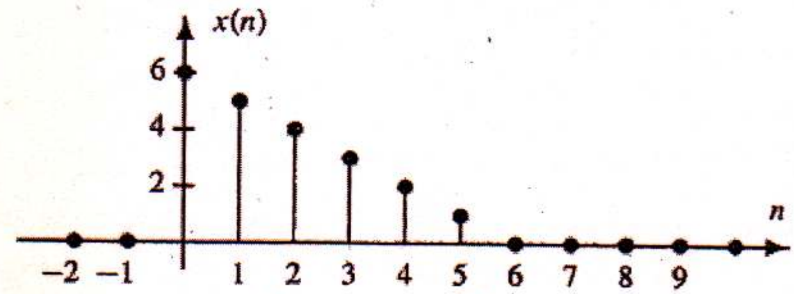
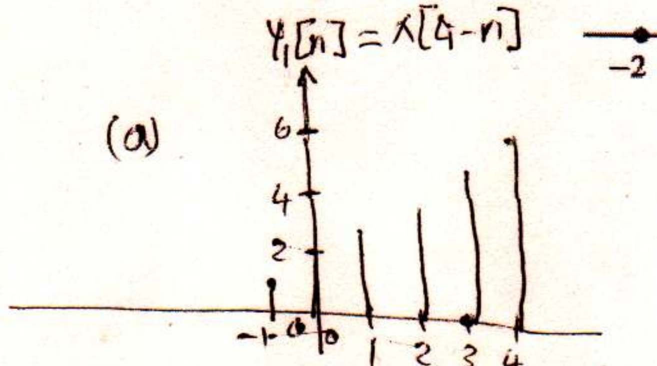
Problem 1 (15 Points)

Given the discrete-time signal shown in the figure below, make a sketch and comment on what happened to the new signal (shifted to right, shifted to left, compressed, expanded,…etc.):

(a) $Y_1[n]=x[4-n]$
(b) $Y_2[n]=x[2n-3]$
(c) $Y_3[n]=x[8-3n]$
(d) $Y_4[n]=x[n^2-2n+1]$
(e) $Y5[n]= Y_1[n]\ Y_2[n]$

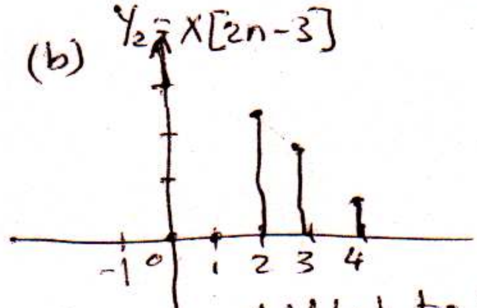


(a)



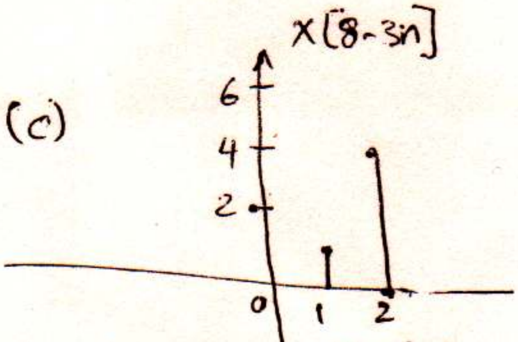
Flipted and shifted to the right 4 samples

(b)

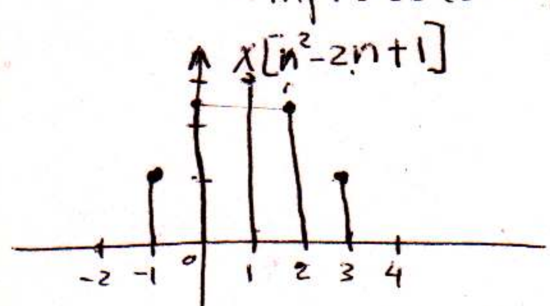


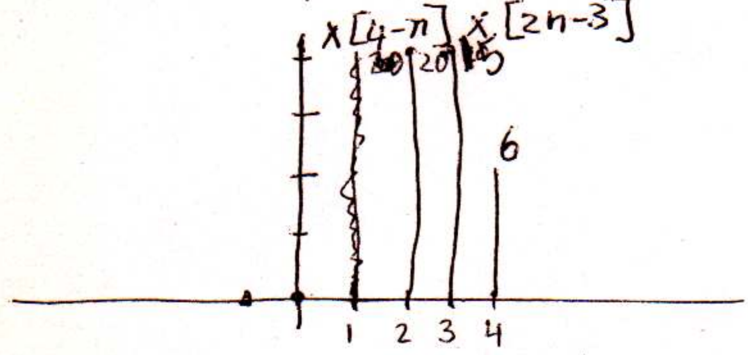
shifted to the right 3 samples then compressed by 2

(c)



Fliped, then shifted to right by 8 - samples then compressed by 3





Problem 2 **(15 points)**

For each of the system below, x[n] is the input and y[n] is the output. Determine which systems are linear and which are not.

(a) Y[n]=log(x[n])
(b) Y[n]=6x[n+2]+4x[n+1]+2x[n]+1
(c) Y[n]=x[n]sin(nπ/2)
(d) Y[n]=Re{x[n]}
(e) Y(t)=x(t/2)

(a) say $x_1[n] = c\,x[n]$

$y_1[n] = \log(x_1[n]) = \log(c\,x[n]) = \log c + \log(x[n])$

which is not equal to $c \cdot \log(x[n])$

Also $\log[x_1[n] + x_2[n]] \neq \log[x_1[n]] + \log[x_2[n]]$

nonlinear

(b) if $x_1[n] = c\,x[n]$

$\therefore y_1[n] = 6x_1[n+2] + 4x_1[n+1] + 2x_1[n] + 1$

$= c\{x[n+2] + 4x[n+1] + 2x[n]\} + 1$

however $c\,y[n] = c\{6x[n+2] + 4x[n+1] + 2x[n] + 1\}$

which is not the same as $y_1[n]$

Simillarly if $x[n] = x_1[n] + x_2[n]$

$\therefore y[n] = 6x[n+2] + 4x[n+1] + 2x[n] + 1$

$= 6\{x_1[n+2] + x_2[n+2]\} + 4\{x_1[n+1] + x_2[n+1]\} +$

$2\{x_1[n] + x_2[n]\} + 1 = y_1[n] + y_2[n] - 1$

Nonlinear

(c) let $y_1[n]$ is response to $x_1[n]$ and $y_2[n]$ is response of $x_2[n]$

$\therefore x[n] = a\,x_1[n] + b\,x_2[n]$

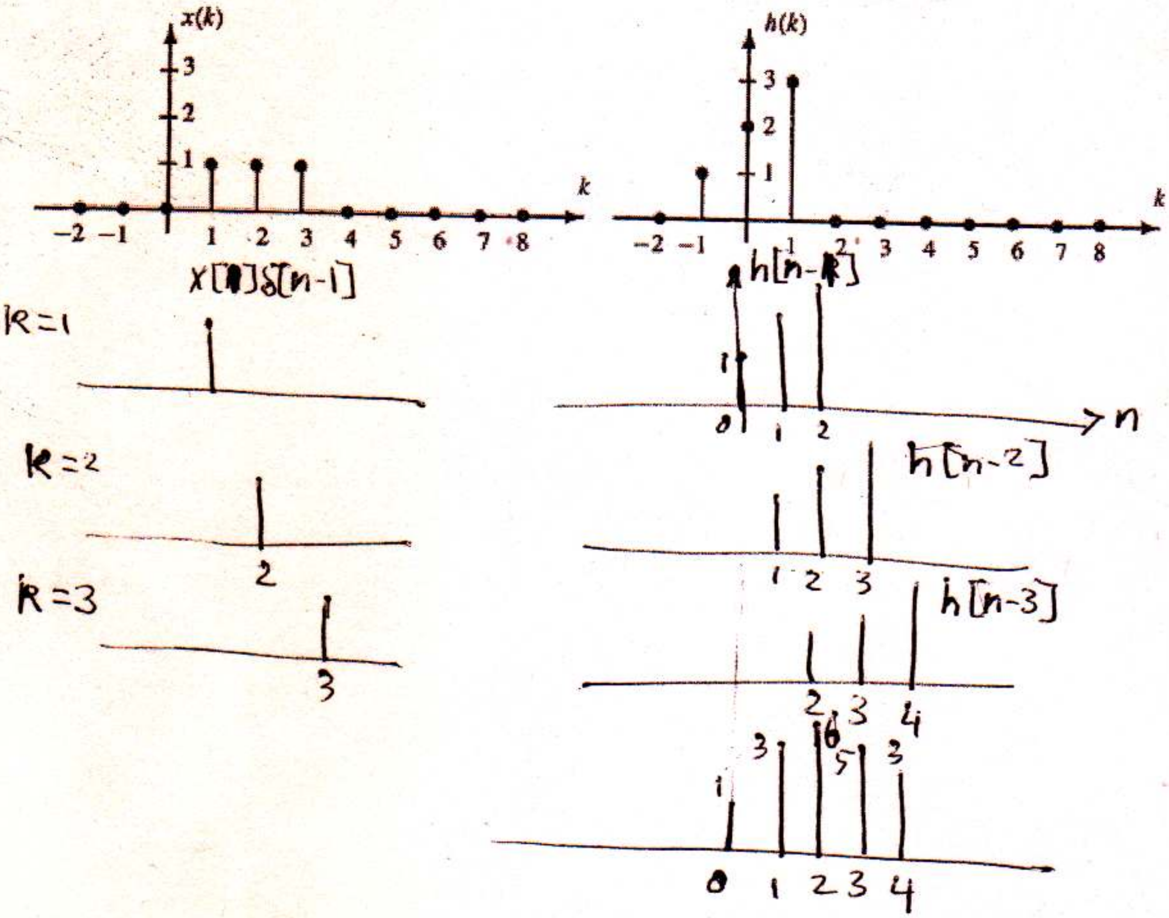
$\therefore y[n] = x[n]\sin\left(\frac{n\pi}{2}\right) = [a\,x_1[n] + b\,x_2[n]]\sin\frac{n\pi}{2}$

$= a\,x_1[n]\sin\frac{n\pi}{2} + b\,x_2[n]\sin\frac{n\pi}{2} = a\,y_1[n] + b\,y_2[n]$

Linear

Problem 3 **(15 points)**

Evaluate y[n] = x[n]*h[n] where x[n] and h[n] are the signals shown in the figure below:



Problem 4 (15 points)

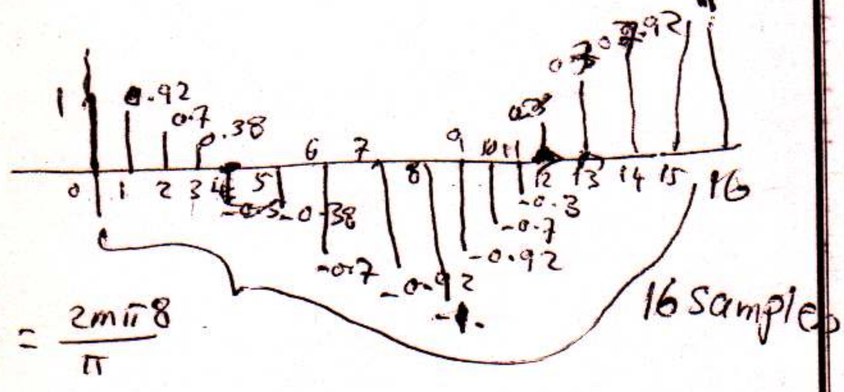
Determine whether or not the signals below are periodic, and, for each signal that is periodic, determine the fundamental period.

(a) $X[n] = \cos(0.125\pi n)$

(b) $X[n] = \text{Re}\{e^{jn\pi/12}\} + \text{Im}\{e^{jn\pi/18}\}$

(c) $X[n] = \sin(\pi + 0.2n)$

(d) $X[n] = e^{jn\pi/16}\cos(n\pi/17)$



(a) $x[n] = \cos(\underbrace{\tfrac{\pi}{8}}_{\Omega_0} n)$

periodic

$N\Omega_0 = 2m\pi \qquad N = \frac{2m\pi}{\Omega_0} = \frac{2m\pi 8}{\pi}$

for $m=1 \quad N=16$

(b) $x[n] = \cos(n \underbrace{\pi/12}_{\Omega_{01}}) + \sin(n \underbrace{\pi/18}_{\Omega_{02}})$ periodic

$\Omega_{01} = \frac{\pi}{12} \quad N_1 = \frac{2m\pi 12}{\pi} \Rightarrow N_1 = 24$

$\Omega_{02} = \frac{\pi}{18} \quad N_2 = \frac{2m\pi 18}{\pi} \Rightarrow N_2 = 36$

$N = \frac{N_1 N_2}{\gcd(N_1, N_2)} = \frac{24 \times 36}{12} = 72$

(c) $x[n] = \sin(\pi + 0.2n)$

$\Omega_0 = \frac{1}{5} \Rightarrow N = \frac{2m\pi \times 5}{\pi} \Rightarrow$ m must be integer and $\pi = 3.14$

so there is no m that make N integer

aperiodic

(d) $x[n] = e^{jn\overbrace{\pi/16}^{}}_{\Omega_{01}} \cos(n\underbrace{\pi/17}_{\Omega_{02}})$

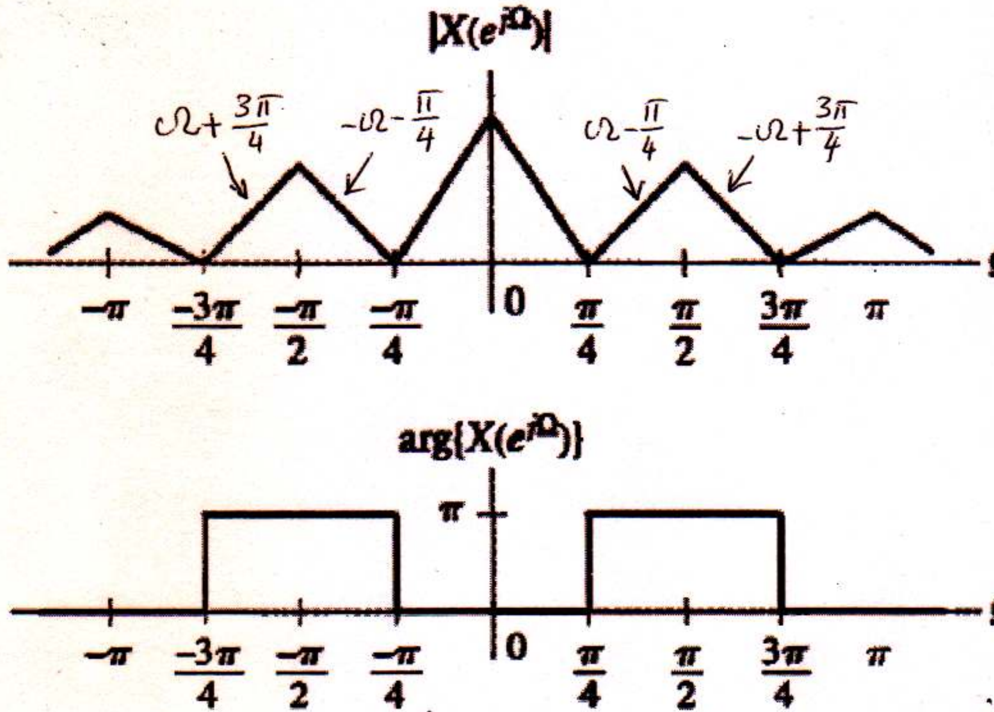
$\Omega_{01} = \frac{\pi}{16} \quad N_1 = \frac{2m\pi \times 16}{\pi} = 32 \qquad \Omega_{02} = \frac{\pi}{17} \Rightarrow N_2 = \frac{2m\pi 17}{\pi} = 34$

periodic with $N = \frac{N_1 N_2}{\gcd(N_1, N_2)} = \frac{32 \times 34}{2} = 544$

Problem 1 (10 Points)

Determine whether the time-domain signals corresponding to the following frequency-domain representations are real or complex valued and even or odd:

(a) $X(e^{j\Omega})$ as depicted in Fig. (b)
(b) $X(j\omega) = \omega^{-3} + j\omega^{-2}$



$e^{j\pi} = -1$

(b)

(a) $$X(e^{j\Omega}) = \left(\Omega+\frac{3\pi}{4}\right)e^{j\pi} + \left(-\Omega-\frac{\pi}{4}\right)e^{j\pi} + \left(\Omega-\frac{\pi}{4}\right)e^{j\pi} + \left(-\Omega+\frac{3\pi}{4}\right)e^{j\pi}$$

$$= -\cancel{\Omega} - \frac{3\pi}{4} + \cancel{\Omega} + \frac{\pi}{4} - \cancel{\Omega} + \frac{\pi}{4} + \cancel{\Omega} - \frac{3\pi}{4} = -\frac{6\pi}{4} + \frac{2\pi}{4} = -\pi$$

As the Imaginary part of $X(e^{j\Omega}) = 0 \Rightarrow x[n]$ is real and even.

(b) $X(j\omega) = \omega^{-3} + j\omega^{-2}$

$X^*(j\omega) = \omega^{-3} - j\omega^{-2}$

$X(-j\omega) = (-\omega)^{-3} + j(-\omega)^{-2} = -\omega^{-3} + j\omega^{-2}$

$X^*(j\omega) \neq X(j\omega) \Rightarrow x(t)$ is not real

But $X^*(j\omega) = -X(-j\omega) \Rightarrow x(t)$ is imaginary

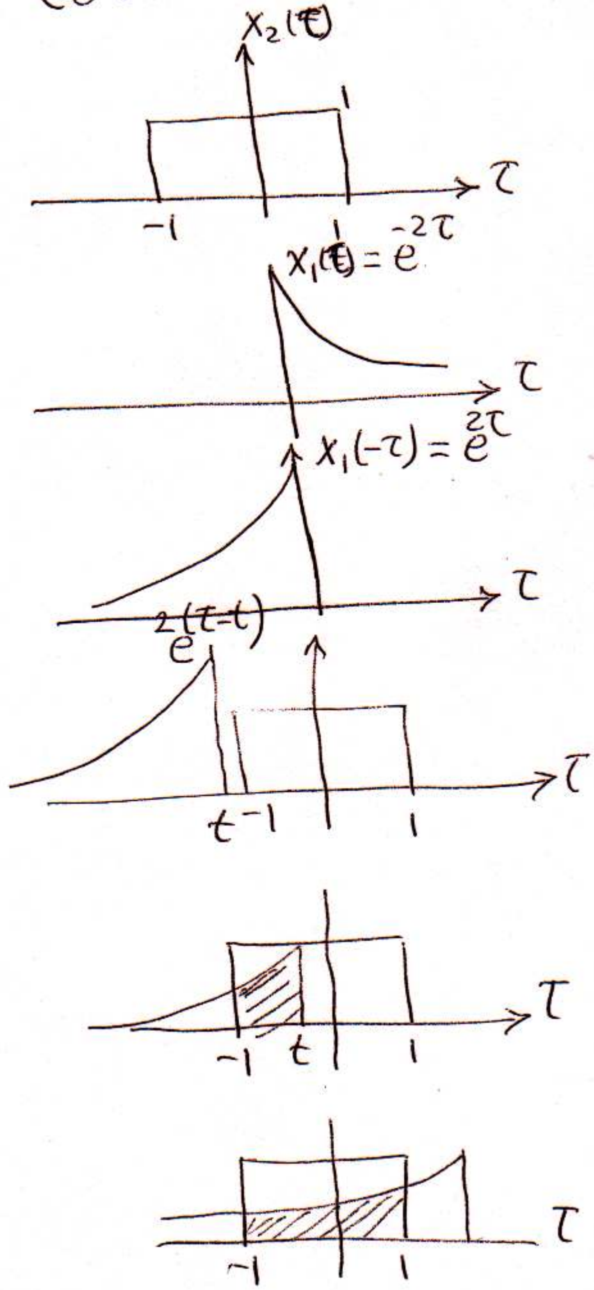
Problem 2 **(10 points)**

Use the convolution property to find the time-domain signals corresponding to the following frequency-domain representation: $X(j\omega) = X_1(j\omega)\, X_2(j\omega)$.

Where: $X_1(j\omega) = (1/(j\omega + 2))$ and $X_2(j\omega) = ((2/\omega) \sin \omega)$.

$$X_1(j\omega) = \frac{1}{j\omega+2} \xrightarrow{\mathcal{F}T} x(t) = e^{-2t} u(t)$$

$$X_2(j\omega) = \frac{2\sin\omega}{\omega} \xrightarrow{\mathcal{F}T} x(t) = \begin{cases} 1 & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



$t < -1 \quad x(t) = 0$

$-1 \leq t < 1$

$$x(t) = \int_{-1}^{t} e^{2(\tau-t)}\, d\tau$$

$$= \frac{e^{2(\tau-t)}}{2}\Big|_{-1}^{t} = \frac{1}{2} - \frac{e^{-2(t+1)}}{2}$$

$$= \frac{1}{2}\left(1 - e^{-2(t+1)}\right)$$

$t \geq 1$

$$x(t) = \int_{-1}^{1} e^{2(\tau-t)}\, d\tau$$

$$= \frac{e^{2(\tau-t)}}{2}\Big|_{-1}^{1}$$

$$= \frac{e^{2(1-t)} - e^{-2(1+t)}}{2}$$

$$x_t(t) = \begin{cases} 0 & t < -1 \\ \frac{1}{2}\left(1 - e^{-2(t+1)}\right) & -1 \leq t < 1 \\ \frac{1}{2}\left(e^{2(1-t)} - e^{-2(1+t)}\right) & t \geq 1 \end{cases}$$

Problem 3 **(10 points)**

Use Parseval's theorem to evaluate the following quantity

$$x_1 = \int_{-\infty}^{\infty} \frac{2}{|j\omega + 2|^2}\, d\omega$$

$$x(t) = e^{-at} u(t) \xleftrightarrow{FT} X(j\omega) = \frac{1}{(a+j\omega)}$$

$$x(t) = \sqrt{2}\, e^{-2t} u(t) \xleftrightarrow{FT} X(j\omega) = \frac{\sqrt{2}}{(2+j\omega)}$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$\text{or } 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$x_1 = 4\pi \int_{-\infty}^{\infty} |e^{-2t} u(t)|^2 dt = \int_{-\infty}^{\infty} \frac{2}{|2+j\omega|^2} d\omega$$

$$x_1 = 4\pi \int_{-\infty}^{\infty} |e^{-2t}|^2 u(t)\, dt = 4\pi \int_0^{\infty} e^{-4t} dt = 4\pi \left. \frac{e^{-4t}}{-4} \right|_0^{\infty}$$

$$= \frac{4\pi}{4} \left[\frac{\cancel{e^{-4\infty}}}{-1} + 1 \right] = \pi$$

Problem 4 **(10 points)**

Use the frequency-differentiation property to find the DTFT of

$$x[n] = (n+1)\alpha^n u[n]. = n\alpha^n u[n] + \alpha^n u[n]$$

$$\alpha^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-\alpha e^{-j\Omega}} \quad \text{——} ①$$

$$\cancel{-j}n\alpha^n u[n] \xleftrightarrow{DTFT} \frac{d}{d\Omega}\frac{1}{1-\alpha e^{-j\Omega}} = \frac{\cancel{-j}\alpha e^{-j\Omega}}{(1-\alpha e^{-j\Omega})^2}$$

$$\therefore n\alpha^n u[n] \xleftrightarrow{DTFT} \frac{\alpha e^{-j\Omega}}{(1-\alpha e^{-j\Omega})^2} \quad \text{——} ②$$

add ①+②

$$n\alpha^n u[n] + \alpha^n u[n] \xleftrightarrow{DTFT} \frac{\alpha e^{-j\Omega}}{(1-\alpha e^{-j\Omega})^2} + \frac{1}{(1-\alpha e^{-j\Omega})}$$

$$= \frac{\cancel{\alpha e^{-j\Omega}} + 1 - \cancel{\alpha e^{-j\Omega}}}{(1-\alpha e^{-j\Omega})^2}$$

$$(n+1)\alpha^n u[n] \xleftrightarrow{DTFT} \frac{1}{(1-\alpha e^{-j\Omega})^2}$$

Problem 5 **(10 points)**

Find the FT of the signal:

$$x(t) = \frac{d}{dt}\{(\underbrace{e^{-3t}u(t)}_{w(t)}) * (\underbrace{e^{-t}u(t-2)}_{v(t)})\}. = \frac{d}{dt}\{w(t) * v(t)\}$$

$$X(j\omega) = j\omega\{W(j\omega)V(j\omega)$$

$$w(t) = e^{-3t}u(t) \xleftrightarrow{FT} W(j\omega) = \frac{1}{j\omega+3}$$

$$v(t) = e^{-t}u(t-2) = e^{-2}e^{-(t-2)}u(t-2)$$

$$\therefore V(j\omega) = e^{-2}\frac{e^{-j2\omega}}{j\omega+1}$$

$$X(j\omega) = j\omega\frac{1}{j\omega+3}e^{-2}\frac{e^{-j2\omega}}{j\omega+1}$$

$$= e^{-2}\frac{j\omega e^{-j2\omega}}{(j\omega+3)(j\omega+1)}$$

Problem 1 **(10 Points)**

(a) Find the FT of the impulse train.
(b) Draw p(t) in the time domain.
(c) Draw the FT of p(t) in the frequency domain.

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

$p(t)$ is periodic with fundamental period T, so $\omega_0 = \frac{2\pi}{T}$

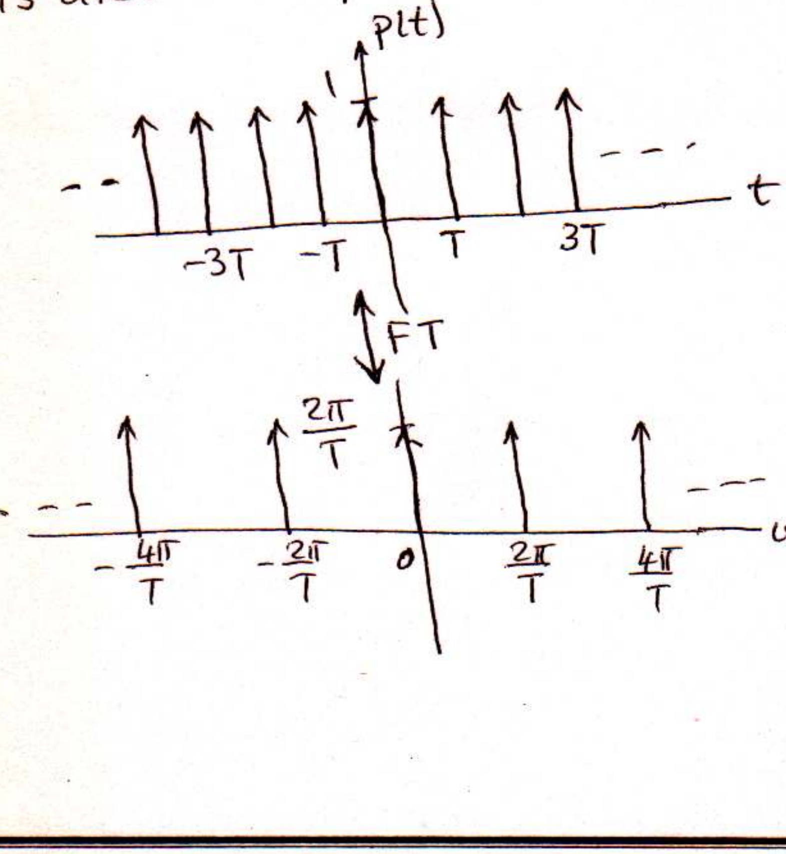
$$P[k] = \frac{1}{T}\int_0^T p(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}$$

$$P(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} P[k]\, \delta(\omega - k\omega_0)$$

$$= \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

Hence, the FT of $p(t)$ is also an impulse train

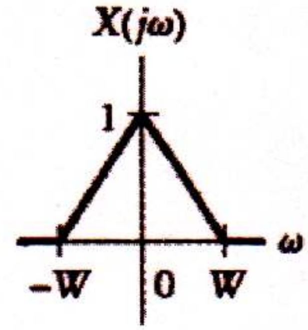


roblem 2 (10 points)

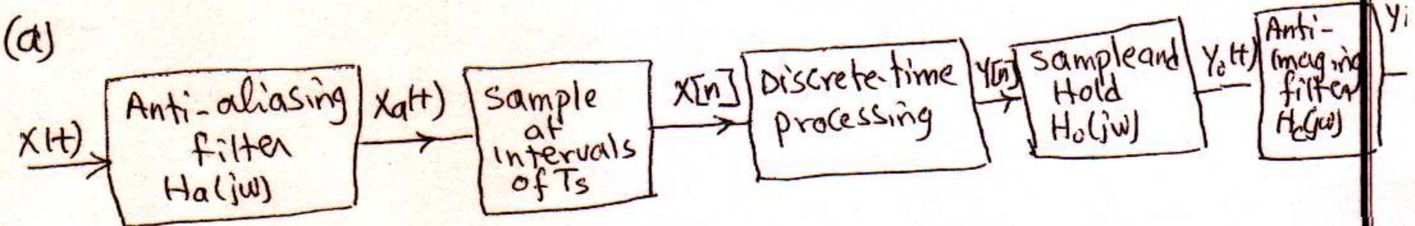
(a) Draw block diagram for discrete-time processing of continuous-time signals.

(b) Explain what the aliasing problem is and when it happens.

(c) Draw the FT of a sampled signal shown in the figure below (spectrum of a continuous-time signal) for different sampling rates: $\omega_s = 1.5W$, $\omega_s = 2W$ and $\omega_s = 3W$.

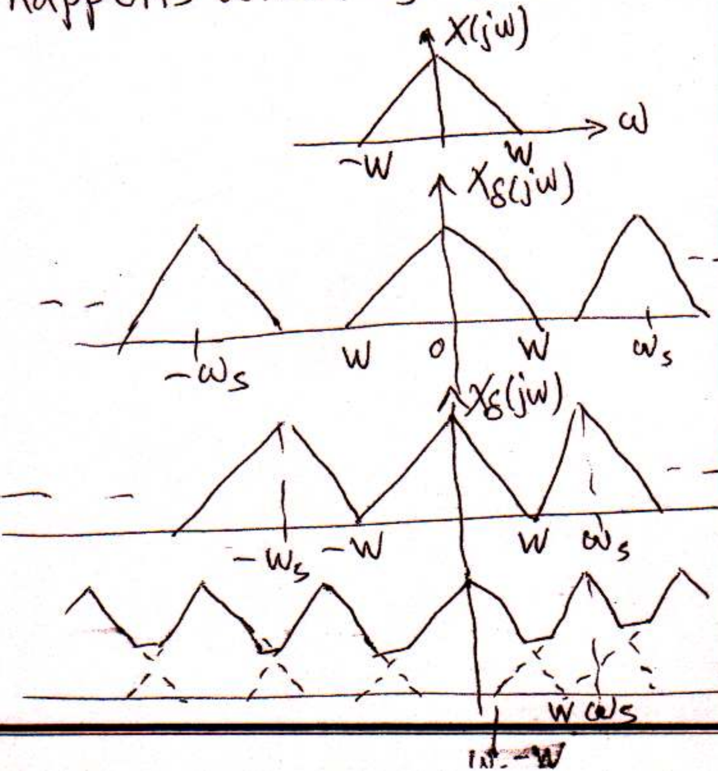


(a)



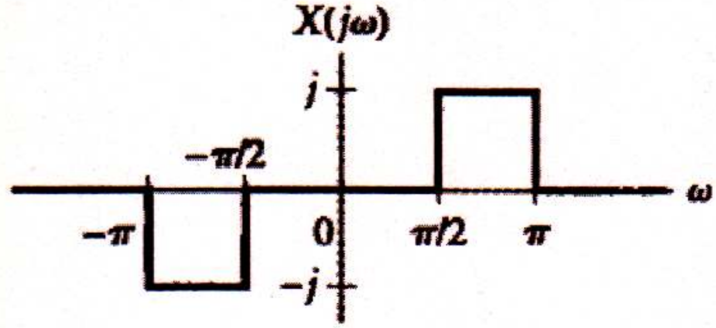
(b) If the continous-time signal is sampled by a frequency ω_s is not large enough compared with the frequency extent or bandwidth of $X(jw)$, the shifted version of $X(jw)$ may overlap. Aliasing happens when $\omega_s < 2W$

(c)



Problem 3 (10 points)

Draw the FT of a sampled version of the continuous-time signal having the FT depicted in the figure shown below for (a) $T_s = ½$ and (b) $T_s = 2$.



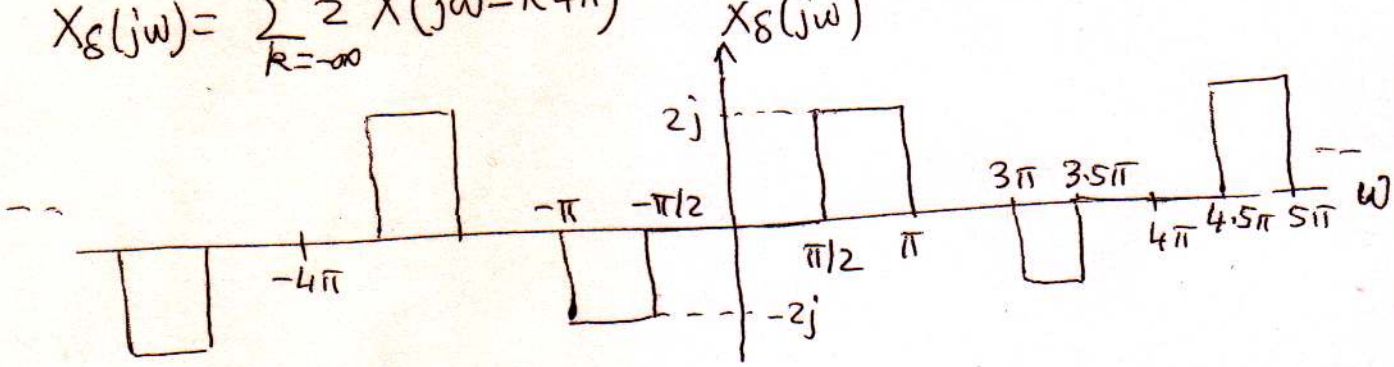
$$X_\delta(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j\omega - k\omega_s)$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X_\delta(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\left(j\omega - k\frac{2\pi}{T_s}\right)$$

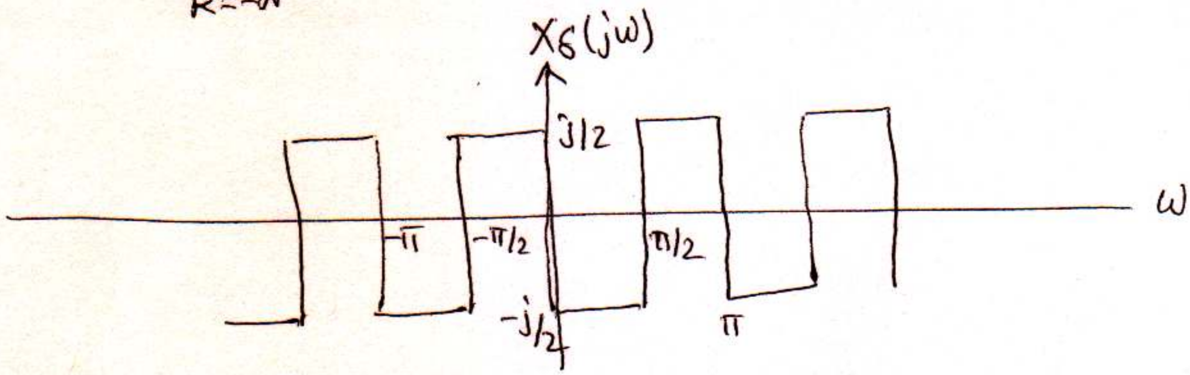
(a) $T_s = \frac{1}{2} \Rightarrow \omega_s = 4\pi$

$$X_\delta(j\omega) = \sum_{k=-\infty}^{\infty} 2\, X(j\omega - k4\pi)$$



(b) $T_s = 2 \Rightarrow \omega_s = \frac{2\pi}{T_s} = \pi$

$$X_\delta(j\omega) = \frac{1}{2}\sum_{k=-\infty}^{\infty} X(j\omega - k\pi) = \sum_{k=-\infty}^{\infty} \frac{1}{2} X(j\omega - k\pi)$$



Problem 4 (10 points)

Determine the z-transform of the signal

X[n] = - α^n u[-n-1]

Depict the ROC and the locations of poles and zeros of X(z) in the z-plane.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-1} = \sum_{-\infty}^{\infty} -\alpha^n u[-n-1] z^{-n}$$

$$= -\sum_{-\infty}^{-1} \left(\frac{\alpha}{z}\right)^n = -\sum_{k=1}^{\infty} \left(\frac{z}{\alpha}\right)^k$$

$$= 1 - \sum_{k=0}^{\infty} \left(\frac{z}{\alpha}\right)^k$$

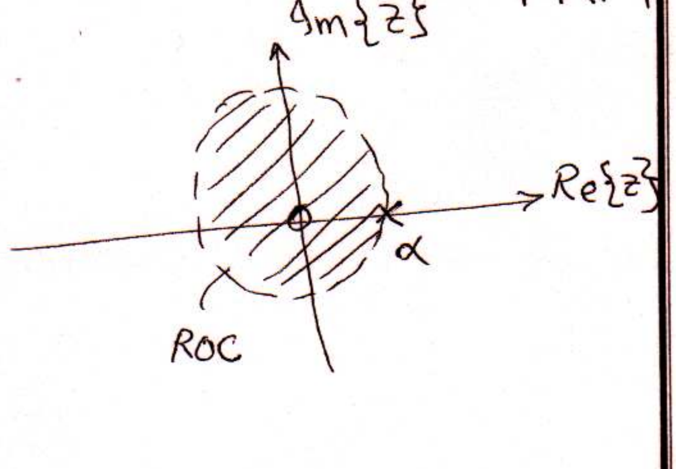
The sum converges provided that $\left|\frac{z}{\alpha}\right| < 1$ or $|z| < |\alpha|$

$$\therefore X(z) = 1 - \frac{1}{1-\alpha^{-1}z^{+1}} \qquad |z| < |\alpha|$$

$$= \frac{1-\alpha^{-1}z-1}{1-\alpha^{-1}z} = \frac{-\alpha^{-1}z}{1-\alpha^{-1}z} = \frac{-z}{\alpha - z} = \frac{z}{z-\alpha} \qquad |z|<|\alpha|$$

zero at 0

pole at $z=\alpha$



Problem 5 **(10 points)**

Find the z-transform of the signal

$$x[n] = \left(n\left(\frac{-1}{2}\right)^n u[n]\right) * \left(\frac{1}{4}\right)^{-n} u[-n].$$

$\left(-\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{z}{z+\frac{1}{2}}$ with ROC $|z| > \frac{1}{2}$

$\alpha^n u[n] \leftrightarrow \frac{1}{1-\alpha z^{-1}}$

$\left(-\frac{1}{2}\right)^n u[n] \leftrightarrow \frac{1}{1+\frac{1}{2}z^{-1}} = \frac{z}{z+\frac{1}{2}}$

$n\left(-\frac{1}{2}\right)^n u[n] \longleftrightarrow -z\frac{d}{dz}\left(\frac{z}{z+\frac{1}{2}}\right)$ with ROC $|z| > \frac{1}{2}$

$$= -z\left(\frac{z+\frac{1}{2}-z}{(z+\frac{1}{2})^2}\right)$$

$$W(z) = \frac{-\frac{1}{2}z}{(z+\frac{1}{2})^2}$$

$\left(\frac{1}{4}\right)^{-n} u[n] \longleftrightarrow \frac{1}{1-\frac{1}{4}z^{-1}} = \frac{z}{z-\frac{1}{4}}$ with ROC $|z| > \frac{1}{4}$

$\left(\frac{1}{4}\right)^{-n} u[-n] \longleftrightarrow = \frac{\frac{1}{z}}{\frac{1}{z}-\frac{1}{4}}$ with $\frac{1}{|z|} > \frac{1}{4}$

$Y(z) = \frac{-4}{z-4}$ with $|z| < 4$

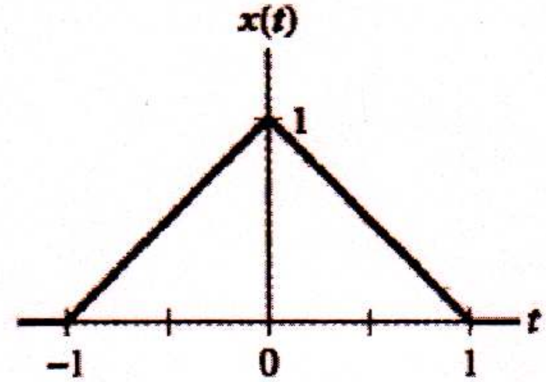
$X(z) = W(z)Y(z) = \frac{-\frac{1}{2}z}{(z+\frac{1}{2})^2}\frac{-4}{(z-4)} = \frac{2z}{(z-4)(z+\frac{1}{2})^2}$

with ROC $\frac{1}{2} < |z| < 4$

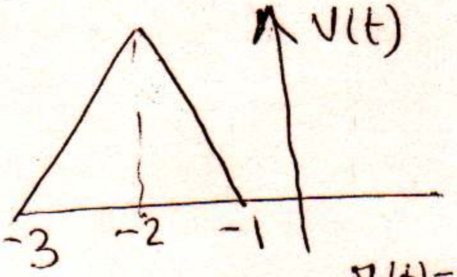
Problem 1 (10 Points)

A triangular pulse signal x(t) is depicted in the figure shown below. Sketch each of the following signals derived from x(t).

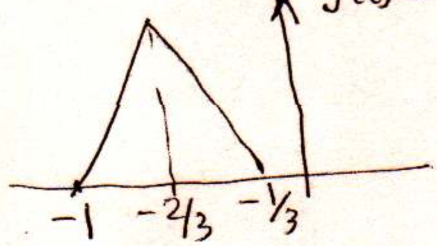
(a) x(3t+2)
(b) x(2(t+2))
(c) x(3t)+x(3t+2)



(a) $v(t) = x(t+2)$



$y(t) = v(3t)$

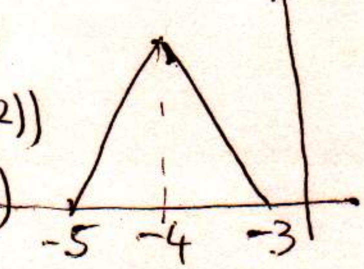


(b)

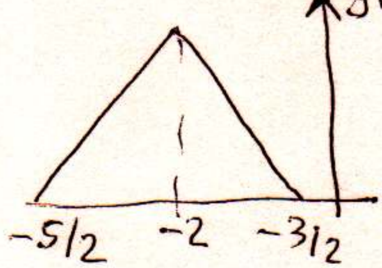
$x(2(t+2))$

$x(2t+4)$

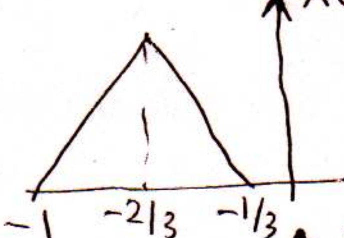
$v(t) = x(t+4)$

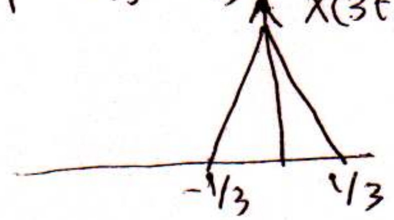


$y(t) = v(2t)$



(c)



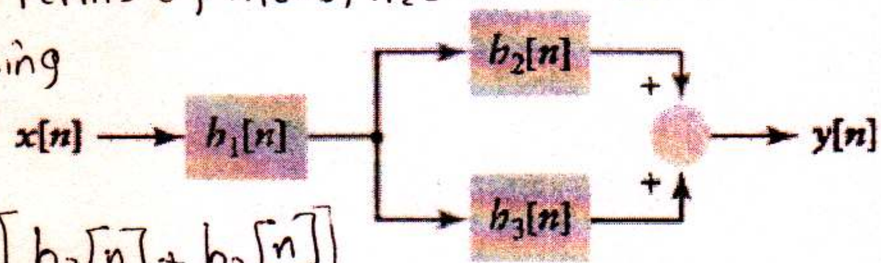


Problem 2 **(10 points)**

An interconnection of LTI systems is depicted in the figure shown below. The impulse responses are $h_1[n] = (1/2)^n u[n+2]$, $h_2[n] = \delta[n$, and $h_3[n] = u[n-1]$. Let the overall impulse response of the system relating y[n] to x[n] be denoted as h[n].

(a) Express $h[n]$ in terms of $h_1[n]$, $h_2[n]$ and $h_3[n]$

(b) Evaluate $h[n]$ using results of pent(a)



(a) $h[n] = h_1[n] * [h_2[n] + h_3[n]]$

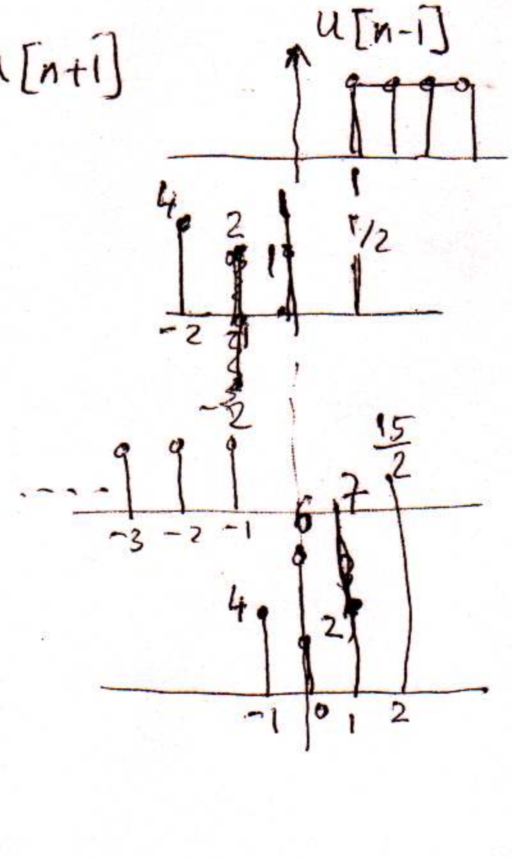
$= h_1[n] * h_2[n] + h_1[n] * h_3[n]$

$(\frac{1}{2})^{-2} = \frac{1}{(\frac{1}{2})^2} = 4$

$h[n] = (\frac{1}{2})^n u[n+2] * \delta[n] + (\frac{1}{2})^n u[n+2] * u[n-1]$

$(\frac{1}{2})^{-1} = \frac{1}{1/2} = 2$

(b) $h[n] = (\frac{1}{2})^n u[n+2] + (8 - (\frac{1}{2})^{n-1}) u[n+1]$



Problem 3 **(10 points)**

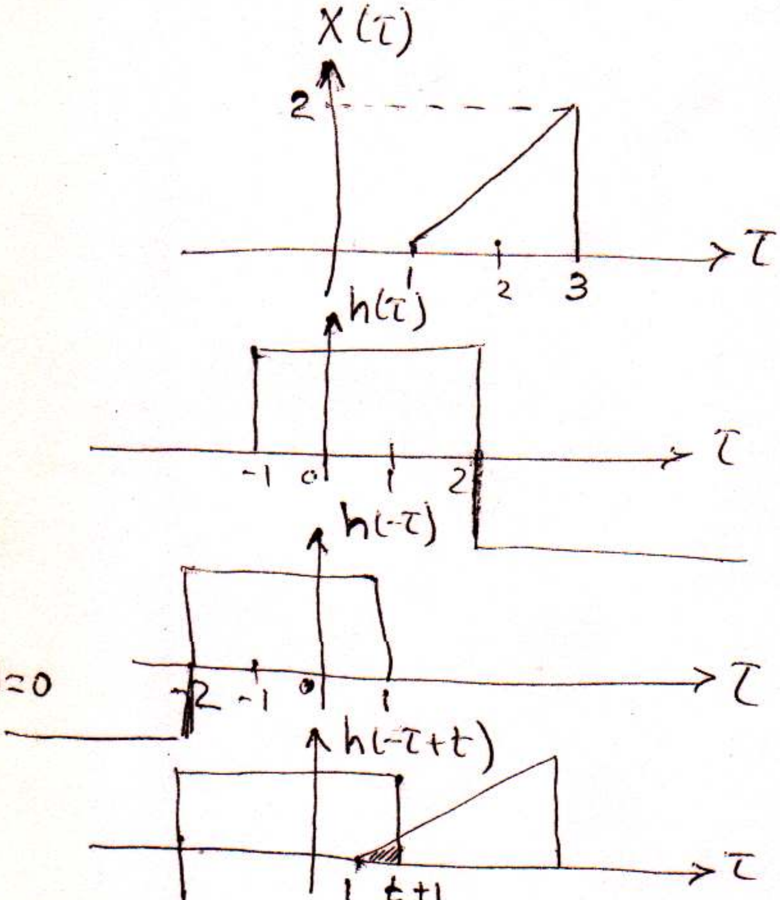
Suppose the input x(t) and impulse response h(t) of an LTI system are, respectively, given by

$$x(t) = (t-1)[u(t-1) - u(t-3)]$$

and

$$h(t) = u(t+1)-2u(t-2)$$

Find the output of this system.



$t<0 \quad y(t)=0$

$1 \leq t+1 < 3$

$0 \leq t < 2$

$$y(t) = \int_1^{t+1} (\tau-1)\,d\tau = \left(\frac{\tau^2}{2} - \tau \Big|_1^{t+1}\right) = \frac{t^2}{2}$$

$3 \leq t+1 < 4$

$y(t) = 2$

$2 \leq t \leq 3$

$4 \leq t+1 < 6$

$3 \leq t < 5 \quad y(t) = -\int_1^{t-2} (\tau-1)\,d\tau \int_{t-2}^{3} (\tau-1)\,d\tau = -t^2+6t-7$

$6 \leq t+1$

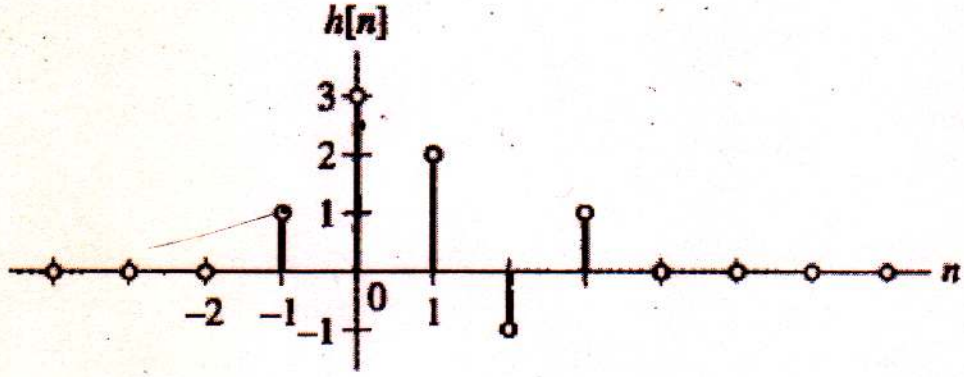
$5 \leq t \quad y(t) = -2$

$$y(t) = \begin{cases} 0 & t<0 \\ \frac{t^2}{2} & 0 \leq t < 2 \\ 2 & 2 \leq t < 3 \\ -t^2+6t-7 & 3 \leq t < 5 \\ -2 & t \geq 5 \end{cases}$$

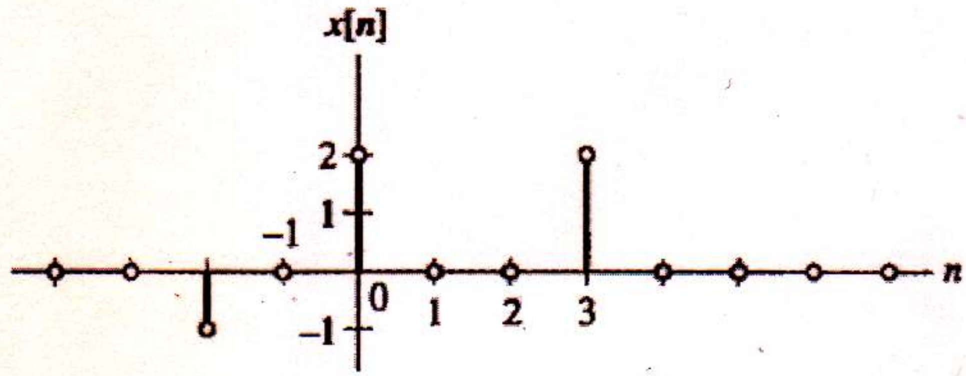
Problem 4 **(10 points)**

A discrete-time LTI system has the impulse response h[n] depicted in Fig. (a). Use linearity and time invariance to determine the system output y[n] if the input is:

(a) x[n] = 3 δ[n] – 2 δ[n-1]
(b) x[n] as given in Fig. (b)



(a)



(b)

(a) $x[n] = 3\,\delta[n] - 2\,\delta[n-1]$

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \{3\,\delta[n] - 2\,\delta[n-1]\} * h[n] \\
&= 3\,\delta[n] * h[n] - 2\,\delta[n-1] * h[n] \\
&= 3\,h[n] - 2\,h[n-1] \\
&= 3\,\delta[n+1] + 9\,\delta[n] + 6\,\delta[n-1] - 3\,\delta[n-2] + 3\,\delta[n-3] \\
&\quad - 2\,\delta[n] - 6\,\delta[n-1] - 4\,\delta[n-2] + 2\,\delta[n-3] \\
&\quad - 2\,\delta[n-4] \\
&= 3\,\delta[n+1] + 7\,\delta[n] - 7\,\delta[n-2] + 5\,\delta[n-3] - 2\,\delta[n-4]
\end{aligned}
$$

(b) From figure (b) we get

$$x[n] = -\delta[n+2] + 2\delta[n] + 2\delta[n-3]$$

$$y[n] = x[n] * h[n]$$

$$= \{-\delta[n+2] + 2\delta[n] + 2\delta[n-3]\} * h[n]$$

$$= -h[n+2] + 2h[n] + 2h[n-3]$$

$$= -\delta[n+3] - 3\delta[n+2] - 2\delta[n+1] + \delta[n] - \delta[n-1] - 2\delta[n-2] + 2\delta[n-3]$$
$$+2\delta[n+1] + 6\delta[n] + 4\delta[n-1]$$
$$+2\delta[n-2] + 6\delta[n-3$$
$$+4\delta[n-4] - 2\delta[n-5] + 2\delta[n-6]$$

$$= -\delta[n+3] - 3\delta[n+2] + 0 + 7\delta[n] + 3\delta[n-1] + 0 + 8\delta[n-3]$$
$$+4\delta[n-4] - 2\delta[n-5] + 2\delta[n-6]$$

Problem 5 **(10 points)**

Find the DTFT and FT, respectively, of the following time-domain signals:

(a) $x[n] = a^{|n|}$ where $|a|<1$
(b) $x(t) = e^{2t}u(t)$

(a) $x[n] = a^{|n|} \quad |a|<1$

$$X[e^{j\Omega}] = \sum_{-\infty}^{\infty} a^{|n|} e^{-j\Omega n} = \sum_{0}^{\infty} a^{n} e^{-j\Omega n} + \sum_{0}^{-\infty} e^{|n|} e^{-j\Omega n} - 1$$

$$= \sum_{0}^{\infty} a^{n} e^{-j\Omega n} + \sum_{0}^{\infty} a^{n} e^{j\Omega n} - 1 = \frac{1}{1-a e^{-j\Omega}} + \frac{1}{1-a e^{j\Omega}} - 1$$

$$= \frac{1-a^2}{1+a^2-2a\cos\Omega}$$

(b) $x(t) = e^{+2t} u(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{2t} u(t)\, e^{-j\omega t}\, dt$$

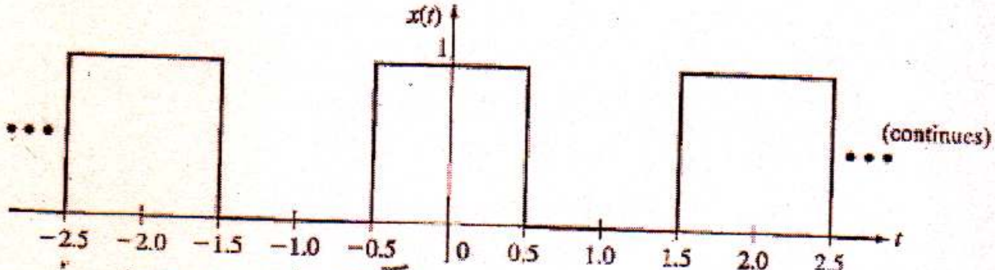
$$= \int_{-\infty}^{0} e^{2t} e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{(-j\omega+2)t}\, dt$$

$$= \frac{e^{(-j\omega+2)t}}{-j\omega+2}\Bigg|_{-\infty}^{0} = \frac{1}{-j\omega+2} - \cancel{\frac{1}{(-j\omega+2)\, e^{\infty}}}^{\,0}$$

$$= \frac{-1}{j\omega-2}$$

Problem 6 **(10 points)**

Consider the rectangular pulse train shown in the figure below. Determine X(jω). Plot the magnitude and phase responses. (Bonus: State the three of the Dirichlet conditions).



periodic with fundamental $T = 2$

$\omega_0 = \frac{2\pi}{T} = \pi$. The signal satisfies Dirichlet conditions and thus it has Fourier series representation.

$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{2}\int_{-1}^{1} x(t)\, e^{-jk\pi t}\, dt = \frac{1}{2}\int_{-0.5}^{0.5} e^{-jk\pi t}\, dt$$

$$= \frac{-1}{j2k\pi}\, e^{-jk\pi t}\Big|_{-0.5}^{0.5} = -\frac{1}{j2k\pi}\left(-j\sin\frac{k\pi}{2} - j\sin\frac{k\pi}{2}\right)$$

$$= \frac{1}{k\pi}\sin\frac{k\pi}{2}, \quad k \neq 0 \quad k \text{ is odd}$$

for $k = 0$, we have

$$X[0] = \frac{1}{2}\int_{-0.5}^{0.5} dt = \frac{1}{2}$$

$$x(t) = \frac{1}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k\pi}\sin\frac{k\pi}{2}\, e^{jk\pi t/2}$$

$$= \frac{1}{2} + \frac{-1}{3\pi}\left(\sin\frac{-3\pi}{2}\right) e^{-j\frac{3\pi}{2}t} + \cancel{\frac{-1}{2\pi}\sin\left(\frac{-2\pi}{2}\right)}^{\,0} e^{-j\pi t} + \frac{-1}{\pi}\sin\left(\frac{-\pi}{2}\right) e^{-j\frac{\pi}{2}t}$$

$$+ \frac{1}{\pi}\sin\frac{\pi}{2}\, e^{j\frac{\pi}{2}t} + \frac{1}{2\pi}\sin\left(\frac{2\pi}{2}\right) e^{j\pi t} + \frac{1}{3\pi}\sin\left(\frac{3\pi}{2}\right) e^{j\frac{3\pi}{2}t}$$