ECE 314 Fall 2012
Homework-10 Solutions

Special Problem 1

(a) The minimum sampling rate necessary to assure perfect reconstruction, i.e., the Nyquist rate, is equal to twice the highest frequency found in the signal spectrum. In the case of $y(t) = \cos(2\pi t)$, all the spectrum is concentrated at the frequencies -2π and 2π, thus the minimum sampling rate is 4π rad/s, or 2 Hertz.

(b) For a sampling rate $f_s = T^{-1}$ greater than the Nyquist rate, the spectrum (Fourier transform) of the sampled signal $y_s(t) = \sum_{n=-\infty}^{\infty} y(nT)\delta(t - nT)$ is composed of a series of copies of the spectrum of $y(t)$ multiplied by f_s and centered in $2\pi n f_s$, $n = \ldots, -1, 0, 1, \ldots$. Considering the signal $y(t) = \cos(2\pi t)$, the reconstruction filter has to filter in the copy that is around 0, while filtering out the other copies. For the signal in question, the copy of the spectrum centered in $2\pi f_s$ has a component in $2\pi f_s - 2\pi$, thus B has to be in the range $2\pi \leq B < 2\pi f_s - 2\pi$. If B is less than this lower limit, than the reconstructed signal would be zero, if it is beyond the upper limit, other frequency components that didn't exist in the original signal will appear in the reconstructed signal.

(c) From the discussion in the answer to the previous question, perfect reconstrution is assured if $1 \leq p < 1/T - 1$.

(d)

- *aliasing*: it doesn't occur in none of the cases, since the sampling rate is $f_s = T^{-1} = 2.5 > 2$.

- *suitability of the bandwidth of the filter*: from the previous discussion, $1 \leq p < 1/T - 1 = 1.5$. Hence, the bandwidth is suitable only in the cases $p = 1.0, 1.2$.

- *discrepancies in the amplitude*: the reconstructed signal should show discrepancies when p is outside of the range $1 \leq p < 1.5$.

- *effects near the ends of the time interval* $[-2, 2]$: Except for the sampling instants, at any instant of time the reconstructed signal receives contributions from all the samples. This happens because the sinc functions that compose the reconstructed signal are non-zero almost everywhere. Hence, to have a perfectly reconstructed signal it would be necessary to take into account all the infinite samples. However, the further away a sample is from the time-instant where the signal is being reconstructed, the lesser contribution it represents. That is why the points closer to the ends of the time interval may present greater discrepancy between the original and reconstructed signal.

- *frequency components of the reconstructed signal*: When $p < 1$, all the frequency components of the sampled signal will be filtered out, so the resulting reconstructed signal should should have no components at all. When $1 \leq p < 1.5$, only the desired frequency component should be present. When $p = 1.5$, there should be 2 frequency components: at 1.0 and 1.5 Hertz. When $p = 4$, there should be frequency components at 1.0, 1.5 and 3.5.

(e) Distortion will begin to occur when $p = 1/T - 1$, i.e., when $T = 1/(1+p) = 0.454545\ldots$.
(f) In this case $p = B/2\pi = 3$, and $p < 1/T - 1$, i. e., $T < 1/(1+p) = 0.25$.

Special Problem 2

(a) We may use the MATLAB code below:

```
t = linspace(-3, 3, 1000);    % Generate time interval
x = sawtooth(t./(2*pi));      % Generate signal
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(b) The Fourier series coefficients of $x(t) = \text{sawtooth}(t/(2\pi))$ are:

$$\begin{aligned}
X[k] &= \int_{-0.5}^{0.5} (4|t| - 1)e^{-j2\pi kt} dt \\
&= 2\int_{0}^{0.5} (4t - 1)\cos(2\pi kt) dt \\
&= 2\left\{4\left[\frac{t\sin(2\pi kt)}{2\pi k} + \frac{\cos(2\pi kt)}{(2\pi k)^2}\right]_0^{0.5} - \left[\frac{\sin(2\pi kt)}{2\pi k}\right]_0^{0.5}\right\} \\
&= 2\frac{(-1)^k - 1}{\pi^2 k^2}.
\end{aligned}$$

The Fourier transform is given by

$$\begin{aligned} X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(t - 2\pi k) \\ &= 4\pi \sum_{k=-\infty}^{\infty} \frac{(-1)^k - 1}{\pi^2 k^2} \, \delta(t - 2\pi k). \end{aligned}$$

$X[k]$ decays with $1/k^2$, but it is not zero for all $|k| \geq K$, hence this signal is not bandlimited.

Set 11 7.17 (a to g), 7.18, 7.22 (d)

7.17 (a) $x(n) = \delta[n-k]$, $k > 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n-k] z^{-n}$$

$$= z^{-k}$$

ROC $= \mathbb{C} \setminus \{0\}$

(b) $x(n) = \delta[n+k]$, $k > 0$

$X(z) = z^{k}$

ROC $= \mathbb{C}$

(c) $x[n] = u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}$$

ROC $= \{ z \in \mathbb{C} ; |z| > 1 \}$

(d) $x[n] = \left(\frac{1}{4}\right)^n (u[n] - u[n-5])$

$$X(z) = \sum_{n=0}^{4} \left(\frac{1}{4} z^{-1}\right)^n = \begin{cases} \dfrac{1 - \left(\frac{1}{4} z^{-1}\right)^5}{1 - \frac{1}{4} z^{-1}} \\ 5, \quad z = \frac{1}{4} \end{cases}$$

ROC $= \mathbb{C} \setminus \{0\}$

(e) $x[n] = \left(\frac{1}{4}\right)^n u[-n]$

$$X(z) = \sum_{n=-\infty}^{0} \left(\frac{1}{4}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (4z)^n = \frac{1}{1-4z}$$

$$|4z| < 1 \Rightarrow |z| < \frac{1}{4}$$

$$ROC = \{ z \in \mathbb{C} ; |z| < \frac{1}{4} \} .$$

(f) $x[n] = 3^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{-1} 3^n z^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{3} z\right)^n = \frac{\frac{1}{3} z}{1 - \frac{1}{3} z} = \frac{-1}{1 - 3z^{-1}}$$

$$\left|\frac{1}{3} z\right| < 1 \Rightarrow |z| < 3$$

$$ROC = \{ z \in \mathbb{C} ; |z| < 3 \} .$$

(g) $x[n] = \left(\frac{2}{3}\right)^{|n|} = \left(\frac{2}{3}\right)^n u(n) + \left(\frac{2}{3}\right)^{-n} u[-n-1]$

$$x_1[n] = \left(\frac{2}{3}\right)^n u[n] \longrightarrow X_1(z) = \frac{1}{1 - \frac{2}{3} z^{-1}} \qquad ROC_1 = \{ |z| > \frac{2}{3} \}$$

$$x_2[n] = \left(\frac{2}{3}\right)^{-n} u[-n-1] \longrightarrow X_2(z) = \frac{\frac{2}{3} z}{1 - \frac{2}{3} z} \qquad ROC_2 = \{ |z| < \frac{3}{2} \}$$

$$X(z) = \frac{1}{1 - \frac{2}{3} z^{-1}} + \frac{\frac{2}{3} z}{1 - \frac{2}{3} z} = \frac{1 - \frac{4}{9}}{(1 - \frac{2}{3} z^{-1})(1 - \frac{2}{3} z)}$$

$$ROC = ROC_1 \cap ROC_2 = \{ \frac{2}{3} < |z| < \frac{3}{2} \} .$$

7.18 The DTFT exists if the unit circle $\{|z|=1\}$ is contained in the ROC.

(a) $\{|z|=1\} \subset \{|z|>\frac{1}{3}\}$, $\exists$ DTFT

$$X(e^{j\Omega}) = \frac{5}{1+\frac{1}{3}e^{-j\Omega}}$$

(b) $\{|z|=1\} \not\subset \{|z|<\frac{1}{3}\}$, $\nexists$ DTFT

(c) $\{|z|=1\} \not\subset \{|z|<\frac{1}{2}\}$, $\nexists$ DTFT

(d) $\{|z|=1\} \subset \{\frac{1}{2}<|z|<3\}$, $\exists$ DTFT

$$X(e^{j\Omega}) = \frac{e^{-j\Omega}}{(1-\frac{1}{2}e^{-j\Omega})(1+3e^{-j\Omega})}.$$

7.22 :

(d) $Y(z) = \frac{z^{2} - z^{-2}}{2} X(z)$

$$Y(z) = \frac{z^{2} X(z)}{2} - \frac{z^{-2} X(z)}{2}$$

$$y[n] = \frac{1}{2} x[n+2] - \frac{1}{2} x[n-2]$$

$$= \frac{1}{2} (n+2)^{2} 3^{n+2} u[n+2] - \frac{1}{2} (n-2)^{2} 3^{n-2} u[n-2] .$$