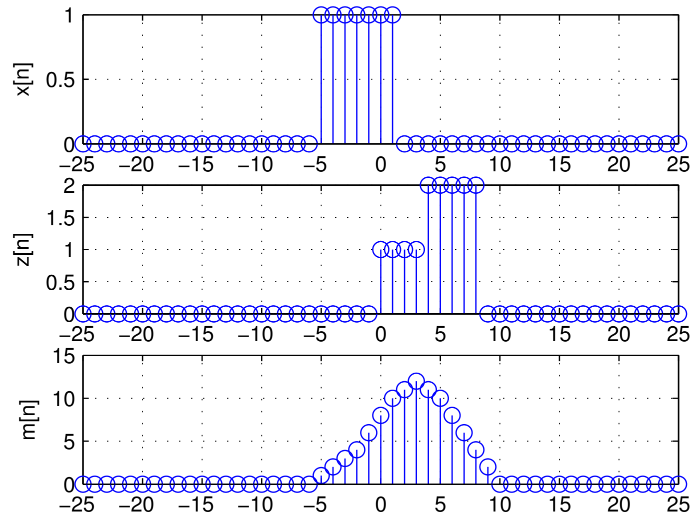
ECE 314 – Signals and Systems Fall/2012

Solutions to Homework 4

Problem 2.34 Consider the discrete-time signals depicted in Fig. P2.34 (textbook). Evaluate the following convolution sums:
(a) $m[n] = x[n] * z[n]$
Solution:

$$\begin{aligned} m[n] &= \sum_{k=-\infty}^{\infty} x[k]z[n-k] \\ &= \sum_{k=-5}^{1} 1 \cdot z[n-k] \end{aligned}$$

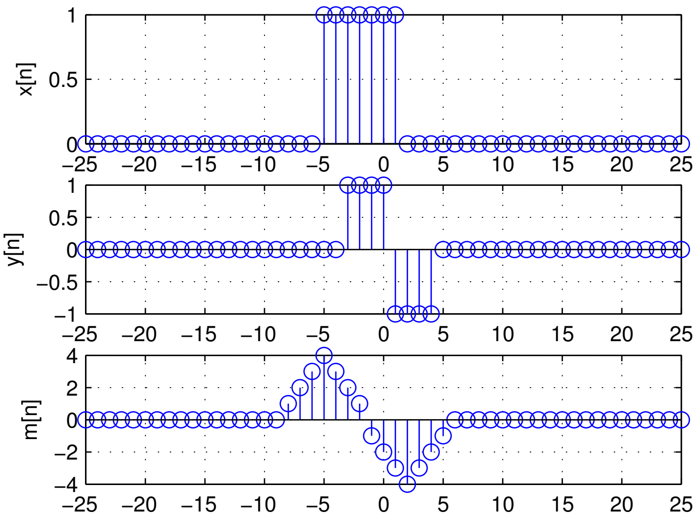
- For $n < -5$, $m[n] = 0$;
- for $-5 \leq n < -1$, $m[n] = n + 6$;
- for $-1 \leq n < 2$, $m[n] = 2n + 8$;
- for $2 \leq n < 4$, $m[n] = n + 9$;
- for $4 \leq n < 5$, $m[n] = 11$;
- for $5 \leq n < 10$, $m[n] = -2n + 20$;
- for $n \geq 10$, $m[n] = 0$.



(b) $m[n] = x[n] * y[n]$
Solution:

$$\begin{aligned} m[n] &= \sum_{k=-\infty}^{\infty} x[k]y[n-k] \\ &= \sum_{k=-5}^{1} 1 \cdot y[n-k] \end{aligned}$$

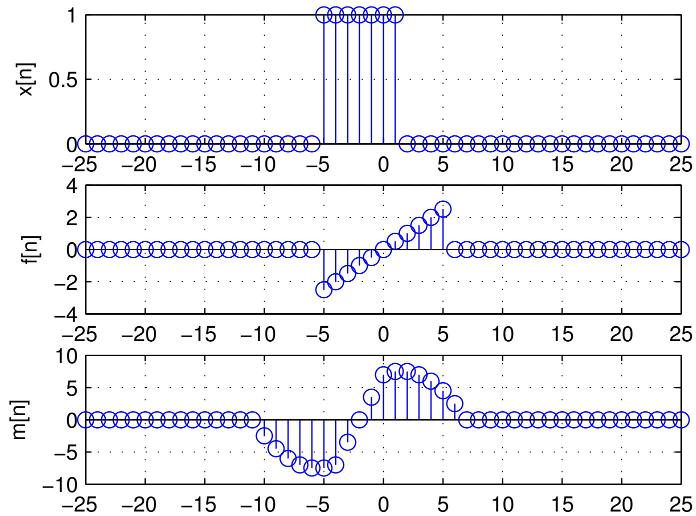
- For $n < -8$, $m[n] = 0$;
- for $-8 \leq n < -4$, $m[n] = n + 9$;
- for $-4 \leq n < -1$, $m[n] = -n - 1$;
- for $-1 \leq n < 0$, $m[n] = -1$;
- for $0 \leq n < 3$, $m[n] = -n - 2$;
- for $3 \leq n < 6$, $m[n] = n - 6$;
- for $n \geq 6$, $m[n] = 0$.



(c) $m[n] = x[n] * f[n]$
Solution:

$$
\begin{aligned}
m[n] &= \sum_{k=-\infty}^{\infty} x[k] f[n-k] \\
&= \sum_{k=-5}^{1} 1 \cdot f[n-k]
\end{aligned}
$$

- For $n < -10$, $m[n] = 0$;
- for $-10 \leq n < -3$, $m[n] = \sum_{k=-5}^{n+5} 0.5k = \frac{(n+11)n}{4}$;
- for $-3 \leq n < 1$, $m[n] = \sum_{k=n-1}^{n+5} 0.5k = \frac{7(2n+4)}{4} = 3.5n + 7$;
- for $1 \leq n < 7$, $m[n] = \sum_{k=n-1}^{5} 0.5k = \frac{(7-n)(n+4)}{4}$;
- for $n \geq 7$, $m[n] = 0$.



(k) $m[n] = x[n] * f[n]$

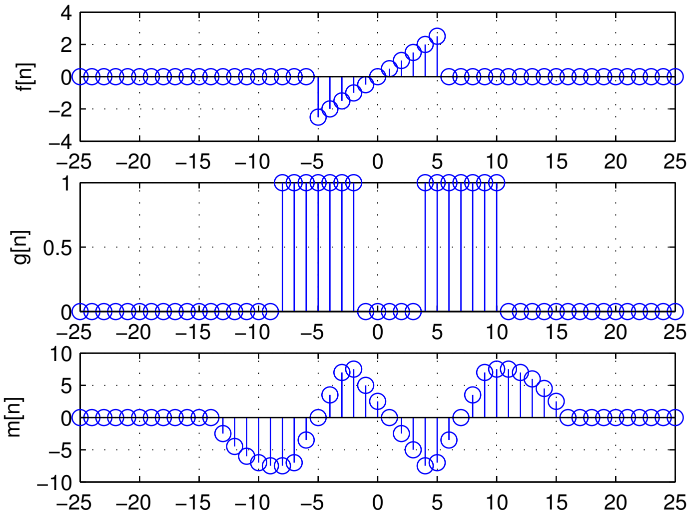
Solution: For this item, we may use the fact that

$$g[n] = x[n+3] + x[n-5].$$

Hence, from the fact that the convolution is both linear and time-invariant, we have

$$\begin{aligned} m[n] &= (x[n+3] + x[n-5]) * f[n] \\ &= (x[n+3] * f[n]) + (x[n-5] * f[n]) \\ &= m_c[n+3] + m_c[n-5], \end{aligned}$$

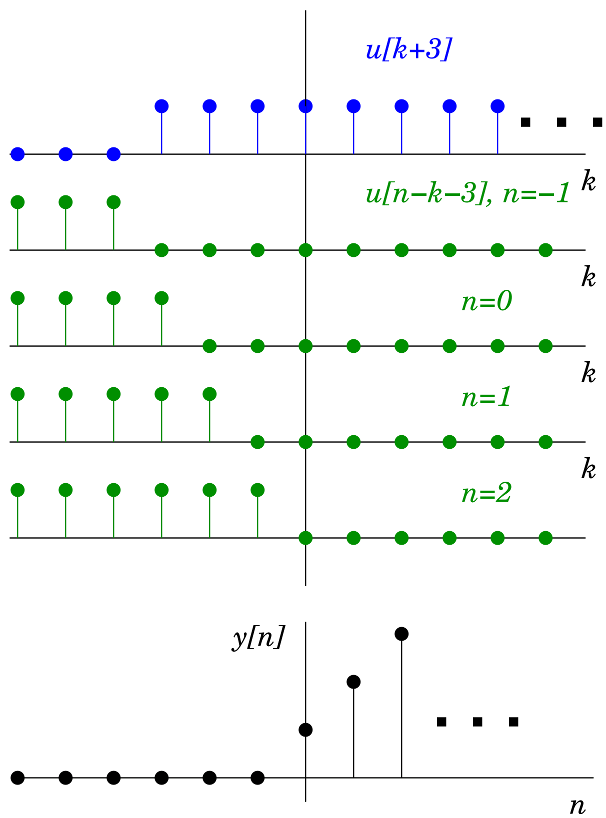
where $m_c[n]$ is the result of item (c).


f[n]
g[n]
m[n]

Problem 2.33 Evaluate the following discrete-time convolution sums:
(a) $y[n] = u[n+3] * u[n-3]$
Solution: By definition

$$y[n] = \sum_{k=-\infty}^{\infty} u[k+3]u[n-k-3].$$

The figure below shows the graph of $u[k+3]$ and $u[n-k-3]$, for some values of n, and the result of the convolution sum.



Analitically, we can evaluate $y[n]$ by considering that $u[k+3] = 0$, for $k < -3$, and $u[n-k-3] = 0$, for $k > n-3$. Hence,

$$y[n] = \begin{cases} 0, & n < 0, \\ \sum_{k=-3}^{n-3} 1 = n - 3 + 3 + 1 = n + 1, & n \geq 0 \end{cases}$$

(l) $y[n] = u[n] * \sum_{p=0}^{\infty} \delta[n - 4p]$

Solution:

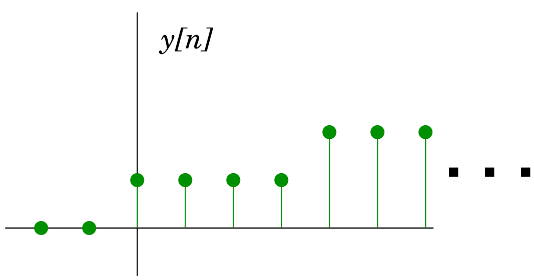
$$y[n] = \sum_{k=-\infty}^{\infty} \left[\sum_{p=0}^{\infty} \delta[k-4p] u[n-k] \right]$$

But, we know that $u[n-k] = 0$, for $k > n$, and $\sum_{p=0}^{\infty} \delta[k-4p] = 0$, for $k < 0$. Thus

$$y[n] = \sum_{k=0}^{n} \left[\sum_{p=0}^{\infty} \delta[k-4p] \right].$$

Therefore, at each multiple of 4, $y[n]$ is incremented by 1, starting on $n = 0$. We can write $y[n]$ as

$$y[n] = \sum_{p=0}^{\infty} u[n-4p].$$



Problem 2.39 Evaluate the following continuous-time convolution integrals:

(a) $y(t) = (u(t) - u(t-2)) * u(t)$

Solution: To evaluate the convolution integral defined as

$$y(t) = \int_{-\infty}^{\infty} [u(\lambda) - u(\lambda - 2)] u(t - \lambda) d\lambda,$$

we point out that $u(t-\lambda)$, as a function of λ, is 1 in the interval $(-\infty, t]$, and is zero elsewhere. While $u(\lambda)-u(\lambda-2)$ corresponds to a rectangular pulse that equals 1 in the interval $[0,2)$. Therefore:

- when $t<0$, the signals do not overlap, and the integral is zero;
- when $0\le t<2$, $y(t)$ corresponds to the integral of 1 between 0 and t;
- and when $t\ge 2$, $y(t)$ corresponds to the integral of 1 between 0 and 2.

Hence,

$$y(t)=\begin{cases}0, & t<0,\\ t, & 0\le t<2,\\ 2, & t\ge 2.\end{cases}$$

(i) $y(t)=(2\delta(t+1)+\delta(t-5))*u(t-1)$
Solution: Using the fact that integration is a linear operation, and that $u(t-\lambda-1)$, as a function of λ is 1 in the semi-interval $(-\infty, t-1]$, we have

$$\begin{aligned}y(t) &= \int_{-\infty}^{\infty}(2\delta(\lambda+1)+\delta(\lambda-5))u(t-\lambda-1)d\lambda,\\ &= 2\int_{-\infty}^{\infty}\delta(\lambda+1)u(t-\lambda-1)d\lambda+\int_{-\infty}^{\infty}\delta(\lambda-5)u(t-\lambda-1)d\lambda\\ &= 2u(t-(-1)-1)+u(t-5-1)\\ &= 2u(t)+u(t-6)\end{aligned}$$

(k) $y(t)=e^{-\gamma t}u(t)*(u(t+2)-u(t))$
Solution: The convolution integral may be expressed as

$$y(t)=\int_{-\infty}^{\infty}e^{-\gamma(t-\lambda)}u(t-\lambda)(u(\lambda+2)-u(\lambda))d\lambda.$$

We point out that $u(\lambda+2)-u(\lambda)$ is a rectangular pulse that equals 1 in the interval $[-2,0)$, while $e^{-\gamma(t-\lambda)}u(t-\lambda)$ is zero for $\lambda>t$. So, when $t<-2$, the convolution integral is zero. When $-2\le t<0$, the convolution corresponds to the integral of $e^{-\gamma(t-\lambda)}$, with respect to λ in the interval $[-2,t)$. And when $t\ge 0$, the convolution corresponds to the same integral in the interval $[-2,0)$. Therefore,

$$y(t)=\begin{cases}0, & t<-2,\\ \frac{1}{\gamma}(1-e^{-\gamma(t+2)}), & -2\le t<0\\ \frac{1-e^{-2\gamma}}{\gamma}e^{-\gamma t}, & t\ge 0\end{cases}$$

Problem 2.40 Consider the continuous time signals depicted in Fig. P2.40 of the text-book. Evaluate the following convolution integrals:
(b) $m(t) = x(t) * z(t)$

Solution: We use the figure below to help us evaluate the integral (also see the figure in the next page):

- For $t < -2$, $m(t) = 0$;
- for $-2 \le t < -1$,
$$m(t) = \int_{-1}^{t+1} (-1) d\lambda = -t - 2;$$
- for $-1 \le t < 0$,
$$m(t) = -1 + \int_{0}^{t+1} d\lambda = t;$$
- for $0 \le t < 1$,
$$m(t) = \int_{t-1}^{0} (-1) d\lambda + 1 = t;$$
- for $1 \le t < 2$,
$$m(t) = \int_{t-1}^{1} d\lambda = -t + 2;$$
- for $t \ge 2$, $m(t) = 0$.

(c) $m(t) = x(t) * f(t)$

Let us flip x since it is even.
For $t < -1$ and $t > 2$, $m(t) = 0$.
For $-1 \le t \le 0$, $m(t) = \int_0^{1+t} e^{-\tau} d\tau = 1 - e^{-t-1}$.
For $0 < t \le 1$, $m(t) = \int_0^1 e^{-\tau} d\tau = 1 - e^{-1}$.
For $1 \le t \le 2$, $m(t) = \int_{-1+t}^1 e^{-\tau} d\tau = e^{1-t} - e^{-1}$.

Problem 2.49 For each of the following impulse responses, determine whether the corresponding system is (i) memoryless, (ii) causal, and (iii) stable.

(a) $h(t) = \cos(\pi t)$

Solution:

(i) This system is **not memoryless**, since $h(t)$ is not zero for $t \neq 0$.
(ii) This system is **not causal**, since $h(t)$ is not zero for $t < 0$.
(iii) This system is **not stable**, since $\int_{-\infty}^{\infty} |h(t)|dt = \infty$ (i.e., it is not *absolutely integrable*).

(b) $h(t) = e^{-2t}u(t-1)$

Solution:

(i) This system is **not memoryless**, since $h(t)$ is not zero for $t \neq 0$.
(ii) This system is **causal**, since $h(t)$ is zero for $t < 0$.
(iii) This system is **stable**, since $\int_{-\infty}^{\infty} |h(t)|dt = e^{-2}/2$.

(c) $h(t) = u(t+1)$

Solution:

(i) This system is **not memoryless**, since $h(t)$ is not zero for $t \neq 0$.
(ii) This system is **not causal**, since $h(t)$ is not zero for $t < 0$.
(iii) This system is **not stable**, since $\int_{-\infty}^{\infty} |h(t)|dt = \infty$.

(d) $h(t) = 3\delta(t)$

Solution:

(i) This system is **memoryless**, since $h(t)$ is zero for $t \neq 0$.
(ii) This system is **causal**, since it is memoryless.
(iii) This system is **stable**, since $\int_{-\infty}^{\infty} |h(t)|dt = 3$.

(e) $h(t) = \cos(\pi t)u(t)$

Solution:

(i) This system is **not memoryless**, since $h(t)$ is not zero for $t \neq 0$.
(ii) This system is **causal**, since $h(t)$ is zero for $t < 0$.
(iii) This system is **not stable**, since $\int_{-\infty}^{\infty} |h(t)|dt = \infty$.

(h) $h[n] = \cos(\pi n/8)\{u[n] - u[n-10]\}$.

(i) The system has memory since $h[1] \neq 0$.

(ii) Since $h[n] = 0$ when n is negative, the system is causal.

(iii) Since $\sum_{n=-\infty}^{\infty} |h[n]|$ is finite (since this is a finite sum, i.e., there is a finite number of terms in the summation), the system is stable.

(i) $h[n] = 2u[n] - 2u[n-5]$.

(i) The system has memory since $h[1] \neq 0$.

(ii) Since $h[n] = 0$ when n is negative, the system is causal.

(iii) Since $\sum_{n=-\infty}^{\infty} |h[n]|$ is finite (since this is a finite sum, i.e., finite number of terms in the summation), the system is stable.

(k) $h[n] = \sum_{p=-1}^{\infty} \delta[n-2p]$.

(i) The system has memory since $h[-2] = 1 \neq 0$.

(ii) Since $h[-2] \neq 0$, the system is noncausal.

(iii) Since $\sum_{n=-\infty}^{\infty} |h[n]| = \infty$, the system is unstable.

(g) $h[n] = (0.5)^{|n|}$.

(i) The system has memory since $h[n] \neq 0$ whenever $n \neq 0$.

(ii) Since $h[n] \neq 0$ when n is negative, the system is noncausal.

(iii) Since $\sum_{n=-\infty}^{\infty} |h[n]|$ is finite (what is it?), the system is stable.

Problem 2.50 Evaluate the step response for the LTI systems represented by the following impulse responses:

(a) $h[n] = (-.5)^n u[n]$. For $n < 0$, $g[n] = 0$. For $n \geq 0$,

$$\begin{aligned}
g[n] &= h[n] * u[n] \\
&= \sum_{k=-\infty}^{\infty} h[n-k]u[k] \\
&= \sum_{k=0}^{\infty} h[n-k] \\
&= \sum_{k=0}^{n} (-0.5)^{n-k} \\
&= (-0.5)^n \frac{1-(-(0.5^{-1}))^{n+1}}{1-(-0.5^{-1})} = \{(-0.5)^n[(1-(-2)^{n+1}]\}/3 \\
&= \frac{(-0.5)^n + 2}{3}.
\end{aligned}$$

(b) $h[n] = \delta[n] - \delta[n-2]$

Solution: The step response is given by the convolution sum between the impulse response and the step function:

$$\begin{aligned}
g[n] &= h[n] * u[n] \\
&= \sum_{k=-\infty}^{\infty} h[n-k]u[k] \\
&= \sum_{k=0}^{\infty} \delta[n-k] - \delta[n-2-k] \\
&= u[n] - u[n-2]
\end{aligned}$$

(d) $h[n] = nu[n]$. For $n < 0$, $g[n] = 0$. For $n \geq 0$,

$$\begin{aligned}
g[n] &= h[n] * u[n] \\
&= \sum_{k=-\infty}^{\infty} h[n-k]u[k] \\
&= \sum_{k=0}^{\infty} h[n-k] \\
&= \sum_{k=0}^{n} (n-k) \\
&= n(n+1) - n(n+1)/2 = n(n+1)/2.
\end{aligned}$$

Hence, $g[n] = 0.5n(n+1)u[n]$.

(e) $h(t) = e^{-|t|}$

Solution:

$$\begin{aligned}
g(t) &= h(t) * u(t) \\
&= \int_{\tau=\infty}^{\infty} h(t-\tau)u(\tau)d\tau \\
&= \int_{\tau=0}^{\infty} e^{-|t-\tau|}d\tau
\end{aligned}$$

We have to consider two cases:

1) $t \geq 0$;

$$\begin{aligned} g(t) &= \int_{\tau=0}^{t} e^{-(t-\tau)} d\tau + \int_{\tau=t}^{\infty} e^{t-\tau} d\tau \\ &= e^{-t}(e^{t}-1) + e^{t}e^{-t} \\ &= 2 - e^{-t}. \end{aligned}$$

2) $t < 0$;

$$\begin{aligned} g(t) &= \int_{\tau=0}^{\infty} e^{t-\tau} d\tau \\ &= e^{t}. \end{aligned}$$