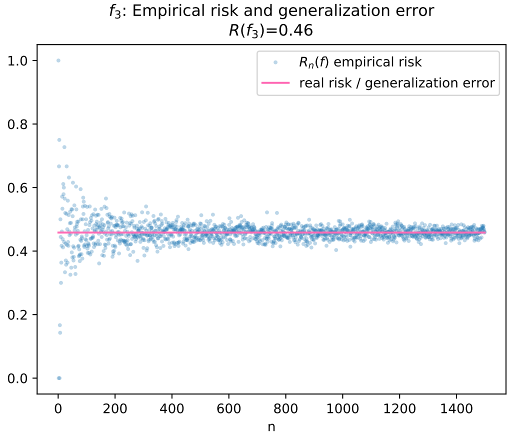
Fondamentaux théoriques du machine learning



Exercice 1 : (Analogous to the penalty shootout example) Consider the following random variable (X, Y).

- $X \sim B(\frac{1}{2})$,
$$Y = \begin{cases} B(p) \text{ if } X = 1 \\ B(q) \text{ if } X = 0 \end{cases}$$
With $B(p)$ a Bernoulli law with parameter p.
- Hence $\mathcal{X} = \{0, 1\}$, $\mathcal{Y} = \{0, 1\}$.

Exercice 1 : Consider the following random variable (X, Y).

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$$Y = \begin{cases} B(p) \text{ if } X = 1 \\ B(q) \text{ if } X = 0 \end{cases}$$

  With $B(p)$ a Bernoulli law with parameter p.
- A predictor $f_1 : \{0, 1\} \to \{0, 1\}$:

$$f_1 = \begin{cases} 1 \text{ if } x = 1 \\ 0 \text{ if } x = 0 \end{cases}$$

With the "$0 - 1$" loss, what is the risk (generalization error) of f_1, $R(f_1)$?

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$$Y = \begin{cases} B(p) \text{ if } X = 1 \\ B(q) \text{ if } X = 0 \end{cases}$$

- $f_1 : \{0, 1\} \rightarrow \{0, 1\}$:

$$f = \begin{cases} 1 \text{ if } x = 1 \\ 0 \text{ if } x = 0 \end{cases}$$

$$\begin{aligned} R(f_1) &= E[l(Y, f(X))] \\ &= 1 \times P(Y \neq f(X)) + 0 \times P(Y = f(X)) \\ &= P(Y \neq f(X)) \end{aligned} \tag{1}$$

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$$\begin{aligned} R(f_1) &= E[l(Y, f(X))] \\ &= 1 \times P(Y \neq f(X)) + 0 \times P(Y = f(X)) \\ &= P(Y \neq f(X)) \\ &= P((Y \neq f(X)) \cap (X = 1)) + P((Y \neq f(X)) \cap (X = 0)) \end{aligned} \tag{2}$$

$$
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R(f_1) &= E[l(Y, f(X))] \\
&= 1 \times P(Y \neq f(X)) + 0 \times P(Y = f(X)) \\
&= P(Y \neq f(X)) \\
&= P((Y \neq f(X)) \cap (X = 1)) + P((Y \neq f(X)) \cap (X = 0)) \\
&= P((Y \neq f(X))|X = 1)P(X = 1) \\
&+ P((Y \neq f(X))|X = 0)P(X = 0)
\end{aligned} \tag{3}
$$

$$
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R(f_1) &= E[l(Y, f(X))] \\
&= 1 \times P(Y \neq f(X)) + 0 \times P(Y = f(X)) \\
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&= P((Y \neq f(X)) \cap (X = 1)) + P((Y \neq f(X)) \cap (X = 0)) \\
&= P((Y \neq f(X))|X = 1)P(X = 1) \\
&+ P((Y \neq f(X))|X = 0)P(X = 0) \\
&= \frac{1}{2}P((Y \neq 1)|X = 1) + \frac{1}{2}P((Y \neq 0)|X = 0)
\end{aligned}
\tag{4}
$$

$$
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R(f_1) &= E[l(Y, f(X))] \\
&= 1 \times P(Y \neq f(X)) + 0 \times P(Y = f(X)) \\
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&= P((Y \neq f(X)) \cap (X = 1)) + P((Y \neq f(X)) \cap (X = 0)) \\
&= P((Y \neq f(X))|X = 1)P(X = 1) \\
&+ P((Y \neq f(X))|X = 0)P(X = 0) \\
&= \frac{1}{2}P((Y = 0)|X = 1) + \frac{1}{2}P((Y = 1)|X = 0) \\
&= \frac{1}{2}(1 - p) + \frac{1}{2}q
\end{aligned}
\tag{5}
$$

Exercice 2 : Now consider

$$f_2 = \begin{cases} 0 \text{ if } x = 1 \\ 1 \text{ if } x = 0 \end{cases}$$

What is $R(f_2)$?

Exercice 2 :

$$\forall x, f_2(x) = 1 - f_1(x) \tag{6}$$

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Hence

$$\begin{aligned} R(f_2) &= P(Y \neq f_2(X)) \\ &= P(Y \neq (1 - f_1(X))) \\ &= P(Y = f_1(X)) \\ &= 1 - R(f_1) \end{aligned} \tag{8}$$

Exercice 3 : Third predictor :

$$\forall x, f_3(x) = 1 \tag{9}$$

What is $R(f_3)$?

Exercice 3 :

$$\begin{aligned} R(f_3) &= P(Y \neq f_3(X)) \\ &= P(Y = 0) \end{aligned} \tag{10}$$

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$$\begin{aligned}
R(f_3) &= P(Y \neq f_3(X)) \\
&= P(Y = 0) \\
&= P(Y = 0 \cap X = 0) + P(Y = 0 \cap X = 1) \\
&= P(Y = 0|X = 0)P(X = 0) + P(Y = 0|X = 1)P(X = 1) \\
&= \frac{1}{2}(1 - p) + \frac{1}{2}(1 - q)
\end{aligned} \tag{11}$$

Exercice 4 :

Now, we observe the following dataset :

$$D_4 = \{(0,1),(0,0),(0,0),(1,0)\} \tag{12}$$

Compute the empirical risks $R_4(f_1)$, $R_4(f_2)$, $R_4(f_3)$.

$$R_n(f) = \frac{1}{n}\sum_{i=1}^{n} l(y_i, f(x_i))$$

$$D_4 = \{(0,1),(0,0),(0,0),(1,0)\} \tag{13}$$

$$\begin{aligned} R_4(f_1) &= \frac{1}{4}\sum_{i=1}^{4} l(f_1(x_i), y_i) \\ &= \frac{1}{4}\Big(l(f_1(0),1) + l(f_1(0),0)) + l(f_1(0),0)) + l(f_1(1),0))\Big) \\ &= \frac{1}{4} \times 2 \\ &= \frac{1}{2} \end{aligned} \tag{14}$$

$$D_4 = \{(0,1),(0,0),(0,0),(1,0)\} \tag{15}$$

$$\begin{aligned} R_4(f_2) &= \frac{1}{4}\sum_{i=1}^{4} l(f_2(x_i), y_i) \\ &= \frac{1}{4}\Big(l(f_2(0), 1) + l(f_2(0), 0)) + l(f_2(0), 0)) + l(f_2(1), 0))\Big) \\ &= \frac{1}{4} \times 2 \\ &= \frac{1}{2} \end{aligned} \tag{16}$$

$$D_4 = \{(0,1),(0,0),(0,0),(1,0)\} \tag{17}$$

$$\begin{aligned}
R_4(f_3) &= \frac{1}{4}\sum_{i=1}^{4} l(f_3(x_i), y_i) \\
&= \frac{1}{4}\Big(l(f_3(0), 1) + l(f_3(0), 0)) + l(f_3(0), 0)) + l(f_3(1), 0))\Big) \\
&= \frac{1}{4} \times 3 \\
&= \frac{3}{4}
\end{aligned} \tag{18}$$