Singular Value Decomposition

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Singular Value Decomposition

- Generalize diagonalization to arbitrary (non-square) matrix
  - $A^k = SD^kS^{-1}$, $k \geq 0$
  - $\|A\|_{op} = \max\{|\lambda_j|\}$

Formula

$$\underset{m \times n}{A} = \underset{\substack{m \times m \\ \text{unitary}}}{U}\ \underset{m \times n \text{ diagonal}}{\Sigma}\ \underset{n \times n \text{ unitary}}{V^*}$$

ex. $\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}} \underset{V^*}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

How to find U, Σ, V given A?

(P1) let $A^*Av = \lambda v$ for $v \neq 0$

$\therefore v^* A^* A v = v^* \lambda v = \lambda v^* v$

$\|Av\|^2 = \lambda \|v\|^2 \longrightarrow \lambda = 0$

(P2) $\therefore A^*A$ & AA^* = same eigen values,

$AA^*(Av) = A(A^*Av) = A\lambda v = \lambda Av$

$\therefore \lambda, Av$ = eigen pair for AA^*

(P3) A^*A & AA^* = Hermitian & Unitarily Diagonalizable

$A^*A = V\Sigma_1^2V^*$, $AA^* = U\Sigma_2^2U^*$

- non-zero entries of Σ_1^2 & Σ_2^2 = same

$\Sigma_1^2 = \begin{bmatrix} \sigma_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n^2 \end{bmatrix}$, $\Sigma_2^2 = \begin{bmatrix} \sigma_2^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix}$

let $\sigma_j = \sqrt{\sigma_j^2}$ & set $\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \end{bmatrix}$

Singular Value Decomposition (SVD)

$A = U\Sigma V^*$

ex. let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$ Find SVD of A

$A^*A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \longrightarrow \lambda_1 = 2,\ \lambda_2 = 3,\ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$

$\lambda_1 = 2$, $u_1 = [1/\sqrt{2}\ \ 1/\sqrt{2}\ \ 0]^T$

[illegible]

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

– Find U: $\quad AA^* = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{array}{l} \lambda_1 = 2, \ u_1 = [1/\sqrt{2} \ \ 1/\sqrt{2} \ \ 0]^T \\ \lambda_2 = 3, \ u_2 = [1/\sqrt{3} \ \ -1/\sqrt{3} \ \ 1/\sqrt{3}]^T \\ \lambda_3 = 0, \ u_3 = [-1/\sqrt{6} \ \ 1/\sqrt{6} \ \ 2/\sqrt{6}]^T \end{array}$

$$\therefore U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{8} & 2/\sqrt{6} \end{bmatrix} = [u_1 \ u_2 \ u_3]$$

$$\therefore A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U \Sigma V^*$$

1. Nullspace of A

As in the example we obtain:

$$x \in N(A) \iff x \perp v_1^*, \dots, v_r^* \iff x \perp \operatorname{span}\{v_1^*, \dots, v_r^*\}$$

Since $\{v_1, \dots, v_n\}$ is ONB : $\boxed{x \in N(A) \iff x \in \operatorname{span}\{v_{r+1}^*, \dots, v_n^*\}}$

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = U \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{so } N(A) = \{\vec{0}\}$$

$$A^* = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}}_{\text{SVD of } A^*}, \quad \text{so } N(A^*) = \operatorname{span}\left\{ \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}.$$

2. Range of A

$$R(A) = \{Ax : x \in \mathbb{R}^n\} = \{U \Sigma \underbrace{V^* x}_{=y} : x \in \mathbb{R}^n\}$$

$$= \{U \Sigma y \ : y \in \mathbb{R}^n\}$$

$$= \left\{ [u_1, u_2 \dots u_m] \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \ 0 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} : y \in \mathbb{R}^n \right\}$$

$$= \left\{ [u_1, u_2 \dots u_m] \begin{bmatrix} \sigma_1 y_1 \\ \sigma_2 y_2 \\ \vdots \\ \sigma_r y_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} : y \in \mathbb{R}^n \right\}$$

$$= \left\{ \sum_{j=1}^{r} \underbrace{\sigma_j y_j}_{\text{scalar}} \underbrace{u_j}_{\text{vector}} : y \in \mathbb{R}^n \right\}$$

$$\Rightarrow \boxed{R(A) = \operatorname{span}\{u_1, \dots, u_r\}}$$