Orthogonal Subspaces, Complement and Relations

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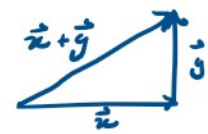
2.7 Orthogonal vectors & subspaces

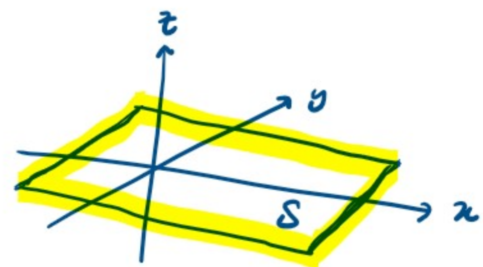
Product Properties

1. $\vec{x}\cdot\vec{y} = \vec{x}^T\vec{y} = [x_1 \cdots x_n]\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$
2. $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$
3. $\langle ax + by, z \rangle = a\langle x, z\rangle + b\langle y, z\rangle$
4. $\langle x, Ay \rangle = \langle A^T x, y \rangle$
5. $\langle \vec{x}, \vec{x} \rangle = \|x\|_2^2 \longrightarrow$ 2-norm
6. $|\vec{x}\cdot\vec{y}| \le \|\vec{x}\|\,\|\vec{y}\| \longrightarrow$ Cauchy-Schwartz
7. $\vec{x}\cdot\vec{y} = \|\vec{x}\|\|y\|\cos\theta$



Orthogonality

- $\vec{x}$ & $\vec{y}$ are $\perp$ orthogonal if $\vec{x}\cdot\vec{y} = 0$



Orthogonal Subspaces

- let S_1 = xy-plane in $\mathbb{R}^3$ = span$\{e_1, \dots, e_2\}$
  S_2 = z-plane = span$\{e_3\}$
- let $\vec{u} \in S_1$ & $\vec{v} \in S_2$ $\therefore \langle \vec{u}, \vec{v} \rangle = 0 \because \vec{u} \perp \vec{v}$
- S_1 & S_2 are orthogonal if:
  - $\forall \vec{u} \in S_1, \forall \vec{v} \in S_2 \longrightarrow \langle \vec{u}, \vec{v} \rangle = 0 \therefore S_1 \perp S_2$

Orthogonality of two subspaces:

- $\therefore$ Two subspaces are orthogonal if their basis vectors are orthogonal
  - ↳ let $B = \{b_1, \dots b_n\} \rightarrow$ basis of S_1 & $C = \{c_1, \dots c_n\} \longrightarrow$ basis of S_2

    $S_1 \perp S_2$ if $\langle b_j, c_j \rangle = \langle B, C \rangle = 0$
  - ↳ $B^T C = \begin{bmatrix} b_1^T c_1 & b_1^T c_2 & \cdots & b_1^T c_K \\ \vdots & \vdots & & \vdots \\ b_n^T c_1 & b_n^T c_2 & \cdots & b_n^T c_K \end{bmatrix}$ if $B^T C = 0$, $S_1 \perp S_2$
- $B = C^\perp$, $B^\perp = C$, $(B^\perp)^\perp = B$
- $N(A) = R(A^T)^\perp$, $N(A^T) = R(A)^\perp$

- **Transpose properties:**
  - $(A^T\vec{x})\cdot\vec{y} = \vec{x}\cdot(A\vec{y})$
  - $(AB)^T = B^T A^T$

ex. $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 0 & 1 \\ 2 & 5 & 1 & 2 \end{bmatrix}$, $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$N(A) = \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, $R(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \right\}$

$N(A^T) = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$, $R(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

$\rightarrow \therefore$ $N(A) \perp R(A^T)$, $N(A^T) \perp R(A)$