MATH 264 Flux

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Flux:

$\vec{D} = \varepsilon \vec{E}$

Gauss: $\oiint_S \vec{D} \cdot \vec{ds} = Q_{enc}$

Flux: scalar that quantifies flow of Q through surface S

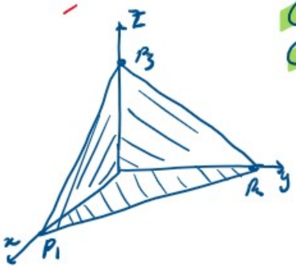
- need surface in 3D (parametrization of surface)
- need flow field
- sign/direction (from problem)

Simple case: Surface = planar patch S, flow field = const. $\vec{G}$
surface normal $= \vec{n}$

$\therefore$ flux: $\Psi = (\vec{G} \cdot \vec{n}) \cdot S \rightarrow$ area of S

Ex. $\vec{G} = \langle A, B, C \rangle$, Find fluxes through faces of a tetrahedron with corners $(0,0,0)$, $(2,0,0)$, $(0,6,0)$ & $(0,0,4)$ using outward normals

1. sketch



2. Calculate fluxes through 4 faces of tetrahedron

1. Triangle: $(0, P_2, P_3)$

- lies on $x=0$ plane, $\therefore \hat{n} = \pm \hat{a}_x$
- $\because$ calculating outward flux, $\therefore \hat{n} = -\hat{a}_x$

$\Psi_1 = (\vec{G} \cdot \vec{n}) \cdot S = -A \cdot 12 = -12A$

2. Triangle: $(0, P_1, P_3)$

- Outward normal $\hat{n} = -\hat{a}_y$

$\Psi_2 = (\vec{G} \cdot \hat{n}) \cdot S = \vec{G} \cdot (-\hat{a}_y) \cdot S = -4B$

3. Triangle: $(0, P_1, P_2)$

- $\hat{n} = -\hat{a}_z \longrightarrow \Psi = -6C$

4. Triangle: $(P_1\, P_2\, P_3)$

- Find normal vector:

a) use $P_1\ P_2\ P_3$ to find plane eqn, find $\hat{n}$

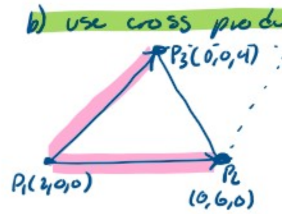
eqn: $\frac{x}{2} + \frac{y}{6} + \frac{z}{4} = 1 \longrightarrow 6x + 2y + 3z = 12$

$\therefore \hat{n} = \pm \langle 6, 2, 3 \rangle \longrightarrow$ but needs to point away from origin $\therefore$ take (+) since outward

- But need to find area using geometry...

OR:

b) use cross product



$\because$ Take $\vec{P_1P_2} \times \vec{P_1P_3}$ $\because$ want $\hat{n}$ outward (RHR)

$(-2,6,0) \times (-2,0,4)$

$= \langle 24, 8, 12 \rangle = 4 \langle 6, 2, 3 \rangle$

$\hat{n} = \dfrac{\vec{P_1P_2} \times \vec{P_1P_3}}{|\vec{P_1P_2} \times \vec{P_1P_3}|}$ & $S = \frac{1}{2} |\vec{P_1P_2} \times \vec{P_1P_3}| \longrightarrow$ A of parallelogram/2

$\therefore \Psi_4 = (\vec{G} \cdot \vec{n}) S = \vec{G} \cdot (\hat{n} \cdot S)$

$= \frac{1}{2} \vec{G} \cdot (\vec{P_1P_2} \times \vec{P_1P_3})$

$= 12A + 4B + 6C$

$\therefore$ Total Flux: $\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 = 0$

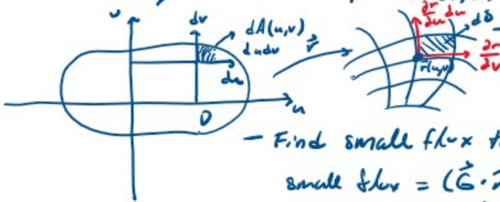
$\rightarrow$ This $\vec{G}$ fits form $\vec{D} = \varepsilon \vec{E}$ where $\vec{D} = \frac{\rho_s}{2} \hat{a}_n$ produced by sheet charge with normal $\hat{a}_n = \dfrac{\langle A, B, C \rangle}{|\langle A, B, C \rangle|}$ at dist. from tetrahedron $\therefore Q_{enc} = 0$

General Surfaces w/ General flow fields

Surface S parametized by $\left.\begin{array}{l} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{array}\right\}$ u,v chosen from set D in u–v plane

– use $\vec{r} = \langle x, y, z \rangle$
$= \vec{r}\langle u, v \rangle$

– Increments in du, dv in u, v plane are mapped to increment tangent to S at pt. $\vec{r}(u,v)$



— For changes in 1 dir. use partial deriv.

– Find small flux through dS

small flux $= (\vec{G}\cdot\hat{n})\cdot dS \longrightarrow$ area of patch

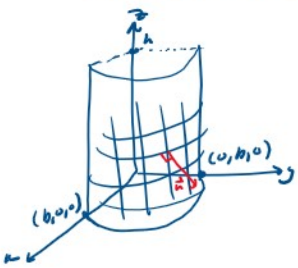
$$\hat{n} = \frac{\pm\left(\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right)}{\left|\left(\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right)\right|} \quad \& \quad dS = \left|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right| du\,dv \longrightarrow \text{parallelogram } A$$

$$(\vec{G}\cdot\hat{n})\cdot dS = \vec{G}\cdot(\hat{n}\,dS) = \pm\left[\vec{G}\cdot\left(\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right)\right]du\,dv$$

$$\Psi = \pm\iint_D\left[\vec{G}\cdot\left(\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right)\right]du\,dv$$

given in question ; $D \longrightarrow$ parametrization domain

ex. Find flux of $\vec{G} = \langle x, y, z \rangle$ into first octet through $S = x^2+y^2 = b^2$
$x \geq 0,\ y \geq 0,\ 0 \leq z \leq h$



① sketch

② parametrization

$$\left.\begin{array}{l} x = b\cos u \\ y = b\sin u \\ z = v \end{array}\right\} \quad \begin{array}{l} 0 \leq u \leq \pi/2 \\ \\ 0 \leq v \leq h \longrightarrow \text{independent} \end{array}$$

③ Partials

$$\frac{\partial r}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle = \langle -b\sin u, b\cos u, 0 \rangle$$

$$\frac{\partial r}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle = \langle 0, 0, 1 \rangle$$

$$\hat{n}\cdot dS = \pm\left(\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right)du\,dv = \pm\begin{vmatrix} -b\sin u & b\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} du\,dv = \pm\langle b\cos u, b\sin u, 0\rangle du\,dv$$

$\because$ in radial direction outward, want (+)

$\therefore$ $\hat{n}\cdot dS = \langle b\cos u, b\sin u, 0 \rangle du\,dv$

④ Flux

$$\Psi = \iint_D \left(\vec{G}\cdot(\hat{n}\cdot dS)\right) = \iint_D \vec{G}\cdot\left(\langle b\cos u, b\sin u, 0\rangle\right)du\,dv$$

$$= \int_0^h\int_0^{\pi/2} \vec{G}(\vec{r}(u,v))\cdot\left(\langle b\cos u, b\sin u, 0\rangle\right)du\,dv$$

$$= \int_0^h\int_0^{\pi/2} \langle b\cos u, b\sin u, 0\rangle\cdot\langle b\cos u, b\sin u, 0\rangle\,du\,dv$$

$$= \int_0^h\int_0^{\pi/2}\left(b^2\cos u + b^2\sin u\right)du\,dv = b^2 h\,\pi/2$$

☆ For cylindrical patches $\rho = b$ if use $\varphi = u,\ z = v$,
get $\hat{n}\cdot dS = b\langle\cos\varphi, \sin\varphi, 0\rangle\,d\varphi\,dz = b\,\hat{\rho}\,d\varphi\,dz$
↳ surface area form for cylindrical surface w/ radius b