Finite Differences

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3. Finite Difference Approximations

— Find approximate solutions to differential eqn. (DE)

$f''(x) + g(x)f(x) = r(x)$ $\quad 0 \leq x \leq 1$

Boundary Conditions: $f(0) = A$, $f(1) = B$ $\longrightarrow$ $g(x)$ & $r(x)$ = known
$f(x)$ = unknown

ex. $g(x) = 0$, $r(x) = 4$ $\longrightarrow f''(x) = 4$

$\longrightarrow$ solve: $f'(x) = 4x + C_1$
$f(x) = 2x^2 + C_1 x + C_2$ $\longrightarrow$ General Solution

Find Unique Soln

① Initial conditions of f & f'

$f''(x) = 4$, $f(0) = 1$, $f'(0) = 2$ $\longrightarrow$ Initial Value Problem (IVP)

for $f(x) = 2x^2 + C_1 x + C_2$:
impose $f(0) = 1 \longrightarrow C_2 = 1$
$f'(0) = 2 \longrightarrow C_1 = 2$

$\therefore f(x) = 2x^2 + 2x + 1$

② Boundary Values of $f(x)$

$f''(x) = 4$, $f(0) = 1$, $f(1) = 0$ $\longrightarrow$ Boundary Value Problem (BVP)

for $f(x) = 2x^2 + C_1 x + C_2$:
impose: $f(0) = 1 \longrightarrow C_2 = 1$
$f(1) = 0 \longrightarrow 2 + C_1 + 1 = 0$, $C_1 = -3$

$\therefore f(x) = 2x^2 - 3x + 1$

$\longrightarrow$ DE's still hard to solve, may not have explicit soln.
$\hookrightarrow$ use Discretization

3.2 Discretization

— Approximate function by vector w/ finite # sample values

— Pick equally spaced pts. $x_k = k/N$
$k = 0 \ldots N$ b/w 0 & 1

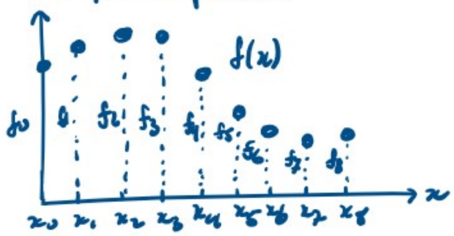
— Represent function $f(x)$ by $f_k = f(x_k)$

Sampled points:

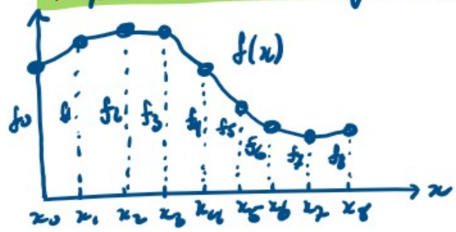
Slope of line segments:

$\vec{F} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \end{bmatrix}$

Sampled points:



Slope of line segments:



$$\vec{F} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix}$$

- $\therefore$ 1 slope for every interval (x_i, x_{i+1}), vector containing slopes has 1 less entry than $\vec{F}$
- Slope of $[x_i, x_{i+1}] = (f_{i+1} - f_i)/\Delta x$, distance $\Delta x = x_{i+1} - x_i = 1/N$
- Vector containing slopes $\vec{F}'$

1st derivative approximation

$$\vec{F}' = \frac{1}{\Delta x}\begin{bmatrix} f_1 - f_0 \\ f_2 - f_1 \\ f_3 - f_2 \\ \vdots \\ f_N - f_{N-1} \end{bmatrix} = (\Delta x)^{-1} \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}}_{D_N} \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{bmatrix}}_{F}$$

$\therefore \vec{F}' = (\Delta x^{-1}) D_N \vec{F} \longrightarrow D_N$ is the $N \times (N+1)$ finite difference matrix above

2nd derivative approximation

$$\vec{F}'' = \frac{1}{(\Delta x)^2}\begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{bmatrix}$$

$\therefore \vec{F}'' = (\Delta x)^{-2} D_{N-1} D_N \vec{F}$

- Let $r_K = r(x_K)$ be sampled points for the load function $r(x)$ & define vector approx for r @ interior pts.:

$\vec{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_{N-1} \end{bmatrix} \longrightarrow$ where $\vec{F}''$ is defined

Finite difference approximation to $f''(x) = r(x)$: $(q(x) = 0)$

- BV: $f(0) = A$, $f(1) = B$

① Discretize LHS of DE:

② Discretize RHS of DE:

③ Incorporate Boundary Values (BV)

- Finite Difference: $(\Delta x)^{-2} D_{N-1} D_N \vec{F} = \vec{r} \longrightarrow D_{N-1} D_N \vec{F} = (\Delta x)^2 \vec{r}$
- BV: $[1, \cdots 0]\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix} = A$, $[0, \cdots, 0, 1]\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix} = B$

BV: $[1,0,\dots,0]\begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} = A$; $[0,\dots,0,1]\begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} = B$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}}_{L} \vec{F} = \begin{bmatrix} A \\ (\Delta x)^2 r_1 \\ (\Delta x)^2 r_2 \\ \vdots \\ (\Delta x)^2 r_{N-1} \\ B \end{bmatrix} \longrightarrow L = D_{N-1} D_N$$

- ∴ To find numerical approx soln. of BV:
  $\longrightarrow$ solve $\left.\begin{cases} f''(x) = r(x) \\ f(0) = A,\ f(1) = B \end{cases}\right] \longrightarrow L\vec{F} = \vec{b}$
- $\det(L) = \pm N$

Finite difference approximation to $f''(x) + q(x) f(x) = r(x)$: $(q(x) \neq 0)$

- Discretize $q(x) f(x)$ at $x_1, x_2, \cdots x_{N-1}$

$$\underbrace{\begin{bmatrix} 0 \\ q_1 f_1 \\ q_2 f_2 \\ \vdots \\ q_{N-1} f_{N-1} \\ 0 \end{bmatrix}}_{q(x)f(x)} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & q_1 & 0 & \cdots & 0 \\ 0 & 0 & q_2 & 0 \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ & & & q_{N-1} & \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}}_{\vec{F}} \longrightarrow \begin{matrix} q_j = q(x_j) \\ f_j = f(x_j) \end{matrix}$$

∴ BVP approximated by: $(L + (\Delta x)^2 Q) F = \begin{bmatrix} A \\ \vdots \\ (\Delta x)^2 R \\ \vdots \\ B \end{bmatrix}$

ex. $f''(x) + (x-1)^3 f(x) = 6x$

$0 \leq x \leq 5$, $f(0) = 0 = A$, $f(5) = 2 = B$

$$(L + (\Delta x)^2 \tilde{Q}) F = \begin{bmatrix} A \\ \vdots \\ (\Delta x)^2 \tilde{R} \\ \vdots \\ B \end{bmatrix} \longrightarrow \vec{F} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix}$$ (finite approx. of $f(x)$ at pts. $x_0, x_1, \dots, x_N$ w/ $N = 10$)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & 0 & & \vdots & \vdots \\ 0 & 1 & -2 & 1 & \ddots & 0 & \vdots \\ \vdots & 0 & 1 & -2 & \ddots & 1 & 0 \\ \vdots & \vdots & 0 & 1 & \ddots 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & & 1 \end{bmatrix} \qquad \tilde{Q} = \begin{bmatrix} 0 & \cdots & & & 0 \\ \vdots & (x_1 - 1)^3 & & & \vdots \\ & & (x_2 - 1)^3 & & \vdots \\ \vdots & & & \ddots & \\ & & & (x_9 - 1)^3 & \vdots \\ 0 & \cdots & & \cdots & 0 \end{bmatrix}$$

$\Delta x = \frac{5-0}{10} = \frac{1}{2}$, $(\Delta x)^2 = \frac{1}{4}$

$$\tilde{R} = \begin{bmatrix} 6x_1 \\ 6x_2 \\ \vdots \end{bmatrix} \longrightarrow \begin{bmatrix} A \\ \vdots \\ (\Delta x)^2 \tilde{R} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 3/2\, x_1 \\ 3/2\, x_2 \\ \vdots \end{bmatrix}$$

$$R = \begin{bmatrix} 6x_2 \\ \vdots \\ 6x_9 \end{bmatrix} \longrightarrow \begin{bmatrix} \vdots \\ (\Delta x)^2 \tilde{R} \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} 3/2\, x_1 \\ 3/2\, x_2 \\ \vdots \\ 3/2\, x_9 \\ 2 \end{bmatrix}$$

$\longrightarrow$ solve for F /MATLAB