Complex Vectorspace, Orthonormal Basis, Unitary Matrices

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2. Complex Vector Spaces

- $\mathbb{C}^n = \left\{ z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} : z_j \in \mathbb{C} \right\} \longrightarrow$ n-dim. complex vector space w/ scalars in $\mathbb{C}$

Complex inner product

- For $z, w \in \mathbb{C}^n$, complex inner product:

$$\langle w, z \rangle = \overline{w}_1 z_1 + \overline{w}_2 z_2 + \dots + \overline{w}_n z_n$$

- $\|z\|_2^2 = \langle \vec{z}, \vec{z} \rangle$

Properties: let $\delta \in \mathbb{C}$, $z, w \in \mathbb{C}^n$

1. $\langle w, z \rangle = \overline{\langle z, w \rangle}$
2. $\langle \delta w, z \rangle = \overline{\delta} \langle w, z \rangle$
3. $\langle w, z \rangle = \overline{w}^T z \longrightarrow \overline{w} = \begin{bmatrix} \overline{w}_1 \\ \vdots \\ \overline{w}_n \end{bmatrix}$
4. $\langle Az, w \rangle = \langle z, \overline{A}^T w \rangle = \langle z, A^* w \rangle$
   $\hookrightarrow A^* = \overline{A}^T \longrightarrow$ Adjoint of A, Conjugate & Transpose

MATLAB

- $A' \longrightarrow$ Adjoint of A
- i = sqrt(-1)
- $A.' \longrightarrow$ transpose of A

Orthonormal Basis

- In n-dim vector space W, any set of n lin. indep. vectors is a basis for W.
  ex. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

- Basis $\{q_1, \dots, q_n\}$ is orthonormal basis (ONB) if:

1. $\langle q_i, q_j \rangle = 0$ when $i \neq j$
2. $\|q_i\|_2 = 1$

- Basis $\{q_1 \dots q_n\}$ is an ONB for vector space V if

① It is a basis for V

② $\langle q_i, q_j \rangle = \delta_{ij} = \begin{cases} 1, & i=j \\ 0 & i \neq j \end{cases}$

ex. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ = ONB for $\mathbb{R}^2$

$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix} \right\}$ = ONB for $\mathbb{C}^2$ but not $\mathbb{R}^2$

- If vector in ONB expanded, find coeffs. in expansion:

$\vec{v} = c_1 \vec{q}_1 + c_2 \vec{q}_2 + \dots + c_n \vec{q}_n$ → take inner product of both sides of q_k

$\langle \vec{q}_k, \vec{v} \rangle = c_1 \langle \vec{q}_k, \vec{q}_1 \rangle + \dots + c_k \langle q_k, q_k \rangle + \dots + c_n \langle q_k, q_n \rangle$

$= 0 + \dots + c_k + \dots + 0 = c_k$

ex. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$c_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}}$

$c_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}}$

$\therefore c_j = \langle \vec{q}_j, \vec{v} \rangle$

3.2 Orthogonal Matrices & Unitary Matrices

- A square matrix Q whose cols form an ONB is:

① orthogonal if all entries real

② Unitary if entries complex

Properties

① If Q unitary or orthogonal,

$QQ^* = I = Q^*Q$ ⟶ $Q^* = Q^{-1}$, $(Q^*)^{-1} = Q = (Q^*)^*$

② Matrix Q unitary if for all $v \in V$

$\|Qv\|_2 = \|v\|_2$

↳ Unitary matrices preserve the length of the vector they are acting on

Parseval's Identity

Q^* unitary ⟶ $\|Q^* v\|_2 = \|v\|_2$

Q^* unitary $\longrightarrow \|Q^* v\|_2 = \|v\|_2$

- let $\{q_j\}_{j=1}^n$ be an ONB of V. For all $v \in V$:

$$\sum_{j=1}^{n} |\langle q_j, v\rangle|^2 = \|v\|_2^2$$