Geometria Diferencial de Curvas e Superfícies
IMPA, Verão 2024 - Prof. Lucas Ambrozio

Lista Grande de Exercícios - Versão 29/02/2024

Os exercícios estão divididos em sete sessões.

A primeira sessão contém exercícios de revisão e podem servir de baliza para os estudantes preocupados com os pré-requisitos do curso.

A segunda sessão contém exercícios sobre curvas, a terceira sobre a definição de superfícies regulares, a quarta sobre a geometria da aplicação de Gauss, a quinta sobre a geometria intrínseca de superfícies regulares e a sexta sobre o Teorema de Gauss-Bonnet. Observem que nem todos os tópicos que serão cobertos estão representados (ainda).

A última sessão contém exercícios de assuntos variados, incluindo de assuntos das outras sessões. Esta sessão crescerá de acordo com o andamento do curso!

Sugerimos que, ao longo de todo curso, os estudantes leiam os enunciados de ***todos*** *os exercícios, refletindo sobre a informação que eles contém, e procurem escrever com detalhes a solução de* ***tantos quanto for possível****, em especial a solução daqueles exercícios que lhe parecerem mais interessantes, úteis ou desafiadores. Sugerimos também que os estudantes procurem outras fontes de exercícios, como o próprio livro do Manfredo, o livro do Montiel e Ros, e quaisquer outras referências que lhe parecerem interessantes.*

Caso encontrem qualquer erro, de um erro de digitação até um erro grave, ficarei grato de ser informado!

1. Aquecimento

Exercise 1.1. Enuncie os axiomas de um espaço vetorial (real). Enuncie a definição de uma norma. Enuncie a definição de produto interno, e de norma induzida. Mostre que, se a norma $|\cdot|$ advém de um produto interno $\langle\cdot,\cdot\rangle$, então vale a identidade do paralelogramo

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2.$$

Exercise 1.2. Considere um espaço vetorial real V munido de produto interno $\langle\cdot,\cdot\rangle$. Defina o que é uma base ortonormal $\{v_1,\ldots,v_n\}$ de V. Expressando um vetor arbitrário $v \in V$ nesta base,

$$v = \sum_{i=1}^{n} a^i v_i = a^1 v_1 + \ldots + a^n v_n,$$

use o produto interno para calcular cada coeficiente a^i, e calcule $|v|^2$ em termos destes a^i's.

Exercise 1.3. Sob as mesmas hipóteses do exercício anterior, seja $v \neq 0$ em V. Usando o produto interno, escreva a fórmula para a projeção ortogonal de um ponto $p \in V$ no subespaço gerado por v, e a fórmula para a projeção ortogonal de um ponto $p \in V$ no complemento ortogonal ao subespaço gerado por v.

Exercise 1.4. A dilatação de $\mathbb{R}^n$ centrada no ponto q por um fator $\lambda > 0$ é a aplicação

$$p \in \mathbb{R}^n \mapsto \lambda(p-q) \in \mathbb{R}^n.$$

Após aplicar uma dilatação por um fator λ, o que ocorre com o comprimento de segmentos? E com a área de quadrados? E com o volume dos objetos k-dimensionais generalizando os anteriores, onde $3 \leq k \leq n$? E o que ocorre com o ângulo formados por três pontos distintos p, q e r em $\mathbb{R}^n$? E o que ocorre com retas, círculos e esferas?

Exercise 1.5. Considere a aplicação linear $J : (x, y) \in \mathbb{R}^2 \mapsto (-y, x) \in \mathbb{R}^2$. Descreva geometricamente o que ela faz. Mostre que

i) $J^2 = -1$; e

ii) $\langle Jx, Jy \rangle = \langle x, y \rangle$ para todo x, $y \in \mathbb{R}^2$.

Existe um endomorfismo linear de $\mathbb{R}^3$ que satisfaz *i)* e *ii)*? E em $\mathbb{R}^4$?

Exercise 1.6. Sejam α, $\beta : I \to \mathbb{R}^n$ funções suaves (= de classe C^∞).

i) Mostre que a aplicação $t \in I \mapsto \langle \alpha(t), \beta(t) \rangle \in \mathbb{R}$ é suave, e calcule a sua derivada.

ii) Mostre que a aplicação $t \in I \mapsto |\alpha(t)| \in \mathbb{R}$ é contínua em I e suave exceto nos pontos $t \in I$ onde $\alpha(t) = 0$. Calcule sua derivada (nos pontos onde ela existe).

iii) Suponha que a imagem de α não contém a origem, e que $\alpha(t_0)$, $t_0 \in I$, é o ponto de $\alpha(I)$ que está mais próximo da origem. Mostre que $\alpha'(t_0)$ é ortogonal a $\alpha(t_0)$.

Exercise 1.7. Seja $L : \mathbb{R}^m \to \mathbb{R}$ uma aplicação linear. Mostre que existe um único vetor $v_0 \in \mathbb{R}^m$ tal que

$$L(w) = \langle v_0, w \rangle \quad \text{para todo} \quad w \in \mathbb{R}^m.$$

(*Observação: trata-se apenas de ser cuidadoso com o significado de ambos os lados da igualdade! Qual é a matrix associada à uma aplicação linear $L : \mathbb{R}^m \to \mathbb{R}$, e como ela age em vetores-coluna*?)

Exercise 1.8. Seja V um espaço vetorial real, *de dimensão finita*, munido de produto interno. A adjunta de uma transformação linear $A : V \to V$ é a transformação linear $A^* : V \to V$ que associa a cada $x \in V$ o único ponto $A^*x \in V$ tal que

$$\langle A^*x, y \rangle = \langle x, Ay \rangle \quad \text{para todo} \quad y \in V.$$

Explique porque A^* esta bem-definida. Mostre que a transformação J do exercício 1.5 satisfaz $J^* = J^{-1} = -J$.

Exercise 1.9. Uma transformação auto-adjunta (ou simétrica) de um espaço vetorial real V *de dimensão finita* munido de produto interno é uma aplicação linear $A : V \to V$ tal que $A = A^*$.

i) O Teorema Espectral para operadores auto-adjuntos afirma que para toda transformação auto-adjunta A existe uma base ortonormal $\{v_1, \ldots, v_n\}$ que a diagonaliza, isto é, existem $\lambda_i \in \mathbb{R}$ tais que $Av_i = \lambda_i v_i$ para todo $i = 1, \ldots, n$. Relacione este enunciado com o enunciado sobre matrizes simétricas normalmente vistos em cursos de Álgebra Linear.

ii) Considerando $A : \mathbb{R}^n \to \mathbb{R}^n$ linear auto-adjunta, mostre que a função

$$Q : (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto \langle Ax, y \rangle \in \mathbb{R}$$

é bilinear e simétrica. Calcule os pontos críticos e os valores críticos da função

$$x \in \mathbb{R}^n \setminus \{0\} \mapsto \frac{Q(x, x)}{\langle x, x \rangle} \in \mathbb{R}.$$

Exercise 1.10. Seja X um subconjunto de $\mathbb{R}^n$. Defina o que significa dizer que X é um subconjunto aberto/fechado/compacto/conexo/conexo por caminhos/limitado. Dê exemplos de subconjuntos X que satisfazem estas propriedades, e que não satisfazem estas propriedades.

Exercise 1.11. Mostre que existe uma coleção enumerável de abertos de $\mathbb{R}^n$ com a seguinte propriedade: todo aberto de $\mathbb{R}^n$ é uma união de abertos desta coleção.

Exercise 1.12. Seja $f : K \subset \mathbb{R}^m \to \mathbb{R}^n$ uma aplicação contínua e injetiva definida em um compacto. Mostre que a inversa $f^{-1} : f(K) \to K$ está bem definida e é também contínua.

Exercise 1.13. Seja $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$ uma função suave definida no aberto U. Explique o que é a derivada de F no ponto $p \in U$, vista como uma transformação linear $DF(p) : \mathbb{R}^m \to \mathbb{R}^n$. Escreva a regra da cadeia para a composição $G \circ F$ de funções suaves $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$ e $G : V \subset \mathbb{R}^n \to \mathbb{R}^p$ com $F(U) \subset V$.

Exercise 1.14. (*Um exercício em Álgebra Linear e Diferenciação*).

i) Uma aplicação linear $L : \mathbb{R} \to \mathbb{R}^n$ é identicamente nula ou injetiva.

ii) Seja $F : I \subset \mathbb{R} \to \mathbb{R}^n$ uma função suave definida no intervalo aberto I, t um ponto de I, e $DF(t) : \mathbb{R} \to \mathbb{R}^n$ a derivada F no ponto t (vista como aplicação linear). Se $F(t) = (F^1(t), \ldots, F^n(t))$, $t \in \mathbb{R}$, é a expressão de F em coordenadas, mostre que

$$DF(t) \cdot 1 = ((F^1)'(t), \ldots, (F^n)'(t)).$$

(*Observação: trata-se apenas de ser cuidadoso com o significado de ambos os lados da igualdade! Observe que* 1 *é um vetor que gera* $\mathbb{R}$, *visto como espaço vetorial real de dimensão um).*

iii) Uma aplicação linear $L : \mathbb{R}^m \to \mathbb{R}$ é identicamente zero ou sobrejetiva.

iv) Seja $F : U \subset \mathbb{R}^m \to \mathbb{R}$ uma aplicação suave definida em um aberto U, $p = (x^1, \ldots, x^m)$ um ponto de U, e $DF(p) : \mathbb{R}^m \to \mathbb{R}$ a derivada de F no ponto p. Mostre que o único vetor associado à aplicação linear $DF(p)$ como no exercício 1.7 é precisamente o vetor

$$grad\, F(p) := \left(\frac{\partial F}{\partial x^1}(p), \ldots, \frac{\partial F}{\partial x^m}(p)\right) \in \mathbb{R}^m,$$

mais conhecido como o vetor *gradiente of F no ponto $p \in U$*.

Exercise 1.15. (*Funções C^∞ de suporte compacto*).

i) Considere a função $\eta : \mathbb{R} \to \mathbb{R}$ que se anula em pontos $t \leq 0$ e que é dada por $\eta(t) = e^{-1/t}$ se $t > 0$. Mostre que esta função é de classe C^∞. Ela é analítica?

ii) Seja $\phi : \mathbb{R}^n \to \mathbb{R}$ a função que se anula fora da bola unitária centrada na origem e é dada por $\phi(p) = e^{-1/(1-|p|^2)}$ se $|p| < 1$. Mostre que ϕ é uma função de classe C^∞, não-negativa, e que o seu *suporte* (= fecho do conjunto onde ela é diferente de zero) é um conjunto compacto não-vazio. *(Dica: observe que $e^{-1/(1-t^2)} = e^{-1/(1+t)}e^{1/(1-t)}$) para pontos $t \in (-1, 1)$, e lembre-se que é possível usar a regra da cadeia para passar à dimensões maiores).*

2. Curves

Exercise 2.1. Let $\alpha : I \to \mathbb{R}^n$ be a regular parametrised curve and p be a point in $\mathbb{R}^n$. The distance from points in the trace of α to p is constant if and only if the vector $\alpha(t) - p$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Exercise 2.2. Let $L \subset \mathbb{R}^n$ be a straight line and $p \in \mathbb{R}^n$ a point that is not contained in L. The distance between p and L is the number

$$d(p, L) := \inf\{|p - q| \in \mathbb{R} \,|\, q \in L\}.$$

Prove that the above infimum is positive and attained at some point $q_0 \in L$. Using some parametrisation of L, prove that the straight line passing through p and q_0 is orthogonal to L, and show that q_0 is the only point in L such that $|p - q_0| = d(p, L)$.

In the particular case $n = 2$, show that

$$|p - q_0| = |\langle p - q_0, N\rangle|,$$

where N is a choice of unit vector normal to the line L.

Exercise 2.3. Let p, $q \in \mathbb{R}^n$. Let $\alpha : I \to \mathbb{R}^n$ be a regular parametrised curve, $[a, b] \subset I$ a compact interval, and assume that $p = \alpha(a)$ and $q = \alpha(b)$. In short, $\alpha([a, b])$ is *a curve joining p to q*.

i) Show that the length of $\alpha([a, b])$ satisfies the inequality

$$|p - q| \leq \ell(\alpha; [a, b]).$$

ii) If the length of $\alpha([a, b])$ is equal to the distance between p and q, show that $\alpha([a, b])$ is the straight line segment joining p to q.

Conclude that the distance between any two given points of the Euclidean space can be computed as the minimum of the length of all curves joining these points.

Exercise 2.4. Let $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \,|\, y = |x|\}$. Please draw this set. Then, justify the following assertions:

i) The map

$$\alpha_1 : t \in \mathbb{R} \mapsto (t, |t|) \in \mathcal{C} \subset \mathbb{R}^2$$

is not a regular parametrised curve.

ii) Construct a smooth map $\alpha_2 : I \to \mathbb{R}^2$ such that $\alpha_2(I) = \mathcal{C}$. (*Hint: define parametrizations of both half-lines that constitute $\mathcal{C}$ in such way they reach the endpoint $(0, 0)$ with zero derivatives of all orders. The well-known function $t \in (0, +\infty) \mapsto \exp(-1/t) \in \mathbb{R}$, extended to be 0 at $t = 0$, might be useful*).

iii) The tangent lines of the parametrised curve you have defined in the previous item do not change locally at points where they are defined. However, the trace of the curve is not contained inside a straight line (*To think about: how is this example related to the usual assumption that a regular parametrised curve has non-zero tangent vector everywhere?*).

iv) The set $\mathcal{C}$ *is not* the trace of a regular parametrised curve. (*Suggestion: first show that $\mathcal{C}$ cannot be the trace of a smooth graphical curve $\beta(u) = (u, f(u))$ near $(0, 0) \in \mathcal{C}$. Then reach a contradiction by using the Inverse Function Theorem to reparametrise a hypothetical regular parametrised curve α with trace $\mathcal{C}$, in a neighbourhood of $t \in I$ such that $\alpha(t) = (0, 0)$, so that it becomes written in the above graphical form*).

Exercise 2.5. Let $a > 0$, $b < 0$ be two real numbers. The *logarithmic spiral* is the trace of the regular parametrised curve

$$\alpha : t \in \mathbb{R} \mapsto (ae^{bt}\cos(t), ae^{bt}\sin(t)) \in \mathbb{R}^2.$$

Sketch this curve. Show that $\alpha(t)$ and $\alpha'(t)$ converge to zero as t goes to $+\infty$, and that, for any given $t_0 \in \mathbb{R}$,

$$\lim_{t\to+\infty} \int_{t_0}^{t} |\alpha'(t)|dt < +\infty.$$

Compute the curvature of α.

Exercise 2.6. Mark a point on the circumference of a circular disc of radius $R > 0$. The figure described by that point as the disk rolls without slipping on the $x-$axis is called a *cycloid*.

i) What can be said about the distance between any two points belonging to the intersection of the cycloid and the $x-$axis?
ii) Find a smooth map $\alpha : \mathbb{R} \to \mathbb{R}^2$ describing the trajectory of the marked point. Is α a regular parametrised curve?
iii) Compute the length of a piece of a cycloid between two consecutive instants $t_1 < t_2 \in \mathbb{R}$ when $\alpha(t_1)$ and $\alpha(t_2)$ are points in the $x-$axis.

Exercise 2.7. Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be given by

$$\alpha(t) = \left(a\cos\left(\frac{t}{c}\right), a\sin\left(\frac{t}{c}\right), \frac{b}{c}t\right) \quad \text{for all} \quad t \in \mathbb{R},$$

where $a, b, c \in \mathbb{R}$, $c \neq 0$, are given constants.

i) Give conditions on a, b and c so that α is a regular parametrised curve not contained in a plane. Under these conditions, the trace of α is called a *circular helix*.
ii) Give conditions on a, b and c so that α is parametrised by arc-length.

From now on, assume α is a circular helix parametrised by arc-length.

iii) Show that the tangent lines to α make a constant angle θ with the $z-$axis, and express θ in terms of the constants a and b.
iv) Show that normal vector to α at $t \in I$ is orthogonal to the $z-$axis.
v) Compute the curvature and the torsion of α.
vi) Show that, for all $t \in I$, the line passing through the point $\alpha(t)$ pointing in the direction of the normal vector of α at $t \in I$ intersects the $z-$axis.
vii) Show that the intersection point $Z(t)$ defined in the previous item moves on the $z-$axis with constant non-zero speed.
viii) Compute the distance between $Z(t)$ and $\alpha(t)$.

Exercise 2.8. Let $\alpha : I \to \mathbb{R}^2$ be a map given by

$$\alpha(u) = (u, f(u)) \quad \text{for all} \quad u \in I$$

for some smooth function $f : I \to \mathbb{R}$. Such α is said to be a curve written as a *graph (over the $x-$axis)*.

i) Show that α is a regular parametrised curve.
ii) Write the formula for the length of $\alpha([a, b])$, $[a, b] \subset I$ a compact interval.

iii) Show that the curvature of α at a point $u \in I$ is given by:

$$k(u) = \frac{f''(u)}{(1+(f'(u))^2)^{3/2}}.$$

How does the graph of f look like at a point $u \in I$ where the curvature of α is positive? And where it is negative?

iv) Compute the curvature of the *catenary*:

$$\alpha : u \in \mathbb{R} \mapsto (u, \cosh(u)) \in \mathbb{R}^2.$$

Exercise 2.9. Let $\alpha : I \to \mathbb{R}^2 \setminus \{0\}$ be a map given by

$$\alpha(\theta) = (r(\theta)\cos(\theta), r(\theta)\sin(\theta))$$

for some smooth positive function $r : I \to (0, +\infty)$. Such α is said to be a curve written in *polar coordinates*.

i) Show that α is a regular parametrised curve.

ii) Compute the length of $\alpha([a, b])$, $[a, b] \subset I$ a compact interval.

iii) Show that the curvature of α at a point $\theta \in I$ is given by:

$$k(\theta) = \frac{2(r'(\theta))^2 - r(\theta)r''(\theta) + r(\theta)^2}{((r(\theta))^2 + (r'(\theta))^2)^{3/2}}.$$

iv) Consider the parabola $\mathcal{P} = \{(x, y) \in \mathbb{R}^2 \,|\, y = (x^2 - 1)/4\}$. Describe this set as the trace of a curve written in polar coordinates.

Exercise 2.10. Let $\alpha : I \to \mathbb{R}^2$ be a regular parametrised plane curve. The straight line passing through $\alpha(t)$ that is orthogonal to the tangent line of α at $t \in I$ is called the *normal line* of the curve α at $t \in I$.

Assume that all normal lines to a regular parametrised curve pass through a fixed point. Prove that the trace of the curve is contained in a circle.

Exercise 2.11. Let $\alpha : I \to \mathbb{R}^2$ be a plane curve parametrised by arc-length. Assume that its curvature never vanishes. The *evolute* of α is the map

$$e : s \in I \mapsto \alpha(s) + \frac{1}{k(s)}N(s) \in \mathbb{R}^2.$$

i) Show that e is a regular parametrised curve if and only if k' never vanishes.

ii) Under the condition of the previous item, compute the length of $e([a, b])$, where $[a, b] \subset I$ is a compact interval.

iii) Still under the conditions of the previous items, show that the tangent line to e at $s \in I$ is parallel to the normal line of α at $s \in I$.

iv) Show that e is a constant map if and only if the trace of α is contained in a circle.

v) Compute the evolute of the logarithmic spiral (see Exercise 5).

Exercise 2.12. (*Distance preserving maps - see Other Materials*).

We say that a map $M : \mathbb{R}^n \to \mathbb{R}^n$ *preserves the Euclidean distance* when

$$|M(p) - M(q)| = |p - q| \quad \text{for all} \quad p, q \in \mathbb{R}^n.$$

i) Show that translations and orthogonal transformations preserve the Euclidean distance.

ii) Define $v = M(0)$. Show that $\widetilde{M} = T_{-v} \circ M$ is a map that preserves Euclidean distances and fixes the origin. In particular, conclude that $|\widetilde{M}(p)| = |p|$ for all $p \in \mathbb{R}^n$.

iii) Keeping the notation of the previous item, show that $\widetilde{M}$ is an orthogonal transformation, and that the derivative of M at any point is precisely the linear map $\widetilde{M}$.

iv) Conclude that $M = T_v \circ \widetilde{M}$, where T_v is a translation and $\widetilde{M}$ is an orthogonal transformation.

v) Show that the set of maps that preserve the Euclidean distance is a group when endowed with the operation of composition of maps.

Exercise 2.13. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parametrised by arc-length, and M a map that preserves the Euclidean distance. Show that $M \circ \alpha : I \to \mathbb{R}^3$ is a regular curve parametrised by arc-length. *Distinguishing between the cases where the derivative of M is orientation-preserving or not*, study the relations between the length, the Frenet frames, the curvature and the torsion of the curves α and $M \circ \alpha$. Perform a similar study for plane curves (in the case of closed curves, study also the area of the region bounded by the curves).

Exercise 2.14. (cf. do Carmo, 1.5, 10). Let $\alpha : (-\sqrt{2/3}, \sqrt{2/3}) \to \mathbb{R}^3$ be the map given by

$$\alpha(t) = \begin{cases} (t, e^{-1/t^2}, 0) & \text{for all} \quad t \in (-\sqrt{2/3}, 0), \\ (0,0,0) & \text{for} \quad t = 0, \\ (t, 0, e^{-1/t^2}) & \text{for all} \quad t \in (0, \sqrt{2/3}). \end{cases}$$

Show that α is a regular parametrised curve whose curvature is positive except at $t = 0$. Observe that the osculating planes of α do not change on $(0, \sqrt{2/3})$ or on $(-\sqrt{2/3}, 0)$, where they are well-defined. However, α is not a plane curve. (*To think about: how is this example related to the usual assumption that a regular parametrised curve in $\mathbb{R}^3$ has positive curvature at all points?*).

Exercise 2.15. Let $\alpha : I \to \mathbb{R}^2$ be a curve parametrised by arc-length, and pick $q_0 = \alpha(s_0)$, $s_0 \in I$, a point on the trace of α. Let H^+ be the half-plane determined by the tangent line of α at $s_0 \in I$ into which the normal vector $N(s_0)$ points, *i.e.*

$$H^+ = \{p \in \mathbb{R}^2 \mid \langle p - q_0, N(s_0) \rangle \geq 0\}.$$

The other half-plane H^- is defined similarly.

The function $f : I \to \mathbb{R}^2$ given by

$$f(s) = \langle \alpha(s) - q_0, N(s_0) \rangle \quad \text{for all} \quad s \in I$$

measures the “signed” distance between the point $\alpha(s)$ and the tangent line (cf. Exercise 3), and the sign of $f(s)$ determines whether the point $\alpha(s)$ belongs to H^+ or to H^-.

i) Show that $f(s_0) = 0$, $f'(s_0) = 0$ and $f''(s_0) = k(s_0)$.

ii) Prove that if $k(s_0) > 0$ then there exists a neighbourhood J of $s_0 \in I$ such that $\alpha(J)$ is contained in H^+.

iii) Show by examples that the same conclusion does not necessarily hold if $k(s_0) = 0$.

iv) Prove that if there exists a neighbourhood J of s_0 in I such that $\alpha(J)$ is contained in H^+, then $k(s_0) \geq 0$.

v) State and prove similar assertions about the case $k(s_0) < 0$ and the half-space H^-.

Exercise 2.16. (*The osculating circle*).

Let $\alpha : I \to \mathbb{R}^2$ be a curve parametrised by arc-length, and pick $q_0 = \alpha(s_0)$, $s_0 \in I$, a point in the trace of α where the curvature does not vanish. Assume without loss of generality that $k(s_0) > 0$. Let B_λ be the (closed) ball of radius $\lambda > 0$ that is tangent to α at q_0 and such that the unit normal $N(s_0)$ of α at $s_0 \in I$ points inside it.

i) Show that the centre of the ball B_λ is the point $c_\lambda = q_0 + \lambda N(s_0) \in \mathbb{R}^2$.

ii) The function $f_\lambda : I \to \mathbb{R}^2$ defined by

$$f_\lambda(s) = |\alpha(s) - c_\lambda|^2 \quad \text{for all} \quad s \in I$$

gives the square of the distance between a point in the trace of α and the centre of the ball B_λ. Show that $f_\lambda(s_0) = \lambda^2$, $f'_\lambda(s_0) = 0$ and $f''_\lambda(s_0) = 2(1 - \lambda k(s_0))$.

iii) Prove that if $\lambda < 1/k(s_0)$, then there exists a neighbourhood J of s_0 in I such that $\alpha(J \setminus \{s_0\})$ is contained *outside* the ball B_λ.

iv) Prove that if $\lambda > 1/k(s_0)$, then there exists a neighbourhood J of s_0 in I such that $\alpha(J \setminus \{s_0\})$ is contained *inside* the ball B_λ.

v) Give an example to show that the behaviour described in the previous two items does not necessarily occur if $\lambda = 1/k(s_0)$.

vi) When $\lambda = 1/k(s_0)$, we say that the circle centred at c_λ with radius λ is the *osculating circle* of α at the point $s_0 \in I$. Conclude from the above the following characterization: *the osculating circle of α at a point $s_0 \in I$ where $k(s_0) > 0$ is the circle with the largest radius among those circles satisfying the following properties: it is tangent to the curve α at s_0, is such that the unit normal $N(s_0)$ of α points inside it, and is such that image by α of a sufficiently small neighbourhood of s_0 lies outside it.*

vii) Conclude the following characterisation of the *evolute* (cf Exercise 11): the evolute of a curve α with non-vanishing curvature is the curve described by the centres of the osculating circles of α.

Exercise 2.17. Let $\alpha : I \to \mathbb{R}^2$ be a regular parametrised plane curve. Assume that the distance from $\alpha(t)$ to the origin attains a *maximum value* at $t_0 \in I$. Show that

$$|k(t_0)| \geq \frac{1}{|\alpha(t_0)|}.$$

Exercise 2.18. If the trace of a closed plane regular curve is contained in a disk of radius $R > 0$, then it contains a point whose curvature is in absolute value greater than or equal to $1/R$.

Exercise 2.19. Let $\alpha : [a, b] \to \mathbb{R}^2$ be a closed plane regular curve parametrised by arc-length. Show that

$$\ell(\alpha; [a, b]) = \int_a^b k(s)(x(s)y'(s) - y(s)x'(s))ds.$$

Suggestion: integrate $(x')^2 + (y')^2 \equiv 1$ by parts.

Exercise 2.20. Let $\alpha : [a, b] \to \mathbb{R}^2$ be a closed plane curve parametrised by arc-length. Show that there exists an integer $m \in \mathbb{Z}$ such that

$$\int_a^b k(s)ds = 2\pi m.$$

Draw examples of curves where m is an arbitrary integer. If the curve is *simple*, what do you expect the number m to be? (*We will discuss this question thoroughly later in the course*).

Exercise 2.21. Let $\alpha : [0, \ell] \to \mathbb{R}^2$ be a closed simple regular curve parametrised by arc-length. As always, orient α so that its unit normal points inside the region it bounds. For a given $r \in \mathbb{R}$, consider the curve

$$\alpha_r : s \in [0, \ell] \mapsto \alpha(s) + rN(s) \in \mathbb{R}^2.$$

i) Show that, if r is small enough, then α_r is also a simple closed regular parametrised curve.

ii) Prove that

$$\ell(\alpha_r) = \ell - \left(\int_0^\ell k(s) ds \right) r.$$

iii) If Ω_r is the (closed) region enclosed by α_r, and Ω is the region enclosed by α, prove that

$$A(\Omega_r) = A(\Omega) - \ell r + \left(\int_0^\ell k(s) ds \right) \frac{r^2}{2}.$$

iv) For $r > 0$ sufficiently small, the tube of radius $r > 0$ around $\alpha([0, \ell])$ is the set $\Omega_{-r} \setminus int(\Omega_r)$. Show that the area of the tube of sufficiently small radius $r > 0$ around $\alpha([0, L])$ is a degree one polynomial expression in r that does not depend on the curvature of α.

Exercise 2.22. (cf. do Carmo, 1.5, 17) A regular parametrised curve $\alpha : I \to \mathbb{R}^3$ with positive curvature and nowhere vanishing torsion is called a *helix* if the tangent lines to α make a constant angle with a fixed direction.

i) α is a helix if and only if k/τ is constant.

ii) α is a helix if and only if the normal lines of α are all parallel to a fixed plane.

iii) α is a helix if and only if the straight lines containing $\alpha(s)$ and parallel to the binormal $B(s)$ makes a constant angle with a fixed direction.

iv) Describe a helix that is not any of the circular helices described in Exercise 8.

Exercise 2.23. Let $\alpha : I \to \mathbb{R}^2$ be a curve parametrised by arc-length. Assume that there exists a point $p \in \mathbb{R}^2$ such that the distance between p and the normal lines of α is constant. Prove that, for some a, $b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$,

$$k(s) = \pm \frac{1}{\sqrt{as + b}}.$$

Can you prove the converse as well?

Exercise 2.24. (*The aim of this exercise is to show that the curvature of a plane curve is related to how the length of a curve varies when we deform it, cf. Exercise 21).*

Let p, $q \in \mathbb{R}^2$ be two distinct points on the plane, and $\alpha : [a, b] \to \mathbb{R}^2$ be a curve parametrised by arc-length such that $\alpha(a) = p$ and $\alpha(b) = q$.

Let $V : [a, b] \to \mathbb{R}^2$ be a smooth map such that $V(a) = V(b) = 0$. Let $F : [a, b] \times (-\epsilon, \epsilon) \to \mathbb{R}^2$ be the smooth map given by

$$F(s, t) = \alpha(s) + tV(s) \quad \text{for all} \quad (s, t) \in [a, b] \times (-\epsilon, \epsilon).$$

Notice that $F(a,t) = p$ and $F(b,t) = q$ for all $t \in (-\epsilon, \epsilon)$. F is called a *variation* of α, and the vector field V is called the *variational vector field* associated to the variation F. We want to compute how the length of the curve α varies infinitesimally under such deformations.

i) Show that, if $\epsilon > 0$ is small enough, then for every fixed $t \in (-\epsilon, \epsilon)$, the map

$$\alpha_t := s \in [a,b] \mapsto F(s,t) \in \mathbb{R}^2$$

is a regular parametrised curve joining p to q.

ii) Show that the length of $\alpha_t([a,b])$, $t \in (-\epsilon, \epsilon)$, is given by

$$\ell(\alpha_t; [a,b]) := \int_a^b \sqrt{(1 + 2t\langle \alpha'(s), V'(s)\rangle + t^2|V'(s)|^2)}ds,$$

and therefore is a smooth function of $t \in (-\epsilon, \epsilon)$.

iii) Show that

$$\frac{d}{dt}\bigg|_{t=0} \ell(\alpha_t; [a,b]) = -\int_a^b k(s)\langle N(s), V(s)\rangle ds.$$

(*Hint: integration by parts*).

iv) Using arbitrary variations $V(s) = \phi(s)N(s)$, where $\phi : [a,b] \to \mathbb{R}$ is a smooth function vanishing at a and at b, prove that

$$\frac{d}{dt}\bigg|_{t=0} \ell(\alpha_t; [a,b]) = 0$$

for all variations of α if and only if $\alpha([a,b])$ is the line segment joining p to q.

Exercise 2.25. (REVISADO) Let $X_1, \ldots, X_n : I \to \mathbb{R}^n$ be smooth functions. Assume that, for every $s \in I$, the set $\{X_1, \ldots, X_n\}$ is an orthonormal basis of $\mathbb{R}^n$, *i.e.*

$$\langle X_i(s), X_j(s)\rangle = \delta_{ij} \quad \text{for all} \quad i, j = 1, \ldots, n.$$

(*Recall that the symbol* δ_{ij} *is equal to* 1 *if* $i = j$ *and zero otherwise.*). Show that, for all $s \in I$, the $n \times n$ matrix

$$B(s) := \langle X_i'(s), X_j(s)\rangle$$

is an *anti-symmetric* matrix, *i.e.* $B(s)_{ij} = -B(s)_{ji}$. How is it related to the derivative of the function $A(s) = \langle X_i(s), X_j(s)\rangle$ for all $s \in I$?

Exercise 2.26. Let $\alpha : I \to \mathbb{R}^n$ be a curve parametrised by arc-length. Under appropriate conditions, can you define a "Frenet frame" for α, satisfying similar "Frenet equations" involving "curvature" functions $k_1, k_2, \ldots, k_{n-2}, k_{n-1} : I \to \mathbb{R}$, $k_i > 0$ on I for each $i = 1, \ldots, n-1$, and prove a "Fundamental Theorem of Curves in $\mathbb{R}^n$"? (*The answer is "yes"! Test your ideas first in* $\mathbb{R}^4$*. Generalise).*

3. Surfaces

Exercise 3.1. In analogy to the definition of *regular surfaces in* $\mathbb{R}^3$, define a *regular curve in* $\mathbb{R}^n$ as a subset of $\mathbb{R}^n$ satisfying further properties. Give examples of subsets of $\mathbb{R}^n$ that are regular curves, and examples of subsets of $\mathbb{R}^n$ that are not regular curves (in each case providing the reasons of why this is the case). Show that *the trace* of *simple closed regular parametrised curves*, as defined in the lectures, are regular curves according to your proposed definition.

Exercise 3.2. By exhibiting a collection of parametrisations whose images cover the given set explicitly, show that:

i) The *cylinder*
$$\{(x,y,z)\in\mathbb{R}^3 \,|\, x^2+y^2=1\}$$
is a regular surface.

ii) More generally, let $\mathcal{Z}\subset\mathbb{R}^2$ be a regular curve as defined in Exercise 1.1. The right cylinder over $\mathcal{Z}$, *i.e.* the set
$$\mathcal{C}:=\{(x,y,z)\in\mathbb{R}^3 \,|\, (x,y)\in\mathcal{Z}\}$$
is a regular surface in $\mathbb{R}^3$.

Exercise 3.3. By exhibiting a collection of parametrisations whose images cover the given set explicitly, show that:

i) The *catenoid*
$$\{(x,y,z)\in\mathbb{R}^3 \,|\, \cosh(z)=\sqrt{x^2+y^2}\}$$
is a regular surface. Draw this set.

ii) More generally, let $\mathcal{C}\subset\mathbb{R}^2$ be a regular curve (as defined in Exercise 1) that does not intersect the z-axis and is contained in the plane orthogonal to the y-axis. Show that the set obtained by rotation of $\mathcal{C}$ around the z-axis is a regular surface in $\mathbb{R}^3$. (This surface is called the *surface of revolution* generated by the curve $\mathcal{C}$.)

iii) Using the above construction, define a regular surface in $\mathbb{R}^3$ that is a *torus* of revolution.

Exercise 3.4. Justify the following assertions: the set $\{(x,y,z)\in\mathbb{R}^3 \,|\, x=0 \text{ and } y^2+z^2<1\}$ *is* a regular surface; its closure $\{(x,y,z)\in\mathbb{R}^3 \,|\, x=0 \text{ and } y^2+z^2\leq 1\}$ *is not* a regular surface.

Exercise 3.5. Draw a picture of the *cone*
$$\{(x,y,z)\in\mathbb{R}^3 \,|\, z=\sqrt{x^2+y^2}\}.$$
Show that the above set *is not a regular surface.* Now, remove the point $p=(0,0,0)$ (the tip of the cone). Show that the set
$$\{(x,y,z)\in\mathbb{R}^3 \,|\, z=\sqrt{x^2+y^2}, z\neq 0\}$$
is a regular surface.

Exercise 3.6. Given a,b,c three positive real numbers, consider the *ellipsoid*
$$\mathcal{E}(a,b,c)=\left\{(x,y,z)\in\mathbb{R}^3 \,|\, \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right\}.$$
Prove that $\mathcal{E}(a,b,c)$ is a regular surface in $\mathbb{R}^3$. Show that the map
$$X:(u,v)\in(0,\pi)\times(0,2\pi)\mapsto(a\sin(u)\cos(v),b\sin(u)\sin(v),c\cos(u))\in\mathbb{R}^3$$
is a parametrisation of $\mathcal{E}(a,b,c)$. Describe in geometric terms what are the coordinate curves of the above parametrisation X (*i.e.* describe the curves $u=$ constant, and $v=$ constant).

Exercise 3.7. As any given subset of $\mathbb{R}^3$, a regular surface can be *compact*, *closed but not compact*, *connected* or *disconnected*. Give examples of regular surfaces satisfying each one of these properties. (*Remark: no open subset of* $\mathbb{R}^3$ *is a regular surface. Why?*).

Exercise 3.8. Show that a regular surface in $\mathbb{R}^3$ can be covered by countably many coordinate neighbourhoods. (*Hint: the topology of the Euclidean space is second-countable*).

Exercise 3.9. Consider the following set

$$S = \bigcup_{n=1}^{+\infty} \{(x, y, z) \in \mathbb{R}^3 \,|\, z = 1/n\},$$

which is a countable union of horizontal planes. Show that S is a regular surface. Show that its closure,

$$\overline{S} = \{(x, y, z) \in \mathbb{R}^3 \,|\, z = 0\} \cup \bigcup_{n=1}^{+\infty} \{(x, y, z) \in \mathbb{R}^3 \,|\, z = 1/n\},$$

which is also a countable union of horizontal planes, *is not* a regular surface.

Exercise 3.10. *(The stereographic projection).* Let $S \subset \mathbb{R}^3$ be the sphere of radius one centred at the origin, and p_0 a point in S. The stereographic projection based at the point p_0 is a function π_{p_0} that maps each point $p \in S \setminus \{p_0\}$ to the point where the straight line joining p_0 and p intersects the plane through the origin that is orthogonal to p_0.

i) Show that the stereographic projection π_N based at the north pole $(0, 0, 1)$ is such that its inverse π_N^{-1} can be written as

$$\pi_N^{-1} : (u, v) \in \mathbb{R}^2 \mapsto \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{u^2 + v^2 - 1}{1 + u^2 + v^2} \right) \in S \setminus \{(0, 0, 1)\}.$$

(*Notice that we have identified* $\mathbb{R}^2$ *and the plane* $z = 0$ *in* $\mathbb{R}^3$).

ii) Show that the inverse of the two stereographic projections π_N and π_S based at the north pole $(0, 0, 1)$ and at the south pole $(0, 0, -1)$, respectively, are two parametrisations of the sphere whose images cover all of its points.

iii) Compute the change of coordinates

$$\pi_S \circ \pi_N^{-1} : \mathbb{R}^2 \setminus \{0, 0\} \to \mathbb{R}^2 \setminus \{0, 0\}$$

Denoting by C the circle $\{(u, v) \in \mathbb{R}^2 \,|\, u^2 + v^2 = 1\}$, show that $\pi_S \circ \pi_N^{-1}$ swaps the exterior region to C and the region bounded by C, minus the origin.

Exercise 3.11. Given a, b, c real numbers, compute the regular values of the function

$$F : (x, y, z) \in \mathbb{R}^3 \mapsto ax^2 + by^2 + cz^2 \in \mathbb{R}.$$

(*Remark: your answer will depend on the parameters* a, b, c. *Analyse at least the cases i*) $a = b = c = 1$*; ii*) $a = b = 1$ *and* $c = 0$*; iii*) $a = b = 1$ *and* $c = -1$*; and iv*) $a = 1$ *and* $b = c = -1$). What are the regular surfaces that are the inverse image of such regular values?

Exercise 3.12. Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular *parametrised* surface, as defined in the lectures. Show that for every point $p \in U$ there exists an open neighbourhood U' of p such that *the set* $X(U') \subset \mathbb{R}^3$ *is a regular surface*. Give an example of a regular parametrised surface $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that the full image of the map X, the set $X(U) \subset \mathbb{R}^3$, *is not a regular surface.*

(*Suggestion: use the same map* Φ *that appeared in the proof that changes of parametrisations are smooth maps).*

Exercise 3.13. Let S be a regular surface, and p a point in S. Prove that there exists an open set W of $\mathbb{R}^3$, W containing the point p, and a smooth function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ defined on an open subset U of one the coordinate xy-, xz- or yz-planes, such that

$$W \cap S = graph(f).$$

(*Hint: given a parametrisation* X *of a neighbourhood of* p*, compose it with the orthogonal projection onto one of the coordinate planes to obtain a smooth map between open subsets of* $\mathbb{R}^2$*. Can you invert this map?).*

Exercise 3.14. Consider the following two parametrisations of the cylinder $S = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\}$:

$$X : (u, v) \in (-1, 1) \times \mathbb{R} \mapsto (u, \sqrt{1 - u^2}, v) \in S,$$
$$\widetilde{X} : (s, \theta) \in \mathbb{R} \times (-\pi, \pi) \mapsto (\cos(\theta), \sin(\theta), s) \in S.$$

Compute the domains of the change of parametrisations $X^{-1} \circ \widetilde{X}$ and $\widetilde{X}^{-1} \circ X$ explicitly, and check that they are smooth maps.

Exercise 3.15. Let S be a regular surface in $\mathbb{R}^3$, and $f : S \subset \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. The *gradient* of f at a point p is defined to be the unique vector

$$grad(f)(p) \in T_pS$$

such that

$$Df(p) \cdot v = \langle grad(f)(p), v\rangle \quad \text{for all} \quad v \in T_pS.$$

i) Fix a non-zero vector $w \in \mathbb{R}^3$, and define the *height function*:

$$f : p \in S \mapsto \langle p, w\rangle \in \mathbb{R}.$$

Show that f is smooth and describe what is the tangent plane of S at a point where the gradient of f vanishes (in other words, at a *critical point of* f).

ii) Let $F : V \subset \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, S a regular surface in $\mathbb{R}^3$, $V \cap S \neq \emptyset$, and $f : V \cap S \to \mathbb{R}$ the restriction of F to $V \cap S$. Show that, at each point $p \in V \cap S$, the vector $grad\, f(p) \in T_pS \subset \mathbb{R}^3$ is the orthogonal projection of the vector $grad\, F(p) \in \mathbb{R}^3$ onto the tangent plane T_pS.

Exercise 3.16. Recall the following theorem from Topology:

Let X *be a topological space that is locally path-connected.* X *is connected if and only if* X *is path-connected.*

Show that a regular surface S in $\mathbb{R}^3$ is a locally path-connected topological space. In particular, conclude that

a regular surface in $\mathbb{R}^3$ *is connected if and only if it is path-connected.*

Prove that if S is a *connected* regular surface in $\mathbb{R}^3$, then a smooth function $f : S \to \mathbb{R}$ is constant if and only if its gradient vanishes at all points of S.

Exercise 3.17. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, and $t \in F(\mathbb{R}^3)$ a regular value of F. Let $grad\, F(p) \in \mathbb{R}^3$ denote the gradient vector of F at a point $p \in F^{-1}(t)$ (*cf.* Exercise 15; observe that $grad\, F(p) \neq 0$ since $p \in F^{-1}(t)$ is a regular point of F).

Show that the regular surface $S = F^{-1}(t)$ is such that, at all points $p \in S$,

$$T_pS = \{v \in \mathbb{R}^3 \,|\, \langle grad\, F(p), v\rangle = 0\}.$$

Exercise 3.18. Let S be a regular surface in $\mathbb{R}^3$ and p_0 a point in $\mathbb{R}^3$ that does not belong to the surface. Show that the map

$$f : p \in S \mapsto |p - p_0| \in \mathbb{R}$$

is smooth and compute its derivative at a point $p \in S$. Show that the tangent plane at a critical point p of f is orthogonal to the vector $p - p_0$.

Exercise 3.19. Let S be a regular surface in $\mathbb{R}^3$, and

$$\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^3$$

a map such that

$$\gamma(t) \in S \quad \text{for all} \quad t \in (-\epsilon, \epsilon).$$

Show that γ is a smooth map if and only if, for every parametrisation $X : U \subset \mathbb{R}^2 \to V \cap S \subset \mathbb{R}^3$ such that $\gamma(t) \in V \cap S$ for all $t \in (-\epsilon', \epsilon') \subset (-\epsilon, \epsilon)$, the map

$$\widetilde{\gamma} := X^{-1} \circ \gamma : (-\epsilon', \epsilon') \to U \subset \mathbb{R}^2$$

is smooth.

(*Suggestion: when proving the implication* $\Rightarrow$*, recall the map* Φ *defined in the proof that changes of parametrisations are smooth*).

Exercise 3.20. Let S_1 and S_2 be regular surfaces in $\mathbb{R}^3$. Check that the derivative of a smooth map $\phi : S_1 \to S_2$ at a point $p \in S_1$, defined as the map

$$D\phi(p) : v \in T_pS_1 \mapsto (\phi \circ \gamma)'(0) \in T_{\phi(p)}S_2,$$

where $\gamma : (-\epsilon, \epsilon) \to S$ is a smooth map such that $\gamma(0) = p$ and $\gamma'(0) = v$, is *well-defined*, and indeed a linear map between the tangent planes of S_1 at p and of S_2 at $\phi(p)$. Using parametrisations of coordinates neighbourhoods of p in S_1 and of $\phi(p)$ in S_2, express it in coordinates.

Exercise 3.21. Let $S_1 = \{p \in \mathbb{R}^3 \,|\, |p| = 1\}$ be the unit sphere centred at the origin, $S_2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1, |z| < 1\}$ the piece of the cylinder over the unit circle on the xy plane centred at the origin whose points are at height $-1 < z < 1$, and

$$\phi : S_1 \setminus \{(0, 0, \pm 1)\} \to S_2$$

the map that sends each point p in the sphere minus the two poles to the point of intersection of the cylinder and the half-line that issues from the z-axis horizontally (*i.e.* parallel to the xy plane) and passes through the point p.

Show that ϕ is a smooth bijective map, and show that its derivative at any given point p in $S_1 \setminus \{(0, 0, \pm 1)\}$ is a linear isomorphism between the tangent planes T_pS_1 and $T_{\phi(p)}S_2$. Conclude that $\phi : S_1 \setminus \{(0, 0, \pm 1)\} \to S_2$ is a diffeomorphism.

Exercise 3.22. Let S be a regular surface in $\mathbb{R}^3$, and denote by $S^2_1(0)$ the sphere of radius one centred at the origin. Assume that there exists a *continuous* map

$$N : S \to S^2_1(0)$$

such that $N(p)$ is orthogonal to T_pS for all p in S. Show that N is *smooth.* In particular, conclude that S must be orientable.

Exercise 3.23. (*On orientation-preserving maps*):

i) Let $(V, \mathcal{P})$ and $(W, \mathcal{Q})$ be *oriented* real vector spaces of finite dimension n. A linear map $L : V \to W$ is said to be *orientation-preserving* when it sends positive bases of V into positive bases of W, and *orientation-reversing* otherwise. Show that the set of orientation-preserving linear endomorphisms of an oriented vector space $(V, \mathcal{P})$ form a group under composition of maps. Prove that the linear map $L : V \to V$ given by

$$L : v \in V \to -v \in V$$

preserves the orientation of $(V, \mathcal{P})$ if and only if n is an *even* integer number.

ii) Let S_1 and S_2 be orientable regular surfaces in $\mathbb{R}^3$, *oriented* by collections of parametrizations $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. A smooth map $\phi : S_1 \to S_2$ is said to be *orientation-preserving* when, for every p in S, the derivative $D\phi(p) : T_pS_1 \to T_{\phi(p)}S_2$ is orientation-preserving. Write this definition as a condition on the expression of ϕ in parametrizations $X_1 \in \mathcal{C}$, $X_2 \in \mathcal{C}_2$.

iv) Let $S^1_1(0) \subset \mathbb{R}^2$ be the unit circle centred at the origin, and choose an orientation for it. Show that the *antipodal map*,

$$A : p \in S^1_1(0) \mapsto -p \in S^1_1(0),$$

preserves the chosen orientation of $S^1_1(0)$.

iv) Let $S^2_1(0) \subset \mathbb{R}^3$ be the unit sphere centred at the origin, and choose an orientation for it. Show that the *antipodal map*,

$$A : p \in S^2_1(0) \mapsto -p \in S^2_1(0),$$

reverses the chosen orientation of $S^2_1(0)$.

Exercise 3.24. Let $S \subset \mathbb{R}^3$ be a *connected* regular surface. Is S *orientable* when:

i) there exists a single parametrisation $X : U \to S$ with $S = X(U)$?

ii) there exists two parametrisations $X : U \to S \cap V$ and $\widetilde{X} : \widetilde{U} \to S \cap \widetilde{V}$, none of them covering the entire surface S, such that $S \subset X(U) \cup \widetilde{X}(\widetilde{U})$?

iii) there exists two parametrisations $X : U \to S \cap V$ and $\widetilde{X} : \widetilde{U} \to S \cap \widetilde{V}$, none of them covering the entire surface S, such that $S \subset X(U) \cup \widetilde{X}(\widetilde{U})$ **and** the intersection $X(U) \cap \widetilde{X}(\widetilde{U})$ is a non-empty *connected* proper subset of S?

If you answer to an item is *yes*, please justify your answer. If it is *no*, please give a counter-example.

Exercise 3.25. Considering *smooth functions* defined on (open subsets of) $\mathbb{R}^m$ and taking values in a regular surface in $\mathbb{R}^3$, or defined on (open subsets of) regular surfaces in $\mathbb{R}^3$ and taking values on regular surfaces in $\mathbb{R}^3$, or defined on (open subsets of) regular surfaces in $\mathbb{R}^3$ and taking values in $\mathbb{R}^n$, can you extend the usual operations and theorems of Differentiation to that setting? (*Hint: the answer is "yes".*) For example, please think for a while on the following concrete questions: a) is the product of two smooth functions $f_1, f_2 : S \to \mathbb{R}$ a smooth function, and what is its derivative? b) what would be the statement (and the proof) of the Chain Rule? c) what can be said about smooth maps $f : S_1 \to S_2$ such that $Df(p) : T_pS_1 \to T_{\phi(p)}S_2$ is a linear isomorphism? d) What is a regular value $t \in \mathbb{R}$ of a function $f : S \to \mathbb{R}$, and what do you expect the set $f^{-1}(t)$ to be (when it is not the empty set)?

Exercise 3.26. In analogy to the definition of *regular surfaces in* $\mathbb{R}^3$, how would you define a "*k-dimensional surface in* $\mathbb{R}^n$", its "*tangent space*" at a given point, "*smooth functions*" and their "*derivatives*" etc etc? (*Remark: The jargon is "a k-dimensional submanifold of* $\mathbb{R}^n$").

4. Geometry of the Gauss Map

Exercise 4.1. Consider the following regular surfaces in $\mathbb{R}^3$:

- i) The paraboloid of revolution $\{(x, y, z) \in \mathbb{R}^3 \,|\, z = x^2 + y^2\}$;
- ii) The ellipsoid $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + 2y^2 + 3z^2 = 1\}$;
- iii) The hyperboloid of revolution $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 - z^2 = 1\}$; and
- iv) The catenoid $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = \cosh^2(z)\}$.

In each case, describe the subset of the unit sphere centred at the origin that is covered by the image of the (chosen) Gauss map of the surface.

Exercise 4.2. (*Geometry of graphs*):

Let S be a regular surface in $\mathbb{R}^3$ described by the graph of a smooth function $\phi : U \subset \mathbb{R}^2 \to \mathbb{R}$,

$$S = \{(x, y, z) \in \mathbb{R}^3 \,|\, (x, y) \in U,\ z = \phi(x, y)\}.$$

Orient S by the unit normal associated to its graphical parametrisation

$$X : (u, v) \in U \mapsto (u, v, \phi(u, v)) \in S.$$

Express, in terms of the function ϕ, the coefficients of the first and second fundamental forms of S in this coordinate chart, and give a formula for the mean and the Gaussian curvatures of S at an arbitrary point $p = X(u, v)$.

Exercise 4.3. Let a, b be real numbers, and consider the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \,|\, z = ax^2 + by^2\}.$$

Classify the origin $(0, 0, 0) \in S$ as an elliptic, hyperbolic, parabolic or planar point, according to the values of a and b. When $(0, 0, 0) \in S$ is an umbilic point of S?

Exercise 4.4. Without doing any computations, answer the following question (and explain why you have answered that way): given arbitrary constants a, b, c and d, what is the mean curvature and the Gaussian curvature of the graphical surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \,|\, z = ax^3 + bx^2y + cxy^2 + dy^3\}$$

at the origin $(0, 0, 0) \in S$?

Exercise 4.5. Let S_1 and S_2 be two regular surfaces in $\mathbb{R}^3$ that intersect tangentially along a regular curve $\mathcal{C}$, *i.e.* for every point p in $\mathcal{C} = S_1 \cap S_2$, the tangent planes of S_1 and S_2 coincide.

Let p be a point in $\mathcal{C}$. Orient (locally) the surfaces S_1 and S_2 near p so that the respective unit normal vectors coincide at p. What can be said about the normal curvatures of $\mathcal{C}$ at the point p when viewed as a curve in S_1 and as a curve in S_2?

Exercise 4.6. Let S be a regular surface in $\mathbb{R}^3$. Prove that its Gaussian curvature and the square of its mean curvature satisfy the inequality

$$H^2 \geq K.$$

Show that equality holds at a point $p \in S$ if and only if p is an umbilical point of S.

Exercise 4.7. Let S be a *non-orientable* regular surface in $\mathbb{R}^3$. Show that the mean curvature vector of S vanishes at some point $p \in S$.

Exercise 4.8. Let S be a regular surface, p a point in S and N a local smooth unit normal vector field on S in a neighbourhood of p. Considering the associated second fundamental form II, describe the sets

$$\{v \in T_pS \,|\, II_p(v) = 1\}$$

when p is an elliptic, a hyperbolic, a parabolic, a planar, and an umbilical but not planar point, respectively.

Exercise 4.9. Let S be a regular surface and λ a positive number. Consider the dilated surface,

$$\lambda \cdot S = \{p \in \mathbb{R}^3 \,|\, p = \lambda p_0 \quad \text{for some} \quad p_0 \in S\},$$

obtained by multiplying all points in S by λ (notice that $\lambda \cdot S$ is a regular surface as well).

What is the relation between the Gaussian curvatures of S and $\lambda \cdot S$? What is the relation between the mean curvature vectors of S and $\lambda \cdot S$?

Exercise 4.10. Let S be a regular surface in $\mathbb{R}^3$ such that $p = (0,0,0)$ belongs to S and T_pS is the xy plane.

i) If p is an elliptic point, then *there exists a neighbourhood* $V \subset S$ of p such that $V \setminus \{p\}$ is contained either in the half-space open $\{z > 0\}$ or in the open half-space $\{z < 0\}$. If you also know the mean curvature vector of S at 0, can you tell in which half-space $V \setminus \{p\}$ is contained?

ii) If p is a hyperbolic point, then *every neighbourhood* $V \subset S$ of p intersects both open half-spaces $\{z > 0\}$ and $\{z < 0\}$.

iii) What happens when $K(p) = 0$? Give examples of several possible behaviours of S near the point p.

Exercise 4.11. Let $w \in \mathbb{R}^3$ be a non-zero unit vector, and S a regular surface.

Consider the function

$$\phi : p \in \mathbb{R}^3 \mapsto \langle p, w \rangle \in \mathbb{R},$$

whose level sets $\phi^{-1}(t)$, $t \in \mathbb{R}^3$, foliate $\mathbb{R}^3$ by parallel planes that are orthogonal to w.

Suppose that the restriction of ϕ to S attains a local minimum at a point $p_0 \in S$. Show that $K(p_0) \geq 0$ and $\langle \vec{H}(p), w \rangle \geq 0$. Similarly, what can be said about a point of local maximum?

Exercise 4.12. Let S be a compact regular surface in $\mathbb{R}^3$. Let p_0 be an arbitrary point in $\mathbb{R}^3$. Show that there exists a point $p_1 \neq p_0$ in S such that

$$|p_1 - p_0| \geq |p - p_0| \quad \text{for all} \quad p \in S.$$

Prove that p_1 is an elliptic point of S.

Exercise 4.13. Let S be a compact connected regular surface in $\mathbb{R}^3$. Let A be the set of all positive real numbers r such that S is strictly contained inside a sphere of radius r.

Prove that the infimum of A is a positive real number $r_0 > 0$. Moreover, show that there exists a point $p_0 \in \mathbb{R}^3$ such that S is contained in the closed ball of radius r_0 centred at p_0, and such that the sphere $S_{r_0}(p_0)$ of radius r_0 centred at p_0 intersects S tangentially at each intersection point.

Show that S intersects $S_{r_0}(p_0)$ at two different points at least. Give an example of a regular surface in $\mathbb{R}^3$ such that its intersection with $S_{r_0}(p_0)$ consists of exactly two points.

Exercise 4.14. Let S_1 and S_2 be regular surfaces in $\mathbb{R}^3$ that intersect tangentially at a point $p \in S_1 \cap S_2$. Assume that S_1 lies on one side of S_2 near the point p, and choose the unit normal N_p of S_2 at p that points towards the side of S_2 where S_1 lies. Show that

$$\langle \vec{H}^{S_1}, N_p \rangle \geq \langle \vec{H}^{S_2}, N_p \rangle.$$

Exercise 4.15. A regular surface in $\mathbb{R}^3$ is called *minimal* when its mean curvature vanishes at every point (*e.g.* planes). Prove that there exists no *compact* minimal regular surfaces in $\mathbb{R}^3$.

Exercise 4.16. A regular surface in $\mathbb{R}^3$ is said to have *constant mean curvature* when its mean curvature vector has constant norm. Give examples of non-compact surfaces of constant mean curvature. Prove that a connected compact regular surface with constant mean curvature *and* positive Gaussian curvature must be a sphere.

(*Remark*: a theorem by *A. D. Alexandrov* improves this result and asserts that *the only compact regular surfaces in* $\mathbb{R}^3$ *with constant mean curvature are spheres*!).

Exercise 4.17. Consider the regular parametrised surface of revolution

$$X : (s, \theta) \in I \times \mathbb{R} \mapsto (\phi(s)\cos(\theta), \phi(s)\sin(\theta), \psi(s)) \in \mathbb{R}^3,$$

where $\phi(s) > 0$ and $(\phi'(s))^2 + (\psi'(s))^2 = 1$ for all $s \in I$. Compute its mean curvature and its Gaussian curvature.

Exercise 4.18. Consider the regular parametrised surface of revolution

$$X : (s, \theta) \in I \times \mathbb{R} \mapsto (\phi(s)\cos(\theta), \phi(s)\sin(\theta), \psi(s)) \in \mathbb{R}^3,$$

where $\phi(s) > 0$ and $(\phi'(s))^2 + (\psi'(s))^2 = 1$ for all $s \in I$. Compute the normal curvature of the coordinate curves $s = const.$ and $\theta = const.$ (*Caution: the formula given in the lectures is the formula that is valid for curves parametrised by arc-length*).

Exercise 4.19. (*Surfaces of revolution with constant Gaussian curvature, c.f. do Carmo, 3.3., Exercise 7 - beware the sketch, it is not quite right in the case $K = 1$! See do Carmo 5.2, Example 1*):

Considering a regular parametrised surface of revolution described by a map X as in the previous exercise, prove that its Gaussian curvature is equal to a constant $c \in \mathbb{R}$ if and only if the function $\phi > 0$ satisfy the second order ODE

$$(*) \quad \phi'' + c\phi = 0 \quad \text{on} \quad I.$$

Assuming for simplicity that $0 \in I$, show that ψ is then determined by the formula

$$(**) \quad \psi(s) = \int_0^s \sqrt{1-(\phi'(u))^2}du \quad \text{for all} \quad s \in I.$$

Let us then suppose that $\phi(s) > 0$ and $(\phi'(s))^2 + (\psi'(s))^2 = 1$ for all $s \in I$, and that I is the maximal interval containing 0 where solutions to $(*)$ and $(**)$ satisfying these two further conditions exist.

Solving $(*)$ for an initial admissible condition $\phi(0) = A > 0$, $\phi'(0) = B \in [-1, 1]$, on the maximal interval I, and considering the corresponding ψ given by $(**)$, study the surfaces obtained on each case $c = 0$, $c = 1$ and $c = -1$. In particular, conclude that

i) If $c = 0$, then X parametrises either a right circular cylinder, or a cone, or a plane.

ii) If $c = 1$, show that all solutions ϕ attain a maximum value at a point in I. Changing parameters so that $0 \in I$ corresponds to that maximum point, show that the solutions are given by

$$\phi(s) = A\cos(s) \quad \text{and} \quad \psi(s) = \int_0^s \sqrt{1 - A^2\sin^2(u)}du.$$

Determine the domain of a solution ϕ. Which value of A correspond to a sphere of curvature 1? For values of $0 < A < 1$, show that the curve $(\phi(s), 0, \psi(s))$, $s \in I$, converges to the rotation axis as s approaches the boundary of the interval I, at a definite angle $\neq \pi/2$. For values $A > 1$, show that $(\phi(s), 0, \psi(s))$, $s \in I$, is such that $\phi(s) > \sqrt{A^2-1}/A > 0$. Finally, show that for all positive $A \neq 1$, the surface of revolution generated by the above curves are regular surfaces in $\mathbb{R}^3$ with no umbilic points that, moreover, are not proper subsets of any regular surface of $\mathbb{R}^3$.

iii) If $c = -1$, show that solutions ϕ are such that $|\phi'(t)|$ eventually grows monotonically to 1 as t approaches a finite number $\bar{t}$. (This means that the image of X *can not* be continued further as a regular surface). Possibly after changing the parameter, show that solutions fall into one of the following types

$$\phi(s) = A\cosh(s) \quad \text{and} \quad \psi(s) = \int_0^s \sqrt{1 - A^2\sinh^2(u)}du, \quad \text{or}$$

$$\phi(s) = D\sinh(s) \quad \text{and} \quad \psi(s) = \int_0^s \sqrt{1 - D^2\cosh^2(u)}du, \quad \text{or}$$

$$\phi(s) = e^s \quad \text{and} \quad \psi(s) = \int_0^s \sqrt{1 - e^{2u}}du.$$

(in the second case, D is a constant such that $0 < D < 1$).

Draw sketches of the solutions obtained (or plot them using a computer).

Exercise 4.20. (*Non-planar minimal surfaces of revolution are catenoids*):

i) Show that the catenoid is a minimal surface.

ii) Let S be the surface of revolution

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z \in \mathbb{R}, \sqrt{x^2+y^2} = f(z)\},$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth positive function. Prove that, if S is a minimal surface, then it is a catenoid (up to a dilation and vertical translation).

Exercise 4.21. (*The helicoid*):

Let S be the set obtained by taking a horizontal straight line (say, the x-axis), and moving it with constant speed in the z direction while the line rotates horizontally at constant speed. Show that S is a regular surface in $\mathbb{R}^3$ parametrised by the map

$$X : (t, \theta) \in \mathbb{R}^3 \mapsto (t\cos(\theta), t\sin(\theta), \theta) \in \mathbb{R}^3.$$

Prove that S is a minimal surface.

Exercise 4.22. Let S be a regular surface in $\mathbb{R}^3$. A *line of curvature* is a curve C in S that is tangent to a principal direction of S at every point p in C.

Assume that S is orientable, with Gauss map N_S. Show that a regular parametrised curve $\alpha : I \to S$ is a line of curvature if and only if there exists a smooth function $\lambda : I \to \mathbb{R}$ such that

$$(N_S \circ \alpha)'(t) = \lambda(t)\alpha'(t) \quad \text{for all} \quad t \in I.$$

Exercise 4.23. Let S be a regular surface in $\mathbb{R}^3$. An *asymptotic curve* is a curve in S whose normal curvature vanishes identically.

i) Show that asymptotic curves do not contain elliptic points of S.

ii) If S contains a straight line L, then L is an asymptotic curve of S.

iii) If L is an asymptotic curve in S that is tangent to a principal direction at all of its points, then L is a plane curve.

iv) Give an example of a regular surface in $\mathbb{R}^3$ that contains straight lines that *are not* tangent to a principal direction at every point.

Exercise 4.24. (*An exercise in Differentiation*):

Let $F : U \subset \mathbb{R}^m \to \mathbb{R}$ be a smooth function, and p be a critical point of F. Let $\gamma : (-\epsilon, \epsilon) \to U$ be a smooth map such that $\gamma(0) = p$. If $\gamma'(0) = (a^1, \ldots, a^m) \in \mathbb{R}^n$, then

$$(F \circ \gamma)''(0) = \sum_{i,j=1}^{m} a^i a^j \frac{\partial^2 F}{\partial x^i \partial x^j}(0).$$

In particular, conclude that if $\gamma_1, \gamma_1 : (-\epsilon, \epsilon) \to U$ are smooth maps such that $\gamma_1(0) = \gamma_2(0) = p$ and $\gamma_1'(0) = \gamma_2'(0)$, then

$$(F \circ \gamma_1)''(0) = (F \circ \gamma_2)''(0).$$

Show that above conclusion does not remain true if the same computation is done at a point p that is a regular point of F.

Exercise 4.25. (*On the Spectral Theorem*):

Let S^1 denote the unit circle in $\mathbb{R}^2$ centred at the origin.

i) If $f : S^1 \to \mathbb{R}$ is a smooth map such that

$$f(-v) = f(v) \quad \text{for all} \quad v \in S^1,$$

then f has at least *four* critical points.

ii) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a self-adjoint linear map. Let E be the function given by

$$E : v \in S^1 \mapsto \langle L(v), v\rangle \in \mathbb{R}.$$

Show that $v \in S^1$ is **a critical point of** E if and only if v is an eigenvector of L. What is the corresponding eigenvalue, **in terms of** E?

iii) Show that eigenvectors of a self-adjoint linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$, corresponding to different eigenvalues, are mutually orthogonal.

iv) Combining the topological lemma in item *i*) with the other two items, can you prove the Spectral Theorem? (*Discussion: item i) can be generalised to higher dimensions (and beyond) via a topological theory called the Lusternik-Schnirelmann Theory, which over years has inspired many remarkable results in Geometry*).

5. Intrinsic Geometry

Exercise 5.1. Compute the coefficients E, F and G of the first fundamental form of the unit sphere centred at the origin in the coordinate system given by the inverse of the stereographic projection from the north pole. In particular, verify that $E = G$ and $F = 0$ (*Remark: such parametrisations are called conformal, or isothermal*).

Exercise 5.2. Let S_1 be the unit sphere in $\mathbb{R}^3$ centred at the origin and S_2 be the cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2 = 1\}$. Show that the map $\phi : S_1 \setminus \{(0, 0, \pm 1)\} \to S_2$ defined on Section 2, Exercise 21, is an *area-preserving map*: for every compact subset K of $S_1 \setminus \{(0, 0, \pm 1)\}$, the area of K is equal to the area of $\phi(K)$.

Exercise 5.3. (*Area of spheres and tori of revolution*):

i) Describe the sphere of radius $R > 0$ centred at the origin as a surface of revolution around the z-axis, and compute its area.

ii) Compute the area of the torus $T_{r,R}$ obtained by revolution around the z-axis of the circle on the plane xz of radius r centred at the point $(R, 0, 0)$, where $R > r > 0$.

Exercise 5.4. Let $f : U \to \mathbb{R}$ be a smooth function, and Ω a compact smooth domain contained in U. Show that the area of $K = f(\Omega) \subset graph(f)$ is given by

$$\int_\Omega \sqrt{1 + |\nabla f|^2} dudv,$$

where $\nabla f(u, v) = grad\, f(u, v) = (f_u(u, v), f_v(u, v)) \in \mathbb{R}^2$ denotes the gradient of f at the point $(u, v) \in \mathbb{R}^2$.

Exercise 5.5. (*Integral of real functions defined on surfaces*):

Let S be a connected regular surface in $\mathbb{R}^3$, $f : S \to \mathbb{R}$ a continuous function, and K a compact subset of S that is contained in a coordinate neighbourhood $V \cap S = X(U)$. The *integral* of f over the set K is the number

$$\int_K f dS := \int_{X^{-1}(K)} (f \circ X)\sqrt{EG - F^2} dudv.$$

(*Remark: this definition is independent of choices of parametrisations*).

Let $S = T_{r,R}$ be the torus of revolution

$$T_{r,R} = \{(x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\},$$

and

$$X : (u, v) \in (0, 2\pi) \times (0, 2\pi) \mapsto ((r \cos(u) + R) \cos(v), (r \cos(u) + R) \sin(v), r \sin(u)) \in T_{r,R}$$

a parametrisation of S. Show that, in these coordinates, the Gaussian curvature of $T_{r,R}$ is given by

$$K = \frac{\cos(u)}{r(r\cos(u)+R)},$$

and compute the integral

$$\int_S K dS.$$

(*Suggestion: use the formula* $K = (eg - f^2)/(EG - F^2)$. *Convince yourself that the subset of* S *missed by the image of parametrisation* X *is negligible for the calculus of the above integral*).

Exercise 5.6. Let S_1 and S_2 be two regular surfaces in $\mathbb{R}^3$, and assume that there exists two parametrisations $X_1 : U \to V_1 \cap S_1$ and $X_2 : U \to V_2 \cap S_2$, defined over the same open subset U of $\mathbb{R}^2$, such that the coefficients of the first fundamental form of S_1 and S_2 in the coordinate systems X_1 and X_2 coincide at each point of U:

$$E_1 = E_2, \quad F_1 = F_2, \quad \text{and} \quad G_1 = G_2.$$

Show that there exists a local isometry $\phi : V_1 \cap S_1 \to V_2 \cap S_2$. Use this criterion to produce local isometries between: a plane and the right cylinder over an horizontal circle, and a plane and a cone. (*Extra: and between the helicoid and the catenoid - ver Parte 3*).

Exercise 5.7. Let S be a regular surface in $\mathbb{R}^3$ and $X : U \subset \mathbb{R}^2 \to V \cap S$ be a parametrisation such that the coefficient F of the first fundamental form vanishes identically on U.

Compute the Christoffel symbols in this parametrisation and show that

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v\right).$$

Use the above formula to compute the Gaussian curvature in coordinates given by a parametrisation such that

$$E(r,\theta) = 1, \quad F(r,\theta) = 0, \quad \text{and} \quad G(r,\theta) = \phi^2(r),$$

for all (r,θ) in its domain, in the following cases:

a) $\phi(r) = \sin(r)$.
b) $\phi(r) = r$.
c) $\phi(r) = \sinh(r)$.

Exercise 5.8. Let S be a regular surface in $\mathbb{R}^3$ and $X : U \subset \mathbb{R}^2 \to V \cap S$ be a parametrisation such that the coefficients of first fundamental satisfy

$$E = G = \lambda \quad \text{and} \quad F = 0$$

on U, where λ is a smooth positive function on U. Parametrisations satisfying the above property are called *isothermal parametrisations*.

Show that

$$K = -\frac{1}{2\lambda}\Delta(\log(\lambda)),$$

where Δ denotes the Laplacian operator in U,

$$\Delta\lambda := \frac{\partial^2\lambda}{\partial u^2} + \frac{\partial^2\lambda}{\partial v^2}.$$

In particular, conclude that, if the coefficients of the first fundamental form are given by

$$E(u,v) = G(u,v) = \frac{1}{v^2} \quad \text{and} \quad F(u,v) = 0,$$

for all (u,v) in its domain, then the Gaussian curvature is constant and equal to -1.

Exercise 5.9. A regular surface S in $\mathbb{R}^3$ is called homogeneous when for every two points p and q in S there exists an isometry $\phi : S \to S$ such that $\phi(p) = q$. Prove that a homogeneous surface has constant Gaussian curvature. What are the connected compact homogeneous regular surfaces in $\mathbb{R}^3$?

Exercise 5.10. Let S be a regular surface in $\mathbb{R}^3$ and let $\alpha : I \to S$ be a regular parametrised curve in S. Show that the following statements are equivalent:

i) The tangent α' is a parallel vector field along α.
ii) For every $t \in I$, the vector $\alpha''(t)$ is orthogonal to S.
iii) $|\alpha'|$ is constant and the geodesic curvature of α vanishes identically on I.

(*Remark: notice that* i) *can be expressed as an instrinsic condition, whereas* ii) *is an extrinsic condition*).

Exercise 5.11. Let S be a regular surface in $\mathbb{R}^3$ and $\mathcal{C}$ a regular curve in S. We may call $\mathcal{C}$ a geodesic when it is covered by parametrisations by arc-length $\beta : I \to \mathcal{C}$ whose geodesic curvature vanishes identically. Let $\alpha : I \to S$ be a parametrisation of $\mathcal{C}$, not necessarily by arc-length. Show that the regular curve $\mathcal{C}_0 = \alpha(I) \subset \mathcal{C}$ is a geodesic in S if and only if $\alpha''(t)$ belongs to the plane generated by the tangent vector $\alpha'(t)$ and a unit normal to the surface at $\alpha(t) \in S$ for all $t \in I$.

(*Suggestion: to solve this exercise, reparametrise $\beta = \alpha \circ \phi : J \to S$ by arc-length, compute its geodesic curvature and check what the condition $k_g(s) = 0$, $s \in J$, for the parametrised curve β, imposes on the vector $\alpha''(t)$, $t = \phi(s) \in I$. Compare with the previous exercise*).

Exercise 5.12. Let S_1 and S_2 be two regular surfaces in $\mathbb{R}^3$ that intersect tangentially along a regular curve $\mathcal{C}$ (*i.e.* $T_pS_1 = T_pS_2$ for every point $p \in \mathcal{C} \subset S_1 \cap S_2$). Prove that $\mathcal{C}$ is a geodesic in S_1 if and only if $\mathcal{C}$ is a geodesic in S_2.

Exercise 5.13. (*Exercise on geodesics, and their extrinsic geometry*):

Let S be a regular surface in $\mathbb{R}^3$.

i) A curve in S is simultaneously a geodesic and an asymptotic line if and only if it is contained in a straight line.
ii) Show that if a curve C in S is a geodesic and a line of curvature, then it must be contained in a plane.
iii) If a geodesic C is a plane curve, then either C is contained in a straight line or is a line of curvature.
iv) Give an example of a regular surface in $\mathbb{R}^3$ that contains line of curvatures that are plane curves but not geodesics.

Exercise 5.14. Let S be a regular surface of revolution in $\mathbb{R}^3$ parametrised by

$$X : (s,\theta) \in I \to (\phi(s)\cos(\theta), \phi(s)\sin(\theta), \psi(s)),$$

where $\phi(s) > 0$ and $(\phi'(s))^2 + (\psi'(s))^2 = 1$ for all $s \in I$. Prove that the coordinate curves $\theta = const.$ are geodesics. When are the coordinate curves $s = const.$ geodesics?

Exercise 5.15. (*Geodesics of the sphere*): Let S be a sphere in $\mathbb{R}^3$. A *great circle* in S is the regular curve in S determined by the intersection of S with a plane that contains the origin.

i) Prove that great circles are geodesics.
ii) Show that for each point p in S and for each tangent vector $v \neq 0$ in T_pS there exists a unique great circle passing through p and tangent to v.
iii) Conclude that the geodesics of a sphere are great circles.

Exercise 5.16. Let S be a regular surfaces in $\mathbb{R}^3$. Show that the set of isometries $\phi : S \to S$, endowed with the operation of composition of maps, is a group. If S is orientable, show that the subset of those isometries of S that preserve a given orientation of S form a subgroup of the group of isometries.

Exercise 5.17. (*Isometries of the sphere*) Let $S = S^2_1(0)$ be the unit sphere centred at the origin.

i) Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation. Show that $R(S) = S$ and that the restriction $R : S \to S$ is an isometry of S.
ii) Let $\phi : S \to S$ be an isometry such that $\phi(p) = p$ and

$$D\phi(p) \cdot v = v \quad \text{for all} \quad v \in T_pS.$$

Show that $\phi(p) = p$ for all $p \in S$ (*i.e.* $\phi : S \to S$ must be the identity map).
iii) Let $\phi : S \to S$ be an isometry of S, and denote by p_0 the north pole. Show that there exists a unique orthogonal transformation $R : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\phi(p_0) = R(p_0)$ and $D\phi(p_0) \cdot v = R(v)$ for every $v \in T_pS \subset \mathbb{R}^3$.
iv) Conclude that isometries of the unit sphere centred of the origin are restrictions of orthogonal transformations of $\mathbb{R}^3$.
v) What are the orientation-preserving isometries of $S^2_1(0)$? Give an example of an orientation-reversing isometry of $S^2_1(0)$.

Exercise 5.18. Let S be a regular surface in $\mathbb{R}^3$, $\alpha : I \to S$ a regular parametrised curve in S starting at $p = \alpha(0)$, $0 \in I$, and W_1, $W_2 : I \to \mathbb{R}^3$ two tangent vector fields to S along α. Prove that

$$\langle W_1, W_2\rangle' = \Big\langle \frac{DW_1}{dt}, W_2 \Big\rangle + \Big\langle W_1, \frac{DW_2}{dt} \Big\rangle.$$

Next, suppose that W_1 and W_2 are parallel along α. If $W_1(0)$ is orthogonal to $W_2(0)$, show that $W_1(t)$ is orthogonal to $W_2(t)$ for every $t \in I$.

Exercise 5.19. Let S be a regular surface in $\mathbb{R}^3$, $\alpha : I \to S$ a regular parametrised curve in S, and $\beta = \alpha \circ \phi : J \to S$ a reparametrisation of α. If $W : I \to \mathbb{R}^3$ is a parallel tangent vector field along α, then $Y = W \circ \phi : J \to \mathbb{R}^3$ is a parallel tangent vector field along β.

Exercise 5.20. (*The holonomy group*):

Let S be a regular surface in $\mathbb{R}^3$, p a point in S and $\alpha : [0,1] \to S$ a piecewise smooth map into S that starts and finishes at the point p, that is, $\alpha(0) = p$ and $\alpha(1) = p$. Let $P_\alpha : T_pS \to T_pS$ be the map that assigns, to each vector $v \in T_{\alpha(0)}S = T_pS$, the vector $P_\alpha(v) \in T_{\alpha(1)}S = T_pS$ that is the parallel transport of v along α at $t = 1$.

Show that $P_\alpha : T_pS \to T_pS$ is a linear map that preserves the first fundamental form of S at p:

$$\langle P_\alpha(v), P_\alpha(w)\rangle = \langle v, w\rangle \quad \text{for all} \quad v, w \in T_pS.$$

Notice that the composition of maps $P_\beta \circ P_\alpha$ corresponds to do parallel transport along the curve $\gamma : [0,1] \to S$ that on $[0,1/2]$ traverses α with double speed, and on $[1/2,1]$ traverses β with double speed. The set of all such transformation P_α, endowed with composition, is a group known as the *holonomy group* of the surface S at the point p.

Let $S = S^2_1(0)$ be the unit sphere centred at the origin, and let $p_0 = (0,0,1)$ be the north pole. Show that the holonomy group of $S^2_1(0)$ at p_0 is the set of orientation-preserving rotations of the (oriented) plane $T_{p_0}S$.

(*Hint: it is convenient to use curves consisting of geodesics, since their tangent vectors are parallel. Exercise 18 is also useful. Use paths consisting of a geodesic that goes from p_0 until the equator, then move along the equator, and then return to the the north pole as a geodesic*).

Exercise 5.21. (*Geometry of regular parametrised curves in spheres*):

Let $S = S^2_1(0)$ denote the unit sphere in $\mathbb{R}^3$, centred at the origin. Orient $S^2_1(0)$ by the outward pointing Gauss map

$$N_S : p \in S \mapsto p \in S.$$

Let $\alpha : I \to S$ be a regular curve in S parametrised by arc-length. Consider the orthonormal frame at $s \in I$ consisting of the vectors

$$\{T := \alpha'(s), N_S(s) := N_S(\alpha(s)), D(s) := N_S(s) \times \alpha'(s)\}.$$

(*Remark: this frame is in general different from the Frenet frame*).

Show that the above adapted frame satisfy the equations

$$\begin{pmatrix} T' \\ N_S' \\ D' \end{pmatrix} = \begin{pmatrix} 0 & -1 & k_g \\ 1 & 0 & 0 \\ -k_g & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} T \\ N_S \\ D \end{pmatrix},$$

where $k_g = \langle \alpha'', N_S \times \alpha' \rangle$ is the geodesic curvature of α.

Can you propose a statement and a proof for the *Fundamental Theorem of Curves in $S^2_1(0)$*?

Classify those curves in the unit sphere that have constant geodesic curvature.

6. The Gauss Bonnet Theorem

Exercise 6.1. Draw several closed piecewise regular parametrised plane curves and compute the total angular variation of their tangent lines (please draw at least four of them with total angular variation equal to 2π, 4π, 0 and -6π, respectively).

Exercise 6.2. Explain the precise meaning of each term in the Gauss-Bonnet formula for a compact piecewise regular region Ω in an oriented surface S in $\mathbb{R}^3$:

$$\int_\Omega K + \int_{\partial\Omega} k_g + \sum_{i=1}^{k} \phi_i = 2\pi\chi(\Omega).$$

Exercise 6.3. Check that a sphere S of radius $R > 0$ in $\mathbb{R}^3$ is such that

$$\int_S K = 4\pi.$$

Exercise 6.4. Compute the area of a geodesic triangle $\mathcal{T}$ in a unit sphere in $\mathbb{R}^3$ in terms of the interior angles of $\mathcal{T}$.

Exercise 6.5. Let S be the unit sphere in $\mathbb{R}^3$, centred at the origin. Use the Gauss-Bonnet Theorem to show that the area of the region bounded by the parallels at height $z = h_1$ and $z = h_2$, where $-1 \leq h_1 < h_2 \leq 1$, is equal to

$$2\pi(h_2 - h_1).$$

Exercise 6.6. Let S be a compact connected regular surface **in** $\mathbb{R}^3$ that has non-negative Gaussian curvature. What is the genus of S?

Exercise 6.7. Let S be a connected compact oriented regular surface in $\mathbb{R}^3$ with positive Gaussian curvature. Any two disjoint closed simple curves in S bound an annular region. Conclude that any two closed simple geodesics in S must intersect.

Exercise 6.8. Let S be an oriented regular surface in $\mathbb{R}^3$ with non-positive Gaussian curvature, and assume that every *simple* closed piecewise regular curve in S bounds a region homeomorphic to a disc. Given p and q two distinct points in S, show that there can be *no more than one* geodesic joining p to q.

Exercise 6.9. Compute the Euler characteristic of at least ten different (not homeomorphic) piecewise regular regions on arbitrary regular surfaces in $\mathbb{R}^3$.

7. Other exercises

Exercise 7.1. Let $\alpha : I \to \mathbb{R}^n$ be a regular parametrized curve. If the trace of α is a straight line, show that there exists a point $p \in \mathbb{R}^n$, a unit vector $v \in \mathbb{R}^n$ and a smooth function $\lambda : I \to \mathbb{R}$ whose derivative never vanishes such that $\alpha(t) = p + \lambda(t)v$ for all $t \in I$. In particular, conclude that $\alpha'(t)$ is parallel to $\alpha''(t)$ for all $t \in I$.

Exercise 7.2. (*Cf. Exercise 2.16*) Let $\alpha : I \to \mathbb{R}^2$ be a curve parametrised by arc-length. Suppose that C_λ is a circle of radius $R > 0$ that is tangent to $\alpha(I)$ at the point $\alpha(s_0)$, and that α is oriented so that its normal at s_0 points inside the circle.

i) If $\alpha(I)$ lies inside the circle, then $k(s_0) \geq 1/R$.
ii) If $\alpha(I)$ lies outside the circle, then $k(s_0) \leq 1/R$.

Exercise 7.3. (*The sign of the torsion, cf. do Carmo section 1.6*).

Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc-length with non-vanishing curvature. Assume that $0 \in I$, $\alpha(0) = 0$ and that the Frenet trihedron at 0 is the canonical basis of $\mathbb{R}^3$. In particular, its osculating plane at $0 \in I$ is the plane $z = 0$.

Writing $\alpha(s) = (x(s), y(s), z(s))$ in Euclidean coordinates, write the Taylor expansion of α at $s = 0$ up to order three,

$$\alpha(s) = \alpha(0) + s\alpha'(0) + \frac{s}{2}\alpha''(0) + \frac{s^3}{6}\alpha'''(0) + o(s^3),$$

in terms of the curvature k, its derivative k', and the torsion τ of α at 0. Analysing the sign of $z(s)$, describe what happens with the point $\alpha(s)$ as s varies from a negative number to a positive number, according to the sign of $\tau(0)$.

Exercise 7.4. (*Curvature of curves in* $\mathbb{R}^3$ *and osculating planes, cf. do Carmo section 1.6*).

Keeping the notation of the previous exercise, show that the curvature of the orthogonal projection of α on the plane $z = 0$ at the point $s = 0$ (which in this case is the osculating plane of α at $s = 0$) is equal to the curvature of the curve α at $s = 0$.

Exercise 7.5. Let $\alpha : [0, \ell] \to \mathbb{R}^3$ be a closed curve parametrized by arc-length whose curvature never vanishes. Let $B : [0, \ell] \to \mathbb{R}^3$ be the binormal. If α has constant torsion, show that B is a regular parametrised curve whose trace lies in the unit sphere centred at the origin in $\mathbb{R}^3$. What can you say about the geodesic curvature of this curve?

Exercise 7.6. Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrised by arc-length and fix $s_0 \in I$. Suppose its curvature is never zero, and let $B : I \to \mathbb{R}^3$ denote the bi-normal vector field. If α has constant torsion $\tau \neq 0$, then

$$\alpha(s) = \alpha(s_0) - \frac{1}{\tau}\int_{s_0}^{s} B(s) \times \frac{dB}{ds}(s)ds \quad \text{for all} \quad t \in I.$$

Exercise 7.7. Show that there exists a subset S of $\mathbb{R}^2$ and maps $X, \widetilde{X} : (-1, 1) \to \mathbb{R}^2$ that satisfy the following properties:

- *i)* X and $\widetilde{X}$ are bijections onto S.
- *ii)* X and $\widetilde{X}$ are smooth.
- *iii)* $X'(t) \neq 0$ and $\widetilde{X}'(t) \neq 0$ for every $t \in (-1, 1)$.
- *iv)* *None of the compositions* $X^{-1} \circ \widetilde{X} : (-1, 1) \to (-1, 1)$ *and* $\widetilde{X}^{-1} \circ X : (-1, 1) \to (-1, 1)$ *is continuous.*

How does this exercise relate to the proof that change of parametrisations of a regular curves in $\mathbb{R}^2$ are difeomorphims?

(*Hint: consider the figure of the number* 8*, map* $0 \in (-1, 1)$ *to the self-intersection point, and try to imagine different ways to map* $(-1, 0]$ *and* $[0, 1)$ *continuously onto the figure* 8 *in such way that near* -1 *and* 1 *the maps are converging to the self-intersection point*).

Exercise 7.8. Let S be a regular surface in $\mathbb{R}^3$, and p a point in S. Show that there is a sufficiently small neighbourhood V of p in $\mathbb{R}^3$ such that $S \cap V$ is a graph of a smooth function on the tangent plane of S at p. (*Hint: without loss of generality, after a rigid motion you may assume* $p = 0$ *and* T_pS *is the xy-plane. Use the Inverse Function Theorem*).

Exercise 7.9. Let $f : U \to \mathbb{R}$ be a smooth function defined on an open neighbourhood of the origin. Assume that $f(0, 0) = 0$ and $Df(0, 0) = 0$.

- *i)* Consider the regular surface $S = graph(f)$. Show that $T_{(0,0)}S$ is the xy plane.
- *ii)* Consider the regular surfaces
$$\lambda \cdot S = \{\lambda \cdot p \in \mathbb{R}^3; p \in S\},$$
obtained by dilation based at the origin by the factor $\lambda > 0$. Show that the regular surfaces $\lambda \cdot S$ are the graphs of smooth functions f_λ on open subsets of the xy plane.
- *iii)* As λ goes to infinity, the domains of the functions f_λ eventually contain a open disc that is the intersection of an open ball B_r in $\mathbb{R}^3$ of radius r centred at the origin with the xy-plane. Show that functions f_λ and its first partial derivatives converge to zero uniformly on each B_r as λ goes to infinity. (*Comment: actually, each partial derivative of any order converge to zero uniformly. This exercise can be interpreted*

as saying that, as one dilates S around a point $p \in S$ by a factor $\lambda > 0$, dilated surfaces converge locally smoothly to T_pS as λ goes to infinity. Thus, in a geometric sense, T_pS is the best infinitesimal approximation of S).

Exercise 7.10. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a self-adjoint endomorphism. Show that

$$\int_{S^{n-1}} \langle L(v), v\rangle dA_{can}(v) = \alpha_n tr(L),$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$ centred at the origin, and α_n is a constant depending only on the dimension n.

(Hint: try first in dimension $n = 2$. Use a basis diagonalising L and then compute the integral over the circle S^1 directly. In higher dimensions, the vector field $p \in \mathbb{R}^n \mapsto L(p) \in \mathbb{R}^n$ might be useful).

Exercise 7.11. Let L, $M : \mathbb{R}^n \to \mathbb{R}^n$ be self-adjoint endomorphisms. Let Q, $R : \mathbb{R}^n \to \mathbb{R}$ be the associate quadratic forms,

$$Q(v) = \langle Lv, v\rangle, \quad R(v) = \langle Mv, v\rangle \quad \text{for all} \quad v \in \mathbb{R}^n,$$

and let $\lambda_1 \leq \ldots \leq \lambda_n$, $\mu_1 \leq \ldots \leq \mu_n$ be the corresponding eigenvalues of L and M.

In this exercise, we assume that:

$$Q(v) \geq R(v) \quad \text{for all} \quad v \in \mathbb{R}^n.$$

i) Show that $trace(L) \geq trace(M)$.

ii) Suppose the dimension is $n = 2$. Give an example of L and M as above such that $det(L) < det(M)$.

iii) Show that $\lambda_1 \geq \mu_1$ and $\lambda_n \geq \mu_n$.

iv) Suppose the dimension is $n = 3$. What can be said about the diference of the middle eigenvalues of L and M, that is, about the number $\lambda_2 - \mu_2$?

(Comment: the reader should try to make the connection between this exercise and the discussion on regular surfaces that intersect tangentially at a point p).

Exercise 7.12. Let $S \subset \mathbb{R}^3$ be a regular surface and let $X : U \subset \mathbb{R}^2 \to V \cap S \subset \mathbb{R}^3$ be a parametrisation of S. Check the following identity:

$$\langle ((X_{uu})^T)_v - ((X_{uv})^T)_u, X_v\rangle = (K \circ X)(|X_u|^2|X_v|^2 - \langle X_u, X_v\rangle^2),$$

where w^T denotes the orthogonal projection of a vector w in $\mathbb{R}^3$ onto the tangent plane T_pS at the point $p = X(q)$, and $K \circ X$ is the expression in coordinates of the Gaussian curvature of S. Interpret the result using the concept of covariant derivative.