Power Method, Recursion Relations, Tight Binding Model

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Power method for computing eigen values

- Gives eigen value w/ biggest abs. value

i) $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$
ii) $\{v_1 \dots v_n\}$ = eigen basis
iii) $\|v_i\|_2 = 1$

- $x_0 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

$\hookrightarrow A x_0 = c_1 A v_1 + \cdots + c_n A v_n = c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n$

$A^2 x_0 = c_1 \lambda_1^2 v_1 + \cdots + c_n \lambda_n^2 v_n$

$\therefore A^k x_0 = c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n$

$\hookrightarrow$ Divide by largest λ_1: $= \lambda_1^k \Big(c_1 v_1 + \underbrace{c_2 \frac{\lambda_2^k}{\lambda_1^k} v_2 + \cdots + c_n \frac{\lambda_n^k}{\lambda_1^k} v_n}_{\varepsilon_k}\Big)$, with $\frac{\lambda_2^k}{\lambda_1^k} \to 0$, $\cdots \to 0$, $\frac{\lambda_n^k}{\lambda_1^k} \to 0$

- As $k \to \infty$, $\varepsilon_k \to \vec{0}$

$\therefore \frac{A^k x_0}{\|A^k x_0\|} \xrightarrow[k \to \infty]{} \pm v_1$ $\longrightarrow$ eigen vector corresponding to largest eigenvalue λ_1

- To find eigen value λ_1: inner product $\langle v_1, A v_1 \rangle = \lambda_1 \|v_1\|^2 = \lambda_1$

Recursion Relation

ex. $a_n = a_{n-1} + 2$; $a_0 = 5$

or w/ formula: $a_n = 5 + 2n$

Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, ...

- $F_{n+1} = F_n + F_{n-1}$; $F_0 = 0$, $F_1 = 1$

- Find formula for F_n:

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} \quad ; \quad \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- $v_1 = A v_0$, $v_2 = A^2 v_0$, ..., $v_n = A^n v_0$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\because A$ = real symmetric Hermitian, $\therefore$ unitary & diagonalizable

$\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$ $\quad v_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$A = SDS^{-1}$, $S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $S^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$

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− $F_n \sim A^n = SD^nS^{-1}$: $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = SD^nS^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \cdots = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^{n} - \lambda_2^{n} \end{bmatrix}$

$\therefore F_n = \frac{1}{\sqrt{5}} (\lambda_1^n - \lambda_2^n)$