Lecture 15
(Applications of Gauss's Law)

Physics 161-01 Spring 2012

Douglas Fields

Gauss's Law

- We have already seen how Gauss's Law can be used to better understand charge distributions on and electric fields around conductors.
- Now we want to use Gauss's Law to calculate the electric field of charge distributions, mainly as a way to introduce the idea of symmetry in physics and how powerful that idea can be.
- First, remember that Gauss's Law:

$$\Phi_{\mathrm{E,Net}} = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\mathrm{enc}}}{\varepsilon_0}$$

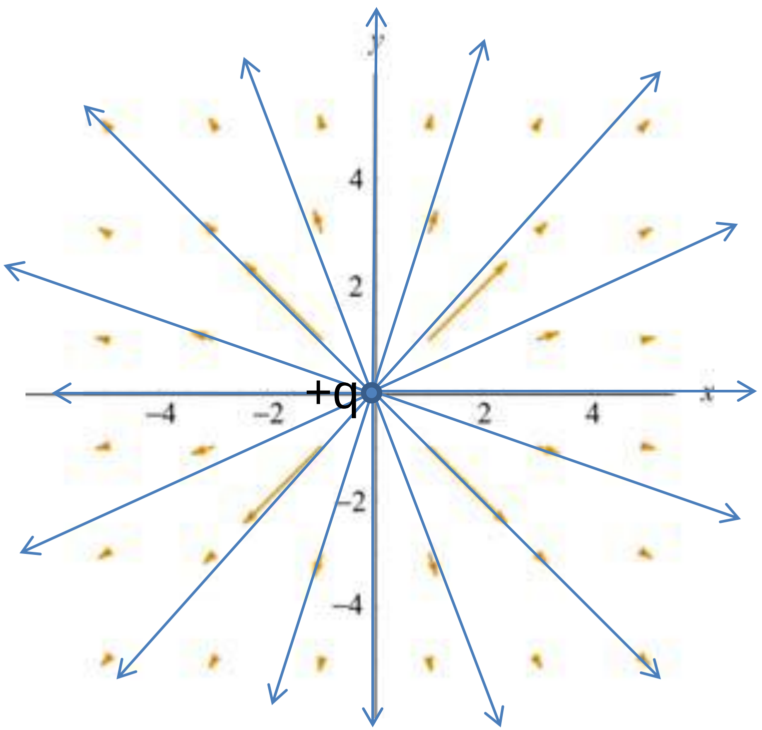
is always true, regardless of what surface you choose or the charge distribution you are talking about.

E-field of an Isolated Point Charge

- Let's now use Gauss's Law to calculate the electric field of a few charge distributions.
- Let's begin with a point charge.

$$\Phi_{\text{E,Net}} = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

- **Step 1:** Find a closed surface over which the electric field is constant (zero is also a constant) and either perpendicular, or parallel to the surface.
- To do this we use the idea of symmetry of the charge distribution.

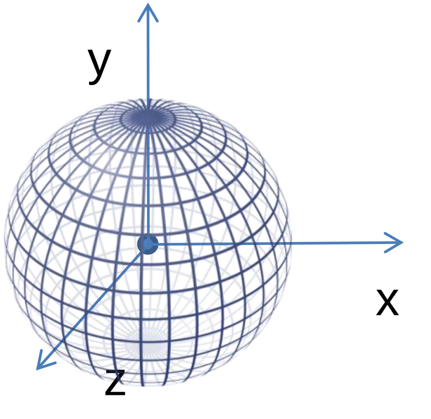


Treatise on Symmetries

- The geometric symmetries of objects (or even physics) can be discovered through the use of operations on space and the effect of these operations on the object (or equation).
- A few possible operations are:
  - Rotations about an axis.
  - Translations along an axis.
  - Reflections about a plane.
- Let's try a few objects, starting with a point charge.

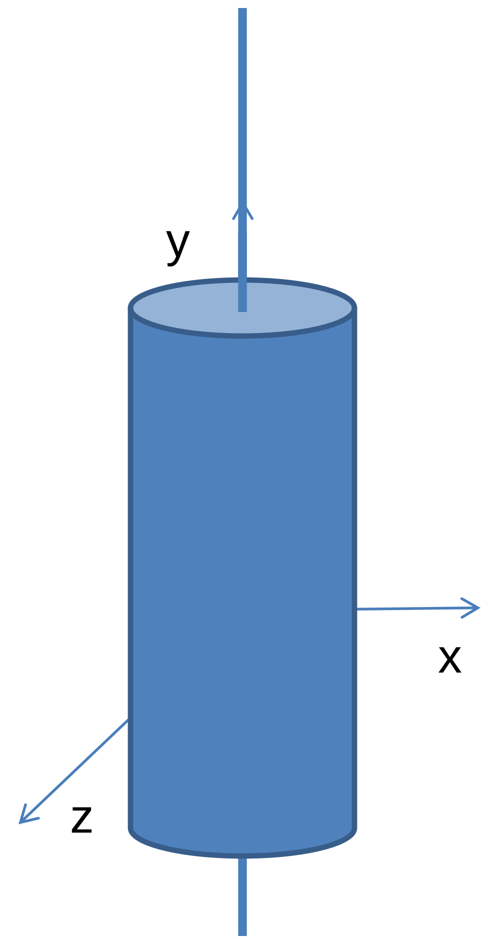
Treatise on Symmetries

- For a point charge, we can rotate at any angle and the situation looks the same.
- If we try to translate, we can tell that the point is moving.
- So, there is spherical symmetry for a point charge.



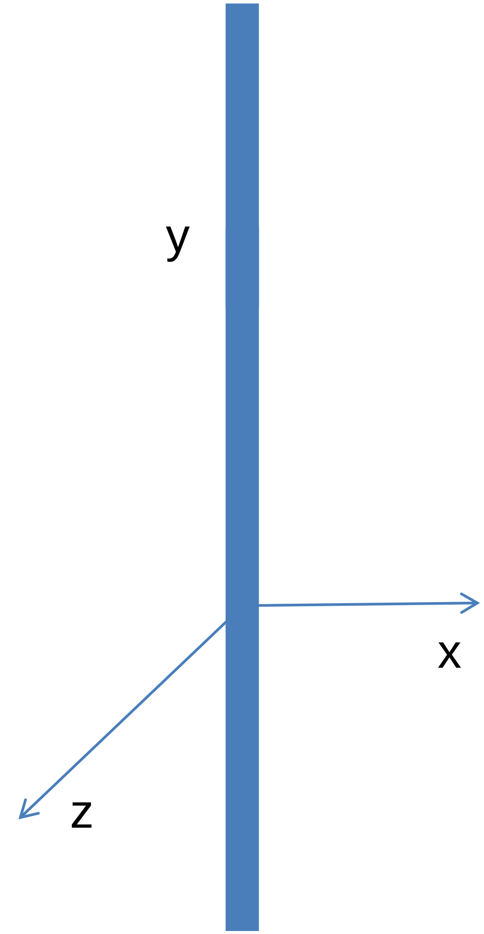
Treatise on Symmetries

- How about a line charge?
- It is symmetric with respect to rotations about its axis (in this case the y-axis).
- This means the field from this charge should also have this symmetry.
- It will have translational symmetry along the y-axis ***only*** if it is an infinite line charge.
- Then, it will have cylindrical symmetry (the field will be the same at every point on the surface of a co-axial cylinder).



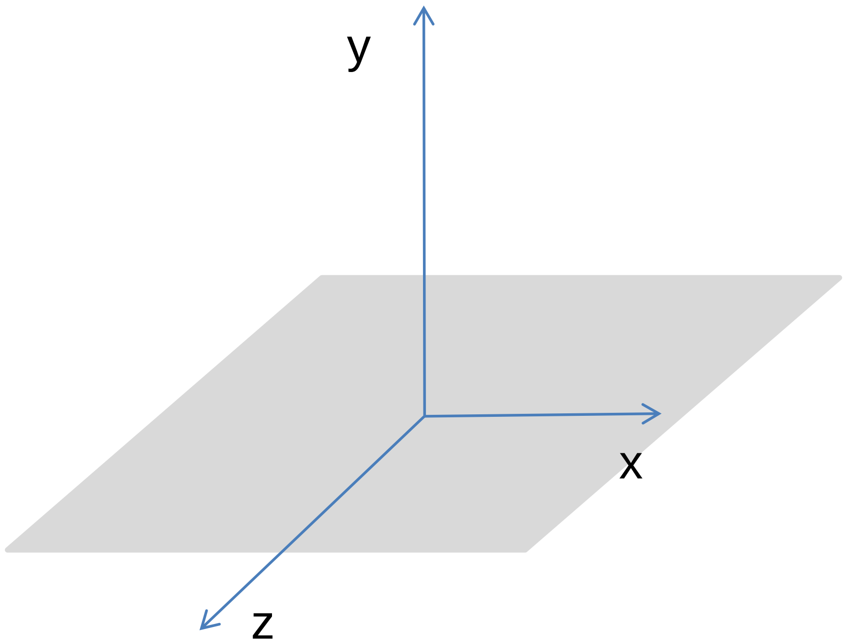
Treatise on Symmetries

- How about a strip charge?
- It is ***no longer*** symmetric with respect to rotations about its axis.
- It will have translational symmetry along the y-axis ***only*** if it is an infinite strip charge.
- Then, the field has no geometric symmetry except that it should be invariant to translations in y.



Treatise on Symmetries

- How about an infinite plane of charge?
- It is symmetric with respect to translations in the x- and z-directions.
- It will have translational symmetry along the x- and z-axis if it is an infinite plane of charge.



CPS 15-1

For which of the following charge distributions would Gauss's law *not* be useful for calculating the electric field?

A. a uniformly charged sphere of radius R

B. a spherical shell of radius R with charge uniformly distributed over its surface

C. a right circular cylinder of radius R and height h with charge uniformly distributed over its surface

D. an infinitely long circular cylinder of radius R with charge uniformly distributed over its surface

E. Gauss's law would be useful for finding the electric field in all of these cases.

CPS 15-1

For which of the following charge distributions would Gauss's law *not* be useful for calculating the electric field?

A. a uniformly charged sphere of radius *R*

B. a spherical shell of radius *R* with charge uniformly distributed over its surface

C. a right circular cylinder of radius *R* and height *h* with charge uniformly distributed over its surface

D. an infinitely long circular cylinder of radius *R* with charge uniformly distributed over its surface

E. Gauss's law would be useful for finding the electric field in all of these cases.

E-field of an Isolated Point Charge

- **Step 1:** Find a closed surface over which the electric field is constant (zero is also a constant) and either perpendicular, or parallel to the surface.
- A sphere centered on the charge with radius r.
- ***Step 2:*** If E is constant over a surface, E and the dot product can come out of the integral.

$$\Phi_{\text{E,Net}} = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

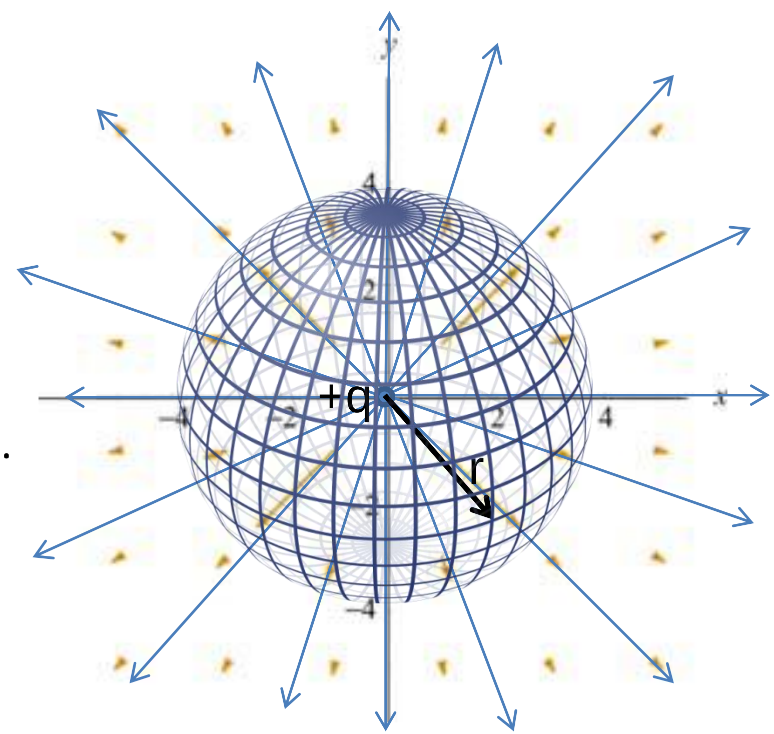
$$= |E|\hat{r} \cdot \hat{r} \oint dA$$

$$= |E| 4\pi r^2 = \frac{q}{\varepsilon_0}$$

- ***Step 3:*** Find charge enclosed and solve for E.

$$|E| 4\pi r^2 = \frac{q}{\varepsilon_0} \Rightarrow$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$



E-field of an Infinite Line Charge

- **Step 1:** Find a closed surface over which the electric field is constant (zero is also a constant) and either perpendicular, or parallel to the surface.
- A cylinder with axis the same as the line charge.
- ***Step 2:*** If E is constant over a surface, E and the dot product can come out of the integral.

$$\oint \vec{E}\cdot d\vec{A} = \int_{A_1} \vec{E}\cdot d\vec{A} + \cancel{\int_{A_2} \vec{E}\cdot d\vec{A}} + \cancel{\int_{A_3} \vec{E}\cdot d\vec{A}}$$

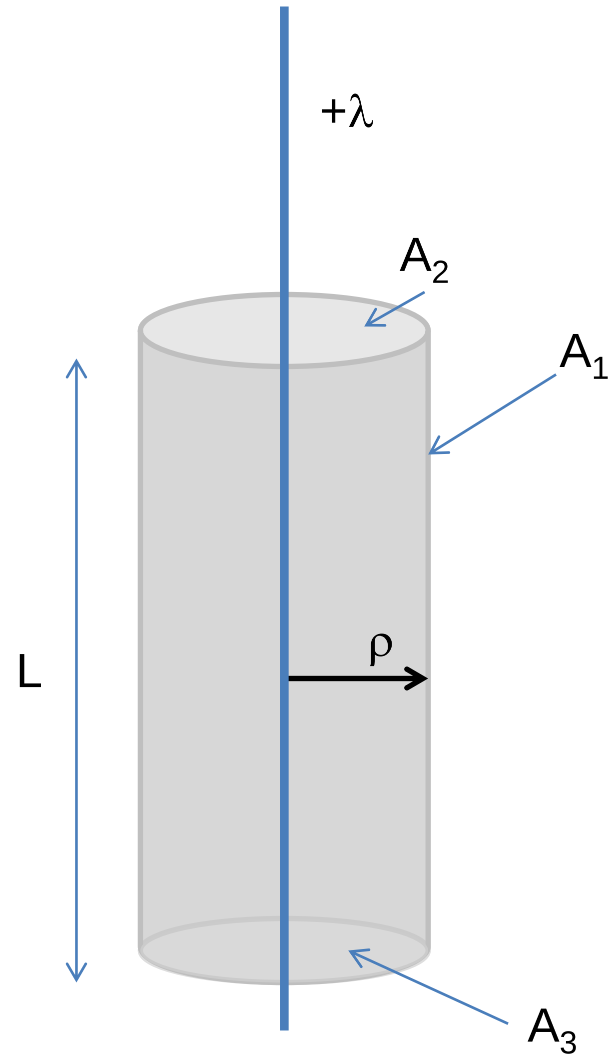
$$= |E|\,\hat{\rho}\cdot\hat{\rho}\int_{A_1} dA$$

$$= |E|\,2\pi r L = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}$$

- ***Step 3:*** Find charge enclosed and solve for E.

$$|E|\,2\pi r L = \frac{\lambda L}{\varepsilon_0} \Rightarrow$$

$$\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\hat{\rho}$$



E-field of an Infinite Sheet of Charge

- **Step 1:** Find a closed surface over which the electric field is constant (zero is also a constant) and either perpendicular, or parallel to the surface.
- A cylinder with the flat "caps" parallel to the sheet.
- ***Step 2:*** If E is constant over a surface, E and the dot product can come out of the integral.

$$\oint \vec{E}\cdot d\vec{A} = \cancel{\int_{A_1} \vec{E}\cdot d\vec{A}} + \int_{A_2} \vec{E}\cdot d\vec{A} + \int_{A_3} \vec{E}\cdot d\vec{A}$$

$$= |E|\,\hat{j}\cdot\hat{j}\int_{A_2} dA + |E|\left(-\hat{j}\right)\cdot\left(-\hat{j}\right)\int_{A_3} dA$$

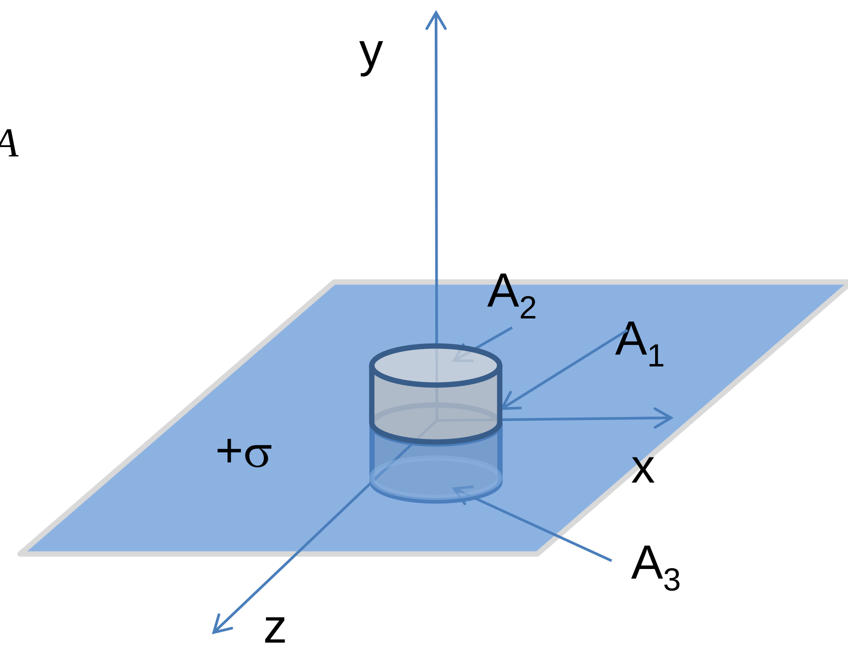
$$= 2|E|A = \frac{q_{\mathrm{enc}}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

- ***Step 3:*** Find charge enclosed and solve for E.

$$2|E|A = \frac{\sigma A}{\varepsilon_0} \Rightarrow$$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{j} \quad (\text{above})$$

$$\vec{E} = \frac{-\sigma}{2\varepsilon_0}\hat{j} \quad (\text{below})$$



E-field of an Infinite Sheet of Charge

- **Step 1:** Find a closed surface over which the electric field is constant (zero is also a constant) and either perpendicular, or parallel to the surface.
- A cube with two faces parallel to the sheet.
- ***Step 2:*** If E is constant over a surface, E and the dot product can come out of the integral.

$$\oint \vec{E}\cdot d\vec{A} = \cancel{\int_{A_1} \vec{E}\cdot d\vec{A}} + \int_{A_2} \vec{E}\cdot d\vec{A} + \int_{A_3} \vec{E}\cdot d\vec{A}$$

$$= |E|\,\hat{j}\cdot\hat{j}\int_{A_2} dA + |E|\left(-\hat{j}\right)\cdot\left(-\hat{j}\right)\int_{A_3} dA$$

$$= 2|E|A = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

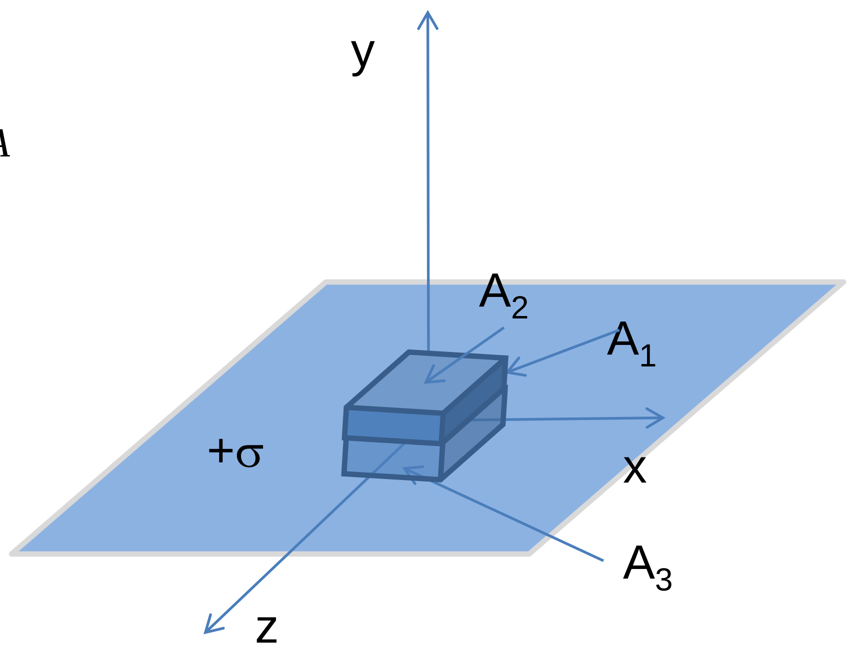
- ***Step 3:*** Find charge enclosed and solve for E.

$$2|E|A = \frac{\sigma A}{\varepsilon_0} \Rightarrow$$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{j} \quad (\text{above})$$

$$\vec{E} = \frac{-\sigma}{2\varepsilon_0}\hat{j} \quad (\text{below})$$

- Note that the result is independent of the exact shape of the Gaussian surface.



Dimensionality of E-field Strength

- Take a look at a point charge as a little particle. As you move away from it, that particle appears smaller to you in **two** dimensions. It appears smaller both in height and in width.

  Now take a look at an infinite line of charge. As you move away from it, that infinite line appears smaller (more precisely "thinner") to you in **one** dimension. It still looks like it's infinitely long, but it does appear thinner and thinner as you move away from it.

  Now take a look at an infinite plane of charge. As you move away from it, how does it appear smaller? If you're in empty space, except for that infinite plane of charge, how can you tell that that infinite plane of charge is 1 kilometer away from you or 1 light-year away from you?
- (http://www.physicsforums.com/showthread.php?t=188661)

E-field of Sphere of Charge

- **Step 1:** Find a closed surface over which the electric field is constant (zero is also a constant) and either perpendicular, or parallel to the surface.
- A sphere centered on the charge with radius r.
- ***Step 2:*** If E is constant over a surface, E and the dot product can come out of the integral.

$$\Phi_{\text{E,Net}} = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

$$= |E| \hat{r} \cdot \hat{r} \oint dA$$

$$= |E| 4\pi r^2 = \frac{1}{\varepsilon_0} \int_V \rho(r) dV$$

- ***Step 3:*** Find charge enclosed and solve for E. (let's assume constant ρ)

$$|E| 4\pi r^2 = \frac{\rho}{\varepsilon_0} \int_V dV \Rightarrow$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\rho}{r^2} \left(\frac{4}{3} \pi r^3 \right) \hat{r}$$