Homework #1 Solutions

1. (a) Differentiating $u(t)$ gives

$$\frac{du}{dt} = \frac{d}{dt}\left[T_0 - ae^{-kt}\right] = ake^{-kt}.$$

Substituting $u(t)$ and $\frac{du}{dt}$ into the equation yields

$$ake^{-kt} = k\left(T_0 - (T_0 - ae^{-kt})\right) = k\left(T_0 - T_0 + ae^{-kt}\right) = ake^{-kt},$$

which shows that $u(t)$ is a solution for any real value of a.

(b) If $u(0) = 100$, then $u = 100$ when $t = 0$. Using these values, we have that

$$100 = T_0 - ae^{-k\cdot 0} = T_0 - a,$$

so that $a = T_0 - 100$.

(c) We are given initial values for u, T_0, and $\frac{du}{dt}$, so we can substitute these values into the equation to get

$$-2°\text{C/min} = k(70°\text{C} - 100°\text{C}) = k(-30°\text{C}),$$

or $k = \frac{1}{15}\text{min}^{-1}$. Using parts (a) and (b), we find that the particular solution in this case is

$$u(t) = T_0 - (T_0 - 100)e^{-k\cdot t} = 70 - (70 - 100)e^{-\frac{1}{15}t} = 70 + 30e^{-\frac{1}{15}t}.$$

(d) Since $e^{-ct} \to 0$ as $t \to \infty$ for any $c > 0$, we see that

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty}(70 + 30e^{-\frac{1}{15}t}) = 70.$$

See the other file for the graph.

2. (a) i. We can factor the right-hand side of the equation as

$$2y^3 + y^2 - 2y - 1 = y^2(2y+1) - (2y+1) = (2y+1)(y^2-1) = (2y+1)(y-1)(y+1) = \dot{y}.$$

For $y > 1$, each of these factors is positive, which shows that on the region of the x, y plane corresponding to $y > 1$, we have that $\dot{y} > 0$, which implies that y is increasing. Since each factor has a multiplicity of one, there will be a sign change as we pass through each root. This means that for the region $-1/2 < y < 1$, we have that $\dot{y} < 0$, so that y is decreasing. For the region $-1 < y < -1/2$, $\dot{y} > 0$, so y is increasing, and for $y < -1$, $\dot{y} < 0$, so that y is decreasing.

ii. The equilibrium solutions are those values of y for which $\dot{y} = 0$, so part (a) shows that $y = 1$, $y = -1/2$, and $y = -1$ are each equilibrium solutions. Since solutions are increasing for $y > 1$ and decreasing for $-1/2 < y < 1$, the curves are tending *away* from the line $y = 1$, which shows that this equilibrium is *unstable*. Similarly, $y = -1/2$ is asymptotically stable, and $y = -1$ is unstable.

iii. See the other file for the sketches.

(b) See the other file.

3. (a) Since the expression on the right is a product of a function of x and a function of t, the equation is separable. Separating the variables gives

$$\int \frac{dx}{x} = \int \sec^2 t\, dt.$$

Integrating both sides gives

$$\ln|x| = \tan t + C.$$

Exponentiating both sides gives the general solution

$$x(t) = Ke^{\tan t},$$

where $K = e^C$. Using the initial condition, we see that

$$x(0) = Ke^{\tan 0} = K = 1,$$

so the particular solution is

$$x(t) = e^{\tan t}.$$

Note that we can write the equation as $\frac{dx}{dt} - x\sec^2 t = 0$, which shows that the equation is in fact linear, so the method of integrating factors works here as well to find the general solution. We have that

$$\mu(t) = e^{-\int \sec^2 t\, dt} = e^{-\tan t},$$

so multiplying through by μ and using the product rule, we get

$$\frac{dx}{dt}e^{-\tan t} - e^{-\tan t}x\sec^2 t = \frac{d}{dt}\left[xe^{-\tan t}\right] = 0.$$

Integrating both sides gives

$$xe^{-\tan t} = K,$$

so the general solution is

$$x = Ke^{\tan t},$$

which agrees with what was found using separation of variables.

(b) We note that we can write the equation either as $\frac{du}{dx} = x(u+1)$ or as $\frac{du}{dx} - xu = u$, so once again the equation is both separable and linear. Separating variables gives

$$\int \frac{du}{u+1} = \int x\, dx,$$

so that

$$\ln|u+1| = \frac{x^2}{2} + C.$$

Exponentiating and solving for u gives the general solution

$$u(x) = Ke^{\frac{x^2}{2}} - 1,$$

where $K = e^C$. The initial condition is $u(2) = 5e^4$, so we have

$$u(2) = Ke^{\frac{2^2}{2}} - 1 = Ke^2 - 1 = 5e^4.$$

Solving for K gives

$$K = 5e^2 + e^{-2},$$

so the particular solution is

$$u(x) = (5e^2 + e^{-2})e^{\frac{x^2}{2}} - 1.$$

Alternatively, we can use an integrating factor to find the general solution. We have that

$$\mu(x) = e^{-\int x\,dx} = e^{-\frac{x^2}{2}}.$$

Multiplying through by μ and using the product rule gives

$$\frac{du}{dx}e^{-\frac{x^2}{2}} - e^{-\frac{x^2}{2}}xu = \frac{d}{dx}\left[ue^{-\frac{x^2}{2}}\right] = xe^{-\frac{x^2}{2}},$$

so integrating gives

$$ue^{-\frac{x^2}{2}} = \int xe^{-\frac{x^2}{2}}\,dx = -e^{-\frac{x^2}{2}} + K,$$

where we have used the substitution $w = \frac{x^2}{2}$ to evaluate the integral. Multiplying through by $e^{\frac{x^2}{2}}$ gives us once again that

$$u(x) = Ke^{\frac{x^2}{2}} - 1.$$

(c) Note that this equation is *NOT* separable. However, if we write it as $\dot{y} + \frac{2y}{t} = e^t$, we see that it *is* linear, so we can use integrating factors. First, we find μ:

$$\mu(t) = e^{2\int \frac{dt}{t}} = e^{2\ln t} = e^{\ln t^2} = t^2.$$

Multiplying through by μ and using the product rule, we find that

$$\dot{y}t^2 + 2ty = \frac{d}{dt}\left[yt^2\right] = t^2e^t.$$

Integrating both sides gives

$$yt^2 = \int t^2e^t\,dt = t^2e^t - 2\int te^t\,dt = t^2e^t - 2(te^t - \int e^t\,dt) = t^2e^t - 2te^t + 2e^t + C,$$

where we have used integration by parts twice to evaluate the integral. Multiplying through by t^{-2} gives the general solution

$$y(t) = e^t - 2t^{-1}e^t + t^{-2}(2e^t + C).$$

Since no initial condition was given, we are done.

(d) This is a Bernoulli equation where $k = 2$, so we make the substitution $u = y^{1-2} = y^{-1}$. Differentiating this equation with respect to x gives

$$u' = -y^{-2}y'.$$

We multiply the equation by $-y^{-2}$ to get

$$-y^{-2}y' + y^{-1} = -x.$$

Substituting yields

$$u' + u = -x,$$

which is a linear equation in u, as expected. This equation is inseparable, so we must use an integrating factor:

$$\mu(x) = e^{\int dx} = e^x,$$

which gives

$$u'e^x + e^xu = (ue^x)' = -xe^x.$$

Integrating using integration by parts gives

$$ue^x = -\int xe^x\,dx = -(xe^x - \int e^x\,dx) = -xe^x + e^x + C.$$

We mulitiply through by e^{-x} to find the general solution:

$$u(x) = 1 - x + Ce^x.$$

This is the solution to the equation in u, so we must find the solution in y. Since $u = y^{-1}$, we have that $y = u^{-1}$, so our general solution in y is

$$y(x) = [u(x)]^{-1} = (1 - x + Ce^x)^{-1} = \frac{1}{1 - x + Ce^x}.$$

No initial condition was given, so we are done.

4. (a) If we let $u = x - y$, then we have

$$\frac{du}{dx} = 1 - \frac{dy}{dx},$$

so that

$$\frac{dy}{dx} = 1 - \frac{du}{dx},$$

(b) Using the equations in part (a), we substitute to find that

$$1 - \frac{du}{dx} = u^2,$$

or

$$\frac{du}{dx} = 1 - u^2,$$

which is a separable equation.

(c) Separating variables gives

$$\int \frac{du}{1-u^2} = \int dx = x + C.$$

To evaluate the integral, I asked you to use partial fractions (if you noticed that this is the derivative of $\tanh x$ and proceeding accordingly, I did not take off any points, provided you did everything correctly). Partial fractions gives the equation

$$1 = A(1-u) + B(1+u).$$

If $u = 1$, then $B = 1/2$, and if $u = -1$, then $A = 1/2$, so we have that

$$\int \frac{du}{1-u^2} = \frac{1}{2}\int \frac{du}{1+u} + \frac{1}{2}\int \frac{du}{1-u} = \frac{1}{2}(\ln|1+u| - \ln|1+u|) = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| = x + C.$$

Multiplying through by 2 and exponentiating gives

$$\frac{1+u}{1-u} = e^{2x+C} = Ke^{2x},$$

where $K = e^C$. Since $1 + u = (1-u)Ke^{2x}$, solving for u gives

$$u(x) = \frac{Ke^{2x} - 1}{Ke^{2x} + 1}.$$

Using the fact that $u = x - y$, so that $y = x - u$, we arrive at

$$y(x) = x - \frac{Ke^{2x} - 1}{Ke^{2x} + 1}.$$

In order to check our solution, we differentiate with respect to x:

$$\begin{aligned}
\frac{dy}{dx} &= 1 - \frac{2Ke^{2x}(Ke^{2x}+1) - 2Ke^{2x}(Ke^{2x}-1)}{(Ke^{2x}+1)^2} \\
&= 1 - \frac{2Ke^{2x} \cdot Ke^{2x} + 2Ke^{2x} - 2Ke^{2x} \cdot Ke^{2x} + 2Ke^{2x}}{(Ke^{2x}+1)^2} \\
&= 1 - \frac{4Ke^{2x}}{(Ke^{2x}+1)^2} \\
&= \frac{(Ke^{2x}+1)^2 - 4Ke^{2x}}{(Ke^{2x}+1)^2} \\
&= \frac{(Ke^{2x}-1)^2}{(Ke^{2x}+1)^2} \\
&= \left(\frac{Ke^{2x}-1}{Ke^{2x}+1}\right)^2 \\
&= \left(x - \left(x - \frac{Ke^{2x}-1}{Ke^{2x}+1}\right)\right)^2 \\
&= (x-y)^2,
\end{aligned}$$

so y is in fact a solution to the equation.

(d) I wanted you to notice that if $u = y^2$, then $\frac{du}{dx} = 2y\frac{dy}{dx}$, both of which appear in this equation. Making these substitutions, we have that

$$x\frac{du}{dx} + 2u = 3x - 6,$$

which is a linear equation in u.

(e) Since $u = \dot{y}$, differentiating both sides with respect to t shows that $\dot{u} = \ddot{y}$. Making these substitutions gives

$$\dot{u} - 8u\frac{1}{t} = 1,$$

which is linear in u. It is inseparable, so we must use an integrating factor:

$$\mu(t) = e^{-8\int \frac{dt}{t}} = e^{-8\ln t} = e^{\ln t^{-8}} = t^{-8}.$$

This gives

$$\dot{u}t^{-8} - 8t^{-9}u = \frac{d}{dt}[ut^{-8}] = t^{-8}.$$

Multiplying through by t^8 and integrating gives

$$u(t) = t^8 \int t^{-8}\, dt = t^8(\frac{t^{-7}}{-7} + C_1) = -\frac{1}{7}t + C_1 t^8.$$

Since $u = \dot{y}$, we integrate one more time to find y:

$$y = \int u(t)\, dt = \int (-\frac{1}{7}t + C_1 t^8)\, dt = -\frac{1}{14}t^2 + \frac{C_1}{9}t^9 + C_2.$$