ECE 314 – Signals and Systems   Fall 2012

Solutions to Homework 9

Problem 3.71
Solution:
(a)

$$x(t) = yr(t) + yl(t) = i(t)R + yl(t) = L^{-1}\int_{-\infty}^{t} yl(\tau)\,\mathrm{d}\tau.$$

Hence, upon differentiation we obtain

$$L\frac{d}{dt}x(t) = yl(t) + L\frac{d}{dt}yl(t)$$

and upon taking Fourier transforms we further obtain

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = j\omega L/(1 + j\omega L).$$

Note that when $\omega \approx 0$, $|H(j\omega)| \approx 0$, and as $\omega \to \infty$, $|H(j\omega)| \to 1$. Hence, this is a high-pass filter.

(c)

$$x(t) = yr(t) + yl(t) = L\frac{d}{dt}yr(t).$$

After taking Fourier transforms we further obtain

$$H(j\omega) = 1/(1 + j\omega L).$$

Here, when $\omega \approx 0$, $|H(j\omega)| \approx 1$, and as $\omega \to \infty$, $|H(j\omega)| \to 0$. Hence, this is a low-pass filter.

(d)

$$\begin{aligned} h(t) &= 1/Le^{-L/t} \\ v(t) = u(t) * h(t) = L^{-1}e^{-L/t}(\int_0^t e^{\tau/L}\,\mathrm{d}\tau)u(t) \\ &= (1 - e^{-t/L})u(t). \end{aligned}$$

Problem 3.73

Solution: **(a)**

$$\begin{aligned}
X(jw) &= \frac{6jw + 16}{(jw)^2 + 5jw + 6} \\
&= \frac{A}{3 + jw} + \frac{B}{2 + jw} \\
6 &= A + B \\
16 &= 2A + 3B \\
X(jw) &= \frac{2}{3 + jw} + \frac{4}{2 + jw}.
\end{aligned}$$

Now by taking inverse Fourier transform (by inspection) we obtain

$$x(t) = (2e^{-3t} + 4e^{-2t})u(t).$$

(d)

$$\begin{aligned}
X(j\omega) &= \frac{(j\omega)^2 4j\omega 6}{((j\omega)^2 + 3j\omega + 2)(j\omega + 4)} \\
&= \frac{A}{2 + j\omega} + \frac{B}{1 + j\omega} + \frac{C}{4 + j\omega} \\
1 &= A + B + C \\
4 &= 5A + 6B + 3C \\
6 &= 4A + 8B + 2C.
\end{aligned}$$

Upon solving for the constants we obtain

$$X(j\omega) = \frac{1}{2 + j\omega} + \frac{1}{1 + j\omega} + \frac{1}{4 + j\omega}$$

Now by taking the inverse Fourier transform by inspection we obtain

$$x(t) = (e^{-2t} + e^{-t} + e^{-4t})u(t).$$

(f)

$$\begin{aligned}
X(j\omega) &= \frac{j\omega + 3}{(j\omega + 1)^2} \\
&= \frac{A}{1 + j\omega} + \frac{B}{(1 + j\omega)^2}.
\end{aligned}$$

Upon solving for the constants we obtain

$$\begin{aligned} 1 &= A \\ 3 &= A + B \\ X(j\omega) &= \frac{1}{1+j\omega} + \frac{2}{(1+j\omega)^2}, \end{aligned}$$

and hence,

$$x(t) = (e^{-t} + 2te^{-t})u(t).$$

Note that we have used the relationship $(\frac{1}{1+j\omega})^2 = \mathcal{F}\{e^{-t} * e^{-t}\} = te^{-t}$.

Problem 3.74

$$\begin{aligned} X(e^{j\Omega}) &= \frac{2e^{j\Omega}}{1 - \frac{1}{4}e^{-j2\Omega}} \\ &= \frac{A}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{B}{1 + \frac{1}{2}e^{-j\Omega}}, \end{aligned}$$

with 2 = 0.5 A - 0.5 B and 0 = A + B, so

$$X(e^{j\Omega}) = \frac{2}{1 - \frac{1}{2}e^{-j\Omega}} - \frac{2}{1 + \frac{1}{2}e^{-j\Omega}}.$$

Now by taking the inverse DTFT by inspection we obtain

$$x[n] = 2\Big((0.5)^n - (-0.5)^n\Big)u[n].$$

Problem 3.75

Solution: **(a)** Note that

$$\frac{2}{1 - \frac{1}{3}e^{-j\Omega}} \leftrightarrow 2(\frac{1}{3})^n u[n],$$

so by Parserval's identity we have

$$\int_{-\pi}^{\pi} |\frac{2}{1 - \frac{1}{3}e^{-j\Omega}}|^2 \, \mathrm{d}\Omega = 2\pi \sum_{n=-\infty}^{\infty} |2(\frac{1}{3})^n u(n)|^2 = 9\pi.$$

(c) From class notes we know that

$$X(j\omega) = \frac{2(2)}{\omega^2 + 2^2} \leftrightarrow x(t) = e^{-2|t|}.$$

Hence, by Parserval's identity we have

$$\int_{-\infty}^{\infty} |(\frac{4}{\omega^2 + 2^2})|^2 \, \mathrm{d}\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t = 4\pi \int_{0}^{\infty} e^{-4t} \, \mathrm{d}t = \pi.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{8}{|\omega^2 + 4|^2} \, \mathrm{d}\omega = \pi/2.$$

Problem 3.76 *Solution:* **(b)** By simply interchanging the roles of ω and t in the relationship $\mathcal{F}\{te^{-2t}u(t)\} = \frac{1}{(2+j\omega)^2}$ we realize that

$$\int_{-\infty}^{\infty} \omega e^{-2\omega} u(\omega) e^{-jt\omega} d\omega = \frac{1}{(2+jt)^2}.$$

Now by taking inverse Fourier transform (with the roles of t and ω interchanged) we obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(2+jt)^2} e^{jt\omega} dt = \omega e^{-2\omega} u(\omega).$$

Next, replace ω by $-\omega$ in the above expression to obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(2+jt)^2} e^{-jt\omega} dt = -\omega e^{2\omega} u(-\omega),$$

and the expression on the right is nothing but $\mathcal{F}\{\frac{1}{(2+jt)^2}\}$.

Problem 3.77 *Solution:*

(a)

$$\int_{-\infty}^{\infty} x(t) \, \mathrm{d}t = X(j0) = 1$$

(b)

$$\int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, \mathrm{d}\omega$$
$$= \frac{1}{2\pi} [\int_{-5}^{-3} (\omega + 5)^2 \, \mathrm{d}\omega + \int_{-3}^{-1} (-\omega - 1)^2 \, \mathrm{d}\omega$$
$$+ \int_{-1}^{1} (\omega + 1)^2 \, \mathrm{d}\omega + \int_{1}^{3} (-\omega + 3)^2 \, \mathrm{d}\omega] = \frac{16}{3\pi}$$

(c)

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, \mathrm{d}w = \int_{-\infty}^{\infty} x(t) e^{j3t} \, \mathrm{d}t = X(j(-3))$$
$$= 2$$

(d) Note that $X(j(\omega - 1))$ is real and even, so $\mathcal{F}^{-1}\{X(j(\omega - 1))\}$ is real. However, by the shift-in-frequency property, $\mathcal{F}^{-1}\{X(j(\omega - 1))\} = e^{jt}x(t)$. Hence, for $e^{jt}x(t)$ to be real the phase of $x(t)$ must be $-t$.

(e)

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega 0} \, \mathrm{d}\omega$$
$$= \frac{1}{2\pi} [\int_{-5}^{-3} (w + 5) \, \mathrm{d}w + \int_{-3}^{-1} (-w - 1) \, \mathrm{d}w +$$
$$\int_{-1}^{1} (w + 1) \, \mathrm{d}w + \int_{1}^{3} (-w + 3) \, \mathrm{d}w] = \frac{4}{\pi}$$

Problem 3.78 *Solution:*

(a)

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = 0$$

(b) Note that $x[n + 2]$ is real and odd function, so its DTFT, $e^{j2\Omega} X(e^{j\Omega})$ is purely imaginary, i.e., its phase is $\pi/2$. Hence, the phase of $X(e^{j\Omega})$ must be $\pi/2 - 2\Omega$.

(c) By Parserval's identity,

$$\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 \, d\Omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 28\pi$$

(d)

$$\int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j3\Omega} \, d\Omega = 2\pi x(3) = -2\pi$$

(e) From part (a) we know that $e^{j2\Omega} X(e^{j\Omega})$ is purely imaginary, so $y[n] = 0$.

Problem 4.24 *Solution:* Note that $x(n) = x_c(nT_s)$, where $x_c(t) = \sin(\pi t/6)/(\pi t/2)$. Now we know from class notes that

$$X_c(j\omega) = \begin{cases} 2 & |\omega| < \frac{\pi}{6} \\ 0 & \text{otherwise} \end{cases}$$

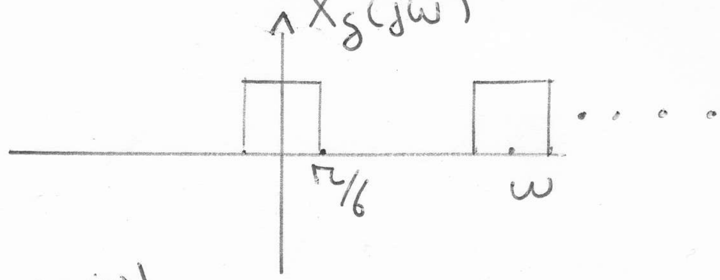
Let $x_s(t) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT_s)$. From class notes we have, $X_s(j\omega) = \sum_{n=-\infty}^{\infty} X_c(j(\omega - 2\pi n/T_s)) = \sum_{n=-\infty}^{\infty} X_c(j(\omega - \pi n))$.

(e) Note that $x(n) = x_s(nT_s)$, where $x_s(t) = \sum_{k=-\infty}^{\infty} \delta(t - 4kT_s)$, which is periodic with period $4T_s$. Now we know from class notes that (see class notes on FT of periodic signals)

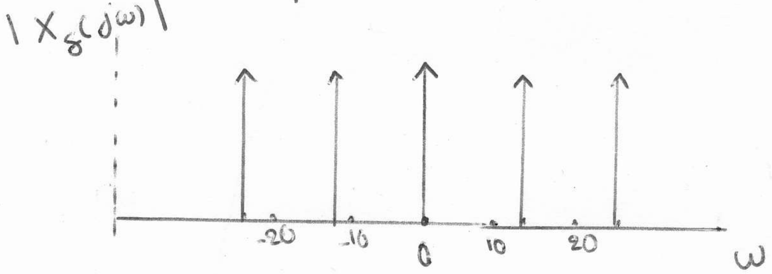
$$X_s(j\omega) = 2\pi/(4T_s) \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi/(4T_s)) = 4\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k4\pi).$$

4.24)

a:



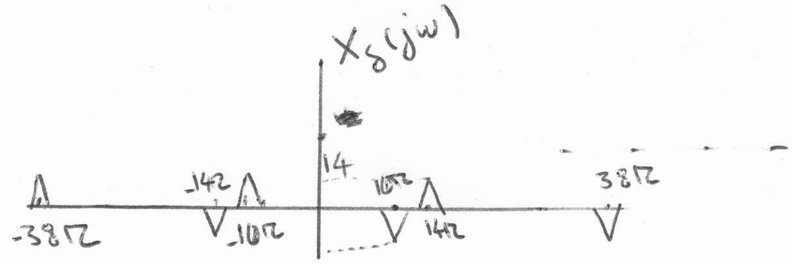
c)



4.26)

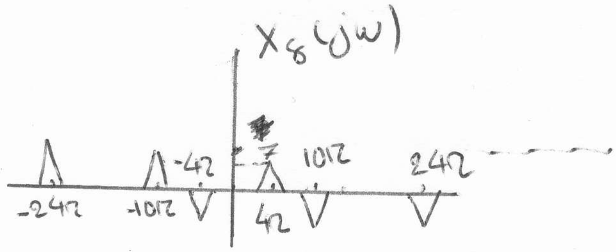
a:

i: $T_s = 1/14$



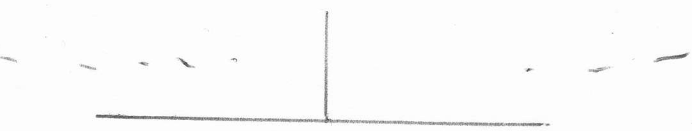
ii: $T_s = 1/7$

aliasing occurs



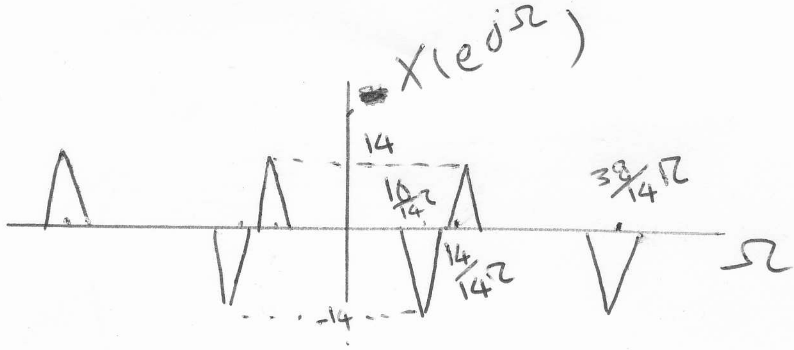
iii: $T_s = 1/5$

aliasing occurs

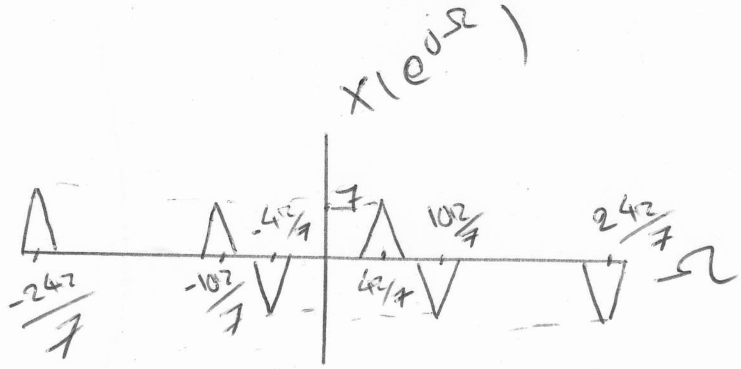


b:

i: $T_s = 1/14$



ii: $T_s = 1/7$



iii: $T_s = 1/5$