Assignment 6

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1. *Balls and bins revisited.* There are 10 bins numbered $1, 2, \ldots, 10$. n balls are thrown into the 10 bins. For each ball, the probability that it falls into bin i is $\frac{1}{10}$ for $i = 1, 2, \ldots, 10$. Different balls are thrown independently of each other. Let Y be the number of balls in bin 1. Let Z be the total number of balls in bins $6, 7, 8, 9, 10$.

   (a) Find $\mathsf{P}(Y = y|Z = z)$. Please specify the range of y, z.

   (b) Given $Z = z$, find the estimator $\hat{y}(z)$ that minimizes conditional MSE $\mathsf{E}[(\hat{y}(z) - Y)^2|Z = z]$.

   (c) Find the conditional MSE $\mathsf{E}[(\hat{y}(z) - Y)^2|Z = z]$ for the estimator in part (b).

   (d) Find the linear LMS estimator of Y given $Z = z$.

   (e) Find $\mathsf{E}[Z]$ and $\mathrm{Var}[Z]$.

   (f) Find $\mathsf{E}[Y]$ and $\mathrm{Var}[Y]$.

   (g) Find $\mathrm{Cov}(Y, Z)$.

   Hint: Use the law of total expectation. Try to first determine the conditional pmf.

   (h) Find the linear LMS estimator of Z given Y.

   (i) Find the corresponding (overall) MSE for the estimator in part (h).

From Assignment 4:
$Y \sim Binom(n, 1/10)$
$Z \sim Binom(n, 1/2)$
$Y|Z=z \sim Binom(n-z, 1/2)$
$Z|Y=y \sim Binom(n-y, 5/9)$

a) $\mathbb{P}(Y=y|Z=z) = \binom{n-z}{y}\left(\frac{1}{5}\right)^y\left(\frac{4}{5}\right)^{n-z-y}$

for $z = 0, 1, \ldots, n$ & $y = 0, 1, \ldots, n-z$

b) $\hat{y}(z)_{LMS} = \mathbb{E}[Y|Z=z] = \sum_y y\, p_{Y|Z}(y|z) = \frac{n-z}{5}$

c) $MSE_{LMS}(z) = Var(Y|Z) = \mathbb{E}[Y^2|Z] - \mathbb{E}[Y|Z]^2$
$= \sum_y y^2 p_{Y|Z}(y|z) - \left(\frac{n-z}{5}\right)^2 = \frac{(n-z)^2}{5} - \left(\frac{n-z}{5}\right)^2$

d) $\hat{Y}_{LMS} = \underset{\hat{y}=aZ+b}{\operatorname{argmin}}\left(\mathbb{E}[(\hat{y}-Y)^2]\right) = \frac{n}{10} - \frac{n-z}{5}$

e) $\mathbb{E}[Z] = \sum^n z(1/2) = \frac{n(n+1)}{4}$
$Var(Z) = \frac{\sum (z - \mathbb{E}[Z])^2}{n}$

f) $\mathbb{E}[Y] = \sum^n y(1/10) = \frac{n(n+1)}{20}$
$Var(Y) = \frac{\sum (y - \mathbb{E}[Y])}{n}$

g) $Cov(Y, Z) = \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z] = -\frac{n}{20}$
$\hookrightarrow \mathbb{E}[YZ] = \mathbb{E}[\mathbb{E}[YZ|Y]] = \mathbb{E}[Y\mathbb{E}[Z|Y]]$

h) $LLMS = \frac{Cov(Y,Z)}{Var(Y)}(Y - \mathbb{E}[Y]) + \mathbb{E}[Z]$
$= \frac{n}{2} - \frac{5}{9}\left(Y - \frac{n}{10}\right)$

i) $MSE = Var(Z) - \frac{Cov(Z,Y)^2}{Var(Y)} = \frac{2n}{9}$

2. *Estimation vs. detection.* Let the signal

$$X = \begin{cases} +1, & \text{with probability } \frac{1}{2} \\ -1, & \text{with probability } \frac{1}{2}, \end{cases}$$

and the noise $Z \sim \text{Unif}[-2, 2]$ be independent random variables. Their sum $Y = X + Z$ is observed.

(a) Find the LMS estimate of X given Y

(b) Find the (overall) MSE for the estimator you find in part (a).

(c) Now suppose we use a decoder to decide whether $X = +1$ or $X = -1$ so that the probability of error is minimized. Find the MAP decoder and its probability of error. Compare the MAP decoder's MSE to the least MSE.

$X = \begin{cases} 1 \ ; & wp\ 1/2 \\ -1 \ ; & wp\ 1/2 \end{cases}$ $\quad Z \sim Unif[-2,2] \rightarrow Y = X + Z$

$\hookrightarrow f_Z(z) = 1/4$

$\mathbb{P}(X=1) = \mathbb{P}(X=-1) = 1/2$

$f(y|x) = f_Z(y-z) = 1/4$

$f(x|y) = \dfrac{f(y|x)\,\mathbb{P}_x(x)}{f_Y(y)} = \dfrac{1/4 \cdot 1/2}{1/4\, 1/2 \cdot 1/4 \cdot 1/2} = \dfrac{1}{2}$

a) $\hat{X}_{LMS}(y) = \mathbb{E}[X|Y] = \sum_{x=-1}^{1} x(1/2) = 0$

b) $MSE_{LMS}(y) = \mathbb{E}[X^2|Y] - \mathbb{E}[X|Y]^2 = \sum_{x=-1}^{1} x^2(1/2) - \left[\sum_{x=-1}^{1} x(1/2)\right]^2 = 1 - 0^2 = 1$

c) $\hat{X}_{MAP}(y) = \underset{x}{\text{argmax}}\, f(x|y) = \dfrac{f(y|x)\,\mathbb{P}_x(x)}{f_Y(y)}$

$\hookrightarrow f_Z(y-x) = \begin{cases} 1/4 \ ; & -2 \le y-x \le 2 \\ 0 \ ; & o/w \end{cases}$

$\hookrightarrow \underset{x}{\text{argmax}}\, f_Z(y-x) = \begin{cases} -1 \ , & y < 0 \\ +1 \ ; & y \ge 0 \end{cases}$

$\mathbb{P}_e = \mathbb{P}(y \ge 0 | X=-1)\,\mathbb{P}(X=-1) + \mathbb{P}(y<0|X=1)\,\mathbb{P}(X=1)$

$= \mathbb{P}(-1+z \ge 0) + \mathbb{P}(1+z<0) = \mathbb{P}(z \ge 1) + \mathbb{P}(z < -1) = \frac{1}{4} + \frac{1}{4} = 1/2$

$MSE_{MAP}(y) = \mathbb{E}[(X - \hat{X}_{MAP})^2|Y] = \mathbb{E}[X^2|Y] - 2\hat{X}_{MAP}(y)\,\mathbb{E}[X|Y] + \hat{X}_{MAP}(y)^2$

$= 1 - 2(-1)(0) + (-1)^2 = 2 \rightarrow y < 0$

$= 1 - 2(1)(0) + (1)^2 = 2 \rightarrow y \ge 0$ $\Big\}$ $MSE_{MAP} = 2$

$\therefore$ MSE_{MAP} is twice MSE_{LMS}

3. *Stick breaking.* Given a stick of length 1, break it into two pieces at a location chosen uniform at random. Denote the breaking location by X, then $X \sim \text{Unif}[0, 1]$. Keep the piece corresponding to the interval $[X, 1]$. Break it again into two pieces at a location chosen uniform at random. Denote the second breaking location by Y, then $Y|\{X = x\} \sim \text{Unif}[x, 1]$.

(a) Find the estimator of X given Y that minimizes the MSE $\mathsf{E}[(\hat{X} - X)^2]$.

(b) Find the conditional MSE given $Y = y$ for the estimator you find in part (a).

(c) Find the covariance $\text{Cov}(X, Y)$.

(d) Find the linear LMS estimator of X given Y.

(e) Find the MSE for the estimator you find in part (d)

$X \sim Unif[0,1]$, $Y|X=x \sim Unif[x,1]$

a) Estimator of X given Y minimizing MSE $\mathbb{E}[(\hat{X}-X)^2]$

$\hat{X}(y) = \mathbb{E}[X|Y=y]$

$f_{X|Y}(x|y) = \dfrac{f(y|x)\,f(x)}{f(y)} = \dfrac{\left(\frac{1}{1-x}\right)\left(\frac{1}{1-0}\right)}{\int_{-\infty}^{\infty} f_x(x)\, f_{Y|X}(y|x=x)\,dx} \rightarrow 0 \le x \le y \le 1$

$= \dfrac{\frac{1}{1-x}}{\int_0^y \frac{1}{1-x}\,dx} = \dfrac{\frac{1}{1-x}}{-\ln(1-y)}$

$\therefore \hat{X}_{LMS} = \mathbb{E}[X|Y=y] = \int x\, f(x|y)\,dx = \int_0^y \dfrac{1}{-\ln(y-1)}\,dx$

$= \dfrac{y}{\ln(y-1)} + 1$

$$= \frac{y}{\ln(y-1)} + 1$$

b) conditional MSE given $Y=y \rightarrow \mathbb{E}[(x-\hat{x})^2 | Y=y]$

$\mathbb{E}[X^2 | Y=y] = \int_0^y \frac{x^2 \cdot 1/x}{-\ln(y-1)} dx = \frac{-y^2}{2\ln(y-1)}$

$\hookrightarrow -2\hat{x}(y)\,\mathbb{E}[X|Y=y] = -2(\hat{x}_{LMS}(y))^2 = -2\left(\frac{y}{-\ln(y-1)}\right)^2$

$(\hat{x}_{LMS}(y))^2 = \left(\frac{y}{-\ln(y-1)}\right)^2$

$\therefore MSE = -\frac{y}{2\ln(y-1)} - 2\left(\frac{y}{-\ln(y-1)}\right)^2 + \left(\frac{y}{-\ln(y-1)}\right)^2$

c) $Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$\mathbb{E}[XY] = \mathbb{E}_Y[\mathbb{E}[XY|X]] = E_Y[X\,\mathbb{E}[Y|X]]$

$= E_Y[\frac{x+1}{2} X] \rightarrow \frac{x^2+x}{2}$

$= 1/2\,[\mathbb{E}_Y(X) + \mathbb{E}_Y(X^2)]$

$= 1/2\,(1/2 + 1/3)$

$= 5/12$

d) $Var(Y) = \mathbb{E}[Var(Y|X)] + Var(\mathbb{E}[Y|X])$

$= \mathbb{E}[\frac{(1-x)^2}{12}] + Var(\frac{1+x}{2})$

$= 1/36 + 1/4 \cdot 1/12$

$= 7/144$

e) $\rho_{xy} = \frac{1/24}{\sqrt{1/12 \cdot 7/144}}$

$MSE = 1/12\left(1 - \frac{\frac{1}{24}}{\sqrt{1/12 \cdot 7/144}}\right) = 1/21$

4. *Estimation based on a function of the observation.* Let Θ be a positive random variable, with known mean μ and variance σ^2, to be estimated on the basis of a measurement X of the form $X = \sqrt{\Theta}W$. We assume that W is independent of Θ with zero mean, unit variance, and known fourth moment $\mathsf{E}[W^4]$. Thus, the conditional mean and variance of X given Θ are 0 and Θ, respectively, so we are essentially trying to estimate the variance of X given an observed value.

   (a) Find the linear LMS estimator of Θ based on $X = x$.

   (b) Let $Y = X^2$. Find the linear LMS estimator of Θ based on $Y = y$.

$Cov(\Theta, X) = \mathbb{E}[\Theta^{3/2} W] - \mathbb{E}[\Theta]\mathbb{E}[X] = 0$

a) $\Theta_{LMS} \rightarrow \hat{\Theta} = \mu$

$\hookrightarrow$ does not make use of observation $X = x$

b) For $Y = X^2 = \Theta W^2$ & $\hat{\Theta} = aY + b$

$\mathbb{E}[Y] = \mathbb{E}[\Theta W^2] = \mathbb{E}[\Theta]\mathbb{E}[W^2] = \mu$

$\mathbb{E}[\Theta Y] = \mathbb{E}[\Theta^2 W^2] = \mathbb{E}[\Theta^2]\mathbb{E}[W^2] = \sigma^2 + \mu^2$

$Cov(\Theta, Y) = \mathbb{E}[\Theta Y] - \mathbb{E}[\Theta]\mathbb{E}[Y] = (\sigma^2 + \mu^2) - \mu^2 = \sigma^2$

$Var(Y) = \mathbb{E}[\Theta^2 W^4] - \mathbb{E}[Y]^2 = (\sigma^2 + \mu^2)\mathbb{E}[W^4] - \mu^2$

$\Theta_{LMS}(\Theta | Y=y) \rightarrow \hat{\Theta} = \mu + \frac{\sigma^2}{(\sigma^2+\mu^2)\mathbb{E}[W^4] - \mu^2}(Y - \sigma^2)$

5. *Neural net.* Let $Y = X + Z$, where the signal $X \sim \text{Unif}[-1, 1]$ and noise $Z \sim N(0, 1)$ are independent. We want to estimate $\text{sgn}(X)$, where

$$\text{sgn}(x) = \begin{cases} -1 & x \le 0 \\ +1 & x > 0. \end{cases}$$

   (a) Find the function $g(y)$ that minimizes

$$\text{MSE} = \mathsf{E}[(\text{sgn}(X) - g(Y))^2]$$

$\{ +1 \quad x > 0.$

(a) Find the function $g(y)$ that minimizes

$$\text{MSE} = \mathsf{E}\left[(\text{sgn}(X) - g(Y))^2\right].$$

Express your answer in terms of the cumulative distribution function of $N(0,1)$

$$\Phi(z) \triangleq \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx.$$

(b) Plot $g(y)$ as a function of y.

$f_x(x) = \begin{cases} 1/2, & -1 \le x \le 1 \\ 0, & o/w \end{cases}$

$f_{Y|X}(y|x) \equiv f_Z(y-x) \rightarrow [Y|(X=x)] \sim N(x,1)$

$f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_x(x)\,dx = \int_{-1}^{1} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}}\right) \cdot 1/2\, dx$

$= 1/2 \left[\int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}} dx - \int_{1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}} dx\right]$

$= 1/2 \left[Q(y-1) - Q(y+1)\right]$

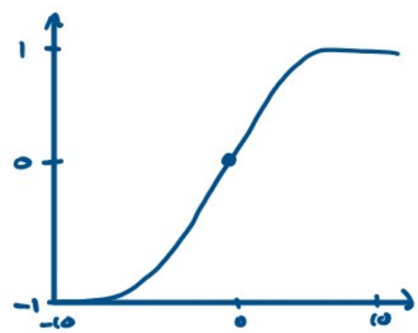
a) $g(y) = E[\text{sgn}(X)|Y=y] = \int_{-\infty}^{\infty} \text{sgn}(x) f(x|y)\,dx$

$= \int_{-1}^{1} \text{sgn}(x) \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}}}{Q(y-1)-Q(y+1)} dx = \frac{1}{Q(y-1)-Q(y+1)}\left[\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}}dx + \int_{-1}^{0}\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}} dx\right]$

$= \frac{1}{Q(y-1)-Q(y+1)}\left[Q(y-1) - Q(y) + Q(y+1) - Q(y)\right]$

$= \frac{Q(y-1)+Q(y+1)-2Q(y)}{Q(y-1)-Q(y+1)}$

b)



6. *Communication in Gaussian noise.* In a communication system, a transmitter wants to send some signal to a receiver over a noisy medium. Suppose the signal Θ is a Gaussian distributed random variable $\Theta \sim \mathcal{N}(0, \sigma_\Theta^2)$. The noise in the medium is modeled as a Gaussian distributed random variable $W \sim \mathcal{N}(0, \sigma_W^2)$ independent of the signal. The receiver observes $X = 2\Theta + W$.

(a) Find the estimator of Θ given X that minimizes the MSE $\mathsf{E}[(\hat{\Theta} - \Theta)^2]$.

(b) Find the MSE for the estimator you found in part (a).

(c) Find the linear LMS estimator of Θ given X.

(d) Find the MSE for the estimator you found in part (c).

(e) Find the LMS estimator of Θ^2 given X.

(f) Find the linear LMS estimator of Θ^2 given X.

a) estimator of Θ given $X \rightarrow$ MSE $E[(\hat\Theta - \Theta)^2]$

LLMS of gaussian noise = LMS

$\hat\theta_{LLMS} = \frac{Cov(X,\Theta)}{Var(X)}(X - E[X]) + E[\Theta]$

$Var(X) = Var(2\Theta + W)$

$Cov(X,\Theta) = E[X\Theta] - E[X]E[\Theta] = E[(2\Theta+W)\Theta] = E[2\Theta^2 + W\Theta]$

$E[\Theta] = 0$, $X = 2\Theta + W$

$E[X] = E[2\Theta + W] = 2\cancel{E[\Theta]} + \cancel{E[W]} = 0$

$E[W] = 0$,

$E[W^2] = Var(W) + 0 = \sigma_W^2$

$\mathbb{E}[w]=0$,
$\mathbb{E}[w^2] = Var(w) + 0 = \sigma_w^2$

$\hookrightarrow \mathbb{E}[\Theta X] = \mathbb{E}[2\Theta^2 + w\Theta] - \cancel{\mathbb{E}[\Theta]}\,\mathbb{E}[X]$
$= 2\mathbb{E}[\Theta^2] + \mathbb{E}[w]\mathbb{E}[\Theta] = 2\sigma_\Theta^2$

$\hookrightarrow \mathbb{E}[\Theta^2] = Var(\Theta) + \cancel{\mathbb{E}^2[\Theta]}$
$= \sigma_\Theta^2$

$\therefore Cov(X,\Theta) = 2\sigma_\Theta^2$

$Var(X) = \mathbb{E}[X^2] - \cancel{\mathbb{E}^2[X]}$

$\hookrightarrow X^2 = 4\Theta^2 + 4\Theta w + w^2$
$\mathbb{E}[X^2] = 4\mathbb{E}[\Theta^2] + 4\cancel{\mathbb{E}[\Theta]}\mathbb{E}[w] + \mathbb{E}[w^2]$
$= 4\sigma_\Theta^2 + \sigma_w^2$

$\therefore Var(X) = 4\sigma_\Theta^2 + \sigma_w^2$

$\therefore LLMS = \dfrac{2\sigma_\Theta^2}{4\sigma_\Theta^2+\sigma_w^2} X = LMS$

b) $MSE = Var(\Theta) - \dfrac{Cov(\Theta, X)^2}{Var(X)}$
$= \sigma_\Theta^2 - \dfrac{(\sigma_\Theta^2 X)^2}{4\sigma_\Theta^2 + \sigma_w^2}$

c) $\because LLMS = LMS$,
$\therefore LLMS = \dfrac{2\sigma_\Theta^2}{4\sigma_\Theta^2+\sigma_w^2} X$

d) $MSE = \sigma_\Theta^2 - \dfrac{(\sigma_\Theta^2 X)^2}{Var(X)}$

e) LMS of Θ^2 given X
$LMS = \mathbb{E}[\Theta^2 | X]$ $\left(\dfrac{Cov(X,\Theta)}{Var(X)}(X - \mathbb{E}[X]) + \mathbb{E}[\Theta],\ Var(\Theta) - \dfrac{Cov(X,\Theta)^2}{Var(X)} \right)$
$\hookrightarrow \Theta | X = x \sim N$

$\hookrightarrow \mathbb{E}[\Theta | X = x]$
$\hookrightarrow Var(\Theta|X) = \mathbb{E}[\Theta^2|X] - \mathbb{E}[\Theta|X]^2$

$\mathbb{E}[\Theta^2|X] = Var(\Theta|X) + \mathbb{E}[\Theta|X]^2$

$\hookrightarrow \mathbb{E}[\Theta|X=x] = \dfrac{\frac{x}{\sigma_w^2}}{1/\sigma_w^2 + 1/\sigma_\Theta^2} = \dfrac{x\sigma_\Theta^2}{\sigma_w^2 + \sigma_\Theta^2}$

$\hookrightarrow Var(\Theta|X=x) = \dfrac{1}{1/\sigma_\Theta^2 + 1/\sigma_w^2} = \dfrac{\sigma_\Theta^2\sigma_w^2}{\sigma_\Theta^2+\sigma_w^2}$

$\therefore LMS = \mathbb{E}[\Theta^2|X] = \dfrac{\sigma_\Theta^2\sigma_w^2}{\sigma_\Theta^2+\sigma_w^2} + \left(\dfrac{X\sigma_\Theta^2}{\sigma_w^2+\sigma_\Theta^2}\right)^2$

f) $LLMS = \dfrac{Cov(\Theta^2, X)}{Var(X)}(X - \mathbb{E}[X]) + \mathbb{E}[\Theta^2]$

$Cov(\Theta^2, X) = \mathbb{E}[\Theta^2 X] + \cancel{\mathbb{E}[\Theta^2]\mathbb{E}[X]}$
$= \mathbb{E}[\Theta^2(2\Theta + w)] = 2\mathbb{E}[\Theta^3] + \mathbb{E}[\Theta^2 w]$
$= 2\int_{-\infty}^{\infty} \Theta^3 f_\Theta(\Theta)\, d\Theta$
$= 0 \longrightarrow$ by symmetry

$\therefore LLMS = 0$