$$

Taking z -transforms:

$$Y(z) = X(z) + \rho z^{-1} Y(z)$$

The transfer function of the filter is therefore

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \rho z^{-1}} \quad (1)$$

For $\rho = 1$, we have

$$H(z) = \frac{1}{1 - z^{-1}} \quad (2)$$

For $z = j\Omega$, the frequency response of the filter is defined by

$$H(e^{j\Omega}) = \frac{1}{1 - e^{-j\Omega}}$$

with

$$\begin{aligned} |H(e^{j\Omega})| &= \frac{1}{|1 - e^{-j\Omega}|} \\ &= \frac{1}{[(1 - \cos\Omega)^2 + \sin^2\Omega]^{1/2}} \\ &= \frac{1}{(2 - 2\cos\Omega)^{1/2}} \end{aligned}$$

which is plotted in Fig. 1 for $0 \leq \Omega \leq \pi$. From this figure we see that the filter defined in Eq. (2) does not deviate from the ideal integrator by more than 1% for $0 \leq \Omega \leq 0.49$.

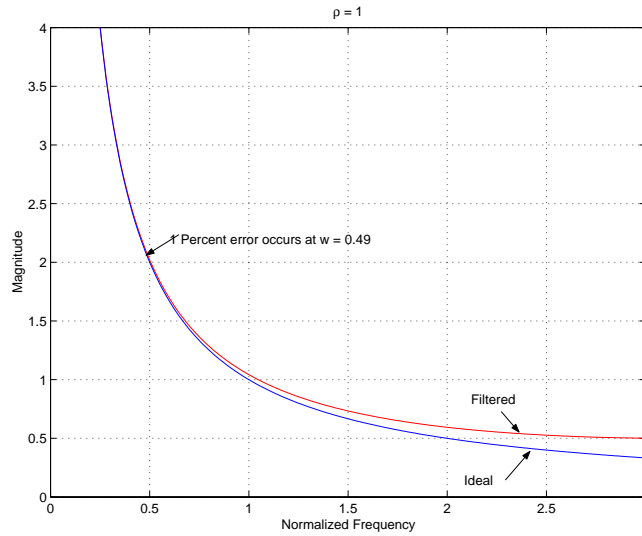


Figure 1

(b) For $\rho = 0.99$, the use of Eq. (1) yields

$$H(z) = \frac{1}{1 - 0.99z^{-1}} \quad (3)$$

for which the frequency response is defined by

$$H(e^{j\Omega}) = \frac{1}{1 - 0.99e^{-j\Omega}}$$

That is,

$$\begin{aligned} |H(e^{j\Omega})| &= \frac{1}{|1 - 0.99e^{-j\Omega}|} \\ &= \frac{1}{[(1 - 0.99\cos\Omega)^2 + (0.99\sin\Omega)^2]^{1/2}} \\ &\approx \frac{1}{(1.98 - 1.98\cos\Omega)^{1/2}} \end{aligned}$$

which is plotted in Fig. 2. From this second figure we see that the usable range of the filter of Eq. (3) as an integrator is reduced to 0.35.

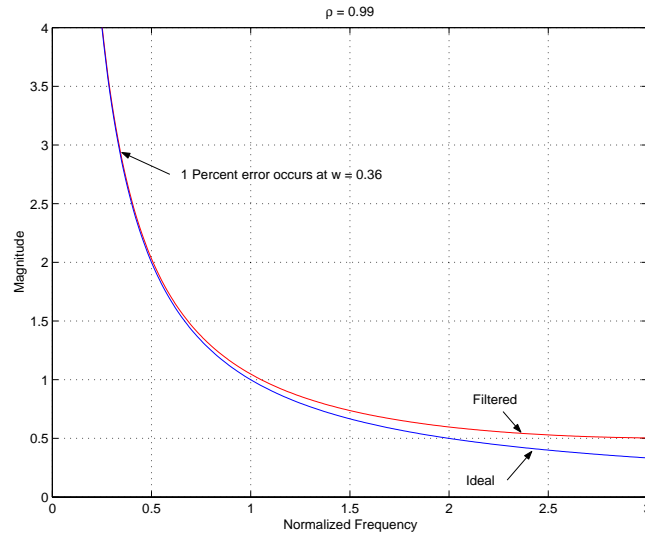


Figure 2

8.29 The transfer function of the digital IIR filter is

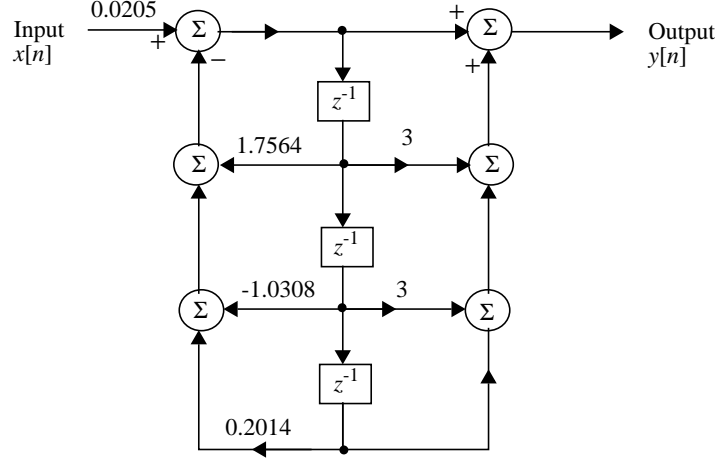
$$H(z) = \frac{0.0181(z + 1)^3}{(z - 0.50953)(z^2 - 1.2505z + 0.39812)}$$

Expanding the numerator and denominator polynomials of $H(z)$ in ascending powers of

z^{-1} , we may write

$$H(z) = \frac{0.0181(1 + 3z^{-1} + 3z^{-2} + z^{-3})}{1 - 1.7564z^{-1} + 1.0308z^{-2} - 0.2014z^{-3}}$$

Hence, the filter may be implemented in direct form II using the following configuration:



- 8.30 (a) The received signal, ignoring channel noise, is given by

$$y(t) = x(t) + 0.1x(t - 10) + 0.2x(t - 15)$$

where $x(t)$ is the transmitted signal, and time t is measured in microseconds. Suppose $y(t)$ is sampled uniformly with a sampling period of $5 \mu s$, yielding

$$y[n] = x[n] + 0.1x[n - 2] + 0.2x[n - 3] \quad (1)$$

- (b) Taking the z -transforms of Eq. (1):

$$Y(z) = X(z) + 0.1z^{-2}X(z) + 0.2z^{-3}X(z)$$

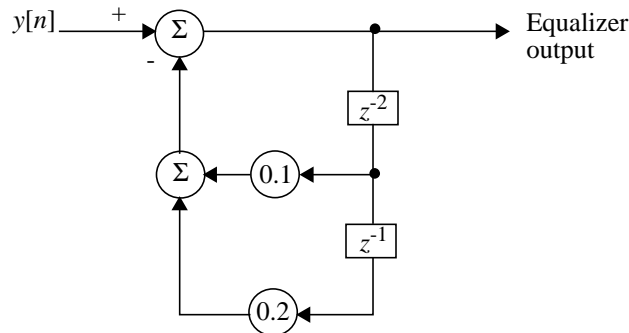
which, in turn, yields the transfer function for the channel:

$$H(z) = \frac{Y(z)}{X(z)} = 1 + 0.1z^{-2} + 0.2z^{-3}$$

The corresponding equalizer is defined by the transfer function

$$H_{eq}(z) = \frac{1}{H(z)} = \frac{1}{1 + 0.1z^{-2} + 0.2z^{-3}}$$

which is realized by the IIR filter:



For $H_{eq}(z)$ to be stable, all of its three poles or, equivalently, all three roots of the cubic equation

$$1 + 0.1z^{-2} + 0.2z^{-3} = 0, \text{ or equivalently,}$$

$$z^3 + 0.1z + 0.2 = 0$$

must lie inside the unit circle in the z -plane. Using MATLAB, we find the roots are

$$z = 0.2640 + j0.5560$$

$$z = 0.2640 - j0.5560$$

$$z = -0.5280$$

which confirm stability of the IIR equalizer.

[Note: The stability of $H_{eq}(z)$ may also be explored using an indirect approach, namely, the Routh-Hurwitz criterion which avoids having to compute the roots. First, we use the bilinear transformation

$$z = \frac{1+s}{1-s}$$

and then construct the Routh array as described in Section 9.12. The stability of $H_{eq}(z)$ is confirmed by examining the coefficients of the first column of the Routh array. The fact that all these coefficients are found to be positive assures the stability of $H_{eq}(z)$.]

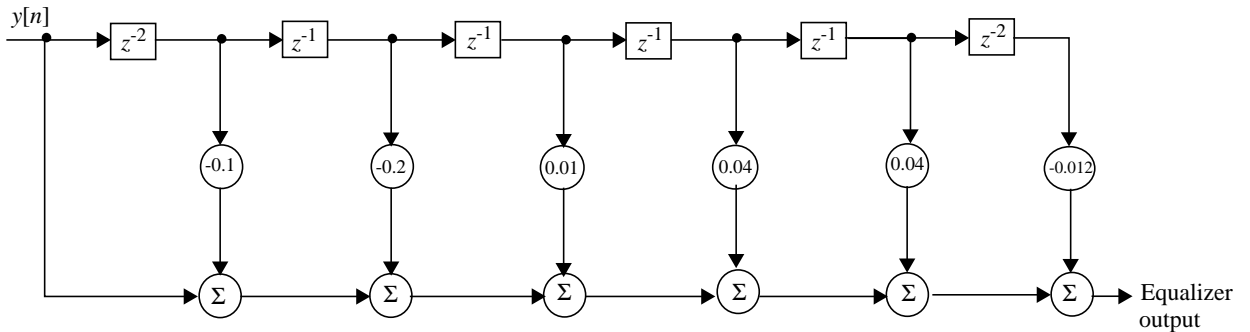
Realizing the equalizer by means of an FIR structure, we have

$$\begin{aligned} H_{eq}(z) &= (1 + 0.1z^{-2} + 0.2z^{-3})^{-1} \\ &= 1 - (0.1z^{-2} + 0.2z^{-3}) + (0.1z^{-2} + 0.2z^{-3})^2 - (0.1z^{-2} + 0.2z^{-3})^3 + \dots \\ &= 1 - (0.1z^{-2} + 0.2z^{-3}) + (0.01z^{-4} + 0.04z^{-5} + 0.04z^{-6}) \\ &\quad - (0.001z^{-6} + 0.006z^{-7} + 0.012z^{-8} + 0.008z^{-9}) + \dots \end{aligned}$$

Ignoring coefficients smaller than 1% as specified, we have the approximate result:

$$H_{eq}(z) \approx 1 - 0.1z^{-2} - 0.2z^{-3} + 0.01z^{-4} + 0.04z^{-5} + 0.04z^{-6} - 0.012z^{-8}$$

which is realized using the following FIR structure:



Advanced Problems

8.31 The integrator output is

$$y(t) = \int_{t-T_0}^t x(\tau) d\tau \quad (1)$$

Let $x(t) \xleftrightarrow{FT} X(j\omega)$. We may therefore reformulate the expression for $y(t)$ as

$$y(t) = \int_{t-T_0}^t \left(\underbrace{\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega\tau} d\omega \right)}_{x(\tau)} \right) d\tau$$

Interchanging the order of integration:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} X(j\omega) \left(\int_{t-T_0}^t e^{j\omega\tau} d\tau \right) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot \frac{T_0}{2\pi} \text{sinc}\left(\frac{\omega T_0}{2\pi}\right) e^{j\omega\left(t - \frac{T_0}{2}\right)} d\omega \end{aligned} \quad (2)$$

(a) Invoking the formula for the inverse Fourier transform, we immediately deduce from Eq. (2) that the Fourier transform of the integrator output $y(t)$ is given by

$$Y(j\omega) = \frac{T_0}{2\pi} \text{sinc}\left(\frac{\omega T_0}{2\pi}\right) e^{-j\omega T_0/2} \quad (3)$$

Examining this formula, we also readily see that $y(t)$ can be equivalently obtained by passing the input signal $x(t)$ through a filter whose frequency response is defined by

$$H(j\omega) = \frac{T_0}{2\pi} \text{sinc}\left(\frac{\omega T_0}{2\pi}\right) e^{-j\omega T_0/2}$$

The magnitude response of the filter is depicted in Fig. 1:

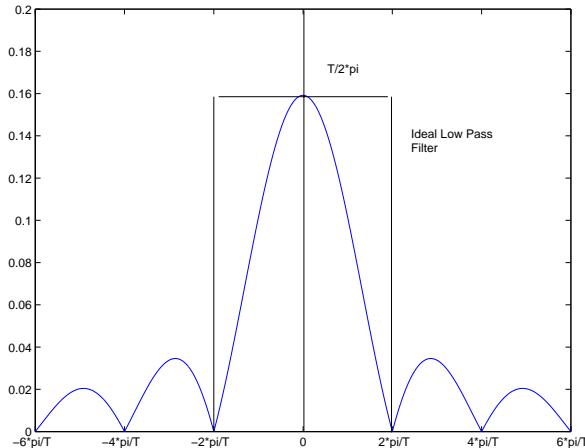


Figure 1

- (b) Figure 1 also includes the magnitude response of an “approximating” ideal low-pass filter. This latter filter has a cutoff frequency $\omega_0 = 2\pi/T_0$ and passband gain of $T_0/2\pi$. Moreover, the filter has a constant delay of $T_0/2$. The response of this ideal filter to a step function applied at time $t = 0$ is given by

$$y'(t) = \frac{T_0}{\pi} \int_{-\infty}^{\frac{2\pi}{T_0}\left(t - \frac{T_0}{2}\right)} \frac{\sin \lambda}{\lambda} d\lambda$$

At $t = T_0$, we therefore have

$$\begin{aligned} y'(T_0) &= \frac{T_0}{\pi} \int_{-\infty}^{\pi} \frac{\sin \lambda}{\lambda} d\lambda \\ &= \frac{T_0}{\pi} \left(\int_{-\infty}^0 \frac{\sin \lambda}{\lambda} d\lambda + \int_0^{\pi} \frac{\sin \lambda}{\lambda} d\lambda \right) \\ &= \frac{T_0}{\pi} (S_i(\infty) + S_i(\pi)) \\ &= 1.09T_0 \end{aligned}$$

From Eq. (1) we find that the ideal integrator output at time $t = T_0$ in response to the step function $x(t) = u(t)$ is given by

$$y(T_0) = \int_0^{T_0} u(\tau) d\tau$$

It follows therefore that the output of the “approximating” ideal low-pass filter exceeds the output of the ideal integrator by 9%. It is noteworthy that this overshoot is indeed a manifestation of the Gibb’s phenomenon.

- 8.32 To transform a prototype low-pass filter into a bandstop filter of midband rejection frequency ω_0 and bandwidth B , we may use the transformation

$$s \rightarrow \frac{Bs}{s^2 + \omega_0^2} \tag{1}$$

An as illustrative example, consider the low-pass filter:

$$H(s) = \frac{1}{s + 1} \tag{2}$$

Using the transformation of Eq. (1) in Eq. (2), we obtain a bandstop filter defined by

$$H(s) = \frac{1}{\frac{Bs}{s^2 + \omega_0^2} + 1}$$

$$= \frac{s^2 + \omega_0^2}{s^2 + B s + \omega_0^2}$$

which is characterized as follows:

$$H(s)|_{s=0} = H(s)|_{s=\infty} = 1$$

$$H(s)|_{s=\pm j\omega_0} = 0$$

- 8.33 An FIR filter of type 1 has an even length M and is symmetric about $M/2$ in that its coefficients satisfy the condition

$$h[n] = h[M-n] \quad \text{for } n = 0, 1, \dots, M$$

The frequency response of the filter is

$$H(e^{j\Omega}) = \sum_{n=0}^M h[n] e^{-jn\Omega}$$

which may be reformulated as follows:

$$\begin{aligned} H(e^{j\Omega}) &= \sum_{n=0}^{\frac{M}{2}-1} h[n] e^{-jn\Omega} + h\left[\frac{M}{2}\right] e^{-jM\Omega/2} + \sum_{n=\frac{M}{2}+1}^M h[n] e^{-jn\Omega} \\ &= \sum_{n=0}^{\frac{M}{2}-1} h[n] e^{-jn\Omega} + h\left[\frac{M}{2}\right] e^{-jM\Omega/2} + \sum_{n=0}^{\frac{M}{2}-1} h[M-n] e^{-j\Omega(M-n)} \\ &= \sum_{n=0}^{\frac{M}{2}-1} h[n] e^{-jn\Omega} + h\left[\frac{M}{2}\right] e^{-jM\Omega/2} + \sum_{n=0}^{\frac{M}{2}-1} h[n] e^{-j\Omega(M-n)} \end{aligned} \quad (1)$$

Define

$$a[k] = 2h\left[\frac{M}{2} - k\right], \quad k = 1, 2, \dots, \frac{M}{2}$$

$$a[0] = h\left[\frac{M}{2}\right]$$

and let

$$n = \frac{M}{2} - k$$

We may then rewrite Eq. (1) in the equivalent form:

$$\begin{aligned}
H(e^{j\Omega}) &= e^{-jM\Omega/2} \left\{ \sum_{k=1}^{\frac{M}{2}} h\left[\frac{M}{2} - k\right] (e^{jk\Omega} + e^{-jk\Omega}) + h\left[\frac{M}{2}\right] \right\} \\
&= e^{-jM\Omega/2} \left\{ \sum_{k=1}^{\frac{M}{2}} 2a[k] (e^{jk\Omega} + e^{-jk\Omega}) + a(0) \right\} \\
&= e^{-jM\Omega/2} \left\{ \sum_{k=0}^{\frac{M}{2}} a[k] \cos(k\Omega) \right\} \tag{2}
\end{aligned}$$

From Eq. (2) we may make the following observations for an FIR filter of type I:

1. The frequency response $H(e^{j\Omega})$ has a linear phase component exemplified by the exponential $e^{-jM\Omega/2}$.
2. At $\Omega = 0$,

$$H(e^{j0}) = \sum_{k=0}^{M/2} a[k]$$

At $\Omega = \pi$,

$$H(e^{j\pi}) = \sum_{k=0}^{M/2} a[k] (-1)^{\frac{M}{2} + k}$$

The implications of these two results are that there are no restrictions on $H(e^{j\Omega})$ at $\Omega = 0$ and $\Omega = \pi$.

- 8.34 For an FIR filter of type II, the filter length M is even and it is antisymmetric in that its coefficients satisfy the condition

$$h[n] = -h[M - n], \quad 0 \leq n \leq \frac{M}{2} - 1$$

The frequency response of the filter is

$$H(e^{j\Omega}) = \sum_{n=0}^M h[n] e^{-jn\Omega}$$

which may be reformulated as follows:

$$\begin{aligned}
H(e^{j\Omega}) &= \sum_{n=0}^{\frac{M}{2}-1} h[n]e^{-jn\Omega} + h\left[\frac{M}{2}\right]e^{-jM\Omega/2} + \sum_{n=\frac{M}{2}+1}^M h[n]e^{-jn\Omega} \\
&= \sum_{n=0}^{\frac{M}{2}-1} h[n]e^{-jn\Omega} + h\left[\frac{M}{2}\right]e^{-jM\Omega/2} + \sum_{n=0}^{\frac{M}{2}-1} h[M-n]e^{-j(M-n)\Omega} \\
&= \sum_{n=0}^{\frac{M}{2}-1} h[n]e^{-jn\Omega} + h\left[\frac{M}{2}\right]e^{-jM\Omega/2} - \sum_{n=0}^{\frac{M}{2}-1} h[n]e^{-j(M-n)\Omega}
\end{aligned} \tag{1}$$

Define

$$a[k] = 2h\left[\frac{M}{2} - k\right], \quad k = 1, 2, \dots, \frac{M}{2}$$

$$a[0] = h\left[\frac{M}{2}\right],$$

and let

$$k = \frac{M}{2} - n$$

We may then rewrite Eq. (1) in the equivalent form

$$\begin{aligned}
H(e^{j\Omega}) &= e^{-jM\Omega/2} \sum_{n=0}^{M/2} h\left[\frac{M}{2} - k\right]e^{jk\Omega} + h\left[\frac{M}{2}\right]e^{-jM\Omega/2} - e^{-jM\Omega/2} \sum_{k=1}^{M/2} h\left[\frac{M}{2} - k\right]e^{-jk\Omega} \\
&= e^{-jM\Omega/2} \left\{ 2 \sum_{k=1}^{M/2} a[k](e^{jk\Omega} - e^{-jk\Omega}) + a[0] \right\} \\
&= e^{-jM\Omega/2} \left\{ j \sum_{k=1}^{M/2} a[k] \sin(k\Omega + a[0]) \right\} \\
&= e^{-jM\Omega/2} \sum_{k=0}^{M/2} a[k] \sin(k\Omega)
\end{aligned} \tag{2}$$

From Eq. (2) we may make the following observations on the frequency response of an FIR filter of Type II

1. The phase response includes a linear component exemplified by the exponential $e^{-jM\Omega/2}$.

2. At $\Omega = 0$,

$$H(e^{j0}) = 0$$

At $\Omega = \pi$, $\sin(k\pi) = 0$ for integer k and therefore,

$$H(e^{j\pi}) = 0$$

8.35 An FIR filter of type III is characterized as follows:

- The filter length M is an odd integer.
- The filter is symmetric about the noninteger midpoint $n = M/2$ in that its coefficients satisfy the condition

$$h[n] = h[M - n] \quad \text{for } 0 \leq n \leq M$$

The frequency response of the filter is

$$H(e^{j\Omega}) = \sum_{n=0}^M h[n] e^{-jn\Omega}$$

which may be reformulated as follows:

$$\begin{aligned} H(e^{j\Omega}) &= \sum_{n=0}^{\frac{M-1}{2}} h[n] e^{-jn\Omega} + \sum_{n=\frac{M+1}{2}}^M h[n] e^{-jn\Omega} \\ &= \sum_{n=0}^{\frac{M-1}{2}} h[n] e^{-jn\Omega} + \sum_{n=0}^{\frac{M-1}{2}} h[M-n] e^{-j(M-n)\Omega} \\ &= \sum_{n=0}^{\frac{M-1}{2}} h[n] (e^{-jn\Omega} + e^{-j(M-n)\Omega}) \\ &= e^{-jM\Omega/2} \sum_{n=0}^{\frac{M-1}{2}} h[n] \left(e^{j\left(\frac{M}{2}-n\right)\Omega} + e^{-j\left(\frac{M}{2}-n\right)\Omega} \right) \end{aligned} \tag{1}$$

Define

$$b[k] = 2h\left[\frac{M+1}{2} - k\right] \quad \text{for } k = 1, 2, \dots, \frac{M+1}{2}$$

and let

$$n = \frac{M+1}{2} - k$$

We may then rewrite Eq. (1) in the equivalent form

$$\begin{aligned} H(e^{j\Omega}) &= e^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} \frac{1}{2} b(k) \left(e^{j\Omega\left(k-\frac{1}{2}\right)} + e^{-j\Omega\left(k-\frac{1}{2}\right)} \right) \\ &= e^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} b(k) \cos\left(\Omega\left(k-\frac{1}{2}\right)\right) \end{aligned} \quad (2)$$

From Eq. (2) we may make the following observations on the frequency response of an FIR filter of type III:

1. The phase response of the filter is linear as exemplified by the exponential factor $e^{-jM\Omega/2}$.
2. At $\Omega = 0$,

$$H(e^{j0}) = \sum_{k=1}^{\frac{M+1}{2}} b(k)$$

which shows that there is no restriction on $H(e^{j0})$.

At $\Omega = \pi$,

$$\begin{aligned} H(e^{j\pi}) &= e^{-jM\pi/2} \sum_{k=1}^{\frac{M+1}{2}} b(k) \cos\left(\pi\left(k-\frac{1}{2}\right)\right) \\ &= e^{-jM\pi/2} \sum_{k=1}^{\frac{M+1}{2}} b(k) \sin(\pi k) \end{aligned}$$

which is zero since $\sin(\pi k) = 0$ for all integer values of k .

8.36 An FIR filter of type IV is characterized as follows:

- The filter length M is an odd integer.
- The filter is antisymmetric about the noninteger midpoint $n = M/2$ in that its coefficients satisfy the condition

$$h[n] = -h[M-n] \quad \text{for } 0 \leq n \leq M$$

The frequency response of the filter is

$$H(e^{j\Omega}) = \sum_{n=0}^M h[n]e^{-jn\Omega}$$

which may be reformulated as follows:

$$\begin{aligned}
H(e^{j\Omega}) &= \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{-jn\Omega} + \sum_{n=\frac{M+1}{2}}^M h[n]e^{-jn\Omega} \\
&= \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{-jn\Omega} + \sum_{n=0}^{\frac{M-1}{2}} h[M-n]e^{-j(M-n)\Omega} \\
&= \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{-jn\Omega} - \sum_{n=0}^{\frac{M-1}{2}} h[n]e^{-j(M-n)\Omega} \\
&= e^{-jM\Omega/2} \sum_{n=0}^{\frac{M-1}{2}} h[n] \begin{pmatrix} e^{-j\left(n-\frac{M}{2}\right)\Omega} & e^{j\left(n-\frac{M}{2}\right)\Omega} \end{pmatrix}
\end{aligned} \tag{1}$$

Define

$$b[k] = 2h\left[\frac{M+1}{2} - k\right] \quad \text{for } k = 1, 2, \dots, \frac{M+1}{2}$$

and let

$$k = \frac{M+1}{2} - n$$

We may then rewrite Eq. (1) in the equivalent form

$$\begin{aligned}
H(e^{j\Omega}) &= e^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} h\left[\frac{M+1}{2} - k\right] \begin{pmatrix} e^{j\left(k-\frac{1}{2}\right)\Omega} & -e^{-j\left(k-\frac{1}{2}\right)\Omega} \end{pmatrix} \\
&= je^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} 2h\left[\frac{M+1}{2} - k\right] \sin\left(\left(k - \frac{1}{2}\right)\Omega\right) \\
&= je^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} b[k] \sin\left(\left(k - \frac{1}{2}\right)\Omega\right)
\end{aligned} \tag{2}$$

From Eq. (2) we may make the following observations on the FIR filter of type IV:

1. The phase response of the filter includes a linear component exemplified by the exponential $e^{-jM\Omega/2}$.

2. At $\Omega = 0$,

$$H(e^{j0}) = 0$$

At $\Omega = \pi$,

$$\begin{aligned} H(e^{j\pi}) &= je^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} b(k) \sin\left(\left(k - \frac{1}{2}\right)\pi\right) \\ &= je^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} (-1)^{k+1} b[k] \end{aligned}$$

which shows that $H(e^{j\pi})$ can assume an arbitrary value.

8.37 The FIR digital filter used as a discrete-time differentiator in Example 8.6 exhibits the following properties:

- The filter length M is an odd integer.
- The frequency response of the filter satisfies the conditions:
 1. At $\Omega = 0$,

$$H(e^{j0}) = 0$$

2. At $\Omega = \pi$,

$$H(e^{j\Omega}) = 0$$

These properties are basic properties of an FIR filter of type III discussed in Problem 8.34. We therefore immediately deduce that the FIR filter of Example 8.6 is antisymmetric about the noninteger point $n = M/2$.

8.38 For a digital IIR filter, the transfer function $H(z)$ may be expressed as

$$H(z) = \frac{N(z)}{D(z)}$$

where $N(z)$ and $D(z)$ are polynomials in z^{-1} . The filter is unstable if any pole of $H(z)$ or, equivalently, any zero of the denominator polynomial $D(z)$ lies outside the unit circle in the z -plane. According to the bilinear transform,

$$H(z) = H_a(s) \Big|_{s = \frac{z-1}{z+1}}$$

where $H_a(s)$ is the transfer function of an analog filter used as the basis for designing the digital IIR filter. The poles of $H(z)$ outside the unit circle in the z -plane correspond to certain poles of $H(s)$ in the right half of the s -plane. Conversely, the poles of $H(s)$ in the right half of the s -plane are mapped onto the outside of the unit circle in the z -plane. Now if any pole of $H_a(s)$ lies in the right-half plane, the analog filter is unstable. Hence if any such filter is used in the bilinear transform, the resulting digital filter is likewise unstable.

8.39 We are given an analog filter whose transfer function is defined by

$$H_a(s) = \sum_{k=1}^N \frac{A_k}{s - d_k}$$

Recall the Laplace transform pair

$$e^{d_k t} \xleftrightarrow{L} \frac{1}{s - d_k}$$

It follows therefore that the impulse response of the analog filter is

$$h_a(t) = \sum_{k=1}^N A_k e^{d_k t} \quad (1)$$

According to the method of impulse invariance, the impulse response of a digital filter derived from the analog filter of Eq. (1) is defined by

$$h[n] = T_s h_a(nT_s)$$

where T_s is the sampling period. Hence, from Eq. (1) we find that

$$h[n] = \sum_{k=1}^N T_s A_k e^{nd_k T_s} \quad (2)$$

Now recall the z -transform pair:

$$e^{nd_k T_s} \xleftrightarrow{z} \frac{1}{1 - e^{d_k T_s} z^{-1}}$$

Hence, the transfer function of the digital filter is deduced from Eq. (2) to be

$$H(z) = \sum_{k=1}^N \frac{T_s A_k}{1 - e^{d_k T_s} z^{-1}}$$

8.40 Consider a discrete-time system whose transfer function is denoted by $H(z)$. By definition,

$$H(z) = \frac{Y(z)}{X(z)}$$

where $Y(z)$ and $X(z)$ are respectively the z -transforms of the output sequence $y[n]$ and input sequence $x[n]$. Let $H_{\text{eq}}(z)$ denote the z -transform of the equalizer connected in cascade with $H(z)$.

Let $x'[n]$ denote the equalizer output in response to $y[n]$ as the input. Ideally,
 $x'[n] = x[n - n_0]$

where n_0 is an integer delay. Hence,

$$H_{\text{eq}}(z) = \frac{X'(z)}{Y(z)} = \frac{z^{-n_0} X(z)}{Y(z)} = \frac{z^{-n_0}}{H(z)}$$

Putting $z = e^{j\Omega}$, we may thus write

$$H_{\text{eq}}(e^{j\Omega}) = \frac{e^{-jn_0\Omega}}{H(e^{j\Omega})}$$

- 8.41 The phase delay of an FIR filter of even length M and antisymmetric impulse response is linear with frequency Ω as shown by

$$\theta(\Omega) = -M\Omega$$

Hence, such a filter used as an equalizer introduces a constant delay

$$\tau(\Omega) = -\frac{\partial\theta(\Omega)}{\partial\Omega} = M \text{ samples}$$

The implication of this result is that as we make the filter length M larger, the constant delay introduced by the equalizer is correspondingly increased. From a practical perspective, such a trend is highly undesirable.

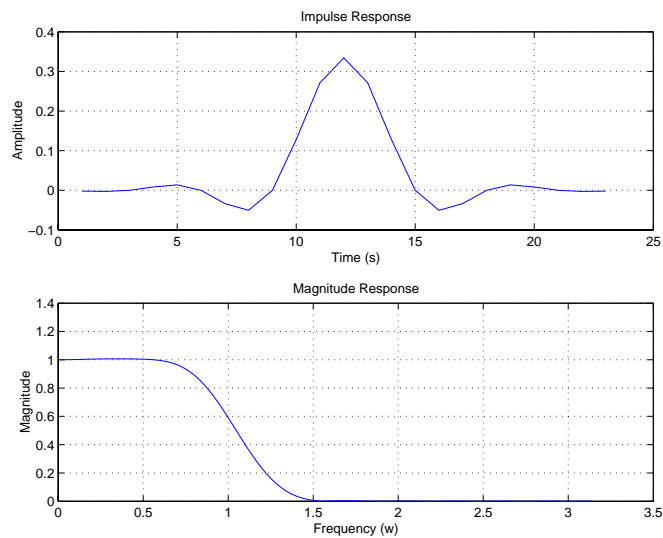
Computer Experiments

%Solution to 8.42

```
b = fir1(22,1/3,hamming(23));
```

```
subplot(2,1,1)
plot(b);
title('Impulse Response')
ylabel('Amplitude')
xlabel('Time (s)')
grid
```

```
[H,w] = freqz(b,1,512,2*pi);
subplot(2,1,2)
plot(w,abs(H))
title('Magnitude Response')
ylabel('Magnitude')
xlabel('Frequency (w)')
grid
```



%Solution to problem 8.43

```
clear;
```

```
M = 100;
n = 0:M;
f = (n-M/2);
```

```

%Integration by parts (see example 8.6)
h = cos(pi*f)./f - sin(pi*f)./(pi*f.^2);
k = isnan(h);
h(k) = 0;
h_rect = h;
h_hamm = h .* hamming(length(h))';

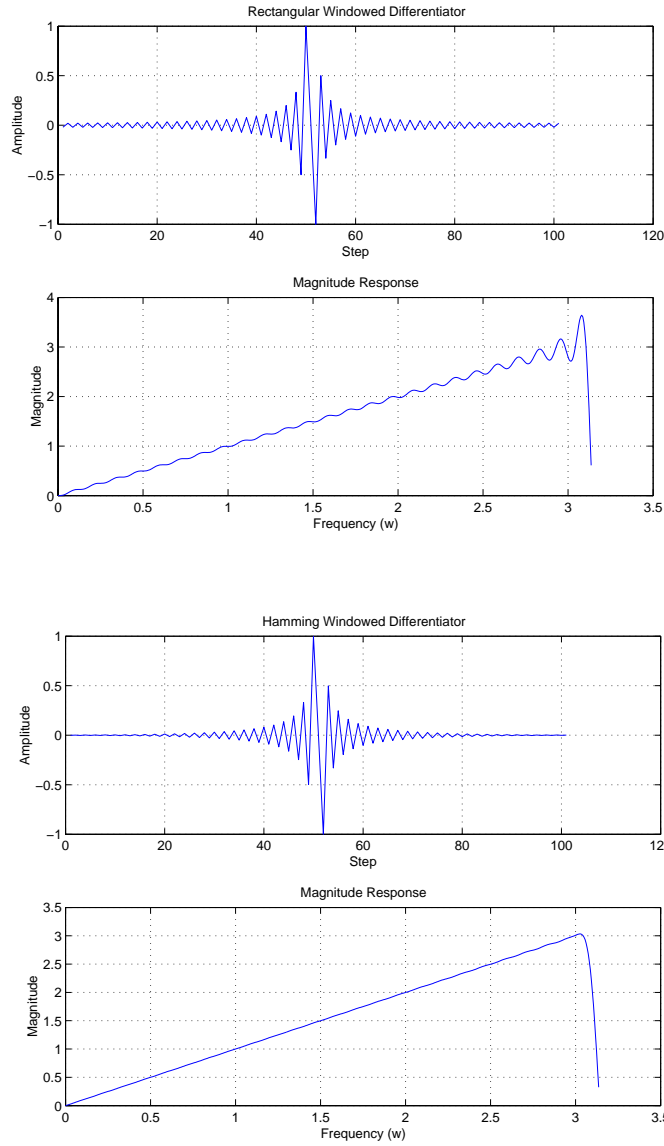
[H,w] = freqz(h_rect,1,512,2*pi);

figure(1)
subplot(2,1,1)
plot(h_rect);
title('Rectangular Windowed Differentiator')
xlabel('Step')
ylabel('Amplitude')
grid
subplot(2,1,2)
plot(w,abs(H))
title('Magnitude Response')
ylabel('Magnitude')
xlabel('Frequency (w)')
grid

[H,w] = freqz(h_hamm,1,512,2*pi);

figure(2);
subplot(2,1,1)
plot(h_hamm);
title('Hamming Windowed Differentiator')
xlabel('Step')
ylabel('Amplitude')
grid
subplot(2,1,2)
plot(w,abs(H))
title('Magnitude Response')
ylabel('Magnitude')
xlabel('Frequency (w)')
grid

```



8.44 For a Butterworth low-pass filter of order N , the squared magnitude response is

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}} \quad (1)$$

where ω_c is the cutoff frequency.

(a) We are given the following specifications:

- (i) $\omega_c = 2\pi \times 800$ rad/s
- (ii) At $\omega = 2\pi \times 1,500$ rad/s, we have

$$10\log_{10}|H(j\omega)|^2 = -15 \text{ dB}$$

or, equivalently

$$|H(j\omega)|^2 = \frac{1}{31.6228}$$

Substituting these values in Eq. (1):

$$\begin{aligned} 31.6228 &= 1 + \left(\frac{2\pi \times 1,200}{2\pi \times 800} \right)^{2N} \\ &= 1 + (1.5)^{2N} \end{aligned}$$

Solving for the filter order N :

$$\begin{aligned} N &= \frac{1}{2}(\log 30.6228 / \log 1.5) \\ &= 4.2195 \end{aligned}$$

So we choose $N = 5$.

%Solution to Problem 8.44

omegaC = 0.2;

N = 5;

wc = tan(omegaC/2);

coeff = [1 3.2361 5.2361 5.2361 3.2361 1]; %(see table 8.1)

ns = wc^N;

ds = coeff .* (wc.^[0:N]);

[nz,dz]=bilinear(ns,ds,0.5);

[H,W]=freqz(nz,dz,512);

subplot(2,1,1)

plot(W,abs(H))

title('Magnitude Response of IIR low-pass filter')

xlabel('rad/s')

ylabel('Magnitude')

grid

phi = 180/pi *angle(H);

subplot(2,1,2)

plot(W,phi)

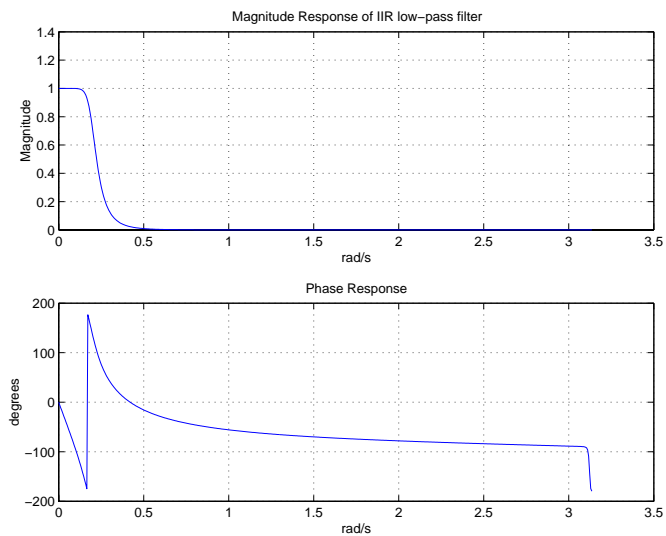
title('Phase Response')

xlabel('rad/s')

ylabel('degrees')

grid

```
%set(gcf,'name',['Low Pass: order=' num2str(N) ' wc=' num2str(omegaC)])
```



%Solution to Problem 8.45

```
omegaC = 0.6;
```

```
N = 5;
```

```
wc = tan(omegaC/2);
```

```
coeff = [ 1 3.2361 5.2361 5.2361 3.2361 1]; %(see table 8.1)
```

```
ns = [1/wc^N zeros(1,N)];
```

```
ds = flipr(coeff ./ (wc.^[0:N]));
```

```
[nz,dz]=bilinear(ns,ds,0.5);
```

```
[H,W]=freqz(nz,dz,512);
```

```
subplot(2,1,1)
```

```
plot(W,abs(H))
```

```
title('Magnitude Response of IIR high-pass filter')
```

```
xlabel('rad/s')
```

```
ylabel('Magnitude')
```

```
grid
```

```
phi = 180/pi *angle(H);
```

```
subplot(2,1,2)
```

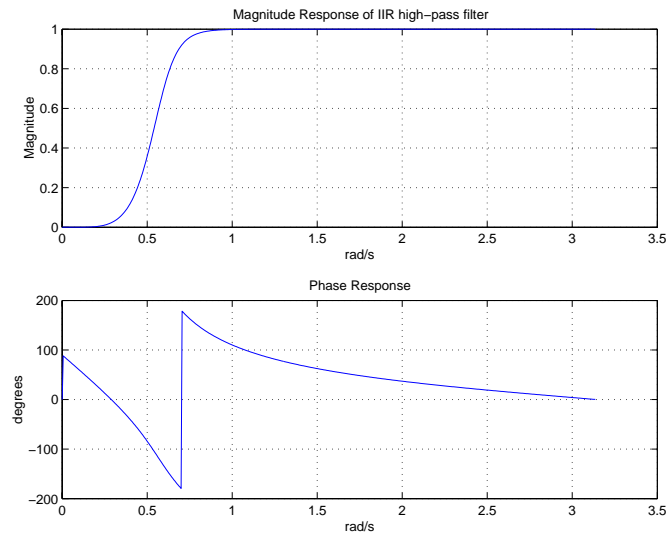
```
plot(W,phi)
```

```
title('Phase Response')
```

```
xlabel('rad/s')
```

```
ylabel('degrees')
```

grid



%Solution to problem 8.46

```
w = 0:pi/511:pi;
den = sqrt(1+2*((w/pi).^2) + (w/pi).^4);
Hchan = 1./den;
taps = 95;
M = taps -1;
n = 0:M;
f = n-M/2;

%Term 1
hh1 = fftshift(ifft(ones(taps,1))).*hamming(taps);

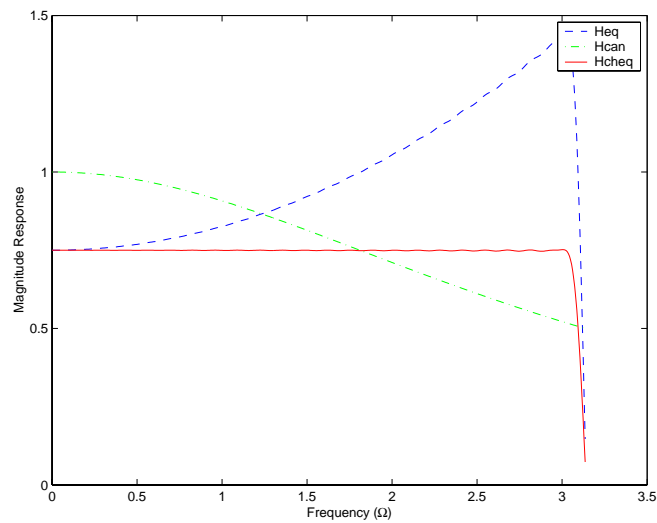
%Term 2
h = cos(pi*f)./f - sin(pi*f)./(pi*f.^2);
k=isnan(h); h(k)=0;
hh2 = 2*(hamming(taps)'.*h)/pi;

%Term 3
hh3a = -(1./(pi*f)) .* sin(pi*f);
hh3b = -(2./(pi*f).^2) .* cos(pi*f);
hh3c = (2./(pi*f).^3) .* sin(pi*f);
hh3 = hamming(taps) .* (hh3a + hh3b + hh3c)';
hh = hh1' + hh2 + hh3';
hh(48)=2/3;
```

```
[Heq,w]=freqz(hh,1,512,2*pi);
```

```
p = 0.7501;
```

```
Hcheq = (p*abs(Heq)).*Hchan';  
plot(w,p*abs(Heq),'b--')  
hold on  
plot(w,abs(Hchan),'g-.')  
plot(w,abs(Hcheq),'r-')  
legend('Heq', 'Hcan', 'Hcheq',1)  
hold off  
xlabel('Frequency (\Omega)')  
ylabel('Magnitude Response')
```



CHAPTER 9

Additional Problems

- 9.21 (a) The closed-loop gain of the feedback amplifier is given by

$$T = \frac{A}{1 + \beta A} \quad (1)$$

We are given the following values for the forward amplification A and feedback factor β :

$$A = 2,500$$

$$\beta = 0.01$$

Substituting these values in Eq. (1):

$$T = \frac{2500}{1 + 0.01 \times 2500} = \frac{2500}{26} = 92.15$$

- (b) The sensitivity of the feedback amplifier to changes in A is given by

$$S_T^A = \frac{\Delta T/T}{\Delta A/A} = \frac{1}{1 + \beta A} = \frac{1}{26}$$

With $(\Delta A/A) = 10\% = 0.10$, we thus have

$$\begin{aligned} \frac{\Delta T}{T} &= S_T^A \left(\frac{\Delta A}{A} \right) \\ &= \frac{1}{26}(0.10) = 0.0038 = 0.38\% \end{aligned}$$

- 9.22 (a) The closed-loop gain of the feedback system is

$$T = \frac{G_a G_p}{1 + H G_a G_p}$$

- (b) The return difference of the system is

$$F = 1 + H G_a G_p$$

Hence the sensitivity of T with respect to changes in G_p is

$$\begin{aligned} S_T^{G_p} &= \frac{\Delta T/T}{\Delta G_p/G_a} \\ &= \frac{1}{F} \\ &= \frac{1}{1 + H G_a G_p} \end{aligned}$$

- (c) We are given: $H = 1$ and $G_p = 1.5$. Hence for $S_T^{G_p} = 1\% = 0.01$, we require

$$F = 100$$

The corresponding value of G_a is therefore

$$\begin{aligned} G_a &= \frac{F-1}{HG_p} \\ &= \frac{99}{1.5} = 66 \end{aligned}$$

- 9.23 The local feedback around the motor has the closed-loop gain $G_p/(1 + HG_p)$. The closed-loop gain of the whole system is therefore

$$\begin{aligned} T &= \frac{K_r G_c G_p / (1 + HG_p)}{1 + K_r G_c G_p / (1 + HG_p)} \\ &= \frac{K_r G_c G_p}{1 + HG_p + K_r G_c G_p} \end{aligned}$$

- 9.24 The closed-loop gain of the operational amplifier is

$$\frac{V_2(s)}{V_1(s)} = - \frac{Z_2(s)}{Z_1(s)}$$

We are given $Z_1(s) = R_1$ and $Z_2(s) = R_2$. The closed-loop gain or transfer function of the operational amplifier in Fig. P9.24 is therefore

$$\frac{V_2(s)}{V_1(s)} = - \frac{R_2}{R_1}$$

- 9.25 (a) From Fig. P9.25, we have

$$Z_1(s) = R_1 + \frac{1}{sC_1}$$

$$Z_2(s) = R_2$$

The transfer function of this operational amplifier is therefore

$$\begin{aligned} \frac{V_2(s)}{V_1(s)} &= - \frac{Z_2(s)}{Z_1(s)} \\ &= - \frac{R_2}{R_1 + \frac{1}{sC_1}} \\ &= - \frac{sC_1 R_2}{1 + sC_1 R_1} \end{aligned} \tag{1}$$

- (b) For positive values of frequency ω that satisfy the condition

$$\frac{1}{\omega C_1} \gg R_1$$

we may approximate Eq. (1) as

$$\frac{V_2(s)}{V_1(s)} \approx -sC_1R_2$$

That is, the operational amplifier acts as a differentiator.

- 9.26 Throughout this problem, $H(s) = 1$, in which case the open-loop transfer equals $G(s)$. In any event, the open-loop transfer function of the feedback control system may be expressed as the rational function $P(s)/(s^p Q_1(s))$, where neither the polynomial $P(s)$ nor $Q_1(s)$ has a zero at $s = 0$. Since $1/s$ is the transfer function of an integrator, it follows that p is the number of free integrators in the feedback loop. The order p is referred to as the *type* of the feedback control system.

- (a) For the problem at hand, we are given

$$G(s) = \frac{15}{(s+1)(s+3)}$$

The control system is therefore type 0. The steady-state error for unit step is

$$\epsilon_{ss} = \frac{1}{1+K_p}$$

where

$$K_p = \lim_{s \rightarrow 0} G(s)H(s)$$

That is,

$$\begin{aligned} K_p &= \lim_{s \rightarrow 0} \frac{15}{(s+1)(s+3)} \\ &= \frac{15}{3} = 5 \end{aligned}$$

Hence,

$$\epsilon_{ss} = \frac{1}{1+5} = \frac{1}{6}$$

For both ramp and parabolic inputs, the steady-state error is infinitely large.

- (b) For

$$G(s) = \frac{5}{s(s+1)(s+4)}$$

the control system is type 1. The steady-state error for a step input is zero. For a ramp of unit slope, the steady-state error is

$$\epsilon_{ss} = \frac{1}{K_v}$$

where

$$K_v = \lim_{s \rightarrow 0} \frac{5}{(s+1)(s+4)}$$

$$= \frac{5}{4}$$

The steady-state error is therefore

$$\epsilon_{ss} = \frac{4}{5}$$

For a parabolic input the steady-state error is infinitely large.

(c) For

$$G(s) = \frac{5(s+1)}{s^2(s+3)}$$

the control system is type 2. Hence, the steady-state error is zero for both step and ramp inputs. For a unit parabolic input the steady-state error is

$$\epsilon_{ss} = \frac{1}{K_a}$$

where

$$K_a = \lim_{s \rightarrow 0} \frac{5(s+1)}{s+3}$$

$$= \frac{5}{3}$$

The steady-state error is therefore 3/5. For a unit parabolic input the steady-state error is infinitely large.

(d) For

$$G(s) = \frac{5(s+1)(s+2)}{s^2(s+3)}$$

the control system is type 2. The steady-state error is therefore zero for both step and ramp inputs. For a unit parabolic input the steady-state error is

$$\epsilon_{ss} = \frac{1}{K_n}$$

where

$$K_n = \lim_{s \rightarrow 0} \frac{5(s+1)(s+2)}{s+3}$$

$$= \frac{10}{3}$$

The steady-state error is therefore 3/10.

9.27 For the results on steady-state errors calculated in Problem 9.26 to hold, all the feedback control systems in parts (a) to (d) of the problem have to be stable.

$$(a) \quad G(s)H(s) = \frac{15}{(s+1)(s+3)}$$

The characteristic equation is

$$\begin{aligned} A(s) &= (s+1)(s+3) + 15 \\ &= s^2 + 4s + 18 \end{aligned}$$

both roots of which are in the left-half plane. The system is therefore stable.

$$\begin{aligned} (b) \quad G(s)H(s) &= \frac{5}{s(s+1)(s+4)} \\ &= \frac{5}{s^3 + 5s^2 + 4s} \end{aligned}$$

$$A(s) = s^3 + 5s^2 + 4s + 5$$

Applying the Routh-Hurwitz criterion:

s^3	1	4
s^2	5	5
s^1	15/5	0
s^0	5	0

There are no sign changes in the first column coefficients of the array; the system is therefore stable.

$$(c) \quad G(s)H(s) = \frac{5(s+1)}{s^2(s+3)}$$

$$A(s) = s^3 + 3s^2 + 5s + 5$$

The Routh-Hurwitz array is:

s^3	1	5
s^2	3	5
s^1	10/3	0
s^0	5	0

Here again there are no sign changes in the first column of coefficients, and the control system is therefore stable.

$$(d) \quad G(s)H(s) = \frac{5(s+1)(s+2)}{s^2(s+3)}$$

$$\begin{aligned} A(s) &= s^3 + 3s^2 + 5s^2 + 15s + 10 \\ &= s^3 + 8s^2 + 15s + 10 \end{aligned}$$

The Routh-Hurwitz array is

s^3	1	15
s^2	8	10
s^1	110/8	0
s^0	10	0

Here again there are no sign changes in the first column of array coefficients, and the control system is therefore stable.

9.28 (a) $s^4 + 2s^2 + 1 = 0$

Equivalently, we may write

$$(s^2 + 1)^2 = 0$$

which has double roots at $s = \pm j$. The system is therefore on the verge of instability.

(b) $s^4 + s^3 + s + 0.5 = 0$

By inspection, we can say that this feedback control system is unstable because the term s^2 is missing from the characteristic equation. We can verify this observation by constructing the Routh array:

s^4	1	0	0.5
s^3	1	1	0
s^2	-1	0.5	0
s^1	+1.5	0	
s^0	9.5	0	

There are two sign changes in the first column of array coefficients, indicating that the characteristic equation has a pair of complex-conjugate roots in the right-half of the s -plane. The system is therefore unstable as previously observed.

(c) $s^4 + 2s^3 + 2s^2 + 2s + 4 = 0$

The Routh array is

s^4	1	3	4
s^3	2	2	0
s^2	2	4	0
s^1	-2	0	
s^0	-4	0	

There are two sign changes in the first column of array coefficients, indicating the presence of two roots of the characteristic equation in the right-half of the s -plane. The control system is therefore unstable.

9.29 The characteristic equation of the control system is

$$s^3 + s^2 + s + K = 0$$

Applying the Routh-Hurwitz criterion:

s^3	1	1
s^2	1	K
s^1	$1-K$	0
s^0	K	0

The control system is therefore stable provided that the parameter K satisfies the condition:

$$0 < K < 1$$

9.30 (a) The characteristic equation of the feedback system is

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

Applying the Routh-Hurwitz criterion:

s^3	a_3	a_1
s^2	a_2	a_0
s^1	$\frac{a_2 a_1 - a_3 a_0}{a_2}$	0
s^0	a_0	

For the system to be stable, the following conditions must be satisfied

$a_3 > 0, a_2 > 0, a_0 > 0$ and

$$\frac{a_2 a_1 - a_3 a_0}{a_2} > 0 \quad (1)$$

(b) For the characteristic equation

$$s^3 + s^2 + s + K = 0$$

we have $a_3 = 1, a_2 = 1$, and $a_1 = 1$ and $a_0 = K$. Hence, applying the condition (1):

$$\frac{1 \times 1 - 1 \times K}{1} > 0$$

or

$$K < 1$$

Also, we require that $K > 0$ since we must have $a_0 > 0$. Hence, K must satisfy the condition $0 > K < 1$ for stability.

9.31 (a) We are given the loop transfer function

$$L(s) = \frac{K}{s(s^2 + s + 2)}, \quad K > 0$$

$L(s)$ has three poles, one at $s = 0$ and the other two at $s = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$.

Hence, the root locus of $L(s)$ has 3 branches. It starts at these 3 pole locations and terminate at the zeros of $L(s)$ at $s = \infty$. The straight-line asymptotics of the root locus are defined by the angles (see Eq. (9.69))

$$\theta_k = \frac{(2k+1)\pi}{3}, \quad k = 0, 1, 2$$

or $\pi/3, \pi$, and $5\pi/3$. Their location point on the real axis of the s -plane is defined by (see Eq. (9.70))

$$\begin{aligned} \sigma_0 &= \frac{0 + \left(-\frac{1}{2} + j\frac{\sqrt{7}}{2}\right) - \left(\frac{1}{2} - j\frac{\sqrt{7}}{2}\right)}{3} \\ &= -\frac{1}{3} \end{aligned}$$

Since $L(s)$ has a pair of complex-conjugate poles, we need (in addition to the rules described in the text) a rule concerning the angle at which the root locus leaves a complex pole (i.e., angle of departure). Specifically, we wish to determine the angle θ_{dep} indicated in Fig. 1 for the problem at hand:

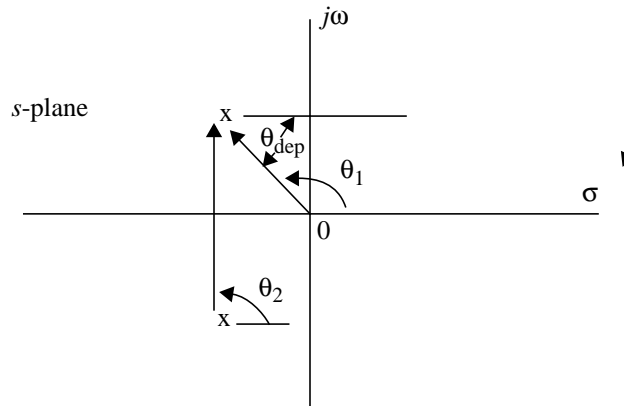


Figure 1

The angle θ_1 and θ_2 are defined by

$$\theta_1 = \tan^{-1} \left(\frac{\sqrt{7/2}}{-1/2} \right) = \tan^{-1}(2.6458) = 111.7^\circ$$

$$\theta_2 = 90^\circ$$

Applying the angle criterion (Eq. (9.....)):

$$\theta_{\text{dep}} + 90^\circ + 111.7^\circ = 180$$

Hence,

$$\theta_{\text{dep}} = -21.70^\circ$$

We may thus sketch the root locus of the given system as in Fig. 2:

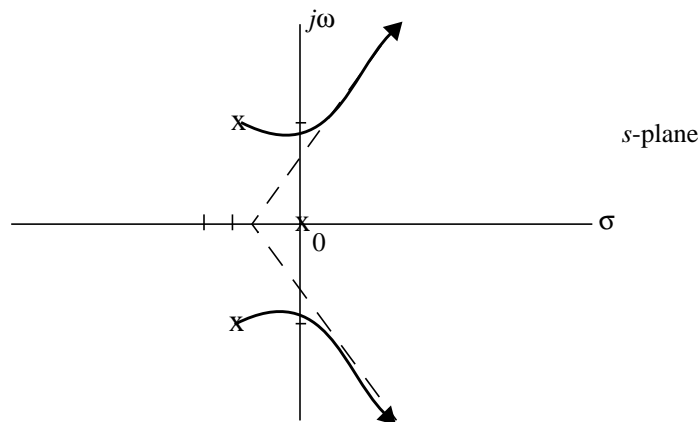


Figure 2

(b) The characteristic equation of the system is

$$s^3 + s^2 + 2s + K = 0$$

Applying the Routh-Hurwitz criterion:

s^3	1	2
s^2	1	K
s^1	$2-K$	0
s^0	K	0

The system is therefore on the verge of instability when $K = 2$. For this value of K the root locus intersects the $j\omega$ -axis at $s^2 + K = 0$ or $s = \pm j\sqrt{K} = \pm j\sqrt{2}$, as indicated in Fig. 2.

9.32 The closed-loop transfer function of a control system with unity feedback is

$$T(s) = \frac{L(s)}{1 + L(s)}$$

We are given

$$L(s) = \frac{K}{s(s+1)} = \frac{K}{s^2 + 3}$$

Hence

$$T(s) = \frac{K}{s^2 + 3 + K}$$

(a) For $K = 0.1$ the system is overdamped

$$T(s) = \frac{0.1}{s^2 + s + 0.1}$$

where $\omega_n = \sqrt{0.1}$ and $\zeta = 0.5/\sqrt{0.1}$.

(b) For $K = 0.25$ the system is critically damped:

$$T(s) = \frac{0.25}{s^2 + s + 0.25}$$

where $\omega_n = 0.5$ and $\zeta = 1$.

(c) For $K = 2.5$ the system is underdamped:

$$T(s) = \frac{0.25}{s^2 + s + 2.5}$$

where $\omega_n = \sqrt{2.5}$ and $\zeta = 0.5/\sqrt{2.5}$.

Figure 1 plots the respective step responses of these three special cases of the feedback system.

%Solution to problem 9.32

t = 0:0.1:40;clc;

%Underdamped System

K = 0.1;

w = sqrt(K);

z = 0.5/w;

fac = w*sqrt(1-z^2)*t + atan(sqrt(1-z^2)/z);

y1 = 1 - 1/sqrt(1-z^2).*exp(-z*w*t).*sin(fac);

figure(1);clf;

plot(t,y1,'-')

%ylim([0 1.5])

xlabel('Time (s)')

ylabel('Response')

%title('Underdamped K = 0.1')

hold on

%Critically Damped System

K = 0.25;

w = sqrt(K);

z = 0.5/w

tau = 1/w

y2 = 1 - exp(-t/tau) - t.*exp(-t/tau);

plot(t,y2,'-')

%Over Damped System

K = 2.50000;

w = sqrt(K);

z = 0.5/w;

t1 = 1 / (z*w - w*sqrt(z^2-1));

t2 = 1 / (z*w + w*sqrt(z^2-1));

k1 = 0.5 * (1 + z/sqrt(z^2+1));

k2 = 0.5 * (1 - z/sqrt(z^2+1));

y3 = 1 - k1*exp(-t/t1) - k2*exp(-t/t2);

plot(t,y3,'--')

hold off

legend('K = 0.1','K = 0.25','K = 2.5')

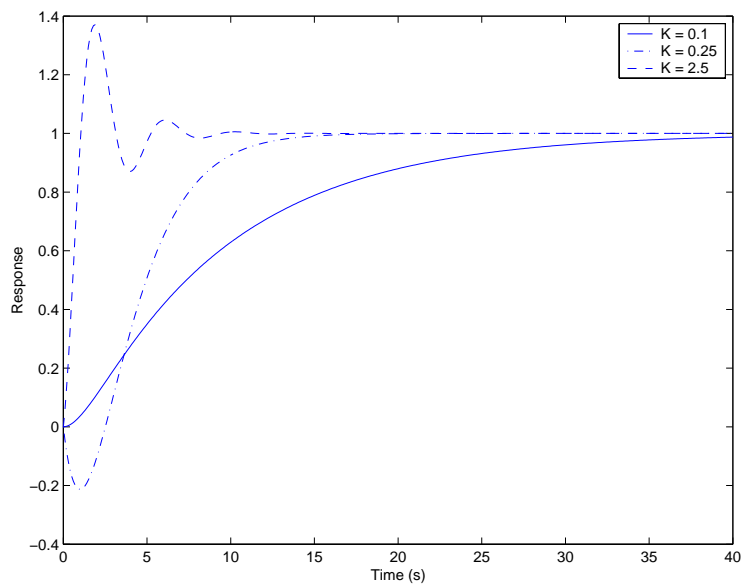


Figure 1

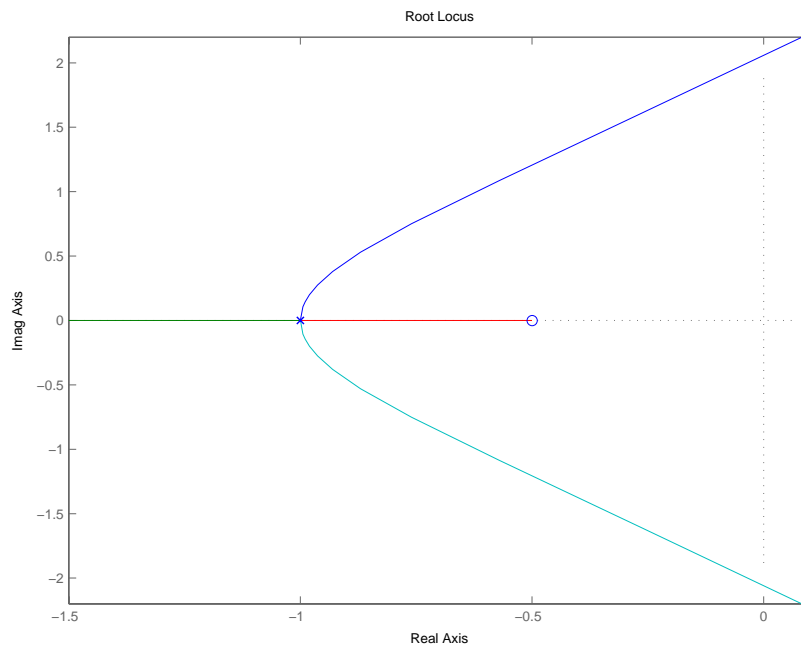
9.33 We are given the loop transfer function

$$L(s) = \frac{K(s + 0.5)}{(s + 1)^4}$$

Figure 1 shows the root locus diagram of the closed-loop feedback system for varying positive values of K .

%Solution to problem 9.33

```
num = [1 0.5];  
den = [1 4 6 4 1];  
rlocus(num,den)  
axis([-1.5 .1 -2.2 2.2])
```



9.34 We are given the loop transfer function

$$L(s) = \frac{K}{(s+1)^2(s+5)}$$

(a) Figure 1 shows the root locus diagram of the closed-loop feedback system for varying K .

Next, putting $s = j\omega$,

$$L(j\omega) = \frac{K}{(j\omega+1)^2(j\omega+5)}$$

Parts (a), (b) and (c) of Fig. 2 show plots of the Nyquist locus of the system for $K = 50$, 72, and 100. On the basis of these figures we can make the following statements:

1. For $K = 50$ the locus does not encircle the critical point $(-1, 0)$ and the system is stable.
2. For $K = 100$ the locus encloses the critical point $(-1, 9)$ and the system is unstable.
3. For $K = 72$ the Nyquist locus passes through the critical point $(-1, 0)$ and the system is on the verge of instability.

(b) The critical value $K = 72$ is determined in accordance with the condition

$$|L(j\omega_p)| = \frac{K_{\text{critical}}}{(\omega_p^2 + 1)(\omega_p^2 + 25)^{1/2}}$$

where ω_p is the phase crossover frequency defined by

$$180^\circ = 2 \tan^{-1}(\omega_p) + \tan^{-1}\left(\frac{\omega_p}{5}\right)$$

For a gain margin of 5 dB:

$$20 \log_{10} K = 20 \log_{10} K_{\text{critical}} - 5$$

Hence

$$K = \frac{72}{\text{antilog}(0.25)} = \frac{72}{1.7783} = 40.49$$

(c) For $K = 40.49$ the gain-crossover frequency is

$$(40.49)^2 = (1 + \omega_g^2)^2 (25 + \omega_g^2)$$

Let $\omega_g^2 = x$, so rewrite this equation in the form of a cubic equation in x :

$$x^3 + 27x^2 + 51x - 1614 = 0$$

The only positive root of x is 6.2429. Hence

$$\omega_g = \sqrt{6.2429} = 2.4986$$

The phase of $L(j\omega)$ at $\omega = 2.4986$ is

$$2 \tan^{-1}(2.4986) + \tan^{-1}\left(\frac{2.4986}{2}\right) = 2 \times 68.2^\circ + 26.5^\circ = 162.9^\circ$$

The phase margin is therefore

$$\phi_m = 180^\circ - 162.9^\circ = 17.1^\circ$$

%Solution to problem 9.34

```
clear; clc;
num = [1];
den = [1 7 11 5];
figure(1);clf;
rlocus(num,den)
figure(2);clf;
nyquist(num,den)
xlim([-1.1 0.22])
```

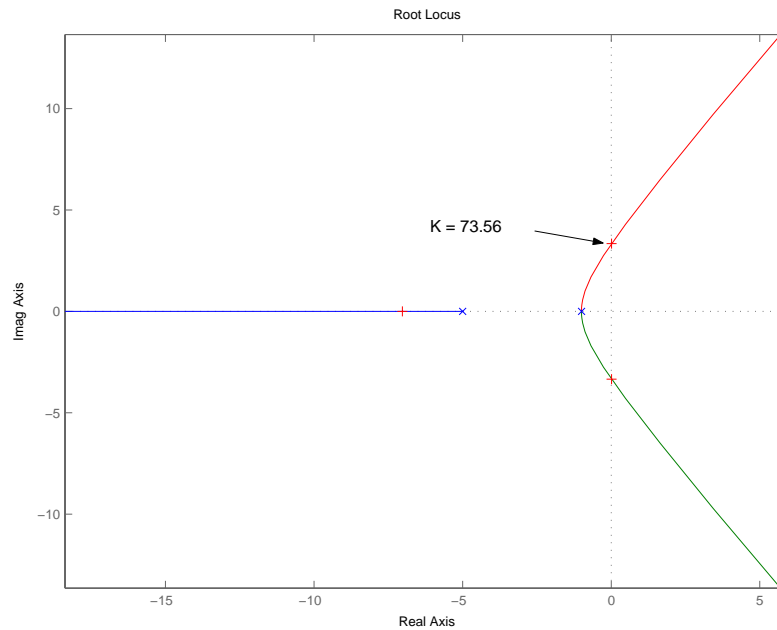


Figure 1

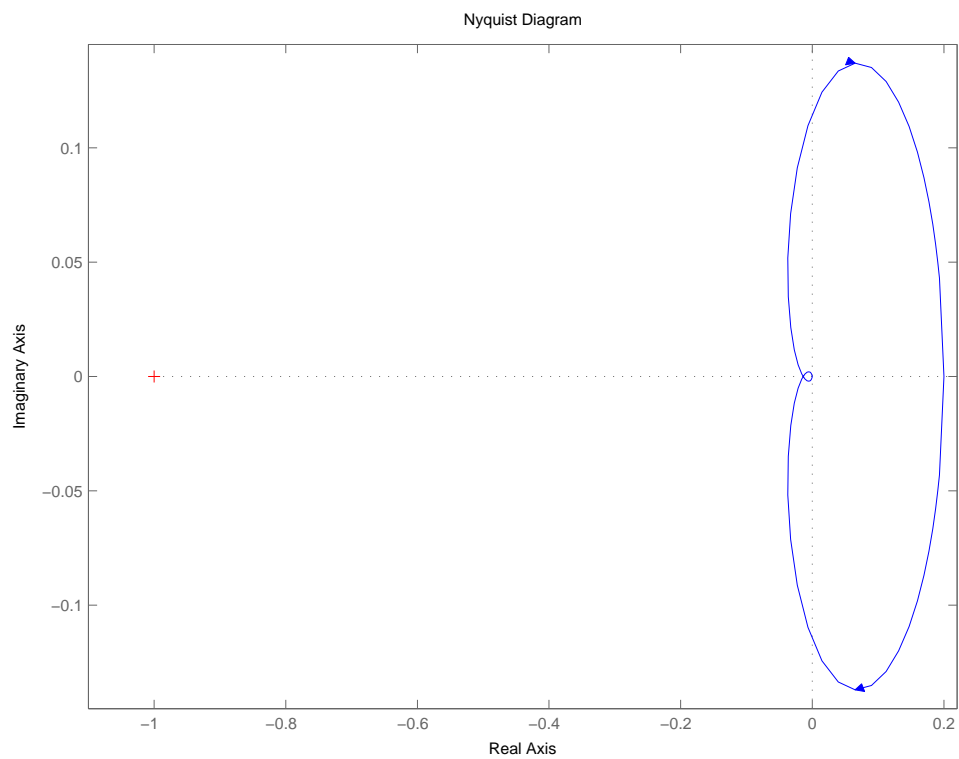


Figure 2

9.35 We are given

$$L(s) = \frac{K}{s(s+1)}$$

For $s = j\omega$,

$$L(j\omega) = \frac{K}{j\omega(j\omega+1)}$$

from which we deduce:

$$|L(j\omega)| = \frac{K}{\omega(\omega^2+1)^{1/2}} \quad (1)$$

$$\arg\{L(j\omega)\} = -90^\circ - \tan^{-1}(\omega) \quad (2)$$

From Eq. (2) we observe that the phase response $\arg\{L(j\omega)\}$ is confined to the range $[-90^\circ, -180^\circ]$. The value of -180° is attained only at $\omega = \infty$. At this frequency we see from Eq. (1) that the magnitude response $|L(j\omega)|$ is zero. Hence, the Nyquist locus will never encircle the critical point $(-1,0)$ for all positive values of K . The feedback system is therefore stable for all $K > 0$.

9.36 We are given

$$L(s) = \frac{K}{s^2(s+1)}$$

For $s = j\omega$,

$$L(j\omega) = \frac{K}{-\omega^2(j\omega+1)}$$

from which we deduce:

$$|L(j\omega)| = \frac{K}{\omega^2(\omega^2+1)^{1/2}}$$

$$\arg\{L(j\omega)\} = 180^\circ - \tan^{-1}(\omega)$$

Hence the Nyquist locus will encircle the critical point $(-1,0)$ for all $K > 0$; that is, the system is unstable for all $K > 0$.

We may also verify this result by applying the Routh-Hurwitz criterion:

s^3	1	0
s^2	1	K
s^1	$-K$	0

$$s^0 \quad K \quad 0$$

There are 2 sign changes in the first column of coefficients in the array, assuming $K > 0$. Hence the system is unstable for all $K > 0$.

9.37 We are given the loop transfer function

$$L(s) = \frac{K}{s(s+1)(s+2)}$$

(a) For $s = j\omega$,

$$L(j\omega) = \frac{K}{j\omega(j\omega+1)(j\omega+2)}$$

Hence

$$|L(j\omega)| = \frac{K}{\omega(\omega^2+1)^{1/2}(\omega^2+4)^{1/2}} \quad (1)$$

$$\arg\{L(j\omega)\} = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) \quad (2)$$

Setting $K = 6$ in Eq. (1) under the condition $|L(j\omega_p)| = 1$:

$$36 = \omega_p^2(\omega_p^2+1)(\omega_p^2+4)$$

where ω_p is the phase-crossover frequency. Putting $\omega_p^2 = x$ and rearranging terms:

$$x^3 + 5x^2 + 4x - 36 = 0$$

Solving this cubic equation for x we find that it has only one positive root at $x = 2$. Hence,

$$\omega_p = \sqrt{2}$$

Substituting this value in Eq. (2):

$$\begin{aligned} \arg\{L(j\omega_p)\} &= -90^\circ - \tan^{-1}(\sqrt{2}) - \tan^{-1}\left(\frac{\sqrt{2}}{2}\right) \\ &= -90^\circ - 54.8^\circ - 35.2^\circ = -180^\circ \end{aligned}$$

This result confirms ω_p as the phase-crossover frequency, and $K = 6$ as the critical value of the scaling factor for which the Nyquist locus of $L(j\omega)$ passes through the critical point $(-1,0)$.

We may also verify this result by applying the Routh-Hurwitz criterion to the characteristic equation:

$$s^3 + 3s^2 + 2s + K = 0$$

Specifically, we write

$$\begin{array}{ccc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \frac{6-K}{3} & 0 \\ s^0 & K & \end{array}$$

The system is therefore on the verge of instability for $K = 6$.

(b) For $K = 2$ the gain margin is

$$20\log_{10}6 - 20\log_{10}2 = 20\log_{10}3 = 9.542 \text{ dB}$$

To calculate the phase margin for $K = 2$ we need to know the gain-crossover frequency ω_g . At $\omega = \omega_g$, $|L(j\omega)| = 1$. Hence for the problem at hand

$$1 = \frac{2}{\omega_g(\omega_g^2 + 1)^{1/2}(\omega_g^2 + 4)^{1/2}}$$

Let $\omega_g^2 = x$, for which we may then rewrite this equation as

$$x^3 + 5x^2 + 4x - 4 = 0$$

The only positive root of this equation is $x = 0.5616$. That is,

$$\omega_g = \sqrt{0.5616} = 0.7494$$

The phase margin is therefore

$$\begin{aligned} 180^\circ - (90^\circ + \tan^{-1}(0.7494) + \tan^{-1}(0.3749)) \\ = 90^\circ - 36.9^\circ - 20.5^\circ \\ = 32.6^\circ \end{aligned}$$

(c) Let ω_g denote the gain-crossover frequency for the required phase margin $\phi_m = 20^\circ$.

We may then write

$$|L(j\omega_g)| = 1$$

$$\arg\{L(j\omega_g)\} = -90^\circ - \tan^{-1}(\omega_g) - \tan^{-1}\left(\frac{\omega_g}{2}\right)$$

With

$$\arg\{L(j\omega_g)\} = 180^\circ - 20^\circ$$

we thus have

$$70^\circ = \tan^{-1}(\omega_g) + \tan^{-1}\left(\frac{\omega_g}{2}\right)$$

Through a process of trial and error, we obtain the solution

$$\omega_g = 0.972$$

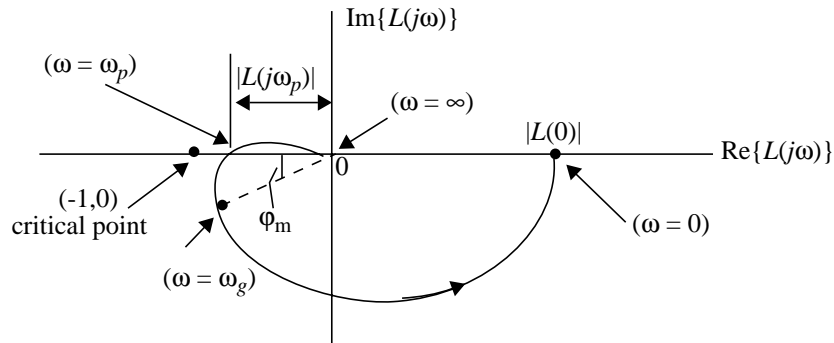
For $|L(j\omega_g)| = 1$ we thus require

$$\begin{aligned} K &= \omega_g(\omega_g^2 + 1)^{1/2}(\omega_g^2 + 4)^{1/2} \\ &= 0.972(1.9448)^{1/2}(4.9448)^{1/2} \\ &= 3.0143 \end{aligned}$$

The gain margin is therefore

$$20\log_{10}\left(\frac{6}{3.0143}\right) \approx 20\log_{10}2 = 6\text{dB}$$

- 9.38 For the purpose of illustration, suppose the loop frequency response $L(j\omega)$ has a finite magnitude at $\omega = 0$. Suppose also the feedback system represented by $L(j\omega)$ is stable. We may then sketch the Nyquist locus of the system for $0 \leq \omega \leq \infty$, including gain margin and phase margin provisions, as follows



The phase margin of the system is ϕ_m (measured at the gain-crossover frequency ω_g). The

gain margin is $20\log_{10}\frac{1}{|L(j\omega_p)|}$ (measured at the phase crossover frequency ω_p).

- 9.39 We are given the loop frequency response

$$\begin{aligned} L(j\omega) &= \frac{6K}{(j\omega + 1)(j\omega + 2)(j\omega + 3)} \\ &= \frac{6K}{(j\omega)^3 + 6(j\omega)^2 + (j\omega) + 6} \end{aligned}$$

- (a) Figure 1 on the next page plots the Bode diagram of $L(j\omega)$ for $K = 1$. Changing the value of K merely shifts the loop gain response by a constant amount equal to $20\log_{10}K$. This constant gain is tabulated as follows:

K	$20\log_{10}K$, dB
7	16.90
8	18.06
9	10.08
10	20.00
11	20.83

Applying the gain adjustments (as shown in this table) to the loop response of Fig. 1, we may make the following observations:

1. The feedback system is stable for $K = 7, 8, 9$.
2. It is on the verge of instability for $K = 10$.
3. It is unstable for $K = 11$.

Hence the system is on the verge of instability for $K = 10$.

- (b) For $K = 7$ the gain margin is

$$20\log_{10}\left(\frac{10}{7}\right) = 3.098 \text{ dB}$$

To calculate the phase margin we need to know the gain-crossover frequency ω_g . At $\omega = \omega_g$ the following condition is satisfied

$$1 = \frac{42}{(\omega_g^2 + 1)^{1/2} (\omega_g^2 + 4)(\omega_g^2 + 9)^{1/2}}$$

Let $\omega_g^2 = x$ and so rewrite this equation as

$$x^3 + 14x^2 + 49x - 1728 = 0$$

The only positive root of this cubic equation is at $x = 7.8432$. Hence,

$$\omega_g = \sqrt{7.8432} = 2.8006$$

The phase margin for $K = 7$ is therefore

$$\begin{aligned} 180^\circ - \tan^{-1}(2.8006) - \tan^{-1}(1.4003) - \tan^{-1}(.9335) \\ = 180^\circ - 70.4^\circ - 54.5^\circ - 43^\circ \\ = 12.2^\circ \end{aligned}$$

Proceeding in a similar fashion we get the following results:

- For $K = 8$, gain margin = 1.938 dB
phase margin = 7.39°

- For $K = 9$, gain margin = 0.915 dB
phase margin = 3.41°

%Solution to problem 9.39

clear; clc;

```
for K = 7:11,
figure(K-6)
num = [6*K];
den = [1 6 11 6];
subplot(1,2,1)
margin(num,den)
subplot(1,2,2)
nyquist(num,den)
text(0.3,0.3,['K = ' num2str(K)])
end
```

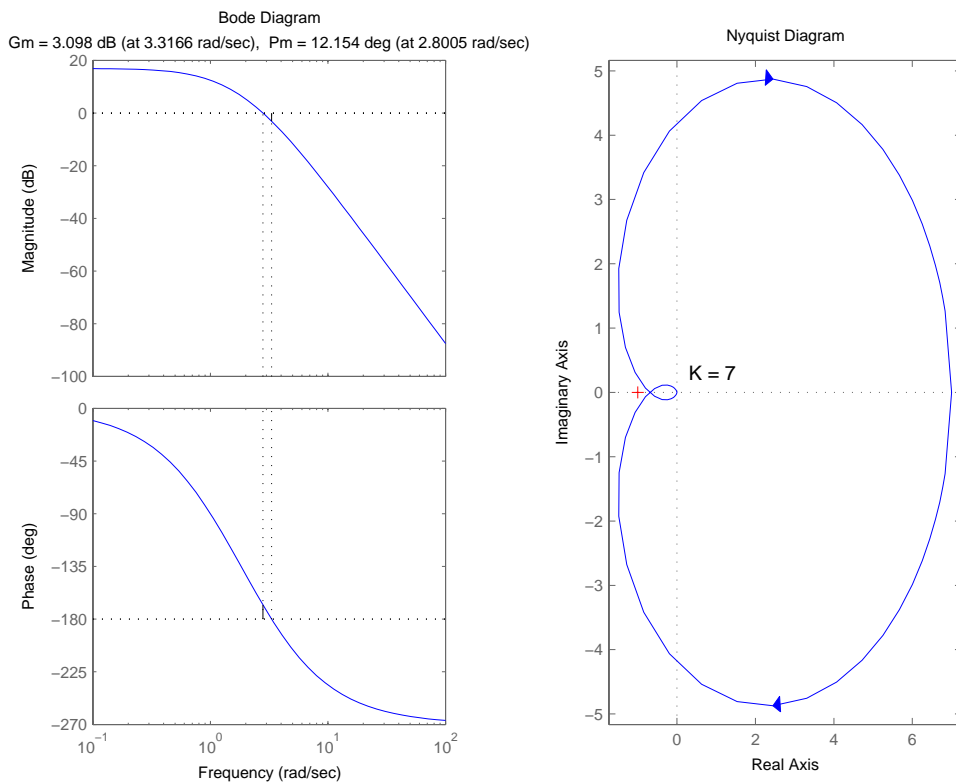


Figure 1

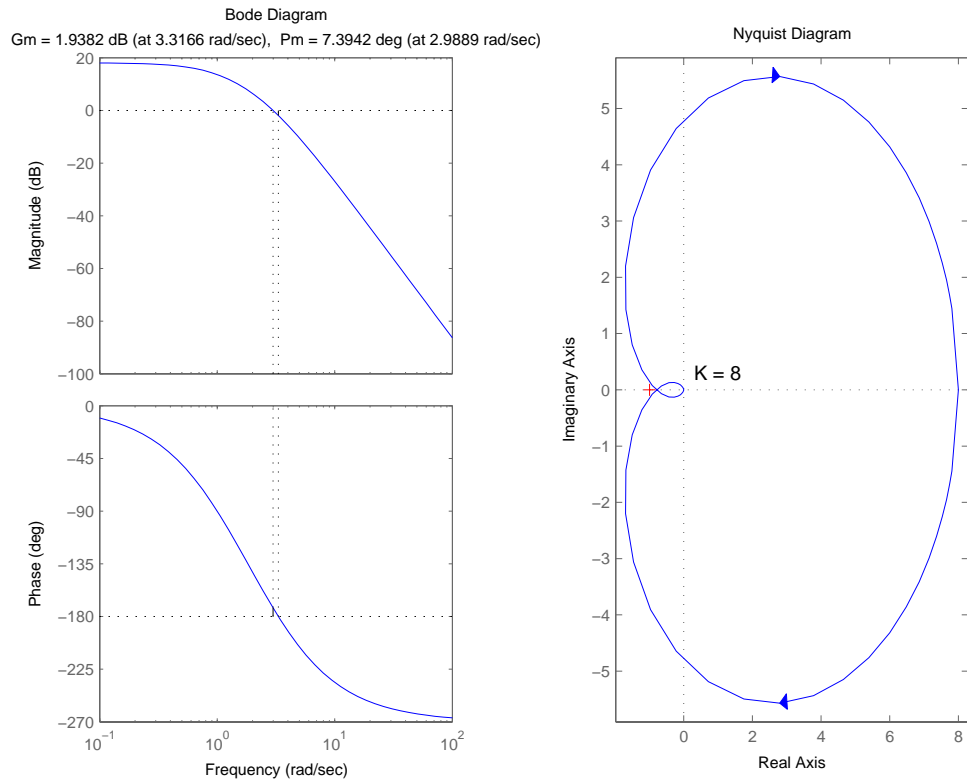


Figure 2

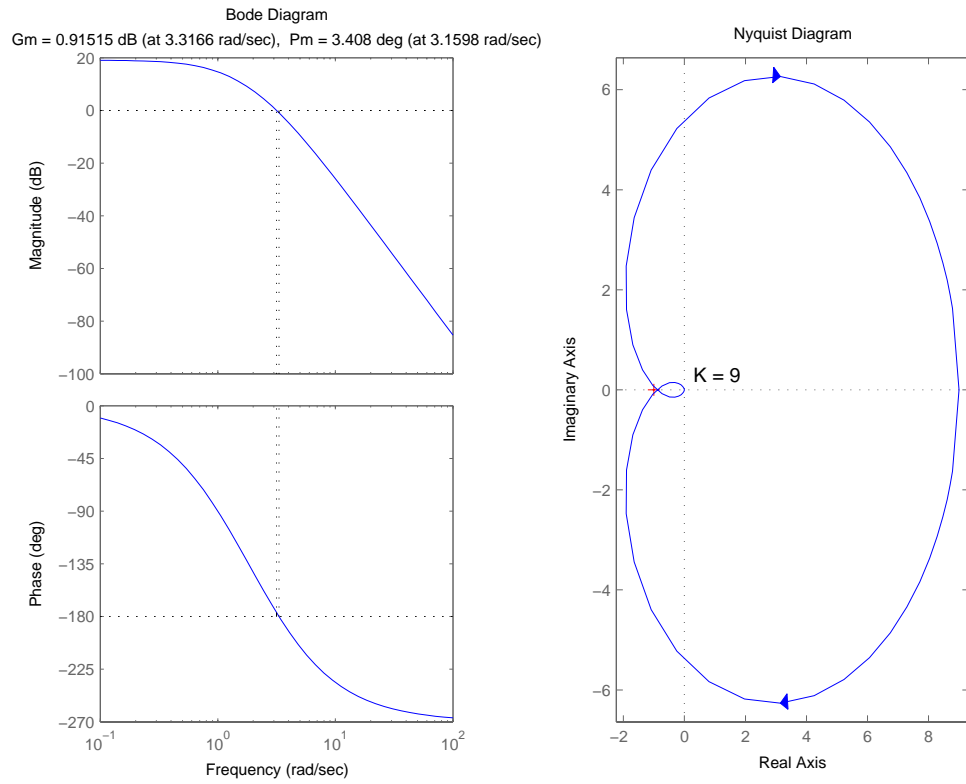


Figure 3

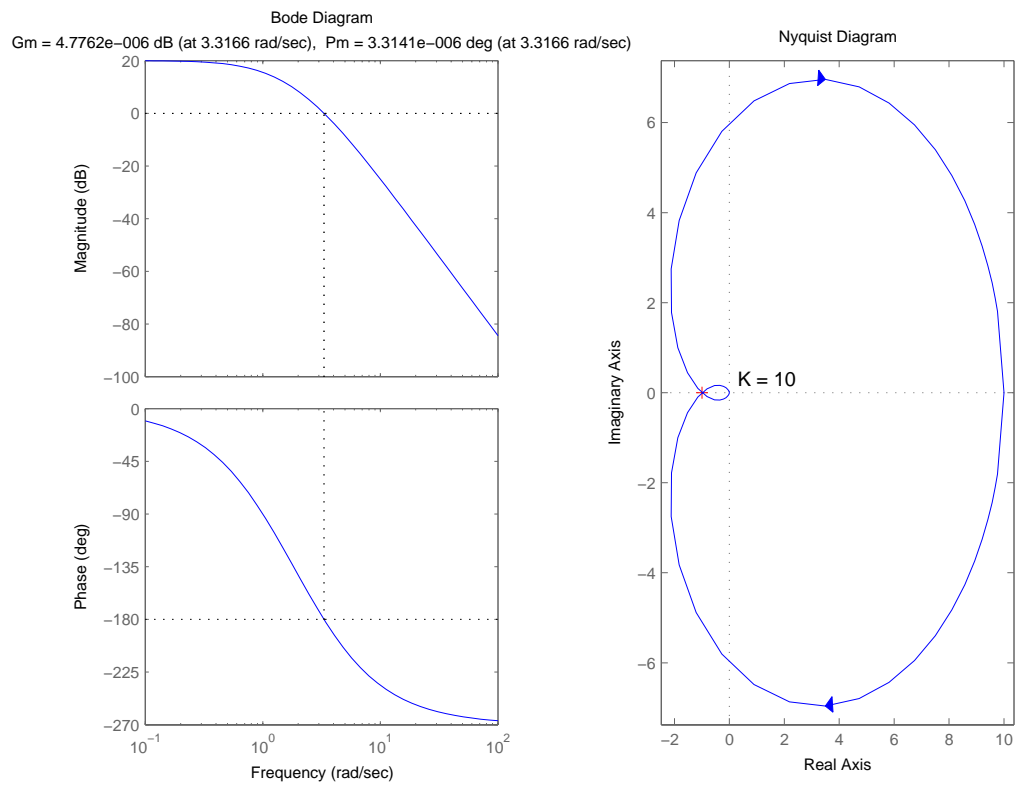


Figure 4

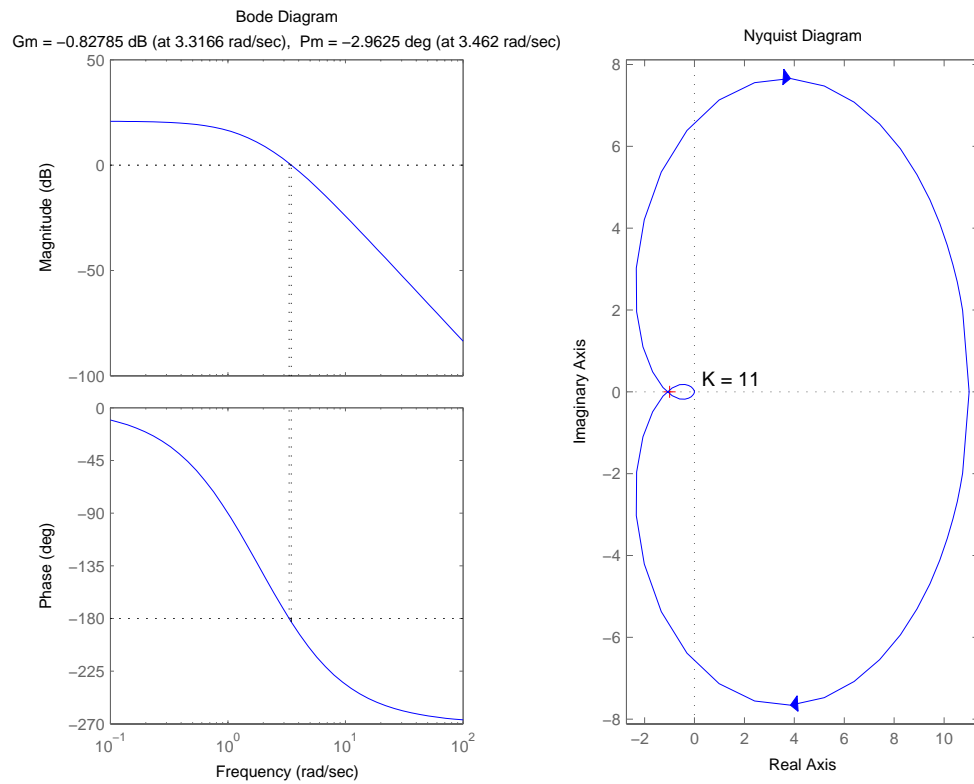


Figure 5

9.40 Figures 1, 2, 3, and 4 on the next four pages plot the Bode diagrams for the following loop transfer functions:

$$(a) L(s) = \frac{50}{(s+1)(s+2)}$$

$$(b) L(s) = \frac{5}{(s+1)(s+2)(s+5)}$$

$$(c) L(s) = \frac{5}{(s+1)^3}$$

$$(d) L(s) = \frac{10(s+0.5)}{(s+1)(s+2)(s+5)}$$

%Solution to problem 9.40

```
clear; clc;
```

```
figure(1)
num = [50];
den = [1 3 2];
margin(num,den)
```

```
figure(2)
num = [5];
den = [1 8 17 10];
margin(num,den)
```

```
figure(3)
num = [5];
den = [1 3 3 1];
margin(num,den)
```

```
figure(4)
num = [10 5];
den = [1 8 17 10];
margin(num,den)
```

```
figure(4)
num = [10 5];
den = [1 8 17 10];
margin(num,den)
```

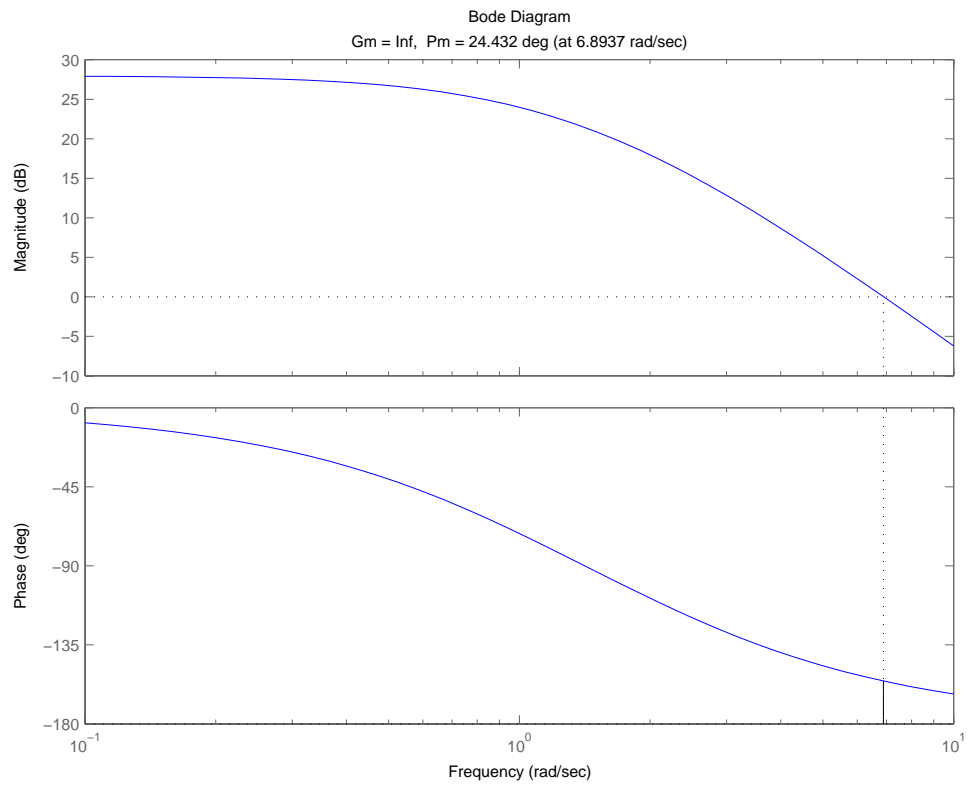


Figure 1

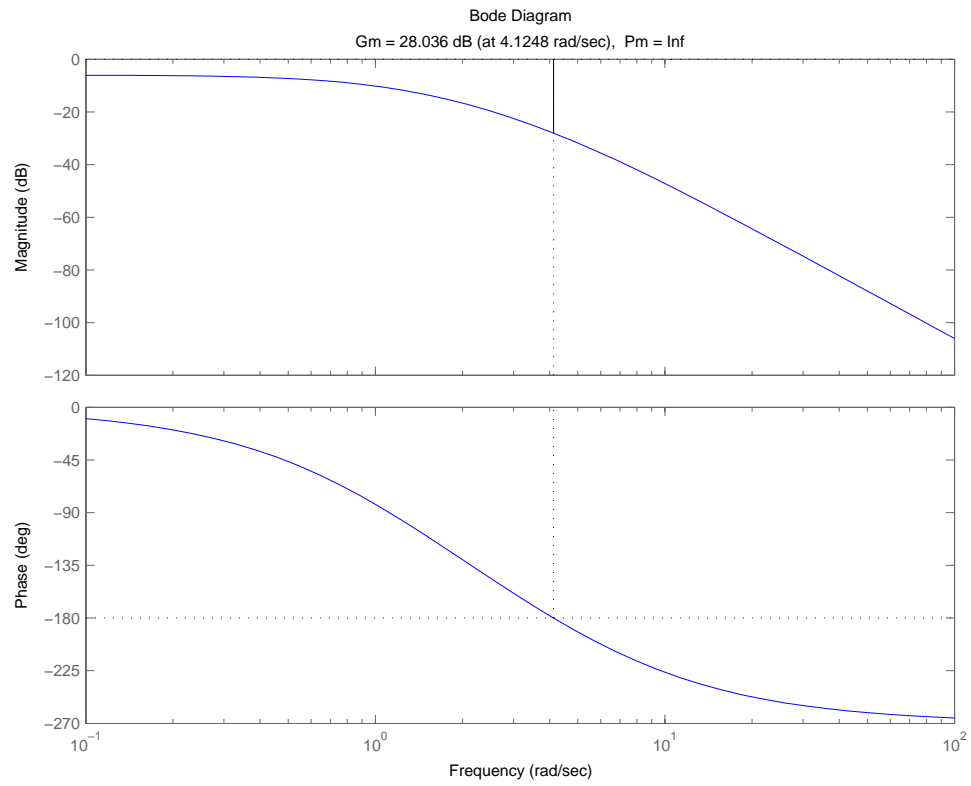


Figure 2

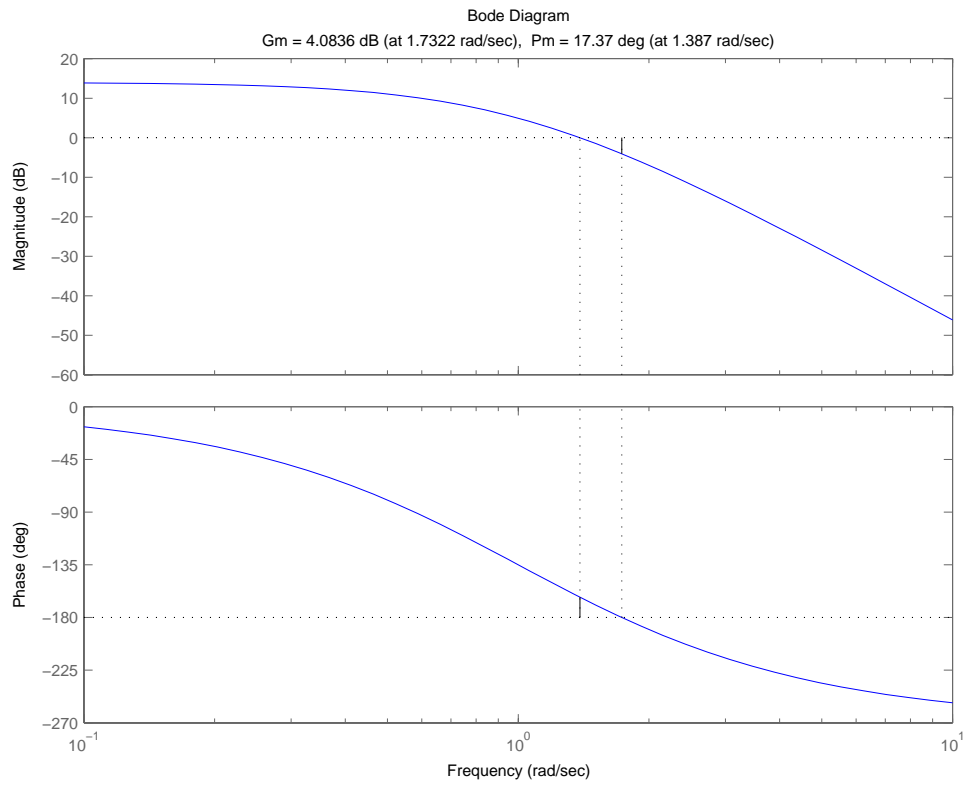


Figure 3

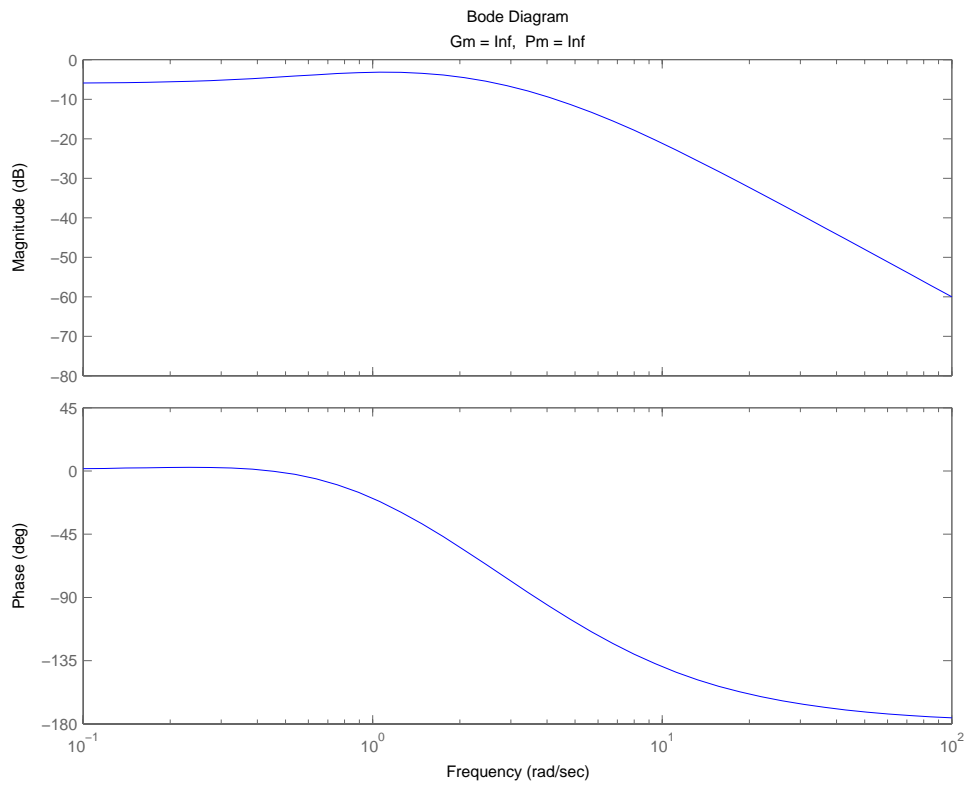


Figure 4

9.41 From Example 9.9 we have the loop transfer function

$$L(s) = \frac{0.5K(s+2)}{s(s+12)(s-4)}$$

For $s = j\omega$,

$$L(j\omega) = \frac{0.5K(j\omega+2)}{j\omega(j\omega+12)(j\omega-4)}$$

Figures 1, 2, and 3 on the next three pages plot the Bode diagrams of $L(j\omega)$ for $K = 100$, 128, and 160, respectively.

From these figures we make the following observations:

- (a) The phase-crossover frequency $\omega_p =$ for all K .
- (b) For $K = 128$ the Bode diagram exactly satisfies the conditions $|L(j\omega)| = 1$ and $\arg\{L(j\omega)\} = 180^\circ$. Hence the system is on the verge of instability for $K = 128$.
- (c) For $K = 100$ the gain margin is -2.14 dB, which is negative. The system is therefore unstable for $K = 100$.
- (d) For $K = 160$ the gain margin is 1.94 dB, which is positive. The system is therefore stable for $K = 160$.

These observations reconfirm the conclusions reached on the stability performance of $L(s)$ in Example 9.9.

%Solution to problem 9.41

clear; clc;

```
figure(1)
K=128;
num = 0.5*K*[1 2];
den = [1 8 -48 0];
[Gm128,Pm128,Wcg128,Wcp128] = margin(num,den)
```

```
figure(2)
K=100;
num = 0.5*K*[1 2];
den = [1 8 -48 0];
[Gm100,Pm100,Wcg100,Wcp100] = margin(num,den)
```

```
figure(3)
K=160;
```



```

num = 0.5*K*[1 2];
den = [1 8 -48 0];
[Gm160,Pm160,Wcg160,Wcp160] = margin(num,den)

```

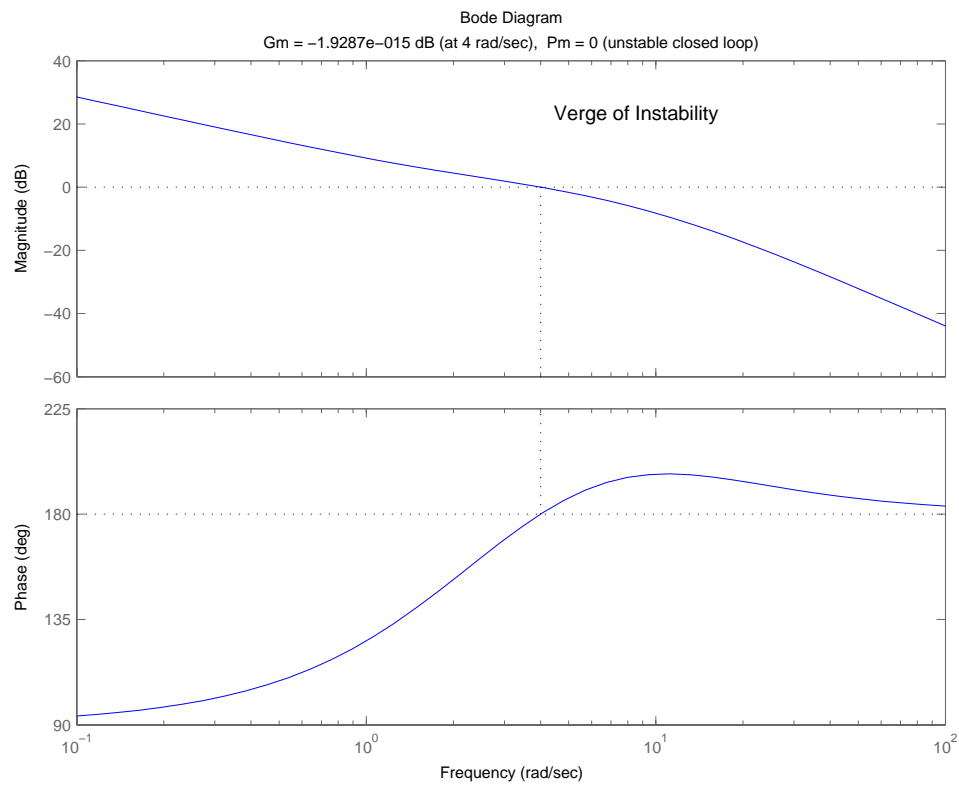


Figure 1

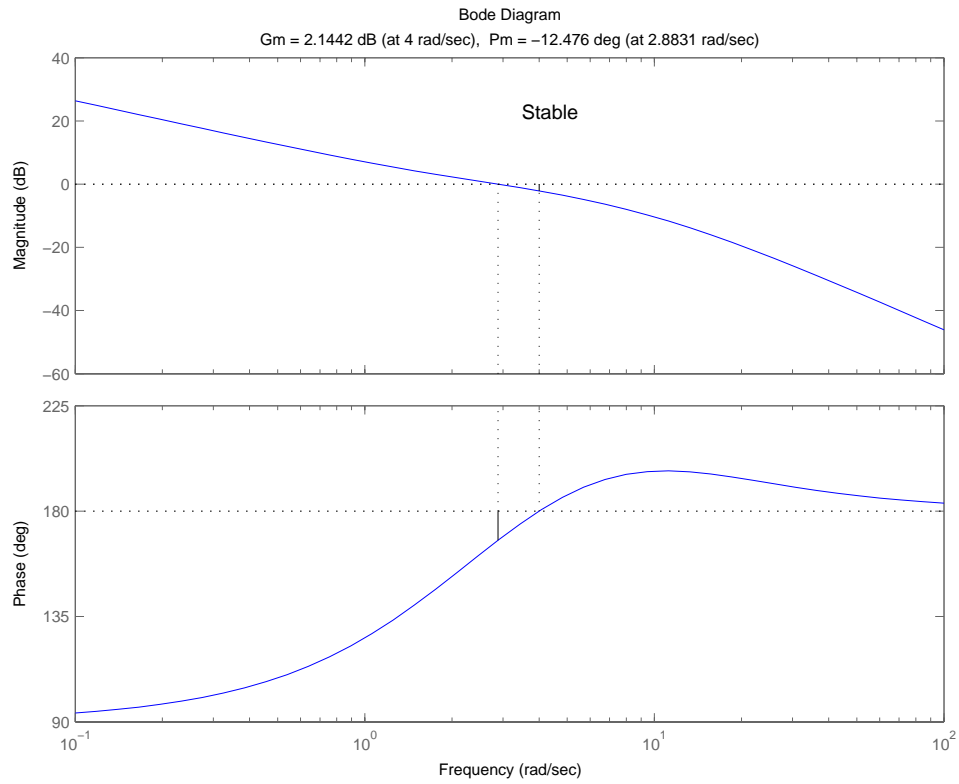


Figure 2

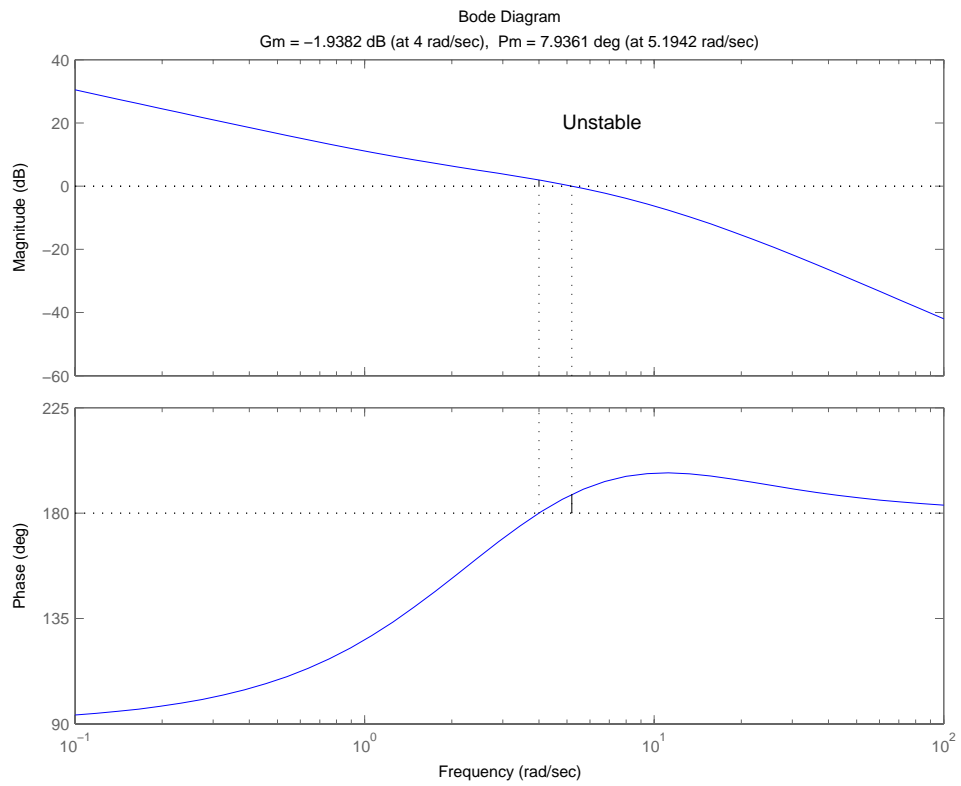


Figure 3

- 9.42 Let $r(t)$ denote the feedback (return) signal produced by the sensor $H(s)$. Then, following the material presented in Section 9.16, we may, insofar as the computation of the Laplace transform $R(s) = \mathcal{L}\{r(t)\}$ is concerned, replace the sampled-data system of Fig. P9.42 with that shown in Fig. 1 below:

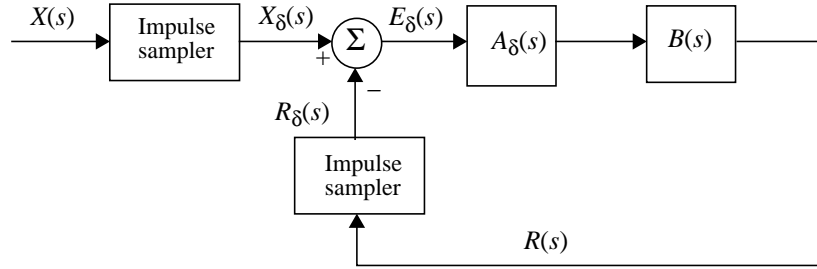


Figure 1

The blocks labelled $A_\delta(s)$ and $B(s)$ are defined as follows, respectively,

1. $A_\delta(s)$ = transfer function of the discrete-time components of the system

$$= D_\delta(s) (1 - e^{-sT_s})$$

where τ is the sampling period and

$$D_\delta(s) = D(z) \Big|_{z = e^{sT_s}}$$

2. $B(s)$ = transfer function of the continuous-time components of the system

$$= \frac{1}{s} G_c(s) G_p(s) H(s)$$

where $G_c(s)$, $G_p(s)$, and $H(s)$ are as indicated in Fig. P9.42.

Adapting Eq. (9.94) of the text to the problem at hand:

$$\frac{R_\delta(s)}{X_\delta(s)} = \frac{L_\delta(s)}{1 + L_\delta(s)} \quad (1)$$

where

$$L_\delta(s) = A_\delta(s) B_\delta(s) \quad (2)$$

The $B_\delta(s)$ is itself defined in terms of $B(s)$ by the formula

$$B_\delta(s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} B(s - jk\omega_s)$$

where $\omega_s = 2\pi/T_s$.

From Fig. P9.42 we note that

$$E(s) = H(s)Y(s)$$

Hence,

$$E_{\delta}(s) = H_{\delta}(s)Y_{\delta}(s) \quad (3)$$

where

$$H_{\delta}(s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} H(s - jk\omega_s)$$

Eliminating $E_{\delta}(s)$ between Eqs. (1) and (3):

$$Y_{\delta}(s) = \frac{L_{\delta}(s)/H_{\delta}(s)}{1 + L_{\delta}(s)} X_{\delta}(s)$$

Changing this result to the z -transform:

$$Y(z) = \frac{L(z)/H(z)}{1 + L(z)} X(z)$$

where

$$Y(z) = Y_{\delta}(s) \Big|_{e^{sT_s} = z}$$

and so on for $L(z)$, $H(z)$, and $X(z)$.

9.43 From the material presented in Section 9.16 we note that (see Eq. (9.94))

$$Y_{\delta}(s) = \frac{L_{\delta}(s)}{1 + L_{\delta}(s)} X_{\delta}(s) \quad (1)$$

where

$$L_{\delta}(s) = A_{\delta}(s)B_{\delta}(s) \quad (2)$$

For the problem at hand, $A_{\delta}(s)$ and $B_{\delta}(s)$ are defined as follows, respectively:

$$1. \quad A_{\delta}(s) = (1 - e^{-sT_s})D_{\delta}(s) \quad (3)$$

where

$$D_{\delta}(s) = D(z) \Big|_{z = e^{sT_s}}$$

$$2. \quad B_{\delta}(s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} B(s - jk\omega_s), \quad \omega_s = \frac{2\pi}{T_s}$$

where

$$\begin{aligned} B(s) &= \frac{1}{s} G(s) \\ &= \frac{K}{s^3} \end{aligned}$$

We are given

$$\begin{aligned} D(z) &= K \left(1.5 + \frac{z-1}{z} \right) \\ &= K(2.5 - z^{-1}) \end{aligned}$$

Hence,

$$D_{\delta}(s) = K(2.5 - e^{-sT_s})$$

The use of Eq. (3) thus yields

$$\begin{aligned} A_{\delta}(s) &= K(1 - e^{-sT_s})(2.5 - e^{-sT_s}) \\ &= K(2.5 - 3.5e^{-sT_s} + e^{-2sT_s}) \end{aligned}$$

Correspondingly, we may write

$$A(z) = K(2.5 - 3.5z^{-1} + z^{-2})$$

Next, with $B(s) = K/s^3$ we may write

$$\begin{aligned}
b(t) &= L^{-1}\{B(s)\} \\
&= \frac{Kt^2}{2}u(t)
\end{aligned}$$

Hence,

$$\begin{aligned}
B_{\delta}(s) &= \sum_{k=-\infty}^{\infty} b[n]e^{-nsT_s} \\
&= \frac{K}{2} \sum_{k=0}^{\infty} n^2 e^{-nsT_s}
\end{aligned}$$

which may be summed to yield

$$\begin{aligned}
B(z) &= B_{\delta}(s) \Big|_{e^{sT_s} = z} \\
&= \frac{KT_s^2}{4} \frac{z(z+1)}{(z-1)^3} = \frac{KT_s^2}{4} \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}
\end{aligned}$$

The use of Eq. (2) yields

$$\begin{aligned}
L(z) &= L_{\delta}(s) \Big|_{e^{sT_s} = z} \\
&= A(z)B(z) \\
&= K^2(2.5 - 3.5z^{-1} + z^{-2}) \frac{T_s}{4} \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} \\
&= \frac{K^2T_s^2}{4} \frac{z^{-1}(2.5 - z^{-1})(1+z^{-1})}{(1-z^{-1})^2}
\end{aligned}$$

Finally, the closed-loop transfer function of the system is

$$T(z) = \frac{L(z)}{1 + L(z)}$$

where $L(z)$ is defined by Eq. (4).

9.44 Here again following the material presented in Section 9.16 we may state that

$$\frac{Y(z)}{X(z)} = \frac{L(z)}{1 + L(z)}$$

where

$$\begin{aligned}
L(z) &= L_{\delta}(s) \Big|_{e^{sT_s} = z} \\
L_{\delta}(s) &= A_{\delta}(s)B_{\delta}(s) \\
A_{\delta}(s) &= (1 - e^{-sT_s}) \\
B_{\delta}(s) &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} B(s - jk\omega_s), \quad \omega_s = \frac{2\pi}{T_s} \\
B(s) &= \frac{1}{s} G(s) \\
&= \frac{5}{s^2(s+2)} \\
&= -\frac{5/4}{s} + \frac{5/2}{s^2} + \frac{5/4}{s+2}
\end{aligned}$$

To calculate $B_{\delta}(s)$ we first note that

$$b(t) = -\frac{5}{4}u(t) + \frac{5}{2}t u(t) + \frac{5}{4}e^{-2t}u(t)$$

Hence,

$$\begin{aligned}
B_{\delta}(s) &= \sum_{n=-\infty}^{\infty} b[n]e^{-nsT_s} \\
&= -\frac{5}{4} \sum_{n=0}^{\infty} e^{-nsT_s} + \frac{5}{2} \sum_{n=0}^{\infty} n e^{-nsT_s} + \frac{5}{4} \sum_{n=0}^{\infty} e^{-2nsT_s} e^{-nsT_s}
\end{aligned}$$

Correspondingly, we may write

$$B(z) = -\frac{5}{4} \frac{1}{1-z^{-1}} + \frac{5}{2} \frac{T_s z^{-1}}{(1-z^{-1})^2} + \frac{5}{4} \frac{1}{1-e^{-2T_s} z^{-1}}$$

From the expression for $A_{\delta}(s)$:

$$A(z) = (1 - z^{-1})$$

The loop transfer function of the system is therefore

$$\begin{aligned}
L(z) &= A(z)B(z) \\
&= (1 - z^{-1}) \left[-\frac{5}{4} \frac{1}{1 - z^{-1}} + \frac{5}{2} \frac{T_s z^{-1}}{(1 - z^{-1})^2} + \frac{5}{4} \frac{1}{1 - e^{-2T_s} z^{-1}} \right]
\end{aligned}$$

Putting this expression on a common denominator:

$$L(z) = \frac{5}{4} \frac{z^{-1}((-1 + 2T_s + e^{-2T_s}) + (1 - e^{-2T_s}(1 + 2\tau)z^{-1}))}{(1 - z^{-1})(1 - e^{-2T_s} z^{-1})}$$

The closed-loop transfer function is

$$T(z) = \frac{L(z)}{1 + L(z)}$$

(a) For sampling period $T_s = 0.1$ s, the loop transfer function takes the value

$$\begin{aligned}
L(z) &= \frac{5}{4} \frac{z^{-1}((-1 + 0.2 + e^{-0.2}) + (1 - e^{-0.2}(1.2)z^{-1}))}{(1 - z^{-1})(1 - e^{-0.2} z^{-1})} \\
&= \frac{5}{4} \frac{z^{-1}(0.019 + 0.017z^{-1})}{(1 - z^{-1})(1 - 0.819z^{-1})}
\end{aligned}$$

The closed-loop transfer function is therefore

$$\begin{aligned}
T(z) &= \frac{\frac{5}{4} \frac{z^{-1}(0.019 + 0.017z^{-1})}{(1 - z^{-1})(1 - 0.819z^{-1})}}{1 + \frac{5}{4} \frac{z^{-1}(0.019 + 0.017z^{-1})}{(1 - z^{-1})(1 - 0.819z^{-1})}} \\
&= \frac{\frac{5}{4} z^{-1}(0.019 + 0.017z^{-1})}{(1 - z^{-1})(1 - 0.819z^{-1}) + \frac{5}{4} z^{-1}(0.019 + 0.017z^{-1})} \\
&= \frac{\frac{5}{4} z^{-1}(0.019 + 0.017z^{-1})}{1 - 1.795z^{-1} + 0.840z^{-2}}
\end{aligned}$$

$T(z)$ has a zero at $z = -0.895$ and a pair of complex poles at $z = 0.898 \pm j0.186$. The poles are inside the unit circle and the system is therefore stable.

(b) For sampling period $T_s = 0.05\text{s}$, $L(z)$ takes the value

$$L(z) = \frac{\frac{5}{4}z^{-1}((-1 + 0.1 + e^{-0.1}) + (1 - e^{-0.1}(1.1))z^{-1})}{(1 - z^{-1})(1 - e^{-0.1}z^{-1})}$$

$$= \frac{5}{4} \frac{z^{-1}(0.005 + 0.005z^{-1})}{(1 - z^{-1})(1 - 0.996z^{-1})}$$

The closed-loop transfer function is therefore

$$T(z) = \frac{\frac{5}{4} \frac{z^{-1}(0.005 + 0.005z^{-1})}{(1 - z^{-1})(1 - 0.996z^{-1})}}{1 + \frac{5}{4} \frac{z^{-1}(0.005 + 0.005z^{-1})}{(1 - z^{-1})(1 - 0.996z^{-1})}}$$

$$= \frac{\frac{5}{4} z^{-1}(0.005 + 0.005z^{-1})}{(1 - z^{-1})(1 - 0.996z^{-1}) + \frac{5}{4} z^{-1}(0.005 + 0.005z^{-1})}$$

$$= \frac{0.006z^{-1}(1 + z^{-1})}{1 - 1.989z^{-1} + 1.001z^{-2}}$$

$T(z)$ has a zero at $z \approx 1$ and a pair of complex poles at $z = 0.995 \pm j0.109$. The poles lie inside the unit circle in the z -plane and the sampled-data system remains stable. Note, however, the reduction in the sampling period has pushed the closed-loop poles much closer to the unit circle.

Advanced Problems

9.45 (a) We are given the loop transfer function

$$\begin{aligned} L(s) &= G(s)H(s) \\ &= \frac{\beta A}{1 + Q\left(\frac{s}{\omega_0} + \frac{\omega_0}{s}\right)} \\ &= \frac{\beta A \omega_0 s}{Qs^2 + \omega_0 s + Q\omega_0^2} \end{aligned}$$

where ω_0 is the center frequency and Q is the quality factor. The poles of $L(s)$ are given by the roots of the quadratic equation:

$$Qs^2 + \omega_0 s + Q\omega_0^2 = 0$$

That is,

$$\begin{aligned} s &= \frac{1}{2Q}(-\omega_0 \mp \sqrt{\omega_0^2 - 4Q^2\omega_0^2}) \\ &= \frac{\omega_0}{2Q}(-1 \pm j\sqrt{4Q^2 - 1}) \end{aligned}$$

With Q assumed to be large, we may approximate the poles as

$$s \approx \frac{\omega_0}{2Q}(-1 \pm j2Q) = -\frac{\omega_0}{2Q} \pm j\omega_0$$

which lie inside the left half of the s -plane and very close to the $j\omega$ -axis. The loop transfer function $L(s)$ has a single zero at $s = 0$. Hence, the root locus of the closed-loop feedback amplifier starts at the two poles, tracing out the path illustrated in Fig. 1 for $Q = 100$ and $\omega_0 = 1$.

%Solution to problem 9.45

```
w = 1;  
Q = 100;
```

```
num = [w 0];  
den = [Q w Q*w^2];  
figure(1)  
rlocus(num,den)
```

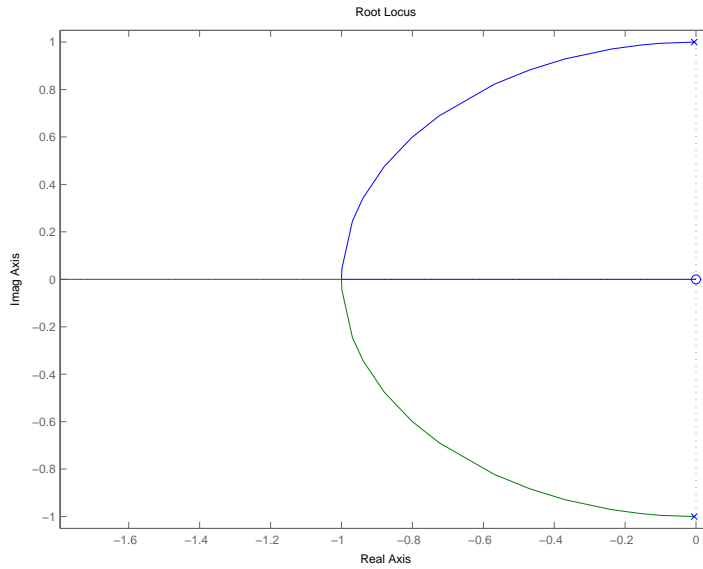


Figure 1

(b) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

which may be expressed in the form

$$T(s) = \frac{\left(\frac{A}{1 + \beta A}\right)}{\left(\frac{Q}{1 + \beta A}\right)\left(\frac{s}{\omega_0}\right) + 1 + \left(\frac{Q}{1 + \beta A}\right)\left(\frac{\omega_0}{s}\right)}$$

From this expression we immediately deduce that the application of feedback produces two effects:

1. It reduces the gain of the amplifier by the factor $1 + \beta A$.
2. It reduces the quality factor Q by the same amount $1 + \beta A$.

By definition, we have

$$Q = \frac{\omega_0}{B}$$

where B is the bandwidth. Therefore, for fixed ω_0 , if the quality factor Q is reduced by the amount $1 + \beta A$ as a result of applying the feedback, the closed-loop bandwidth is increased by the same amount.

9.46 (a) The “sensor” transfer function is

$$H(s) = \frac{1}{s}$$

Hence the closed-loop gain (i.e., transfer function) of the phase-locked loop model shown in Fig. P9.46 is

$$\begin{aligned}
 \frac{V(s)}{\Phi_1(s)} &= \frac{G(s)}{1 + G(s)H(s)} \\
 &= \frac{K_0 H(s)(1/K_v)}{1 + \frac{1}{s} K_0 H(s)} \\
 &= \frac{s K_0 H(s)(1/K_v)}{s + K_0 H(s)} \quad (1)
 \end{aligned}$$

(b) For positive frequencies that satisfy the condition $|K_0 H(j\omega)| \gg \omega$, we may approximate Eq. (1) as

$$\begin{aligned}
 \frac{V(s)}{\Phi_1(s)} &\approx \frac{s K_0 H(s)(1/K_v)}{K_0 H(s)} \\
 &= \frac{s}{K_v}
 \end{aligned}$$

Correspondingly, we may write

$$v(t) \approx \frac{1}{K_v} \frac{d}{dt} \phi_1(t)$$

9.47 The error signal $e(t)$ is defined as the difference between the actuating or target signal $y_d(t)$ and the controlled signal (i.e., actual response) $y(t)$. Expressing these signals in terms of their respective Laplace transforms, we may write

$$\begin{aligned}
 E(s) &= Y_d(s) - Y(s) \\
 &= [1 - T(s)] Y_d(s) \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 &= \left[1 - \frac{G(s)H(s)}{1 + G(s)H(s)} \right] Y_d(s) \\
 &= \left[\frac{1}{1 + G(s)H(s)} \right] Y_d(s) \quad (2)
 \end{aligned}$$

The *steady-state error* of a feedback control system is defined as the value of the error signal $e(t)$ when time t approaches infinity. Denoting this quantity by ϵ_{ss} , we may thus write

$$\epsilon_{ss} = \lim_{s \rightarrow 0} e(t) \quad (3)$$

Using the final value theorem of Laplace transform theory described in Chapter 6, we may redefine the steady-state error ϵ_{ss} in the equivalent form

$$\epsilon_{ss} = \lim_{s \rightarrow 0} sE(s) \quad (4)$$

Hence, substituting Eq. (2) into (4), we get

$$\epsilon_{ss} = \lim_{s \rightarrow 0} \frac{sY_d(s)}{1 + G(s)H(s)} \quad (5)$$

Equation (5) shows that the steady-state error ϵ_{ss} of a feedback control system depends on two quantities:

- The open-loop transfer function $G(s)H(s)$ of the system
- The Laplace transform $Y_d(s)$ of the target signal $y_d(t)$

However, for Eq. (5) to be valid, the closed-loop control system of Fig. 9.14 must be stable.

In general, $G(s)H(s)$ may be written in the form of a rational function as follows:

$$G(s)H(s) = \frac{P(s)}{s^p Q_1(s)} \quad (6)$$

where neither the polynomial $P(s)$ nor $Q_1(s)$ has a zero at $s = 0$. Since $1/s$ is the transfer function of an integrator, it follows that p is the number of free integrators in the loop transfer function $G(s)H(s)$. The order p is referred to as the *type* of the feedback control system. We thus speak of a feedback control system being of type 0, type 1, type 2, and so on for $p = 0, 1, 2, \dots$, respectively. In light of this classification, we next consider the steady-state error for three different input functions.

Step Input

For the step input $y_d(t) = u(t)$, we have $Y_d(s) = 1/s$. Hence, Eq. (5) yields the steady-state error

$$\begin{aligned} \epsilon_{ss} &= \lim_{s \rightarrow 0} \frac{1}{1 + G(s)H(s)} \\ &= \frac{1}{1 + K_p} \end{aligned} \quad (7)$$

where K_p is called the *position error constant*, defined by

$$\begin{aligned} K_p &= \lim_{s \rightarrow 0} G(s)H(s) \\ &= \lim_{s \rightarrow 0} \frac{P(s)}{s^p Q_1(s)} \end{aligned} \quad (8)$$

For $p \geq 1$, K_p is unbounded and therefore $\epsilon_{ss} = 0$. For $p = 0$, K_p is finite and therefore $\epsilon_{ss} \neq 0$. Accordingly, we may state that the steady-state error for a step input is zero for a feedback control system of type 1 or higher. On the other hand, for a system of type 0, the steady-state error is not zero, and its value is given by Eq. (7).

Ramp Input

For the ramp input $y_d(t) = tu(t)$, we have $Y_d = 1/s^2$. In this case. Eq. (5) yields

$$\begin{aligned}\epsilon_{ss} &= \lim_{s \rightarrow 0} \frac{1}{s + sG(s)H(s)} \\ &= \frac{1}{K_v}\end{aligned}\tag{9}$$

where K_v is the *velocity error constant*, defined by

$$\begin{aligned}K_v &= \lim_{s \rightarrow 0} sG(s)H(s) \\ &= \lim_{s \rightarrow 0} \frac{P(s)}{s^{p-1}Q_1(s)}\end{aligned}\tag{10}$$

For $p \geq 2$, K_v is unbounded and therefore $\epsilon_{ss} = 0$. For $p = 1$, K_v is finite and $\epsilon_{ss} \neq 0$.

For $p = 0$, K_v is zero and $\epsilon_{ss} = \infty$. Accordingly, we may state that the steady-state error for a ramp input is zero for a feedback control system of type 2 or higher. For a system of type 1, the steady-state error is not zero, and its value is given by Eq. (9). For a system of type 0, the steady-state error is unbounded.

Parabolic Input

For the parabolic input $y_d(t) = (t^2/2)u(t)$, we have $Y_d = 1/s^3$. The use of Eq. (5) yields

$$\begin{aligned}\epsilon_{ss} &= \lim_{s \rightarrow 0} \frac{1}{s^2 + s^2G(s)H(s)} \\ &= \frac{1}{K_a}\end{aligned}\tag{11}$$

where K_a is called the *acceleration error constant*, defined by

$$K_a = \lim_{s \rightarrow 0} \frac{P(s)}{s^{p-2}Q_1(s)}\tag{12}$$

For $p \geq 3$, K_a is unbounded and therefore $\epsilon_{ss} = 0$. For $p = 2$, K_a is finite and therefore $\epsilon_{ss} = 0$. For $p = 0, 1$, K_a is zero and therefore $\epsilon_{ss} = \infty$. Accordingly, we may state that for a parabolic input, the steady-state error is zero for a type 3 system or higher. For a system of type 2, the steady-state error is not zero, and its value is given by Eq. (11). For a system of type 0 or type 1, the steady-state error is unbounded.

In Table 1, we present a summary of the steady-state errors according to system type as determined above.

Additional Notes

A type 0 system is referred to as a *regulator*. The object of such a system is to maintain a physical variable of interest at some prescribed constant value despite the presence of external disturbances. Examples of regulator systems include:

- Control of the moisture content of paper, a problem that arises in the paper-making process
- Control of the chemical composition of the material outlet produced by a reactor
- Biological control system, which maintains the temperature of the human body at approximately 37°C despite variations in the temperature of the surrounding environment

Table 1: Steady-State Errors According to System Type

	<i>Step</i>	<i>Ramp</i>	<i>Parabolic</i>
Type 0	$1/(1 + K_p)$	∞	∞
Type 1	0	$1/K_v$	∞
Type 2	0	0	$1/K_a$

Type 1 and higher systems are referred to as *servomechanisms*. The objective here is to make a physical variable follow, or track, some desired time-varying function. Examples of servomechanisms include:

- Control of a robot, in which the robot manipulator is made to follow a preset trajectory in space
- Control of a missile, guiding it to follow a specified flight path
- Tracking of a maneuvering target (e.g., enemy aircraft) by a radar system

Example: Figure P9.47 shows the block diagram of a feedback control system involving a *dc motor*. The dc motor is an electromechanical device that converts dc electrical energy into rotational mechanical energy. Its transfer function is approximately defined by

$$G(s) = \frac{K}{s(\tau_L s + 1)}$$

where K is the gain and τ_L is the load time constant. Here it is assumed that the field time constant of the dc motor is small compared to the load time constant. The transfer function of the controller is

$$H(s) = \frac{\alpha\tau s + 1}{\tau s + 1}$$

The requirement is to find the steady-state errors of this control system.

The loop transfer function of the control system in Fig. P9.47 is

$$G(s)H(s) = \frac{K(\alpha\tau s + 1)}{s(\tau_L s + 1)(\tau s + 1)} \quad (13)$$

which represents a type 1 system. Comparing Eq. (13) with Eq. (6), we have for the example at hand

$$P(s) = \frac{\alpha K}{\tau_L} \left(s + \frac{1}{\alpha \tau} \right)$$

$$Q_1(s) = \left(s + \frac{1}{\tau_L} \right) \left(s + \frac{1}{\tau} \right)$$

$$p = 1$$

Hence the use of these values in Eqs. (8), (10), and (12) yields the following steady-state errors for the control system of Fig. P9.47:

(a) Step input: $\epsilon_{ss} = 0$.

(b) Ramp input: $\epsilon_{ss} = 1/K = \text{constant}$.

(c) Parabolic input: $\epsilon_{ss} = \infty$.

The effect of making the gain K large is to reduce the steady-state error of the system due to a ramp input. This, in turn, improves the behavior of the system as a velocity control system.

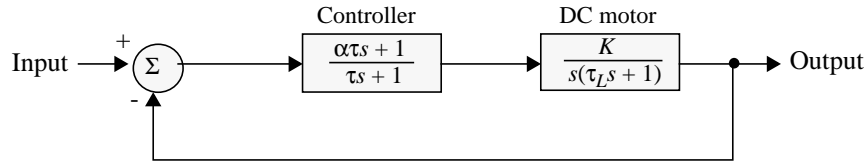


Figure P9.47

9.48 From Fig. P9.20 the closed-loop transfer function of the feedback system is

$$\begin{aligned} T(s) &= \frac{K/(s^2 + 2s)}{1 + K/(s^2 + 2s)} \\ &= \frac{K}{s^2 + 2s + K} \end{aligned} \quad (1)$$

In general, we may express $T(s)$ in the form

$$T(s) = \frac{\omega_n^2 T(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (2)$$

Hence, comparing Eqs. (1) and (2):

$$T(0) = 1 \quad (3)$$

$$\omega_n = \sqrt{K} \quad (4)$$

$$\zeta = 1/\omega_n = 1/\sqrt{K} \quad (5)$$

We are given $K = 20$, for which the use of Eqs. (4) and (5) yields

$$\omega_n = \sqrt{20} = 4.472 \text{ rad/second}$$

$$\zeta = 0.224$$

The time constant of the system is

$$\tau = \frac{1}{\zeta\omega_n} = 1 \text{ second}$$

9.49 The loop transfer function of the feedback control system is

$$\begin{aligned} L(s) &= G(s)H(s) \\ &= \frac{0.2K_p}{(s+1)(s+3)} \end{aligned}$$

The closed-loop transfer function of the system is

$$\begin{aligned} T(s) &= \frac{L(s)}{1+L(s)} \\ &= \frac{0.2K_p}{s^2 + 4s + (3 + 0.2K_p)} \end{aligned} \tag{1}$$

For a second-order system defined by

$$T(s) = \frac{T(0)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2}$$

the damping factor is ζ and the natural frequency is ω_n . Comparing Eqs. (1) and (2):

$$2\zeta\omega_n = 4$$

$$\omega_n^2 = 3 + 0.2K_p$$

$$T(0)\omega_n^2 = 0.2K_p$$

Hence,

$$\omega_n = \sqrt{3 + 0.2K_p} \tag{3}$$

$$\zeta = 2/\sqrt{3 + 0.2K_p} \tag{4}$$

$$T(0) = 0.2K_p/(3 + 0.2K_p) \tag{5}$$

We are required to have

$$\omega_n = 2 \text{ rad/s}$$

Hence the use of Eq. (3) yields

$$\sqrt{3 + 0.2K_p} = 2$$

That is,

$$K_p = \frac{4-3}{0.2} = 5$$

Next, the use of Eq. (4) yields

$$\begin{aligned}\zeta &= \frac{2}{\sqrt{3 + 0.2 \times 5}} \\ &= \frac{2}{\sqrt{4}} = 1\end{aligned}$$

which means that the system is critically damped.

The time constant of the system is defined by

$$\begin{aligned}\tau &= \frac{1}{\zeta \omega_n} \\ &= \frac{1}{1 \times 2} = 0.5 \text{ seconds}\end{aligned}$$

(Note: The use of Eq. (5) yields

$$T(0) = \frac{0.2 \times 5}{3 + 0.2 \times 5} = \frac{1}{4}$$

9.50 The loop transfer function of the system is

$$\begin{aligned}L(s) &= \left(K_P + \frac{K_I}{s} \right) \left(\frac{0.25}{s+1} \right) \\ &= 0.25 K_P \left(1 + \frac{K_I/K_P}{s} \right) \left(\frac{1}{s+1} \right)\end{aligned}$$

With $K_I/K_P = 0.1$ we have

$$L(s) = \frac{0.25 K_P (s + 0.1)}{s(s + 1)} \quad (1)$$

The root locus of the system is therefore as shown in Fig. 1:

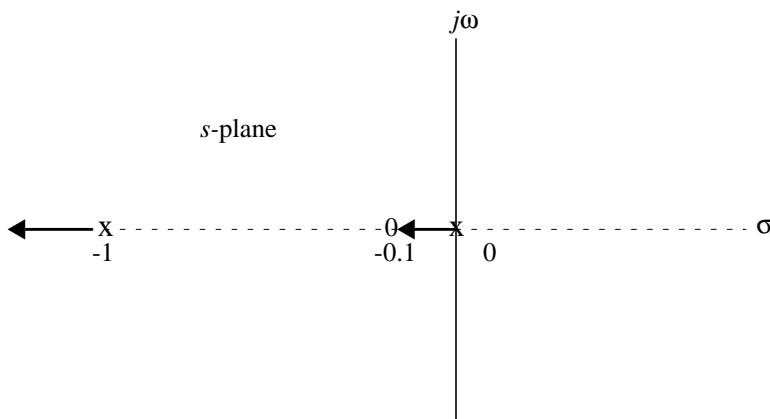


Figure 1

The closed-loop transfer function of the system is

$$T(s) = \frac{L(s)}{1 + L(s)} \quad (2)$$

Substituting Eq. (1) into (2):

$$T(s) = \frac{0.25K_p(5 + 0.1)}{s(s + 1) + 0.25K_p(s + 0.1)} \quad (3)$$

For a closed-loop at $s = -5$ we require that the denominator of $T(s)$ satisfy the following equation

$$(s + 5)(s + a) = 0 \quad (4)$$

where $s = -a$ is the other closed-loop pole of the system. Comparing the denominator of Eq. (3) with Eq. (4):

$$5 + a = 1 + 0.25K_p$$

$$5a = 0.1 \times 0.25K_p$$

Solving this pair of equations for K_p and a , we get

$$K_p = \frac{20}{1.225} = 16.33$$

$$a = 0.005 \times K_p = 0.08165$$

9.51 The loop transfer function of the PD controller system is

$$\begin{aligned} L(s) &= (K_p + K_D s) \left(\frac{1}{s(s + 1)} \right) \\ &= K_D \left(s + \frac{K_p}{K_D} \right) \left(\frac{1}{s(s + 1)} \right) \end{aligned}$$

We are given $K_p/K_D = 4$. Hence,

$$L(s) = \frac{K_D(s + 4)}{s(s + 1)}$$

We may therefore sketch the root locus of the system as follows:

%Solution to problem 9.51

clear; clc;

figure(1)

K = 1;

num = K*[1 4];

den = [1 (1+K) K*4];

rlocus(num,den)

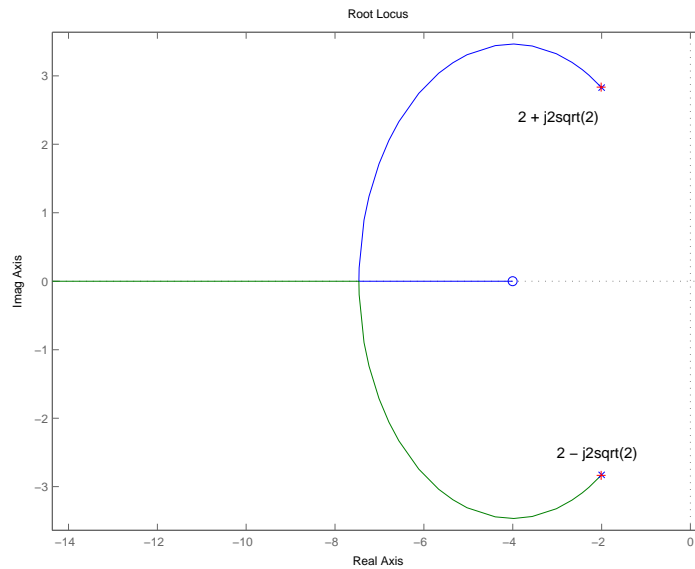
figure(2)

K = 3;

num = K*[1 4];

den = [1 (1+K) K*4];

rlocus(num,den)



The closed-loop transfer function of the system is

$$\begin{aligned} T(s) &= \frac{L(s)}{1 + L(s)} \\ &= \frac{K_D(s + 4)}{s^2 + s + K_D(s + 4)} \end{aligned} \quad (1)$$

We are required to choose K_D so as to locate the closed-loop poles at $s = -2 \pm j2\sqrt{2}$. That is, the characteristic equation of the system is to be

$$\begin{aligned} (s + 2 + j2\sqrt{2})(s + 2 - j2\sqrt{2}) &= 0 \\ \text{or} \\ s^2 + 4s + 12 &= 0 \end{aligned} \quad (2)$$

Comparing the denominator of Eq. (1) with Eq. (2):

$$\begin{aligned} 1 + K_D &= 4 \\ 4K_D &= 12 \end{aligned}$$

Both of these conditions are satisfied by choosing $K_D = 3$

9.52 (a) For a feedback system using PI controller, the loop transfer function is

$$L(s) = \left(K_P + \frac{K_I}{s} \right) L'(s)$$

where $L'(s)$ is the uncompensated loop transfer function. For $s = j\omega$ we may write

$$K_P + \frac{K_I}{j\omega} = \frac{1}{j\omega} (K_I + jK_P\omega)$$

The contribution of the PI controller to the loop phase response of the feedback system is

$$\phi = -90^\circ + \tan^{-1} \left(\frac{K_P\omega}{K_I} \right)$$

For all positive values of K_P/K_I , the angle $\tan^{-1}(K_P\omega/K_I)$ is limited to the range $[0, 90^\circ]$. It follows therefore that the use of a PI controller introduces a phase lag into the loop phase response of the feedback system, as shown in Fig. 1.

(b) For a feedback system using PD controller, the loop transfer function of the system may be expressed as

$$L(s) = (K_P + K_D s) L'(s)$$

where $L'(s)$ is the uncompensated loop transfer function. For $s = j\omega$, the contribution of the PD controller to the loop phase response of the system is $\tan^{-1}(K_D\omega/K_P)$. For all

positive values of K_D/K_P this contribution is limited to the range $[0, 90^\circ]$. We therefore conclude that the use of PD controller has the effect of introducing a phase lead into the loop phase response of the system, as illustrated in Fig. 2.

%Solution to problem 9.52

```
clear; clc;
w = 0:0.01:5*pi;
Kp = 1.2;
Ki = 1;

y = -pi/2 + atan(w*Kp/Ki);
plot(w,y)
xlabel('\omega')
ylabel('Phase
hi')
title('-
i/2 + tan^{-1}(\omega Kp/Ki)')

figure(2)
y = -pi/2 + atan(w*Kp/Ki);
plot(w,y)
xlabel('\omega')
ylabel('Phase
hi')
title('tan^{-1}(\omega Kp/Ki)')
```

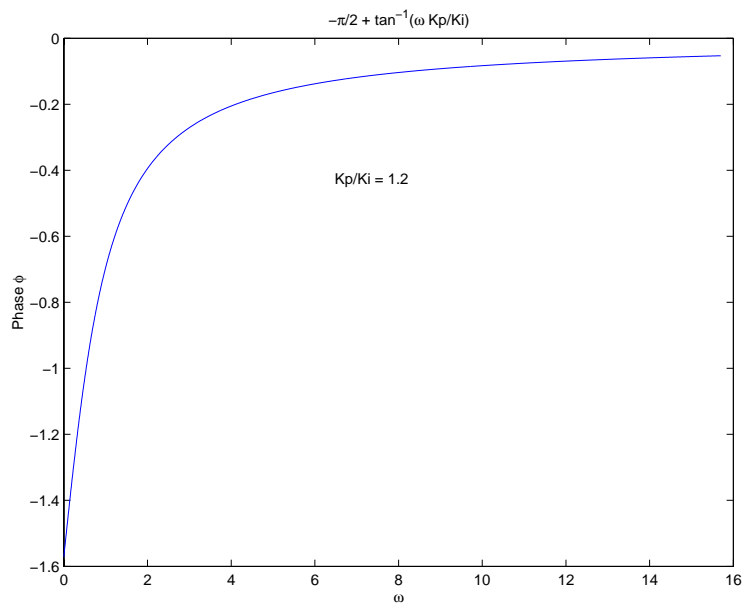


Figure 1

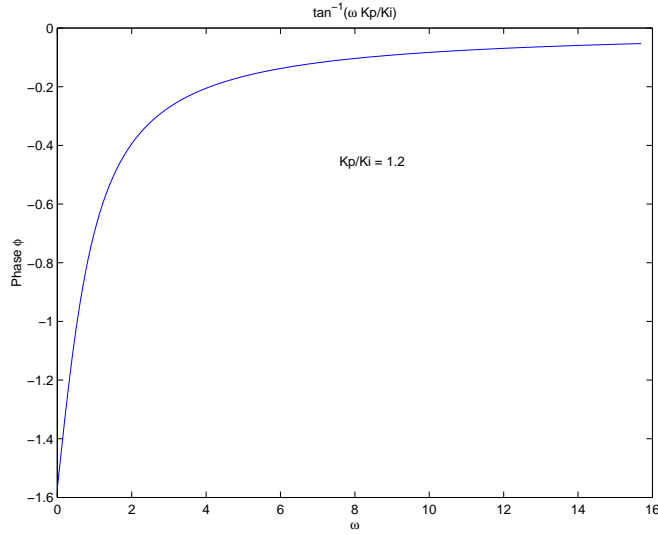


Figure 2

- 9.53 The transfer function of the PID controller is defined by $K_P + \frac{K_I}{s} + K_D s$. The requirement is to use this controller to introduce zeros at $s = -1 \pm j2$ into the loop transfer function of the feedback system. Hence,

$$\begin{aligned} s^2 + (K_P/K_D)s + (K_I/K_D) &= (s + 1)^2 + (2)^2 \\ &= s^2 + 2s + 5 \end{aligned}$$

Comparing terms:

$$\frac{K_P}{K_D} = 2$$

$$\frac{K_I}{K_D} = 5$$

The loop transfer function of the compensated feedback system is therefore

$$\begin{aligned} L(s) &= \frac{K_P + \frac{K_I}{s} + K_D s}{(s + 1)(s + 2)} \\ &= \frac{K_D(s^2 + 2s + 5)}{s(s + 1)(s + 2)} \end{aligned}$$

Figure 1 displays the root locus of $L(s)$ for varying K_D . From this figure we see that the root locus is confined to the left half of the s -plane for all positive values of K_D . The feedback system is therefore stable for $K_D > 0$.

%Solution to problem 9.53

```
clear; clc;
num = [1 2 5];
den = [1 3 2 0];
rlocus(num,den)
```

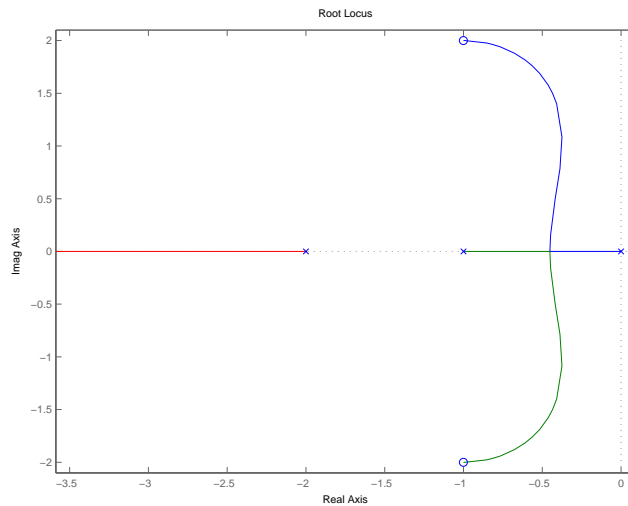


Figure 1

9.54 Let K_P denote the action of the proportional controller. We may then express the loop transfer function of the controlled inverted pendulum as

$$L(s) = \frac{K_P(s + 3.1)(s - 3.1)}{s^2(s + 4.4)(s - 4.4)}$$

which has zeros at $s = \pm 3.1$ and poles at $s = 0$ (order 2) and $s = \pm 4.4$. The resulting root locus is sketched in Fig. 1. This diagram shows that for all positive values of K_P , the

closed-loop transfer function of the system will have a pole in the right-half plane and a pair of poles on the $j\omega$ -axis.

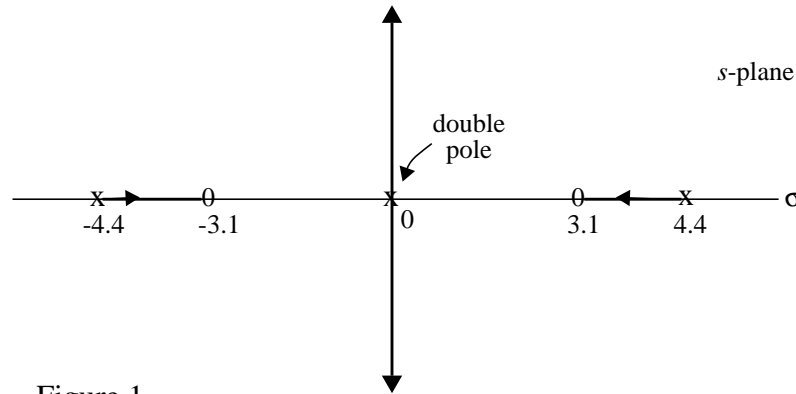


Figure 1

Accordingly, the inverted pendulum cannot be stabilized using a proportional controller for $K_p > 0$. The reader is invited to reach a similar conclusion for $K_p < 0$.

The stabilization of an inverted pendulum is made difficult by the presence of two factors in the loop transfer function $L(s)$:

1. A double pole at $s = 0$.
2. A zero at $s = 3.1$ in the right-half plane.

We may compensate for (1) but nothing can be done about (2) if the compensator is itself to be stable. Moreover, we have to make sure that the transfer function of the compensator is proper for it to be realizable. We may thus propose the use of a compensator (controller) whose transfer function is

$$C(s) = \frac{Ks^2(s - 4.4)}{(s + 1)(s + 2)(s + 3.1)}$$

The compensated loop transfer function is therefore (after performing pole-zero cancellation)

$$\begin{aligned} L'(s) &= C(s)L(s) \\ &= \frac{K(s - 3.1)}{(s + 1)(s + 2)(s + 4.4)} \end{aligned}$$

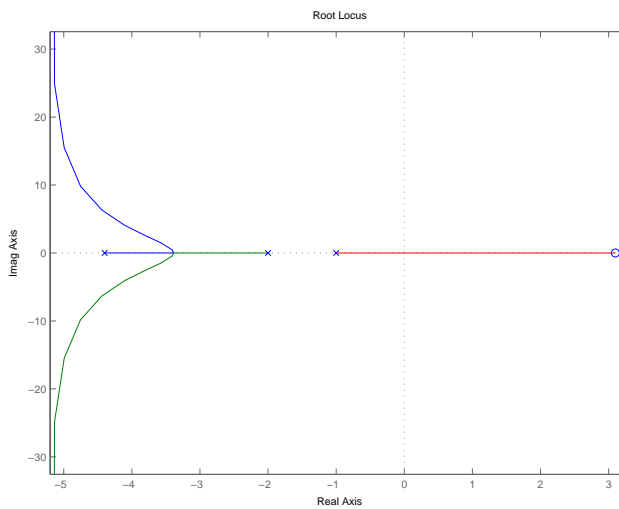
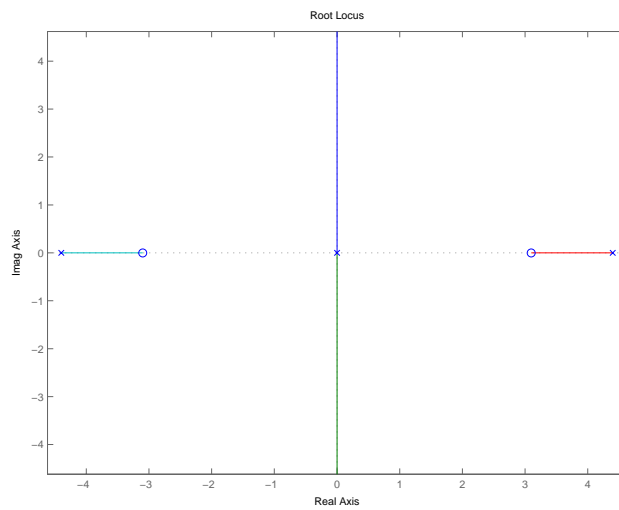
Figure 2 shows a sketch of the root locus of $L'(s)$. From this figure we see that the compensated system is stable provided that the gain factor K satisfies the condition

$$0 < K < \frac{4 \times 2}{3.1} = 2.839$$

%Solution to problem 9.54

```
clear; clc;  
num = [1 0 -961/100];  
den = [1 0 -484/25 0 0];  
rlocus(num,den)
```

```
figure(2); clc;  
num = [1 -3.1];  
den = [1 37/5 76/5 44/5];  
rlocus(num,den)
```



9.55 For linear control systems, the transient response is usually measured in terms of the step response. Typically, the step response, denoted by $y(t)$, is oscillatory as illustrated in Fig. P9.55. In describing such a response, we have two conflicting criteria: swiftness of the response, and closeness of the response to the desired response. Swiftness of the response is measured in terms of the rise time and peak time. Closeness of the response to the desired response is measured in terms of the percentage overshoot and settling time. these four quantities are defined as follows:

- *Rise time*, T_r , is defined as the time taken by the step response to rise from 10% to 90% of its final value $y(\infty)$.
- *Peak time*, T_p , is defined as the time taken by the step response to reach the overshoot (overall) maximum value y_{\max} .
- *Percentage overshoot*, P.O., is defined in terms of the maximum value y_{\max} and final value $y(\infty)$ by

$$\text{P.O.} = \frac{y_{\max} - y(\infty)}{y(\infty)} \times 100$$

- *Settling time*, T_s , is defined as the time required by the step response to settle within $\pm\delta\%$ of the final value $y(\infty)$, where δ is user specified.

Figure P9.55 illustrates the definitions of these four quantities, assuming that $y(\infty) = 1.0$. They provide an adequate description of the step response $y(t)$. Most importantly, they lend themselves to measurement. Note that in the case of an overdamped system, the peak time and percentage overshoot are not defined. In such a case, the step response of the system is specified simply in terms of the rise time and settling time.

For reasons that will become apparent later, the underdamped response of a second-order system to a step input often provides an adequate approximation to the step response of a linear feedback control system. Accordingly, it is of particular interest to relate the above-mentioned quantities to the parameters of a second-order system.

Example: Consider an underdamped second-order system of damping ratio ζ and natural frequency ω_n . Determine the rise time, peak time, percentage overshoot, and settling time of the system. For settling time, use $\delta = 1$.

Solution: Unfortunately, it is difficult to obtain an explicit expression for the rise time T_r in terms of the damping ratio ζ and natural frequency ω_n . Nevertheless, it can be determined by simulation. Table 9.2 presents the results of simulation for the range $0.1 \leq \zeta \leq 0.9$. In this table we have also included the results obtained by using the approximate formula:

$$T_r \approx \frac{1}{\omega_n} (0.60 + 2.16\zeta)$$

This formula yields fairly accurate results for $0.3 \leq \zeta \leq 0.8$, as can be seen from Table 1.

To determine the peak time T_p , we may differentiate Eq. (...) with respect to time t and then set the result equal to zero. We thus obtain the solutions $t = \infty$ and

$$t = \frac{n\pi}{\omega_n \sqrt{1 - \zeta^2}}, \quad n = 0, 1, 2, \dots$$

Table 1: Normalized Rise Time as a Function of the Damping Ratio ζ for a Second-Order System

$\omega_n T_r$		
ζ	<i>Simulation</i>	<i>Approximate Value</i>
0.1	1.1	0.82
0.2	1.2	1.03
0.3	1.3	1.25
0.4	1.45	1.46
0.5	1.65	1.68
0.6	1.83	1.90
0.7	2.13	2.11
0.8	2.5	2.33
0.9	2.83	2.54

The solution $t = \infty$ defines the maximum of the step response $y(t)$ only when $\zeta \geq 1$ (i.e., the critically damped or overdamped case). We are interested in the underdamped case. The first maximum of $y(t)$ occurs for $n = 1$; hence the peak time is

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

The first maximum of $y(t)$ also defines the percentage overshoot. Thus, putting $t = T_p$ in Eq. (...) and simplifying the result, we get

$$y_{\max} = 1 + e^{-\pi\zeta/(\sqrt{1 - \zeta^2})}$$

The percentage overshoot is therefore

$$\text{P.O.} = 100e^{-\pi\zeta/(\sqrt{1 - \zeta^2})}$$

Finally, to determine the settling time T_s we seek the time t for which the step response $y(t)$ defined in Eq. (...) decreases and stays within $\pm\delta\%$ of the final value $y(\infty) = 1.0$. For

$\delta = 1$, this time is closely approximated by the decaying exponential factor $e^{-\zeta\omega_n t}$, as shown by

$$e^{-\zeta\omega_n T} \approx 0.01$$

or

$$T_s \approx \frac{4.6}{\zeta\omega_n}$$

9.56 We often find that the poles and zeros of the closed-loop transfer function $T(s)$ of a feedback system are grouped in the complex s -plane roughly in the manner illustrated in Fig. P9.56. In particular, depending on how close the poles and zeros are to the $j\omega$ -axis we may identify two groupings:

1. *Dominant poles and zeros*, which are those poles and zeros of $T(s)$ that lie close to the $j\omega$ -axis. They are said to be dominant because they exert a profound influence on the frequency response of the system. Another way of viewing this situation is to recognize that poles close to the $j\omega$ -axis correspond to large time constants of the system. The contributions made by these poles to the transient response of the system are slow and therefore dominant.
2. *Insignificant poles and zeros*, which are those poles and zeros of $T(s)$ that are far removed from the $j\omega$ -axis. They are said to be insignificant because they have relatively little influence on the frequency response of the system. In terms of time-domain behavior, poles that are far away from the $j\omega$ -axis correspond to small time constants. The contributions of such poles to the transient response of the system are much faster and therefore insignificant.

Let $s = -a$ and $s = -b$ define the boundaries of the dominant poles and the insignificant poles, as indicated in Fig. P9.56. As a rule of thumb, the grouping of poles into dominant poles and insignificant poles is justified if the ratio b/a is greater than 4.

Given that we have a high-order feedback system whose closed-loop transfer function fits the picture portrayed in Fig. P9.56, we may then approximate the system by a *reduced-order model* simply by retaining the dominant poles and zeros of $T(s)$. The use of a reduced-order model in place of the original system is motivated by the following considerations:

- Low-order models are simple; they are therefore intuitively appealing in system analysis and design.
- Low-order models are less demanding in computational terms than high-order ones.

A case of particular interest is when the use of a first-order or second-order model as the reduced-order model is justified, for then we can exploit the wealth of information available on such low-order models.

When discarding poles and zeros in the derivation of a reduced-order model, it is important to rescale the gain of the system. Specifically, the reduced-order model and the original system should have the same gain at zero frequency.

Example: Consider again the linear feedback amplifier. Assuming that $K = 8$, approximate this system using a second-order model. Use the step response to assess the quality of the approximation.

Solution: For $K = 8$, the characteristic equation of the feedback amplifier is given by

$$s^3 + 6s^2 + 11s + 54 = 0$$

Using the computer, the roots of this equation are found to be

$$s = -5.7259, \quad s = -0.1370 \pm j3.0679$$

The locations of these three roots are plotted in Fig. 2. We immediately observe from the pole-zero map of Fig. 2 that the poles at $s = -0.1370 + j3.0679$ are the dominant poles, and the poles at $s = -5.7259$ is an insignificant pole. The numerator of the closed-loop transfer function $T(s)$ consists simply of $6K$. Accordingly, the closed-loop gain of the feedback amplifier may be approximated as

$$\begin{aligned} T'(s) &\approx \frac{6K'}{(s + 0.1370 + j3.0679)(s + 0.1370 - j3.0679)} \\ &= \frac{6K'}{s^2 + 0.2740s + 9.4308} \end{aligned}$$

To make sure that $T'(s)$ is scaled properly relative to the original $T(s)$, the gain K' is chosen as follows:

$$K' = \frac{9.4308}{54} \times 8 = 1.3972$$

for which we thus have $T'(0) = T(0)$.

In light of Eq. (9.41), we set

$$2\zeta\omega_n = 0.2740$$

$$\omega_n^2 = 9.4308$$

from which we readily find that

$$\zeta = 0.0446$$

$$\omega_n = 3.0710$$

The step response of the feedback amplifier is therefore underdamped. The time constant of the exponentially damped response is from Eq. (9.44)

$$\tau = \frac{1}{\zeta\omega_n} = \frac{1}{0.137} = 8.2993s$$

The frequency of the exponentially damped response is

$$\begin{aligned} \omega_n\sqrt{1 - \zeta^2} &= 3.0710\sqrt{1 - (0.0446)^2} \\ &= 3.0679 \text{ rad/s} \end{aligned}$$

Figure 3 shows two plots, one displaying the step response of the original third-order feedback amplifier with $K = 8$, and the other displaying the step response of the

approximating second-order model with $K' = 1.3972$. The two plots are very similar, which indicates that the reduced-order model is adequate for the example at hand.

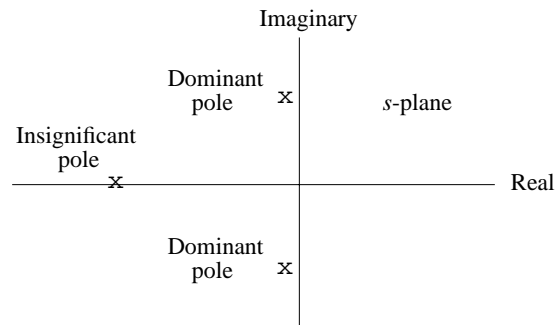


Figure 1: Pole-zero map for feedback amplifier with gain $K = 8$.

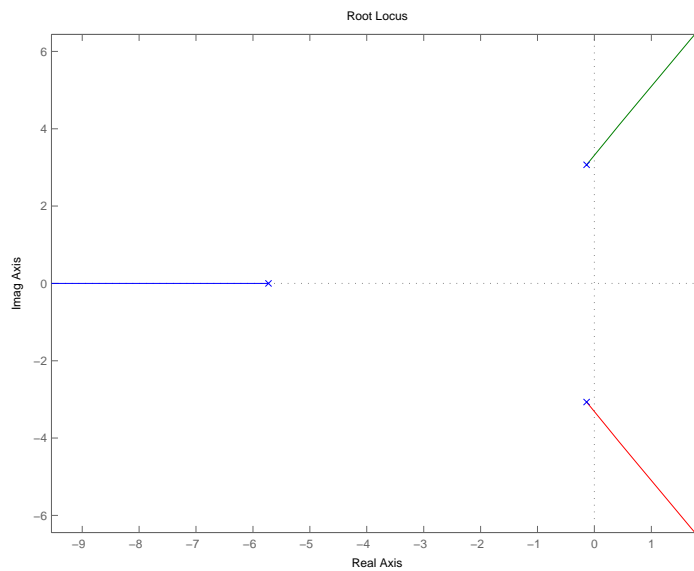


Figure 2: Root locus of original 3rd order system

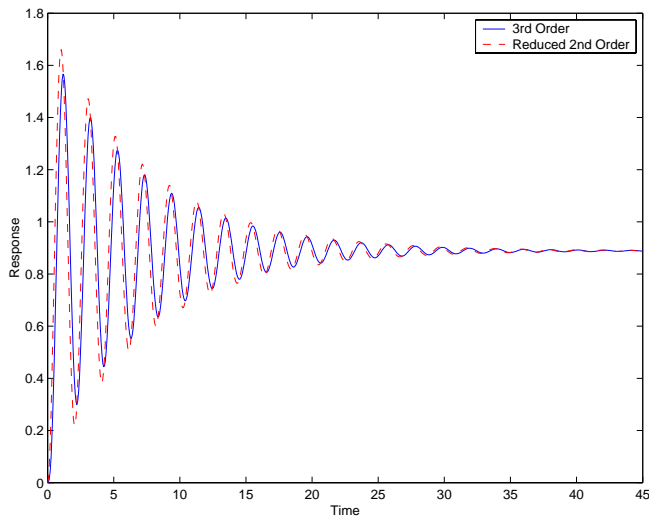


Figure 3: Step response for third-order system shown as dashed curve, and step response for (reduced) second-order system shown as solid curve.

%Solution to problem 9.56

%Original 3rd order system

num = 6*8;

den = [1 6 11 54];

figure(2)

rlocus(num,den)

figure(3)

[order3,T] = step(tf(num,den));

%Reduced 2nd order system

num = 6*1.3972;

den = [1 0.2740 9.4308];

[order2,T] = step(tf(num,den));

plot(T,order3,'b-',T,order2,'r--')

legend('3rd Order','Reduced 2nd Order')

xlabel('Time')

ylabel('Response')

9.57 Consider first the original system described by the transfer function

$$T(s) = \frac{48}{(s + 5.7259)(s^2 + 0.2740s + 9.4308)}$$

The corresponding Bode diagram, obtained by putting $s = j\omega$, is plotted in Fig. 1 on the next page.

Consider next the reduced-order model described by the transfer function

$$T'(s) = \frac{8.3832}{s^2 + 0.2740s + 9.4308}$$

which is obtained from $T(s)$ by ignoring the distant pole at $s = -5.7259$ and readjusting the constant gain factor. Figure 2, on the page after the next one, plots the Bode diagram of the approximating system.

Comparing these two figures, 1 and 2, we see that the frequency responses of the original feedback system and its reduced-order approximate model are close to each other over the frequency range $0 \leq \omega \leq 4.0$.

%Solution to problem 9.57

```
clear; clc;
num = [8.3832];
den = [1 0.2740 9.4308];
margin(num,den)
```

```
figure(2); clc;
num = [48];
den = [1 6 11 54];
margin(num,den)
```

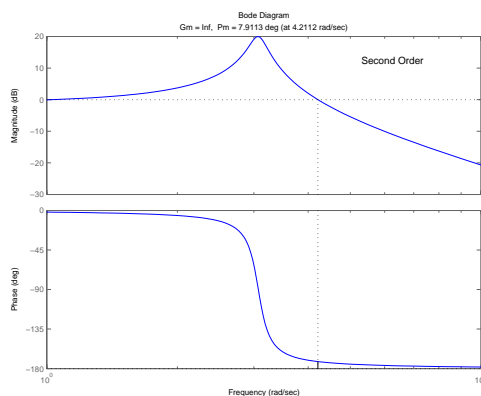


Figure 1: Frequency response of original system

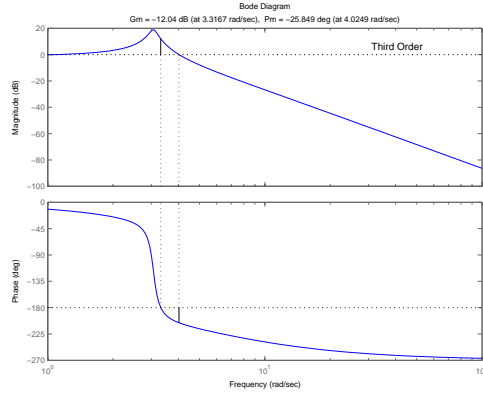


Figure 2: Frequency response of reduced-order model

- 9.58 The guidelines used in the classical (frequency-domain) approach to the design of a linear feedback control system are usually derived from the analysis of second-order servomechanism dynamics. Such an approach is justified on the following grounds. First, when the loop gain is large the closed-loop transfer function of the system develops a pair of dominant complex-conjugate poles. Second, a second-order model provides an adequate approximation to a higher-order model whose transfer function is dominated by a pair of complex-conjugate poles.

Consider then a second-order system whose loop transfer function is given by

$$T(s) = \frac{K}{s(\tau s + 1)}$$

This system was studied in Section 9.11. When the loop gain K is large enough to produce an underdamped step response (i.e., the closed-loop poles form a complex-conjugate pair), the damping ratio and natural frequency of the system are defined by, respectively (see Drill Problem 9.8),

$$\zeta = \frac{1}{2\sqrt{\tau K}} \quad \text{and} \quad \omega_n = \sqrt{\frac{K}{\tau}}$$

Accordingly, we may redefine the loop transfer function in terms of ζ and ω_n as follows:

$$L(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \tag{1}$$

By definition, the gain crossover frequency ω_g is determined from the relation

$$|L(j\omega_g)| = 1$$

Therefore, putting $s = j\omega_g$ in Eq. (1) and solving for ω_g , we get

$$\omega_g = \omega_n \sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2} \quad (2)$$

The phase margin, measured in degrees, is therefore

$$\begin{aligned} \phi_m &= 180^\circ - 90^\circ - \tan^{-1} \left(\frac{\omega_g}{2\zeta\omega_n} \right) \\ &= \tan^{-1} \left(\frac{2\zeta\omega_n}{\omega_g} \right) \\ &= \tan^{-1} \left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}} \right) \end{aligned} \quad (3)$$

Equation (3) provides an exact relationship between the phase margin ϕ_m , a quantity pertaining to the open-loop frequency response, and the damping ratio ζ , a quantity pertaining to the closed-loop step response. This exact relationship is plotted in Fig. 1. For values of the damping ratio in the range $0 \leq \zeta \leq 0.6$, Eq. (3) is closely approximated by

$$\phi_m \approx 100\zeta, \quad 0 \leq \zeta \leq 0.6$$

Equivalently, we may write

$$\zeta = \frac{\phi_m}{100}, \quad 0 \leq \zeta \leq 0.6 \quad (4)$$

where, as mentioned previously, the phase margin ϕ_m is measured in degrees. This approximation is included as the dashed line in Fig. 1.

Once we have obtained a value for the damping ratio ζ , we can use Eq. (2) to determine the corresponding value of the natural frequency ω_n in terms of the gain cross-over frequency ω_g . With the values of ζ and ω_n at hand, we can then go on to determine the rise time, peak time, percentage overshoot, and settling time as descriptors of the step response using the formulas derived in

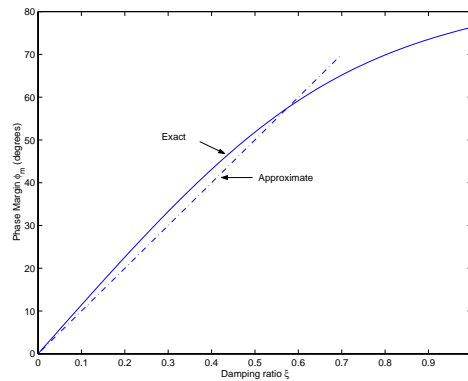


Figure 1

Computer Experiments

9.59 We are given

$$L(s) = \frac{K}{(s+1)^3}$$

The MATLAB command for finding the value of K corresponding to a desired pair of closed-loop poles, and therefore damping factor ζ , is given below.

The root locus of the unity feedback system is presented in Fig. 1. A damping factor $\zeta = 0.5$ corresponds to a pair of dominant closed-loop poles lying on the angles $\pm 120^\circ$. Hence using the command `rlocfind`, we obtain the value $K = 1$.

The corresponding values of the complex closed-loop poles are $s = -0.5 \pm j0.866$.

Hence the natural frequency $\omega_n = \sqrt{0.5^2 + 0.866^2} = 1$.

%Solution to Problem 9.59

```
num = [1];
K = 0;
den = [1 3 3 1];
rlocus(num,den)
K = rlocfind(num,den)
den = [1 3 3 1];
[Wn z]=damp(den)
```

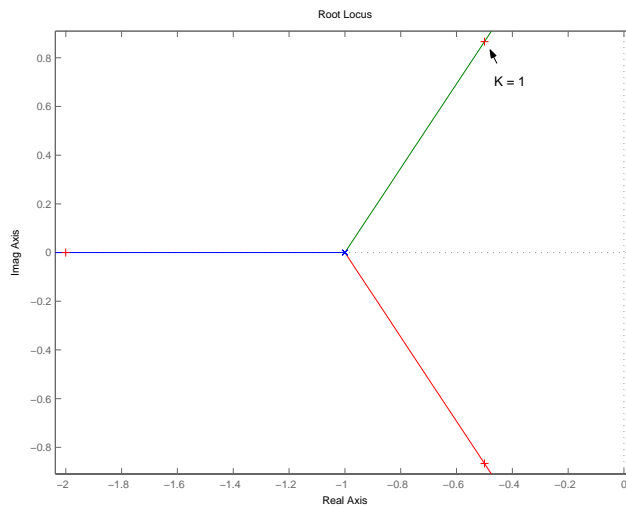


Figure 1

9.60 The MATLAB command for this experiment is presented below.

Figure 1 displays the root locus for the loop transfer function

$$L(s) = \frac{K}{s(s^2 + s + 2)}$$

The value of K , corresponding to a pair of complex poles with damping factor $\zeta = 0.0707$, is computed to be $K = 1.46$ using `rlocfind`.

Figure 2 presents the Bode diagram of $L(s)$ for $K = 1.5$. From this figure we find the following:

- (1) Gain margin - 6.1 dB
Phase crossover frequency = 1.4 rad/sec
- (2) Phase margin = 71.3°
Gain crossover frequency = 0.6 rad/sec

%Solution to Problem 9.60

```
figure(1); clf;
num = [1];
den = [1 1 2 0];
rlocus(num,den)
axis([-1 1 -2 2])
K = rlocfind(num,den)
```

```
figure(2); clf;
K = 1.5;
num = K*[1];
margin(num,den)
```

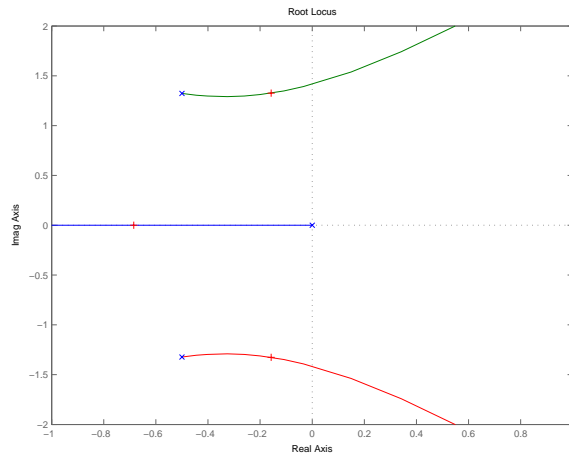


Figure 1

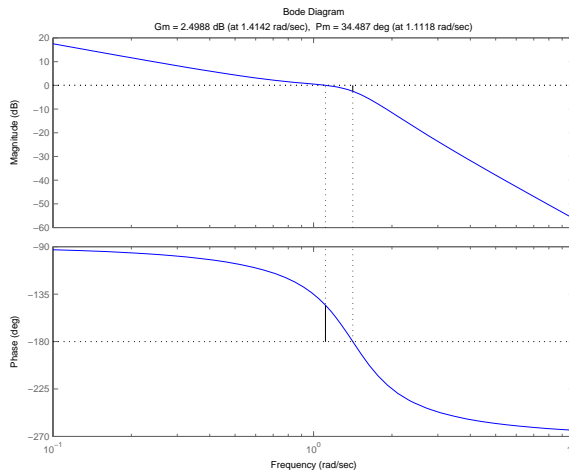


Figure 2

9.61 The MATLAB code for this experiment is presented below.

Figure 1 displays the root locus of the feedback system defined by the loop transfer function

$$L(s) = \frac{K(s-1)}{(s+1)(s^2+s+1)}$$

Using the command `rlocfind`, we find that the value of K for which the only real closed-loop pole lies on the $j\omega$ -axis (i.e., the system is on the verge of instability) is $K = 1$. We may verify this result by constructing the Routh array for the characteristic equation

$$(s + 1)(s^2 + s + 1) + K(s - 1) = 0$$

as shown by

$$\begin{array}{r|rr} s^3 & 1 & 2 + K \\ s^2 & 2 & -K \\ s^1 & \frac{4 + 3K}{2} & 0 \\ s^0 & -K & 0 \end{array}$$

For $K = 1$ there is only one sign change in the first column of array coefficients.

Figure 2 displays the Nyquist locus of $L(j\omega)$ for $K = 0.8$. We see that the critical point is not enclosed, and therefore the feedback system is stable.

Figure 3 displays the Bode diagram of $L(j\omega)$ for $K = 0.8$. From this figure we obtain the following values:

Gain margin = 2.0 dB

Phase-crossover frequency = 0 rad/sec.

Note that the gain margin is exactly equal to

$$-20\log_{10}K = 20\log_{10}\frac{1}{0.8} = 2\text{dB}$$

%Solution to Problem 9.61

```
figure(1); clf;
K = 1;
num = K*[1 -1];
den = [1 2 2 1];
rlocus(num,den)
ylim([-5 5])
K = rlocfind(num,den)
```

```
figure(2); clf;
K = 0.8;
num = K*[1 -1];
nyquist(num,den)
```

```
return
```



```
figure(3); clf;
K = 0.8;
num = K*[1 -1];
margin(num,den)
```

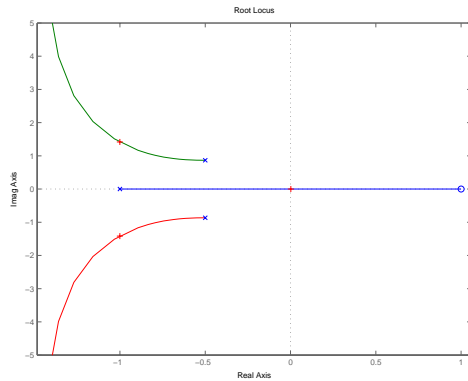


Figure 1

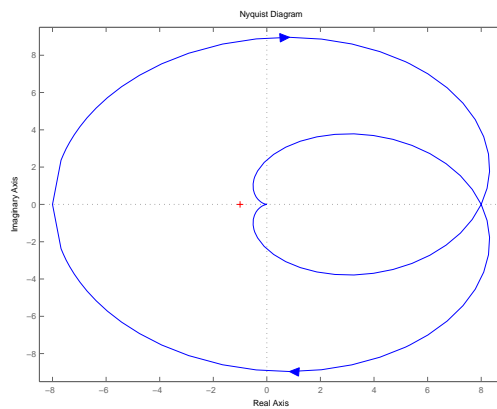


Figure 2

9.62 The MATLAB code for this experiment is below.

We are given a feedback system with the loop transfer function

$$L(s) = \frac{K(s+1)}{s^4 + 5s^3 + 6s^2 + 2s - 8}$$

This $L(s)$ is representative of a conditionally stable system in that for it to be stable, K must lie inside a certain range of values.

The characteristic equation of the system is

$$s^4 + 5s^3 + 6s^2 + (K+2)s + (K-8) = 0$$

Constructing the Routh array:

$$\begin{array}{ccc}
 s^4 & 1 & 6 \quad K - 8 \\
 s^3 & 5 & K + 2 \quad 0 \\
 s^2 & 5.6 - 0.2K & K - 8 \quad 0 \\
 s^1 & \frac{51.2 + 0.2K - 0.2K^2}{5.6 - 0.2K} & 0 \\
 s^0 & K - 8 & 0
 \end{array}$$

From this array we can make the following deductions:

1. For stability, $K > 8$, which follows from the last entry of the first column of array coefficients. For $K = 8$, the system is on the verge of instability. From the fourth row, it follows that this occurs for the last row: s^0 .

2. From the fourth row of the array, it also follows that K must satisfy the condition

$$51.2 + 0.2K - 0.2K^2 > 0$$

for the system to be stable. That is,

$$(K - 16.508)(K + 15.508) < 0$$

from which it follows that for stability

$$K < 16.508$$

If this condition is satisfied, then $5.6 - 0.2K > 0$.

When $K = 16.508$ the system is again on the verge of instability. From the third row of the Routh array, this occurs when

$$(5.6 - 0.2K)s^2 + (K - 8) = 0$$

or

$$s = \pm j \sqrt{\frac{8.508}{2.302}} = \pm j1.923$$

Accordingly, we may state that the feedback system is conditionally stable provided that K satisfies the condition

$$8 < K < 16.508$$

The result is confirmed using the following constructions:

- (a) Root locus. Figure 1 shows the root locus of the system. Using the command `rlocfind`, we obtain the following values of K :

$$K_{\min} = 8$$

$$K_{\max} = 16.5$$

For $K < 8$, the system has a single closed-loop pole in the right-half plane. For

$K > 16.5$, the complex closed-loop poles of the system move to the right-half plane. Hence for stability, we must have $8 < K < 16.5$.

(b) Bode diagram. Figure 2(a) shows the Bode diagram of $L(s)$ for

$$\begin{aligned} K &= \frac{K_{\max} + K_{\min}}{2} \\ &= \frac{16.5 + 8}{2} \\ &= 12.25 \end{aligned}$$

From this figure we obtain the following stability margins:

Gain margin = -3.701 dB; phase-crossover frequency = 0 rad/s

Phase margin = 14.58°, gain-crossover frequency = 1.54 rad/s

Figure 2(b) shows the Bode diagram for $K = 8$. From this second Bode diagram we see that the gain margin is zero, and the system is therefore on the verge of instability. Figure 2(c) shows the Bode diagram for $K = 16.508$. From this third Bode diagram we see that the phase margin is zero, and the feedback system is again on the verge of instability.

(c) Nyquist diagram. Figure 3(a) shows the Nyquist diagram of $L(j\omega)$ for $K = 12.25$, which lies between $K_{\min} = 8$ and $K_{\max} = 16.5$. The critical point $(-1,0)$ is not encircled for this value of K and the system is therefore stable. Figure 3(b) shows the Nyquist locus of $L(j\omega)$ for $K = 8$. The locus of Fig. 3(b) passes through the critical point $(-1,0)$ exactly and the system is therefore on the verge of instability. Figure 3(c) shows the Nyquist locus for $K < 8$, namely $K = 5$. The locus of Fig. 3(c) encircles the critical point $(-1,0)$ and the system is again unstable. Figure 3(d) shows the Nyquist locus for $K > 16.508$, namely, $K = 20$. The locus of Fig. 3(d) encircles the critical point $(-1,0)$ and the system is therefore unstable. Finally, Fig. 3(e) shows the Nyquist locus for $K = 16.508$. Here again we see that the locus passes through the critical point $(-1,0)$ exactly, and so the system is on the verge of instability.

Care has to be exercised in the interpretation of these Nyquist loci, hence the reason for the use of shading.

%Solution to Problem 9.62

```
figure(1); clf;
K = 1;
num = K*[1 1];
den = [1 5 6 2 -8];
rlocus(num,den);
Kmin = rlocfind(num,den)
Kmax = rlocfind(num,den)
```

```
figure(2); clf;
Kmid = (Kmin+Kmax)/2;
num = Kmid*[1 1];
margin(num,den)
```

```
figure(3); clf;
nyquist(num,den)
```

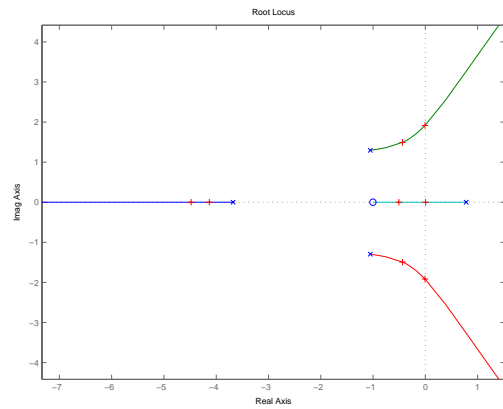


Figure 1

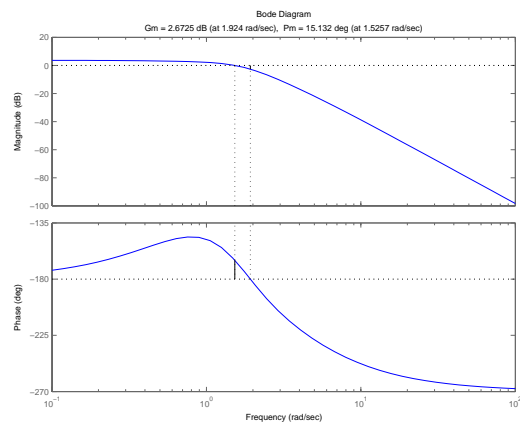


Figure 2

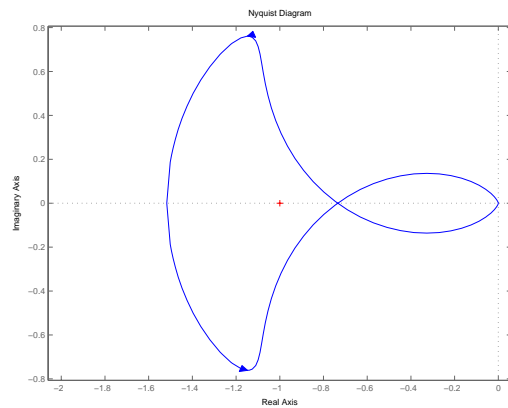


Figure 3

%Solution to Problem 9.63

```
figure(1); clf;
K = 1;
num1 = 0.5*K*[1 2];
den1 = [1 16 48 0];
subplot(3,1,1)
rlocus(num1,den1)
```

%Make figure 9.22

```
K = 1;
num2 = K;
den2 = [1 1 0];
subplot(3,1,2)
rlocus(num2,den2)
```

%Make figure 9.28

```
K = 1;
num3 = 0.5*K*[1 2];
den3 = [1 8 -48 0];
subplot(3,1,3)
rlocus(num3,den3)
```

```
figure(2); clf;
subplot(2,2,1)
nyquist(num1,den1)
subplot(2,2,3)
nyquist(num2,den2)
subplot(2,2,4)
nyquist(num3,den3)
```

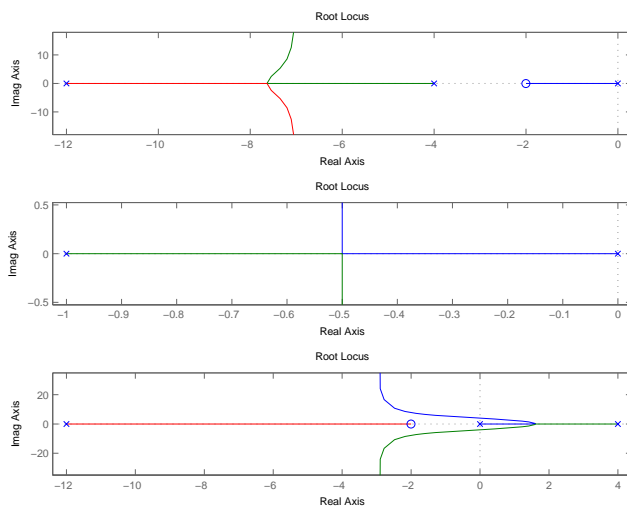


Figure 1

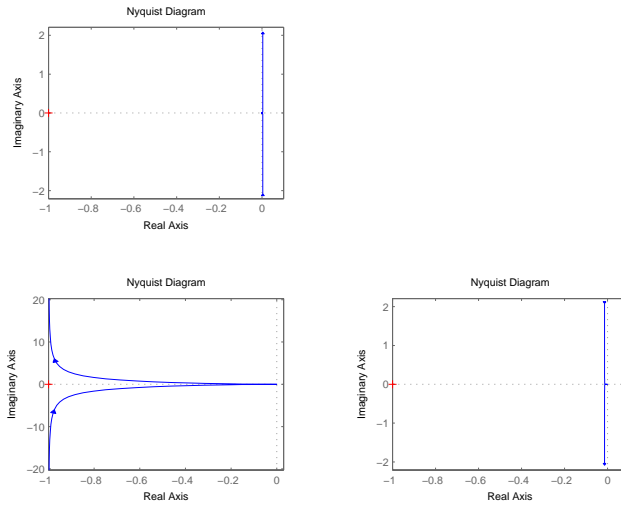


Figure 2

9.64 We are given a unity feedback system with loop transfer function

$$\begin{aligned} L(s) &= \frac{K}{s(s+1)} \\ &= \frac{K}{s^2 + s} \end{aligned} \quad (1)$$

The closed-loop transfer function of the system is

$$\begin{aligned} T(s) &= \frac{L(s)}{1 + L(s)} \\ &= \frac{K}{s^2 + s + K} \end{aligned} \quad (2)$$

(a) With $K = 1$, $T(s)$ takes the value

$$T(s) = \frac{1}{s^2 + s + 1} \quad (3)$$

In general, the transfer function of a second-order system is described as

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta_n\omega_n s + \omega_n^2} \quad (4)$$

Comparing Eqs. (3) and (4):

$$\omega_n = 1$$

$$\zeta_n = 0.5$$

The closed-loop poles of the uncompensated system with $K = 1$ are located at

$$s = -\frac{1}{2} \pm j\sqrt{\frac{3}{2}}.$$

Figure 1 shows the root locus of the uncompensated system.

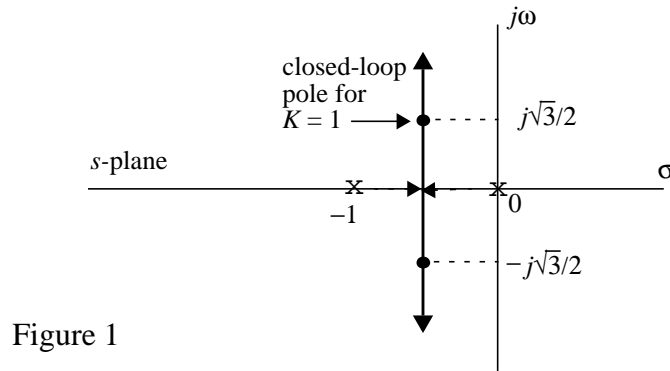


Figure 1

- (b) The feedback system is to be compensated so as to produce a dominant pair of closed-loop poles with $\omega_n = 2$ and $\zeta_n = 0.5$. That is, the damping factor is left unchanged but the natural frequency is doubled. This set of specifications is equivalent to pole locations at $s = -1 \pm j\sqrt{3}$, as indicated in Fig. 2, shown below.

It is clear from the root locus of Fig. 1 that this requirement cannot be satisfied by a change in the gain K of the uncompensated loop transfer function $L(s)$. Rather, we may have to use a phase-lead compensator as indicated in the problem statement.

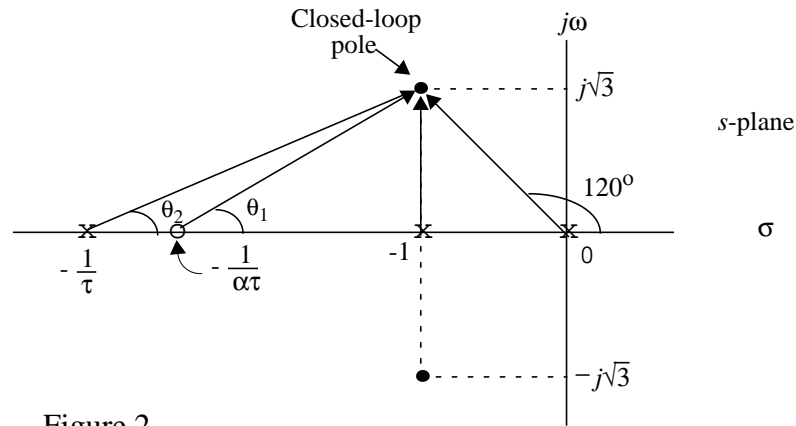


Figure 2

We may restate the design requirement:

- Design a phase-lead compensated system so that the resulting root locus has a dominant pair of closed-loop poles at $s = -1 \pm j\sqrt{3}$ as indicated in Fig. 2.

For this to happen, the angle criterion of the root locus must be satisfied at $s = -1 \pm j\sqrt{3}$. With the open-loop poles of the uncompensated system at $s = 0$ and $s = -1$, we readily see from Fig. 2 that the sum of contributions of these two poles to the angle criterion is

$$-120^\circ - 90^\circ = -210^\circ$$

We therefore require a phase advance of 30° to satisfy the angle criterion.

Figure 2 also includes the pole-zero pattern of the phase-lead compensator, where the angles θ_1 and θ_2 are respectively defined by

$$\theta_1 = \tan^{-1} \left(\frac{\sqrt{3}}{\frac{1}{\alpha\tau} - 1} \right) \quad (5)$$

$$\theta_2 = \tan^{-1} \left(\frac{\sqrt{3}}{\frac{1}{\tau} - 1} \right) \quad (6)$$

where the parameters α and τ pertain to the compensator.

To realize a phase advance of 30° , we require

$$\theta_1 - \theta_2 = 30^\circ$$

or

$$\theta_1 = 30^\circ + \theta_2$$

Taking the tangents of both sides of this equation:

$$\begin{aligned} \tan \theta_1 &= \frac{\tan 30^\circ + \tan \theta_2}{1 - \tan 30^\circ \tan \theta_2} \\ &= \frac{\frac{1}{\sqrt{3}} + \tan \theta_2}{1 - \frac{1}{\sqrt{3}} \tan \theta_2} \end{aligned} \quad (7)$$

Using Eqs. (5) and (6) in (7) and rearranging terms, we find that with $\alpha > 1$ the time constant τ is constrained by the equation

$$\tau^2 - 0.95\tau + 0.025 = 0$$

Solving this equation for τ :

$$\tau = 0.923 \quad \text{or} \quad \tau = 0.027$$

The solution $\tau = 0.923$ corresponds to a compensator whose transfer function has a zero to the right of the desired dominant poles and a pole on their left. On the other hand, for the solution $\tau = 0.027$ both the pole and the zero of the compensator lie to the left of the dominant closed-loop poles, which conforms to the picture portrayed in Fig. 2. So we choose $\tau = 0.027$, as suggested in the problem statement.

(c) The loop transfer function of the compensated feedback system for $\alpha = 10$ is

$$\begin{aligned} L_c(s) &= G_c(s)L(s) \\ &= \frac{\alpha\tau s + 1}{\tau s + 1} \frac{K}{s(s + 1)} \end{aligned}$$

$$= \frac{K(0.27s + 1)}{s(s + 1)(0.027s + 1)}$$

Figure 3(a) shows a complete root locus of $L(s)$. An expanded version of the root locus around the origin is shown in Fig. 3(b). Using the RLOC FIND command of MATLAB, the point $-1 + j\sqrt{3}$ is located on the root locus. The value of K corresponding to this closed-loop pole is 3.846.

%Solution to Problem 9.64

```
figure(1); clf;
K = 1;
t = 0.027;
num = K*[10*t 1];
den = [t 1+t 1 0];
rlocus(num,den)
```

```
figure(2); clf;
rlocus(num,den)
axis([-5 1 -4 4])
K = rlocfind(num,den)
```

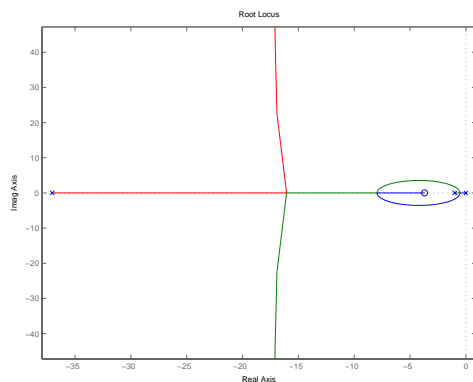


Figure 1

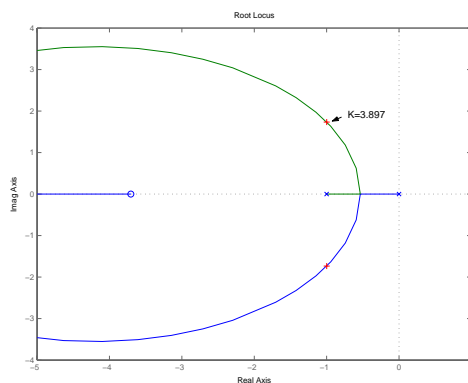


Figure 2

9.65 The loop transfer function of the compensated feedback system is

$$\begin{aligned} L(s) &= G(s)H(s) \\ &= \frac{10K(\alpha\tau s + 1)}{s(0.2s + 1)(\tau s + 1)}, \quad \alpha < 1 \end{aligned} \quad (1)$$

We start the system design with the requirement for a steady-state error of 0.1 to a ramp input of unit slope. The system under study is a type-1 system; see Problem 9.47. The velocity error constant of such a system is given by

$$K_v = \lim_{s \rightarrow 0} sG(s)H(s)$$

Hence,

$$\begin{aligned} K_v &= \lim_{s \rightarrow 0} \frac{10K(\alpha\tau s + 1)}{s(0.25s + 1)(\tau s + 1)} \\ &= 10K \end{aligned}$$

From the definition of K_v :

$$\frac{1}{K_v} = 0.1 \quad \text{or} \quad K_v = 10$$

Hence, $K = 1$.

Next, we use the prescribed value of percentage overshoot to calculate the minimum permissible phase margin. We do this in two steps:

1. We use the relation between percentage overshoot P.O. and damping ratio ζ (see Problem 9.55)

$$\text{P.O.} = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} \quad (2)$$

with P.O. = 10% in response to a step input, the use of Eq. (2) yields

$$\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{1}{\pi} \log_e 10 = 0.7329$$

Hence, solving for ζ :

$$\zeta = 0.5911$$

We may thus set $\zeta = 0.6$.

2. We use the relation between phase margin ϕ_m and damping ratio ζ (see Problem 9.55)

$$\phi_m = \tan^{-1} \left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}} \right)$$

Using the value $\zeta = 0.6$:

$$\begin{aligned}
\phi_m &= \tan^{-1} \left(\frac{1.2}{\sqrt{\sqrt{4 \times 0.6^4 + 1} - 2 \times 0.6^2}} \right) \\
&= \tan^{-1}(1.6767) \\
&= 59.19^\circ
\end{aligned}$$

Hence, we may set $\phi_m = 59^\circ$.

The next step in the design is to calculate the minimum value of the gain-crossover frequency ω_g . To do this, we use the requirement that the 5% settling time of the step response should be less than $2s$. We may again proceed in two stages:

1. Using the formula for the settling time T_{setting} with $\delta = 5\% = 0.05$ (see Problem 9.55)

$$e^{-\zeta \omega_n T_s} = 0.05$$

Solving this equation for ω_n with $T_{\text{setting}} = 2$ seconds and $\zeta = 0.6$:

$$\begin{aligned}
\omega_n &= \frac{\log_e 20}{2 \times 0.6} \\
&= 2.5 \text{ rad/s}
\end{aligned}$$

2. Using the formula for the gain-crossover frequency (see Problem 9.58):

$$\begin{aligned}
\omega_g &= \omega_n \sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2} \\
&= 2.5 \sqrt{\sqrt{4 \times 0.6^4 + 1} - 2 \times 0.6^2} \\
&= 1.79 \text{ rad/s}
\end{aligned}$$

So we may set $\omega_g = 1.8 \text{ rad/s}$.

The stage is now set for calculating the parameters of the compensating network. Figures 1(a) and 1(b) show the uncompensated loop gain response and loop phase response, respectively. From Fig. 1(a) we see that at $\omega = \omega_g = 1.8 \text{ rad/s}$, the uncompensated loop gain is 14.4 dB, which corresponds to the numerical value 5.23. Hence,

$$\alpha = \frac{1}{5.23} = 0.19$$

We also note that the corner frequency $1/\alpha\tau$ should coincide with $\omega_g/10$, and so we may set

$$\frac{\omega_g}{10} = \frac{1}{\alpha\tau}$$

With $\omega_g = 1.8 \text{ rad/s}$ and $\alpha = 0.19$, we thus get

$$\begin{aligned}
\tau &= \frac{10}{\alpha\omega_g} \\
&= \frac{10}{0.19 \times 1.8} \\
&= 29 \text{ seconds}
\end{aligned}$$

The transfer function of the compensator is therefore

$$\begin{aligned}
H(s) &= K \left(\frac{\alpha\tau s + 1}{\tau s + 1} \right) \\
&= \frac{5.6s + 1}{20s + 1}
\end{aligned}$$

Performance evaluation and Fine Tuning

The compensated loop transfer function is

$$\begin{aligned}
L(s) &= G(s)H(s) \\
&= \frac{10(5.6s + 1)}{s(0.2s + 1)(29s + 1)}
\end{aligned}$$

Figure 2 shows the compensated loop response of the feedback system. According to this figure, the phase margin ϕ_m is 65.46° and the gain-crossover frequency ω_g is 1.823 rad/s, both of which are within the design specifications.

Figure 3 shows the response of the closed-loop system to a step input. The overshoot is less than the prescribed value of 10%. But the 5% settling time is greater than the prescribed value of 2 seconds. We therefore need to fine tune the compensator design.

We propose to move the pole-zero pattern of the compensator's transfer function $H(s)$ away from the $j\omega$ -axis so as to reduce the settling time. Specifically, we modify $H(s)$ as

$$H(s) = \frac{5.2s + 1}{20s + 1}$$

Hence the modified loop transfer function is

$$L(s) = \frac{10(5.2s + 1)}{s(0.2s)(20s + 1)} \quad (3)$$

Figures 4 and 5 show the modified loop frequency response and closed-loop step response of the feedback system, respectively. From these figures we observe the following:

1. The phase margin is 61.3° and the gain-crossover frequency is 2.359, both of which are within the design specifications.
2. The percentage overshoot of the step response is just under 10% and the 5 percent settling time is just under 2 seconds.

We can therefore say that the modified transfer function of Eq. (3) does indeed meet all the prescribed design specifications.

Design of the Compensator

Figure 6 shows an operational amplifier circuit for implementing the phase-lead compensator, characterized by the transfer function

$$H(s) = \frac{5.2s + 1}{20s + 1}$$

The transfer function of this circuit is

$$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R_3} \left(\frac{R_1 C_1 s + 1}{(R_1 + R_2) C_1 s + 1} \right)$$

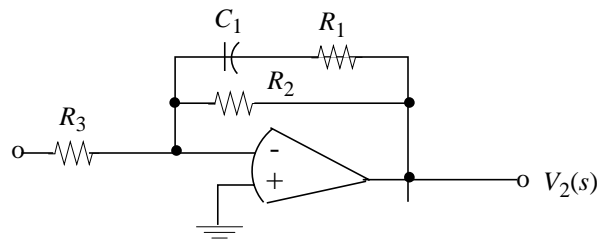


Figure 6

Hence,

$$\frac{R_2}{R_3} = 1$$

$$R_1 C_1 = 5.2$$

$$(R_1 + R_2) C_1 = 20$$

Choose $C_1 = 10 \mu\text{F}$. We may then solve these three equations for the resistive elements of the circuit:

$$R_1 = 0.52 M\Omega$$

$$R_2 = 1.48 M\Omega$$

$$R_3 = 1.48 M\Omega$$

With this, the system design is completed.

%Solution to Problem 9.65

%Uncompensated

```
figure(1); clf;  
num = [10];  
den = [0.2 1 0];  
margin(num,den)
```

%Compensated

```
figure(2); clf;  
a = 5.6;  
b = 0.2;  
num = [10*a 10];  
den = [29*b (b+29) 1 0];  
margin(num,den)
```

%Fine Tuned

```
figure(4); clf;  
a = 5.2;  
b = 0.2;  
num = [10*a 10];  
den = [20*b (b+20) 1 0];  
margin(num,den)
```

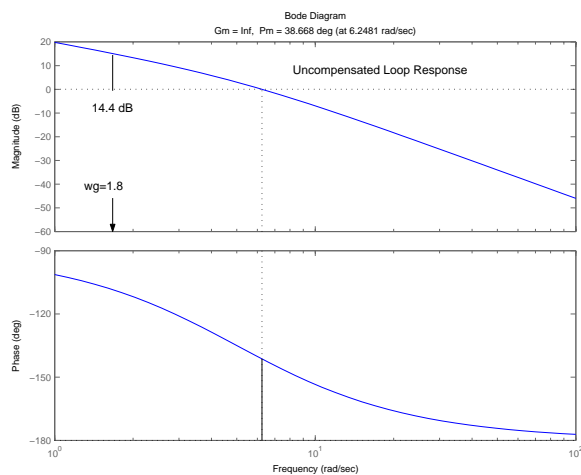


Figure 1

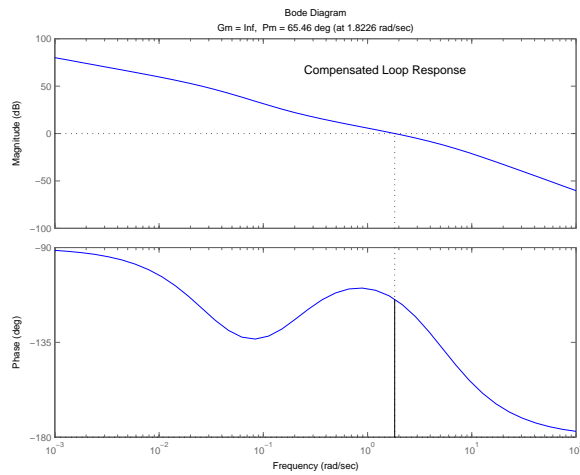


Figure 2

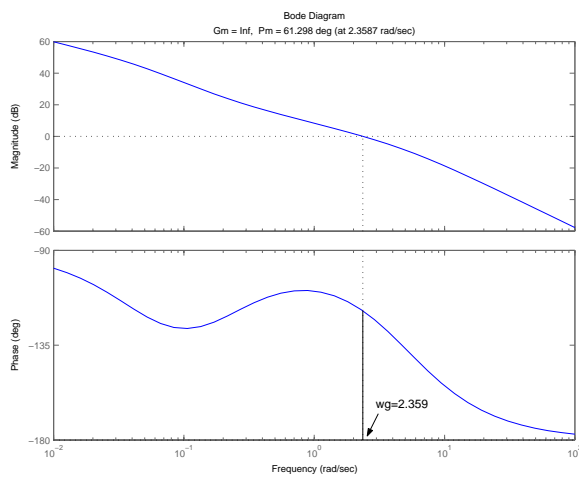


Figure 3

9.66 The MATLAB code is attached.

(a) We are given the loop transfer function

$$L(s) = \frac{K(s-1)}{(s+1)(s^2+s+1)}$$

This is the same as the $L(s)$ considered in Problem 9.61, except for the fact that this time the gain factor K is negative. Let

$$K = -K' \quad K' \geq 0$$

We may then rewrite $L(s)$ as

$$L(s) = \frac{K'(-s+1)}{(s+1)(s^2+s+1)}$$

Now we can proceed in exactly the same way as before, treating K' as a positive gain factor.

Figure 1 shows the root locus of $L(s)$. Using the command `rlocfind`, we find that the system is on the verge of instability (i.e., the closed-loop poles reside exactly on the $j\omega$ -axis) when $K' = 1$ or $K = -1$.

We may verify this special value of K' using the Routh-Hurwitz criterion. The characteristic equation of the system is

$$s^3 + 2s^2 + (2 - K')s + (1 + K') = 0$$

Hence, constructing the Routh array:

$$\begin{array}{ccc} s^3 & 1 & 2 - K' \\ s^2 & 2 & 1 + K' \\ s^1 & \frac{3(1 - K')}{2} & 0 \\ s^0 & 1 + K' & 0 \end{array}$$

The third element of the first column of array coefficients is zero when $K' = 1$ or $K = -1$. This occurs when

$$2s^2 + (1 + K') = 0$$

or

$$s = \pm j$$

as shown on the root locus in Fig. 1.

(b) We are given the loop transfer function

$$L(s) = \frac{K(s+1)}{s^4 + 5s^3 + 6s^2 + 2s - 8}$$

which is the same as the $L(s)$ considered in Problem 9.62, except this time K is negative.

Set $K = -K'$, where K' is nonnegative. Hence we may rewrite $L(s)$ as

$$L(s) = \frac{-K'(s+1)}{s^4 + 5s^3 + 6s^2 + 2s - 8}$$

and thus proceed in the same way as before.

Figure 3 shows the root locus of $L(s)$ for six different values of K , namely, $K = -1, -2, -3$ and $K = -0.1, -0.2, -0.3$. We see that for all these values of K or the corresponding values of K' , we see that the root locus has a pole in the right-half plane. Indeed, the feedback system described here will always have a closed-loop pole in the right-half plane. Hence, the system is unstable for all $K < 0$.

%Solution to Problem 9.66

```
figure(1); clf;
num = -1*[1 -1];
den = [1 2 2 1];
rlocus(num,den)

figure(2)
K = [-3 -2 -1 -0.4 -0.2 -0.1];
for n = 1:length(K),
    num = K(n)*[1 1];
    den = [1 5 6 2 -8];
    subplot(2,3,n)
    rlocus(num,den)
    title(['K = ' num2str(K(n))])
end
```

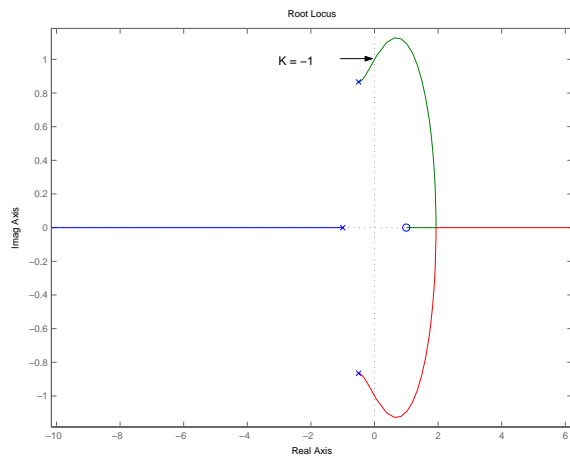


Figure 1

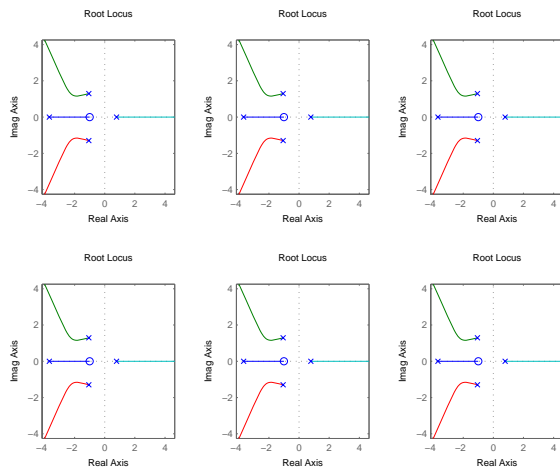


Figure 2