Chapter 9 Time-Varying Fields and Maxwell's Equations

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9.1 Faraday's Law
- Current produced by magnetism

9.1.1 Point & Integral Form
- Time-varying B field produces emf → establish current
- Voltage produced by conductors moving in B-field

$$emf = -\frac{d\varphi}{dt} \rightarrow \frac{d\varphi}{dt} = \text{rate of change of flux}$$

- Lenz's Law
  - emf produces opposing flux → closed loop
  - For N-turns: $emf = -N\frac{d\varphi}{dt}$

$$emf = \oint E \cdot dL \longrightarrow \text{Voltage about closed path}$$
$$= -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S}$$

- For stationary path: $emf = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$
- Stoke's theorem → integral point form:

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{S} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

- Differential form: $(\nabla \times \vec{E}) \cdot d\vec{S} = -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

9.1.2 EMF from time-varying Fields
- Increasing B-field in cylindrical region $\rho < b$:

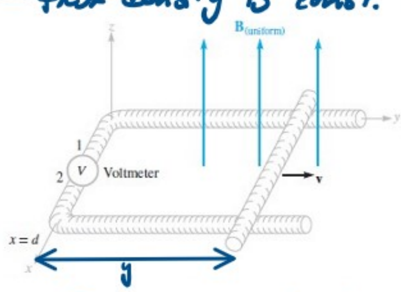
$\vec{B} = B_0 e^{kt}\hat{a}_z =$

$emf = 2\pi a E_\varphi = -kB_0 e^{kt}\pi a$

↳ emf around closed path

- E field intensity @ ρt.: $\vec{E} = -\frac{1}{2}kB_0 e^{kt}\rho\,\hat{a}_\varphi$

9.1.3 Motional EMF
- flux density B const. & ⊥ to plane of closed path



- Flux through surface: $\varphi = Byd \rightarrow y = \text{bar pos.}$

$$emf = -\frac{d\varphi}{dt} = -B\frac{dy}{dt}d = -Bvd$$

- Force on charge Q @ velocity $\vec{v}$ in $\vec{B}$ field

$$\vec{F} = Q\vec{v} \times \vec{B}$$

- Sliding conducting bar → E-field $\vec{E}_m = \vec{v} \times \vec{B}$

$$\varphi = \iint_S \vec{B} \cdot d\vec{S}$$

- If moving conductor off rails, e^- forced at opp. end until static field balance field induced by bar motion
- For conductor moving in uniform B-field, motional E field $\vec{E}_m = \vec{v} \times \vec{B}$
  - ↳ resulting $emf = \oint \vec{E} \cdot d\vec{L} = \oint \vec{E}_m \cdot d\vec{L} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{L}$
- If mag. flux density changing w/ time:

$$emf = \oint \vec{E} \cdot d\vec{L} = \underbrace{-\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}}_{\text{transformer emf}} + \underbrace{\oint (\vec{v} \times \vec{B}) \cdot d\vec{L}}_{\text{motional emf}}$$

9.2 Displacement Current
- Maxwell's eq. in differential form: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

transformer emf   motional emf

9.2 Displacement Current

- Maxwell's eqn. in differential form: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
  - ↳ time changing $\vec{B}$ field produces $\vec{E}$ field

9.21 Ampere's Law for time-varying Field

- Ampere circuital law (steady $\vec{B}$ field):
  $\nabla \times \vec{H} = \vec{J}$
  - ↳ Point form: $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ → current density
- Displacement current density: $\nabla \times \vec{H} = \vec{J} + \vec{J}_d \longrightarrow \vec{J}_d = \frac{\partial \vec{D}}{\partial t}$
- Conduction Current Density: $\vec{J} = \sigma \vec{E}$
  - ↳ Motion of charge
- Convection current density: $\vec{J} = \rho_v \vec{v}$
  - ↳ Motion of volume charge density
- In non-conducting medium ($\vec{J} = 0$): $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$

Figure 9.3 A filamentary conductor forms a loop connecting the two plates of a parallel-plate capacitor. A time-varying magnetic field inside the closed path produces an emf of $V_0 \cos \omega t$ around the closed path. The conduction current I is equal to the displacement current between the capacitor plates.

→ filament loop & parallel plate capacitor
- ↳ varying $\vec{B}$ field sinusoidally
- Current in loop: $I = -\omega C V_0 \sin\omega t = -\omega \frac{\varepsilon S}{d} V_0 \sin\omega t$
- Ampere's Law about closed path k: $\oint_k \vec{H} \cdot d\vec{L} = I_k$
  - ↳ I_k → current through surfaces w/ perimeter k
- b/w capacitor plates: $D = \varepsilon E = \varepsilon \left(\frac{V_0}{d} \cos\omega t \right)$
  $I_d = \frac{\partial D}{\partial t} S = -\omega \frac{\varepsilon S}{d} V_0 \sin\omega t$
  - ↳ I_d → displacement current b/w capacitor plates
- Conduction Current:
  $I_c = \int_S \vec{J}_c \cdot d\vec{S}$

- Total displacement current crossing surface:
  $I_d = \int_S \vec{J}_d \cdot d\vec{S} = \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$
  - ↳ Ampere's circuital law (time-varying): $\int_S (\nabla \times \vec{H}) \cdot d\vec{S} = \int_S \vec{J} \cdot d\vec{S} + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$
  - ↳ Apply Stoke's Theorem: $\oint \vec{H} \cdot d\vec{L} = I + I_d = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$

9.3 Maxwell's Equations – Point Form

- Time varying fields:

① $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

② $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

- Non-time varying:

③ $\nabla \cdot \vec{D} = \rho_v$ → charge density is a source of electric flux lines

④ $\nabla \cdot \vec{B} = 0$ → Magnetic charges/poles DNE, flux in closed loops, doesn't diverge

- Auxiliary Eqns:
  - $\vec{D}$ & $\vec{E}$: $\vec{D} = \varepsilon \vec{E}$
  - $\vec{B}$ & $\vec{H}$: $\vec{B} = \mu \vec{H}$
  - Conduction Current Density: $\vec{J} = \sigma \vec{E}$
  - Convection Current Density & Volume charge density: $\vec{J} = \rho_v \vec{v}$
- With Polarization & B-fields (less "nice" materials)
  $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$
  $\vec{B} = \mu_0 (\vec{H} + \vec{M})$

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- Linear Materials:
  $\vec{P} = \chi_e \varepsilon_0 \vec{E}$
  $\vec{M} = \chi_m \vec{H}$
- Lorentz Force Equation (Point Form)
  $\vec{f} = \rho_v (\vec{E} + \vec{v} \times \vec{B})$

9.4 Maxwell's Equations – Integral Form

- Faraday's Law – Integrate over surface + Stoke's Theorem:

① $\oint \vec{E} \cdot d\vec{L} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$

- Ampere's Circuital Law

② $\oint \vec{H} \cdot d\vec{L} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$

- Gauss's Law for E & $\vec{B}$ Fields – Integrate through Volume + Divergence Theorem

③ $\oint_S \vec{D} \cdot d\vec{S} = \int_{vol} \rho_v \, dv$

④ $\oint_S \vec{B} \cdot d\vec{S} = 0$

- Use 4 integral equations to find boundary conditions on $\vec{B}, \vec{D}, \vec{H}, \vec{E}$
- B/w 2 physical media:
  - Tangential: $\vec{E}$-field: $\vec{E}_{t1} = \vec{E}_{t2}$
    $\vec{H}$-field: $\vec{H}_{t1} = \vec{H}_{t2}$
  - Normal: $\vec{D}_{N1} - \vec{D}_{N2} = \rho_S$
    $\vec{B}_{N1} = \vec{B}_{N2}$

- Perfect Conductor:
  ↳ $\sigma \to \infty$, $\vec{J} \to 0$
  ↳ $\vec{E} = 0$
  ↳ $\vec{H} = 0$
  - Current carried on conductor surface as surface current K
  - If region 2 = perfect conductor:
    $E_{t1} = 0$
    $H_{t1} = K$ $(H_{t1} = K \times \hat{a}_N) \to \hat{a}_N$ = outward normal to conductor surface
    $D_{N1} = \rho_S$
    $B_{N1} = 0$

9.5 Retarded Potentials

- Retarded Potentials → Time varying Potentials
- Fundamental Fields:
  ↳ gradient (static): $\vec{E} = -\nabla V$
  ↳ curl (dc): $\vec{B} = \nabla \times \vec{A}$
  $\vec{N} = -\frac{\partial \vec{A}}{\partial t} \to \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$

- electromagnetic disturbances travel at: $v = \frac{1}{\sqrt{\mu\varepsilon}}$
  ↳ homogenous medium

- Scalar Electric potential (static): $V = \int_{vol} \frac{[\rho_v]}{4\pi\varepsilon R} dv \to \nabla^2 V = -\frac{\rho_v}{\varepsilon}$
  ↳ $[\rho_v] = e^{-r} \cos[\omega(t - R/v)]$
  ↳ ρ_v indicates that every t → replaced by retarded time

vol $4\pi\varepsilon R$

↳ $[\rho_v] = e^{-r}\cos[\omega(t - R/v)]$

↳ ρ_v indicates that every t → replaced by retarded time $t' = t - R/v$

$\therefore [\rho_v] = e^{-r}\cos[\omega(t - R/v)]$

– Vector Magnetic potential (dc): $\vec{A} = \int_{vol} \frac{\mu \vec{J}}{4\pi R} dv \longrightarrow \nabla^2 \vec{A} = -\mu \vec{J}$