ECE 314 – Signals and Systems Fall/2012

Solutions to Homework 8

3.54

(a) $x(t) = e^{-2t}u(t-3)$

Solution: We can write

$$x(t) = e^{-6}e^{-2(t-3)}u(t-3).$$

The Fourier transform of $y(t) = e^{-2t}u(t)$ is

$$Y(j\omega) = \frac{1}{2+j\omega}.$$

Hence,

$$\begin{aligned} X(j\omega) &= e^{-6}Y(j\omega)e^{-j3\omega} \\ &= \frac{e^{-6-j3\omega}}{2+j\omega}. \end{aligned}$$

(b) $x(t) = e^{-4|t|}$

Solution: We may write

$$x(t) = e^{-4t}u(t) + e^{4t}u(-t).$$

The Fourier transform of $y(t) = e^{-4t}u(t)$ is

$$Y(j\omega) = \frac{1}{4+j\omega}.$$

Hence,

$$\begin{aligned} X(j\omega) &= Y(j\omega) + Y(-j\omega) \\ &= \frac{1}{4+j\omega} + \frac{1}{4-j\omega} \\ &= \frac{8}{16+\omega^2}. \end{aligned}$$

(f) $x(t)$ as depicted in Figure P3.54(b).

Solution: Since $x(t)$ is real and even, thus

$$\begin{aligned} X(j\omega) &= 2\int_0^2 e^{-t}\cos(\omega t)dt \\ &= 2\left[\frac{e^{-t}\omega\sin(\omega t) - e^{-t}\cos(\omega t)}{1+\omega^2}\right]_{t=0}^{2} \\ &= 2\left[\frac{e^{-2}\omega\sin(2\omega) - e^{-2}\cos(2\omega)+1}{1+\omega^2}\right]. \end{aligned}$$

An alternative way is to consider that

$$x(t) = e^{-|t|}w_2(t),$$

where

$$w_2(t) = \begin{cases} 1, & |t| \le 2, \\ 0, & |t| > 2, \end{cases}$$

Therefore,

$$\begin{aligned} X(j\omega) &= \frac{1}{2\pi}\left\{\frac{2}{1+\omega^2}\right\} * \mathcal{F}\left\{w_2(t)\right\} \\ &= \frac{1}{2\pi}\left\{\frac{2}{1+\omega^2}\right\} * \left\{2\frac{\sin(2\omega)}{\omega}\right\} \\ &= \frac{4}{\pi}\left\{\frac{1}{1+\omega^2}\right\} * \operatorname{sinc}(2\omega), \end{aligned}$$

where

$$\operatorname{sinc}(\omega) \triangleq \frac{\sin(\omega)}{\omega}.$$

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(c) $X(j\omega) = e^{-2|\omega|}$

Solution: We can use the results of previous exercises if we take into account the property:

$$\mathcal{F}\left\{X(jt)\right\} = \int_{-\infty}^{\infty} X(jt)e^{-j\omega t}dt = 2\pi x(-\omega).$$

Since the Fourier transform of $y(t) = X(jt) = e^{-2|t|}$ is

$$Y(j\omega) = \frac{4}{4+\omega^2} = 2\pi x(-\omega).$$

Then

$$x(t) = \frac{1}{2\pi}\frac{4}{4+t^2}.$$

(d) $X(j\omega)$ as depicted in Figure P3.55(a).

Solution: $X(j\omega)$ is

$$X(j\omega) = e^{-j2\omega}P_2(j\omega),$$

where

$$P_{\omega_o}(j\omega) = \begin{cases} 1, & |\omega| \le \omega_o, \\ 0, & |\omega| > \omega_o. \end{cases}$$

The time-domain signal corresponding to $P_{\omega_o}(j\omega)$ is

$$p_{\omega_o}(t) = \frac{\sin \omega_o t}{\pi t}.$$

Hence,

$$x(t) = p_2(t-2) = \frac{\sin 2(t-2)}{\pi(t-2)}.$$

3.56

(a) $x(t) = e^{-t}\cos(2\pi t)u(t)$.

Solution: We can use the fact that $x(t)$ results from the modulation of $y(t) = e^{-t}u(t)$ by $\cos 2\pi t$. Thus,

$$X(j\omega) = \frac{1}{2}\left[Y(j(\omega - 2\pi) + Y(j(\omega + 2\pi)\right],$$

while

$$Y(j\omega) = \frac{1}{1+j\omega}.$$

(b) $x[n] = \begin{cases} \cos\left(\frac{\pi}{10}n\right) + j\sin\left(\frac{\pi}{10}n\right), & |n| < 10, \\ 0, & \text{otherwise.} \end{cases}$

Solution: In this case, $x[n] = e^{j\frac{\pi}{10}n} w_{10}[n]$. Thus, by the frequency shift property,

$$X\left(e^{j\Omega}\right) = W_{10}\left(e^{j\left(\Omega - \frac{\pi}{10}\right)}\right),$$

where the result for $W_N\left(e^{j\Omega}\right)$ has been found in Problem 3.52 (c).

(c) $x[n]$ as depicted in Figure P3.56(a).

Solution: From the definition:

$$X[k] = \frac{1}{7} - \frac{2j}{7}\sin\left(4\frac{2\pi}{7}k\right).$$

(d) $x(t) = e^{1+t}u(-t+2)$

Solution: We can write

$$x(t) = e^3 e^{t-2}u(-t+2) = e^3 y(-(t-2)),$$

where $y(t) = e^{-t}u(t)$, whose FT is $1/(1+j\omega)$.

Using homogeinity, time-shift, and time inversion properties,

$$X(j\omega) = e^3 Y(-j\omega)e^{-j2\omega}.$$

(e) $x(t) = |\sin 2\pi t|$

Solution: $x(t)$ may be written as

$$x(t) = \left[2s_{1,1/4}\left(t - \frac{1}{4}\right) - 1\right]\sin(2\pi t),$$

where $s_{T,T_o}(t)$ is the square wave of Figure 3.21. Using the Fourier series representation of $s_{1,1/4}$ and $\sin 2\pi t$, we have

$$\begin{aligned}
x(t) &= \left[2\sum_{k=-\infty}^{\infty} S_{1,1/4}[k]e^{j2\pi k(t-1/4)} - 1\right]\left(e^{j2\pi t} - e^{-j2\pi t}\right)\frac{1}{2j} \\
&= \left[\frac{2}{2j}\sum_{k=-\infty}^{\infty} S_{1,1/4}[k]e^{-j\pi k/2}e^{j2\pi(k+1)t} - S_{1,1/4}[k]e^{-j\pi k/2}e^{j2\pi(k-1)t}\right] \\
&\quad - \left(e^{j2\pi t} - e^{-j2\pi t}\right)\frac{1}{2j} \\
&= \sum_{k=-\infty}^{\infty}\left[\frac{2}{2j}\left(S_{1,1/4}[k-1]e^{-j\pi(k-1)/2} - S_{1,1/4}[k+1]e^{-j\pi(k+1)/2}\right)\right. \\
&\quad \left. - \frac{1}{2j}(\delta[k-1] - \delta[k+1])\right]e^{j2\pi kt} \\
&= \sum_{k=-\infty}^{\infty}\left[S_{1,1/4}[k-1]e^{-j\pi k/2} + S_{1,1/4}[k+1]e^{-j\pi k/2}\right. \\
&\quad \left. - \frac{1}{2j}(\delta[k-1] - \delta[k+1])\right]e^{j2\pi kt}
\end{aligned}$$

(f) $x[n]$ as depicted in Figure P3.56(b).
Solution:

$$x[n] = nw_4[n]$$

where $w_N[n]$ is the same as defined in 3.52 (c).

Then, we may use the property

$$\mathcal{F}\{nx[n]\} = j\frac{dX\left(e^{j\Omega}\right)}{d\Omega}$$

(g) $x(t)$ as depicted in Figure P3.56(c).
Solution: Let $s_{T,T_o}(t)$ be the square wave depicted in Fig. 3.21, whose Fourier coefficients are derived in Example 3.13. $x(t)$ may be written as

$$x(t) = s_{4,1}(t-2) + 2s_{4,0.5}(t-2.5).$$

Using the linearity and time-shift properties of the FT, we can derive $X[k]$ and $X(j\omega)$ based on the result of Example 3.13.