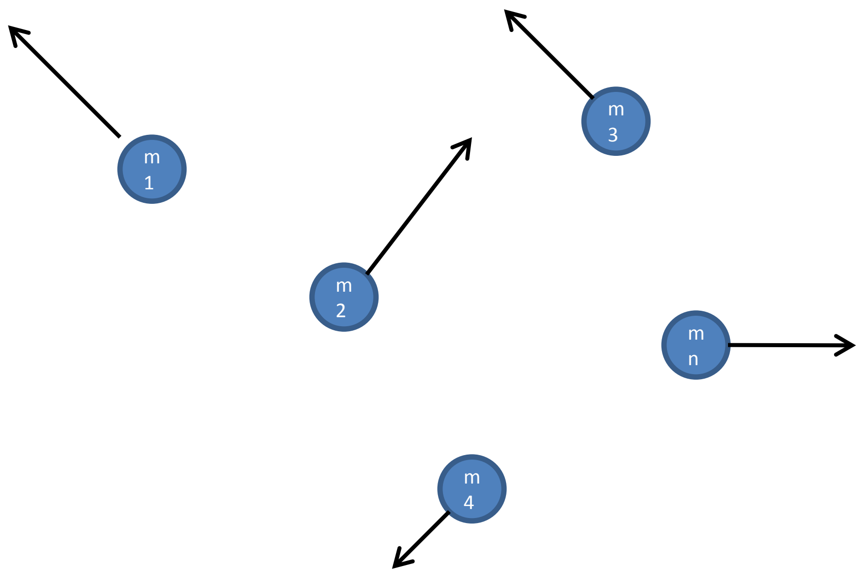
Lecture 24
(Equations of Angular Motion)

Physics 160-01 Fall 2012

Douglas Fields

Rotational Kinetic Energy

- Let's examine a system of n particles:



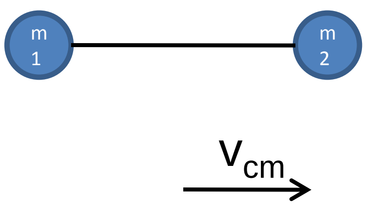
- The total kinetic energy of the system is:

$$KE = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \cdots \frac{1}{2}m_nv_n^2$$

$$= \sum_{i=1}^{n} \frac{1}{2}m_iv_i^2$$

Rotational Kinetic Energy

- But, for a rigid body, there are constraints on what the velocities can be.
- Let’s examine a system of 2 constrained particles (not rotating):



- The total kinetic energy of the system is:

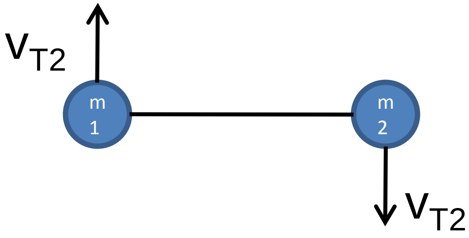
$$KE_{cm} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

$$but, \quad v_1 = v_2 = v_{cm}$$

$$KE_{cm} = \frac{1}{2}(m_1 + m_2)v_{cm}^2 = \frac{1}{2}Mv_{cm}^2$$

Rotational Kinetic Energy

- Now let's examine a system of 2 constrained particles (no cm motion, but rotating):



- Then the total kinetic energy of the system is:

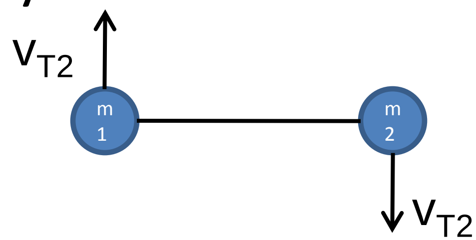
$$KE_{rot} = \frac{1}{2} m_1 v_{T1}^2 + \frac{1}{2} m_2 v_{T2}^2$$

$$KE_{rot} = \frac{1}{2} m_1 \left(\omega r_1\right)^2 + \frac{1}{2} m_2 \left(\omega r_2\right)^2$$

$$= \frac{1}{2} \omega^2 \left(m_1 r_1^2 + m_2 r_2^2\right)$$

Rotational Kinetic Energy

- Notice that the term in parenthesis is only a function of the geometry of the system!



$$KE_{rot} = \frac{1}{2}\omega^2\left(m_1r_1^2 + m_2r_2^2\right) = \frac{1}{2}I\omega^2,$$

$$where,$$

$$I = \left(m_1r_1^2 + m_2r_2^2\right)$$

- Or, in general,

$$I = \sum_i m_i r_i^2$$

- This is called the moment of inertia.

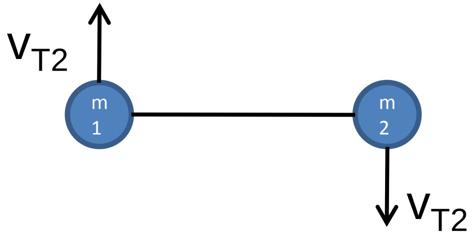
Rotational Kinetic Energy

- But what is r_i ? Remember where it came from:

$$KE_{rot} = \frac{1}{2} m_1 v_{T1}^2 + \frac{1}{2} m_2 v_{T2}^2$$

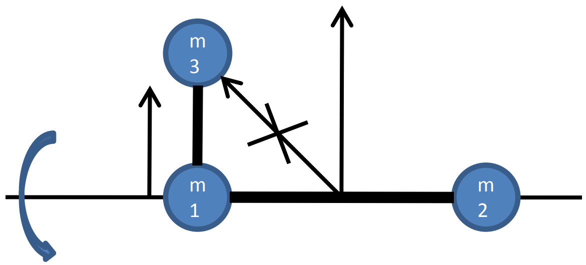
$$KE_{rot} = \frac{1}{2} m_1 \left(\omega r_1\right)^2 + \frac{1}{2} m_2 \left(\omega r_2\right)^2$$

$$= \frac{1}{2} \omega^2 \left(m_1 r_1^2 + m_2 r_2^2\right)$$

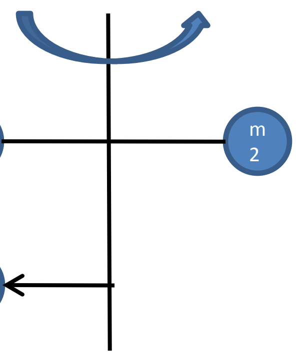


Rotational Kinetic Energy

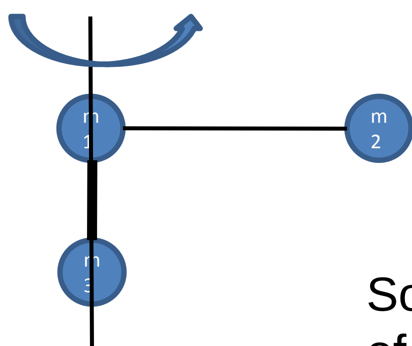
- So the r_i are the distance of the masses to the axis of rotation



$$I = \sum_i m_i r_i^2 = m_1 (0)^2 + m_2 (0)^2 + m_3 (d)^2$$
$$= m_3 d^2$$



$$I = \sum_i m_i r_i^2 = m_1 (d)^2 + m_2 (d)^2 + m_3 (d)^2$$
$$= (m_1 + m_2 + m_3) d^2$$

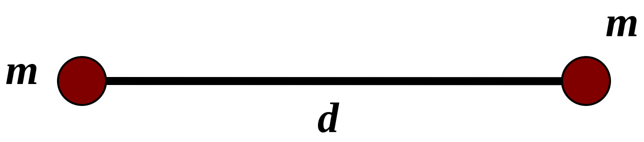


$$I = \sum_i m_i r_i^2 = m_1 (0)^2 + m_2 (2d)^2 + m_3 (0)^2$$
$$= (m_2) 4d^2$$

So, the same object can have a different moment of inertia depending upon the axis of rotation!

CPS Question 23-1

- **Two point masses of mass *m* are attached to a long massless rod of length *d*. What is the moment of inertia that you would use in the calculation of the system's kinetic energy?**



A) $2md^2$.

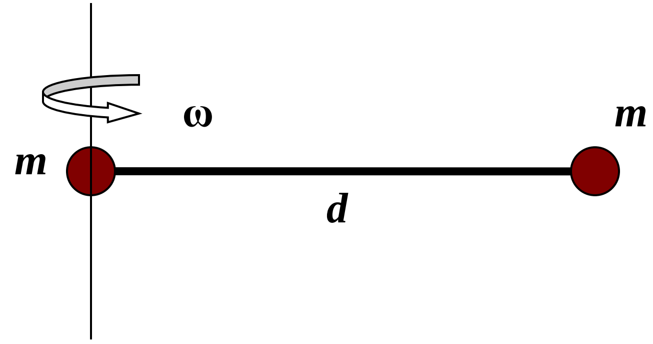
B) md^2.

C) $\frac{1}{2}\omega^2 md^2$.

D) Not enough information to solve.

CPS Question 23-2

- **Two point masses of mass *m* are attached to a long massless rod of length *d*. The system is rotating about one end. What is the moment of inertia that you would use in the calculation of the system's kinetic energy?**



A) $2md^2$.

B) md^2.

C) ½ $\omega^2 md^2$.

D) Not enough information to solve.

Continuous Distributions

- Generalization of the sum

$$I = \sum_i m_i r_i^2 \to \int r^2 dm$$

- But what is dm???

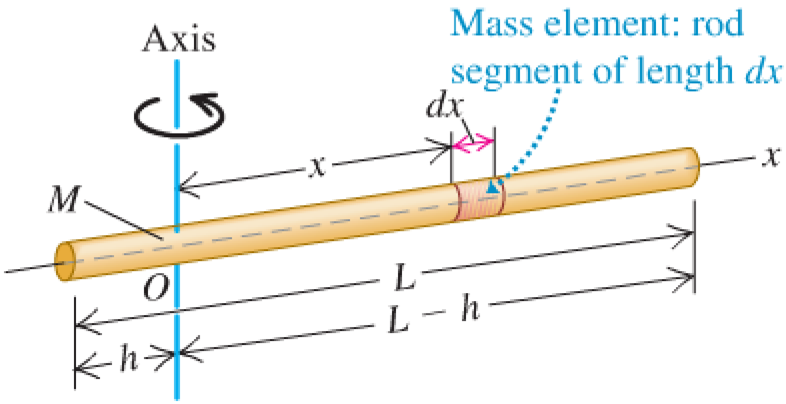
$$dm = \rho dV$$

- In Cartesian coordinates then we can write

$$I = \int r^2 \rho dV = \int r^2 \rho dxdydz$$

Example

- Thin rod of uniform density (mass/unit length)



$$dm = \frac{M}{L}dx$$

$$I = \int r^2 dm = \int x^2 \frac{M}{L} dx$$

$$= \frac{M}{L}\int x^2 dx$$

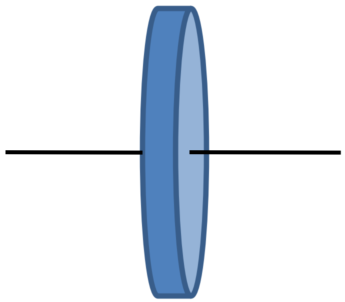
$$= \frac{M}{L}\int_{-h}^{L-h} x^2 dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-h}^{L-h} = \frac{1}{3}M\left(L^2 - 3Lh + 3h^2\right)$$

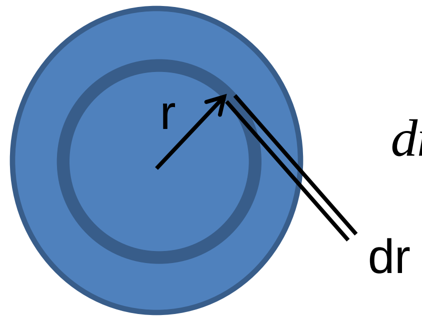
when $h = 0, \quad I = \frac{1}{3}ML^2$

when $h = \frac{L}{2}, \quad I = \frac{1}{12}ML^2$

Example

- Thin disk of uniform density (mass/unit area)





$$dm = \frac{M}{A}dxdy = \frac{M}{A}2\pi r dr$$

$$I = \int r^2 dm = \int r^2 \frac{M}{A} 2\pi r dr = 2\pi \frac{M}{A} \int r^3 dr$$

$$= 2\pi \frac{M}{A} \left[\frac{r^4}{4} \right]_0^R = 2\pi \frac{M}{A} \frac{R^4}{4}$$

$$but,$$

$$A = \pi R^2$$

$$I = 2\pi \frac{M}{\pi R^2} \frac{R^4}{4} = \frac{1}{2} MR^2$$

Example

- Sphere of uniform density (mass/unit volume)

$$dm = \rho dV = \rho\pi r^2 dx = \rho\pi\left(R^2 - x^2\right)dx$$

$$dI = \frac{1}{2}r^2 dm = \frac{1}{2}\left(R^2 - x^2\right)\rho\pi\left(R^2 - x^2\right)dx$$

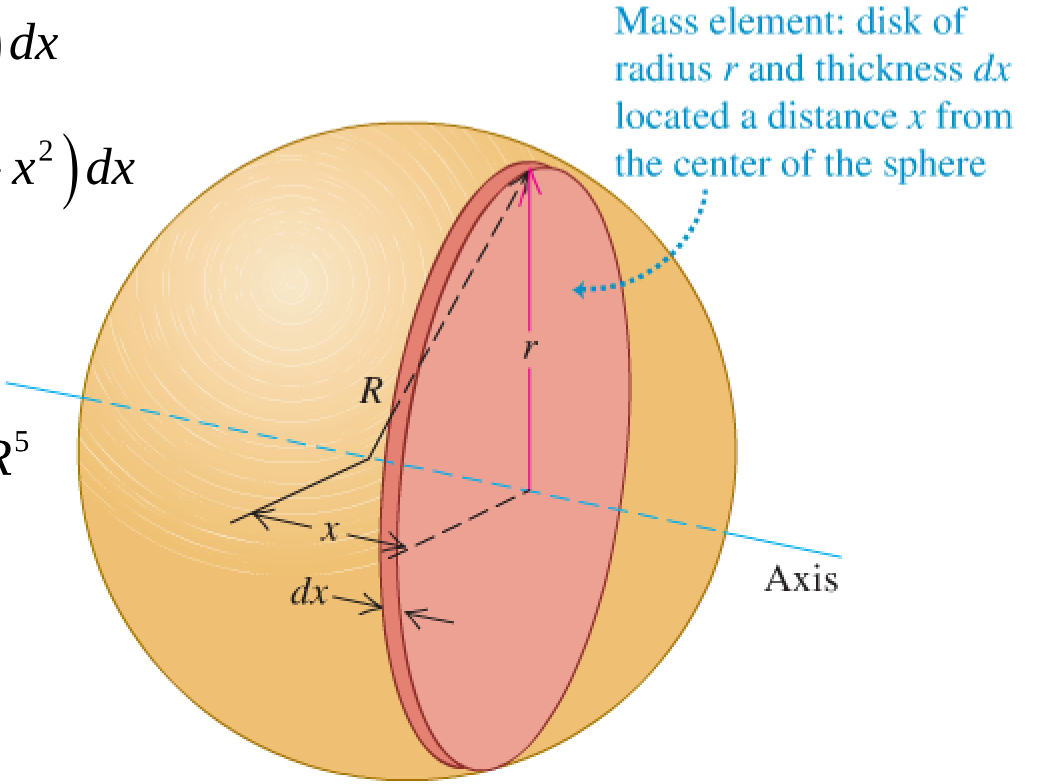
$$= \frac{\rho\pi}{2}\left(R^2 - x^2\right)^2 dx \Rightarrow$$

$$I = (2)\frac{\rho\pi}{2}\int_0^R \left(R^2 - x^2\right)^2 dx = \frac{8\pi\rho}{15}R^5$$

but,

$$M = \rho V = \rho\frac{4}{3}\pi R^3 \Rightarrow$$

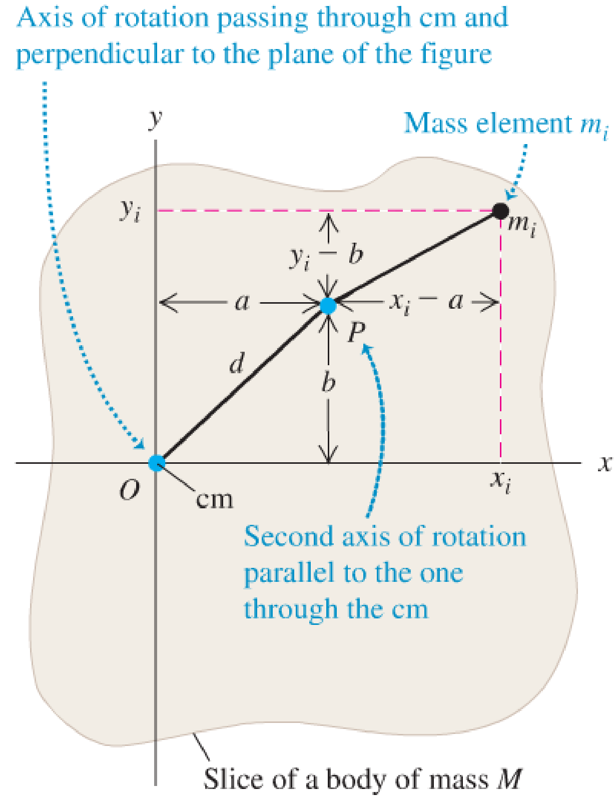
$$I = \frac{2}{5}MR^2$$



Parallel Axis Theorem

- If we know the moment of inertia about an axis that passes through the center of mass of an object, then the moment of inertia about any axis parallel to that a distance d away is given by:

$$I_P = I_{cm} + Md^2$$



$$I_{cm} = \sum_i m_i \left(x_i^2 + y_i^2 \right)$$

$$I_P = \sum_i m_i \left[\left(x_i - a \right)^2 + \left(y_i - b \right)^2 \right]$$

$$= \sum_i m_i \left(x_i^2 + y_i^2 \right) - 2a \cancelto{0}{\sum_i m_i x_i} - 2b \cancelto{0}{\sum_i m_i y_i} + \cancelto{d^2}{\left(a^2 + b^2 \right)} \cancelto{M}{\sum_i m_i}$$

Review

- Linear
  - For const a:

$$x_f = x_i + v_i t + \frac{1}{2} a t^2$$

$$v_f = v_i + at$$

  - Kinetic energy:

$$KE = \frac{1}{2} M v^2$$

  - Comes from:

$$W = \int \vec{F} \cdot d\vec{s}$$

- Rotational
  - For const α:

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2} \alpha t^2$$

$$\omega_f = \omega_i + \alpha t$$

  - Kinetic energy:

$$KE = \frac{1}{2} I \omega^2$$

  - Comes from:

???