ECE340 spring 2011
Homework-7 Solutions

Problems: 3-1.2, 3-1.3(b), 3-2.1, 3-2.2, 3-2.3, 3-2.4, 3-3.1, 3-3.3

3-1.2

a) In this problem our joint pdf is:

$$f_{XY}(x,y)=\begin{cases}kxy & 0\le x\le 1, 0\le y\le 1\\ 0 & elsewhere\end{cases}$$

To determine the value k, we need the following condition:
$\iint f_{XY}(x,y)dxdy=1.$

Or, set

$$k\int_0^1 x\,dx\int_0^1 y\,dy=1$$

and from this condition above we find

$$k=4$$

The joint pdf is therefore:

$$f_{XY}(x,y)=\begin{cases}4xy & 0\le x\le 1, 0\le y\le 1\\ 0 & elsewhere\end{cases}$$

b) To determine the cdf $F_{XY}(x,y)$, we use the following equation:

$$F_{XY}(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)\,dudv$$

We find the cdf as the following:

$$F_{XY}(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)\,dudv$$

$$=\begin{cases}\int_0^x\int_0^y 4uv\,dudv & 0\le x\le 1, 0\le y\le 1\\ \begin{cases}\int_0^1\int_0^x 4uv\,dudv & 0\le y\le 1, x>1\\ \int_0^x\int_0^1 4uv\,dudv & 0\le x\le 1, y>1\end{cases}\\ \begin{cases}1 & x>1, y>1\\ 0 & otherwise\end{cases}\end{cases}$$

$$=\begin{cases}x^2y^2 & 0\le x\le 1, 0\le y\le 1\\ \begin{cases}y^2 & 0\le y\le 1, x>1\\ x^2 & 0\le x\le 1, y>1\end{cases}\\ \begin{cases}1 & x>1, y>1\\ 0 & otherwise\end{cases}\end{cases}$$

c) We formulate the probability as the following:

$$P\left\{X\le\frac{1}{2}, Y>\frac{1}{2}\right\}=F_{XY}\left(\frac{1}{2},1\right)-F_{XY}\left(\frac{1}{2},\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2 1^2-\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2=\frac{3}{16}=0.1875$$

d) The marginal density function $f_X(x)$ is

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy & 0 \le x \le 1 \\ 0 & otherwise \end{cases} = \begin{cases} \int_{0}^{1} 4xy\,dy & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} 2x & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

3-1.3 (b)

For the random variables of Problem **3-1.2**, the expected value of the product of X and Y is

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x,y)\,dxdy = \int_{0}^{1} \int_{0}^{1} xy4xy\,dxdy = \frac{4}{9}$$

3-2.1

a) Since the signal X is Rayleigh distributed, we know the pdf of X as

$$f_X(x) = \begin{cases} \frac{x}{\sigma_X^2} \exp\left(-\frac{x^2}{2\sigma_X^2}\right) & x \ge 0 \\ 0 & otherwise \end{cases}$$

Since the mean value of X is 10, according to the page 80, we have

$$\bar{X} = \sqrt{\frac{\pi}{2}}\sigma_X = 10$$

So, we know that

$$\sigma_X = 10\sqrt{\frac{2}{\pi}}, \quad \sigma_X^2 = \frac{200}{\pi}$$

The pdf of signal X is

$$f_X(x) = \begin{cases} \frac{x}{200/\pi} \exp\left(-\frac{x^2}{400/\pi}\right) & x \ge 0 \\ 0 & otherwise \end{cases}$$

Now, the noise N is uniformly distributed, it is straightforward to find the pdf of N as

$$f_N(n) = \begin{cases} \frac{1}{12} & -6 \le n \le 6 \\ 0 & otherwise \end{cases}$$

We also know that X and N are statistically independent and we have the observation

$$Y = X + N$$

Since we know that the conditional pdf

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}$$

Now, since N = Y – X,

$$f_{Y|X}(y|x) = f_N(n = y - x) = f_N(y - x)$$

Thus, we can write the conditional pdf $f_{X|Y}(x|y)$ as the following:

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} = \frac{f_N(y-x) f_X(x)}{f_Y(y)} = \frac{f_N(y-x) f_X(x)}{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx} = \frac{f_N(y-x) f_X(x)}{\int_{-\infty}^{\infty} f_N(y-x) f_X(x)\,dx}$$

Note that in our case,

$$-6 \le n \le 6$$
$$-6 \le y - x \le 6$$

$$-6 \leq x - y \leq 6$$
$$-6 + y \leq x \leq 6 + y$$

The limits for x should be reflected in the integration $\int_{-\infty}^{\infty} f_N(y-x) f_X(x)\, dx$, so we can write the conditional pdf $f_{X|Y}(x|y)$ as the following:

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f_N(y-x) f_X(x)}{\int_{-6+y}^{6+y} f_N(y-x) f_X(x)\, dx} & -6+y \geq 0 \\ \dfrac{f_N(y-x) f_X(x)}{\int_{0}^{6+y} f_N(y-x) f_X(x)\, dx} & -6+y < 0 \end{cases}$$

$$= \begin{cases} \dfrac{(1/12) f_X(x)}{(1/12)\int_{-6+y}^{6+y} f_X(x)\, dx} & -6+y \geq 0 \\ \dfrac{(1/12) f_X(x)}{(1/12)\int_{0}^{6+y} f_X(x)\, dx} & -6+y < 0 \end{cases}$$

$$= \begin{cases} \dfrac{f_X(x)}{\int_{-6+y}^{6+y} f_X(x)\, dx} & -6+y \geq 0 \\ \dfrac{f_X(x)}{\int_{0}^{6+y} f_X(x)\, dx} & -6+y < 0 \end{cases}$$

For the case when *y* =0,

$$f_{X|Y}(x|y=0) = \frac{f_X(x)}{\int_0^{6+y} f_X(x)\, dx} = \frac{f_X(x)}{\int_0^{6} f_X(x)\, dx}$$

$$= \begin{cases} \dfrac{\frac{x}{200/\pi} \exp\left(-\frac{x^2}{400/\pi}\right)}{\int_0^6 \frac{x}{200/\pi} \exp\left(-\frac{x^2}{400/\pi}\right) dx} & x \geq 0 \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \dfrac{\frac{x}{200/\pi} \exp\left(-\frac{x^2}{400/\pi}\right)}{0.2463} & x \geq 0 \\ 0 & otherwise \end{cases}$$

Now,

$$f_{X|Y}(x|y=0) = \begin{cases} \dfrac{x}{15.68} \exp\left(-\dfrac{x^2}{400/\pi}\right) & x \geq 0 \\ 0 & otherwise \end{cases}$$

For the case when *y* =6,

$$f_{X|Y}(x|y=6) = \frac{f_X(x)}{\int_{-6+y}^{6+y} f_X(x)\, dx} = \frac{f_X(x)}{\int_0^{12} f_X(x)\, dx} = \begin{cases} \dfrac{x}{43.12} \exp\left(-\dfrac{x^2}{400/\pi}\right) & x \geq 0 \\ 0 & otherwise \end{cases}$$

For the case when y =12,

$$f_{X|Y}(x|y=12) = \frac{f_X(x)}{\int_{-6+y}^{6+y} f_X(x)\, dx} = \frac{f_X(x)}{\int_6^{18} f_X(x)\, dx} = \begin{cases} \dfrac{x}{42.99} \exp\left(-\dfrac{x^2}{400/\pi}\right) & x \geq 0 \\ 0 & otherwise \end{cases}$$

b) If the observation yields a value of y = 12, we know from Part 1 that

$$f_{X|Y}(x|y=12) = \begin{cases} \frac{x}{42.99}\exp\left(-\frac{x^2}{400/\pi}\right) & x \geq 0 \\ 0 & otherwise \end{cases}$$

According to the book on page 127, the value of x for which $f_{X|Y}(x|y=12)$ is a maximum is a good estimate for the true value of *X*. So the value of *x* can be determined by equating the derivative (with respect to *x*) of the conditional pdf to zero:

$$\frac{df_{X|Y}(x|y=12)}{dx} = 0$$

Or,

$$\frac{d\frac{x}{42.99}\exp\left(-\frac{x^2}{400/\pi}\right)}{dx} = 0$$

We find the solution as

$$x = 10\sqrt{\frac{2}{\pi}} = \sigma_x$$

3-2.2

a) The joint pdf of X and Y in problem **3-1.2** is

$$f_{XY}(x,y) = \begin{cases} 4xy & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & elsewhere \end{cases}$$

To find the conditional pdf $f_{X|Y}(x|y)$ we do the following.
Since

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

We need to find $f_Y(y)$ first:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^1 4xy\,dx = 2y, \qquad 0 \leq y \leq 1$$

So now,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} \frac{4xy}{2y} & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & elsewhere \end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} 2x & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & elsewhere \end{cases}$$

b) By noticing that we can change x to y, and y to x in the joint pdf of X and Y, we get the result as

$$f_{Y|X}(y|x) = \begin{cases} 2y & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & elsewhere \end{cases}$$

3-2.3

A dc signal X has a uniform distribution over the range of -5V to +5V, then the pdf for r.v. X is

$$f_X(x) = \begin{cases} \frac{1}{10} & -5 \leq x \leq 5 \\ 0 & otherwise \end{cases}$$

Since the noise N is a Gaussian distributed with zero mean and variance of $2V^2$, we know the pdf for r.v. N is

$$f_N(n) = \frac{1}{\sqrt{2\pi}\sigma_N} exp\left[\frac{-n^2}{2\sigma_N{}^2}\right], \quad -\infty < n < \infty$$

Where $\sigma_N{}^2 = 2$.

a) Now, our observation is Y = X + N. The conditional pdf of the signal x given the value of measurement y $f_{X|Y}(x|y)$ is

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} = \frac{f_N(y-x)f_X(x)}{f_Y(y)} = \frac{f_N(y-x)f_X(x)}{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}$$

$$= \frac{f_N(y-x)f_X(x)}{\int_{-\infty}^{\infty} f_N(y-x)f_X(x)\,dx}$$

$$= \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi}\sigma_N} exp\left[\frac{-(y-x)^2}{2\sigma_N{}^2}\right](1/10)}{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_N} exp\left[\frac{-(y-x)^2}{2\sigma_N{}^2}\right](1/10)\,dx} & -5 \le x \le 5 \\ 0 & elsewhere \end{cases}$$

So,

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{exp\left[\frac{-(x-y)^2}{2\sigma_N{}^2}\right]}{\int_{-5}^{5} exp\left[\frac{-(x-y)^2}{2\sigma_N{}^2}\right]dx} & -5 \le x \le 5 \\ 0 & elsewhere \end{cases}$$

b) When we have a measurement value 6 (X + N = Y = 6),

$$f_{X|Y}(x|y=6) = \begin{cases} \dfrac{exp\left[\frac{-(x-6)^2}{2\sigma_N{}^2}\right]}{\int_{-5}^{5} exp\left[\frac{-(x-6)^2}{2\sigma_N{}^2}\right]dx} & -5 \le x \le 5 \\ 0 & elsewhere \end{cases}$$

The best estimate of x is 5 (and not 6; sketch and verify this), because we have a constraint that $-5 \le x \le 5$.

c) Same as part b) the best estimate of x is 5 V (also sketch and verify this).

3-2.4

a) The observation Y = X +N, where X is the random signal and N is the noise, we know the signal and the noise are independent. From the results we found in problem 3-2.1, we know that $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$,

where the joint pdf of X and Y are given as the following:

$$f_{XY}(x,y) = Kexp[-(x^2+y^2+4xy)] \qquad all\ x\ and\ y$$

Now we can write

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{Kexp[-(x^2+y^2+4xy)]}{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}$$

$$= \frac{Kexp[-(x^2+y^2+4xy)]}{\int_{-\infty}^{\infty} Kexp[-(x^2+y^2+4xy)]\,dx}$$

$$= \frac{exp[-(x^2+y^2+4xy)]}{\int_{-\infty}^{\infty} exp[-(x^2+y^2+4xy)]\,dx}$$

Now, our focus is to find the best estimate for X as a function of the observation Y = y, the best estimate is found by equation the following derivative to zero:

$$\frac{df_{X|Y}(x|y)}{dx} = 0$$

which is equivalent to solve x in the following equation

$$\frac{d(exp[-(x^2 + y^2 + 4xy)])}{dx} = 0$$

Since the denominator of $\frac{f_{XY}(x,y)}{f_Y(y)}$ has no x terms (whatever the value it is after the integration).

Now our problem is simplified as finding the x in the following equation

$$\frac{d(exp[-(x^2 + y^2 + 4xy)])}{dx} = 0$$

$$-(2x + 4y)\exp[-(x^2 + y^2 + 4xy)] = 0$$

So, given the observation Y = y, we have the best estimate of x such that x satisfies

$2x + 4y = 0.$

Or,

$$\hat{x} = -2y$$

So, $\hat{x} = -2y$ is our best estimate of X given observation Y = y.

b) Now if the observed value of Y is y=3, the best estimate of X is -6.

3-3.1

To see whether two random variables are statistically independent, we need to check if the following is true:

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

If two random variables are statistically independent, we can use the following to compute E[XY]:

$$E[XY] = E[X]E[Y]$$

a)

$$f_{XY}(x,y) = \begin{cases} kx/y & 0 \le x \le 1, 1 \le y \le 2 \\ 0 & elsewhere \end{cases}$$

We use the following to find $f_X(x)$ and $f_Y(y)$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

In this case

$$f_X(x) = \int_1^2 kx/ydy = \begin{cases} kln2 \cdot x & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}$$

Where $k = \frac{2}{ln2}$, if found by doing the following:

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

So,

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}$$

Similarly we find:

$$f_Y(y) = \begin{cases} \frac{1}{ln2 \cdot y} & 1 \le x \le 2 \\ 0 & elsewhere \end{cases}$$

Now, we know

$$f_{XY}(x,y) = \begin{cases} \dfrac{kx}{y} & 0 \le x \le 1, 1 \le y \le 2 \\ 0 & elsewhere \end{cases}$$

$$= \begin{cases} \dfrac{2x}{ln2 \cdot y} & 0 \le x \le 1, 1 \le y \le 2 \\ 0 & elsewhere \end{cases} = f_X(x)f_Y(y)$$

So, the two random variables X and Y are statistically independent and

$$E[XY] = E[X]E[Y] = \int_{-\infty}^{\infty} x f_X(x)dx \int_{-\infty}^{\infty} y f_Y(y)dy = \frac{2}{3ln2} = 0.9618$$

b) Following the same steps shown in part a), we find that

$$f_X(x) = \begin{cases} \dfrac{3}{2}x^2 + \dfrac{1}{2} & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}$$

$$f_Y(y) = \begin{cases} \dfrac{3}{2}y^2 + \dfrac{1}{2} & 0 \le y \le 1 \\ 0 & elsewhere \end{cases}$$

$$f_{XY}(x,y) = \begin{cases} \dfrac{3}{2}(x^2 + y^2) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & elsewhere \end{cases}$$

$$f_{XY}(x,y) \ne f_X(x)f_Y(y)$$

So, X and Y are not statistically independent and

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x,y)\, dxdy = \int_0^1 \int_0^1 xy \frac{3}{2}(x^2 + y^2)\ dxdy = \frac{3}{8}$$

c) Following the same steps shown in part a), we find that

$$f_X(x) = \begin{cases} \dfrac{2}{7}(x+3) & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}$$

$$f_Y(y) = \begin{cases} \dfrac{2}{5}(y+2) & 0 \le y \le 1 \\ 0 & elsewhere \end{cases}$$

$$f_{XY}(x,y) = \begin{cases} \dfrac{4}{35}(xy + 2x + 3y + 6) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & elsewhere \end{cases}$$

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

So, X and Y are statistically independent and

$$E[XY] = E[X]E[Y] = \int_{-\infty}^{\infty} x f_X(x)dx \int_{-\infty}^{\infty} y f_Y(y)dy = 0.2794$$

3-3.3

Two independent r.v.s X and Y are both Gaussian distributed:

$$\bar{X} = 1, \qquad \sigma_X^2 = 1$$
$$\bar{Y} = 2, \qquad \sigma_Y^2 = 4$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} = exp\left[\frac{-(x-\bar{X})^2}{2\sigma_X^2}\right], \quad -\infty < x < \infty$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} = exp\left[\frac{-(x-\bar{Y})^2}{2\sigma_Y^2}\right], \quad -\infty < y < \infty$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} = exp\left[\frac{-(x-1)^2}{2}\right], \quad -\infty < x < \infty$$

$$f_Y(y) = \frac{1}{2\sqrt{2\pi}} = exp\left[\frac{-(y-2)^2}{8}\right], \quad -\infty < y < \infty$$

Now

$$P\{XY > 0\} = P\{X > 0\}P\{Y > 0\} + P\{X < 0\}P\{Y < 0\}$$

$$P\{X > 0\} = \int_0^\infty f_X(x)dx = 1 - F_X(0) = 1 - P\{X \leq 0\}$$

$$P\{Y > 0\} = \int_0^\infty f_Y(y)dy = 1 - F_Y(0) = 1 - P\{Y \leq 0\}$$

So,

$$\begin{aligned} P\{X > 0\} = 1 - F_X(0) &= 1 - \left[1 - Q\left(\frac{x-\bar{X}}{\sigma_X}\right)\right] \\ &= Q(-1) \\ &= 1 - Q(1) \\ &= 1 - 0.1587 \\ &= 0.8413 \end{aligned}$$

$$\begin{aligned} P\{Y > 0\} = 1 - F_Y(0) &= 1 - \left[1 - Q\left(\frac{y-\bar{Y}}{\sigma_Y}\right)\right] \\ &= Q(-1) \\ &= 0.8413 \end{aligned}$$

$$P\{XY > 0\} = P\{X > 0\}P\{Y > 0\} + P\{X < 0\}P\{Y < 0\} = 0.8413^2 + 0.1587^2 = 0.7330$$