Operator Norm and Condition Number

April 12, 2020 12:27 AM

1.7 Condition Number

- A almost never singular $\longrightarrow$ close to being singular
- A^{-1} exists, but has large entries, $\therefore x = A^{-1}b$ is very sensitive to small changes in $\vec{b}$
  $\hookrightarrow$ less accurate $b + \Delta b \longrightarrow x' = A^{-1}(b + \Delta b)$

Check if linear system is well-conditioned

- Use ratio of largest to smallest stretch factor of $A \longrightarrow$ cond(A)
  - Largest stretch factor of A : $\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \|A\|_2$
  - Smallest stretch factor of A : $\min_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \frac{1}{\|A^{-1}\|}$
- Condition Number $= \frac{\text{largest stretch}}{\text{smallest stretch}}$

  $\therefore \text{cond}(A) = \|A\| \, \|A^{-1}\|$

  - Measures how well can solve
    $x' = A^{-1}(b + \Delta b) = x + \Delta x$
    $\hookrightarrow$ original soln : $x = A^{-1}b$
    $\hookrightarrow$ change in soln : $\Delta x = A^{-1}\Delta b$
  - Relative errors $\longrightarrow$ Bound $\frac{\|\Delta b\|}{b}$ & $\frac{\|\Delta x\|}{x}$
    $\frac{\|\Delta x\|}{\|x\|} \leq \text{cond}(A) \frac{\|\Delta b\|}{\|b\|}$

- For square matrix A cond$(A) = \|A\|_{op}\|A^{-1}\|_{op}$ if invertible, ∞ if not

  ① cond ≥ 1 for square matrix

  ② Want cond # $\longrightarrow 1$

- For diagonal matrix A: $\text{cond}(A) = \frac{\max\{|d_1| \cdots |d_n|\}}{\min\{|d_1| \cdots |d_n|\}}$

MATLAB:

```
cond(A)
```