UNM Physics 262, Fall 2006
Midterm Exam 2: Relativity

Name and/or CPS number: Dr. Landahl - Solution Key

Show all your work for full credit. Remember that quantities have units and vectors have components (or magnitude and direction). ASK if anything seems unclear.

CALCULATORS AND CELL PHONES ARE PROHIBITED.
USE OF THESE WILL RESULT IN A ZERO FOR THE EXAM.
Keep any factors of π, e, $\sqrt{2}$, etc. in your answers.

You may use a single 8.5" × 11" paper containing notes you have prepared ahead of time to assist you.

Apportion your time sensibly. Spend about 10–12 minutes per problem.

Please put a box around your final answers.

Problem 1: 25

Problem 2: 25

Problem 3: 25

Problem 4: 25

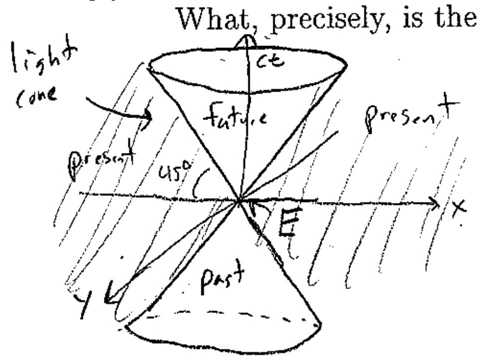
100

1. Short answer [25 points]

The questions below should be answered with no more than five lines of text and no (or very limited) calculations. Please be brief and to the point.

[5] *a)* Define proper time. The time measured by an object's own instantaneous rest frame. For a nonaccelerating object, proper time is the time between two events that occur at the $\boxed{\text{same place}}$ in the object's IRF. (E.g., the time between clicks of a clock on the lattice of clocks establishing the IRF.)

[5] *b)* Relative to any event E, spacetime can be decomposed into *future*, *past*, and *elsewhere*. What, precisely, is the significance of the region called "elsewhere?"



It is the $\boxed{\text{causal present.}}$

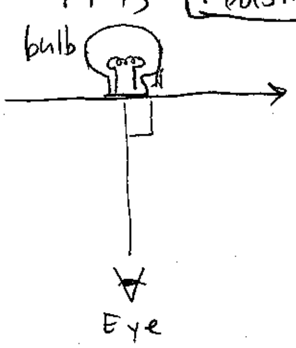
I.e. the set of events that can neither cause nor be caused by E.

It is the set of events strictly outside E's lightcone.

Note that the causal present of E contains events that can cause or be caused by one another.

[5] *c)* A lightbulb is moving perpendicular to your line of sight at a velocity close to the speed of light. Is the light you see emitted from the bulb redshifted, blueshifted, or unshifted? What is your reasoning?

It is $\boxed{\text{redshifted}}$.



In special relativity, unlike Galilean relativity, there is a transverse Doppler effect coming from time dilation. It states that $f = f_0/\gamma$, where f_0 is the emitted frequency, f is the observed frequency, and $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} > 1$. Since $\gamma > 1$, $f < f_0$ and $\lambda > \lambda_0$, which is a redshift.

[5] *d*) Evaluate the following statement: "A photon has energy but does not have momentum since it has no mass."

$\boxed{\text{False.}}$ A photon has both energy and momentum. One way to see this is via the relation $E^2 - p^2c^2 = m^2c^4$. For $m=0$, this says $p = E/c \neq 0$ when $E \neq 0$. Another way to see this is recalling results we discussed in optics, wherein we saw that Maxwell's Equations assigned both energy and momentum to (massless) EM radiation,

[5] *e*) What is the correct statement of Newton's second law, according to special relativity? Specify in words any symbols you use. (Remember: vectors!)

$$\boxed{\begin{aligned} \vec{F} &= \frac{d\vec{p}}{dt} \\ \vec{p} &= \gamma m \vec{v} \end{aligned}}$$

(Same as before SR)

(The γ is new in SR.)

$\vec{F}$ = force

$\vec{p}$ = momentum

t = time

$\vec{v}$ = velocity

$\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

c = speed of light

d = infinitesimal differential

As measured in a single IRF

2. Spacetime geometry [25 points]

Alice records two events in her inertial reference frame. Event 1 has coordinates $(ct_1, x_1) = (4\text{ m}, 3\text{ m})$ and event 2 has coordinates $(ct_2, x_2) = (5\text{ m}, 6\text{ m})$.

[4] *a*) What is the spacetime interval between these events?

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2$$
$$= [(5-4)^2 - (6-3)^2]\ m^2$$
$$= (1-9)\ m^2$$
$$= \boxed{-8\ m^2}$$

Also, $|\Delta s| = \sqrt{8}\ m = 2\sqrt{2}\ m$, but Δs is imaginary.

[4] *b*) Is this interval timelike, spacelike, or lightlike? Why?

$\boxed{\text{Spacelike.}}$ Because $\Delta s^2 < 0$, and that is the definition of spacelike. Equivalently, because $\Delta x > c\Delta t$, which is another way of defining spacelike.

[2] *c*) There is a special inertial reference frame (sharing the same origin event and rotational orientation as Alice's inertial reference frame) moving at a uniform velocity relative to Alice such that these events are measured to occur either at the same time or the same place. Which case is it and what is the time (or space) separation between the events in this frame?

$\boxed{\text{Same time.}}$ obviously this means $\Delta t = 0$. This means the question is asking for $(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 = -8m^2$

$\Rightarrow \Delta x = \pm\sqrt{8}\ m$. Space ordering is preserved in all IRFs, + this, so $\boxed{x_2 - x_1 = \sqrt{8}m}$

[5] *d*) Bob is in this special inertial reference frame. With what velocity v/c is he moving with respect to Alice? (Remember: velocity is a vector so indicate both its magnitude and direction.) (Hint: You may find answering part *e* first to be easier.)

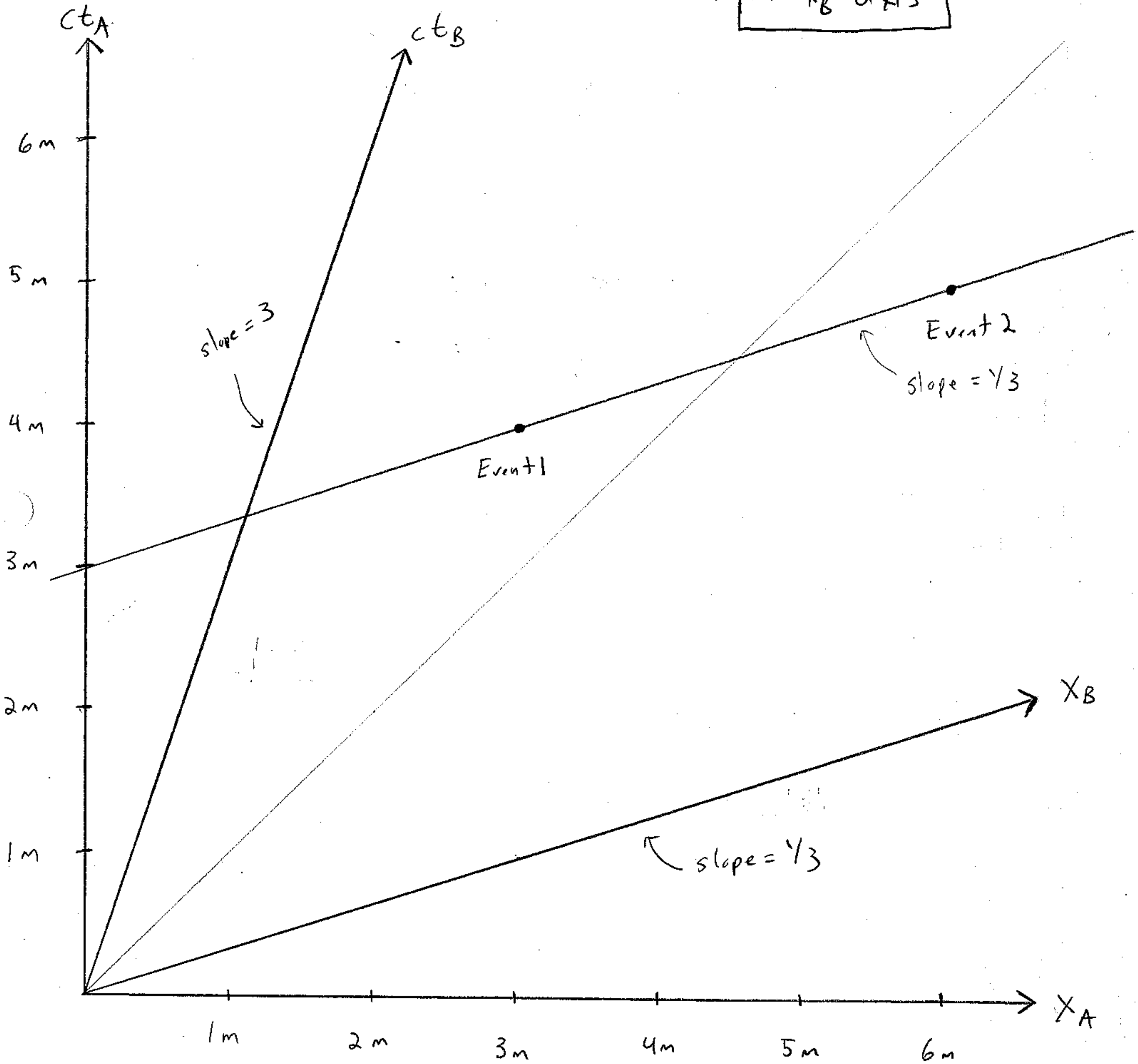
The slope of his x_B axis in Alice's IRF is

$$\frac{v}{c} = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{c\Delta t}{\Delta x} = \frac{(5m - 4m)}{(6m - 3m)} = 1/3$$

Since this is positive, it means Bob is moving to the right

$$\therefore \boxed{\frac{v}{c} = \frac{1}{3} \text{ to the } \underline{\text{right}} \text{ wrt. Alice}}$$

[10] *e*) Draw the ct and x axes for Alice's and Bob's inertial reference frame in a single spacetime diagram below. Let Alice's axes be the pair that form a right angle. Label both sets of axes clearly. Indicate a scale on Alice's axes, and plot events 1 and 2 in this diagram. Draw the line of constant space or time in which these two events lie in Bob's inertial reference frame. Which one of Bob's axes is this line parallel to? $\rightarrow$ $\boxed{\text{His } x_B \text{ axis}}$



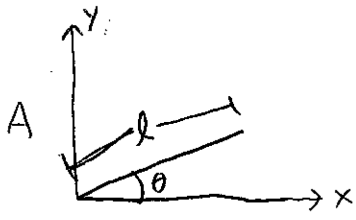
3. Relativistic kinematics [25 points]

A stick of proper length ℓ lies at rest in Alice's inertial reference frame, lying in the xy-plane at an angle of $\theta = \tan^{-1}(3/4)$ with respect to her x-axis. Bob moves with velocity $\vec{v} = v\hat{x}$ with respect to Alice. In Bob's inertial reference frame, the stick is angled at 45° with respect to his x'-axis.

[12] *a*) What is v/c?

Length contraction is longitudinal only.

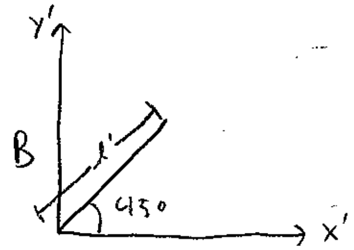
$\ell_y' = \ell_y \,,\ \ell_x' = \ell_x/\gamma$



$\ell' \sin 45° = \ell \sin\theta$

$\ell' \cos 45° = (\ell/\gamma)\cos\theta$

$\tan 45° = \gamma \tan\theta$

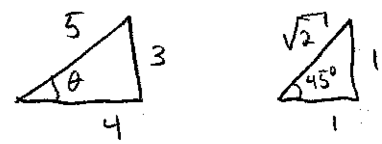


$\gamma = \dfrac{\tan 45°}{\tan\theta} = \dfrac{1}{3/4} = 4/3$

$\sqrt{1-\frac{v^2}{c^2}} = \frac{3}{4} \rightarrow \frac{v^2}{c^2} = 1 - \frac{9}{16} = \frac{7}{16} \Rightarrow \boxed{\frac{v}{c} = \frac{\sqrt{7}}{4}}$

[13] *b*) What is the length ℓ' of the rod as measured by Bob? Express your answer as a fraction times ℓ.

From above, $\ell_y' = \ell_y$, so



$\ell' = \ell \dfrac{\sin\theta}{\sin 45°} = \dfrac{3/5}{1/\sqrt{2}}$

$\boxed{\ell' = \frac{3\sqrt{2}}{5}\,\ell}$

4. Relativistic dynamics [25 points]

The average lifetime of muons at rest is $\tau = 2\ \mu$s. A particle accelerator experiment generates a beam of muons which have an average lifetime of $t = 6\ \mu$s in the laboratory.

[4] *a*) What is the speed v/c of these muons in the laboratory frame?

Time dilation method : $\gamma = \frac{t}{\tau} = \frac{6\,\mu s}{2\,\mu s} = 3 = \frac{1}{\sqrt{1-v^2/c^2}} \rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{9} = \frac{8}{9} \rightarrow \boxed{\frac{v}{c} = \frac{2\sqrt{2}}{3}}$

Interval method : $(c\tau)^2 = (ct)^2 - (vt)^2 = (c^2 - v^2)t^2$

$\tau^2 = \left(1 - \frac{v^2}{c^2}\right) t^2 \Rightarrow 1 - \frac{v^2}{c^2} = \frac{\tau^2}{t^2} \rightarrow \frac{v^2}{c^2} = 1 - \left(\frac{2\,\mu s}{6\,\mu s}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9}$

$\boxed{v/c = \frac{2\sqrt{2}}{3}}$

[4] *b*) The mass of the muon is $m = 100$ MeV/c^2. What is the energy of the muon in its rest frame, in MeV?

$E = mc^2 = \boxed{100\,\text{MeV}}$ in the <u>rest</u> <u>frame</u>.

[4] *c*) What is the energy of the muons in the laboratory frame, in MeV?

$E = \gamma mc^2 = \boxed{300\,\text{MeV}}$

[4] *d*) What is the kinetic energy of the muons in the laboratory frame, in MeV?

$KE = mc^2(\gamma - 1) = \boxed{200\,\text{MeV}}$

[4] *e*) What is the magnitude of the (3-)momentum of the muons in the laboratory frame, in MeV/c?

Method 1: $p = \gamma m v = \gamma m c^2 \frac{v}{c}\frac{1}{c} = (300\,\text{MeV})\left(\frac{2\sqrt{2}}{3}\right)\frac{1}{c}$

$\boxed{p = 200\sqrt{2}\ \text{MeV}/c}$

Method 2: $E^2 - p^2c^2 = m^2c^4 \rightarrow p = \frac{\sqrt{E^2 - m^2c^4}}{c} = \frac{\left[(300\,\text{MeV})^2 - (100\,\text{MeV})^2\right]^{1/2}}{c}$

$= \frac{100}{c}\sqrt{9-1}\ \text{MeV}$

$\boxed{p = 200\sqrt{2}\ \text{MeV}/c}$

[4] f) How far on average do the muons travel before decaying in the laboratory frame, in meters? (Hint: You may need the result of part g.)

$$\Delta x = v \Delta t$$
$$= \frac{v}{c} c \Delta t$$
$$= \left(\frac{2\sqrt{2}}{3}\right)(3 \times 10^8 \tfrac{m}{s})(6 \times 10^{-6} s) \quad = \boxed{12\sqrt{2} \times 10^2 \ m}$$

[1] g) Freebie point: Write down $c = 3 \times 10^8$ m/s if you have read this far.

$$\boxed{c = 3 \times 10^8 \ m/s}$$

or for the literalists,

$$\boxed{c = 3 \times 10^8 \ m/s \text{ if you have read this far}}$$