Conditional Probability and Bayes Rule

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Notation

- Probability Mass Function **pmf** – **discrete** RV
  $p_X(x) = \mathbb{P}(X=x)$
- Cumulative distribution Function **cdf** – **discrete** RV
  $F_X(x) = \mathbb{P}(X \le x) = \sum_{t \le x} p_X(t)$
- Probability distribution function **pdf** – **continuous** RV
  $f_X(x)$
- Cumulative distribution function **cdf** – **continuous** RV
  $F_X(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$

Conditional Probability & Baye's Rule

Baye's Rule

$\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \dfrac{\mathbb{P}(A)\,\mathbb{P}(B \mid A)}{\mathbb{P}(B)}$ → for events

↳ discrete: $\mathbb{P}(X=x \mid Y=y) = p(x|y) = \dfrac{p(x)\,p(y|x)}{p(y)}$

↳ $p(y)$ → Total probability (Marginal pdf) (eg. $P(1)P(1|1) + P(0)P(1|0)$)

↳ $\sum_\theta p_{X|\Theta}(1 \mid \theta)\,p(\theta)$

↳ $\underset{c\ c}{f(x|y)} = \dfrac{f(x)\,f(y|x)}{f(y)}$, $\underset{d\ c}{P(x|y)} = \dfrac{P(x)\,f(y|x)}{f(y)}$, $\underset{c\ d}{f(x|y)} = \dfrac{f(x)\,P(y|x)}{P(y)}$

↳ $f(x|y) = \dfrac{f(x,y)}{f(y)}$

Example: Three cards.

- There are 3 cards
  ① green on both sides
  ② yellow on both sides
  ③ green on one side and yellow on the other

- Pick a card and a side uniform at random. Let X be the color you get
- Let Y be the color on the back.
- Q: What is $\mathbb{P}(Y=\text{green} \mid X=\text{green})$? A. $> \frac{1}{2}$ ✓ B. $< \frac{1}{2}$ C. $= \frac{1}{2}$.
- card number $\Theta \sim \text{Unif}\{1,2,3\}$.
- $P_{X|\Theta}(\text{green}|1) = 1$, $P_{X|\Theta}(\text{green}|2) = 0$, $P_{X|\Theta}(\text{green}|3) = \frac{1}{2}$
- $P_{\Theta|X}(1|\text{green}) = \dfrac{P_\Theta(1)\,P_{X|\Theta}(\text{green}|1)}{\sum_\theta P_{X|\Theta}(\text{green}|\theta)\,P(\theta)} = \dfrac{1 \cdot \frac{1}{3}}{1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3}} = \dfrac{2}{3}$

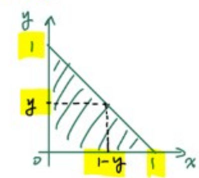
$\Rightarrow \mathbb{P}(Y=\text{green} \mid X=\text{green}) = \mathbb{P}(\Theta = 1 \mid X=\text{green}) = \frac{2}{3}$.

Example: Find conditional pdf for continuous r.v.s.

- Let $X, Y \sim f(x,y)$, where

$$f(x,y) = \begin{cases} 2, & x \ge 0,\ y \ge 0,\ x+y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

Find $f(x|y)$.

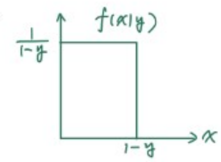


Solution: We first find the marginal pdf

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{1-y} 2\,dx = \begin{cases} 2(1-y), & 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

$$\Rightarrow f(x|y) = \frac{f(x,y)}{f(y)} = \begin{cases} \frac{1}{1-y}, & 0 \le y \le 1,\ 0 \le x \le 1-y \\ 0, & \text{o.w.} \end{cases}$$

In other words, $X \mid \{Y=y\} \sim \text{Unif}[0, 1-y]$



Binary Symmetric Channel (BSC)

$z \in \{0,1\}$

$x \in \{0,1\} \longrightarrow \oplus \longrightarrow y \in \{0,1\}$

eg. $X \sim \text{Bern}(p)$ (↦ prob.) $0 \le p \le 1$, $Z \sim \text{Bern}(\varepsilon)$ (↦ prob.), $Y = \underbrace{X + Z}_{\text{independent}}$

↦ mod2 → $|x+z|$ → binary addition

↳ $p(x|y) = \dfrac{p(y|x)\,p(x)}{\sum_{x' \in \mathcal{X}} P_{Y|X}(y|x')\,P_X(x')}$

↳ Find $p(y|x)$:

$p(y|x) = \mathbb{P}(Y=y \mid X=x) = \mathbb{P}(X \oplus Z = y \mid X=x)$

$= \mathbb{P}(Z = x \oplus y \mid X=x)$ ⟶ ∵ indep.

∴ $= \mathbb{P}(Z = x \oplus y) = p_Z(y \oplus x)$ ⟼ ∵ Bernoulli

Baye's Rule: $p_{X|Y}(0|0) = \dfrac{p_{Y|X}(0|0)\,p_X(0)}{p_{Y|X}(0|0)p_X(0) + p_{Y|X}(0|1)p_X(1)} = \dfrac{(1-\varepsilon)(1-p)}{(1-\varepsilon)(1-p) + \varepsilon p}$

$p_{X|Y}(1|0) = 1 - p_{X|Y}(0|0)$

$p_{X|Y}(0|1) = \dfrac{p_{Y|X}(1|0)\,p_X(0)}{p_{Y|X}(1|0)\,p_X(0) + p_{Y|X}(1|1)\,p_X(1)} = \dfrac{\varepsilon(1-p)}{(1-\varepsilon)p + \varepsilon(1-p)}$

$p_{X|Y}(1|1) = 1 - p_{X|Y}(0|1)$

↳ Find $P_Y(y)$:

$P_Y(y) = p_{Y|X}(y|0)\,p_X(0) + p_{Y|X}(y|1)\,p_X(1)$ → Total Probability

$$= \begin{cases} (1-\varepsilon)(1-p) + \varepsilon p, & y=0 \\ \varepsilon(1-p) + (1-\varepsilon)p, & y=1 \end{cases}$$

- Find $p_Y(y)$:

$p_Y(y) = p_{Y|X}(y|0)\,p_X(0) + p_{Y|X}(y|1)\,p_X(1)$ → Total Probability

$$= \begin{cases} (1-\varepsilon)(1-p) + \varepsilon p, & y = 0 \\ \varepsilon(1-p) + (1-\varepsilon)p, & y = 1 \end{cases}$$

↳ **Probability of Error:**

$$\mathbb{P}(X \neq Y) = p_{X,Y}(0,1) + p_{X,Y}(1,0)$$
$$= p_{Y|X}(1|0)\,p_X(0) + p_{Y|X}(0|1)\,p_X(1)$$
$$= \varepsilon(1-p) + \varepsilon p = \varepsilon$$

Additive Gaussian Noise Channel

$Z \sim N(0, \sigma^2)$

$\Theta \longrightarrow \oplus \longrightarrow Y$

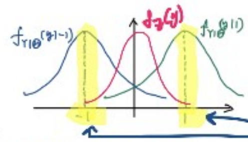
↳ Binary RV: $\begin{cases} 1, & \text{prob. } p \\ -1, & \text{prob. } 1-p \end{cases}$

$Y = \underbrace{\Theta + Z}$ → Indep.

- **Find $p_{\Theta|Y}(\theta|y)$:**

Solution: We use the Bayes rule

$$p_{\Theta|Y}(\theta|y) = \frac{f_{Y|\Theta}(y|\theta)\,p_\Theta(\theta)}{\sum_{\theta'} f_{Y|\Theta}(y|\theta')\,p_\Theta(\theta')}$$



We know $p_\Theta(1) = p$ and $p_\Theta(0) = 1-p$.

$$\boxed{f_{Y|\Theta}(y|\theta) = f_{Z|\Theta}(y-\theta|\theta) = f_Z(y-\theta)}$$

recall the trick in the BSC example

Alternative explanation: Given $\Theta = 1$, $Y = Z+1$. (Y is a shift of Z to the right by 1)

Given $\Theta = -1$, $Y = Z-1$ (Y is a shift of Z to the left by 1)

$$f_Z(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$$

$\Rightarrow Y|\{\Theta = +1\} \sim N(+1, \sigma^2)$ and $Y|\{\Theta = -1\} \sim N(-1, \sigma^2)$

$$\Rightarrow p_{\Theta|Y}(1|y) = \frac{\overbrace{\frac{p}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-1)^2}{2\sigma^2}}}^{\to f_Z(y)}}{\frac{p}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-1)^2}{2\sigma^2}} + \frac{(1-p)}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y+1)^2}{2\sigma^2}}} = \frac{p e^{\frac{y}{\sigma^2}}}{p e^{\frac{y}{\sigma^2}} + (1-p) e^{-\frac{y}{\sigma^2}}}$$

$$p_{\Theta|Y}(-1|y) = 1 - p_{\Theta|Y}(1|y) = \frac{(1-p) e^{-\frac{y}{\sigma^2}}}{p e^{\frac{y}{\sigma^2}} + (1-p) e^{-\frac{y}{\sigma^2}}}$$