MT 1 Bonus Questions

February 28, 2020 3:51 PM

ELEC211_...
Q5 to 7

Problem 5 [15 marks]

Let $\mathcal{C}$ be the curve from $(0,0,0)$ to $(1,1,1)$ along the intersection of the surfaces $y = x^2$ and $z = x^3$ with the parametrization $L(t)$.

1. [5 points] Find $\int \vec{F} \cdot d\vec{L}$ given $\vec{F}(x,y,z) = (xz - y)\hat{a_x} - (z + x)\hat{a_y} + y\hat{a_z}$.
2. [5 points] Find the total charge on the curve given a charge density function $\rho_L(x,y,z) = 8x + 36z$.
3. [5 points] Find $\int \vec{G} \cdot d\vec{L}$ given $\vec{G}(x,y,z) = \sin(y)\hat{a_x} + (x\cos(y) + z)\hat{a_y} - (y + z)\hat{a_z}$.

1. $\vec{F}(x,y,z) = \langle xz - y,\ z + x,\ y \rangle$

① Parametrize: $x = t,\ y = t^2,\ z = t^3 \longrightarrow 0 \le t \le 1$

$\therefore L(t) = \langle t,\ t^2,\ t^3 \rangle$

② $\vec{F}(\vec{L}(t)) = \langle (t)(t^3) - t^2,\ (t^3) + (t),\ t^2 \rangle$

$= \langle t^4 - t^2,\ t^3 + t,\ t^2 \rangle$

③ $\vec{L}'(t) = \langle 1,\ 2t,\ 3t^2 \rangle$

④ $\vec{F}(\vec{L}(t)) \cdot \vec{L}'(t) = \langle t^4 - t^2,\ t^3 + t,\ t^2 \rangle \cdot \langle 1,\ 2t,\ 3t^2 \rangle$

$= 6t^4 + t^2$

⑤ $\int \vec{F}(\vec{L}(t)) \cdot \vec{L}'(t) = \int_0^1 6t^4 + t^2 = \frac{23}{15}\ J \longrightarrow 0 \le t \le 1$

2. $\rho_L(x,y,z) = 8x + 36z$

$Q = \int \rho_L \cdot dL = \int_0^1 8t + 36t^3\ dt = 13\ C$

Additional workspace for problem 5.

3. $\vec{G}(x,y,z) = \langle \sin(y),\ x\cos(y) + z,\ y + z \rangle$

3. $\vec{G}(x,y,z) = \langle \sin(y), x\cos(y) + z, y+z \rangle$

$L(t) = \langle t, t^2, t^3 \rangle$

$\vec{L}'(t) = \langle 1, 2t, 3t \rangle$

$\vec{G}(\vec{L}(t)) = \langle \sin(t^2), t\cos(t^2) + t^3, t^2 + t^3 \rangle$

$\vec{G}(\vec{L}(t)) \cdot \vec{L}'(t) = \langle \sin(t^2), t\cos(t^2) + t^3, t^2 + t^3 \rangle \cdot \langle 1, 2t, 3t \rangle$

$= \sin(t^2) + 2t^2\cos(t^2) + 2t^4 + 3t^3 + 3t^4$

$\int_0^1 \vec{G}(\vec{L}(t)) \cdot \vec{L}(t) = \int_0^1 \sin(t^2) + 2t^2\cos(t^2) + 5t^4 + 3t^3\, dt$

IBP: $u = \sin(t^2)$ $dv = 1$
$du = 2t\cos(t^2)$ $v = t$

$= \left[t\sin(t^2) \right]_0^1 - \cancel{\int_0^1 2t^2\cos(t^2)\,dt} + \cancel{\int_0^1 2t^2\cos(t^2)\,dt} + \int_0^1 5t^4 + 3t^3\, dt$

$= \sin(1) + \frac{7}{4}$

d

Problem 6 [10 Marks]

Let $\vec{F}(x,y,z) = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, x^3 \right\rangle$. Let $\mathcal{C}$ be the curve at the intersection of the surfaces $x^2 + y^2 = 1$ and $xe^z = 1$ that starts at $(1,0,0)$ and ends at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \ln(\sqrt{2}))$.

Find the work done by the force field $\vec{F}$ along the curve $\mathcal{C}$.

$W = \int \vec{F} \cdot d\vec{r} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$

Parametrize: $x = t$, $y = \sqrt{1-t^2}$, $z = \ln(\frac{1}{x})$

$\vec{r}(t) = \langle t, \sqrt{1-t^2}, \ln(\frac{1}{t}) \rangle$

$-1/2$

$\vec{r}(t) = \langle t, \sqrt{1-t^2}, \ln(\frac{1}{t}) \rangle$

$\vec{r}'(t) = \langle 1, -t(1-t^2)^{-1/2}, -\frac{1}{t} \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \langle t, \sqrt{1-t^2}, t^3 \rangle \cdot \langle 1, -t(1-t^2)^{-1/2}, -\frac{1}{t} \rangle$

$= -t^2$

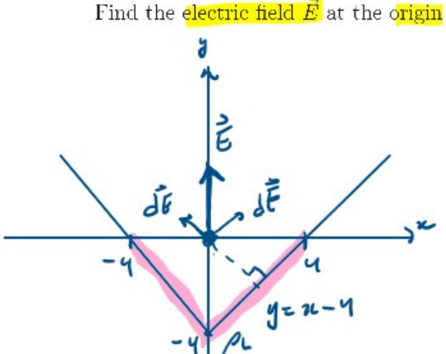
$W = \int_1^{\frac{\sqrt{2}}{2}} -t^2 \, dt \longrightarrow 1 \le t \le \frac{\sqrt{2}}{2} \quad \therefore (1,0,0) \to (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \ln(\sqrt{2}))$

$= \left[-\frac{1}{3} t^3 \right]_1^{\frac{\sqrt{2}}{2}} = -\frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3 + \frac{1}{3} = \frac{-\sqrt{2}+4}{12}$

Problem 7 [15 Marks]

Suppose we have a finite line of charge defined by $f(x) = |x| - 4$ for $-4 < x < 4$ on the xy-plane. Assume, moreover, that the line charge density is given as $\rho_L(x, y) = \alpha\sqrt{x^2 - y^2}$, for $\alpha \in \mathbb{R}$.

Find the electric field $\vec{E}$ at the origin due to the line segment.



$\rho_L = \alpha\sqrt{x^2+y^2}$

Parametrize line segment:

$\vec{L}(x) = \langle x, x-4 \rangle$

$\vec{L}'(x) = \langle 1, 1 \rangle \quad |\vec{L}'(x)| = \sqrt{2} \quad 0 \le x \le 4$

$\vec{E} = \frac{Q}{4\pi\varepsilon_0 \rho^2} \quad dQ = \rho_L \, dL \quad \rho = \sqrt{x^2+y^2}$

$d\vec{E} = \frac{dQ}{\pi\varepsilon_0 \rho^2} \hat{a}_\rho = \frac{\alpha\cancel{\sqrt{x^2+y^2}} \cdot \sqrt{2}\, dx}{4\pi\varepsilon_0 (x^2+y^2)} \cdot \frac{-\langle x, y \rangle}{\cancel{\sqrt{x^2+y^2}}} \rightarrow$ only want y component

x2 for 2 segments $\leftarrow d\vec{E}_y = \frac{-\alpha\sqrt{2}\, y}{2\pi\varepsilon_0(x^2+y^2)} dx = \frac{-\alpha\sqrt{2}(x-4)}{2\pi\varepsilon_0(x^2+(x-4)^2)}$

$\vec{E}_y = \frac{-\alpha\sqrt{2}}{2\pi\varepsilon_0} \int_0^4 \frac{x-4}{2x^2-4x+8} dx$

$\to$ complete the square

For $0 \le x \le 4$:

$L = \sqrt{x^2+y^2}$

$= \sqrt{1+(y/x)^2}$

$dL = \sqrt{1+(dy/dx)^2}\, dx$ slope

$$= \sqrt{1 + (\cdot/x)^2}$$

$$dL = \sqrt{1 + \underbrace{(dy/dx)}_{\text{slope}}{}^2}\, dx$$

$$= \sqrt{1 + (1)^2}\, dx$$

$$= \sqrt{2}\, dx$$

$$dQ = \rho_L\, dL$$

$$= \alpha \sqrt{x^2 + (x+4)^2} \cdot \sqrt{2}\, dx$$

$$E_y = \frac{\cdots}{2\pi\varepsilon_0} \int_0 \frac{\cdots}{2x^2 - 4x + 8}\, dx$$

$\longrightarrow$ complete the square

$$= \frac{-\alpha\sqrt{2}}{2\pi\varepsilon_0} \int_0^4 \frac{x-4}{(x-2)^2 + 4}\, dx$$

IBP: $u = x^2 - 4x + 8 \qquad v = \frac{x-2}{2}$

$du = 2x - 4\, dx \qquad dv = \frac{1}{2}\, dx$

$$\vec{E}_y = \frac{-\alpha\sqrt{2}}{2\pi\varepsilon_0} \left[\int_8^8 \frac{1}{2u}\, du - \int_{-1}^{1} \frac{2}{4v^2 + 4}\, dv \right] \hat{a}_y$$

$$= \frac{-\alpha\sqrt{2}}{4\pi\varepsilon_0} \left[\frac{\ln u}{2} \Big|_8^8 - \arctan v \Big|_{-1}^{1} \right] \hat{a}_y$$

$$= \boxed{\frac{\alpha\sqrt{2}}{8\varepsilon_0}\, \hat{a}_y}$$

Additional workspace for problem 7.