ECE340 spring 2011
Homework-9 Solutions

Problems: 3-7.1, 3-7.2, 3-7.3, 3-7.4, 3-7.5

3-7.1 Firstly, the characteristic functions for random variables X and Y are found as follows:

$$\Phi_X(u) = \int_{-\infty}^{\infty} f_X(x) e^{jux} dx = \int_{-\infty}^{\infty} e^{-x} u(x) e^{jux} dx = \int_{0}^{\infty} e^{-x} e^{jux} dx = \int_{0}^{\infty} e^{(ju-1)x} dx = \frac{1}{1-ju}$$

$$\Phi_Y(u) = \int_{-\infty}^{\infty} f_Y(y) e^{juy} dy = \int_{-\infty}^{\infty} 3e^{-3y} u(y) e^{juy} dx = \int_{0}^{\infty} 3e^{-3y} e^{juy} dy = \int_{0}^{\infty} 3e^{(ju-3)y} dy = \frac{3}{3-ju}$$

Since Z = X + Y, and X and Y are independent, the characteristic function of *Z* is

$$\Phi_Z(u) = \Phi_X(u)\Phi_Y(u)$$

So,

$$f_Z(z) = \mathcal{F}^{-1}\{\Phi_Z(u)\} = \mathcal{F}^{-1}\{\Phi_X(u)\Phi_Y(u)\} = \mathcal{F}^{-1}\left\{\frac{1}{1-ju}\frac{3}{3-ju}\right\} = \frac{3}{2}\mathcal{F}^{-1}\left\{\frac{1}{1-ju} - \frac{1}{3-ju}\right\}$$

$$= \frac{3}{2}(e^{-z} - e^{-3z})u(z)$$

Here we define the notation F^{-1} as the inverse of the characteristic function. That is, the pdf that yields the characteristic function. If you have studied Fourier transforms, this is simply the inverse Fourier transform of the characteristic function; the latter being the Fourier transform of the pdf.

3-7.2

a) Firstly we notice that for a Gaussian random variable X with zero mean and variance σ^2, we have the following pdf:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}, \qquad -\infty < x < \infty$$

The characteristic function of the random variable X is found via

$$\Phi_X(u) = \int_{-\infty}^{\infty} f_X(x) e^{jux} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} e^{jux} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{jux} dx$$

Now recall the Fourier-transform pair:

$$e^{-\alpha x^2} \xleftrightarrow{\mathcal{F}} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{u^2}{4\alpha}}$$

In our case:

$$\alpha = \frac{1}{2\sigma^2}$$

$$e^{-\frac{1}{2\sigma^2}x^2} \xleftrightarrow{\mathcal{F}} \sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}} e^{-\frac{u^2}{4\frac{1}{2\sigma^2}}} = \sqrt{2\pi\sigma^2} e^{-\frac{u^2\sigma^2}{2}}$$

So, we have the characteristic function can be written as the following:

$$\Phi_X(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{jux} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{2\pi\sigma^2} e^{-\frac{u^2\sigma^2}{2}} = e^{-\frac{u^2\sigma^2}{2}}$$

Now the characteristic function for random variable X (Gaussian) is

$$\Phi_X(u) = e^{-\frac{u^2\sigma^2}{2}}$$

b) On page 72 of the book, we have the equation for the nth central moment for a Gaussian random variable (equation (2-27)) as the following:

$$E[(X - E[X])^n] = \begin{cases} 0, & n\ odd \\ 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n, & n\ even \end{cases}$$

To verify this with the characteristic function of X, we do the following:
Since in our case the mean E[X] = 0, our nth central moment becomes:

$$E[X^n]$$

Also according to the equation (3-53) on page 150 of the book, we have the following:

$$E[X^n] = \frac{1}{j^n}\left[\frac{d^n\Phi_X(u)}{du^n}\right]_{u=0}$$

So, the first central moment of X is:

$$E[X] = \frac{1}{j}\left[\frac{d\Phi_X(u)}{du}\right]_{u=0} = \frac{1}{j}\left[\frac{de^{-\frac{u^2\sigma^2}{2}}}{du}\right]_{u=0} = \frac{1}{j}(-u\sigma^2)e^{-\frac{u^2\sigma^2}{2}}|_{u=0} = 0$$

The second central moment of X is:

$$E[X^2] = \frac{1}{j^2}\left[\frac{d^2\Phi_X(u)}{du^2}\right]_{u=0} = \frac{1}{j^2}\left[\frac{d^2e^{-\frac{u^2\sigma^2}{2}}}{du^2}\right]_{u=0} = \frac{1}{-1}\frac{d(-u\sigma^2)e^{-\frac{u^2\sigma^2}{2}}}{du}|_{u=0}$$

$$= \frac{\sigma^2 d[ue^{-\frac{u^2\sigma^2}{2}}]}{du}|_{u=0} = \sigma^2\left(e^{-\frac{u^2\sigma^2}{2}} + u(-u\sigma^2)e^{-\frac{u^2\sigma^2}{2}}\right)|_{u=0} = (2-1)\sigma^2$$

$$= \sigma^2$$

Following these steps for n = 3, 4, 5 …, we see the nth central moment is indeed:

$$E[X^n] = \begin{cases} 0, & n\ odd \\ 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n, & n\ even \end{cases}$$

3-7.3

The characteristic function is given as:

$$\Phi_X(u) = 1 - p + pe^{ju}$$

a) We can obtain the mean value of the random variable X as:

$$E[X] = \frac{1}{j}\left[\frac{d\Phi_X(u)}{du}\right]_{u=0}$$

We have:

$$\frac{d\phi(u)}{du} = jpe^{ju}$$

Therefore:

$$E[X] = \frac{1}{j}\left[jpe^{ju}\right]_{u=0} = p$$

b) For the mean square value of the random variable we have:

$$E[X^2] = \frac{1}{j^2}\left[\frac{d^2\Phi_X(u)}{du^2}\right]_{u=0} = -\left[\frac{d^2\Phi_X(u)}{du^2}\right]_{u=0}$$

$$\frac{\mathrm{d}^2\Phi_{\mathrm{X}}(u)}{du^2}=\frac{\mathrm{d}}{du}\left[\frac{d\Phi_{\mathrm{X}}(u)}{du}\right]=j^2pe^{ju}=\ -pe^{ju}$$

Then :

$$E[X^2]=-\left[-pe^{ju}\right]_{u=0}=p$$

c) For the third moment one can write:

$$E[X^3]=\frac{1}{j^3}\left[\frac{\mathrm{d}^3\Phi_{\mathrm{X}}(u)}{du^3}\right]_{u=0}=j\left[\frac{\mathrm{d}^3\Phi_{\mathrm{X}}(u)}{du^3}\right]_{u=0}$$

In this case we have:

$$\frac{\mathrm{d}^3\Phi_{\mathrm{X}}(u)}{du^3}=\frac{\mathrm{d}}{du}\left[\frac{\mathrm{d}^2\Phi_{\mathrm{X}}(u)}{du^2}\right]=-pje^{ju}$$

Hence:

$$E[X^3]=j\left[\frac{\mathrm{d}^3\Phi_{\mathrm{X}}(u)}{du^3}\right]_{u=0}=j\left[-pje^{ju}\right]_{u=0}=p$$

Therefore the third central moment of the random variable can obtained as:

$$\begin{aligned}E\ [(X-E[X])^3] &=\ E\ [X^3-3X^2E[X]+3X(E[X])^2-(E[x])^3]\\ &=E\ [X^3]-3E[X^2]E[X]+3E[X](E[X])^2-(E[x])^3]\\ &=E\ [X^3]-3E[X^2]E[X]+2(E[X])^3]\\ &=p-3p.p{+}p^3=p-3p^2+2p^3\end{aligned}$$

3-7.4

a) The probability density functions are given as:

$$f_X(x)=5e^{-5x},$$
$$f_Y(x)=2e^{-2y},$$

The characteristic function for random variable X can be obtained as:

$$\begin{aligned}\Phi_X(u)&=\int_{-\infty}^{\infty}f_X(x)e^{jux}dx=5\int_{-\infty}^{\infty}e^{-5x}u(x)e^{jux}dx=5\int_{0}^{\infty}e^{-5x}e^{jux}dx\\ &=5\int_{0}^{\infty}e^{(ju-5)x}dx=\frac{5}{5-ju}\end{aligned}$$

For random variable Y The characteristic function can be calculated as:

$$\begin{aligned}\Phi_Y(u)&=\int_{-\infty}^{\infty}f_Y(y)e^{juy}dy=2\int_{-\infty}^{\infty}e^{-5y}u(y)e^{juy}dy=2\int_{0}^{\infty}e^{-2y}e^{juy}dy\\ &=2\int_{0}^{\infty}e^{(ju-2)y}dy=\frac{2}{2-ju}\end{aligned}$$

Then the characteristic function of the random variable $Z=X+Y$ can be obtained as:

$$\Phi_Z(u)=\Phi_X(u)\Phi_Y(u).$$

Therefore in this case we have:

$$\Phi_Z(u)=\frac{5}{5-ju}.\frac{2}{2-ju}$$

Using partial fraction expansion, one obtains:

$$\Phi_Z(u) = \frac{-10/3}{5-ju} + \frac{10/3}{2-ju}$$

Then

$$f_Z(z) = \int_{-\infty}^{\infty} \Phi_Z(u) e^{-juz} dz$$

Using inverse Fourier transformation, we obtain:

$$f_Z(z) = -\frac{10}{3}(e^{-5z} - e^{-2z})\, u(z)$$

b) The mean value of Z can be obtained via:

$$E[Z] = \frac{1}{j}\left[\frac{d\Phi_Z(u)}{du}\right]_{u=0}$$

$$\frac{d\Phi_Z(u)}{du} = \frac{d}{du}\left[\frac{5}{5-ju}.\frac{2}{2-ju}\right] = \frac{d}{du}\left[\frac{10}{10-7ju-u^2}\right]$$
$$= \frac{0-10(-j7-2u)}{(10-7ju-u^2)^2} = \frac{10(j7+2u)}{(10-7ju-u^2)^2}$$

Then

$$E[Z] = \frac{1}{j}\left[\frac{d\Phi_Z(u)}{du}\right]_{u=0} = \frac{1}{j}\left[\frac{10(j7+2u)}{(10-7ju-u^2)^2}\right]_{u=0} = \frac{1}{j}\left[\frac{10(j7)}{(10)^2}\right] = 0.7$$

The second moment of Z can be obtained as follows:

$$E[Z^2] = \frac{1}{j^2}\left[\frac{d^2\Phi_Z(u)}{du^2}\right]_{u=0} = -\left[\frac{d^2\Phi_Z(u)}{du^2}\right]_{u=0}$$

$$\frac{d^2\Phi_Z(u)}{du^2} = \frac{d}{du}\left[\frac{d\Phi_Z(u)}{du}\right] = \frac{d}{du}\left[\frac{10(j7+2u)}{(10-7ju-u^2)^2}\right]$$
$$= \frac{20(10-7ju-u^2)^2 - 20(-j7-2u)(10-7ju-u^2)(j7+2u)}{(10-7ju-u^2)^4}$$

Then

$$E[Z^2] = -\left[\frac{20(10-7ju-u^2)^2 - 20(-j7-2u)(10-7ju-u^2)(j7+2u)}{(10-7ju-u^2)^4}\right]_{u=0}$$
$$= -\left[\frac{20(10)^2 - 20(-j7)(10)(j7)}{(10)^4}\right] = 0.78$$

3-7.5

The probability density function is given as:

$$f_X(x) = 2e^{-4|x|}$$

The characteristic function of random variable X can be obtained as

$$\Phi_X(u) = \int_{-\infty}^{\infty} f_X(x)e^{jux}dx = 2\int_{-\infty}^{\infty} e^{-4|x|}e^{jux}dx = 2\int_{-\infty}^{0} e^{4x}e^{jux}dx + 2\int_{0}^{\infty} e^{-4x}e^{jux}dx$$
$$= 2\int_{-\infty}^{0} e^{(ju+4)x}dx + 2\int_{0}^{\infty} e^{(ju-4)x}dx = 2\int_{0}^{\infty} e^{(-ju-4)x}dx + 2\int_{0}^{\infty} e^{(ju-4)x}dx$$

$$= \frac{2}{4 + ju} + \frac{2}{4 - ju} = \frac{16}{16 + u^2}$$

One can obtain the mean value of the random variable X as:

$$\mathrm{E[X]} = \frac{1}{\mathrm{j}}\left[\frac{\mathrm{d}\Phi_{\mathrm{X}}(\mathrm{u})}{\mathrm{du}}\right]_{\mathrm{u}=0}$$

We have:

$$\frac{d\phi_X(u)}{du} = \frac{0 - (16)(2u)}{(16 + u^2)^2} = \frac{-32u}{(16 + u^2)^2}$$

Hence

$$E[X] = \frac{1}{j}\left[\frac{-32u}{(16 + u^2)^2}\right]_{u=0} = \frac{1}{j}(0) = 0$$

As for the second moment of the random variable, we have:

$$E[X^2] = -\left[\frac{d^2\Phi_X(u)}{du^2}\right]_{u=0}$$

$$\frac{d^2\Phi_X(u)}{du^2} = \frac{d}{du}\left[\frac{d\Phi_X(u)}{du}\right] = \frac{d}{du}\left[\frac{-32u}{(16 + u^2)^2}\right]$$
$$= \frac{-32(16 + u^2)^2 - (2)(2u)(16 + u^2)(-32u)}{(16 + u^2)^4}$$

Therefore:

$$E[X^2] = -\left[\frac{-32(16 + u^2)^2 - (2)(2u)(16 + u^2)(-32u)}{(16 + u^2)^4}\right]_{u=0}$$
$$= -\left[\frac{-32(16)^2}{(16)^4}\right] = \frac{1}{8}$$