Fourier Transform

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① Fourier Transform (for Bounded signals w/ finite time support)

- Aperiodic signals have a fourier transform

Fourier Transform: $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$

Inverse Fourier Transform: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\Omega t}\, d\omega$

Fourier Transform Pair

ex. $x_1(t) = u(t+1) - u(t-1)$

$$X(\omega) = \int_{-1}^{1} x_1(t) e^{-j\omega t} = \int_{-1}^{1} e^{-j\omega t} = \frac{j}{\omega}\left(e^{-j\omega} - e^{j\omega}\right)$$

Table 5.1 Basic Properties of Fourier Transform

		Time Domain	Frequency Domain
	Signals and constants	$x(t), y(t), z(t), \alpha, \beta$	$X(\omega), Y(\omega), Z(\omega)$
P1	Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\omega) + \beta Y(\omega)$
P2	Expansion/contraction in time	$x(\alpha t),\ \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X\left(\frac{\omega}{\alpha}\right)$
P3	Reflection	$x(-t)$	$X(-\omega)$
P4	Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
P5	Duality	$X(t)$	$2\pi x(-\omega)$
P6	Time differentiation	$\frac{d^n x(t)}{dt^n},\ n \geq 1$, integer	$(j\omega)^n X(\omega)$
P7	Frequency differentiation	$-jtx(t)$	$\frac{dX(\omega)}{d\omega}$
P8	Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
P9	Time shifting	$x(t-\alpha)$	$e^{-j\alpha\omega} X(\omega)$
P10	Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
P11	Modulation	$x(t)\cos(\omega_c t)$	$0.5[X(\omega-\omega_c) + X(\omega+\omega_c)]$
P12	Periodic signals	$x(t) = \sum_k X_k e^{jk\omega_0 t}$	$X(\omega) = \sum_k 2\pi X_k \delta(\omega - k\omega_0)$
P13	Symmetry	$x(t)$ real	$\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$
			$\angle X(\omega) = -\angle X(-\omega)$
P14	Convolution in time	$z(t) = [x * y](t)$	$Z(\omega) = X(\omega)Y(\omega)$
P15	Windowing/Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\omega)$
P16	Cosine transform	$x(t)$ even	$X(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt$, real
P17	Sine transform	$x(t)$ odd	$X(\omega) = -j\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt$, imaginary

Table 5.2 Fourier Transform Pairs

	Function of Time	Function of ω
(1)	$\delta(t)$	1
(2)	$\delta(t-\tau)$	$e^{-j\omega\tau}$
(3)	$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
(4)	$u(-t)$	$\frac{-1}{j\omega} + \pi\delta(\omega)$
(5)	$\text{sign}(t) = 2[u(t) - 0.5]$	$\frac{2}{j\omega}$
(6)	$A,\ -\infty < t < \infty$	$2\pi A\delta(\omega)$
(7)	$Ae^{-at}u(t),\ a > 0$	$\frac{A}{j\omega + a}$
(8)	$Ate^{-at}u(t),\ a > 0$	$\frac{A}{(j\omega + a)^2}$
(9)	$e^{-a\lvert t\rvert},\ a > 0$	$\frac{2a}{a^2 + \omega^2}$
(10)	$\cos(\omega_0 t),\ -\infty < t < \infty$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
(11)	$\sin(\omega_0 t),\ -\infty < t < \infty$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
(12)	$p(t) = A[u(t+\tau) - u(t-\tau)],\ \tau > 0$	$2A\tau\frac{\sin(\omega\tau)}{(\omega\tau)}$ → pulse
(13)	$\frac{\sin(\omega_0 t)}{\pi t}$	$P(\omega) = u(\omega + \omega_0) - u(\omega - \omega_0)$
(14)	$x(t)\cos(\omega_0 t)$	$0.5[X(\omega - \omega_0) + X(\omega + \omega_0)]$

Dirichlet Conditions for Convergence

- $x(t)$ is integrable (area under is finite)
- $x(t)$ has finite # of discontinuities & minima/maxima

② FT from LT (For infinite time signals, contain $j\omega$-axis)

- If $\hat{X}(s) = \mathcal{L}\{x(t)\}$ **contains $j\omega$-axis**, FT of $x(t)$:

$$X(\omega) = \mathcal{F}\{x(t)\} = \hat{X}(s)\big|_{s=j\omega}$$

ex. $x_2(t) = e^{-2t}u(t)$: $\hat{X}_2(s) = \frac{1}{s+2}$

ROC: $\text{Re}(s) > 2$ ∴ includes $j\omega$-axis $\longrightarrow X_2(\omega) = \mathcal{F}\{x_2(t)\} = \frac{1}{j\omega+2}$

③ FT from FS (for periodic signals)

Fourier Pair: $x(t) = \sum_k X_k e^{jk\omega_0 t} \longrightarrow X_k(\omega) = \sum_k 2\pi X_k \delta(\omega - k\omega_0)$

ex. $x_1(t) = A \longrightarrow X_1(\omega) = 2\pi A\delta(\omega)$ (**table 5.2**)

ex. $x_2(t) = \underbrace{4\cos\left(\frac{6\pi}{7}t\right)}_{T_1 = 7/3} + \underbrace{\cos\left(\frac{3\pi}{5}t - \frac{\pi}{2}\right)}_{T_2 = 10/3} = \underbrace{2\left(e^{j\frac{6\pi}{7}t} + e^{-j\frac{6\pi}{7}t}\right)}_{k=10,\ X_{10} = 2} + \underbrace{\frac{1}{2}\left(e^{j\frac{3\pi}{5}t}e^{-j\frac{\pi}{2}} + e^{-j\frac{3\pi}{5}t}e^{j\frac{\pi}{2}}\right)}_{k=7,\ X_7 = \frac{1}{2}e^{-j/2} = j/2}$

$T_0 = \text{LCM}(7/3, 10/3) = 70/3$, $\omega_0 = 6\pi/30$

$$\therefore X_2(\omega) = 2\pi\left[X_7\left(\delta(\omega + 7\omega_0) + \delta(\omega - 7\omega_0)\right) + X_{10}\left(\delta(\omega + 10\omega_0) + \delta(\omega - 10\omega_0)\right)\right]$$

$$= 2\pi\left[\frac{j}{2}\left(\delta\left(\omega + \frac{3\pi}{5}\right) - \delta\left(\omega - \frac{3\pi}{5}\right)\right) + 2\left(\delta\left(\omega + \frac{6\pi}{7}\right) + \delta\left(\omega - \frac{6\pi}{7}\right)\right)\right]$$

④ Duality (For unclassified signals)

$= 2\pi i \left[\frac{1}{2}(\delta(\omega+5) - \delta(\omega-5)) + 2(\delta(\omega+7) + \delta(\omega-7))\right]$

④ Duality (For unclassified signals)

– When $f = \omega/2\pi$ in H3: $\hat{X}(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \longleftrightarrow x(t) = \int_{-\infty}^{\infty} \hat{X}(f) e^{j2\pi f t} dt$

→ use table 5.1

ex. Heaviside: $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \longrightarrow \mathcal{F}\{u(t)\} = \frac{D(\omega)}{j\omega} + \pi D(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$

ex. sinc: $x_1(t) = u(t+1) - u(t-1) \xrightarrow{\mathcal{F}} X_1(\omega) = 2\mathrm{sinc}\left(\frac{\omega}{\pi}\right)$

$X(\omega) = \mathrm{sinc}(t) \xrightarrow{\mathcal{F}} 2\pi x(-\omega) = 2\pi\left[\frac{u(-\frac{\omega}{\pi}+1) - u(-\frac{\omega}{\pi}-1)}{2\pi}\right] = u(\omega+\pi) - u(\omega-\pi)$

Parseval's Energy Relation for Energy Signals

$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

Frequency Response

– System has freq. Response: $H(j\omega) = \mathcal{F}\{h(t)\}$ where $h(t)$ is the impulse response

– Output of LTI system is $y(t) = (x * h)(t)$ w/ FT $Y(\omega) = X(\omega)H(j\omega)$

– If input $x(t)$ periodic: output has FT: $Y(\omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k H(jk\omega_0)\delta(\omega - k\omega_0)$

☆ webwork 5

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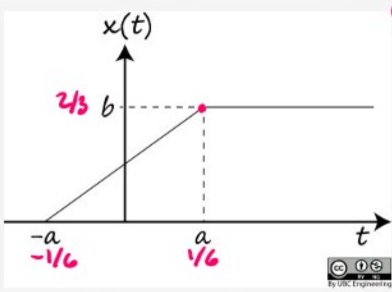
Consider the signal $x(t)$ given in the figure below. Assume $a = \frac{1}{6}$ and $b = 0.666666666666667$.



a) FT of a pulse (table 5.2 (12)) ⟶ derivative of ramp = pulse

$P(t) = A[u(t+\tau) - u(t-\tau)] \xrightarrow{\mathcal{F}} P(\omega) = 2A\tau \frac{\sin(\omega\tau)}{\omega\tau}$

$x(t) = 2[u(t+1/6) - u(t-1/6)] \xrightarrow{\mathcal{F}} X'(\omega) = \frac{4\sin(\omega\, 1/6)}{6\cdot\omega\, 1/6}$ → pulse

$X(\omega) = \int_{-\infty}^{t} \frac{4\sin(\omega/6)}{\omega} d\omega = \frac{4\sin(\omega/6)}{j\omega^2} + \frac{4\pi D(\omega)}{6}$ → ramp

a) Find a closed form expression for the Fourier transform $X(\omega)$ of the signal $x(t)$.

Hint: Use the integration and differentiation properties, as well as the Fourier transform pair of a rectangular pulse.

$X(\omega) =$

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A system is described by the differential equation $\frac{d}{dt}y(t) + 2y(t) = 7\frac{d}{dt}x(t) + x(t)$, where $x(t)$ is the input and $y(t)$ is the output of the system.

In your answers below, enter $D(t)$ instead of $\delta(t)$, and "w" for ω.

a) Find the frequency response of the system, $H(\omega)$.

$H(\omega) =$

b) Find the impulse response of the system, $h(t)$.

$h(t) =$

c) Find the output signal $y(t)$ that corresponds to the input signal $x(t) = e^{-8t}u(t)$. Solve the problem in the time as well as the frequency domain.

$y(t) =$

a) $Ys + 2Y = 7Xs + X$

$H(s) = \frac{7s+1}{s+2}$

$H(\omega) = \frac{7j\omega+1}{j\omega+2}$

b) $h(t) = \mathcal{L}^{-1}\{H(s)\}$

$= 7\delta(t) - (13e^{-2t})u(t)$

c) $Y(s) = X(s)H(s)$

$y(t) = \left(-\frac{13}{6}e^{-2t} + \frac{55}{6}e^{-8t}\right)u(t)$

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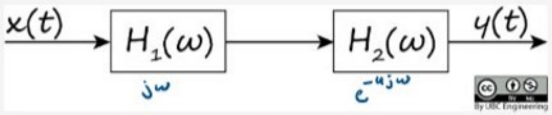
Two filters with frequency responses $H_1(\omega) = jw$ and $H_2(\omega) = e^{-4jw}$ for $-\infty < \omega < \infty$, are cascaded together so that the output of the first filter is fed as the input to the second, as shown in the figure below.



Suppose that the input to this cascaded system is the signal $x(t) = cos\left(\frac{\pi t}{6}\right)[u(t+4) - u(t-4)]$.

a) Find the output, $y(t)$, of this cascaded system.

$y(t) =$

(Table 5.1): Time diff: $\frac{d^n x(t)}{dt^n} \rightarrow (j\omega)^n X(\omega)$

Time delay: $(t-\tau) \rightarrow e^{-\tau j\omega}$

$x(t) = \cos\left(\frac{\pi t}{6}\right)[u(t+4) - u(t-4)]$

$\therefore y(t) = -\frac{\pi}{6}\sin\left(\frac{\pi(t-4)}{6}\right)[u(t-4+4) - u(t-4-4)]$

$= -\frac{\pi}{6}\sin\left(\frac{\pi(t-4)}{6}\right)[u(t) - u(t-8)]$

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NB: In this Webwork problem, take $sinc(t) = sin(t)/t$ (in contrast, in Signal Processing literature, $sinc(t) = sin(\pi t)/\pi t$).

Find the Fourier transform $X_1(\omega)$, $X_2(\omega)$, and $X_3(\omega)$ of the signals $x_1(t)$, $x_2(t)$, and $x_3(t)$, using the Fourier transform pair $x(t) = u(t+1) - u(t-1) \longleftrightarrow X(\omega) = 2sinc(\omega)$. Then select the Fourier transform property you used for each signal, from the corresponding drop-down menu.

In your answers, enter "w" for omega.

a) $x_1(t) = -2u(t+2) + 8u(t) - 6u(t-2)$

$X_1(\omega) =$? Time shift

a) Time shift: $x(t-a) \xrightarrow{\mathcal{F}} e^{-ja\omega}$

$x_1 = -2[u(t+2) + u(t)] + 6[u(t) - u(t-2)]$

$X_1 = -2[e^{j\omega} 2sinc(\omega)] + 6[e^{-j\omega} 2sinc(\omega)]$

$\hookrightarrow x(t) = u(t+1) - u(t-1) \longleftrightarrow X(\omega) = 2sinc(\omega)$