MATH 264 Flux 2

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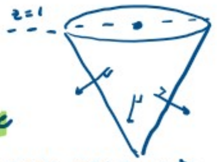
ex. Calculate flux : $\iint_S \vec{F}\cdot\vec{dS} = \iint_S \vec{F}\cdot\hat{n}\,dS$

$\vec{F} = \langle x, y, -2z\rangle$ & S is the cone $z = \sqrt{x^2+y^2}$

with $z \le 1$ oriented outward.

① sketch $z=1$  S  → cross sec. = z = u

② Parametrize

$\vec{r}(u,v) = \langle u\cos v, u\sin v, u\rangle$ $\quad 0 \le u \le 1, \quad 0 \le v \le 2\pi$

$\vec{n}$ surface ← ③ Calculate $\pm\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)$ $\quad \because z = \sqrt{x^2+y^2} = \sqrt{(u\cos v)^2+(u\sin v)^2} = |u| = u$

$$= \pm\begin{array}{r} \\ \frac{\partial\vec{r}}{\partial u} \leftarrow \\ \frac{\partial\vec{r}}{\partial v} \leftarrow \end{array}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \pm\langle -u\cos v, -u\sin v, u\rangle$$

↳ which sign ? $\because$ z components $\hat{n}$ pointing down,

$\therefore (-) \to$ downward normal $\hat{n}$

④ Flux integral

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \int_0^{2\pi}\int_0^1 \overset{\vec{F}\text{ along }S}{\vec{F}(\vec{r}(u,v))}\cdot\left(-\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)du\,dv$$

$$= \int_0^{2\pi}\int_0^1 \left(\langle u\cos v, u\sin v, -2u\rangle\cdot\langle u\cos v, u\sin v, -u\rangle\right)du\,dv$$

$$= \int_0^{2\pi}\int_0^1 3u^2\,du\,dv = \int_0^{2\pi}\left(u^3\Big|_0^1\right)dv = 2\pi$$

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ex. flux of $\vec{F}(x,y,z) = \langle y^2z, y^3, xz+e^y\rangle$ through boundaries of cube : $-1 \le x \le 1$, $-1 \le y \le 1$, $0 \le z \le 2$ in outward dir.

→ Calculate flux through each of cube & sum

→ Too much work.

Divergence : let $\vec{F} = \langle F_x, F_y, F_z\rangle$, define divergence as :

$$\vec{\nabla}\cdot\vec{F} = \left\langle\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle\cdot\langle F_x, F_y, F_z\rangle = \mathrm{div}(\vec{F})$$

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \to$$

→ vector to scalar

ex. $\vec{F}(x,y,z) = (xy, xz, y^2) \to \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(y^2)$

$$= y + 0 + 0 = y$$

Divergence Theorem: $R \subseteq \mathbb{R}^3 \to$ region bounded by surface S $(= \partial R)$ oriented with outward normals, if $\vec{F} \to$ continuously diff. then } Need closed surface !

$$\iint_{\partial R} \vec{F}\cdot\hat{n}\,dS = \iiint_R (\mathrm{div}\,\vec{F})\,dV$$

→ ex. cube → flux of $\vec{F}(x,y,z) = \langle y^2z, y^3, xz+e^y\rangle$ through cube

$-1 \le x \le 1$, $-1 \le y \le 1$, $0 \le z \le 2$ outwards

→ Apply Divergence Thm:

$$\vec{\nabla}\cdot\vec{F} = \mathrm{div}(\vec{F}) = \frac{\partial}{\partial x}(y^2z) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(xz+e^y)$$

→ Apply Divergence Thm:

$$\vec{\nabla}\cdot\vec{F} = \operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(y^2 z) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(xz + e^y)$$

$$= 3y^2 + x$$

$$\iint_{\partial C} \vec{F}\cdot\hat{n}\,dS = \iiint_C (3y^2 + x)\,dV = \int_0^2\int_{-1}^1\int_{-1}^1 (3y^2+x)\,dx\,dy\,dz = 8$$

cube ← bounds

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ex. Let T be disk $T = \{(x,y,1) : x^2+y^2 \le 1\}$

Calculate $\iint_T \vec{F}\cdot d\vec{S}$, T dir. outward (+) z dir.

$\vec{F} = \langle x, y, -2z\rangle$

→ Can apply regular flux formula w/ $\hat{n} = \hat{a}_z$

OR: apply Divergence Thm (but not a closed surface!)

→ Need bounded volume in $\mathbb{R}^3$

→ Flux through disk B = flux through surface of closed cone − flux through open cone

$$\iint_{\text{disk}} \vec{F}\cdot d\vec{S} = \iint_{\text{closed cone}} \vec{F}\cdot d\vec{S} - \iint_{\text{open cone}} \vec{F}\cdot d\vec{S}$$

closed cone → Divergence Thm

open cone → previous ex. $= 2\pi$

Calculate: $\iint_{\text{closed cone}} \vec{F}\cdot d\vec{S} = \iiint_{\text{cone}} (\operatorname{div}(\vec{F}))\,dV$

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(-2z) = 0$$

$$\iint_T \vec{F}\cdot d\vec{S} = -2\pi$$

Gauss' Divergence Theorem

$$\oiint_S \vec{D}\cdot\hat{n}\,dS = Q_{enc} \xrightarrow[\text{continuous}]{\text{if Q density } \rho} = \iiint_R \rho_v\,dV$$

← non singular Q density

$(S = \partial R)$

$$\xrightarrow{\text{div.}} = \iiint_R (\vec{\nabla}\cdot\vec{D})\,dV = \longrightarrow$$

→ If cnts charge density, this is true for every R

$\therefore\ \vec{\nabla}\cdot\vec{D} = \rho_v$

ex. pt. charge at $\vec{O}$ w/ charge Q.

Then $\vec{D} = \varepsilon_0\vec{E} = \frac{Q}{4\pi}\cdot\frac{1}{r^2}\hat{e}_r = \frac{Q}{4\pi}\cdot\frac{\langle x,y,z\rangle}{(x^2+y^2+z^2)^{3/2}}$

$$\frac{\partial}{\partial x}(D_x) = \frac{Q}{4\pi}\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{Q}{4\pi}\left[\frac{(x^2+y^2+z^2)^{3/2} - x\cdot\frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2x}{((x^2+y^2+z^2)^{3/2})^2}\right]$$

$$= \frac{Q}{4\pi}\cdot\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial}{\partial y}(D_y) = \frac{Q}{4\pi}\frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}} \quad \& \quad \frac{\partial}{\partial z}(D_z) = \frac{Q}{4\pi}\frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$$

→ $\operatorname{div}(\vec{D}) = 0$, not defined @ zero

→ For R = Ball of radius a > 1 centered at $\vec{O}$

→ $\operatorname{div}(\vec{D}) = 0$, not defined @ zero

→ For R = Ball of radius $a > 1$ centered at $\vec{0}$

But: $\oiint_{\partial R} \vec{D}\cdot\hat{n}\,dS \stackrel{?}{=} \iiint_{R} (\operatorname{div}\vec{D})\,dV \stackrel{?}{=} 0$?

↳ Does this contradict Div. Thm?

→ No, ∵ $\vec{D}$ is not conts'ly diff'l.

→ For arbitrarily closed surface w/ bounded region R, calculate flux $\iint_{\partial R} \vec{D}\cdot\hat{n}\,dS = \begin{cases} 0 & \text{if } \vec{0} \text{ not in } R \\ Q & \text{if } \vec{0} \text{ is in } R \end{cases}$

→ Take out a small sphere B_{r_0} around $\vec{0}$:

$$\oiint_{\partial R} \vec{D}\cdot\hat{n}\,dS = \oiint_{\substack{\text{surface}\\\text{except}\\\text{sphere}}} \vec{D}\cdot\hat{n}\,dS + \oiint_{\substack{\text{sphere}\\B_{r_0}}} \vec{D}\cdot\hat{n}\,dS = 0 + Q$$

calculate flux