ECE 345: Introduction to Control Systems
Midterm #1

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This midterm is closed-note, closed book, and no electronic devices are allowed. A one-sided, one-page cheat-sheet is allowed and *must be handed in with your midterm.* The cheat-sheet will be returned you with your graded midterm.

For full credit, show all your work.

SOLUTIONS

Student Name   Student ID #

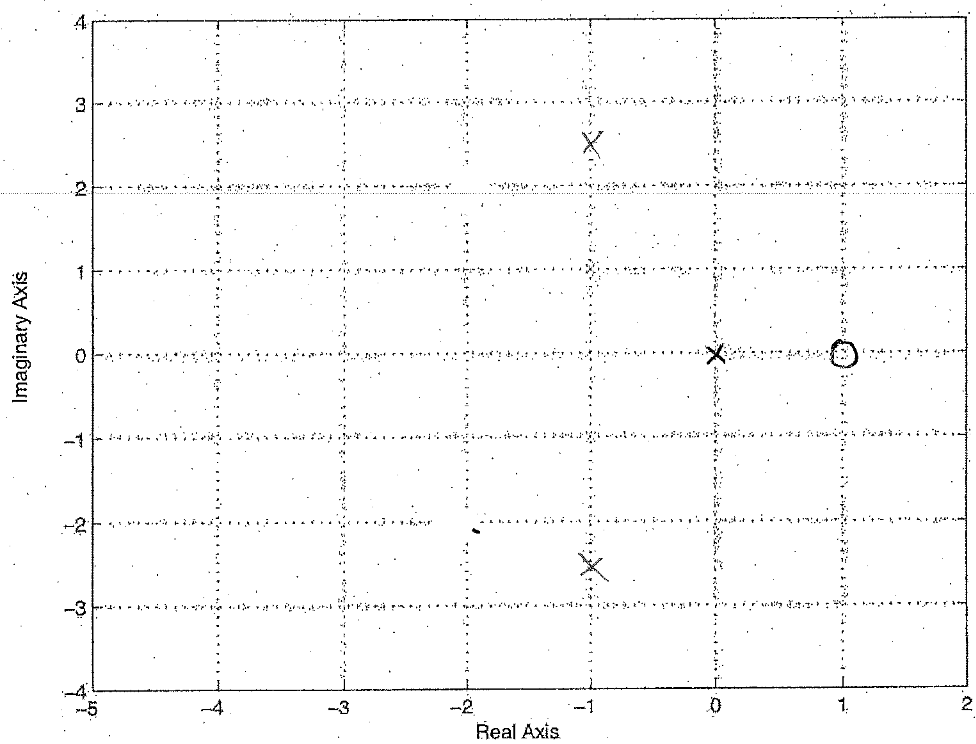
Problem #	Actual points	Possible points
1		10
2		30
3		50
Total:		90

1 Warm-Up (10 points)

Consider the transfer function

$$G(s) = \frac{s-1}{s(s^2+2s+8)} \tag{1}$$

1. Sketch the poles and zeros of the transfer function on the complex plane plot below. Mark the zeros with '∘' and the poles with '×'.



$s^2 + 2s + 8 = (s+1)^2 + \sqrt{7}^2$

$\sqrt{2} = 4 < \sqrt{7} < \sqrt{9} = 3$

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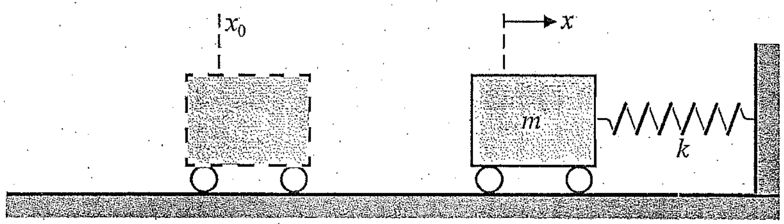
2 Wind-up Toy Car (30 points)

Consider a toy car mounted on spring-loaded wheels (e.g., a wind-up toy). We model the car as a mass m with friction coefficient b. We define the position of the mass by $x(t)$, where $x(t) = 0$ indicates the position at which the car is at rest. Winding up the wheels generates a spring force that acts on the car body, such that the more that the wheels are wound up, the more propulsive force the car will initially experience. Hence initially winding up the wheels is incorporated in the non-zero initial condition $x(0) = x_0$, with $\dot{x}(0) = 0$.



Rolling friction constant = b

The equations of motion are:

$$m\ddot{x} = -b\dot{x} - kx \tag{2}$$

1. With state vector $z(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$ and output that is the position of the car, which of the following system matrices satisfy the state-space form $\dot{z}(t) = Az(t)$, $y = Cz(t)$? (Note that without an input, matrices B and D are zero-valued.)

   (a) $A = \begin{bmatrix} 0 & 1 \\ -b & -k \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1/m \end{bmatrix}$

   (b) $A = \begin{bmatrix} 0 & 1 \\ -b/m & -k/m \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}$, $C = \begin{bmatrix} m & 0 \end{bmatrix}$

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} z_2 \\ -b/m\, z_2 - k/m\, z_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} z$$

$$y = z_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} z$$

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2. Assume that the parameters m, b, k are chosen such that $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

   (a) What is the state transition matrix $\Phi(s)$?

   (b) Solve for the natural response $x(t)$ with initial condition $z(0) = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$. (Full points for computing this using the state transition matrix, partial points otherwise.)

3. In designing parameters m, b, k for such a toy, it is desirable for the wound-up car to move only forward after release, and to never move backwards. Given a choice between creating a system that is (a) underdamped, (b) critically damped, and/or (c) overdamped, which would be most likely to satisfy the desired criteria? Provide a 1–2 sentence explanation of your choice(s).

2(a). $$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ +2 & s+3 \end{bmatrix}^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

(b). $$X(s) = C(sI-A)^{-1} z_0$$

$$= \frac{1}{s^2+3s+2}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 0 \end{bmatrix} = \frac{-4(s+3)}{s^2+3s+2} = \frac{-4s-12}{(s+2)(s+1)}$$

$$= \frac{A}{s+2} + \frac{B}{s+1}$$

$$= \frac{A(s+1) + B(s+2)}{(s+1)(s+2)} \Rightarrow$$

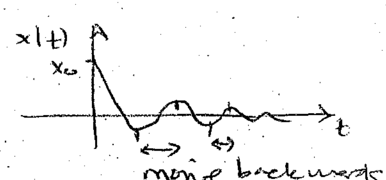
$A + B = -4$, $A + 2B = -12$

$A = -4 - B$ $\quad$ $B = -8 \Rightarrow A = 4$

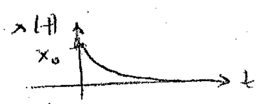
$$x(t) = \mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left\{\frac{4}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{-8}{s+1}\right\}$$

$$= (4e^{-2t} - 8e^{-t})\,u(t)$$

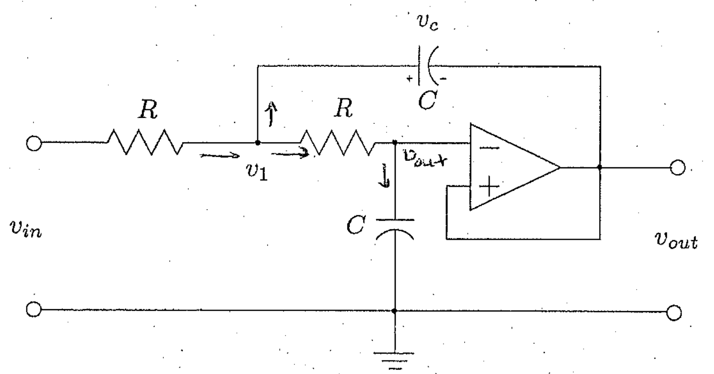
3. An underdamped system will oscillate & hence may end up moving backwards.



Critically damped & overdamped systems will not oscillate from any initial condition & hence will result in only forward movement.



3 RLC Op-Amp Circuit (50 points)



Consider the circuit in the above figure.

1. Show that the transfer function is $G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{\left(\frac{1}{RC}\right)^2}{\left(s+\frac{1}{RC}\right)^2}$. *Hint: Notice that* $v_c = v_1 - v_{out}$.

Node at v_1: KCL:

$$\frac{v_{in} - v_1}{R} = C\frac{d}{dt}(v_1 - v_{out}) + \frac{v_1 - v_{out}}{R}$$

$$V_{in}(s) - V_1(s) = RCs\,(V_1(s) - V_{out}(s)) + V_1(s) - V_{out}(s)$$

$$(*) \qquad V_{in}(s) = V_1(s)\left(RCs + 2\right) + V_{out}(s)\left(-RCs - 1\right)$$

Node at negative terminal of op-amp:

$$\frac{v_1 - v_{out}}{R} = C\frac{d}{dt}(v_{out})$$

$$V_1(s) = (RCs + 1)V_{out}(s)$$

Plug into $(*)$:

$$V_{in}(s) = V_{out}(s)\Big((RCs+1)(RCs+2) - (RCs+1)\Big)$$

$$V_{in}(s) = V_{out}(s)\Big((RCs+1)^2\Big)$$

$$\frac{1}{(RCs+1)^2} = \frac{V_{out}(s)}{V_{in}(s)}$$

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Recall $G(s) = \frac{\left(\frac{1}{RC}\right)^2}{\left(s+\frac{1}{RC}\right)^2}$.

2. Put the system into phase variable form. What are the system matrices A, B, C, D?

3. Which one of the following statements is correct?

   (a) The eigenvectors of A solve the characteristic equation.

   (b) The poles of the transfer function are equivalent to the eigenvalues of A.

   (c) The characteristic equation is equivalent to $|A| = 0$.

   (d) For a pole at some complex number $-p$, the transfer function has the value $G(-p) = 0$.

4. In the step response, what value does v_{out} approach in steady-state?

5. For a suitably fast response, we want the system to have a settling time of less than 2 seconds. Determine whether this system meets the desired criteria with $RC = 1$.

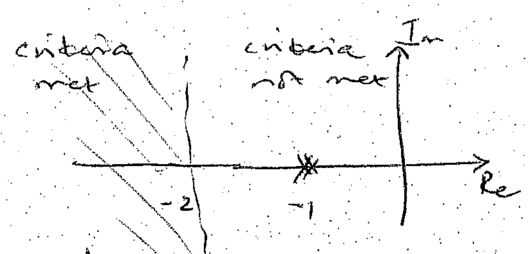
2. $A = \begin{bmatrix} 0 & 1 \\ -\left(\frac{1}{RC}\right)^2 & -\frac{2}{RC} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \left[\left(\frac{1}{RC}\right)^2 \;\; 0 \right]$, $D = 0$

3. (b)

4. $\lim_{s\to 0} s V_{out}(s) = \lim_{s\to 0} s \cdot \frac{1}{s} G(s) = \frac{(1/RC)^2}{(1/RC)^2} = 1$.

5. $T_s = \frac{4}{\zeta\omega_n} < 2$

$\Rightarrow 2 < \zeta\omega_n$



For RC=1, poles are at $\zeta\omega_n = 1$

hence settling time is 4s, & does NOT meet criteria.

End of exam.