Cubic Splines

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2.3 Cubic Spline

- Lagrange Interpolation = impossible w/ too many pts.
- Given data pts $(x_1, y_1) \cdots (x_n, y_n)$, fit data w/ piece-wise

$$f(x) = \begin{cases} p_1(x) & : x_1 \le x \le x_2 \\ p_2(x) & : x_2 \le x \le x_3 \\ \vdots & \quad \vdots \\ p_{n-1}(x) & : x_{n-1} \le x \le x_n \end{cases}$$

→ $f(x)$ is continuous at all points

→ $f(x)$ passes through all pts

Conditions of Smoothness

① $f(x)$ is continuous & $f(x_i) = y_i$ for $i = 1 \ldots n$

② 1st derivative $f'(x)$ is continuous

③ For x b/w data pts x_i, higher order derivatives $f''(x), f'''(x), \ldots$ exist & have left & right limits as x approaches x_i

- Cubic Spline shape makes $E[f] = \int_{x_1}^{x_n} (f''(x))^2 dx$ as small as possible
- $F(x)$ satisfies ①, ②, ③ & ⓐ, ⓑ, ⓒ

Conditions for F(x)

ⓐ In each interval $[x_i, x_{i+1}]$, $F(x)$ → cubic polynomial

- In each interval, coeffs A_i, B_i, C_i, D_i s.t. $F(x) = A_i x^3 + B_i x^2 + C_i x + D_i$

ⓑ $F''(x)$ is continuous

ⓒ When x is an endpoint (x_1 or x_n), $F''(x) = 0$

2.6 Impose Conditions of Smoothness

- $A_j, B_j, C_j, D_j \longrightarrow$ $4(n-1)$ unknowns (n = # data pts.)

∴ Find $4(n-1)$ equations

Condition ①: $f(x)$ is continuous

$$\left.\begin{aligned} p_j(x_j) &= y_j \\ p_j(x_{j+1}) &= y_{j+1} \end{aligned}\right\}$$

fitting data for $1 \le j \le n-1$

→ $2(n-1)$ equations

Condition ②: $f'(x)$ is continuous

$p_j'(x_{j+1}) = p'_{j+1}(x_{j+1})$ for $1 \le j \le n-2$

$\longrightarrow$ $(n-2)$ equations

Condition ③: $f''(x)$ exist

$p_j''(x_{j+1}) = p''_{j+1}(x_{j+1})$ for $1 \le j \le n-2$

$\longrightarrow$ $(n-2)$ equations

Condition ④: When x is an endpoint (x_1 or x_n), $F''(x)=0$

- Impose $f''(x_1) = f''(x_n) = 0$

$$\left.\begin{array}{l} p_1''(x_1)=0 \\ p_{n-1}''(x_n)=0 \end{array}\right\} \longrightarrow 2 \text{ equations}$$

Equations: $2(n-1) + (n-2) + (n-2) + 2 = 4n-4$ for $4(n-1)$ unknowns

Examples

ex. $n = 3$ (data pts),
$A_j, B_j, C_j, D_j \longrightarrow j = 1,2$ ($p(x)$ degree $j = n-1$)

- $4(n-1)$ unknowns $= 8$

C1
$$\left.\begin{array}{l} p_j(x_j) = y_j \\ p_j(x_{j+1}) = y_{j+1} \end{array}\right] \xrightarrow{4 \text{ eqn.}} \begin{array}{ll} p_1(x_1) = y_1 : & A_1x_1^3 + B_1x_1^2 + C_1x_1 + D_1 = y_1 \\ p_1(x_2) = y_2 : & A_1x_2^3 + B_1x_2^2 + C_1x_2 + D_1 = y_2 \\ p_2(x_2) = y_2 : & A_2x_2^3 + B_2x_2^2 + C_2x_2 + D_2 = y_2 \\ p_2(x_3) = y_3 : & A_2x_3^3 + B_2x_3^2 + C_2x_3 + D_2 = y_3 \end{array}$$

C2 $p_1'(x_2) = p_2'(x_2)$

For $p_j'(x) = 3A_jx^2 + 2B_jx + C_j$:

$3A_1x_2^2 + 2B_1x_2 + C_1 = 3A_2x_2^2 + 2B_2x_2 + C_2$

$\xrightarrow{1 \text{ eqn.}} 3A_1x_2^2 + 2B_1x_2 + C_1 - 3A_2x_2^2 + 2B_2x_2 + C_2 = 0$

C3 $p_1''(x_2) = p_2''(x_2)$

For $p''_j(x) = 6A_jx + 2B_j$

$6A_1x_2 + 2B_1 = 6A_2x_2 + 2B_2$

$\xrightarrow{1 \text{ eqn}} 6A_1x_2 + 2B_1 - 6A_2x_2 + B_2 = 0$

C4
$$\left.\begin{array}{l} p_1''(x_1)=0 \\ p_2''(x_3)=0 \end{array}\right] \xrightarrow{2 \text{ eqn.}} \begin{array}{l} 6A_1x_1 + 2B_1 = 0 \\ 6A_2x_3 + 2B_2 = 0 \end{array}$$

System: 8 eqn. 8 unknowns

$$\begin{bmatrix} x_1^3 & x_1^2 & x_1 & 1 & 0 & 0 & 0 & 0 \\ x_2^3 & x_2^2 & x_2 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$
\underset{\substack{S \\ (\text{known})}}{\begin{bmatrix}
x_1^3 & x_1^2 & x_1 & 1 & 0 & 0 & 0 & 0 \\
x_2^3 & x_2^2 & x_2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_2^3 & x_2^2 & x_2 & 1 \\
0 & 0 & 0 & 0 & x_3^3 & x_3^2 & x_3 & 1 \\
3x_2^2 & 2x_2 & 1 & 0 & -3x_2^2 & -2x_2 & -1 & 0 \\
6x_2 & 2 & 0 & 0 & -6x_2 & -2 & 0 & 0 \\
6x_1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6x_3 & 2 & 0 & 0
\end{bmatrix}}
\underset{\substack{a \\ (\text{unknown})}}{\begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \\ A_2 \\ B_2 \\ C_2 \\ D_2 \end{bmatrix}}
=
\underset{\substack{b \\ (\text{known})}}{\begin{bmatrix} y_1 \\ y_2 \\ y_2 \\ y_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}
$$

$\longrightarrow$ Solve $S\vec{a} = \vec{b}$ $\longrightarrow$ $\vec{a} = S^{-1}\vec{b}$ to find $p_1(x)$ & $p_2(x)$

Explicit Example

Data pts $(n=3)$: $(0,0), (1,3), (3,1)$

$x_1 = 0, \quad x_2 = 1, \quad x_3 = 3$
$y_1 = 0, \quad y_2 = 3, \quad y_3 = 1$

$\longrightarrow$ Plug into $S\vec{a} = \vec{b}$

$$
\underset{S}{\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 27 & 9 & 3 & 1 \\
3 & 2 & 1 & 0 & -3 & -2 & -1 & 0 \\
6 & 2 & 0 & 0 & -6 & -2 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 18 & 2 & 0 & 0
\end{bmatrix}}
\underset{\vec{a}}{\begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \\ A_2 \\ B_2 \\ C_2 \\ D_2 \end{bmatrix}}
=
\underset{\vec{b}}{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}
$$

$\longrightarrow$ solve $\vec{a} = S^{-1} b$

$$
\vec{a} = \begin{bmatrix} -2/3 \\ 0 \\ 11/3 \\ 0 \\ 1/3 \\ -3 \\ 20/3 \\ -1 \end{bmatrix} \longrightarrow f(x) = \begin{cases} p_1(x) = -\frac{2}{3}x^3 + \frac{11}{3}x \\ p_2(x) = \frac{1}{3}x^3 - 3x^2 + \frac{20}{3}x - 1 \end{cases}
$$

2.5 Efficient & Numerically stable method 2

$$p_j(x) = A_j(x - x_j)^3 + B_j(x - x_j)^2 + C_j(x - x_j) + D_j$$

$\longrightarrow$ $D_j = y_j$ $\longrightarrow$ $3(n-1)$ unknowns

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