Homework #1 - Due Friday, June 7

1. In many circumstances, the rate at which an object cools is directly proportional to the difference between the temperature of the object and the temperature of its surroundings (which we assume to be cooler than the object itself). More specifically, if u is the temperature of the object, T_0 is room temperature, and k is a positive constant of proportionality, then

$$\frac{du}{dt} = k(T_0 - u).$$

   This is known as *Newton's Law of Cooling*.

   (a) Verify that $u(t) = T_0 - ae^{-kt}$ satisfies Newton's Law of Cooling for any real number a.

   (b) Determine a given that $u(0) = 100$.

   (c) Suppose that a cup of coffee with an initial temperature of 100°C is placed in a room where the ambient temperature is 70°C. If the initial rate at which the cup of coffee cools is -2°C/min, find k and write an expression for the temperature of the coffee over time $u(t)$. Give units for both!

   (d) Find $\lim_{t\to\infty} u(t)$ for the function that you found in part (c). Use this to sketch a graph of $u(t)$. Interpret this limit in physical terms.

2. For the following, please be sure to label everything clearly!

   (a) Consider the differential equation

$$\dot{y} = 2y^3 + y^2 - 2y - 1,$$

   where $\dot{y}$ means differentiation with respect to time (this was the original notation introduced by Newton). In other words,

$$\dot{y} = \frac{dy}{dt}.$$

   Do the following for the region $t \geq 0$:

      i. Indicate the regions where the solutions are increasing and where they are decreasing.

      ii. Find and sketch the equilibrium solutions, and classify them as asymptotically stable, unstable, or semi-stable.

      iii. Sketch the solution curves corresponding to the initial conditions $y(0) = 0$, $y(0) = 2$, and $u(0) = -2$.

   (b) Sketch the direction field for the differential equation

$$x(y-1) - \frac{dy}{dx} = 0$$

   in all four quadrants. Sketch the solution curves corresponding to $y(0) = 0$, $y(0) = 1$ and $y(0) = 2$.

3. Solve the following differential equations subject to the given initial conditions, if any.

   (a) $\frac{dx}{dt} = x\sec^2 t$, $x(0) = 1$

   (b) $\frac{du}{dx} = ux + x$, $u(2) = 5e^4$

   (c) $\dot{y} = e^t - \frac{2y}{t}$

   (d) $y' - y = xy^2$

4. Sometimes it is possible to solve inseparable, non-linear differential equations via a substitution. Consider the equation

$$\frac{dy}{dx} = (x-y)^2,$$

   which is clearly both inseparable and non-linear, so none of the methods we have learned apply in this situation. However, we can still find the solution by doing something clever.

(a) Define a new funtion by $u(x) = x - y$. Differentiate u with respect to x and solve for $\frac{dy}{dx}$.

(b) Using part (a), make appropriate substitutions to transform the given inseparable differential equation in y to a separable differential equation in u.

(c) Solve the equation in (b). Using the definition in (a), write the solution to the original equation. Verify that it is, in fact, a solution.

(d) Using an appropriately clever substitution, transform the inseparable, non-linear differential equation

$$2xy\frac{dy}{dx} + 2y^2 = 3x - 6$$

into a linear differential equation. You do not need to solve the resulting equation. (Hint: Do you see a function *and* its derivative in the equation?)

(e) Using the substitution $u = \dot{y}$, transform the second-order linear differential equation

$$\ddot{y} = 8\dot{y}\frac{1}{t} + 1$$

into a first-order linear differential equation. Solve the resulting equation and integrate to solve the original equation (your solution will have *two* constants in it). This sort of substitution will become significant later on.