# Assignment 5

dimanche 27 mars 2022 10:12

1. *Three coins.* There are three identically looking coins. For coin 1, the probability of heads is 0.3 and the probability of tails is 0.7; for coin 2, the probability of heads is 0.7 and the probability of tails is 0.3; for coin 3, the probability of heads is 0.5 and the probability of tails is 0.5.

   (a) Suppose Alice was given one of the three coins uniform at random. She flips the given coin twice (independent of each other) and got heads on the first flip and tails on the second flip. Based on this outcome, Alice wants to estimate which coin was given to her. Find the optimal estimate that minimizes the probability of error.

   (b) Suppose Alice was given coin 1 with probability 0.6 and coin 2 with probability 0.4 (which implies she was given coin 3 with probability 0). She wants to estimate which coin was given to her from the outcome of flipping the given coin once. Find the MAP estimator given the outcome of a single flip. Find the corresponding (overall) probability of error.

a) Observation $X = \{H, T\}$; Unknown $\Theta = C$ $\Big]$ $\Theta \to \mathbb{P}(x|\Theta) \to p(\theta|x) = \frac{\mathbb{P}(\theta)\mathbb{P}(x|\theta)}{\mathbb{P}(x)} \to \hat{\Theta}$

$\mathbb{P}(HT|C_1) = (0.3)(0.7) = 0.21$ $\qquad \mathbb{P}(C_1) = \mathbb{P}(C_2) = \mathbb{P}(C_3) = 1/3$

$\mathbb{P}(HT|C_2) = (0.7)(0.3) = 0.21$

$\mathbb{P}(HT|C_3) = (0.5)(0.5) = 0.25 \to \hat{\Theta}_{MAP}(HT) = p(HT|\Theta=3) = \text{Coin 3}$

$\mathbb{P}(HT) = \frac{1}{3}\mathbb{P}(C_1) + \frac{1}{3}\mathbb{P}(C_2) + \frac{1}{3}\mathbb{P}(C_3) = 0.223$

Bayes' Rule:

$\mathbb{P}(C_1|HT) = \frac{\mathbb{P}(C_1)\mathbb{P}(HT|C_1)}{\mathbb{P}(HT)} = \frac{1/3\,(0.21)}{0.223} = 0.314$

$\mathbb{P}(C_2|HT) = \frac{\mathbb{P}(C_2)\mathbb{P}(HT|C_2)}{\mathbb{P}(HT)} = \frac{1/3\,(0.21)}{0.223} = 0.314$

$\mathbb{P}(C_3|HT) = \frac{\mathbb{P}(C_3)\mathbb{P}(HT|C_2)}{\mathbb{P}(HT)} = \frac{1/3\,(0.25)}{0.223} = 0.374$

$\mathbb{P}(\hat{\Theta} \neq \Theta \mid X = C_1) = \mathbb{P}(\hat{\Theta} \neq \Theta \mid X = C_2) = 0.686$
$\mathbb{P}(\hat{\Theta} \neq \Theta \mid X = C_3) = 0.626$
$\arg\min\{\mathbb{P}(\hat{\Theta} \neq \Theta \mid X = x)\} = 0.626$ $\Big]$ Probability of Error

$\therefore$ Coin 3

b) $\mathbb{P}(C_1) = 0.6$, $\mathbb{P}(C_2) = 0.4$, $\mathbb{P}(C_3) = 0$

$\mathbb{P}(H) = \mathbb{P}(C_1)\mathbb{P}(H|C_1) + \mathbb{P}(C_2)\mathbb{P}(H|C_2) = 0.46$

$\mathbb{P}(T) = \mathbb{P}(C_1)\mathbb{P}(T|C_1) + \mathbb{P}(C_2)\mathbb{P}(T|C_2) = 0.54$

$\mathbb{P}(C_1|H) = \frac{\mathbb{P}(C_1)\mathbb{P}(H|C_1)}{\mathbb{P}(H)} = 0.391$

$\mathbb{P}(C_1|T) = \frac{\mathbb{P}(C_1)\mathbb{P}(T|C_1)}{\mathbb{P}(T)} = 0.778$

$\mathbb{P}(C_2|H) = \frac{\mathbb{P}(C_2)\mathbb{P}(H|C_2)}{\mathbb{P}(H)} = 0.609$

$\mathbb{P}(C_2|T) = \frac{\mathbb{P}(C_2)\mathbb{P}(T|C_2)}{\mathbb{P}(T)} = 0.222$

MAP Estimator: $\hat{\Theta}_{MAP}(X) = \arg\max_{\theta} \mathbb{P}(\theta|x)$

$\hat{\Theta}_{MAP}(H) \to C_2$

$\hat{\Theta}_{MAP}(T) \to C_1$

Overall Probability of Error:

$\mathbb{P}(\hat{\Theta} \neq \Theta \mid X = H) = 1 - \mathbb{P}(\Theta = C_2 \mid X = H) = 0.391$
$\mathbb{P}(\hat{\Theta} \neq \Theta \mid X = T) = 1 - \mathbb{P}(\Theta = C_1 \mid X = T) = 0.222$
$\mathbb{P}(\hat{\Theta} \neq \Theta) = (0.391)\,\mathbb{P}(H) + (0.222)\,\mathbb{P}(T) = 0.299$ $\Big]$ Probability of Error

2. *Iocane or Sennari.* An absent-minded chemistry professor forgets to label two identically looking bottles. One contains a chemical named "Iocane" and the other contains a chemical named "Sennari". It is well known that the radioactivity level of "Iocane" has the Unif[0, 1] distribution, while the radioactivity level of "Sennari" has the Exp(1) distribution.

   (a) Let $X$ be the radioactivity level measured from one of the bottles. What is the optimal decision rule (based on the measurement $X$) that maximizes the chance of correctly identifying the content of the bottle?

   (b) What is the associated probability of error?

Let $\Theta = 0 \to$ Iocane & $\Theta = 1 \to$ Sennari

$\mathbb{P}(\Theta = 0) = \mathbb{P}(\Theta = 1) = 1/2$

(b) What is the associated probability of error?

Let $\Theta = 0 \rightarrow$ Jocum & $\Theta = 1 \rightarrow$ Semari

$\mathbb{P}(\Theta=0) = \mathbb{P}(\Theta=1) = 1/2$

a) Let $\Theta = \begin{cases} 0, & \text{Jocama bottle} \\ 1, & \text{Semari bottle} \end{cases}$ $\rightarrow$ $x|\Theta=0 \sim \text{Unif}(0,1)$, $x|\Theta=1 \sim \text{Exp}(1)$

$$\mathbb{P}_{\Theta|X}(\theta|x) = \frac{f_{X|\Theta}(x|0)\,p(\Theta=0)}{\sum_{\theta\in\{0,1\}} f_{X|\Theta}(x|\theta)\,p(\theta)} = \frac{\frac{1}{1-0}\cdot\frac{1}{2}}{\frac{1}{1-0}\frac{1}{2}+e^{-x}\frac{1}{2}} = \frac{1}{1+e^{-x}}$$

$$p(\Theta=0|X=x) = \begin{cases} \frac{1}{1+e^{-x}}, & x\in[0,1] \\ 0, & o/w \end{cases}$$

$\hookrightarrow$ $\hat{\Theta}_{MAP}$ for Jocama $x \in [0,1]$

$\hat{\Theta}_{MAP}$ for Semari otherwise

b) Probability of Error

$$\mathbb{P}(\hat{\Theta} \neq \Theta) = 1/2\, \mathbb{P}(\hat{\Theta} \neq \Theta | X=0) + 1/2\, \mathbb{P}(\hat{\Theta} \neq \Theta | X=1)$$
$$= 1/2\, \mathbb{P}(x>1 | \Theta=0) + 1/2\, \mathbb{P}(x>1|\Theta=1)$$
$$= 0 + 1/2(1-e^{-1})$$
$$= 1/2(1-e^{-1})$$

3. *Ternary signaling.* Let the signal $S$ be a random variable defined as follows:

$$S = \begin{cases} -1 & \text{with probability } \frac{1}{3} \\ 0 & \text{with probability } \frac{1}{3} \\ +1 & \text{with probability } \frac{1}{3}. \end{cases}$$

The signal is sent over a channel with additive Laplacian noise $Z$, i.e., $Z$ is a Laplacian random variable with pdf

$$f_Z(z) = \frac{\lambda}{2}e^{-\lambda|z|}, \quad -\infty < z < \infty.$$

The signal $S$ and the noise $Z$ are assumed to be independent and the channel output is their sum $Y = S + Z$.

(a) Find $f_{Y|S}(y|s)$ for $s = -1, 0, +1$.

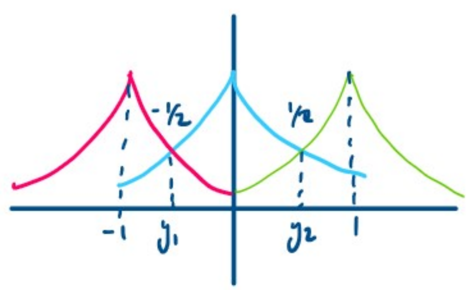
(b) Find the optimal decoding rule $\hat{S}_{\text{MAP}}(y)$ for deciding whether $S$ is $-1$, $0$ or $+1$. Give your answer in terms of ranges of values of $y$.

(c) Find the probability of decoding error for $\hat{S}_{\text{MAP}}(y)$ in terms of $\lambda$.

a) $f_{Y|S}(y|s) = \lambda/2\, e^{-\lambda(s+z)}$

$= f_{Z|S}(y-s|s)$

$$\therefore f_{Y|S}(y|s) = \begin{cases} \lambda/2\, e^{-\lambda|y+1|}, & s=-1 \\ \lambda/2\, e^{-\lambda|y|}, & s=0 \\ \lambda/2\, e^{-\lambda|y-1|}, & s=1 \end{cases}$$




b) By graphing the 3 plots above, we see that:

For $S=-1$ : $y \in (-\infty, -1/2)$

For $S=0$ : $y \in [-1/2, 1/2]$

For $S=1$ : $y \in (1/2, \infty)$

c) $\mathbb{P}(\hat{S} \neq S) = \mathbb{P}(\hat{S} \neq s, Y < -1/2) + \mathbb{P}(\hat{S} \neq s, -1/2 \le Y \le 1/2) + \mathbb{P}(\hat{S} \neq s, Y > 1/2)$

$= \mathbb{P}(S \neq 1, Y < -1/2) + \mathbb{P}(S \neq 0)$

$= 1 - \mathbb{P}(\hat{S} = S)$

$\mathbb{P}(\hat{S} = S) = \mathbb{P}(\hat{S}=s, Y<-1/2) + \mathbb{P}(\hat{S}=s, -1/2 \le Y \le 1/2) + \mathbb{P}(\hat{S}=s, Y=1/2)$

$= \mathbb{P}(S=-1, Y<-1/2) + \mathbb{P}(S=0, 1/2 \le Y \le 1/2) + \mathbb{P}(S=1, Y>1/2)$

$= \mathbb{P}(Y<-1/2 | S=-1)\,\mathbb{P}(S=-1) + \mathbb{P}(-1/2 \le Y \le 1/2 | S=0)\,\mathbb{P}(S=0) + \mathbb{P}(Y>1/2|S=1)\,\mathbb{P}(S=1)$

$\mathbb{P}(Y<-1/2 | S=1) = \int_{-\infty}^{-1/2} f_{Y|S}(y|s=-1)\,dy$

$= \frac{\lambda}{2}\int_{-\infty}^{-1/2} e^{-\lambda|y+1|}\,dy$

$= 1/2 + 1/2(1-e^{-\lambda/2})$

$\mathbb{P}(-1/2 \le Y \le 1/2 | S=0) = \int_{-1/2}^{1/2} f_{Y|S}(y|s=0)\,dy$

$= \lambda\int_0^{1/2} e^{-\lambda y}\,dy$

$= 1 - e^{-\lambda/2}$

$\mathbb{P}(Y>1/2 | S=1) = \int_{1/2}^{\infty} \lambda/2\, e^{-\lambda|y-1|}\,dy$

$= 1/2(1-e^{-\lambda/2}) + 1/2$

$\therefore \mathbb{P}(S=\hat{S}) = 1/3(1/2 + 1/2(1-e^{-\lambda/2}) + 1 - e^{-\lambda/2} + 1/2 + 1/2(1-e^{-\lambda/2}))$

$= 1 - 2/3\, e^{-\lambda/2}$

$\therefore \mathbb{P}(S \neq \hat{S}) = 1 - (1 - 2/3\, e^{-\lambda/2})$

$$\therefore \mathbb{P}(s=\hat{s}) = \tfrac{1}{3}\left(\tfrac{1}{2}+\tfrac{1}{2}(1-e^{-\lambda/2}) + 1-e^{-\lambda/2} + \tfrac{1}{2}+\tfrac{1}{2}(1-e^{-\lambda/2})\right)$$
$$= 1-\tfrac{2}{3}e^{-\lambda/2}$$
$$\therefore \mathbb{P}(s\neq\hat{s}) = 1-(1-\tfrac{2}{3}e^{-\lambda/2})$$
$$= \tfrac{2}{3}e^{-\lambda/2}$$

4. *Signal or no signal.* Consider a communication system that is operated only from time to time. When the communication system is in the "normal" mode (denoted by $M = 1$), it transmits a random signal $S = X$ with

$$X = \begin{cases} +1, & \text{with probability } 1/2, \\ -1, & \text{with probability } 1/2. \end{cases}$$

When the system is in the "idle" mode (denoted by $M = 0$), it does not transmit any signal ($S = 0$). Both normal and idle modes occur with equal probability. Thus

$$S = \begin{cases} X, & \text{with probability } 1/2, \\ 0, & \text{with probability } 1/2. \end{cases}$$

The receiver observes $Y = S + Z$, where the ambient noise $Z \sim \text{Unif}[-1, 1]$ is independent of $S$.

(a) Find and sketch the conditional pdf $f_{Y|M}(y|1)$ of the receiver observation $Y$ given that the system is in the normal mode.

(b) Find and sketch the conditional pdf $f_{Y|M}(y|0)$ of the receiver observation $Y$ given that the system is in the idle mode.

(c) Find the optimal decoder $\hat{M}_{\text{MAP}}(y)$ for deciding whether the system is normal or idle. Provide the answer in terms of intervals of $y$.

(d) Find the associated probability of error.

a) $f_{Y|M}(y|m=1) = f_{Y|S}(y|s=x) = f_{Z|S}(y-s|s=x) = f_Z(y-x)$

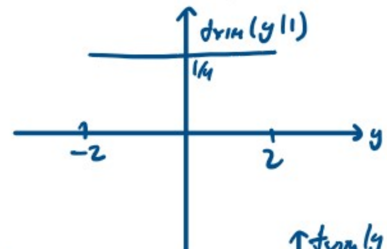
$x = \begin{cases} +1 & \mathbb{P}=1/2 \\ -1 & \mathbb{P}=1/2 \end{cases}$, $Y = X+Z$

$f_Z(y-x) = \tfrac{1}{2}f_Z(y-1) + \tfrac{1}{2}f_Z(y+1)$ for $y = \begin{cases} z+1 & \mathbb{P}=1/2 \to 0\le y\le 2 \\ z-1 & \mathbb{P}=1/2 \to -2\le y\le 0 \end{cases}$ $\because Z \sim U[-1,1]$

For $0\le y\le 2$: $f_Z(y-x) = \tfrac{1}{2}f_Z(y-1) = \tfrac{1}{2}(\tfrac{1}{2}) = \tfrac{1}{4}$

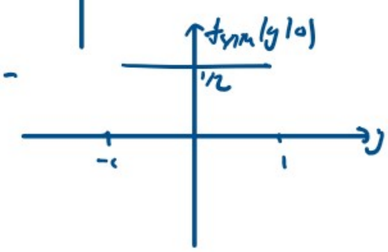
For $-2\le y\le 0$: $f_Z(y-x) = \tfrac{1}{2}f_Z(y+1) = \tfrac{1}{2}(\tfrac{1}{2}) = \tfrac{1}{4}$

$\therefore f_{Y|M}(y|m=1) = \begin{cases} 1/4, & -2\le y\le 2 \\ 0, & \text{o/w} \end{cases}$




b) $f_{Y|M}(y|m=0) = f_{Y|M}(y|s=0) = f_Z(y-s|s=0) = f_Z(y)$ $\because Y=Z$

$\therefore f_{Y|M}(y|m=0) = \begin{cases} 1/2, & -1\le y\le 1 \\ 0, & \text{o/w} \end{cases}$




c) $\because$ diff b/w idle & normal mode = diff values of $f_{Y|M}(y|m)$ for $y = \dots$

$\hookrightarrow M = \begin{cases} 1, & \mathbb{P}=1/2 \\ 0, & \mathbb{P}=1/2 \end{cases}$ $\therefore \hat{m}_{MAP}(y) = \begin{cases} 1, & \text{o/w} \\ 0, & -1\le y\le 1 \end{cases}$

d) $P_e = 1-[\mathbb{P}(-1\le y|m=0) + \mathbb{P}(y\le 1|m=0) + \mathbb{P}(-2\le y|m=1) + \mathbb{P}(y\le 2|m=1)]$

$= 1-[\mathbb{P}(-1\le y\le 1|m=0) + \mathbb{P}(-2\le y\le 2|m=1)]$

$= 1-[\tfrac{1}{2}+\tfrac{1}{4}]$

$= \tfrac{1}{4}$

5. *Optical communication channel.* Let the signal input to an optical channel be:

$$X = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ 10 & \text{with probability } \frac{1}{2}. \end{cases}$$

The conditional pmf of the output of the channel $Y|\{X = 1\} \sim \text{Poisson}(1)$, i.e., Poisson with intensity $\lambda = 1$ and $Y|\{X = 10\} \sim \text{Poisson}(10)$.

Show that the MAP rule reduces to:

$$\hat{X}_{\text{MAP}}(y) = \begin{cases} 1, & y < y^* \\ 10, & \text{otherwise.} \end{cases}$$

Find $y^*$ and the corresponding probability of error.

For $p_{X|Y}(x|y) = \dfrac{p_{Y|X}(y|x)p_X(x)}{\sum_{x\in X} p_{Y|X}(y|x)p_X(x)}$

$\hookrightarrow p_{X|Y}(1|y) = \dfrac{e^{-1}/y!}{e^{-1}/y! + \frac{e^{-10}10^y}{y!}}$

$= \dfrac{e^{-1}}{e^{-1}+e^{-10}10^y}$

$\hookrightarrow p_{X|Y}(10|y) = \dfrac{e^{-10}10^y}{e^{-1}+e^{-10}10^y}$

$\therefore$ MAP $\hat{x}(y) = 1$ if $p_{X|Y}(1|y) > p_{X|Y}(10|y) \to e^{-1} > e^{-10}10^y \to y < \frac{9}{\ln 10}$

$\hookrightarrow p_{X|Y}(10|y) = \dfrac{e^{-10}10^y}{e^{-1}+e^{-10}10^y}$

$\therefore$ MAP $D(y)=1$ if $p_{X|Y}(1|y) > p_{X|Y}(10|y) \rightarrow e^{-1} > e^{-10}10^y \rightarrow y < \frac{9}{\log_e 10}$

$$\therefore D(y) = \begin{cases} 1, & y < \frac{9}{\log_e 10} \\ 10, & \text{o/w} \end{cases}$$

$$\begin{aligned} \mathbb{P}_{Error} &= P_{X|Y}\left(x=1 \mid y > \tfrac{9}{\log_e 10}\right) + P_{X|Y}\left(x=10 \mid y < \tfrac{9}{\log_e 10}\right) \\ &= P_{X|Y}\left(Y > \tfrac{9}{\log_e 10} \mid x=10\right) P(x=10) + P_{Y|X}\left(y < \tfrac{9}{\log_e 10} \mid x=1\right) p(x=1) \\ &= 1/2 \left[p\left(pois(1) > \tfrac{9}{\log_e 10}\right)\right] p(x=0) + 1/2 \left[p\left(pois(10) < \tfrac{9}{\log_e 10}\right)\right] p(x=1) \\ &= 0.0147 \end{aligned}$$

6. *Shopping cart.* The number $\Theta$ of shopping carts in a store is uniformly distributed between 101 and 200. Cart are sequentially numbered between 1 and $\Theta$. You enter the store, observe the number $X$ on the first cart you encounter, assumed uniformly distributed over the range $1, \ldots, \Theta$, and use this information to estimate $\Theta$. Find the MAP estimator of $\Theta$ given $x$.

$\Theta \sim \mathrm{Unif}[101, 200] \quad \mathbb{P}_\Theta(\theta) = \frac{1}{100}$ for $101 \le \theta \le 200$

$X \mid \Theta = \theta \sim \mathrm{Unif}[1, \theta] \quad \mathbb{P}_{X|\Theta}(x|\theta) = \frac{1}{\theta}$ for $1 \le x \le \theta$

$$\begin{aligned} \mathbb{P}_{\Theta|X}(\theta|x) &= \frac{\mathbb{P}_{X|\Theta}(x|\theta)\,\mathbb{P}_\Theta(\theta)}{\mathbb{P}_X(x)} \\ &= \frac{\mathbb{P}_{X|\Theta}(x|\theta)\,\mathbb{P}_\Theta(\theta)}{\sum_{\theta'=101}^{200} \mathbb{P}_{X|\Theta}(x|\theta')\cdot\mathbb{P}_\Theta(\theta')} \rightarrow \begin{array}{l} 1 \le x \le \theta \\ 101 \le \theta \le 200 \end{array} \end{aligned}$$

Case 1: $x < 101$

$\mathbb{P}_{\Theta|X}(\theta|x) = \dfrac{\frac{1}{\theta}}{C(101)} \rightarrow C(x) = \sum_{\theta=x}^{200} \frac{1}{\theta}$

Case 2: $x \geq 101$

when $\theta < x$: $\mathbb{P}_{\Theta|X}(\theta|x) = 0$

when $\theta \geq x$: $\mathbb{P}_{\Theta|X}(\theta|x) = \dfrac{\frac{1}{\theta}}{C(x)}$

$$\therefore \mathbb{P}_{\Theta|X}(\theta|x) = \begin{cases} 0, & \theta < x \\ \frac{1/\theta}{C(x)}, & \theta \geq x \end{cases}$$

$\hat{\Theta}_{MAP} = x$