ECE340 Spring 2011
Homework-6 Solutions

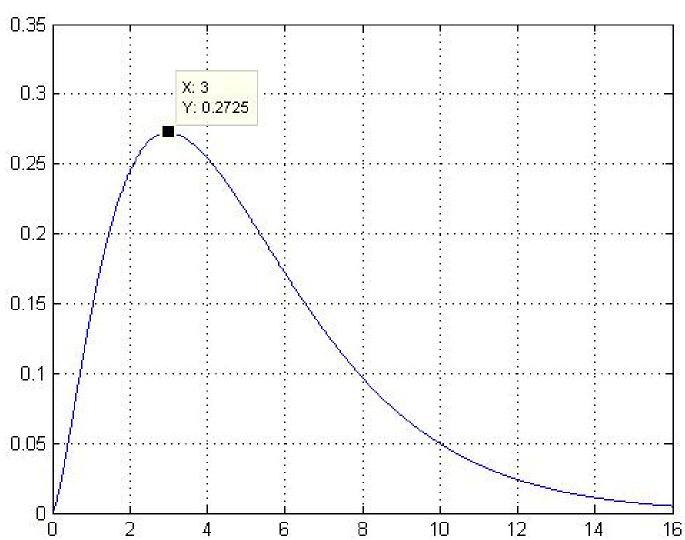
Problems: 2-6.6, 2-7.1, 2-7.4, 2-8.2, 2-8.3, 2-9.2

2-6.6

a) According the book on Page 84, X^2 has a Chi-square distribution, so the mean value of a chi-square random variable X^2 is $\overline{X^2} = E[X^2] =$ n = 5
b) $(\sigma_{X^2})^2 = 2n = 10$
c) If we write a Matlab code as the following to plot the probability distribution function for $f_X(x^2)$:

```
clc
clear all
close all
n = 5;
f_x2 = zeros(40000,1);
x = zeros(40000,1);
for i = 1:1:40000
    x(i) = (i-1)/10000;
    f_x2(i) = (((x(i)^2).^(n/2-1))/(2^(n/2))*gamma(n/2))*exp(-(x(i)^2)./2);
end
plot(x.^2,f_x2)
grid on
```

We can see the plot of the pdf for x^2 with a maximum value at $x^2 = 3$



The most probable value of x^2 should be 3, since at 3, the pdf have the maximum value. By hand, we can firstly find the first derivative of the pdf and set it to equal to zero and we should find the same answer 3:

$$f_X(x^2) = \frac{(x^2)^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\exp\left(-\frac{x^2}{2}\right), \qquad x^2 \geq 0$$

$$= \frac{(x^2)^{\frac{3}{2}}}{2^{\frac{5}{2}}\Gamma(5/2)} \exp\left(-\frac{x^2}{2}\right)$$

So,

$$\frac{df_X(x^2)}{dx^2} = \frac{\frac{3}{2}(x^2)^{\frac{1}{2}}}{2^{\frac{5}{2}}\Gamma\left(\frac{5}{2}\right)} \exp\left(-\frac{x^2}{2}\right) + \frac{(x^2)^{\frac{3}{2}}}{2^{\frac{5}{2}}\Gamma\left(\frac{5}{2}\right)}\left(-\frac{1}{2}\right) \exp\left(-\frac{x^2}{2}\right)$$

$$= [\frac{\frac{3}{2}(x^2)^{\frac{1}{2}}}{2^{\frac{5}{2}}\Gamma\left(\frac{5}{2}\right)} + \frac{(x^2)^{\frac{3}{2}}}{2^{\frac{5}{2}}\Gamma\left(\frac{5}{2}\right)}\left(-\frac{1}{2}\right)]\exp\left(-\frac{x^2}{2}\right)$$

Now, if we set the first derivative equal to 0 (respect to x^2), we find the value of x^2 is 3 as well.

So we know the most probable value of x^2 is 3.

2-7.1

a) Since Θ is uniformly distributed over a range of 0 to 2π, we have the following pdf for Θ:

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \leq \theta \leq 2\pi \\ 0 & elsewhere \end{cases}$$

Now another random variable X is related to Θ by

$$X = cos\Theta$$

Then

$$F_X(x) = P\{X \leq x\} = P\{cos\Theta \leq x\} = P\{\cos^{-1}\mathrm{x} \leq \Theta \leq 2\pi - \cos^{-1}\mathrm{x}\} = \int_{\cos^{-1}\mathrm{x}}^{2\pi-\cos^{-1}\mathrm{x}} f_\Theta(\theta)d\Theta$$

$$= \int_{\cos^{-1}\mathrm{x}}^{2\pi-\cos^{-1}\mathrm{x}} \frac{1}{2\pi}d\Theta = 1 - \frac{\cos^{-1}\mathrm{x}}{\pi}$$

Then we know the pdf of X is the following

$$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 0 - \frac{1}{\pi}(\cos^{-1}\mathrm{x})' & -1 < x < 1 \\ 0 & otherwise \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & -1 < x < 1 \\ 0 & otherwise \end{cases}$$

b) Find the mean value of X.

$$\bar{X} = E[X] = \int_{-1}^{1} xf_X(x)dx = \int_{-1}^{1} \frac{x}{\pi\sqrt{1-x^2}}dx = 0$$

c) Find the variance of X

$$\sigma_X{}^2 = E[X^2] - E[X]^2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_{-1}^{1} x^2 f_X(x)dx$$

$$= \int_{-1}^{1} x^2 \frac{1}{\pi\sqrt{1-x^2}}dx = \frac{1}{2\pi}\left(\sin^{-1}x - x\sqrt{1-x^2}\right)|_{-1}^{1} = \frac{1}{2}$$

d) Find the probability that X > 0.5

$$P\{X > 0.5\} = \int_{0.5}^{1} f_X(x)\,dx = \int_{0.5}^{1} f_X(x)\,dx = \int_{0.5}^{1} \frac{1}{\pi\sqrt{1-x^2}}\,dx = \frac{1}{\pi}\sin^{-1} x\,\Big|_{0.5}^{1} = \frac{1}{3}$$

2-7.4

We assume that the probability distribution function of the sum of the life times of the four bulbs is modeled as an Erlang pdf with $k = 4$. Therefore we have:

$$f_k(\tau) = \left(\frac{1}{\bar{\tau}}\right)^k \left(\frac{1}{(k-1)!}\right)\tau^{k-1}e^{-\tau/\bar{\tau}} =$$
$$= \left(\frac{1}{2000}\right)^4 \left(\frac{1}{3!}\right)\tau^3 e^{-\frac{\tau}{2000}}$$

Then we have:

$$F_X(x) = \int_0^{\tau} = \left(\frac{1}{2000}\right)^4 \left(\frac{1}{3!}\right) x^3 e^{-\frac{\tau}{2000}}\,dx$$
$$= 1 - \frac{1}{6}e^{-\tau/2000}\left[\left(\frac{\tau}{2000}\right)^3 + 3\left(\frac{\tau}{2000}\right)^2 + 6\left(\frac{\tau}{2000}\right) + 6\right]$$

a)

$$E\{\tau_1 + \tau_2 + \tau_3 + \tau_4\} = 4\left(\frac{1}{2000}\right)^{-1} = 8000$$

b)

$$P\{X \geq 10000\} = \frac{1}{6}e^{-5}[5^3 + 3 \times 25 + 6 \times 5 + 6]$$
$$= 0.265$$

c)

$$P\{X < 4000\} = 1 - \frac{1}{6}e^{-2}[2^3 + 3 \times 4 + 6 \times 2 + 6]$$
$$= 0.865$$

2-8.2

a) The event M is defined as $0 \leq \theta \leq \pi/2$, then it is equivalent as the event $M'\colon 0 \leq x \leq 1$

$$f_X(x|M) = f_X(x|M') = \frac{dF_X(x|M')}{dx} = \frac{1}{P\{0 \leq X \leq 1\}}\frac{dF_X(x)}{dx} = \frac{f_X(x)}{\frac{1}{2}} = 2f_X(x)$$

$$= \begin{cases} \dfrac{2}{\pi\sqrt{1-x^2}}, & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

b) $E[X|M] = \int_{-\infty}^{\infty} x\, f_X(x|M)dx = \int_0^1 x\frac{2}{\pi\sqrt{1-x^2}}dx = -\frac{2}{\pi}\int_0^1 d\sqrt{1-x^2} = \frac{2}{\pi}$

2-8.3

Referring to the example in the textbook, the aiming problem in XY coordinate system can be described by a Rayleigh distribution function. Considering the position at which the laser weapon strikes the target is a random variable having an X-component and a Y-component. Then the distribution of the random variable r, where $R = \sqrt{X^2 + Y^2}$ is described by:

$$F_R(r) = 1 - e^{-\frac{r^2}{2\sigma^2}}$$

If one tenth of the shots miss the target entirely, we have:

$$1 - F(1) = \frac{1}{10} = e^{-\frac{r^2}{2\sigma^2}}$$

which yields to $\sigma = 0.2330$.

a)

$$F_R(r|R \le 0.5) = \frac{P\{R \le r, R \le 0.5\}}{P\{R \le 0.5\}}$$

$$= \frac{P\{R \le r\}}{P\{R \le 0.5\}} = \frac{1 - e^{-\frac{r^2}{2(0.232)^2}}}{1 - e^{-\frac{0.5^2}{2(0.232)^2}}}$$

$$= \frac{1}{0.9}\left(1 - e^{-\frac{r^2}{2(0.232)^2}}\right)$$

$$P\{r \le 0.1|r < 0.5\} = \frac{1}{0.9}\left(1 - e^{-\frac{0.1^2}{2(0.232)^2}}\right) = 0.0978$$

b)

$$P\{0.5 < r \le 0.8|r \ge 0.5\} = \frac{F(0.8) - F(0.5)}{1 - F(0.5)}$$

$$= \frac{e^{-\frac{0.5^2}{2(0.232)^2}} - e^{-\frac{0.8^2}{2(0.232)^2}}}{e^{-\frac{0.5^2}{2(0.232)^2}}} = 0.9725$$

2-9.2

a) Referring to the conditional probabilities we have:

$$F(r|R > r_0) = \frac{P\{R \le r, R > r_0\}}{P\{R > r_0\}} = \frac{P\{r_0 < R \le r\}}{P\{R > r_0\}} = \frac{F(r) - F(r_0)}{1 - F(r_0)}$$

Then the conditional pdf is:

$$f(r|R > r_0) = \frac{\partial}{\partial r}F(r|R > r_0) = \frac{f(r)}{1 - F(r_0)} = \frac{\frac{r}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}}}{1 - [1 - e^{-\frac{{r_0}^2}{2\sigma^2}}]}, \quad r > 0.$$

The mean value of random variable R is calculated as $\overline{R} = \sqrt{\frac{\pi}{2}}\sigma = \sqrt{\frac{\pi}{2}}$, then we have: $\sigma = 1$.

$$\therefore f(r|R > r_0) = \begin{cases} r^2 e^{-0.5(r^2 - r_0^2)} & r \geq r_0 \\ 0 & r \leq r_0 \end{cases}$$

b)

$$E\{R|R > 0.5\} = \int_{0.5}^{\infty} r^2 e^{-0.5(r^2 - 0.5^2)}\, dr$$

If we let $u = r$ and $dv = re^{-r^2/2}$, then one have $v = e^{-r^2/2}$. Therefore:

$$\begin{aligned} E\{R|R > 0.5\,\} &= e^{0.125}\left[-r\, e^{-\frac{r^2}{2}} \Big|_{0.5}^{\infty} - \int_{0.5}^{\infty} -e^{-\frac{r^2}{2}} dr \right] \\ &= 0.5 + e^{0.125} \int_{0.5}^{\infty} e^{-\frac{r^2}{2}} dr\; 0.5 + e^{0.125}\sqrt{2\pi}\; \Phi(-0.5) \\ &= 0.5 + 2.84[1 - \Phi(-0.5)] = 1.376 \end{aligned}$$