Chapter 12 Wave Reflection and Dispersion

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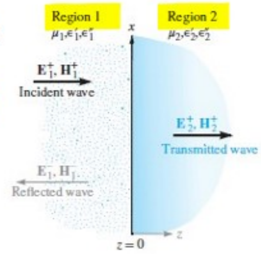
12.1 Reflection of Uniform Plane waves @ normal incidence

- uniform plane wave incident on boundary b/w 2 diff. regions of material
- normal incidence: wave propagation ⊥ boundary
  - ↳ reflected wave
  - ↳ transmitted wave through region 2



12.1.1 Reflected & Transmitted Waves @ boundary

- incident wave Region 1 (lin. pol.):
  - Field wave $\hat{x}$:
    $E^+_{x1}(z,t) = E^+_{x10} e^{-\alpha_1 z}\cos(\omega t - \beta_1 z)$
    phasor: $E^+_{xs1}(z) = E^+_{x10} e^{-jk_1 z}$
  - Magnetic Field $\hat{y}$:
    phasor: $H^+_{ys2}(z) = \frac{1}{\eta_2} E^+_{x20} e^{-jk_2 z}$
- Transmitted wave Region 2:
  $E^+_{xs2}(z) = E^+_{x20} e^{-jk_2 z}$
  $H^+_{ys2}(z) = \frac{1}{\eta_2} E^+_{x20} e^{-jk_2 z}$
- @ Boundary $z=0$:
  - $\vec{E}$ fields ($\hat{x}$) in regions: $E^+_{x10} = E^+_{x20}$
  - $\vec{H}$ fields ($\hat{y}$) in regions: $H_{ys1} = H_{ys2}$
  - $\eta_1 = \eta_2 \rightarrow \frac{E^+_{x10}}{\eta_1} = \frac{E^+_{x20}}{\eta_2}$
  - Satisfy Boundary conditions w/ reflected wave
- Reflected wave $z=0$:
  $E^-_{xs1}(z) = E^-_{x10} e^{jk_1 z}$
  $H^-_{xs1}(z) = -\frac{E^-_{x10}}{\eta_1} e^{jk_1 z}$
  - ↳ field in $-\hat{z}$: $E^-_{xs1} = -\eta_1 H^-_{ys1}$
  - ↳ Poyn vector: $\langle \vec{S} \rangle = \vec{E}_1^- \times \vec{H}_1^- \rightarrow -\hat{a}_z$

12.1.2 Reflection & Transmission Coefficients

- Boundary Conditions for $\vec{E}$ & $\vec{H}$ tangential continuity:
  $E^+_{x10} + E^-_{x10} = E^+_{x20}$
  $H^+_{ys1} + H^-_{ys1} = H^+_{ys2}$
  ↳ $\frac{E^+_{x10}}{\eta_1} - \frac{E^-_{x10}}{\eta_1} = \frac{E^+_{x20}}{\eta_2}$
  $\therefore E^-_{x10} = E^+_{x10} \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$
- Reflection Coefficient Γ:
  $\Gamma = \frac{E^-_{x10}}{E^+_{x10}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = |\Gamma| e^{j\phi_r}$
  - ↳ if η_1 or η_2 complex, Γ complex
    $\therefore$ include reflective phase shift ϕ
- Transmission coefficient τ:
  $\tau = \frac{E^+_{x20}}{E^+_{x10}} = \frac{2\eta_2}{\eta_1 + \eta_2} = 1 + \Gamma = |\tau| e^{j\phi_t}$

12.1.3 Total Reflection: Standing Wave Ratio

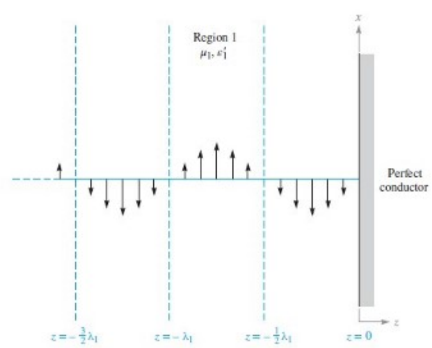


Figure 12.2 The instantaneous values of the total field E_{x1} are shown at $\omega t = \pi/2$. $E_{x1} = 0$ for all time at multiples of one half-wavelength from the conducting surface.

- Region 1 → perfect dielectric
  Region 2 → perfect conductor
  ↳ $\epsilon''_2 = \frac{\sigma_2}{\omega} \rightarrow \eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\epsilon'_2}} = 0 \quad \because \sigma_2 \rightarrow \infty$
  $\therefore E^+_{x20} = 0,\ \Gamma = -1,\ E^-_{x10} = -E^+_{x10}$
  - ↳ Skin depth $\delta = 0$
  - ↳ All incident energy is reflected by perfect conductor
  - ↳ Reflected field shifted in phase by 180° relative to incident field
  $\therefore$ $\vec{E}$ field in R1: $E_{xs1} = E^+_{xs1} + E^-_{xs1} = E^+_{x10} e^{-j\beta_1 z} - E^+_{x10} e^{j\beta_1 z}$

- Skin depth $\delta = 0$
  - ↳ All **incident** energy is **reflected** by perfect conductor
  - ↳ Reflected field shifted in phase by 180° relative to incident field
  - ∴ **$\vec{E}$ field in R1**: $E_{xs1} = E^+_{xs1} + E^-_{xs1} = E^+_{x10} e^{-j\beta_1 z} - E^+_{x10} e^{j\beta_1 z}$
    - $jk_1 = 0 + j\beta_1 \rightarrow \quad = -j2\sin(\beta_1 z) E^+_{x10}$
  - ∴ **Real Inst. Form**: $\mathcal{E}_{x1}(z,t) = 2E^+_{x10}\sin(\beta_1 z)\sin(\omega t)$] $Re\{E_{xs1} \cdot e^{j\omega t}\}$
- **Incident wave**: $\mathcal{E}_{x1}(z,t) = E^+_{x10}\cos(\omega t - \beta_1 z)$
  - ↳ Null locations occur @ $z = m\lambda_1/2$
    - ∴ $E_{x1} = 0$ @ boundary $z=0$ & every $\frac{1}{2}\lambda$ from bound to $z<0$
  - ∴ **Magnetic Field**: $H_{ys1} = \frac{E^+_{x10}}{\eta_1}(e^{-j\beta_1 z} + e^{j\beta_1 z})$
  - ∴ **Real Inst Form**: $H_{y1}(z,t) = 2\frac{E^+_{x10}}{\eta_1}\cos(\beta_1 z)\cos(\omega t)$

12.1.4 **Partial Reflection & Power Reflectivity**
- **Region 1 & 2 → Perfect dielectrics**
  - ↳ η_1 & $\eta_2 \rightarrow (+)$, $\sigma_1 = \sigma_2 = 0$
- **Reflection coeff**: $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \rightarrow E^-_{x10} = \Gamma E^+_{x10}$
- **Magnetic Field Intensities**: $H^+_{y10} = \frac{E^+_{x10}}{\eta_1}$
  $H^-_{y10} = -\frac{E^-_{x10}}{\eta_1}$
- **Power density**:
  - ↳ Region 1: **Incident** → $\langle S_{1i}\rangle = \left|\frac{1}{2}Re\{\vec{E}_s \times \vec{H}^*_s\}\right| = \frac{1}{2}E^+_{x10}H^+_{y10}$
    Reflected → $\langle S_{1r}\rangle = -\frac{1}{2}E^-_{x10}H^-_{y10}$
  - ↳ Region 2: $E^+_{x20} = \tau E^+_{x10}$
    $H^+_{y20} = \frac{E^+_{x20}}{\eta_2}$
  - ↳ Avg power density transmitted to R2:
    $\langle S_2\rangle = \frac{1}{2}E^+_{x20}H^+_{y20}$
    $\langle S_{1i}\rangle = \langle S_{1r}\rangle + \langle S_2\rangle$
- General Relation b/w reflected & incident power
  $\langle S_{1r}\rangle = |\Gamma|^2\langle S_{1i}\rangle$
  $\langle S_2\rangle = (1-|\Gamma|^2)\langle S_{1i}\rangle$

12.2 **Standing Wave Ratio**
- $|\Gamma| < 1$ → Energy transmitted into R2 & reflected
- **Region 1 → Perfect dielectric** ($\sigma_1 = 0$)
  Region 2 → Any
  - ↳ Region 1:
    $\vec{E}$-field phasor: $E_{x1T} = E^+_{x1} + E^-_{x1} = E^+_{x10}e^{-j\beta_1 z} + \Gamma E^+_{x10}e^{j\beta_1 z}$
    - ↳ $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = |\Gamma|e^{j\varphi}$ → possible reflection coeff.
      ∴ include phase φ
      - ↳ lossless: $\eta_1 \rightarrow$ Real & (+), $\eta_2 \rightarrow$ complex
      - ↳ perfect conductor: $\eta_2 = 0$, $\varphi = \pi$
      - ↳ If $\eta_2 \rightarrow$ Real & $< \eta_1$, $\varphi = \pi$
      - ↳ If $\eta_1 \rightarrow$ Real & $> \eta_1$, $\varphi = 0$
  - ∴ **Total Field**: $E_{x1T} = (e^{-j\beta_1 z} + |\Gamma|e^{j(\beta_1 z + \varphi)})E^+_{x10}$
- **Maximum**: $|E_{x1T}|_{max} = (1+|\Gamma|)E^+_{x10}$ where $z_{max} = -\frac{1}{2\beta_1}(\varphi + 2m\pi)$
- **Minimum**: $|E_{x1T}|_{min} = (1-|\Gamma|)E^+_{x10}$ where $z_{min} = -\frac{1}{2\beta_1}(\varphi + (2m+1)\pi)$

- **Total field in Region 1**: $\mathcal{E}_{x1T}(z,t) = \underbrace{(1-|\Gamma|)E^+_{x10}\cos(\omega t - \beta_1 z)}_{\text{traveling wave}} + \underbrace{2|\Gamma|E^+_{x10}\cos(\beta_1 z + \varphi/2)\cos(\omega t + \varphi/2)}_{\text{standing wave}}$
- **Standing Wave Ratio**
  - ↳ Ratio of max : min Amplitudes
    $s = \frac{|E_{x1T}|_{max}}{|E_{x1T}|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$

12.3 **Wave Reflection from multiple Interfaces**

12.3.1 **Two Interface Problem**

- Uniform plane wave $\hat{z}$ incident
- 3rd region of impedance → 2nd interface @ $z=0$
  - ↳ 1st interface @ $z=-l$
- @ 1st interface: Reflect & transmit
  @ 2nd interface: transmit into R3 & Reflected to 1st

↳ 1st interface @ $z = -l$

- @ 1st interface: Reflect & transmit
  @ 2nd interface: transmit into R3 & Reflected to 1st
  ↳ Again partially reflected then combine w/ transmitted Energy from R1, etc.
  ↳ Steady state eventually reached

12.3.2 Wave Impedance

- All Regions → lossless media
- In Region 2 (x-polarized):

$$E_{xs2} = E_{x20}^{+} e^{-j\beta_2 z} + E_{x20}^{-} e^{j\beta_2 z} \rightarrow \beta_2 = \frac{\omega\sqrt{\varepsilon_{r2}}}{c}$$

$$H_{ys2} = H_{y20}^{+} e^{-j\beta_2 z} + H_{y20}^{-} e^{j\beta_2 z}$$

- Reflection Coefficient

$$\Gamma_{23} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} \longrightarrow E_{x20}^{-} = \Gamma_{23} E_{x20}^{+}$$

$$H_{y20}^{+} = \frac{E_{x20}^{+}}{\eta_2}$$

$$H_{y20}^{-} = -\frac{E_{x20}^{-}}{\eta_2} = -\Gamma_{23}\frac{E_{x20}^{+}}{\eta_2}$$

- Wave impedance: Ratio of total $\vec{E}$-Field

$$\eta_w(z) = \frac{E_{xs2}}{H_{ys2}} = \frac{E_{x20}^{+} e^{-j\beta_2 z} + E_{x20}^{-} e^{j\beta_2 z}}{H_{y20}^{+} e^{-j\beta_2 z} + H_{y20}^{-} e^{j\beta_2 z}}$$

$$= \eta_2 \left[\frac{e^{-j\beta_2 z} + \Gamma_{23} e^{j\beta_2 z}}{e^{-j\beta_2 z} - \Gamma_{23} e^{j\beta_2 z}}\right] = \eta_2 \frac{\eta_3 \cos\beta_2 z - j\eta_2 \sin\beta_2 z}{\eta_2 \cos\beta_2 z - j\eta_3 \sin\beta_2 z}$$

$$= \eta_2 \frac{\eta_3 \cos(\beta_2 z) - j\eta_2 \sin(\beta_2 z)}{\eta_2 \cos(\beta_2 z) - j\eta_3 \sin(\beta_2 z)}$$

$$\therefore \frac{E_{x10}^{-}}{E_{x10}^{+}} = \Gamma = \frac{\eta_{in} - \eta_1}{\eta_{in} + \eta_1}$$

↳ Input impedance: $\eta_{in} = \eta_2 \dfrac{\eta_3 \cos(\beta_2 l) + j\eta_2 \sin(\beta_2 l)}{\eta_2 \cos(\beta_2 l) + j\eta_3 \sin(\beta_2 l)}$

12.3.3 Half & Quarter Waves (Total Transmission/No reflection)

- Half wave Matching

  $\Gamma = 0$ or $\eta_{in} = \eta_1$ → input impedance matched to incident medium

  ↳ $\eta_3 = \eta_2$ → $l = \dfrac{m\lambda_2}{2}$ → int multiple of ½ wavelengths

  ↳ Matched input impedance: $\eta_3 = \eta_1$ → $\eta_{in} = \eta_3$

  ↳ choosing R2 thickness → half-wave matching

- Refractive index: $n = \sqrt{\varepsilon_r}$
- Lossless Media:

$$\beta = k = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r} = \frac{n\omega}{c}$$

$$\eta = \frac{1}{\sqrt{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{\eta_0}{n}$$

$$v_p = \frac{c}{n}$$

$$\lambda = \frac{v_p}{f} = \frac{\lambda_0}{n}$$

- Quarter wave Matching

$$l = (2m-1)\frac{\lambda_2}{4} \longrightarrow \eta_{in} = \frac{\eta_2^2}{\eta_3}$$

  ↳ choose η_2 for matching b/w η_1 & η_3

  ↳ total transmission: $\eta_{in} = \eta_1$

  $\therefore \eta_2 = \sqrt{\eta_1 \eta_3}$

  ↳ Antireflective coating for optical devices

12.3.4 Multilayer Problem: Impedance transformation

- transform R4 impedance → input impedance @ boundary R2 & R3

$$\eta_{in,b} = \eta_3 \frac{\eta_{in,b}\cos\beta_3 l_b + j\eta_3 \sin\beta_3 l_b}{\eta_3 \cos\beta_3 l_b + j\eta_{in,b}\sin\beta_3 l_b}$$

- Reflected power fraction $|\Gamma|^2$

$$\Gamma = \frac{\eta_{in,a} - \eta_1}{\eta_{in,a} + \eta_1}$$

- Fraction of power transmitted into R4: $1 - |\Gamma|^2$

12.4 Plane Wave Propagation in General Directions

- Lossless medium, prop. const $\beta = k = \omega\sqrt{\mu\varepsilon}$

- phase $@ (x, z) = \vec{k}\cdot\vec{r}$
  - $\hookrightarrow$ k = prop. const. vector
- Wave (phasor): $\vec{E}_s = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}}$
  - $\hookrightarrow$ position vector: $\vec{r} = x\hat{a}_x + z\hat{a}_z$
  - $\hookrightarrow$ $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{k_x^2 + k_z^2}}$
  - $\hookrightarrow$ $v_p = \frac{\omega}{k} = \frac{\omega}{\sqrt{k_x^2 + k_z^2}}$
  - $\hookrightarrow$ $f = \frac{v_p}{\lambda}$

12.5 Plane Wave Reflection @ Oblique Incidence Angles

- Relation b/w incident, reflected, transmitted angles

- Incident wave dir. & phase $\rightarrow \vec{k}_1^+$
- angle of incidence θ_1
- Reflected wave $\vec{k}_1^-$ propagate away @ θ_1'
- Transmitted wave $\vec{k}_2$ @ θ_2
- Incident & reflected angles are equal ($\theta_1 = \theta_1'$)

- p-polarized (TM)
  - $\vec{E}$ polarized in page plane
  - $\vec{H}$ $\perp$ page pointing outward, parallel/transverse to interface $\rightarrow$ transverse magnetic
  - $\vec{E}$ lying in the plane of incidence $\rightarrow$ parallel polarization
- s-polarized (TE)
  - fields rotated by 90° I
  - $\vec{H}$ in incidence plane
  - $\vec{E}$ $\perp$ plane $\rightarrow$ perpendicular polarization
    - $\hookrightarrow$ parallel to interface $\rightarrow$ transverse electric

- Wave vector magnitudes:

$$k_1 = \frac{\omega\sqrt{\varepsilon_{r1}}}{c} = \frac{n_1\omega}{c}$$

$$k_2 = \frac{\omega\sqrt{\varepsilon_{r2}}}{c} = \frac{n_2\omega}{c}$$

- @ $x=0$: $E_{zs1}^+ = E_{zs1}^- = E_{zs2}$
- $\theta_1 = \theta_1' \rightarrow k_1 \sin\theta_1 = k_2 \sin\theta_2$
- Snell's Law of Refraction
  $n_1 \sin\theta_1 = n_2 \sin\theta_2$
- Relation b/w Amplitudes
  - @ $x=0$: $H_{10}^+ = H_{10}^- = H_{20}$
  - effective impedances:

$$\eta_{1p} = \eta_1 \cos\theta_1$$

$$\eta_{2p} = \eta_2 \cos\theta_2$$

$$\Gamma_p = \frac{E_{10}^-}{E_{10}^+} = \frac{\eta_{2p} - \eta_{1p}}{\eta_{2p} + \eta_{1p}}$$

$$\tau_p = \frac{E_{20}}{E_{10}^+} = \frac{2\eta_{2p}}{\eta_{2p} + \eta_{1p}}\left(\frac{\cos\theta_1}{\cos\theta_2}\right)$$

p-polarization

$\Gamma_p = \frac{E_{10}^-}{E_{10}^+} = \frac{\eta_{2p}-\eta_{1p}}{\eta_{2p}+\eta_{1p}}$ p-polarization

$\tau_p = \frac{E_{20}}{E_{10}^+} = \frac{2\eta_{2p}}{\eta_{2p}+\eta_{1p}}\left(\frac{\cos\theta_1}{\cos\theta_2}\right)$

$\eta_{1s} = \eta_1 \sec\theta_1$

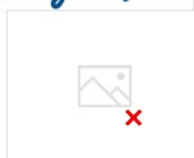
$\eta_{2s} = \eta_2 \sec\theta_2$

$\Gamma_s = \frac{E_{y10}^-}{E_{y10}^+} = \frac{\eta_{2s}-\eta_{1s}}{\eta_{2s}+\eta_{1s}}$ s-polarization

$\tau_s = \frac{E_{y20}}{E_{y10}^+} = \frac{2\eta_{2s}}{\eta_{2s}+\eta_{1s}}$

12.6 Total Reflection & Total transmission of obliquely incident waves

- Total Reflection → total Power Reflection
  - $|\Gamma|^2 = \Gamma\Gamma^* = 1 \rightarrow \Gamma = \Gamma_p$ or Γ_s
  - $\sin\theta_1 \geq \frac{n_2}{n_1}$
  - Critical Angle: $\sin\theta_c = \frac{n_2}{n_1}$
    - ↳ For total Reflection: $\theta_1 \geq \theta_c$
  - Optical waveguides: 3 layers of glass, mid layer $n >$ outer two



- Total transmission
  - $\Gamma = 0$
  - s-polarization:
    - ↳ $\Gamma_s = 0$: $\eta_{2s} = \eta_{1s}$ or $\eta_2 \sec\theta_2 = \eta_1 \sec\theta_1$
    - ↳ $\Gamma_p = 0$ → Snell's law solve
    - ↳ $\sin\theta_1 = \sin\theta_B = \frac{n_2}{\sqrt{n_1^2+n_2^2}}$ → $\eta_1 = \frac{\eta_0}{n_1}$, $\eta_2 = \frac{\eta_0}{n_2}$
  - Brewster Angle (polarization Angle)
    - ↳ if incident @ $\theta_1 = \theta_B$, p-component transmitted & s-polarized reflected
    - ↳ Polaroid Sunglasses → block transmission of horizontally polarized light passing vertically polarized light