134

UNIVERSITY OF GHANA

BSc/BA, FIRST SEMESTER EXAMINATIONS: 2017/2018

DEPARTMENT OF MATHEMATICS

MATH 121: TITLE (3 credits)

INSTRUCTION:

ANSWER ANY 4 (FOUR) OUT OF THE FOLLOWING 6 (SIX) QUESTIONS

TIME ALLOWED:

TWO HOURS AND THIRTY MINUTES $\left(2\frac{1}{2}\text{ hours}\right)$

---

1. (a) By rationalizing the denominator, prove that

$$\frac{1}{\sqrt{3}+\sqrt{2}-\sqrt{5}} = \frac{2\sqrt{3}+3\sqrt{2}+\sqrt{30}}{12}.$$

[15 Marks]

(b) Solve the equations

$$\begin{cases} 2^x + 3^y = 43 \\ 2^{x+3} + 3^{y+2} = 371. \end{cases}$$

[15 Marks]

(c) Prove by induction that for any integer $n \geq 1$, $5^{3n} - 11^{2n}$ is divisible by 4. [20 Marks]

2. (a) Solve the following inequalities and then express the solution sets as an interval or union of disjoint intervals

(i) $|x+2| \geq 2|x+3|$.

(ii) $3 \leq |x+2| < 5$.

[16 Marks]

(b) If P, Q and R are mathematical statements, by drawing a truth table, show that the following compound statements are logically equivalent: $\sim (P \wedge R) \Rightarrow Q$ and $\sim Q \Rightarrow (P \wedge R)$. [20 Marks]

(c) Prove that if $\cos 2\theta \neq \frac{1}{2}$, then

$$\cos\theta = \frac{\cos 3\theta}{2\cos 2\theta - 1}.$$

Hence show that $\cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$. [14 Marks]

135

3. (a) Prove that $\log_a x = \frac{1}{\log_x a}$. Deduce that for any positive reals a, b such that $ab \neq 1$,

$$\frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)} = 1.$$

[23 Marks]

(b) Consider the function f defined by

$$f(x) = \begin{cases} -x-1 & if\ -4 \leq x < -1 \\ \sqrt{1-x^2} & if\ -1 \leq x \leq 1 \\ \sqrt{x-1} & if\ x \in [1,4). \end{cases}$$

(i) Sketch the graph of f.

(ii) Find $f(-2)$, $f(0)$, and $f(2)$. Then using the graph of f, express the following set as interval or union of disjoint intervals

$$\{x \in \mathbb{R} : 0 \leq f(x) < 1\}.$$

[27 Marks]

4. (a) Consider the equation

$$mx^2 - 2(m-1)x + m = 0.$$

Find the values of the parameter m for which the roots x_1 and x_2 of the above equation satisfy $\frac{x_1}{x_2} + \frac{x_2}{x_1} = 7$. [20 Marks]

Hint: you may use the sum and product of the roots and notice that $a^2+b^2 = (a+b)^2 - 2ab$.

(b) Let f be the function defined on $\mathbb{R}$ by $f(x) = ax^2 + bx + c$, with a,b,c $\in \mathbb{R}$.

(i) Find the coefficients a,b,c so that the graph of f intercepts the x-axis at 2 and the y-axis at 0, and the minimum value of f is -2 and it is realized at $x = 1$. [16 Marks]

(ii) Sketch the graphs of $h(x) = 2x^2 - 4x$ and $g(x) = | 2x^2 - 4x |$ and give in terms of interval or union of intervals, the solution of the inequalities $h(x) \geq 0$; $h(x) = g(x)$; $g(x) > h(x)$. [14 Marks]

5 (a) Express $\cos 2x - \sin 2x$ in the form $R\cos(2x + \alpha)$ giving values of R and α. Hence find the general solution of the equation $\cos 2x - \sin 2x = 1$. [18 Marks]

(b) Resolve into partial fractions

$$\frac{3x+2}{x^2(x+2)}.$$

[20 Marks]

(c) If the minimum value of $f(x) = x^2 + 2x + k$ is -4, then find the value of k. Find the set of solutions of the inequality of $f(x) > 0$. [12 Marks]

6. (a) The polynomial $ax^3 - 3x^2 + bx - 3$ has a factor of $x - 3$, and when it is divided by $x - 1$ there is a remainder of -12. Find the values of the constants a and b. Find all the values of x for which the polynomial is zero. [23 Marks]

(b) Let $f(x) = \frac{x-1}{x+1}$, $x \neq -1$, and $g(x) = 1 - \frac{1}{x}$, $x \neq 0$.

(i) Find $(f \circ g)(x)$ and determine its domain. [7 Marks]

(ii) By choosing an appropriate codomain, prove that f is bijective and find the expression of its inverse $f^{-1}(x)$. [15 Marks]

(iii) Evaluate $g[g(\frac{1}{x})]$ and give its domain. [5 Marks]

EXAMINERS: Dr. B.F.Sehba, Mrs. G.A.Botchway, Dr. R.Twum
Mr. J.D.G.Mensah, Mr. A.Abubakar, Mr. C.E.K.Aziedu, and Dr. A.Asare-Tuah