Conditional Expectation

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Expectation w/ 2 RVs

- $\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$
  - ↳ $g(X,Y)$ could be X, Y, X^2, etc.
- $\mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)] = \mathbb{E}[XY] - \mathbb{E}X\cdot\mathbb{E}Y$
  - ↳ X, Y uncorrelated if $\mathrm{Cov}(X,Y)=0$
  - ↳ **Correlation Coefficient**: $\rho_{X,Y} = \dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$

Find $\mathbb{E}X$, $\mathrm{Var}(X)$, and $\mathrm{Cov}(X,Y)$ for $(X,Y)\sim f(x,y)$ where

$$f(x,y) = \begin{cases} 2 & ; \quad x\geq 0,\ y\geq 0,\ x+y\leq 1. \\ 0 & ; \quad \text{o.w.} \end{cases}$$

Solution

$$\mathbb{E}X = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy = \int_0^1\int_0^{1-x} 2x\,dy\,dx$$

$$= 2\int_0^1 (1-x)x\,dx = 2\left(\tfrac{1}{2}-\tfrac{1}{3}\right) = \tfrac{1}{3}$$

To find $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2$, we first find $\mathbb{E}(X^2)$.

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(x,y)\,dx\,dy = \int_0^1\int_0^{1-x} 2x^2\,dy\,dx$$

$$= 2\int_0^1 (1-x)x^2\,dx = 2\left(\tfrac{1}{3}-\tfrac{1}{4}\right) = \tfrac{1}{6}$$

$$\Rightarrow \mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \tfrac{1}{6} - \tfrac{1}{9} = \tfrac{1}{18}$$

To find $\mathrm{Cov}(XY) = \mathbb{E}[XY] - \mathbb{E}X\cdot\mathbb{E}Y$, we find $\mathbb{E}Y$ and $\mathbb{E}[XY]$.

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\,dx\,dy = \int_0^1\int_0^{1-y} 2y\,dx\,dy$$

$$= 2\int_0^1 y(1-y)\,dy = 2\left(\tfrac{1}{2}-\tfrac{1}{3}\right) = \tfrac{1}{3} \quad \text{(Or by symmetry, } \mathbb{E}Y = \mathbb{E}X.)$$

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^{1-y} 2xy\,dx\,dy$$

$$= \int_0^1 y(1-y)^2\,dy$$

$$= \tfrac{1}{12}$$

$$\Rightarrow \mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}X\,\mathbb{E}Y = \tfrac{1}{12} - \tfrac{1}{3}\cdot\tfrac{1}{3} = -\tfrac{1}{36}.$$

- If X, Y indep., $\therefore$ Uncorrelated
  - ↳ $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}X\,\mathbb{E}Y = 0 \Rightarrow X, Y$ Uncorrelated
  - ↳ But Uncorrelation $\not\Rightarrow$ Independence

Conditional Probability given event

- Conditionning on event:
  - X = RV w/ pmf $p_X(x)$
    - A = event $\mathbb{P}(X\in A)\neq 0$
    - **Conditional pmf of X given A:**

$$p_{X|A}(x) = \mathbb{P}(X=x \mid X\in A) = \frac{\mathbb{P}(X=x \mid X\in A)}{\mathbb{P}(X\in A)} = \begin{cases} \dfrac{p_X(x)}{\mathbb{P}(X\in A)} & ,\ x\in A \\ 0 & ;\ \text{o/w} \end{cases}$$

$$f_{X|A}(x) = \cdots$$

Conditional Expectation $g(x)$ given event

- $\mathbb{E}[g(X)\mid A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\,dx$
- Law of Total Expectation
  - $X\sim f_X(x)$ $\exists$ $A_1, A_2, \ldots = \phi$
    - $\therefore \mathbb{E}[g(x)] = \sum_{i=1}^{n} \mathbb{P}(X\in A_i)\,\mathbb{E}[g(x)\mid A_i]$
- Conditioning on a RV
  - Conditional expectation of func $g(x,y)$ wrt $f_{X|Y}(x,y)$:

$$\mathbb{E}[g(X,Y)\mid Y=y] = \int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x|y)\,dx$$

- Conditional Expectation

Let $f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } x\geq 0,\ y\geq 0,\ x+y\leq 1 \\ 0 & \text{o.w.} \end{cases}$

Find $\mathbb{E}[X\mid Y=y]$ and $\mathbb{E}[XY\mid Y=y]$

Solution.

We already know that $f_{X|Y}(x|y) = \begin{cases} \frac{1}{1-y} & \text{if } x\geq 0,\ y\geq 0,\ x+y\leq 1. \\ 0. & \text{o.w.} \end{cases}$

Thus $\mathbb{E}[X\mid Y=y] = \int_0^{1-y} x f_{X|Y}(x|y)\,dx = \int_0^{1-y} \frac{x}{1-y}\,dx = \frac{1-y}{2}$, $0\leq y\leq 1$.

$\mathbb{E}[XY\mid Y=y] = y\,\mathbb{E}[X\mid Y=y] = \frac{y(1-y)}{2}$, $0\leq y\leq 1$.

- Law of Total Expectation (Conditional)
  - For func $g(x,y)$: $\mathbb{E}[g(X,Y)] = \mathbb{E}[\mathbb{E}\ [g(x,y)\mid Y]]$

$\mathbb{E}[XY \mid Y=y] = y\,\mathbb{E}[X \mid Y=y] = \frac{y(1-y)}{2}$, $0 \le y \le 1$.

- **Law of Total Expectation (Conditional)**
  - For func $g(x,y)$: $\mathbb{E}[g(X,Y)] = \mathbb{E}_Y[\mathbb{E}_{X|Y}[g(X,Y) \mid Y]]$

Conditional Variance

- $\mathrm{Var}[X \mid Y=y] = \mathbb{E}[X^2 \mid Y=y] - (\mathbb{E}[X \mid Y=y])^2$
- **Law of Total Variance**

$\mathrm{Var}[X] = \mathbb{E}[\mathrm{Var}[X \mid Y]] + \mathrm{Var}[\mathbb{E}[X \mid Y]]$

↳ $\mathbb{E}_Y[\mathrm{Var}[X \mid Y]] = \mathbb{E}_Y[\mathbb{E}[X^2 \mid Y]] - \mathbb{E}_Y[(\mathbb{E}[X \mid Y])^2]$

$= \mathbb{E}[X^2] - \mathbb{E}_Y[(\mathbb{E}[X \mid Y])^2]$

Total Expectation

↳ $\mathrm{Var}[\mathbb{E}[X \mid Y]] = \mathbb{E}_Y[(\mathbb{E}[X \mid Y])^2] - (\mathbb{E}_Y[\mathbb{E}[X \mid Y]])^2$

$= \mathbb{E}_Y[(\mathbb{E}[X \mid Y])^2] - (\mathbb{E}X)^2$