Notes on Dialogues Concerning Two New Sciences

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Abstract

This article examines several theorems presented by *Galileo Galilei* in his seminal work on dynamics, *Dialogues Concerning Two New Sciences*[1]. While Galileo's original demonstrations relied on classical geometric reasoning, this work aims to reinterpret and reconstruct his results using modern mathematical notation and techniques. By reformulating his theorems with contemporary methods, we provide a clearer, more accessible understanding of his groundbreaking contributions to the science of motion.

Introduction

Galileo Galilei (1564–1642) is widely regarded as one of the greatest minds in the history of science. His contributions to physics, astronomy, and mathematics laid the groundwork for modern scientific thought. Born in Pisa, Italy, Galileo's early studies in medicine quickly shifted toward mathematics, where his true genius emerged. He revolutionized observational astronomy with his improvements to the telescope, famously discovering the moons of Jupiter and supporting the heliocentric model of the solar system. However, his challenges to Aristotelian physics were perhaps just as groundbreaking, setting the stage for classical mechanics. Despite facing opposition from the Catholic Church and even enduring house arrest in his later years, Galileo's legacy remains foundational to modern science.

One of his most important works, Dialogues on the Two New Sciences (1638), was written during his house arrest and is often considered his final major contribution. This book is structured as a conversation between three fictional characters—Salviati, Sagredo, and Simplicio—who debate and discuss the principles of mechanics and motion. The "two new sciences" in question are the science of material strength and the science of motion, both of which became essential foundations for classical physics. The book masterfully blends rigorous mathematical reasoning with experimental observations, marking one of the earliest examples of a true scientific method.

In particular, the "Third Day" of the Dialogues focuses on Galileo's mathematical treatment of motion. Here, he introduces key concepts such as the uniform acceleration of falling bodies, the parabolic trajectory of projectiles, and the law of free fall. He formulates his discoveries using geometric reasoning, constructing theorems, corollaries, lemmas, and propositions in the style of classical Greek mathematics, reminiscent of Euclid and Archimedes. While this method was rigorous for its time, it is far removed from the symbolic notation and modern formalism used in contemporary mathematics and physics.

The purpose of this article is to bridge this gap. Galileo's theorems, corollaries, lemmas, and propositions will be reformulated using modern mathematical notation and expressions, making his results more accessible to today's mathematicians and physicists. By translating his arguments into the language of calculus and modern algebra, we will highlight the enduring relevance of his insights and provide a clearer understanding of his revolutionary contributions to the science of motion.

1 On the Salviati lemma

On the *Third day, Change of position, [De Motu Locali]*, Galileo explores the fundamental principles of motion and acceleration, laying the foundations of *dynamical mechanics*.

One of the most significant theorems he presents in this *Third day* is stated by *Salviati* immediately after the *Scholium* of the *Corollary II* on page 183-184

Theorem 1. *If a body falls freely along smooth planes inclined at any angle whatsoever, but of the same height, the speeds with which it reaches the bottomare the same.*

Galileo's demonstration of this theorem follows an old mathematical approach, with expressions and notations that may be difficult to grasp. In the following, I will reformulate and prove the theorem using modern notation and clearer expressions, making the demonstration more accessible and easier to understand.

Proof. Let ABC be a rectangle triangle.

Let (AB) and (AE) be inclined plane where the body may fall from the vertex A to B or E according to the inclination of the plane.

In what follows, let α be the angle of inclination of the plane (AB) and β that of (AE). Let v_1 represents the speed of the body falling from A to B according to time t_1 and v_2 the speed of the body falling from A to E according to time t_2.

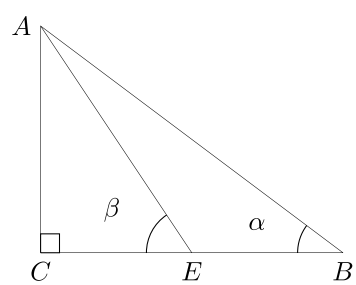

Figure 1: Inclined plane

Let us lay down :

$$\begin{cases} AC = h \\ AB = x_1 \\ AE = x_2; \end{cases}$$

As we know :

$$\begin{cases} \sin\alpha = \frac{h}{x_1} \\ \sin\beta = \frac{h}{x_2} \end{cases}$$

That gives us :

$$\begin{cases} h = x_1 \sin\alpha \\ h = x_2 \sin\beta \end{cases}$$

As we know $x_1 = \frac{1}{2} \cdot a_1 \cdot t_1^2$ where a_1 is the acceleration on the plane AB, and t_1 the time required to travers it; and $x_2 = \frac{1}{2} \cdot a_2 \cdot t_2^2$. where a_2 the acceleration on the plane AE, and t_2 the time required to travers that plane.

Now :

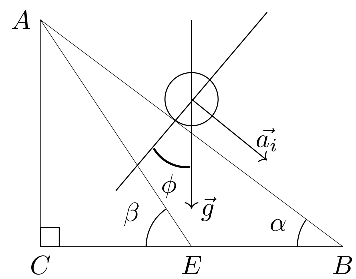

Figure 2: Falling body on the inclined plane

So, as we have : $a_i = g \sin\phi$, we have :

$$\begin{cases} x_1 = \frac{1}{2} a_1 t_1^2 = \frac{1}{2} g t_1^2 \sin\alpha \\ x_2 = \frac{1}{2} a_2 t_2^2 = \frac{1}{2} g t_2^2 \sin\beta \end{cases}$$

$$\Longrightarrow \begin{cases} h = \left(\frac{1}{2} g t_1^2 \sin\alpha\right) \cdot \sin\alpha = \frac{1}{2} g t_1^2 \sin^2\alpha \\ h = \left(\frac{1}{2} g t_2^2 \sin\beta\right) \cdot \sin\beta = \frac{1}{2} g t_2^2 \sin^2\beta \end{cases}$$

$$\Longrightarrow \quad \frac{1}{2} g t_1^2 \sin^2\alpha = \frac{1}{2} g t_2^2 \sin^2\beta$$

$$\Longleftrightarrow \quad t_1^2 \sin^2\alpha = t_2^2 \sin^2\beta$$

$$\Longleftrightarrow \quad t_1 \sin\alpha = t_2 \sin\beta$$

because time and distance (since t and h are both positive)

$$\iff \quad \frac{t_1}{t_2} = \frac{\sin\beta}{\sin\alpha} \tag{1}$$

Now, in the other hand, the speeds of the body at the bottoms of each inclined plane is given by :

$$\begin{cases} v_1 = a_1 t_1 \\ v_2 = a_2 t_2 \end{cases} \quad \Longrightarrow \quad \begin{cases} v_1 = g t_1 \sin\alpha \\ v_2 = g t_2 \sin\beta \end{cases}$$

Now, seing that each term of those equation are not null, by dividing term by term we have :

$$\Longrightarrow \quad \frac{v_1}{v_2} = \frac{g t_1 \sin\alpha}{g t_2 \sin\beta} = \frac{t_1 \sin\alpha}{t_2 \sin\beta}$$

According to equation (1) we get :

$$\frac{v_1}{v_2} = \frac{\sin\alpha}{\sin\beta} \times \frac{\sin\beta}{\sin\alpha} = 1$$

$$\text{Therefore,} \qquad v_1 = v_2 \tag{2}$$

□

2 On the theorem III - Proposition III

This next section, we will deal with the *Theorem III, Proposition III* of the Third Day which taks about the natural accelerated motion.

Theorem 2. *If one and the same body, starting from rest, falls along an inclined plane and also a vertical, each having the same height, the times of descent will be to each oher as the lengths of the inclined plane and the vertical.*

This theorem states that if a body, starting from rest, descends along both an inclined plane and a vertical drop of the same height, the time taken to reach the bottom in each case will be proportional to the respective distances traveled.

The following demonstration lies on the previous theorem.

Proof. According to the last theorem, equation (2), we have $v_1 = v_2$.

If we come back to the definition of speed, the speed of a body traveling a distance d during a precise time t is given by :

$$v = \frac{d}{t}$$

Thus, we have :

$$v_1 = v_2 = \frac{AB}{t_1} = \frac{AE}{t_2}$$
$$\implies \quad \frac{AB}{AE} = \frac{t_1}{t_2}$$

Now, we can notice that the last theorem is true $\forall\ \beta$ and α.

So, taking $\beta = \frac{\pi}{2}$, therefore, the theorem is prooved. $\square$

References

[1] Galileo Galilei. *Dialogues Concerning Two New Sciences.* Macmillan, New York, 1914. Originally published in 1638.