ECE 340: PROBABILISTIC METHODS IN ENGINEERING

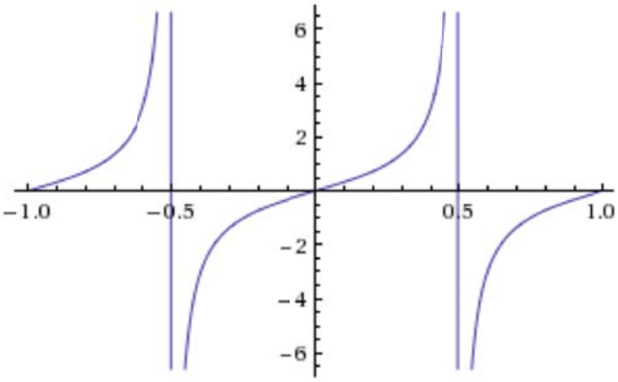
SOLUTIONS TO HOMEWORK #9

4.94

X is a uniformly distributed in the range of (-1,1), thus X has the following pdf

$$f_X(x) = \begin{cases} \frac{1}{2}, & if -1 < x < 1 \\ 0, & otherwise \end{cases}$$

Since $Y = \alpha \tan(\pi X)$, the plot of function $y = \alpha \tan(\pi x)$ is shown below:



Notice that for any given value y≠0, we have two solutions of x in range (-1,1), and only one solution for x if y=0. Let us denote by k the index of the solutions of x to the function $= \alpha \tan(\pi x)$. Now according the equation (4.74) on page 179 in book, we know

$$f_Y(y) = \sum_k f_X(x) \left|\frac{dx}{dy}\right|\Bigg|_{x=x_k}.$$

Since $y = \alpha \tan(\pi x)$, we know $x = \frac{1}{\pi}\tan^{-1}\frac{y}{\alpha}$, and $\frac{dx}{dy} = \frac{\frac{1}{\pi}\alpha}{\alpha^2+y^2}$

So,

$$f_Y(y) = \sum_k f_X(x) \left|\frac{dx}{dy}\right|\Bigg|_{x=x_k} = \frac{1}{2}\frac{\frac{1}{\pi}\alpha}{\alpha^2+y^2}\Bigg|_{x=x_1} + \frac{1}{2}\frac{\frac{1}{\pi}\alpha}{\alpha^2+y^2}\Bigg|_{x=x_2} \quad , if\ y \neq 0$$

For y=0, although there is only one solution at x=0, follow the steps on page 179 in book, we know that the values close to -1 and 1 also contribute to the probability of a portion of y close to 0. Due to the symmetry of the tan function, we can obtain

$$f_Y(0) = \frac{\frac{1}{2}\alpha}{\pi(\alpha^2+0^2)} \times 2 = \frac{\frac{1}{2}\alpha}{\pi(\alpha^2+0^2)}$$

To sum up, we have

$$f_Y(y) = \frac{\alpha}{\pi(\alpha^2+y^2)}, \quad -\infty < y < \infty$$

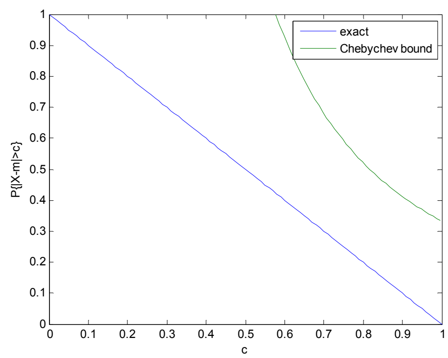
So, Y is a Cauchy random variable.

4.99

(a) Here, the mean, m, is 0. To avoid trivial cases, assume that 0<*c*<*b* (why?). Now, P{|*X*-*m*|>*c*}= P{*X*>*c*+*m*} + P{*X*<-*c*+*m*} = $\int_{c+m}^{\infty} f(x)dx + \int_{-\infty}^{-c+m} f(x)dx$=(b-c)/2b + (b-c)/2b =1- c/b.

On the other hand, Chebychev's inequality gives P{|*X*-*m*|>*c*} ≤ var(*X*)/c^2 = $((2b)^2/12)/c^2 = b^2 / 3c^2$.

In the example below, with *b*=1, we compare the exact probability with its estimate from Chebychev's inequality. Note that when *c*<*b*/sqrt(3), Chebychev's inequality gives a value that is higher than unity, which makes it useless.

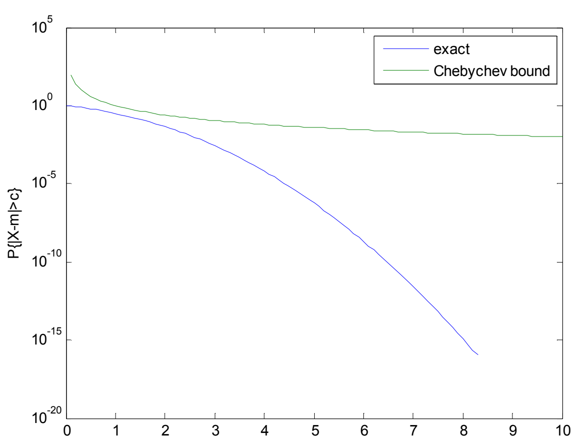



```
clear all
b=1;
b1=[0:.01:1];
c=[1/sqrt(3):0.01:1];
p1=1-(b1 / b);
p2=(b^2)./(3*(c.^2));
plot(b1,p1,c,p2)
xlabel('c')
ylabel('P\{|X-m|>c\}')
legend('exact','Chebychev bound')
```

(c) The Gaussian case: Assume *c*>0. Now P{|*X*-*m*|>*c*}= P{*X*>*c*+*m*} + P{*X*<-*c*+*m*} = $\int_{c+m}^{\infty} f(x)dx + \int_{-\infty}^{-c+m} f(x)dx$= *Q*((*c*+*m*-*m*)/σ) + 1-*Q*((-*c*+*m*-*m*)/σ)= *Q*(*c*/σ) + 1-*Q*(-*c*/σ)= *Q*(*c*/σ) + Q(*c*/σ)=2*Q*(*c*/σ) = 2 {0.5 – 0.5 *erf*(*c*/sqrt(2)σ)}

Note: *Q*(*x*)=0.5 – 0.5 *erf*[*x*/sqrt(2)]

From Chebychev's inequality, P{|X-m|>c} ≤ var(X)/c^2 = σ^2/ c^2. In the example below, with σ=1 and m=0, we compare the exact probability with its estimate from Chebychev's inequality. Note that when c<1, Chebychev's inequality gives a value that is higher than unity, which makes the upper bound useless.



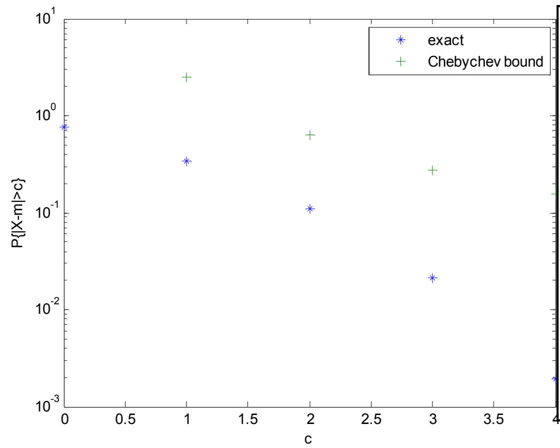

```
clear all
s=1;
c=[0:.1:10];
p1=2*(0.5-0.5*(erf(c/(s*sqrt(2)))));
p2=s^2./(c.^2);
semilogy(c,p1,c,p2)
xlabel('c')
ylabel('P\{|X-m|>c\}')
legend('exact','Chebychev bound')
```

(d) The binomial rv case: Assume *c*<*np*. Now P{|*X*-*m*|>*c*}= P{*X*>*c*+*m*} + P{*X*<-*c*+*m*} =

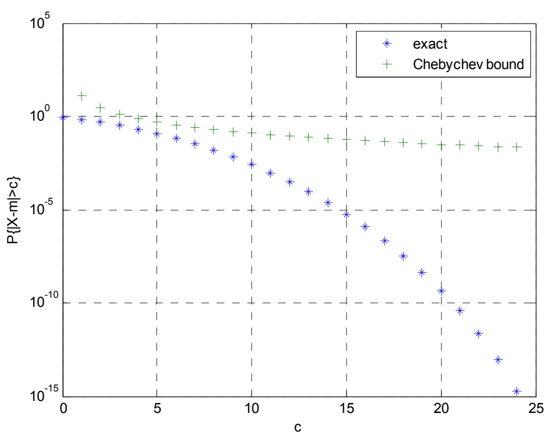
$\sum_{k=0}^{-c+np-1}\binom{n}{k}p^k(1-p)^{n-k}+\sum_{k=c+np+1}^{n}\binom{n}{k}p^k(1-p)^{n-k}$

Here, $m=np$.

From Chebychev's inequality, P{|X-m|>c} ≤ var(X)/c^2 = $np(1-p)/c^2$. In the examples below, with n=10 (50) and p=0.5, we compare the exact probability with its estimate from Chebychev's inequality.




```
clear all
p=0.5;
n=10;
m=n*p;
c=[0:1:4];
n1=-c+m;
n2=c+m;
p1=cdf('bino',n1-1,10,0.5)+1-cdf('bino',n2,10,0.5);
p2=p*(1-p)*n./(c.^2);
semilogy(c,p1,'*',c,p2,'+')
xlabel('c')
ylabel('P\{|X-m|>c\}')
legend('exact','Chebychev bound')
```




```
clear all
p=0.5;
n=50;
m=n*p;
c=[0:1:24];
n1=-c+m;
n2=c+m;
p1=cdf('bino',n1-1,50,0.5)+1-cdf('bino',n2,50,0.5);
p2=p*(1-p)*n./(c.^2);
semilogy(c,p1,'*',c,p2,'+')
xlabel('c')
ylabel('P\{|X-m|>c\}')
legend('exact','Chebychev bound')
grid
```

MATLAB Assignment

1. We have the transformation Z=g(x) where

$$g(x) = -\mu \log(1-x)$$

$$\therefore f_Z(z) = f_X\big(g^{-1}(z)\big)\frac{1}{g'\big(g^{-1}(z)\big)}$$

now $g^{-1}(z) = 1 - e^{-\frac{z}{\mu}}$ and $g'(x) = \frac{\mu}{1-x}$

$$\therefore f_Z(z) = f_X\left(1 - e^{-\frac{z}{\mu}}\right)\frac{1}{\frac{\mu}{1-\left(1-e^{-\frac{z}{\mu}}\right)}}$$

$$= f_X\left(1 - e^{-\frac{z}{\mu}}\right)e^{-\frac{z}{\mu}}\frac{1}{\mu}$$

Now if z>0, then $1 - e^{-\frac{z}{\mu}} < 0$ and hence $f_X\left(1 - e^{-\frac{z}{\mu}}\right) = 0$

$$\therefore f_Z(z) = \begin{cases} \frac{1}{\mu}e^{-\frac{z}{\mu}}, & if\ z \geq 0 \\ 0, & if\ z < 0 \end{cases}$$

To simulate an exponentially distributed rv in Matlab, we simply generate a uniform [0,1] rv, *X*, and then apply the transformation

$$g(X) = -\mu \log(1-X)$$