Stoke Theorem

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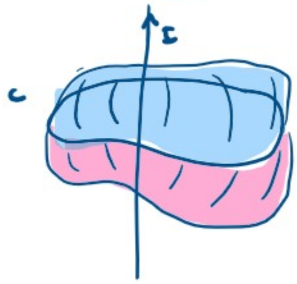
Stoke's Theorem:

$I_{enc} = \oint_c \vec{H} \cdot \vec{dl}$ for loop

For current: $I_{enc} = \iint_S \vec{J} \cdot \hat{n}\, dS$

— true for surface that "caps" w/ steady current
— "caps" = takes curve as its boundary



$$I_{top} - I_{bot} = \iint_{S_1+S_2} \vec{J} \cdot \hat{n}\, dS$$

$$= \iiint_R (\vec{\nabla} \cdot \vec{J})\, dV$$

$\overset{\shortparallel}{0}$ → continuity of current

$$I_{top}^{up} = I_{bot}^{up}$$

Continuity of current

$$-\frac{d}{dt} Q_{enc} = \oiint_S \vec{J} \cdot \hat{n}\, dS \quad \rightarrow \text{closed surface } S$$

$$-\frac{d}{dt} Q_{enc} = \iiint_R \rho_v\, dV = -\iiint \frac{d}{dt} \rho_v\, dV = \iiint_R \vec{\nabla} \cdot \vec{J} \cdot dV$$

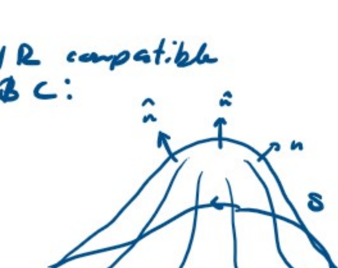
↳ region enclosed by S

$\therefore \boxed{\vec{\nabla} \cdot \vec{J} = -\frac{d}{dt} \rho_v}$ → continuity of current eqn.

$-\frac{d}{dt} \rho_v = 0 \longrightarrow \vec{\nabla} \cdot \vec{J} = 0$

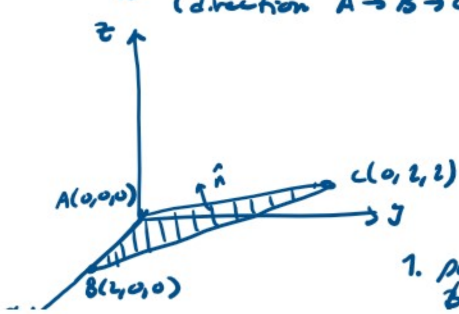
Theorem: Given loop C & capping surface S w/ RHR compatible orientations and for $\vec{G}$ non-singular S & C:

$$\boxed{\oint_c \vec{G} \cdot \vec{dl} = \iint_S (\vec{\nabla} \times \vec{G}) \cdot \hat{n}\, dS}$$



ex. $\vec{G} = \langle y, z, ax \rangle$, $a \rightarrow$ const.

Find $\oint_c \vec{G} \cdot \vec{dl}$ where $C = \overset{\triangle}{ABC}$ w/ $A = (0,0,0)$
(direction $A \rightarrow B \rightarrow C \rightarrow A$) $B = (2,0,0)$
$C = (0,2,2)$



1. parametrize line segments & integrate for each seg

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OR.

2. Stokes Theorem

- choose simple capping surface
- Find compatible orientation ($\hat{n} \to$ RHR)
- $\hat{n}$ is constant, triangle lays on plane $y = z$ $(0x + y - z = 0)$
  $\therefore \hat{n} = \pm \langle 0, 1, -1 \rangle$
- For RHR compatible, $\hat{n} = \langle 0, -1, 1 \rangle \longrightarrow \hat{n} = \dfrac{\langle 0,-1,1 \rangle}{\sqrt{2}}$
  $\because y \to$ pos, $z \to$ neg
- $\vec{\nabla} \times \vec{G} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & ax \end{vmatrix} = \langle -1, a, -1 \rangle$

$$\oint_C \vec{G} \cdot d\vec{L} = \iint_S (\vec{\nabla} \times \vec{G}) \cdot \hat{n} \, dS = \iint_S \left(\langle -1, -a, -1 \rangle \cdot \frac{\langle 0,-1,1 \rangle}{\sqrt{2}} \right) dS$$

$$= \frac{a-1}{\sqrt{2}} \iint_S dS = \frac{a-1}{\sqrt{2}} \text{Area}(\widehat{ABC}) = \frac{a-1}{\sqrt{2}} \cdot 2\sqrt{2}$$

$$= 2(a-1)$$

- For $a = 1 \rightarrow \oint_C \vec{G} \cdot d\vec{L} = 0 \longrightarrow$ not implying $\vec{G}$ = conservative

ex. $\iint_S (\vec{\nabla} \times \vec{G}) \cdot \hat{n} \, dS$, $S = x^2 + y^2 + z^2 = a$, $z \geq 0$ oriented upward

$G = \langle 3y, -2xz, x^2 - y^2 \rangle$

$\vec{\nabla} \times \vec{G} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y & -2xz & x^2 - y^2 \end{vmatrix} = \langle -2y + 2x, -2x, -2z - 3 \rangle$

- Instead of param. & finding flux, use Stokes Thm.

  $\iint_S (\vec{\nabla} \times \vec{G}) \cdot \hat{n} \, dS = \oint_C \vec{G} \cdot d\vec{L}$ $\quad$ $C \to$ loop capped by S (∂S)

- C satisfies surface eqn when $z = 0$
  $C \to$ circle $x^2 + y^2 = a^2$ $(z = 0)$
  $\hookrightarrow$ only look at parametrization of curve, not S.

- Parametrize C : $x = a\cos t$, $y = a\sin t$, $z = z = 0$

  bounds : $0 \leq t \leq 2\pi$
  $\to$ but need to take direction into account

$\dfrac{d\vec{L}}{dt} = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt} \right\rangle$
$= \langle -a\sin t, a\cos t, 0 \rangle$

$\oint \vec{G} \cdot d\vec{L} = \int^{2\pi} \underline{\langle 3a\sin t, 0, a^2(\cos^2 t - \sin^2 t) \rangle} \cdot \langle -a\sin t, a\cos t, 0 \rangle \, dt$

$= \langle -a\sin t, a\cos t, 0 \rangle$

$$\oint_C \vec{G}\cdot d\vec{c} = \int_0^{2\pi} \underbrace{\langle 3a\sin t, 0, a^2(\cos^2 t - \sin^2 t)\rangle}_{\vec{G}(\vec{c}(t))} \cdot \langle -a\sin t, a\cos t, 0\rangle \, dt$$

$$= \int_0^{2\pi} -3a\sin^2 t\, dt = -3a\int_0^{2\pi} \sin^2 t \, dt \overset{\text{half angle formula}}{=} -3\pi a^2$$

OR (Another Stokes way):

$$\iint_S (\vec{\nabla}\times\vec{G})\cdot\hat{n}\, dS = \oint_C \vec{G}\cdot d\vec{c} = \iint_D (\vec{\nabla}\times\vec{G})\cdot\hat{n}\, dS$$

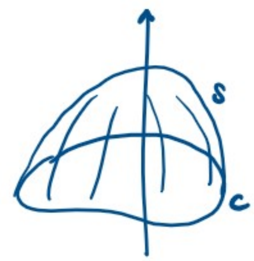
→ $\hat{n} = \hat{a}_z$ (RHR compatible)

$$\iint_S (\vec{\nabla}\times\vec{G})\cdot\hat{n}\, dS = \iint_D (\vec{\nabla}\times\vec{G})\cdot\hat{a}_z\, dS = \iint_D (-2z-3)\, dS$$

z-comp. of $\vec{\nabla}\times\vec{G}$ ∴ $=0$ ∵ $z=0$ on D

$$\text{Flux} = \iint_D (-3)\, dS = -3\iint dS = -3\pi a^2$$

Electromagnetics ex.



- loop C capped by S

$$\iint_S \vec{J}\cdot\hat{n}\, dS = I_{enc.} = \oint_C \vec{H}\cdot d\vec{c} \overset{\text{stokes thm}}{=} \iint_S (\vec{\nabla}\times\vec{H})\cdot\hat{n}\, dS \quad \text{where } C = \partial S$$

∵ works for any capping surfaces: $\vec{J} = \vec{\nabla}\times\vec{H}$

- If have $\vec{G} = \langle G_x, G_y, G_z\rangle$,

$$\vec{\nabla}\times\vec{G} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ G_x & G_y & G_z \end{vmatrix}$$

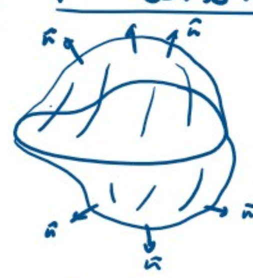
$$= \left\langle \frac{\partial}{\partial y}G_z - \frac{\partial}{\partial z}G_y, \frac{\partial}{\partial z}G_x - \frac{\partial}{\partial x}G_z, \frac{\partial}{\partial x}G_y - \frac{\partial}{\partial y}G_x \right\rangle$$

$(\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{G})) = \cdots = 0$ → divergence of curl $= 0$

∴ $\vec{J} = \vec{\nabla}\times\vec{H}$, $\quad \vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{H}) = 0$

$\boxed{\text{☆ Divergence: } \vec{\nabla}\cdot\vec{F} \quad \text{Curl: } \vec{\nabla}\times\vec{F}}$

- If have loop C & 2 capping surfaces (top, bot) S_1 S_2



$$\oiint_{S_{top}+S_{bot}} (\vec{\nabla}\times\vec{G})\cdot\hat{n}\, dS = \iiint_R \vec{\nabla}\cdot(\vec{\nabla}\times\vec{G})\, dV = 0$$

$$\iint_{S_{top}} (\vec{\nabla}\times\vec{G})\cdot\hat{n}_{top}\, dS = \iint_{S_{bot}} (\vec{\nabla}\times\vec{G})\cdot(-\hat{n}_{bot})\, dS$$

☆ Check formula sheet for curl in other coord. system