Eigen Values, Vectors, Diagonalization, Hermitian Matrices

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Eigen values & Eigen vectors

- Scalar λ & non-zero vector $\vec{v}$ = eigen pair if:

  $A\vec{v} = \lambda\vec{v}$ ⟶ $\vec{v} \neq 0$ but $\lambda = 0$ ✓

  ↳ $(A - \lambda I_n)\vec{v} = \vec{0}$ ⟶ $\vec{v} \in N(A - \lambda I)$

- ∴ If $A\vec{v} = \lambda\vec{v}$, $\det(A - \lambda I_n) = 0$

  ↳ Characteristic polynomial of A: $p(\lambda) = \det(A - \lambda I_n) = 0$

Finding eigen values & eigen vectors

- The roots of $p(\lambda)$ = eigen values λ of A

- let λ_j be an eigen value of A, any non-zero vector satisfying $(A - \lambda_j I)\vec{v} = \vec{0}$ are eigen vectors

ex. Find eigen vectors corresponding to eigen values $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 6$ of $A = \begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & -2 & 1 \end{bmatrix}$

$\lambda_1 = 2$: solve system (GE) $(A - \lambda_1 I)\vec{v} = \vec{0}$: $\begin{bmatrix} 3-2 & -6 & -7 \\ 1 & 8-2 & 5 \\ -1 & -2 & 1-2 \end{bmatrix}$

$\left\{ s \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ ∴ $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 4$: $v_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = 6$: $v_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

- If all eigen values distinct, corresponding eigenvectors for basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$)

- Eigen vectors corresponding to diff. eigen values = L.I.

If not all eigen values distinct:

ex. If $p(\lambda) = (\lambda - 2)^2(\lambda - 3)$ ⟶ $\lambda_1 = \lambda_2 = 2$, $\lambda = 3$

- Algebraic multiplicity: $m_1 = 2$, $m_2 = 1$ ⟶ $d_j = \dim(E_{\lambda_j})$

- There exists an eigen basis to A if $d_j = m_j$

ex. $A = \begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ \ldots \end{bmatrix}$ ⟶ $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 6$

- There exists an eigenbasis to A if $d_j = m_j$

ex. $A = \begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & -2 & 1 \end{bmatrix} \rightarrow \lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 0$

$m_1 = m_2 = m_3 = 1$, $d_1 = d_2 = d_3 = 1$

$\therefore$ there is an eigenbasis

ex. $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow p(\lambda) = (\lambda - 1)^2 \rightarrow \lambda_1 = \lambda_2 = 1 \rightarrow m = 2$

$\rightarrow$ geometric multiplicity (dimension of eigenspace) of λ_1

$B - \lambda I = B - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow$ solve $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{v} = \vec{0}$

$\vec{v} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow E_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \rightarrow d_1 = 1$

$\because m_1 \neq d_1$, no eigenbasis corresponding to B

- If all entries real & λ_1 & v_1 eigenpair of A,

$\overline{\lambda_1}$ & $\overline{v_1}$ also eigenpair of A

Diagonalization

- For $\lambda_1 \ldots \lambda_n$ and eigenbasis $\{v_1 \ldots v_n\}$ s.t. $Av_j = \lambda_j v_j$

$S = [v_1 \ldots v_n] \rightarrow AS = [Av_1 \ldots Av_n] = [\lambda_1 v_1 \ldots \lambda_n v_n]$

$\hookrightarrow AS = SD$ $\qquad = \underbrace{[v_1 \ldots v_n]}_{S} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\therefore A = SDS^{-1} \rightarrow$ Diagonalization of A

$\hookrightarrow D = S^{-1}AS$

① Determinant & Trace

Determinant

- $\det(BC) = \det(B)\det(C)$
- $\det(B^{-1}) = 1/\det(B)$

$\hookrightarrow \det(A) = \det(S)\det(D)\det(S^{-1}) = \det(D)$

$\therefore \det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

- Determinant of diagonalizable matrix = product of eigenvalues

Trace

- $tr(BC) = tr(CB)$

$\hookrightarrow tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

- Trace of diagonalizable matrix = sum of eigenvalues

- Trace of diagonalizable matrix = sum of eigenvalues

② Powers of diagonalizable matrices

- $A = SDS^{-1}$

$$\left.\begin{aligned} A &= SDS^{-1} \\ A^2 &= SD\underbrace{S^{-1}S}_{I}DS^{-1} = SD^2S^{-1} \\ A^3 &= \cdots\cdots = SD^3S^{-1} \end{aligned}\right\} A^k = SD^kS^{-1}$$

where $D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$

③ Hermitian Matrices

- Square Matrix = Hermitian if: $A^* = A$ $(\bar{A}^T = A)$

ex. $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 1+i \\ 1-i & 5 \end{bmatrix}$

i) All diagonal entries = real

ii) Hermitian matrix that only has real entries = symmetric

Properties

① A is Hermitian if: $\langle \vec{v}, A\vec{w} \rangle = \langle A\vec{v}, \vec{w} \rangle$

② Eigenvalues of Hermitian matrices are real

③ Eigenvectors correspond to distinct eigenvalues

- If A Hermitian w/ distinct eigenvalues, eigenbasis $\{v_1, \dots, v_n\}$ can be chosen to be an ONB

- Diagonalization of Hermitian Matrices

let $S = [v_1 \dots v_n] = U \longrightarrow$ unitary matrix $\therefore (U^{-1} = U^*)$

$A = UDU^*$

- $\|A\|_{op} = \max\{|\lambda_j|, j = 1, \dots, n\}$

④ Any Hermitian is unitarily diagonalizable

U is unitary even if A has repeated eigenvalues

⑤ Powers & other func of Hermitian matrices

$A = UDU^*$ $U = [v_1 \cdots v_n]$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $U^* = [\bar{v}_1^T, \dots, \bar{v}_n^T]^T$

then $A = \lambda_1 v_1 \bar{v}_1^T + \dots + \lambda_n v_n \bar{v}_n^T$