LMS Estimation

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Estimation Criteria

- Close Estimates may be better
- Find $\hat{\Theta}$ that minimizes Mean Square Error
  $MSE \triangleq \mathbb{E}[(\hat{\Theta}-\Theta)^2]$

- LMS – Absence of Observations
  - Unknown Θ, prior $p(\Theta)$, Find $\hat{\Theta}$ minimizing $\mathbb{E}[(\Theta-\hat{\Theta})^2]$
    ↳ set func $g(\hat{\Theta}) = \mathbb{E}[\hat{\Theta}^2 - 2\Theta\hat{\Theta} + \Theta^2] = \hat{\Theta}^2 - 2\mathbb{E}[\Theta]\hat{\Theta} + \mathbb{E}[\Theta^2]$
    ↳ set $g'(\hat{\Theta}) = 2\hat{\Theta} - 2b = 0 \Rightarrow \hat{\Theta} = \mathbb{E}[\Theta]$

  $\therefore \hat{\Theta}_{LMS} = \mathbb{E}[\Theta]$
  $\hat{\Theta}_{MAP} = \arg\max_{\Theta} f_{\Theta}(\Theta)$
  - $\mathbb{E}[(\Theta-\hat{\Theta})^2] = Var(\Theta-\hat{\Theta}) + (\mathbb{E}[\Theta-\hat{\Theta}])^2$ $\hat{\Theta} \to$ const
    $= Var(\Theta) + (\mathbb{E}[\Theta]-\hat{\Theta})^2$

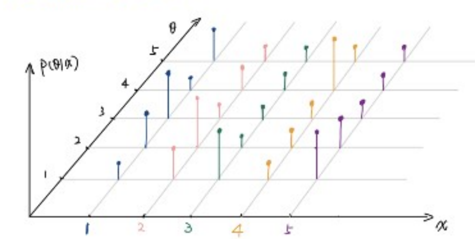
    $(\mathbb{E}[\Theta]-\hat{\Theta})^2 \geqslant 0 \quad \geqslant Var(\Theta)$
    ↳ Equal when $\hat{\Theta}_{LSE} = \mathbb{E}[\Theta]$
  - Corresponding MSE:
    $MSE = \mathbb{E}[(\Theta-\mathbb{E}[\Theta])^2] = Var(\Theta)$

- LMS – based on observation $X=x$
  - For observation $X=x$, find each $\hat{\Theta}(x)$ minimizing conditional MSE
    $\mathbb{E}[(\Theta-\hat{\Theta}(x))^2 \mid X=x] = Var(\Theta \mid X=x) + (\mathbb{E}[\Theta \mid X=x] - \hat{\Theta}(x))^2$
    $\geqslant Var(\Theta \mid X=x)$
    ↳ Equal when $\hat{\Theta}_{LMS}(x) = \mathbb{E}[\Theta \mid X=x]$
  - Conditional MSE:
    $MSE = Var(\Theta \mid X=x)$

$$\xrightarrow{\Theta} \boxed{p_{X|\Theta}} \xrightarrow{x} \boxed{\hat{\Theta}_{LMS}(x)} \xrightarrow{\hat{\Theta}}$$

  - LMS Estimator: $\hat{\Theta} = \hat{\Theta}_{LMS}(X) = \mathbb{E}[\Theta \mid X]$
  - MSE $= \mathbb{E}[(\Theta-\hat{\Theta})^2] = \sum_x p(x)\, \mathbb{E}[(\Theta-\hat{\Theta})^2 \mid X=x]$
    $= \sum_x p(x) Var(\Theta \mid X=x)$
    $= \mathbb{E}[Var(\Theta \mid X)]$

Illustration of LMS estimation.



$\hat{\theta}_{LMS}(x) = \mathbb{E}[\Theta \mid X=x]$

$MSE(x) = Var[\Theta \mid X=x]$

$\hat{\theta}_{MAP}(x) = \arg\max_{\theta} p(\theta|x)$

$\mathbb{P}(\hat{\Theta} \neq \Theta \mid X=x) = 1 - p_{\Theta|X}(\hat{\theta}_{MAP}(x) \mid x)$