worked out the details of propositional, or Boolean, logic (Boole, 1847). In 1879, Gottlob Frege (1848–1925) extended Boole's logic to include objects and relations, creating the first-order logic that is used today.[4] Alfred Tarski (1902–1983) introduced a theory of reference that shows how to relate the objects in a logic to objects in the real world.

ALGORITHM

INCOMPLETENESS THEOREM

The next step was to determine the limits of what could be done with logic and computation. The first nontrivial **algorithm** is thought to be Euclid's algorithm for computing greatest common divisors. The word *algorithm* (and the idea of studying them) comes from al-Khowarazmi, a Persian mathematician of the 9th century, whose writings also introduced Arabic numerals and algebra to Europe. Boole and others discussed algorithms for logical deduction, and, by the late 19th century, efforts were under way to formalize general mathematical reasoning as logical deduction. In 1930, Kurt Gödel (1906–1978) showed that there exists an effective procedure to prove any true statement in the first-order logic of Frege and Russell, but that first-order logic could not capture the principle of mathematical induction needed to characterize the natural numbers. In 1931, Gödel showed that limits on deduction do exist. His **incompleteness theorem** showed that in any formal theory as strong as Peano arithmetic (the elementary theory of natural numbers), there are true statements that are undecidable in the sense that they have no proof within the theory.

COMPUTABLE

This fundamental result can also be interpreted as showing that some functions on the integers cannot be represented by an algorithm—that is, they cannot be computed. This motivated Alan Turing (1912–1954) to try to characterize exactly which functions *are* **computable**—capable of being computed. This notion is actually slightly problematic because the notion of a computation or effective procedure really cannot be given a formal definition. However, the Church–Turing thesis, which states that the Turing machine (Turing, 1936) is capable of computing any computable function, is generally accepted as providing a sufficient definition. Turing also showed that there were some functions that no Turing machine can compute. For example, no machine can tell *in general* whether a given program will return an answer on a given input or run forever.

TRACTABILITY

Although decidability and computability are important to an understanding of computation, the notion of **tractability** has had an even greater impact. Roughly speaking, a problem is called intractable if the time required to solve instances of the problem grows exponentially with the size of the instances. The distinction between polynomial and exponential growth in complexity was first emphasized in the mid-1960s (Cobham, 1964; Edmonds, 1965). It is important because exponential growth means that even moderately large instances cannot be solved in any reasonable time. Therefore, one should strive to divide the overall problem of generating intelligent behavior into tractable subproblems rather than intractable ones.

NP-COMPLETENESS

How can one recognize an intractable problem? The theory of **NP-completeness**, pioneered by Steven Cook (1971) and Richard Karp (1972), provides a method. Cook and Karp showed the existence of large classes of canonical combinatorial search and reasoning problems that are NP-complete. Any problem class to which the class of NP-complete problems can be reduced is likely to be intractable. (Although it has not been proved that NP-complete

[4] Frege's proposed notation for first-order logic—an arcane combination of textual and geometric features—never became popular.