Four Fundamental Spaces for Matrix

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2. Four fundamental spaces for a Matrix

① $N(A) \longrightarrow$ null space of A
② $R(A) \longrightarrow$ range of A
③ $N(A^T) \longrightarrow$ nullspace of A^T
④ $R(A^T) \longrightarrow$ range of A^T

① N(A) Nullspace of A

– $N(A) = \{ x \in \mathbb{R}^n : Ax = \vec{0} \} \longrightarrow$ $N(A)$ is the soln. set of $A\vec{x} = \vec{0}$

ex. let A be an invertible Matrix:

$A x = \vec{0} \longrightarrow \vec{x} = A^{-1}\vec{0} = 0 \quad \therefore N(A) = \{\vec{0}\}$ (trivial nullspace)

ex. $C = \begin{bmatrix} 2 & -4 & -3 & -1 \\ 1 & -2 & 4 & 5 \\ -1 & 2 & 1 & 0 \end{bmatrix}$ $N(C) = ?$ $\quad rref(C) = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_2 = s, \quad x_4 = t, \quad x_1 = 2s - t, \quad x_3 = -t$

$\therefore N(C) = \left\{ s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$

– Dimension: $\dim(N(C)) = \#\text{cols} - \#\text{pivots} = \#\text{free vars} = 4 - 2 = 2$

② R(A) Range of A

– $R(A) = \{ x \in \mathbb{R}^n : Ax = \vec{0} \} \longrightarrow$ colum space of A, $R(A)$ are the pivot cols of A

– $R(A) = \text{span}\{a_1, \dots a_n\}$

ex. $C = \begin{bmatrix} 2 & -4 & -3 & -1 \\ 1 & -2 & 4 & 5 \\ -1 & 2 & 1 & 0 \end{bmatrix}$ $rref(C) = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$ pivot cols = col #1 & col #3

$\therefore \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \right\}$ is a basis for $R(C)$

– Dimension: $\dim(R(C)) = \#\text{pivots} = \text{Rank}(C) = 2$
$\hookrightarrow \dim(R(C)) + \dim(N(C)) = \#\text{col } C$

③ R(Aᵀ) Range of Aᵀ

– $R(A^T) = \{ x \in \mathbb{R}^m : A^T x \} \longrightarrow$ colum space of A^T, $R(A^T)$ are the non-zero cols of $(rref(A))^T$

ex. $rref(C) = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $(rref(C))^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$R(C^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

– Dimension: $\dim(R(C^T)) = \#\text{pivots} = \text{Rank}(C^T) = \text{Rank}(C) = 2$ (non zero rows)

④ N(Aᵀ) Nullspace of Aᵀ

– $N(A^T) = \{ x \in \mathbb{R}^m : A^T x = \vec{0} \} \longrightarrow$ $N(A^T)$ is the soln. set of $A^T\vec{x} = 0$

– Dimension: $\dim(N(A^T)) = \#\text{col} - \#\text{pivots}$

Example

$A = \begin{bmatrix} 1 & 3 & 4 & 5 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 4 & 7 \end{bmatrix}$

a) $\text{rank}(A) = ?$
b) $\dim(N(A))$, $\dim(R(A))$, $\dim(N(A^T))$, $\dim(R(A^T))$
c) basis for $N(A)$, basis for $R(A)$, basis for $(R(A^T))$

a) $\text{rank}(A)$: $rref(A) = \begin{bmatrix} 1 & 1 & 0 & -2 & -5 \\ 0 & 1 & 0 & 0 & 18 \\ 0 & 0 & 1 & -2 & 12 \end{bmatrix}$ $\quad \therefore \text{rank}(A) = 3$

b) $\dim(N(A)) = \#\text{cols} - \text{rank}(A) = 5 - 3 = 2$

a) $\text{rank}(A)$: $\text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & -2 & -5 \\ 0 & 1 & 0 & 0 & 18 \\ 0 & 0 & 1 & -2 & 12 \end{bmatrix}$ $\therefore \text{rank}(A) = 3$

b) $\dim(N(A)) = \#\text{cols} - \text{rank}(A) = 5 - 3 = 2$

$\dim(R(A)) = \text{rank}(A) = 3$

$\dim(N(A^T)) = \#\text{col} - \#\text{pivots} = 3 - 3 = 0$

$\dim(R(A^T)) = \dim(R(A)) = 3$

c) basis $N(A)$: $x_4 = s$, $x_5 = t$, $x_1 = 2s + 5t$, $x_2 = -5s - 18t$, $x_3 = 2s - 12t$

basis $N(A) = A\vec{x} = \left\{ s \begin{bmatrix} 2 \\ -5 \\ 7 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ -18 \\ -12 \\ 0 \\ 1 \end{bmatrix} \right\}$

basis $R(A)$ = pivot cols of A = $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \right\}$

basis $R(A^T)$ = non-zero cols of rref = $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 12 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 18 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \\ -5 \end{bmatrix} \right\}$