Z Transform

April 16, 2020 4:53 PM

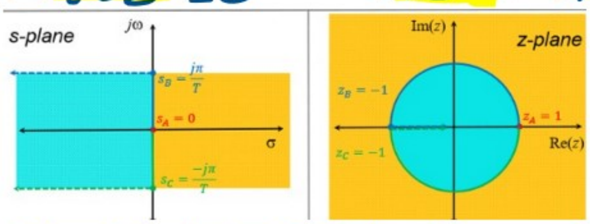
Z - Domain

$Y(z) = H(z)X(z)$

- Laplace Transform: $X_\delta(s) = \sum_n x(nT_s)e^{-ns6}$
  - ↳ let $z = e^{sT_s}$
- Z-Transform: $Z\{x_s[n]\} = \mathcal{L}\{x_\delta(6)\}\big|_{z=e^{sT_s}} = \sum_n x_s[n] \cdot z^{-n}$

Z-Plane

- $z = e^{sT} = e^{(\sigma + j\omega)T} = re^{j\Omega} \longrightarrow r = e^{\sigma},\ e^{j\Omega} = e^{j\omega T}$



Two-Sided Z-Transform

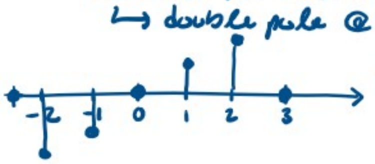
$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

ex. z-transform of $x[n] = \sum_{k=-2}^{2} k\delta[n-k]$

$x[n] = -2\delta[n+2] - \delta[n+1] + \delta[n-1] + 2\delta[n-2]$

$X(z) = -2z^2 - z + z^{-1} + 2z^{-2} = \dfrac{-2z^4 - z^3 + z + 2}{z^2}$

↳ double pole @ $z=0$



$ROC(x[n]) = \{\mathbb{C} \setminus z=0\}$

One-Sided Z-Transform

- For signals made causal by multiplying them with $u[n]$

  $X_1(z) = Z(x[n]u[n]) = \sum_{n=0}^{\infty} x[n]u[n]z^{-n}$

- Two-sided z-Transform can be expressed in terms of one-sided transform

  $X(z) = Z(x[n]u[n]) + Z(x[-n]u[n])\big|_z - x[0]$

  ↳ $z^{-1} \to z,\ z \to z^{-1}$

Region of convergence (ROC)

$ROC(x) = \{z = re^{j\Omega} \in \mathbb{C} \mid x[n]r^{-n}$ is absolutely summable$\}$

- poles cannot belong to the ROC

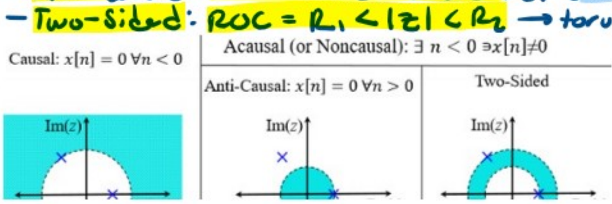
Finite Support ROC:

- For $[N_0, N_1]$, ROC is the z-plane excluding $z=0$ & $z=\pm\infty$ depending on N_0, N_1.

Infinite Support ROC:

- Causal signal: $|z| > r_{min} = \max\{|p_i|\} \longrightarrow p_i$ = poles of $X(z)$

  ROC = outside circle r_{min}
- Anti-Causal: ROC = inside circle $|z| < r_{max} = \min\{|p_i|\}$
- Two-Sided: ROC = $R_1 < |z| < R_2$ → torus inside min & max poles



Causal: $x[n] = 0 \; \forall n < 0$	Acausal (or Noncausal): $\exists \; n < 0 \ni x[n] \neq 0$	
	Anti-Causal: $x[n] = 0 \; \forall n > 0$	Two-Sided

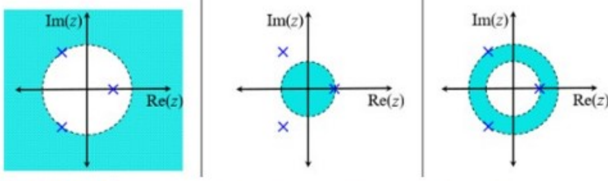


Table 10.1 One-sided Z-transforms of Common Signals

One-sided Z-transforms		
	Function of Time	**Function of z, ROC**
(1)	$\delta[n]$	1, Whole z-plane
(2)	$u[n]$	$\frac{1}{1-z^{-1}}$, $\lvert z\rvert > 1$
(3)	$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$, $\lvert z\rvert > 1$
(4)	$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$, $\lvert z\rvert > 1$
(5)	$\alpha^n u[n]$, $\lvert\alpha\rvert < 1$	$\frac{1}{1-\alpha z^{-1}}$, $\lvert z\rvert > \lvert\alpha\rvert$
(6)	$n\alpha^n u[n]$, $\lvert\alpha\rvert < 1$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$, $\lvert z\rvert > \lvert\alpha\rvert$
(7)	$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$
(8)	$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$
(9)	$\alpha^n\cos(\omega_0 n)u[n]$, $\lvert\alpha\rvert < 1$	$\frac{1-\alpha\cos(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2 z^{-2}}$, $\lvert z\rvert > 1$
(10)	$\alpha^n\sin(\omega_0 n)u[n]$, $\lvert\alpha\rvert < 1$	$\frac{\alpha\sin(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2 z^{-2}}$, $\lvert z\rvert > \lvert\alpha\rvert$

Table 10.2 Basic Properties of One-sided Z-transform

	Causal signals and constants	$\alpha x[n]$, $\beta y[n]$	$\alpha X(z)$, $\beta Y(z)$
P1	Linearity	$\alpha x[n] + \beta y[n]$	$\alpha X(z) + \beta Y(z)$
P2	Convolution sum	$(x * y)[n] = \sum_k x[n]y[n-k]$	$X(z)Y(z)$
P3	Time shifting – causal	$x[n-N]$ N integer	$z^{-N}X(z)$
P4	Time shifting – non-causal	$x[n-N]$ $x[n]$ non-causal, N integer	$z^{-N}X(z) + x[-1]z^{-N+1}$ $+x[-2]z^{-N+2} + \cdots + x[-N]$
P5	Time reversal	$x[-n]$	$X(z^{-1})$
P6	Multiplication by n	$nx[n]$	$-z\frac{dX(z)}{dz}$
P7	Multiplication by n^2	$n^2x[n]$	$z^2\frac{d^2X(z)}{dz^2} + z\frac{dX(z)}{dz}$
P8	Finite difference	$x[n] - x[n-1]$	$(1-z^{-1})X(z) - x[-1]$
P9	Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{X(z)}{1-z^{-1}}$
P10	Initial value	$x[0]$	$\lim_{z\to\infty} X(z)$
P11	Final value	$\lim_{n\to\infty} x[n]$	$\lim_{z\to 1}(z-1)X(z)$

ex. Determine all possible impulse responses for DT Filter w/ $H(z) = \frac{1+2z^{-1}+z^{-2}}{(1-0.5z^{-1})(1-z^{-1})}$

PFE: $\because H(z)$ not proper: $K_0 + \frac{K_1}{1-0.5z^{-1}} + \frac{K_2}{1-z^{-1}}$

$K_0 = \lim_{z^{-1}\to\infty} H(z) = \frac{1}{(-1/2)(-1)} = 2$, $K_1 = \lim_{z^{-1}\to 2}(1-0.5z^{-1})H(z) = \frac{1+4+4}{1-1/2} = -9$

$K_2 = \lim_{z^{-1}\to 1}(1-z^{-1})H(z) = \frac{1+2+1}{1-1/2} = 8 \longrightarrow z^{-1}\{K_0\} = K_0\,\delta[n]$

$H(z) = 2 + \frac{-9}{1-0.5z^{-1}} + \frac{8}{1-z^{-1}} \longrightarrow$ 3 possible ROCs

R1: $|z| > 1$ (causal, outside circle): $h_1[n] = 2\delta[n] - 9(1/2)^n u[n] + 8u[n]$

R2: $1/2 < |z| < 1$ (two sided): $h_2[n] = 2\delta[n] - \underbrace{9(1/2)^n u[n]}_{\text{causal}} - \underbrace{8u[-n-1]}_{\text{anti-causal}}$ anti-causal causal

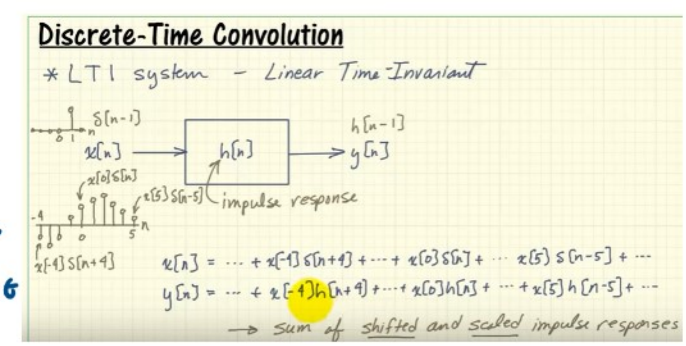
R3: $|z| < 1/2$ (anti-causal): $h_3[n] = 2\delta[n] + [9(1/2)^n - 8]u[-n-1]$

– None are BIBO stable $\because$ none contain unit circle

Convolution Sum

$y[n] = [x * h][n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} h[k]x[n-k]$

$Y(z) = X(z)H(z)$



Steady State & Transient Responses

$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \longrightarrow$ difference eqn.

– Simple/multiple poles inside unit circle $\longrightarrow$ transient

☆ Webwork 9

→ Sum of shifted and scaled impulse responses

✿ Webwork 9

Homework Sets
Problem Set 8
Prob 1
User Settings
Grades

Problems

Problem 1
Problem 2
Problem 3
Problem 4
Problem 5
Problem 6
Problem 7
Problem 8

Problem Set 8: Problem 1

Previous Problem | Problem List | Next Problem

A causal discrete-time LTI system is described by the difference equation, $y[n] - \frac{14}{48}y[n-1] + \frac{1}{48}y[n-2] = 9x[n]$, where $x[n]$ and $y[n]$ are the input and output of the system respectively.

a) Find the system transfer function $H(z)$, and indicate the region of convergence in interval notation (e.g. (-INF,a), (a,b) or (b, INF)).

$H(z) =$ ______ $RoC:$ ______

b) Find the impulse response, $h[n]$, of the system.

$h[n] =$ ______

c) Find the step response, $s[n]$, of the system.

$s[n] =$ ______

In your answers, enter $x(n)$ for a discrete-time function $x[n]$ and enter $D(n)$ instead of $\delta[n]$. WebWork is unable to parse a function that uses square brackets.

a) $y[n] - \frac{14}{48}y[n-1] + \frac{1}{48}y[n-2] = 9x[n]$

z-transform: $x[n-N] \rightarrow z^{-N}X(z)$

$Y(z) - \frac{14}{48}z^{-1}Y(z) + \frac{1}{48}z^{-2}Y(z) = 9X(z)$

$H(z) = \frac{Y(z)}{X(z)} = \frac{9}{1 - \frac{14}{48}z^{-1} + \frac{1}{48}z^{-2}}$

ROC: $\{z = re^{j\Omega} : r > \max\{|p_i|\}\}$

Poles: $1/8$, $1/6$ $\therefore$ ROC $= [0.1666, \infty]$

b) Impulse Response $h[n]$: invztrans(H(z), z, n)

$h[n] = [36(1/6)^n - 27(1/8)^n]u(n)$

c) Step Response $s[n]$

$x[n] = u[n] \longrightarrow$ convolute w/ $h[n]$

In z-domain: $S(z) = H(z) \cdot \frac{1}{1 - z^{-1}} \longrightarrow$ invztrans( )

$s[n] = 1/35(-252(1/6)^n + 135(1/8)^n + 432)u[n]$

Problem Set 8: Problem 4

Previous Problem | Problem List | Next Problem

For the two discrete-time LTI systems described below, find the transfer function $H(z)$, if:

a) In system A, where an input-output signal pair is given by:

$$x[n] = \begin{cases} 4 & n = 0, 1 \\ 0 & otherwise \end{cases}$$

$$y[n] = \begin{cases} 5 & n = 0, 4 \\ 7 & n = 1, 3 \\ 0 & otherwise \end{cases}$$

$H(z) =$ ______

$x[n] = 4(\delta[n] + \delta[n-1])$

$y[n] = 5(\delta[n] + \delta[n-1]) + 7(\delta[n-1] + \delta[n-3])$

z-transform: $X(z) = 4 + 4z^{-1}$, $Y(z) = 5 + 5z^{-4} + 7z^{-1} + 7z^{-3}$

$H(z) = \frac{Y(z)}{X(z)}$

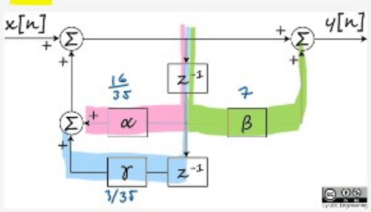
Problem 5
User Settings
Grades

Problems

Problem 1
Problem 2
Problem 3
Problem 4
Problem 5
Problem 6
Problem 7
Problem 8

JY Note Apr 5, 2020: One student has had parts (b) & (c) mistakenly graded as wrong. If you are confident that you were similarly mistakenly penalized, please contact me directly.

Consider a causal discrete-time LTI system whose input, $y[n]$, and output, $x[n]$, are related by the block diagram given in the figure below. Assume $\alpha = \frac{16}{35}$, $\beta = 7$, and $\gamma = \frac{3}{35}$



a) Find the difference equation that describes this system

______

In your answers, enter $x(n)$ for a discrete-time function $x[n]$. WebWork is unable to parse a function that uses square brackets.

b) Find the transfer function, $H(z)$, of this system

$H[z] =$ ______

c) M/C Question: Is the system stable?

?

a) $w[n] = x[n] + (\alpha w[n-1] + \gamma w[n-2])$

$y[n] = x[n] + \alpha y[n-1] + \gamma w[n-2] + \beta x[n-1]$

b) $Y(z) = X(z) + \beta z^{-1}X(z) + \alpha z^{-1}Y(z) + \gamma z^{-2}Y(z)$

$H(z) = \frac{1 + 7z^{-1}}{1 - \frac{16}{35}z^{-1} - \frac{3}{35}z^{-2}}$

c) $(1 - \frac{16}{35} - \frac{3}{35}z^{-2} = 0) \times z^2 \longrightarrow z^2 - \frac{16}{35}z - \frac{3}{35} = 0$

$z = \frac{7}{70} \pm \frac{2\sqrt{41}i}{70}$

$\because$ inside unit circle $\therefore$ stable