Example of the Improved Euler's Method

PROBLEM: Given that

$$\frac{dy}{dx} = xy,$$

and $y(0) = 1$, use Heun's formula to approximate $y(0.9)$ with a step-size of 0.3. What is the absolute error?

SOLUTION: We are asked to approximate $y(0.9)$ with a step-size of 0.3, so we will need to apply Heun's formula three times: $x_0 = 0$, $x_1 = 0.3$, $x_2 = 0.6$, and $x_3 = 0.9$.

1. STEP ONE: Since $F(y,t) = ty$ and $y(0) = 1$, we see that

$$F(x_0, y_0) = F(0,1) = 0 \cdot 1 = 0,$$

so that

$$y_1^* = y_0 + h \cdot F(x_0, y_0) = 1 + 0.3 \cdot 0 = 1.$$

Note that if we were using the basic Euler method, then we would be done at this point. However, the improved Euler method requires that we now use Heun's formula. We first calculate

$$F(x_1, y_1^*) = F(0.3, 1) = 0.3 \cdot 1 = 0.3,$$

so Heun's Formula gives

$$y_1 = y_0 + h\frac{F(x_0,y_0) + F(x_1,y_1^*)}{2} = 1 + 0.3\frac{F(0,1) + F(0.3,1)}{2} = 1 + 0.2\frac{0+0.3}{2} = 1 + 0.2 \cdot 0.15 = 1.03.$$

This is the approximate value for $y(0.2)$.

2. STEP TWO: Now we have that

$$F(x_1, y_1) = F(0.3, 1.03) = 0.3 \cdot 1.03 = 0.309.$$

Then

$$y_2^* = y_1 + h \cdot F(x_1, y_1) = 1.03 + 0.3 \cdot 0.309 = 1.1227$$

and

$$F(x_2, y_2^*) = F(0.6, 1.1227) = 0.6 \cdot 1.1227 = 0.67362.$$

Heun's formula gives

$$\begin{aligned} y_2 = y_1 + h\frac{F(x_1,y_1) + F(x_2,y_2^*)}{2} &= 1.03 + 0.3\frac{F(0.3,1.03) + F(0.6,1.1227)}{2} \\ = 1.03 + 0.3\frac{0.309 + 0.67362}{2} &= 1.03 + 0.147393 = 1.177393. \end{aligned}$$

This is the approximate value of $y(0.6)$.

3. STEP THREE: We make the relevant calculations as follows:

$$F(x_2, y_2) = F(0.6, 1.177393) = 0.6 \cdot 1.177393 = 0.7064358$$

$$y_3^* = y_2 + h \cdot F(x_2, y_2) = 1.177393 + 0.3 \cdot 0.7064358 = 1.38932374$$

$$F(x_3, y_3^*) = F(0.9, 1.177393) = 0.9 \cdot 1.38932374 = 1.250391366$$

$$\begin{aligned} y_3 &= y_2 + h\frac{F(x_2, y_2) + F(x_3, y_3^*)}{2} = 1.177393 + 0.3\frac{F(0.6, 1.177393) + F(0.9, 1.38932374)}{2} \\ &= 1.03 + 0.3\frac{0.7064358 + 1.250391366}{2} = 1.03 + 0.2935240749 = 1.323524075. \end{aligned}$$

This is the approximate value of $y(0.9)$, which we wanted. We conclude that $y(0.9) \approx 1.323524075$. It is not difficult to find that the particular solution to this IVP is

$$y(x) = e^{\frac{t^2}{2}},$$

so that $y(0.9) = 1.4993025$. So the absolute error is $|1.4993025 - 1.323524075| = 0.1757784252$.