ANSWERS TO PROBLEMS ON FINAL REVIEW

I. BASICS

1. (b) negative (c) out of plane
2. (a) (a) $\mathbf{u}\cdot\mathbf{v} = |u||v|\cos\theta = 3\sqrt{2}$, $|\mathbf{u}\times\mathbf{v}| = |u||v|\sin\theta = 3\sqrt{2}$, $\mathbf{u}\times\mathbf{v}$ points out of the plane
   (b) $\mathbf{u}\cdot\mathbf{v} = -3\sqrt{3}$, $|\mathbf{u}\times\mathbf{v}| = 3$, $\mathbf{u}\times\mathbf{v}$ points out of the plane
3. $\mathbf{u}\cdot\mathbf{v} = 0$, $|\mathbf{u}\times\mathbf{v}| = 6$, out of plane, $\mathbf{u}\cdot\mathbf{u} = 4$, $\mathbf{u}\times\mathbf{u} = \mathbf{0}$.
5. (a) length=$5/\sqrt{2}$, vector $proj_{\mathbf{a}}\mathbf{b} = -(5/2)\langle 1,0,-1\rangle$
   (b) length=$5/\sqrt{33}$, vector $proj_{\mathbf{a}}\mathbf{b} = -(5/33)\langle -1,4,4\rangle$
   (c) $\sqrt{41}/\sqrt{33}$
   (d) $(\sqrt{41}/\sqrt{33})\cdot\sqrt{33}/2 = \sqrt{41}/2$
   (e) $\cos^{-1}(-5/\sqrt{66}) = 2.2338 = 127^o$
   (f) $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$
6. (a) $7x + 2y - z = 0$ (b) $\mathbf{r}(t) = \langle 7,2,-1\rangle t$
7. $x - 1 - z = 0$
8. $(x-4) + 2(y-1) + 5(z-2) = 0$
9. $\mathbf{r}(t) = \langle -6,-1,0\rangle + t\langle 8,-2,5\rangle$
10. (b) $5/\sqrt{53}$
11. (a) intersect at (3,-1,4) (b) skew (c) parallel, not intersecting
12. Intersects xy-plane at (-2,4,0), xz-plane at (2,0,4), yz-plane at (0,2,2)
13. (in the exam you need to sketch a graph, here we give only an indication of the answer)
    (a) paraboloid about x-axis
    (b) single cone about z-axis
    (c) double cone about z-axis
    (d) plane parallel to yz-plane
    (e) the plane obtained by translating the curve $y = 4/x$ in the z-direction unchanged
    (f) the yz-plane and the xz-plane
    (g) half-plane
    (h) cone opening downward
    (i) z-axis
    (j) sphere of radius 4
    (k) sphere of radius 2 centered at (0,0,2)
    (l) torus (scratch this problem, you are not responsible for this)
    (m) plane $y = 3$
    (n) cylinder centered on z-axis of radius 3
    (o) cylinder centered on the line $x = 1/2, y = 0$ of radius 1/2
    (p) a plane independent of y, making a 45 degree angle with the yz plane.
    (q) a plane independent of z
    (r) parabolic cylinder independent of x
    (s) ellipsoid centered at (0,0,1)

(t) hyperboloid of one sheet
(u) plane

14. (a) $\sqrt{z^2+y^2}$ (b) x (c) $y^2+z^2=4x^2$

II. VECTOR FUNCTIONS

1. (a) helix (b) ellipse (c) line (d) graph of $x=y^4+1$ (e) oscillating z-coordinate in the plane $x=y$

2. (a) circle, $\mathbf{T}=\langle 1,-1\rangle/\sqrt{2}$
   (b) graph of $y=\pm x^{3/2}$, traversed in the upward direction, $\mathbf{T}=\dfrac{\langle 2,\frac{3}{2}\rangle}{\sqrt{13}}$
   (c) graph of $y=(x-1)^2$, $\mathbf{T}=\langle 1,2\rangle/\sqrt{5}$

3. (a) $\mathbf{r}(t)=\langle t,0,0\rangle$ (b) $\theta=\cos^{-1}(1/\sqrt{6})\approx 65^o$

4. velocity $\mathbf{r}'(t)=\langle 2t,1/t,1\rangle$, acceleration $\mathbf{r}''(t)=\langle 2,-1/t^2,0$, speed $\sqrt{4t^2+1/t^2+1}$

5. velocity $\mathbf{v}(t)=\mathbf{k}t+(\mathbf{i}-\mathbf{j}-\mathbf{k})$, position $\mathbf{s}(t)=\mathbf{k}(t^2/2)+(\mathbf{i}-\mathbf{j}-\mathbf{k})t+(-\mathbf{i}+\mathbf{j}+\mathbf{k}/2)$

III. SCALAR FUNCTIONS

1. (a) elliptical paraboloid
   (b) circular one-sided cone
   (c) plane
   (d) parabolic cylinder ("trough", parabola in the yz plane, translated unchanged in the x direction)
   (e) saddle

2. (a) spheres
   (b) planes
   (c) hyperbolic cylinders (hyperbolas in the xy-plane, translated unchanged in the z direction)

3. $df/dt=1$

4. $\frac{\partial u}{\partial s}=t^3+(st+t^2+1)te^{st}\Big|_{s=0,t=1}=3$ $\frac{\partial u}{\partial t}\Big|_{s=0,t=1}=2$

5. (a) $-9/(2\sqrt{14})$, (b)$1/2$, (c) $\langle 1/2,-1,-1\rangle$, (d) $3/2$, (e) $\langle -1/2,1,1\rangle$, (f) $\langle 1/2,-1,-1\rangle$, (g) $(x-4)-2(y-1)-2(z-1)=0$

6. (a) parabolas, (b) $\langle 1,-2\rangle$, (c) it increases, (d) $\langle 1,-2\rangle$, (e) $\sqrt{5}$

8. (a) $\mathbf{F}(x,y,z)=z-x^2-y^2$, $\mathbf{n}=\nabla\mathbf{F}\Big|_P=\langle -2,-2,1\rangle$, $-2(x-1)-2(y-1)+(z-2)=0$
   (b) $\mathbf{F}(x,y,z)=x+y-2z$, $\mathbf{n}=\nabla\mathbf{F}\Big|_P=\langle 1,1,-2\rangle$, $(x-2)+1(y-1)-2(z+1)=0$
   (c) $\mathbf{F}(x,y,z)=xy+xz+yz$, $\mathbf{n}=\nabla\mathbf{F}\Big|_P=\langle 2,2,2\rangle$, $(x-1)+(y-1)+(z-1)=0$
   (d) $\mathbf{F}(x,y,z)=x^2-2y^2-3z^2+xyz$, $\mathbf{n}=\nabla\mathbf{F}\Big|_P=\langle 4,11,-12\rangle$, $4(x-3)+11(y+2)-12(z-1)=0$

(e) $\mathbf{F}(x,y,z) = z - f(x,y) = z - xe^y - 3y$, $\mathbf{n} = \nabla \mathbf{F}\Big|_P = \langle -1,-4,1\rangle$, $-(x-1)-4y+(z-1)=0$

9. (a) Local minimum at (0,0), saddles at $(\pm\sqrt{2},-1)$
   (b) Local minimum at (0,0), local maximum at (-5/3,0), saddles at $(-1,\pm 2)$

10. Maximum $f(\pm 1,1)=7$, Minimum $f(0,0)=4$.

11. $\Delta z \approx \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y$

12. (a) $\frac{\partial h}{\partial v}(40,20) \approx \frac{v(50,20)-v(30,20)}{20} = 1.15$ $\frac{\partial h}{\partial t}(40,20) \approx \frac{v(40,20)-v(40,15)}{5} = 0.6$
    (b) $\Delta h \approx 1.15\Delta v + 0.6\Delta h = 5.85$
    (c) $h(43,24) = h(40,20) + \Delta h \approx 33.85$

IV. INTEGRALS

1. (a) $f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$ (b) $\frac{1}{2\sin 1}$

2. (a) $\int\int_D f(x,y)\,dA \approx 10\pi$ (b) mass$\approx 3\cdot\frac{\pi(10)^2 10}{3} = 1000\pi$ grams

3. $\pi/4 - 1/2$

4. (a) Cartesian $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dzdydx$

   Cylindrical $\int_0^{2\pi}\int_0^a\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} dzrdrd\theta$

   Spherical $\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^2\sin\phi d\rho d\phi d\theta$

   (b) $4\pi a^3/3$

5. 11/20

6. 2/15

7. (a) $\pi/a^2 + \int_0^{2\pi}\int_{\frac{1}{a}}^{\frac{2}{a}}\int_{ar}^{2} dzrdrd\theta$

   (b) $\int_0^{2\pi}\int_1^2\int_0^{z/a} rdrdzd\theta$

   (c) $7\pi/(3a^2)$

Problems 8-11,13-18,20. Answers on review sheet.

19. Yes. $f(x,y,z) = x^3yz - 3xy + z^2$

21. Notice that $\mathbf{F}$ is conservative since $Q_x - P_y = 0$. Therefore the line integral over a closed curve $= 0$.

V-VI. VECTOR FIELDS AND GREENS THEOREM

1. (a) constant vector field
   (b) vectors point radially outward, with magnitude 1/r
   (c) vectors point normal to position vector (rotation), with magnitude = r

2. curl $\mathbf{F} = \langle -2z\sin y, x^2, 2\sin y\rangle$, div $\mathbf{F} = 2xz + 2x\cos y + 2\cos y$

3. (a) x-component=1, y-component=0 on axis, otherwise always positive, and increasing with increasing distance from the x-axis.
   (b) curl $\mathbf{u} = \mathbf{0}$, div$\mathbf{u} = 2y$.
   (c) compression for $y < 0$, expansion for $y > 0$ (this can also be seen from graph in (a))
   (d) =0, since $\mathbf{F}$ conservative.
4. (a) Show that curl$\nabla f = \mathbf{0}$. (See Stewart, p.1111)
   (b) No, since curl$\mathbf{F} \neq \mathbf{0}$
5. (a) Show that div(curl$\mathbf{F}$) = 0. (similar to 4a)
   (b) No, since div$\mathbf{F} \neq 0$
6. (a) zero (b) positive (c) negative (d) zero ($\mathbf{F}$ constant, therefore conservative) (e) negative
7. No, since there is a closed curve such that $\int_C \mathbf{F} \cdot \mathbf{T} ds \neq 0$ (show such a curve on graph)