Vector and Matrix Norms

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1.5 Norms of vectors

① Euclidean norm (2-norm)

$\vec{a} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \longrightarrow \|a\|_2 = \sqrt{(-4)^2 + 3^2} = 5$

— Distance b/w tip & tail vector

MATLAB: vector 2-norm

norm(a)

② 1-norm

— Shortest distance to walk from tail to tip

$\|a\|_1 = |-4| + |3| = 7$

MATLAB: vector 1-norm

norm(a, 1)

③ Infinity norm

— Largest component in absolute value

$\vec{a} = [-4, 3]^T \longrightarrow \|a\|_\infty = \max\{|-4|, |3|\} = 4$

MATLAB: vector ∞-norm

norm(a, inf)

☆ For any vector: $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$

Cauchy-Schwartz inequality for dot product:

$|\vec{a} \cdot \vec{b}| \le \|a\|_2 \|b\|_2$

Vector Norm Properties

① Norms = non-negative, $\|x\| \geq 0$

— $\|x\| = 0$ ONLY IF $\vec{x} = 0$

② $\|s\vec{x}\| = |s| \|\vec{x}\|$

③ $\|x+y\| \le \|x\| + \|y\| \longrightarrow$ Triangle Inequality

③ $\|x+y\| \le \|x\| + \|y\|$ → Triangle Inequality
↳ length of longest side of triangle < sum of two shorter sides

1.6 Matrix Norms

Hilbert-Schmidt norm (2-norm)

$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \longrightarrow \|A\|_{HS} = \sqrt{1^2+2^2+0^2+2^2} = 3$

- measures size of matrix

Operator Norm (Matrix Norm)

- A matrix = big if increases size of vectors $A\vec{x}$
  $\therefore \|A\vec{x}\|$ is big compared to $\|\vec{x}\|$
- Consider stretching ratio $\frac{\|A\vec{x}\|}{\|\vec{x}\|}$ → No stretch for $A\vec{x}=0$
- Matrix norm = largest ratio:
  $\|A\|_{op} = \max_{\|\vec{x}\| \neq 0} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2} = \max_{\|\vec{x}\| \neq 0} \left\| A\left(\frac{\vec{x}}{\|\vec{x}\|_2}\right)\right\|_2$
- measures maximum factor by which A can stretch a vector
  $\therefore \|A\|_{op} \geq \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2} \longrightarrow \|A\vec{x}\|_2 \le \underbrace{\|A\|_{op}\|\vec{x}\|_2}$ upper bound on norm of $A\vec{x}$
  ↳ max of a collection of stretching ratios

  $\|A\|$ = smallest number bigger than $\|A\vec{x}\| / \|\vec{x}\|$

  $\therefore \|A\| = \max_{\|\vec{x}\|=1} \|A\vec{x}\|$

- Diagonal Matrix $\begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & d_2 & \vdots \\ 0 & \cdots & d_3 \end{bmatrix}$

  ex. Find D s.t. $\|A\|_{op} \le D$ & $\|A\|_{op} \geq D$

  ① $\|Ax\|_2^2 = |d_1|^2|x_1|^2 + \cdots + |d_n|^2|x_n|^2$

  let $D = \max\{|d_1|, \cdots |d_n|\} \le D^2 (|x_1|^2 + \cdots |x_n|^2) \rightarrow \|x\|_2^2$

  $\|Ax\|_2^2 \le D^2\|x\|_2^2, \quad \frac{\|Ax\|_2}{\|x\|_2} \le D \longrightarrow \|A\|_{op} \le D$

  ② Find $\tilde{x}$ s.t. $\|A\tilde{x}\|_2 = D \cdot \|\tilde{x}\|_2$

  - $D = |d_k|$, set $\tilde{x} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ → k^{th} place, $\|\tilde{x}\|_2 = 1$
  - $A\tilde{x} = \begin{bmatrix} 0 \\ \vdots \\ d_k \\ \vdots \end{bmatrix} \longrightarrow \|A\tilde{x}\| = |d_k| = D$
  - $\|A\tilde{x}\|_2 = D \cdot \|\tilde{x}\|_2 \quad \therefore \|A\|_{op} \geq D$

MATLAB:

Aaug = [A b] , rref, eye(n), randi(#, n, m)

inv(A) or A^(-1) or rref[A eye(n)]

x = inv(A) · b , x = A \ b

Fastest Method: tic; ... toc;