Linear Dependence

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1.2 Linear Dependence

- Linear combination of vectors $v_1 \to v_k$ is sum $\sum_{j=1}^{k} c_j v_j$ where $c_1, \dots c_k \in \mathbb{R}$
- $\{v_1 \dots v_k\}$ is linear dependent if exist c_j not all 0 such that $\sum_{j=1}^{k} c_j v_j = \vec{0}$
- Else, $\{v_1 \dots v_k\}$ is linear independent

$\longrightarrow$ Check for linear dependence:

- Do coeffs $c_1, c_2, \dots c_k$ exist s.t.:
  1. At least one $c_i \neq 0$ for $i \dots k$
  2. $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$

$$[v_1, v_2, \dots v_k]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \vec{0} \longrightarrow Vc = \vec{0}$$

$\therefore$ Find all solutions to system $Vc = \vec{0}$

Possibility ①: - Unique soln
- $c_1 = c_2 = \dots = c_k = 0$
- $\therefore$ $\{v_1 \dots v_k\}$ is linearly independent

Possibility ②: - Infinitely many solutions
- $\{v_1 \dots v_k\}$ is linear dependent
- $\hookrightarrow$ let A be a $n \times k$ matrix, $\exists$ an invertible matrix Q s.t.: $QA = \text{rref}(A)$, $A = Q^{-1}\,\text{rref}(A)$
- $\hookrightarrow$ For any $\vec{x} \in \mathbb{R}^k$, $V\vec{x} = \vec{0} \longrightarrow \text{rref}(V)\vec{x} = \vec{0}$
  - Soln set of $Vc = \vec{0}$ & $\text{rref}(V)c = \vec{0}$
  - let $V\vec{x} = \vec{0}$, for $QV = \text{rref}(V)$, $Q(V\vec{x}) = \vec{0} \to (QV)\vec{x} = \vec{0}$, $\text{rref}(V)\vec{x} = 0$
  - let $\text{rref}(V)\vec{x} = \vec{0}$ then $Q^{-1}(\text{rref}(V)\vec{x}) = \vec{0} \longrightarrow \underbrace{Q^{-1}\text{rref}(V)}_{=V}\vec{x} = 0$, $V\vec{x} = \vec{0}$

ex1. $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\} \to$ lin dep/indep?

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \text{rref}(V) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\longrightarrow$ $\text{rref}(V)c = \vec{0}$ has unique soln. $\therefore$ vectors are lin. indep.

ex2: $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}8\\11\\14\end{bmatrix}\right\} \longrightarrow V = \begin{bmatrix} 1 & 2 & 8 \\ 2 & 2 & 11 \\ 3 & 2 & 14 \end{bmatrix}$

$\xrightarrow{\text{rref}}$ $\text{rref}(V) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2.5 \\ 0 & 0 & 0 \end{bmatrix}$, # pivots = 2, # col/vectors = 3

$\because$ # col > # pivots $\therefore$ inf. soln $\longrightarrow$ linearly dependent

$\longrightarrow$ col # 3: $3\vec{v}_1 + 2.5\vec{v}_2$ $\therefore$ $\vec{v}_3 = 3\vec{v}_1 + 2.5\vec{v}_2$

ex3. $\text{rref}(V) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow$ know that $[v_1\, v_2\, v_3] \to$ lin dep