Review for Exam #1

For the exam, you should know how to

1. Find equilibrium solutions, determine their stability and classify them, find the regions where the solutions are increasing/decreasing, sketch solution curves corresponding to initial values, and sketch slope fields;
2. Solve 1st order ODEs by separation of variables and the method of integrating factors;
3. Solve Bernoulli equations and use implicit differentiation to make substitutions;
4. Use Euler's Method to approximate the value of a solution at a point;
5. Solve half-life/doubling-time problems;
6. Interpret a population model by finding limits and inflection points.

You will not need to know Heun's Formula or the equations for Runge-Kutta, nor will I have you do a chemical concentration or chemical reaction problem. Below are some sample problems. The problems on the exam will be no more difficult than these. Make sure you can solve these in a reasonable amount of time. I will post some solutions tomorrow.

1. Sketch direction fields for the following differential equations. In your sketches, indicate the regions on which the solutions are increasing/decreasing, draw the equilibrium solutions and indicate their stability, and sketch the solution curve corresponding to $x(0) = 1/16$ and $y(0) = 1$, respectively. Label everything (including the axes)!

   (a) $\frac{dx}{dt} = 10x - 80x^2$

   (b) $\dot{y} = y - y \ln y$

2. Sketch a direction field for the differential equation

$$\frac{du}{dx} = u^2 x + u x^2.$$

   Include at least three horizontal lines on either side of the x-axis and three vertical lines on either side of the y-axis. Where do solutions have zero slope? Sketch the solution curve corresponding to $y(0) = 0$. Label everything (including the axes)!

3. Solve the following differential equations subject to the given initial conditions (if any) using whichever method you please (some may be solved in more than one way):

   (a) $\dot{x} = x^3/t^2$

   (b) $t^2\dot{y} + ty = 1$, $t(1) = 1$, for $t > 0$

   (c) $\frac{dy}{dx} = 4(y^2 + 1)$, $y(\pi/4) = 1$

   (d) $(1 + e^x)y' + e^x y = 0$

   (e) $au' + bu = c$, $u(0) = u_0$, for constants $a, b, c > 0$

   (f) $\frac{dP}{dx} - P = e^x P^2$

4. Use the substitution $u = \ln y$ to solve the equation $\dot{y} + y \ln y = ye^t$.

5. Use Euler's method to approximate $y(4)$ for the following IVPs. Use a step size of $h = 2$. (You shouldn't need a calculator.)

   (a) $\dot{y} = yt + 2$, $y(0) = 1$

   (b) $\dot{x} = (tx)^2$, $y(0) = 2$

6. Solve the following problems. You may use a calculator for these. On the exam, I will give you values that do not require one.

   (a) The population of bacteria in a culture is observed to be 400 after 3 hours. After 10 hours, there are 2000 bacteria present. What is the initial number of bacteria? Use the equation $P(t) = P_0 e^{rt}$.

   (b) The radioactive isotope of lead, Pb-209, has a half-life of 3.3 hours. If 1 gram of lead is present initially, how long will it take for 90% of the lead to decay? Use the equation $A(t) = A_0 e^{-rt}$.

7. Another version of the Gompertz equation is

$$\frac{dP}{dt} = rP \ln(K/P),$$

   where $r, K > 0$. Assume that $0 \leq P_0 \leq K$. Answer the following questions without solving the equation:

   (a) (Equilibrium Points) Is there a non-zero stable population? If so, what is it?

   (b) (Inflection Points) At what population is maximum growth acheived?

   (c) (Limits) What does the model predict will happen to the population as $t \to \infty$?

   (d) (Initial Value) If $P_0 = K/3$, will the population growth increase or decrease initially? What about if $P_0 = 2K/3$?