ECE340 Spring 2011

Homework-4 Solutions

Problems: 2-1.1, 2-1.2, 2-2.1, 2-2.2, 2-2.3, 2-3.1, 2-3.2, 2-3.3

2-1.1

a) Every one of the quantities mentioned in the question can be considered as a random variable. Here is the list of these random variables:
   a. X_1 : the forecasted high temperature for July 4^{th}, continuous, [-30, 170] (this interval is arbitrary , since it depends on where the temperature is meant for)
   b. X_2 : the forecasted low temperature for July 4^{th}, continuous, [-80, 120] (also arbitrary)
   c. X_3 : the forecasted humidity for July 4^{th}, continuous, [0%, 100%]
   d. X_4 : the forecasted THI (Temperature Humidity Index) for July 4^{th}
   e. X_5 : the forecasted sunrise time for July 4^{th}, continuous , reasonable interval is arbitrary, since it depends on where you are.
   f. X_6 : the forecasted sunset time for July 4^{th}, continuous , reasonable interval is arbitrary, since it depends on where you are.

b) Every one of the quantities mentioned in the question can be considered as a random variable. Here is the list of these random variables:
   a. X_1 : the number of vehicles per minute is a discrete random variable. Reasonable sample spaces are {0, 1, 2, …, 300}, or {0, 1, 2, …}.
   b. X_2 : average speed, continuous, [0, 70] (mph)
   c. X_3 : the ratio of cars to trucks, continuous, [0, ∞)
   d. X_4 : the average weight, continuous, [2000, 10000]
   e. X_5 : the number of accidents per day is a discrete random variable. Examples of ranges are {0,1,2,3,… 10000} or {0,1,2,3,… }

c) Note: a random variable can have only one value. Thus, all the quantities can be considered as a random variable, in this case though, they can only be discrete random variables.

2-1.2

a) Discrete random variable; a reasonable range of values: the set A = {2, 3, 4, 5, …, 12};
b) Continuous random variable; a reasonable range of values: [0, 14] (there is no fixed answer for the upper bound);
c) Discrete random variable; a reasonable range of values:

the set B={000 000 0000, …, 999 999 9999};

d) Continuous random variable; a reasonable range of values: [50, 300] (both the lower bound and the upper bound are arbitrary, no unique answer)

2-2.1

a) Define the discrete random variable *X* to be the number of heads you observed when you flip 10 coins. Because the probability you see a head when you flip a coin is 0.5, we have

$$P\{X=0\} = \binom{10}{0} \times (0.5^0) \times (0.5^{10-0}) = 9.76 \times 10^{-4}$$

$$P\{X=1\} = \binom{10}{1} \times (0.5^1) \times (0.5^{10-1}) = 0.0098$$

$$P\{X=2\} = \binom{10}{2} \times (0.5^2) \times (0.5^{10-2}) = 0.0439$$

$$\vdots$$

$$\vdots$$

$$P\{X=8\} = \binom{10}{8} \times (0.5^8) \times (0.5^{10-8}) = 0.0439$$

$$P\{X=9\} = \binom{10}{9} \times (0.5^9) \times (0.5^{10-9}) = 0.0098$$

$$P\{X=10\} = \binom{10}{10} \times (0.5^{10}) \times (0.5^{10-10}) = 9.76 \times 10^{-4}$$

Here is a list of the probabilities:

P{ X = 0}=0.0009765625

P{ X = 1}=0.009765625

P{ X = 2}=0.0439453125

P{ X = 3}=0.1171875

P{ X = 4}=0.205078125

P{ X = 5}=0.24609375

P{ X = 6}=0.205078125

P{ X = 7}=0.1171875

P{ X = 8}=0.0439453125

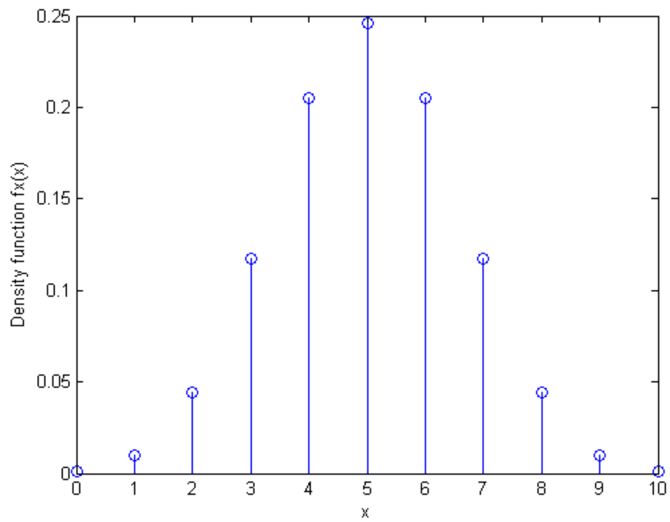
P{ X = 9}=0.009765625

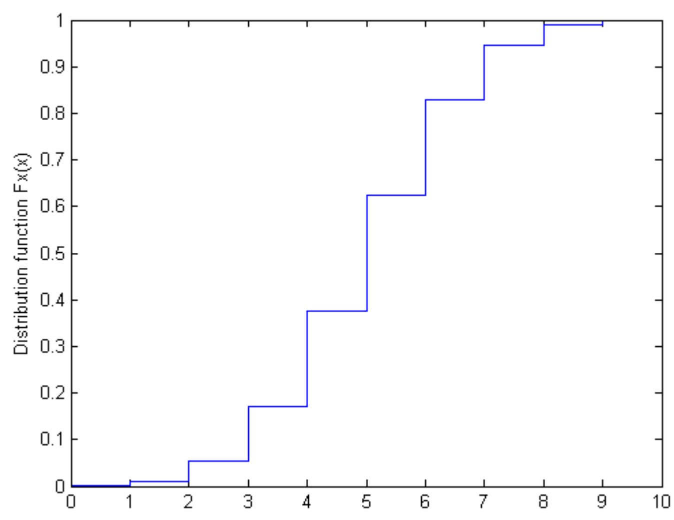
P{ X = 10}=0.0009765625

And, here is the code in Matlab to compute these probabilities and plot the density function and the distribution function of random variable *X*:

```
clc
clear all
close all
N = 10;
f = zeros(N+1,1);
F = zeros(N+1,1);
for i = 0:1:N
    f(i+1) = nchoosek(N,i)*(0.5^i)*(0.5^(10-i));
```

```
end
F(1) = f(1);
for i = 2:1:N+1
    F(i) = F(i-1) + f(i);
end
x = 0:1:N;
figure(1)
stem(x,f, 'o');%plot the density function
xlabel('X = x');
ylabel('Density function fx(x)');
figure(2)
stairs(x,F);
xlabel('X = x');
ylabel('Distribution function Fx(x)');
```





b) To compute the probability that X is between six and nine inclusively, we can use the density function as the following:

$$P\{6 \le X \le 9\} = f_X(6) + f_X(7) + f_X(8) + f_X(9)$$
$$= 0.205078125 + 0.1171875 + 0.0439453125 + 0.009765625 = 0.3760$$

Or we can use the distribution function as the following:

$$P\{6 \le X \le 9\} = F_X(9) - F_X(5) = 0.9990234375 - 0.623046875 = 0.3760$$

c) Using the distribution function of X is straightforward as the following:

$$P\{X \ge 8\} = 1 - P\{X \le 7\} = 1 - F_X(7) = 1 - 0.9453125 = 0.0547$$

2-2.2

a) $\mathrm{P}\{\mathrm{X} = \frac{1}{4}\} = 0$, because X is a continuous random variable.

b) $\mathrm{P}\{\mathrm{X} > \frac{3}{4}\} = 1 - \mathrm{P}\{\mathrm{X} \le \frac{3}{4}\} = 1 - F_X\left(\frac{3}{4}\right) = 1 - \left(0.5 + 0.5 \times \frac{3}{4}\right) = \frac{1}{8}$

c) $\mathrm{P}\{-0.5 < x \le 0.5\} = F_X(0.5) - F_X(-0.5) = (0.5 + 0.5 \times 0.5) - [0.5 + 0.5 \times (-0.5)] = 0.5$

2-2.3

a) In order to make the function as a valid probability distribution function, we need to make sure at least that $F_X(\infty) = 1$. So, we have the following:

$$F_X(\infty) = 1 = A\{1 - \exp[-(\infty - 1)]\} = A(1 - \exp(-\infty)) = A$$

So, we know A=1. This is the answer. Now the distribution function is:

$$F_X(x) = \begin{cases} 1 - \exp[-(x-1)] & 1 < x < \infty \\ 0 & -\infty < x \le 1 \end{cases}$$

Note that we have two defined intervals and the function is different in those two intervals

Note: In order to make sure a function is indeed a distribution function, we need to check all the constraints which you can find on Page 54 of the book, there are 4 of them.

b) $F_X(2) = 1 - \exp[-(2-1)] = 1 - e^{-1} = 0.6321$

c) $P\{2 < X < \infty\} = 1 - P\{X \le 2\} = 1 - F_X(2) = e^{-1} = 0.3679$

d) $P\{1 < X \le 3\} = F_X(3) - F_X(1) = \{1 - \exp[-(3-1)]\} - 0 = 1 - e^{-2} = 0.8647$

2-3.1

a) We already showed the plot of the density function in the solution of Problem 2-2.1.

b) Using the density function to compute the probability that r.v. *X* is in the range between four and seven inclusive is shown below:

$$P\{4 \le X \le 7\} = f_X(4) + f_X(5) + f_X(6) + f_X(7)$$
$$= 0.205078125 + 0.24609375 + 0.205078125 + 0.1171875 = 0.7734$$

c) $P\{X < 4\} = P\{X \le 3\} = f_X(0) + f_X(1) + f_X(2) + f_X(3) =$
$0.0009765625 + 0.009765625 + 0.0439453125 + 0.1171875 = 0.1718$

2-3.2

a) Given a distribution function $F_X(x)$, we use the following to get the density function:

$$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} \frac{d\{1-\exp[-(x-1)]\}}{dx} = \exp[-(x-1)] & 1 < x < \infty \\ 0 & -\infty < x \leq 1 \end{cases}$$

b) Now we have the density function $f_X(x)$, $P\{2 < x \leq 3\} = \int_2^3 f_X(x) \cdot dx = \int_2^3 e^{-(x-1)} \cdot dx =$ 0.2325

c) $P\{x < 2\} = 1 - P\{2 \leq x < \infty\} = 1 - \int_2^\infty f_X(x) \cdot dx = 1 - \int_2^\infty e^{-(x-1)} \cdot dx = 1 - e^{-1} =$ 0.632

2-3.3

a) First, note that

$$f_X(x) = \begin{cases} \exp(2x) & x < 0 \\ \exp(-2x) & x \geq 0 \end{cases}$$

A second r.v. Y is related to X by $Y = X^2$.

Method: If we find the distribution function of Y, then by differentiating the distribution function we get the density function of r.v. Y. Now firstly we know that:

$$F_Y(y) = P\{Y \leq y\} = P\{X^2 \leq y\} = P\{-\sqrt{y} \leq X \leq \sqrt{y}\} = F_X(\sqrt{y}) - F_X(-\sqrt{y}), \quad y \geq 0$$

We derive the distribution function of X as follows:

Since $F_X(x_1) = \int_{-\infty}^{x_1} f_X(x)\,dx = \begin{cases} \int_{-\infty}^{0} f_X(x)\,dx + \int_0^{x_1} f_X(x)\,dx & if\ x_1 \geq 0 \\ \int_{-\infty}^{x_1} f_X(x)\,dx & if\ x_1 < 0 \end{cases}$

$$= \begin{cases} \int_{-\infty}^{0} \exp(2x)\,dx + \int_0^{x_1} \exp(-2x)\,dx & if\ x_1 \geq 0 \\ \int_{-\infty}^{x_1} \exp(2x)\,dx & if\ x_1 < 0 \end{cases}$$

$$= \begin{cases} 1 - \frac{1}{2}\exp(-2x_1) & if\ x_1 \geq 0 \\ \frac{1}{2}\exp(2x_1) & if\ x_1 < 0 \end{cases}$$

We can write $F_X(x) = \begin{cases} 1 - \frac{1}{2}\exp(-2x) & if\ x \geq 0 \\ \frac{1}{2}\exp(2x) & if\ x < 0 \end{cases}$

Hence, $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = 1 - \frac{1}{2}\exp(-2\sqrt{y}) - \frac{1}{2}\exp\left(2(-\sqrt{y})\right)$

$$= 1 - \exp(-2\sqrt{y})$$

Now, $f_Y(y) = \frac{d(F_Y(y))}{dy} = \frac{d(1-\exp(-2\sqrt{y}))}{dy} = \frac{\exp(-2\sqrt{y})}{\sqrt{y}}, \quad if\ y \geq 0.$

When $y < 0$, $F_Y(y) = 0$. So $f_Y(y) = \frac{d(F_Y(y))}{dy} = 0$. So we have the density function of r.v. Y as follows:

$$f_Y(y) = \begin{cases} \dfrac{\exp\left(-2\sqrt{y}\right)}{\sqrt{y}} & when\ y \geq 0 \\ 0 & when\ y < 0 \end{cases}$$

b) We know:

$P\{Y > 2\} = 1 - P\{Y \leq 2\} = 1 - \int_{-\infty}^{2} f_Y(y)\, dy = 1 - \left(\int_{-\infty}^{0} f_Y(y)\, dy + \int_{0}^{2} f_Y(y)\, dy\right) = 1 - \left(\int_{-\infty}^{0} 0\, dy + \int_{0}^{2} \frac{\exp(-2\sqrt{y})}{\sqrt{y}} dy\right) = 1 - \int_{0}^{2} \frac{\exp(-2\sqrt{y})}{\sqrt{y}} dy$

We can either do the integration above to find the answer or we can follow the strategy shown below:

$P\{Y > 2\}$

$= P\{x < -\sqrt{2}\} + P\{x > \sqrt{2}\}$

$= P\{x \leq -\sqrt{2}\} + \left(1 - P\{x \leq \sqrt{2}\}\right)$

$= \int_{-\infty}^{-\sqrt{2}} f_X(x) dx + 1 - \int_{-\infty}^{\sqrt{2}} f_X(x) dx$

$= \int_{-\infty}^{-\sqrt{2}} \exp\,(2x) dx + 1 - (\int_{-\infty}^{0} \exp(2x)\, dx\ + \int_{0}^{\sqrt{2}} \exp(-2x)\, dx)$

$= \frac{1}{2} \cdot \exp\left(-2\sqrt{2}\right) + 1 - \left(\frac{1}{2} + \frac{1}{2} - \frac{1}{2}\exp\left(-2\sqrt{2}\right)\right) = \exp\left(-2\sqrt{2}\right)$

So $P\{Y > 2\} = \exp\left(-2\sqrt{2}\right)$