Fourier Series

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Fourier Series

– Undamped frequency: $\omega_0 = \frac{2\pi}{T_0} \longrightarrow T_0$ = fundamental period

Fourier Series of periodic signal $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$$

– Fourier coefficient: $X_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) e^{-jk\omega_0 t} dt$

– X_0 = DC value avg. of $x(t)$

Parseval's Power Relation

– Power P_x of periodic signal $x(t)$ of fundamental period T_0:

$$P_x = \underbrace{\frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt}_{\text{time domain}} = \underbrace{\sum_{k=-\infty}^{\infty} |X_k|^2}_{\text{frequency domain}}$$

Trigonometric Representations

– The trigonometric fourier series uses sinusoids rather than complex exponentials as basis functions

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega_0 t + \theta_k)$$
$$= c_0 + 2\sum_{k=1}^{\infty} [c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)]$$

– dc-component: $X_0 = c_0$, k^{th} harmonic $= \{2|X_k|\cos(k\omega_0 t + \theta_k)\}$ for $k = 1, 2, \ldots$

$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\omega_0 t) dt \longrightarrow Re(k)$, even component of $x(t)$

$d_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\omega_0 t) dt \longrightarrow -Im(k)$, odd component of $x(t)$

$X_k = |X_k| e^{j\theta_k} \longrightarrow |X_k| = \sqrt{c_k^2 + d_k^2}$

$\theta_k = -\tan^{-1}\left[\frac{d_k}{c_k}\right] = \angle X_k$

linear time invariant

LTI System Frequency Response

– If input $x(t)$ w/ impulse response $h(t)$, steady-state response is:

$$y(t) = X_0|H(j0)| + 2\sum_{k=1}^{\infty} |X_k||H(jk\omega_0)| \cos(k\omega_0 t + \angle X_k + \angle H(jk\omega_0))$$

– frequency response of the system at $k\omega_0$

$$H(jk\omega_0) = |H(jk\omega_0)| e^{t\angle H(jk\omega_0)} = \int_0^{\infty} h(\tau) e^{-jk\omega_0 \tau} d\tau = H(s)\Big|_{s=jk\omega_0}$$

ex. $x_1(t) = e^{j\omega t}$, $x_2(t) = e^{-j\omega t} \longrightarrow \boxed{h(t) \text{ System } H(s)} \longrightarrow y_1(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j(\omega t + \angle H(j\omega))}$

$y_2(t) = H(-j\omega)e^{-j\omega t} = H(j\omega)e^{-j\omega t}$

Even/Odd Decomposition

– If Fourier coefficients of aperiodic signal $x(t)$ are $\{X_k\}$, then the fourier coefficients of $x(-t)$ are $\{X_{-k}\}$

– Even $x(t)$: Fourier coeffs. X_k are real.
Trig. Fourier Series:

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} X_k \cos(k\omega_0 t)$$

– Odd $x(t)$: Fourier coeffs. X_k are imaginary
Trig. Fourier Series:

$$x(t) = 2\sum_{k=1}^{\infty} j X_k \sin(k\omega_0 t)$$

– Fourier coeffs: $X_k = X_{ek} + X_{ok}$

$\hookrightarrow X_{ek} = 0.5[X_k + X_{-k}]$
$X_{ok} = 0.5[X_k - X_{-k}]$

Operations of Periodic Signals

– Addition $z(t) = \alpha x(t) + \beta y(t)$

– Same ω_0
$\longrightarrow$ Fourier Coeff. : $Z_k = \alpha x(t) + \beta y(t)$

– diff. ω_0
– If $x(t)$ has period T_1 & $y(t)$ has period T_2 s.t $\frac{T_2}{T_1} = \frac{N}{M}$, then $z(t)$ has period $T_0 = MT_2 = NT_1$
$\longrightarrow$ Fourier Coeff. : $Z_k = \alpha X_{k/N} + \beta Y_{k/M}$

ex. $\left.\begin{array}{l} x(t) = \cos 2\pi t \longrightarrow \omega_1 = 2\pi, T_1 = 1 \\ y(t) = \sin 3\pi t \longrightarrow \omega_2 = 3\pi, T_2 = 2/3 \end{array}\right] \frac{T_2}{T_1} = \frac{2}{3} = \frac{N}{M} \quad \therefore T_0 = 2, \omega_0 = \pi$

$X_k = \begin{cases} 1/2, & k = \pm 1 \\ 0, & \text{otherwise} \end{cases}$, $Y_k = \begin{cases} \mp 1/2 & k = \pm 1 \\ 0 & \text{, otherwise} \end{cases}$

$z(t) = 4\cos 2\pi t + 5\sin 3\pi t$, $z_0 = 0$, $z_{\pm 1} = 0$, $z_{\pm 2} = 4(1/2) = 2$, $z_{\pm 3} = 5(\mp\frac{1}{2})j = \mp 5/2 j$

- **Product** $z(t) = x(t)y(t)$
  - Fourier coefficients are the convolution sum of fourier coeffs. of $x(t)$ & $y(t)$:

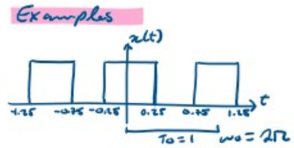
    $$Z_K = \sum_m X_m Y_{K-m}$$

FS Basic Properties (c.f. Slide 4.5)

Table 4.1 Basic Properties of Fourier Series

Basic Properties of Fourier Series		
	Time Domain	Frequency Domain
Signals and constants	$x(t)$, $y(t)$ periodic with period T_0, α, β	X_k, Y_k
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X_k + \beta Y_k$
Parseval's power relation	$P_x = \frac{1}{T_0}\int_{T_0} \lvert x(t)\rvert^2 dt$	$P_x = \sum_k \lvert X_k\rvert^2$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 X_k$
Integration	$\int_{-\infty}^{t} x(t')dt'$ only if $X_0 = 0$	$\frac{X_k}{jk\omega_0}$ $k \neq 0$
Time shifting	$x(t-\alpha)$	$e^{-jk\omega_0\alpha}X_k$
Frequency shifting	$e^{jM\omega_0 t}x(t)$	X_{k-M}
Symmetry	$x(t)$ real	$\lvert X_k\rvert = \lvert X_{-k}\rvert$ even function of k $\angle X_k = -\angle X_{-k}$ odd function of k
Convolution in time	$z(t) = [x*y](t)$	$Z_k = X_k Y_k$

Examples



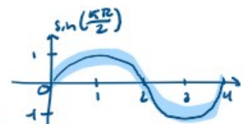
- **Complex Exponential Fourier Series**

$$X_K = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{jK\omega_0 t}\,dt$$

$$X_0 = \frac{1}{1}\int_{-0.25}^{0.25} 2\,dt = 1 \longrightarrow K=0$$

$$X_K = \int_{-0.25}^{0.25} 2e^{-jK\omega_0 t} = \frac{2e^{-jK\omega_0 t}}{-jK\omega_0}\Bigg|_{-0.25}^{0.25}$$

$$= \sin\left(\frac{K}{2}\right) \longrightarrow K \neq 0$$



$$\begin{cases} 0 & \text{if even} \\ (-1)\left(\frac{K-1}{2}\right) & \text{if odd} \end{cases}$$

- **Trigonometric Fourier Series**
  - **even** Function

$$c_K = \int_{-0.25}^{0.25} 2\cos(K\omega_0 t) = \frac{2\sin(K\omega_0 t)}{K\omega_0}\Bigg|_{-0.25}^{0.25} = \frac{\sin\left(\frac{K\pi}{2}\right)}{\left(\frac{K\pi}{2}\right)} = X_K$$

  - **odd** Function

$$d_K = \int_{-0.25}^{0.25} 2\sin(K\omega_0 t)\,dt = \frac{-2\cos(K\omega_0 t)}{K\omega_0}\Bigg|_{-0.25}^{0.25} = 0 \quad \therefore \text{ function is even}$$

Fourier Series Coefficient from Laplace Transform

- Fourier coefficients of Laplace Transform:

$$x_1(t) = x(t)\left[u(t-t_0) - u(t-t_0-T_0)\right] \longrightarrow \text{signal}$$

$$X_K = \frac{1}{T_0}\mathcal{L}\left[x_1(t)\right]\Big|_{s=jK\Omega} \longrightarrow \omega_0 = \frac{2\pi}{T_0}$$

ex. for pulse train: $x_1(t) = 2\left[u(t) - u(t-0.25) + u(t-0.75) - u(t-1)\right]$

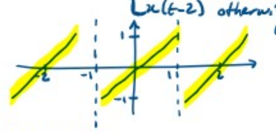
$$X_1(s) = 2/s\left[1 - e^{-1/4 s} - e^{-3/4 s} - e^{-s}\right]$$

$$X_K = X_1(jK2\pi) = \frac{2}{jK2\pi}\left[1 - e^{-\frac{jK\pi}{2}} + e^{-jK\left(\frac{3\pi}{2}\right)} - 1\right]$$

$$= \frac{2\sin(K\pi/2)}{K\pi}$$

Fourier Series of Triangle Wave (MT Q!)

ex. $x(t) = \begin{cases} t & \text{for } -1 < t \leq 1 \\ x(t-2) & \text{otherwise} \end{cases}$ $\quad \omega_0 = \frac{2\pi}{T_0} = \pi$



Method A

$$X_K = \frac{1}{T_0}\int_{-1}^{1} te^{-jK\omega_0 t}\,dt \longrightarrow X_0 = 0 \text{ (by inspection - avg. of graph)}$$

$$= \frac{1}{2}\int_{-1}^{1} te^{-jK\pi t}\,dt \quad \text{IBP: let } u = t,\ du = dt,\ dv = e^{-jK\pi t},\ v = \frac{e^{-jK\pi t}}{-jK\pi}$$

$$= \frac{1}{2}\left[\underbrace{\frac{te^{-jK\pi t}}{-jK\pi}\Bigg|_{-1}^{1}}_{=\frac{j}{K\pi}\left(e^{-jK\pi} - (-1)e^{jK\pi}\right)} - \underbrace{\int_{-1}^{1}\frac{e^{-jK\pi t}}{-jK\pi}\,dt}_{=0 \quad K \in \mathbb{Z}}\right]$$

$\therefore$ For $K \neq 0$, $X_K = \frac{j(-1)^K}{K\pi}$

Method B

$$x_1(t) = t\left[\underset{e^{s}/s}{u(t+1)} - \underset{e^{-s}/s}{u(t-1)}\right]$$

$$\mathcal{L}\{x_1(t)\} = \frac{d}{ds}\left[\frac{e^{s}}{s}\right] + \frac{d}{ds}\left[\frac{e^{s}}{s}\right] = \frac{1}{s^2}\left[e^{s}(1-s) - e^{-s}(s+1)\right]$$

$$X_K = \frac{\mathcal{L}\{x_1(t)\}}{T}\Big|_{s=jK\omega_0} = \frac{1}{-2(K\pi)^2}\left[e^{jK\pi}(1-jK\pi) - e^{-jK\pi}(jK\pi+1)\right]$$

$\qquad e^{s}/s \qquad\qquad e^{-s}/s$

$\mathcal{L}\{x_1(t)\} = \frac{d}{ds}\left[\frac{e^{s}}{s}\right] + \frac{d}{ds}\left[\frac{e^{s}}{s}\right] = \frac{1}{s^2}\left[e^{s}(1-s) - e^{-s}(s+1)\right]$

$X_k = \frac{\mathcal{L}\{x_1(t)\}}{T_0}\Big|_{s=jk\omega_0} = \frac{1}{-2(k\pi)^2}\left[e^{jk\pi}(1-jk\pi) - e^{-jk\pi}(jk\pi+1)\right]$

$= \frac{(-1)^2}{-2(k\pi)^2}(-2jk\pi) = \frac{(-1)^k j}{k\pi}$

Example (MT Q!)

- For $x(t) = 4\cos\left(\frac{6\pi}{7}t\right) + \cos\left(\frac{3\pi}{5}t - \frac{\pi}{2}\right)$, Find ω_0 & complex exponential fourier coeffs. $\{X_k\}$

$\omega_0 = GCF(\omega_1, \omega_2)$: $\left.\begin{matrix}\omega_1 = 6\pi/7 & T_1 = 7/3 \\ \omega_2 = 3\pi/5 & T_2 = 10/3\end{matrix}\right]$ GFC $\rightarrow$ $T_0 = \frac{70}{3}$, $\omega_0 = 3\pi/35$

$x(t) = 4\cos\left(\frac{6\pi t}{7}\right) + \sin\left(\frac{3\pi t}{5}\right)$

exp. form: $\underbrace{4/2\left[e^{\frac{j6\pi t}{7}} + e^{\frac{-j6\pi t}{7}}\right]}_{k \quad \omega_1 = 10\omega_0} + \underbrace{\frac{1}{2j}\left[e^{\frac{j3\pi t}{5}} - e^{\frac{-j3\pi t}{5}}\right]}_{\omega_2 = 7\omega_0}$

$X_k = \begin{cases} \pm\frac{1}{2j} = \mp\frac{j}{2} & \text{for } k = \pm 7 \\ 2 & \text{for } k = \pm 10 \\ 0 & \text{otherwise} \end{cases}$

☆ webwork 4

Problem Set 4: Problem 1

Previous Problem | Problem List | Next Problem

For each of the signals given in the table below, indicate whether it is periodic or not. For the periodic signals, write the following seven coefficients in a comma seperated list as: $X_0, X_1, X_2, X_3, X_4, X_5, X_6$ and enter 0 for each coefficient that you find to be zero. If it is not possible to calculate the Fourier Series coefficients, enter NA.

	Signal	Periodic/Aperiodic	Fourier Series Coefficients
1	$12 + 10cos(10\pi t) + 4cos(30t + \frac{\pi}{5})$	?	
2	$[14 + cos(2\pi t)]sin(10\pi t + \frac{\pi}{8})$	?	
3	$2 + sin(3t + \frac{\pi}{6}) + 8cos(5t) + 14cos(3t) + 13sin(6t)$	?	

① $\omega_1 = 10\pi$ $\omega_2 = 30$, $\longrightarrow$ Aperiodic $\because$ No rational common period

② $\omega_1 = 2\pi$, $\omega_2 = 10\pi$, $T_1 = 1$, $T_2 = 1/5$

Fundamental Period: $T_0 = LCM(1, 1/5) = 1$
Fundamental Frequency: $\omega_0 = 2\pi$

$14\sin(10\pi t + \pi/8) + \cos(2\pi t)\sin(10\pi t + \pi/8) \longrightarrow \sin(\omega t) = \cos(\omega t - \pi/2)$

$= 14\sin(10\pi t + \pi/8) + \cos(2\pi t)\cos(10\pi t - 3\pi/8) \longrightarrow \cos(a)\cos(b) = 1/2(\cos(a+b) + \cos(a-b))$

$= 14\sin(10\pi t + \pi/8) + 1/2[\cos(12\pi t - 3\pi/8) + \cos(-8\pi t + 3\pi/8)] \longrightarrow x(t) = A\cos(\omega t + \theta) = A\frac{e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}}{2}$ OR: $x(t) = A\sin(\omega t + \theta) = A\frac{e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}}{2j}$

$= \underbrace{7(e^{j10\pi t}e^{j\pi/8} + e^{-j10\pi t}e^{-j\pi/8}}_{X_5 \,\because\, jk\omega_0 t \,\rightarrow\, \omega_0 = 2\pi,\, k=5} + \underbrace{1/4(e^{j12\pi t}e^{\frac{-j3\pi}{8}} + e^{j8\pi t}e^{-j\pi/8}}_{X_6 \,\because\, jk\omega_0 t \,\rightarrow\, \omega_0 = 2\pi,\, k=6} + \underbrace{1/4(e^{-j8\pi t}e^{j3\pi/4} + e^{j8\pi t}e^{-j3\pi/8}}_{X_4 \,\because\, jk\omega_0 t \,\rightarrow\, \omega_0 = 2\pi,\, K=-4}$

$\longrightarrow$ By inspection $t=0$: $X_5 = \frac{14}{2j}e^{j\pi/8}$, $X_6 = 1/4\, e^{\frac{-j3\pi}{8}}$, $X_4 = 1/4\, e^{-j3\pi/8}$

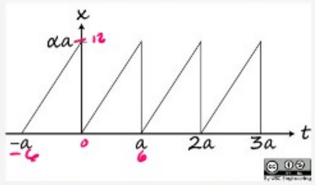
$X_0 = X_1 = X_2 = X_3 = 0$

Homework Sets
Problem Set 4
Problem 3
User Settings
Grades
Problems
Problem 1
Problem 2
Problem 3
Problem 4
Problem 5
Problem 6
Problem 7

Problem Set 4: Problem 3

Previous Problem | Problem List | Next Problem

A periodic signal, $x(t)$ is given in the figure below, where $a = 6$, and $\alpha = 2$.



$T_0 = 6$
$\omega_0 = \frac{\pi}{3}$

a) Find an equation for $x_c(t)$, the signal that describes one cycle of $x(t)$, in terms of the unit step function $u(t)$.

$x_c(t) =$ $2t\,[u(t) - u(t-6)]$ (slope)

$-e^{-6s}\mathcal{L}\{2(t-6)\}$

b) Find the Laplace transform, $X_c(s)$ of the signal in part **a**.

$X_c(s) =$ $u_c(t)g(t) \rightarrow e^{-cs}\mathcal{L}\{g(t+c)\}$ $\therefore$ $\frac{2}{s^2} - e^{-6s}\frac{2}{s^2} - e^{-6s}\frac{12}{s}$

c) Calculate the Fourier Series coefficients of the signal $x(t)$, X_k for $k \neq 0$ using the Laplace transform from part **b**.

$X_k =$ let $s = jk\pi/3$ $\longrightarrow$ $X_k = \frac{1/6 \cdot 36j}{k\pi 2}$

d) Is it possible to find the Fourier Series coefficient, X_0 using the Laplace transform method? ? NO $\because$ divide by 0

e) Compute the Fourier Series coefficient, X_0, using the integral definition.

$X_0 =$ $X_0 = \frac{1}{T_0}\int_0^T x_c(t)\,dt \longrightarrow \frac{1}{6}\int_0^6 2t[u(t)-u(t-6)]\,dt = 6$

Part d will only be marked correct if part c is correct.

Courses
Homework Sets
Problem Set 4
Problem 6
User Settings
Grades
Problems
Problem 1
Problem 2

Problem Set 4: Problem 6

Previous Problem | Problem List | Next Problem

The transfer function of an LTI system is given by:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+10}{s^2+5s+8}$$

Given the input $x(t) = 7.5 + cos(t + \frac{\pi}{5})$, use the eigenfunction property of the LTI system to find the steady-state output.

$y_{ss}(t) =$

$H(s) = \frac{s+10}{s^2+5s+8} \longrightarrow s = j\omega \quad H(\omega) = \frac{j\omega + 10}{-\omega^2 + 5j\omega + 8}$

$x(t) = 7.5 + \cos(t + \pi/5) \longrightarrow \omega = 1$

$H(1) = \frac{-75+43j}{-74} = 1.1682743 \angle -0.520581$

@ $\omega = 0$, $H_0 = \frac{10}{8}$

$\therefore$ $y_{ss}(t) = 7.5 \cdot \frac{10}{8} + 1.1682743\cos(t + \frac{\pi}{5} - 0.520581)$