ECE340 Spring 2011
Recitation class
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Problem 1

Two random variables, X and Y, have a joint probability density function of the form:

$$f_{XY}(x,y) = \begin{cases} ke^{-(x+y-1)} & 0 \le x \le \infty, 1 \le y \le \infty \\ 0 & elsewhere \end{cases}$$

Find

a) The values of k and a for which the random variables X and Y are statistically independent.
b) The expected value of XY.
c)

<u>Solution</u>

a) If the two random variables are statistically independent, the following equality holds:

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

We use the following to find $f_X(x)$ and $f_Y(y)$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

In this case

$$f_X(x) = \int_1^{\infty} ke^{-(x+y-1)}dy = ke^{-(x-1)}\int_1^{\infty} e^{-y}dy = ke^{-(x-1)}e^{-1} = ke^{-x}$$

$$f_Y(y) = \int_0^{\infty} ke^{-(x+y-1)}dx = ke^{-(y-1)}\int_0^{\infty} e^{-x}dx = ke^{-(y-1)}.1 = ke^{-(y-1)}$$

Then we have:

$$f_X(x)f_Y(y) = ke^{-x}.ke^{-(y-1)} = k^2e^{-(x+y-1)}$$

Therefore we must have $k = 1$ in order to have X and Y statistically independent.

b) If two random variables are statistically independent, we can use the following to compute E[XY]:

$$E[XY] = E[X]E[Y]$$

$$E\{X\} = \int_1^{\infty} xe^{-x}dx = -e^{-x}(x+1)|_0^{\infty} = 2e^{-1}$$

$$E\{Y\} = \int_0^{\infty} ye^{-(y-1)}dy = -e^{-(y-1)}(y+1)\Big|_0^{\infty} = e^{+1}$$

Therefore,

$$E[XY] = E[X]E[Y] = 2$$

Problem 2

Two independent random variables, X and Y, have the following probability density functions.

$$f(x) = 0.5e^{-|x-1|} \qquad -\infty < x < \infty$$
$$f(y) = 0.5e^{-|y-1|} \qquad -\infty < y < \infty$$

Find the probability that XY>0.

Solution

Since two random variables are independent, we have:

$$f(x,y) = f(x)f(y) = 0.25e^{-|x-1|}e^{-|y-1|}$$

Then the probability that XY>0 is calculated as follows:

$$P\{XY > 0\} = P\{X > 0, Y > 0\} + P\{X < 0, Y < 0\}$$
$$= \int_0^\infty \int_0^\infty 0.25e^{-|x-1|}e^{-|y-1|}dxdy + \int_{-\infty}^0 \int_{-\infty}^0 0.25e^{-|x-1|}e^{-|y-1|}dxdy$$
$$= 0.25\left[\int_0^1 \int_0^1 e^{(x-1)}e^{(y-1)}dxdy + \int_1^\infty \int_1^\infty e^{-(x-1)}e^{-(y-1)}dxdy \right.$$
$$\left. + \int_{-\infty}^0 \int_{-\infty}^0 e^{(x-1)}e^{(y-1)}dxdy\right]$$
$$= 0.25[e^{-2}(e-1)^2 + e^2.e^{-2} + e^{-2}] = 0.666$$

Problem 3

let X and Y be statistically independent random variables. let $W = g(X)$ and $V = h(Y)$ be any transformations with continous derivatives on X and Y. Show that W and V are also statistically independent.

$f(x,y) = f(x) \cdot g(y)$

$W = g(X)$, $V = h(Y) \Rightarrow X = g^{-1}(W)$, $Y = h^{-1}(V)$

$dw = \dot{g}(x)\,dx$, $dv = \dot{h}(y)\,dy$

$f_X(x) = f_W(w) \left| \frac{dw}{dx} \right|$ and $f_Y(Y) = f_V(V) \left| \frac{dv}{dy} \right|$

$f_X(x) = f_W(w) \left| \dot{g}(g^{-1}(w)) \right|$ and $f_Y(Y) = f_V(v) \cdot \left| \dot{h}(h^{-1}(v)) \right|$

$\therefore$ W and V are statiscally independent.

Problem 4

X is a zero mean random variable having a variance of 9 and Y is another zero mean random variable. The sum of X and Y has a variance of 29 and the difference has a variance of 21.

a) Find the varaiance of Y.

b) Find the correlation coefficient of X and Y.

c) Find the variance of $U = 3X - 5Y$.