Notes on Galileo Galilei Theorems

Abegà Razafindratelo

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Abstract

The purpose of this article is to explore some of *Galileo Galilei* theorems in his famous book on dynamic mechanics *Dialogues Concerning Two New Sciences*[1] and do their demonstrations in a modern way.

Introduction

In his book *The two New Sciences*

1 On the Salviati lemma

On the *Third day, Change of position, [De Motu Locali]*, Galileo deals with the notions of motions, acceleration, and more precisely, he fonded the *Dynamical mechanics*.

One of the important theorem he stated in this *Third day* is the one that *Salviati* states after the *Scholium* of the *Corollary II.* on page 183-184

Theorem 1 *If a body falls freely along smooth planes inclined at any angle whatsoever, but of the same height, the speeds with which it reaches the bottomare the same.*

Galileo's demonstration of this theorem is very old and might be difficult to apprehend easily, i.e, the expressions and the notations are very old. But hereafter, I'm going to proove that theorem in a modern notation and modern expression which will make the demonstration easy to understand and to apprehend easily.

Proof.

Let ABC be a rectangle triangle.

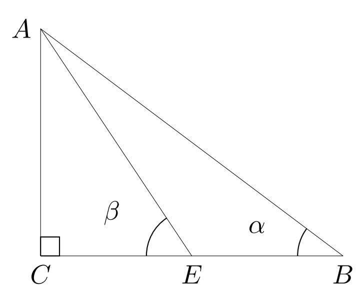

Figure 1: Inclined plane

Let us lay down :

$$\begin{cases} AC = h \\ AB = x_1 \\ AE = x_2; \end{cases}$$

As we know :

$$\begin{cases} \sin\alpha = \frac{h}{x_1} \\ \sin\beta = \frac{h}{x_2} \end{cases}$$

That gives us :

$$\begin{cases} h = x_1 \sin\alpha \\ h = x_2 \sin\beta \end{cases}$$

As we know $x_1 = \frac{1}{2} \cdot a_1 \cdot t_1^2$ where a_1 is the acceleration on the plane AB, and t_1 the time required to travers it; and $x_2 = \frac{1}{2} \cdot a_2 \cdot t_2^2$. where a_2 the acceleration on the plane AE, and t_2 the time required to travers that plane.

Now :

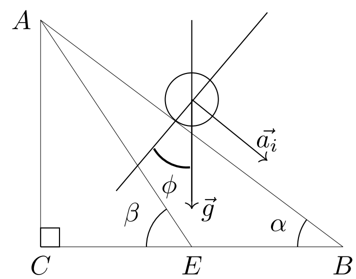

Figure 2: Falling body on the inclined plane

So, as we have : $a_i = g \sin\phi$, we have :

$$\begin{cases} x_1 = \frac{1}{2} a_1 t_1^2 = \frac{1}{2} g t_1^2 \sin\alpha \\ x_2 = \frac{1}{2} a_2 t_2^2 = \frac{1}{2} g t_2^2 \sin\beta \end{cases}$$

$$\implies \begin{cases} h = \left(\frac{1}{2} g t_1^2 \sin\alpha\right) \cdot \sin\alpha = \frac{1}{2} g t_1^2 \sin^2\alpha \\ h = \left(\frac{1}{2} g t_2^2 \sin\beta\right) \cdot \sin\beta = \frac{1}{2} g t_2^2 \sin^2\beta \end{cases}$$

$$`\implies \quad \frac{1}{2} g t_1^2 \sin^2\alpha = \frac{1}{2} g t_2^2 \sin^2\beta$$

$$\iff \quad t_1^2 \sin^2\alpha = t_2^2 \sin^2\beta$$

$$\iff \quad t_1 \sin\alpha = t_2 \sin\beta$$

because time and distance (since t and h are both positive)

$$\iff \quad \frac{t_1}{t_2} = \frac{\sin\beta}{\sin\alpha} \tag{1}$$

Now, in the other hand, the speeds of the body at the bottoms of each inclined plane is given by :

$$\begin{cases} v_1 = a_1 t_1 \\ v_2 = a_2 t_2 \end{cases} \implies \begin{cases} v_1 = g t_1 \sin\alpha \\ v_2 = g t_2 \sin\beta \end{cases}$$

Now, seing that each term of those equation are not null, by dividing term by term we have :

$$\Longrightarrow \qquad \frac{v_1}{v_2} = \frac{gt_1 \sin\alpha}{gt_2 \sin\beta} = \frac{t_1 \sin\alpha}{t_2 \sin\beta}$$

According to equation (1) we get :

$$\frac{v_1}{v_2} = \frac{\sin\alpha}{\sin\beta} \times \frac{\sin\beta}{\sin\alpha} = 1$$

$$\therefore \qquad v_1 = v_2$$

References

[1] Galileo Galilei. *Dialogues Concerning Two New Sciences*. Macmillan, New York, 1914. Originally published in 1638.