ECE 340: PROBABILISTIC METHODS IN ENGINEERING

SOLUTIONS TO HOMEWORK #8

4.54

a) Solution:

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$
$$= -a\int_{-\infty}^{-a} f_X(x)dx + \int_{-a}^{a} x f_X(x)dx + a\int_{a}^{\infty} f_X(x)dx$$
$$= -aF_X(-a) + \int_{-a}^{a} x f_X(x)dx + a(1 - F_X(a))$$

$$E[Y^2] = a^2 F_X(-a) + \int_{-a}^{a} x^2 f_X(x)dx + a^2(1 - F_X(a))$$

$$VAR[Y] = E[Y^2] - E[Y]^2$$

d) Solution:

First, we need to obtain the pdf of *X*. We do so by first obtaining the cdf of *X*. Notice that $X=U^3$ and *U* is uniformly distributed in [-1,1]. Since *U* is a random variable, *X* will also be a random variable.

Also, $P\{X \le x\} = P\{U^3 \le x\} = P\{U \le \sqrt[3]{x}\}$. Note that the range of *X* is also [-1,1].

Now, for $x < -1$,

$$F_X(x) = P\{X \le x\} = P\{U \le \sqrt[3]{x}\} = 0 \quad .$$

For $-1 \le x \le 1$,

$$F_X(x) = P\{X \le x\} = P\{U^3 \le x\} = P\{U \le \sqrt[3]{x}\} = F_U(\sqrt[3]{x})$$

We know that the cdf of *U* in the range between -1 and 1 is
$F_U(u) = \frac{1}{2}u + \frac{1}{2}$. Thus,
$F_X(x) = F_U(\sqrt[3]{x}) = \frac{1}{2}\sqrt[3]{x} + \frac{1}{2}$, for $-1 \le x \le 1$.

And, for $x>1$, $F_X(x) = 1$.

Now we have completed calculating the cdf of *X*.
To obtain the pdf of *X*, we differentiate $F_X(x)$ with respect to *x*. For the range of [-1, 1],

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{1}{6}x^{-\frac{2}{3}}$$

Note that the $f_X(x)$=0 otherwise.

From part a) and by using a=1/2 we have:

$$\begin{aligned}
E[Y] &= -\frac{1}{2}F_X\left(-\frac{1}{2}\right) + \int_{-\frac{1}{2}}^{\frac{1}{2}} x f_X(x)dx + \frac{1}{2}\left(1 - F_X\left(\frac{1}{2}\right)\right) \\
&= -\frac{1}{2}F_U\left(\sqrt[3]{-\frac{1}{2}}\right) + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{\left(6 * x^{\frac{2}{3}}\right)}dx + \frac{1}{2}\left(1 - F_U\left(\sqrt[3]{\frac{1}{2}}\right)\right) \\
&= -\frac{1}{2} * \left(\frac{1}{2} * \sqrt[3]{-\frac{1}{2}} + \frac{1}{2}\right) + \frac{1}{6} * \frac{3}{4} * x^{\frac{4}{3}}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{2} * \left(1 - \left(\frac{1}{2} * \sqrt[3]{\frac{1}{2}} + \frac{1}{2}\right)\right) \\
&= -0.0516 + \frac{1}{8} * (0) + 0.0516 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
E[Y^2] &= \frac{1}{4}F_X\left(-\frac{1}{2}\right) + \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 f_X(x)dx + \frac{1}{4}\left(1 - F_X\left(\frac{1}{2}\right)\right) \\
&= \frac{1}{4}F_U\left(\sqrt[3]{-\frac{1}{2}}\right) + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^2}{\left(6 * x^{\frac{2}{3}}\right)}dx + \frac{1}{4}\left(1 - F_U\left(\sqrt[3]{\frac{1}{2}}\right)\right) \\
&= 0.02578 + \frac{1}{6}\left(\frac{3}{7} * x^{\frac{7}{3}}\Big|_{-\frac{1}{2}}^{\frac{1}{2}}\right) + 0.02578 \\
&= 0.05156 + \frac{1}{14}(0.198 + 0.198) \\
&= 0.0798
\end{aligned}$$

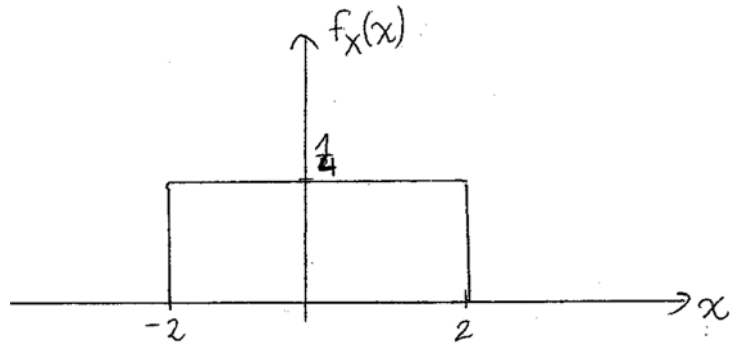
$$VAR[Y] = E[Y^2] - E[Y]^2 = 0.0798$$

4.59 Solution

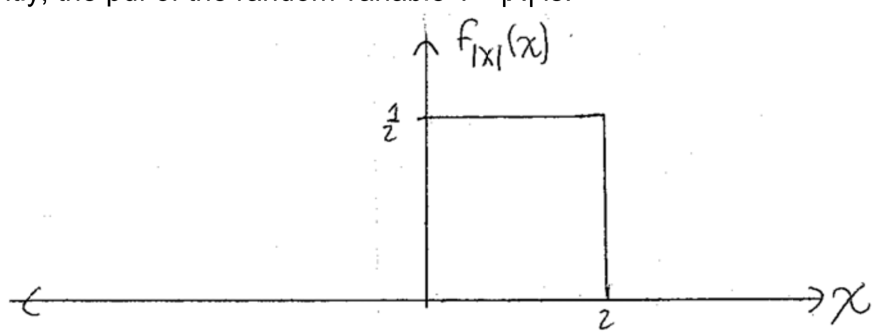
Assuming that we define the random variable Y=|X|.

$$P\{|X| > x\} = P\{Y > x\} = 1 - P\{Y \le x\} = 1 - F_Y(x)$$

Since *X* is uniform between [-2,2], its pdf is:



The random variable Y=|X| is then uniformly distributed in the range from 0 to 2. Consequently, the pdf of the random variable Y= |*X*| is:



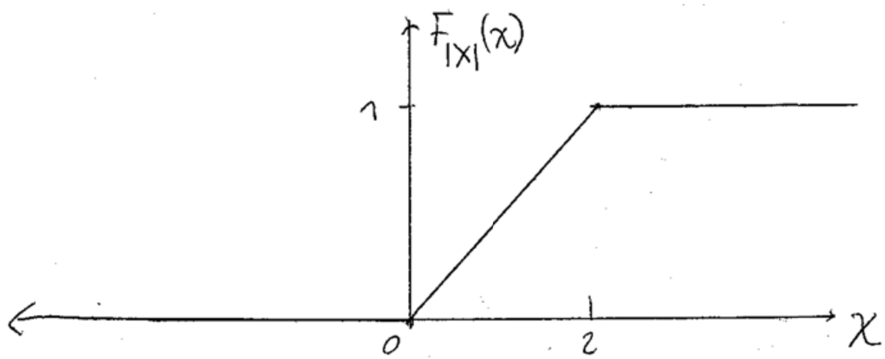
From that, we can get $F_{|X|}(x)$ by integrating the pdf of Y or |X|

$$F_{|X|}(x) = \int_{-\infty}^{x} f_{|X|}(t)\,dt = \begin{cases} 0, & if\ x < 0 \\ \frac{x}{2}, & if\ 0 \le x < 2 \\ 1, & if\ x \ge 2 \end{cases}$$

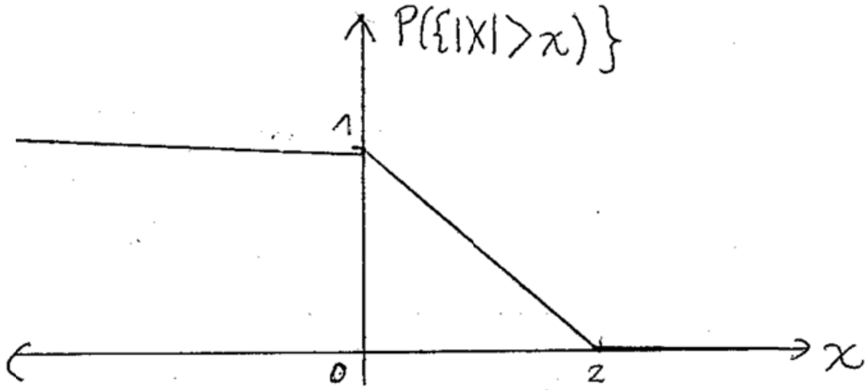
So

$$P\{|X| > x\} = 1 - F_Y(x) = 1 - F_{|X|}(x) = \begin{cases} 1, & if\ x < 0 \\ 1 - \frac{x}{2}, & if\ 0 \le x < 2 \\ 0, & if\ x \ge 2 \end{cases}$$

and we can draw the CDF of |X| in the following figure



$P\{(|X|>x)\}= 1-P\{(|X|<=x)\}=1-F_{|X|}(x)$ is shown here:



4.62

a) Since X is exponential with parameter λ, its cdf is:
$F_X(x)=1-e^{-\lambda x}$ $x\geq 0$

$P\{X \leq \pi(r)\} = 1 - e^{-\lambda\pi(r)} = \frac{r}{100}$

$$1 - \frac{r}{100} = e^{-\lambda\pi(r)}$$

So,

$$\pi(r) = -\frac{1}{\lambda}\ln\left(1 - \frac{r}{100}\right)$$
$$= \frac{1}{\lambda}\ln\left(\frac{100}{100 - r}\right)$$

We can compute the percentiles by plugging in r in the above expression, so:

$$\pi(90) = \frac{2.3026}{\lambda}$$

$$\pi(95) = \frac{2.9957}{\lambda}$$
$$\pi(99) = \frac{4.6025}{\lambda}$$

b) Since *X* is Gaussian random variable with *m*=0 and σ

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{r^2}{2\sigma^2}} dr = \phi(\frac{x-m}{\sigma}) = \phi(\frac{x}{\sigma})$$

Note that we also have $\phi(x) = 1 - Q(x)$.

Thus we can solve the value of x with the help of **table 4.2,** or use the **'qfuncinv'** in matlab**.** We know $F_X(\pi(90)) = 1 - Q(\pi(90)/\sigma) = 90/100$ then

$$Q(\frac{\pi(90)}{\sigma}) = 1 - 0.90 = 0.1$$

By using 'qfuncinv(0.1)' in matlab, we have

$\frac{\pi(90)}{\sigma} = 1.2816$ so $\pi(90) = 1.2816\sigma$

Similarly, we obtain

$$\pi(95) = 1.6449\sigma$$

$$\pi(99) = 2.3263\sigma$$

4.63 *X* is Gaussian r.v. with *m*=5 and σ^2=16

a) $P\{X > 4\} = Q\left(\frac{4-m}{\sigma}\right) = Q\left(-\frac{1}{4}\right) = 1 - Q\left(\frac{1}{4}\right) \approx 0.5987$

You can use either matlab function 'qfunc(.25)' or Talbe 4.2, when using the table, using linear interpolation is recommended. For example, in table 4.2 you have Q(.2) ≈0.421, and Q(.3) ≈0.382, but Q(.25) is not given, then you can do the following linear interpolation

Q(.25) ≈ 0.421+(0.382-0.421)*[(0.25-0.2)/(0.3-0.2)]=0.4015.)

$P\{X \geq 7\} = Q\left(\frac{7-m}{\sigma}\right) = Q\left(\frac{2}{4}\right) = Q(0.5) \approx 0.3085$

$$P\{6.72 < X < 10.16\} = P\{X < 10.16\} - P\{X < 6.72\} = \Phi\left(\frac{10.16 - 5}{4}\right) - \Phi\left(\frac{6.72 - 5}{4}\right)$$
$$= \left(1 - Q(1.29)\right) - \left(1 - Q(0.43)\right) \approx 0.2351$$
$$P\{2 < X < 7\} = \Phi\left(\frac{7 - 5}{4}\right) - \Phi\left(\frac{2 - 5}{4}\right) = \left(1 - Q(0.5)\right) - \left(1 - Q(-0.75)\right) \approx 0.4648$$
$$P\{6 \leq X \leq 8\} = \Phi\left(\frac{8 - 5}{4}\right) - \Phi\left(\frac{6 - 5}{4}\right) = \left(1 - Q(0.75)\right) - \left(1 - Q(0.25)\right) \approx 0.1747$$

b)

$$P\{X < a\} = 0.8869$$
$$0.8869 = 1 - P\{X \geq a\}$$
$$0.8869 = 1 - Q\left(\frac{a - m}{\sigma}\right)$$
$$Q\left(\frac{a - m}{\sigma}\right) = 0.1131$$

Using ‘qfuncinv(0.1131)’,

$$\frac{a - m}{\sigma} \approx 1.2102$$
$$a \approx 1.2102 * 4 + 5 = 9.8408$$

c)

$$P\{X > b\} = 0.11131$$
$$Q\left(\frac{b - m}{\sigma}\right) = 0.1131$$

Using ‘qfuncinv(0.1131)’,

$$\frac{b - m}{\sigma} = 1.2102$$
$$b = 9.8408$$

d)

$$P\{13 < X \leq c\} = 0.0123$$
$$0.0123 = P\{X > 13\} - P\{X > c\}$$
$$0.0123 = Q\left(\frac{13 - 5}{4}\right) - Q\left(\frac{c - 5}{4}\right)$$
$$0.0123 = 0.0228 - Q\left(\frac{c - 5}{4}\right)$$
$$Q\left(\frac{c - 5}{4}\right) = 0.0228 - 0.0123 = 0.0105$$
$$\frac{c - 5}{4} = 2.3080$$
$$c = 14.2319$$

4.64

$$Q(-x) = \frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{\frac{t^2}{2}} dt = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x} e^{-\frac{t^2}{2}} dt$$
$$= 1 - \frac{1}{\sqrt{2\pi}}\int_{\infty}^{x} e^{-\frac{t^2}{2}}(-dt') \qquad where\ t' = -t$$
$$= 1 - \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-\frac{t^2}{2}} dt' = 1 - Q(x)$$

4.66

a) $P\{X \le m\} = 1 - P\{X > m\} = \; 1 - Q\left(\frac{m-m}{\sigma}\right) = 1 - Q(0) = \frac{1}{2}$

b) $P\{|X-m| \le k\sigma\} = P\{-k\sigma \le X - m \le k\sigma\} = P\{-k\sigma + m \le X \le k\sigma + m\}$

$$= P\{X \le k\sigma + m\} - P\{X \le -k\sigma + m\}$$
$$= 1 - P\{X > k\sigma + m\} - (1 - P\{X > -k\sigma + m\})$$
$$= Q(-k) - Q(k)$$
$$= 1 - Q(k) - Q(k) = 1 - 2Q(k)$$

$\boldsymbol{k}$	1	2	3	4	5	6
$\boldsymbol{P\{\lvert X-m\rvert \le k\sigma\}}$	0.6827	0.9545	0.9973	0.9999	≈1	≈1

c) Find *k* such that,

$$Q(k) = P\{X > m + k\sigma\} = 10^{-j}$$

Using table 4.3:

$\boldsymbol{j}$	1	2	3	4	5	6
$\boldsymbol{k}$	1.2815	2.3263	3.0902	3.7190	4.2649	4.7535