Orthogonality

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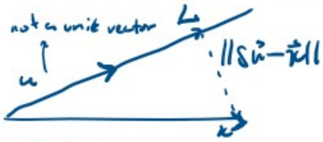
1.1 III Orthogonality

$L \rightarrow$ line span$\{u\}$



→ projection of vector x onto $L = \text{span}\{u\} = \{su\}$ $s \in \mathbb{R}$

is defined to be the vector in L that is closest to x

→ $P_u x$ or Px

Find $P_u x$:

→ $P_u x \in \text{span}\{u\}$ $P_u x = su$ for some $s \in \mathbb{R}$

→ Find the correct s, s minimizes the distance b/w x & $P_u x$

The square distance between x and $P_u x$ is: $\|x - P_u x\|_2^2 = \|x - su\|_2^2 = f(x)$ (shortest distance)

→ Minimize $d(s)$ to find $P_u x$: $d(s) = \|x - su\|_2^2$

$$\begin{aligned} &= \langle x - su, x - su\rangle \\ &= \langle x, x\rangle - \langle x, su\rangle - \langle su, x\rangle + \langle su, su\rangle \\ &= \|x\|^2 - s\langle x, u\rangle - s\langle u, x\rangle + s^2\|x\|^2 \\ &= \|x\|^2 - 2s\langle u, x\rangle + s^2\|u\|^2 \\ &= s^2\|x\|^2 - 2s\langle u, x\rangle + \|x\|^2 \end{aligned}$$

→ quadratic eqn in s

→ Differentiate in s to find min.

$$d'(s) = 2s\|u\|^2 - 2\langle u, x\rangle = 0$$

$$s^* = \frac{\langle u, x\rangle}{\|u\|^2} \qquad \therefore P_u x = s^* u = \underbrace{\frac{\langle u, x\rangle}{\|u\|^2}}_{\text{scalar}} \underbrace{u}_{\text{vector}}$$

= projection of x onto line spanned by vector u

Represent $P_u x$:

① $P_u x = \left\langle \frac{u}{\|u\|}, x \right\rangle \frac{u}{\|u\|}$

② $P_u x = u\frac{\langle u, x\rangle}{\|u\|^2} = \frac{u^T x}{\|u\|^2} = uu^T x \frac{1}{\|u\|^2} = \frac{uu^T}{\|u\|^2} x$ $\quad (\langle u, x\rangle = u^T x)$

uu^T: u (3×1), u^T (1×3) → uu^T (3×3)

$P_u x = (\text{orthogonal projector} \xrightarrow{} \text{matrix}) x$

→ Matrix $P_u = \frac{uu^T}{\|u\|^2}$ → orthogonal projection onto span$\{u\}$

ex. $u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $P_u = ?$

$\|u\|^2 = 1^2 + (-1)^2 = 2$ $\quad uu^T = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}[1, -1, 0] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$P_u = \frac{uu^T}{\|u\|^2} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

ex. $x = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \rightarrow P_u x = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Properties of P:

① $P(Px) = Px$ for all x → $P^2 = P$

② $P^T = P$

Consequences of these properties

a) $\langle y, Pz\rangle = \langle Py, z\rangle$

b) $\langle y, Pz\rangle = \langle y, P^2 z\rangle = \langle Py, Pz\rangle = \langle Py, z\rangle$

c) $R(P_u) = L = \text{span}\{u\}$

d) $N(P) = [R(P^T)]^\perp = [R(P)]^\perp = L^\perp$

c) $R(P_u) = L = \text{span}\{u\}$

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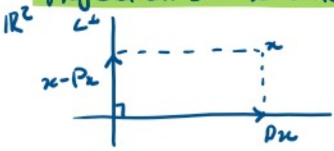
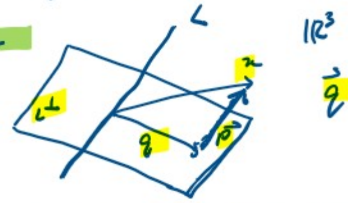
Projection of $\vec{x}$ onto L





$\vec{q} = \vec{x} - \vec{p} \longrightarrow \vec{p} = P_x$ (proj. $\vec{x}$ onto L)

$q = x - Px = (I - P)x$, $Q = (I - P) \rightarrow$ matrix that projects onto $L^\perp$

ex. projection of $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ onto orthogonal complement of $L = \text{span}\{u\}$ where $u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$Q = I - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Qx = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

1.2 Orthogonal Projection Matrix (Summary)

- P is an orthogonal Matrix if:
  1. $P^2 = P$
  2. $P^T = P$
- $P_u = \frac{uu^T}{\|u\|^2}$ $\longrightarrow$ orthogonal proj. matrix
- let $Q = I - P$
  1. $Q \rightarrow$ orthogonal proj. matrix
  2. $P + Q = I$, $PQ = QP = 0$
  3. P proj. orthogonally onto $R(P)$
  4. Q proj. orthogonally onto $N(P) = R(P)^\perp$

Properties of Q

1. a) $Q = Q^2$
   $Q^2 = I - P = Q$

   b) $Q = Q^T$
   $Q^T = (I - P)^T = I - P = Q$

2. $Q = I - P \rightarrow P + Q = I$
   $QP = 0$

ex. find closest vector in $R(P)$ to x

$\rightarrow$ need to find y such that $\|Py - x\|^2$ is as small as possible

$\|Py - x\|^2 = \|P(y - x) - Qx\|^2 = \langle (P(y-x) - Qx), (P(y-x) - Qx) \rangle$ ↗ project y & x onto image of P ($R(P)$)

$x = Px + Qx$ $\quad$ $PQ = QP = 0 \longrightarrow = \|P(y - x)\|^2 + \|Qx\|^2$

$\rightarrow$ this minimizes if $y = x$ $\therefore$ Px is the closest vector in $R(P)$ to x

ex. Q projects orthogonally on $R(Q)$. $\because R(P)^\perp = N(P)$ it remains to show that $R(Q) = N(P)$

$x \in R(Q) \longrightarrow Qx = x$

$\rightarrow$ let $x \in R(Q)$ for some y $x = Qy$, $Qx = Q(Qy) = Q^2 y = Qy = x$

$\therefore x \in R(Q) \rightarrow Qx = x \longrightarrow (I - P)x = x \longrightarrow Px = 0 \rightarrow x \in N(P)$