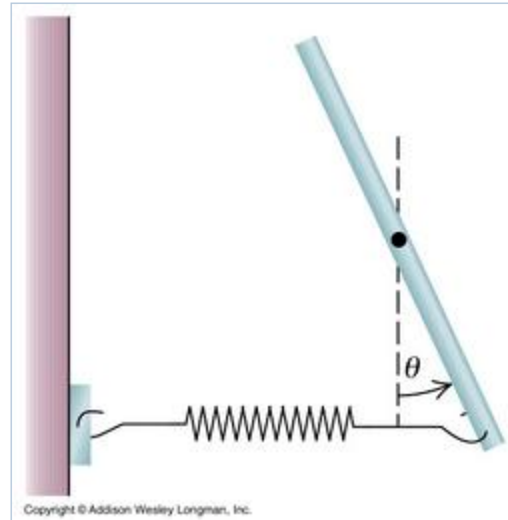
#37 Simple Harmonic Motion Post-class

Due: 11:00am on Monday, November 19, 2012

Note: You will receive no credit for late submissions. To learn more, read your instructor's Grading Policy

A Pivoting Rod on a Spring

A slender, uniform metal rod of mass M and length l is pivoted without friction about an axis through its midpoint and perpendicular to the rod. A horizontal spring, assumed massless and with force constant k, is attached to the lower end of the rod, with the other end of the spring attached to a rigid support.



Part A

We start by analyzing the torques acting on the rod when it is deflected by a small angle θ from the vertical. Consider first the torque due to gravity. Which of the following statements most accurately describes the effect of gravity on the rod?

Choose the best answer.

ANSWER:

- ○ Under the action of gravity alone the rod would move to a horizontal position. But for small deflections from the vertical the torque due to gravity is sufficiently small to be ignored.
- ○ Under the action of gravity alone the rod would move to a vertical position. But for small deflections from the vertical the restoring force due to gravity is sufficiently small to be ignored.
- ◉ There is no torque due to gravity on the rod.

Correct

Assume that the spring is relaxed (exerts no torque on the rod) when the rod is vertical. The rod is displaced by a small angle θ from the vertical.

Part B

Find the torque τ due to the spring. Assume that θ is small enough that the spring remains effectively horizontal and you can approximate $\sin(\theta) \approx \theta$ (and $\cos(\theta) \approx 1$).

Express the torque as a function of θ and other parameters of the problem.

Hint 1. Find the change in spring length

Deflecting the rod will stretch or compress the spring by a length δ. The spring will react with a restoring force given by Hooke's law:

$$F = -k\delta.$$

What is δ?

Express your answer in terms of l and θ.

Hint 1. Relation between displacement and rotation angle

Consider an object at a distance r from the pivot it is rotating about. If it rotates through an angle ϕ, then its displacement Δx is given by

$$\Delta x = r\phi.$$

ANSWER:

$\delta = \frac{l}{2}\theta$

Hint 2. Find the moment arm

The torque τ about a point is defined as the product of the force F acting on a body times the moment arm (perpendicular distance d from the line of action of the force to the center point):

$$\tau = Fd.$$

What is d for the given situation?

Express your answer in terms of quantities given in the problem introduction.

ANSWER:

$d = \frac{l}{2}$

ANSWER:

$\tau = -k\left(\frac{l}{2}\right)^2\theta$

Correct

Since the torque is opposed to the deflection θ and increases linearly with it, the system will undergo angular simple harmonic motion.

Part C

What is the angular frequency ω of oscillations of the rod?

Express the angular frequency in terms of parameters given in the introduction.

Hint 1. How to find the oscillation frequency

ω can be found from the equation of motion by comparing it to the standard form

$$\frac{d^2\theta(t)}{dt^2} = -\omega^2\theta(t).$$

This equation describes any angular simple harmonic motion. Bring your equation of motion into this standard form, and hence extract the expression for ω in terms of the parameters of this specific problem.

Hint 2. Solve the angular equation of motion

The angular equation of motion for this problem relates the change in angular momentum L to the torque τ:

$$\tau = \frac{dL(t)}{dt} = I\frac{d^2\theta(t)}{dt^2},$$

where I is the moment of inertia of the rod about its center.

Solve this equation for $\frac{d^2\theta(t)}{dt^2}$.

You should already have found the expression for the torque in Part B.

Express your answer in terms of k, l, I, and θ.

ANSWER:

$$\frac{d^2\theta(t)}{dt^2} = \frac{-kl^2}{4I}\theta$$

Hint 3. Determine the moment of inertia of the rod

What is the moment of inertia I of a rod of length l about its midpoint?

Express your answer in terms of l and M.

ANSWER:

$$I = \frac{1}{12}Ml^2$$

ANSWER:

$$\omega = \sqrt{\frac{3k}{M}}$$

Correct

Note that if the spring were simply attached to a mass m, or if the mass of the rod were concentrated at its ends, ω would be $\sqrt{k/m}$. The frequency is greater in this case because mass near the pivot point doesn't move as much as the end of the spring. What do you suppose the frequency of oscillation would be if the spring were attached near the pivot point?

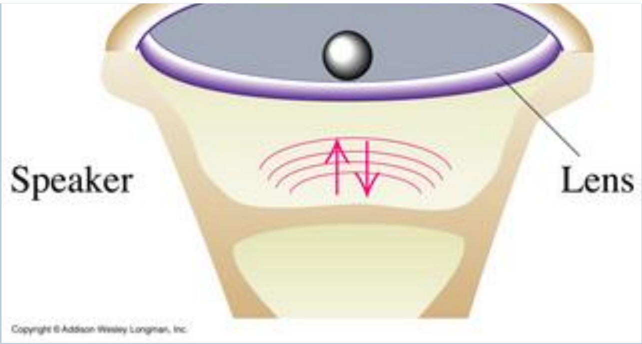
Measuring the Acceleration Due to Gravity with a Speaker

To measure the magnitude of the acceleration due to gravity g in an unorthodox manner, a student places a ball bearing on the concave side of a flexible speaker cone . The speaker cone acts as a simple harmonic oscillator whose amplitude is A and

whose frequency f can be varied. The student can measure both A and f with a strobe light. Take the equation of motion of the oscillator as

$$y(t) = A\cos(\omega t + \phi),$$

where $\omega = 2\pi f$ and the y axis points upward.



Part A

If the ball bearing has mass m, find $N(t)$, the magnitude of the normal force exerted by the speaker cone on the ball bearing as a function of time.

Your result should be in terms of A, either f (or ω), m, g, a phase angle ϕ, and the constant π.

Hint 1. Determine the total force on the ball bearing

What is the net force $\sum F_y$ on the ball bearing?

Your answer should include $N(t)$, the normal force as it varies with time.

ANSWER:

$\sum F_y = ma_y =$ $N(t) - mg$

Hint 2. Find the acceleration of the ball bearing

What is $a(t)$, the acceleration of the ball bearing as a function of time?

Express your answer in terms of given variables. Your answer should not contain N.

Hint 1. How to approach the problem

Acceleration is the second derivative of the displacement:

$$a(t) = \frac{d^2y}{dt^2}.$$

Hint 2. Find $y\,(t)$

What is $y\,(t)$, the vertical displacement as a function of time?

Give your answer in terms of A, f, and ϕ.

ANSWER:

$y\,(t) =$ $A\cos\left(2\pi f t + \phi\right)$

ANSWER:

$a\,(t) =$ $-A\left(2\pi f\right)^2 \cos\left(2\pi f t + \phi\right)$

ANSWER:

$N\,(t) =$ $mg - m\omega^2 A\cos\left(\omega t + \phi\right)$

Correct

Part B

The frequency is slowly increased. Once it passes the critical value f_b, the student hears the ball bounce. There is now enough information to calculate g. What is g?

Express the magnitude of the acceleration due to gravity in terms of f_b and A.

Hint 1. Determine the force on the ball bearing when it loses contact

What is the net force $\sum F_y$ on the ball bearing the instant it loses contact with the speaker and is thrown into the air?

Hint 1. Specify the possible forces

What forces are acting on the ball bearing at that moment?

ANSWER:

- There are no forces acting on the ball bearing.
- weight only
- the normal force only
- the normal force and weight

ANSWER:

$\sum F_y$ = $-mg$

Hint 2. Find the value of $\cos(2\pi f_b t + \phi)$ when the ball loses contact

In the first part of this problem you obtained an equation for the normal force:

$$N(t) = mg - mA(2\pi f)^2 \cos(2\pi f t + \phi).$$

At the critical value of f_b, the ball just loses contact with the speaker when the speaker is completely convex (i.e., when it is at the top of its oscillation). What is the value of $\cos(2\pi f_b t + \phi)$ at this moment?

ANSWER:

- -1
- 0
- 1

Hint 3. Relation between ω and f

Recall that $\omega = 2\pi f$.

ANSWER:

g = $A(2\pi f_b)^2$

Correct

Matching Initial Position and Velocity of Oscillator

Learning Goal:

Understand how to determine the constants in the general equation for simple harmonic motion, in terms of given initial conditions.

A common problem in physics is to match the particular initial conditions - generally given as an initial position x_0 and velocity v_0 at $t = 0$ - once you have obtained the general solution. You have dealt with this problem in kinematics, where the formula

1. $x(t) = x_0 + v_0 t + \frac{at^2}{2}$

has two arbitrary constants (technically constants of integration that arise when finding the position given that the acceleration is a constant). The constants in this case are the initial position and velocity, so "fitting" the general solution to the initial conditions is very simple.

For simple harmonic motion, it is more difficult to fit the initial conditions, which we take to be

x_0, the position of the oscillator at $t = 0$, and

v_0, the velocity of the oscillator at $t = 0$.

There are two common forms for the general solution for the position of a harmonic oscillator as a function of time t:

2. $x(t) = A\cos(\omega t + \phi)$ and
3. $x(t) = C\cos(\omega t) + S\sin(\omega t)$,

where A, ϕ, C, and S are constants, ω is the oscillation frequency, and t is time.

Although both expressions have two *arbitrary constants*--parameters that can be adjusted to fit the solution to the initial conditions--Equation 3 is much easier to use to accommodate x_0 and v_0. (Equation 2 would be appropriate if the initial conditions were specified as the total energy and the time of the first zero crossing, for example.)

Part A

Find C and S in terms of the initial position and velocity of the oscillator.

Give your answers in terms of x_0, v_0, and ω. Separate your answers with a comma.

Hint 1. The only good way to start

Which of the following procedures would solve this problem in the most straightforward manner?

ANSWER:

- Differentiate $x(t)$ twice to find $a(t)$. Then integrate it twice. Plug in v_0 and x_0 as the constants of integration.
- Differentiate $x(t)$ once to find $v(t)$. Evaluate $x(t=0)$ and $v(t=0)$ and then solve for the desired quantities.
- Dimensional analysis suffices since x_0 and v_0 have different dimensions.
- Use Equation 1. Plug in $a = -kx(t)/m$ where $k/m = \omega^2$.

Hint 2. Using kinematic relationships

Find $v(t)$, the velocity as a function of time from Equation 3.

Hint 1. Derivative of a trig function

From the chain rule of calculus, find the derivative of $\cos(at)$ with respect to time.

ANSWER:

$\frac{d\cos(at)}{dt}$ = $-a\sin(at)$

ANSWER:

$v(t)$ = $-C\omega\sin(\omega t) + S\omega\cos(\omega t)$

Hint 3. Initial position

Now you have general expressions for $x(t)$ and $v(t)$. Find the position at $t = 0$.

ANSWER:

$x(t=0)$ = C

Hint 4. Initial velocity

Find the velocity at time $t=0$.

ANSWER:

$v(t=0)$ = $S\omega$

ANSWER:

C, S = x_{0},\frac{v_{0}}{{\omega}}

Correct

± The Fish Scale

A vertical scale on a spring balance reads from 0 to 185N . The scale has a length of 14.5cm from the 0 to 185N reading. A fish hanging from the bottom of the spring oscillates vertically at a frequency of 2.80Hz .

Part A

Ignoring the mass of the spring, what is the mass m of the fish?

Express your answer in kilograms.

Hint 1. How to approach the problem

Calculate the spring constant for the fish scale, then use this with the angular frequency of the bouncing fish to calculate its mass.

Hint 2. Calculate the spring constant

Calculate the spring constant k for the spring in the fish scale.

Express your answer in newtons per meter.

Hint 1. Using the reading

At rest, the weight of the object will be counteracted by the restoring force in the spring, which can be seen by drawing a force diagram of the fish on the spring. Hence $F=-w=-kx$. Because we know both the maximum weight the scale can show and the length the spring is stretched at that weight, the spring constant can be calculated from this equation.

ANSWER:

k = 1280 $\mathrm{N/m}$

Hint 3. Calculate the angular frequency

Calculate the angular frequency ω for the fish oscillating on the spring.

Express your answer numerically in radians per second.

Hint 1. Relating frequency and angular frequency

Recall that, for a given oscillation, $\omega = 2\pi f$, where ω is the angular frequency and f = 2.80 Hz is the frequency.

ANSWER:

ω = 17.6 $\mathrm{radians/s}$

Hint 4. Formula for the angular frequency of a mass on a spring

An object of mass m on the end of a spring with spring constant k will oscillate with frequency $\omega = \sqrt{k/m}$.

ANSWER:

m = 4.12 kg

Correct

Score Summary:

Your score on this assignment is 106.7%.
You received 42.67 out of a possible total of 40 points.