IMEX spectral method for three-dimensional incompressible Hall-MHD system

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目录

In 1879, the American physicist Edwin Hall (1855-1938) found that:

Hall effect: A voltage difference (the Hall voltage) is produced across an electrical conductor that is transverse to an electric current in the conductor and to an applied magnetic field perpendicular to the current.

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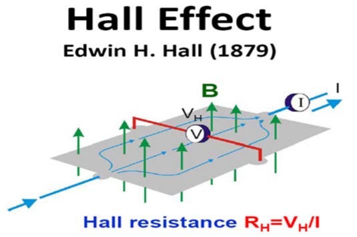

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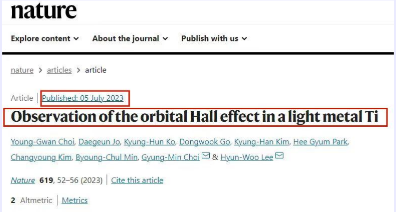

(b)

Figure 1.1: Hall effect.

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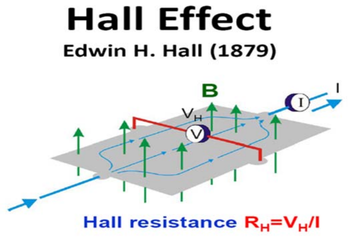

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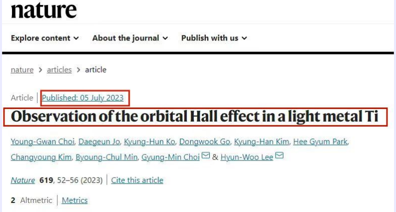
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Observation of the orbital Hall effect in a light metal Ti

Young-Gwan Choi, Daegeun Jo, Kyung-Hun Ko, Dongwook Go, Kyung-Han Kim, Hee Gyum Park, Changyoung Kim, Byoung-Chul Min, Gyung-Min Choi & Hyun-Woo Lee

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(b)

Figure 1.1: Hall effect.

The Hall effect plays an important role in magnetohydrodynamics (MHD) with small scale appearing in astrophysical plasma:

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Successful applications of Hall-MHD system: Understanding solar wind at ion-cyclotron scales

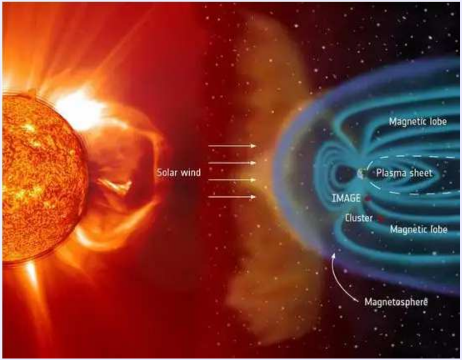

Figure 1.2: Solar wind.

In this talk, we consider the following incompressible Hall-MHD system in three-dimensional domain: For $(\mathbf{x}, t) \in [-1,1]^3 \times (0,T]$

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - (\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{0}, \tag{1.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \eta \nabla \times (\nabla \times \mathbf{B}) + \nabla \times (\mathbf{B} \times \mathbf{u}) + \lambda \nabla \times \big((\nabla \times \mathbf{B}) \times \mathbf{B}\big) = \mathbf{0}, \tag{1.2}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{B} = 0. \tag{1.3}$$

supplemented with the initial-boundary conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0(\mathbf{x}). \tag{1.4}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{n} \times \mathbf{B}|_{\partial\Omega} = \mathbf{0}, \quad \nabla \cdot \mathbf{B}|_{\partial\Omega} = 0. \tag{1.5}$$

- $\mathbf{x} = (x_1, x_2, x_3)' \in \mathbb{R}^3$: spatial variable
- $\mathbf{u} = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))'$: velocity field
- $\mathbf{B} = (B_1(\mathbf{x}, t), B_2(\mathbf{x}, t), B_3(\mathbf{x}, t))'$: magnetic field
- $p = p(\mathbf{x}, t)$: pressure
- ν: fluid viscosity
- η: magnetic resistivity
- λ: strength of the Hall effect

L^2-energy estimate for Hall-MHD system

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|^2 + \|\mathbf{B}\|^2) = -\nu\|\nabla\mathbf{u}\|^2 - \mu\|\nabla\mathbf{B}\|^2. \tag{1.6}$$

Research Status

Mathematical analysis of Hall-MHD system:

- Partial regularity of weak solutions of nonstationary Hall-MHD system in $\mathbb{R}^2$ [*Chae-Wolf, SIAM J. Math. Anal., 2016*]
- Convergence rate of strong solutions from the incompressible Hall-MHD system to the MHD system [*Wan-Zhou, J. Differ. Equ., 2019*]
- Global well-posedness of three-dimensional incompressible Hall-MHD system in critical Besov spaces [*Liu-Tan, J. Differ. Equ., 2021*]
- Well-posedness results for incompressible Hall-MHD system with fractional dissipation in $\mathbb{R}^3$ [*Ye, J. Differ. Equ., 2022*]
- ......

Research Status

Numerical solutions of Hall-MHD system:

- Direct numerical simulations for 3D incompressible Hall-MHD system to study the spontaneous chiral symmetry breaking in solar wind [*Meyrand-Galtier, Phys. Rev. Lett., 2012*]
- Numerical investigation of wave turbulence in 3D incompressible Hall magnetohydrodynamics [*Meyrand et al., Phys. Rev. X, 2012*]
- Sub-grid-scale model for a large eddy simulation of homogeneous and isotropic Hall-MHD system within incompressible regime [*Miura-Hamba, J. Comput. Phys., 2022*]
- Newton-Krylov-type iterative methods for incompressible Hall-MHD system [*Chacón-Knoll, J. Comput. Phys., 2003*]
- ......

Difficulties in the scientific computing:

- Coupling of $(\mathbf{u}, p)$ via $\nabla \cdot \mathbf{u} = 0$
- L^2-energy dissipation
- Nonlinearity, especially the Hall term $\nabla \times \Big((\nabla \times \mathbf{B}) \times \mathbf{B} \Big)$

Let $\mathbf{B} = (B_1, B_2, B_3)$ and $\nabla \times \Big((\nabla \times \mathbf{B}) \times \mathbf{B} \Big) = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)^{\mathrm{T}}$. We have

$$
\begin{aligned}
\mathcal{H}_1 = {} & \left(\frac{\partial^2 B_3}{\partial x_2^2} - \frac{\partial^2 B_2}{\partial x_3 \partial x_2} \right) B_2 + \left(\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} \right) \frac{\partial B_2}{\partial x_2} - \left(\frac{\partial^2 B_1}{\partial x_3 \partial x_2} - \frac{\partial^2 B_3}{\partial x_2 \partial x_1} \right) B_1 - \left(\frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1} \right) \frac{\partial B_1}{\partial x_2} \\
& + \left(\frac{\partial^2 B_3}{\partial x_3 \partial x_2} - \frac{\partial^2 B_2}{\partial x_3^2} \right) B_3 + \left(\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} \right) \frac{\partial B_3}{\partial x_3} - \left(\frac{\partial^2 B_2}{\partial x_3 \partial x_1} - \frac{\partial^2 B_1}{\partial x_3 \partial x_2} \right) B_1 \\
& - \left(\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} \right) \frac{\partial B_1}{\partial x_3}, \\
\mathcal{H}_2 = {} & \left(\frac{\partial^2 B_1}{\partial x_3^2} - \frac{\partial^2 B_3}{\partial x_3 \partial x_1} \right) B_3 + \left(\frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1} \right) \frac{\partial B_3}{\partial x_3} - \left(\frac{\partial^2 B_2}{\partial x_3 \partial x_1} - \frac{\partial^2 B_1}{\partial x_2 \partial x_3} \right) B_2 - \left(\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} \right) \frac{\partial B_2}{\partial x_3} \\
& - \left(\frac{\partial^2 B_3}{\partial x_1 \partial x_2} - \frac{\partial^2 B_2}{\partial x_1 \partial x_3} \right) B_2 - \left(\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} \right) \frac{\partial B_2}{\partial x_1} + \left(\frac{\partial^2 B_1}{\partial x_3 \partial x_1} - \frac{\partial^2 B_3}{\partial x^2} \right) B_1 \\
& + \left(\frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1} \right) \frac{\partial B_1}{\partial x_1}, \\
\mathcal{H}_3 = {} & - \left(\frac{\partial^2 B_3}{\partial x_1 \partial x_2} - \frac{\partial^2 B_3}{\partial x_3 \partial x_1} \right) B_3 - \left(\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} \right) \frac{\partial B_3}{\partial x_1} + \left(\frac{\partial^2 B_2}{\partial x_1^2} - \frac{\partial^2 B_1}{\partial x_1 \partial x_2} \right) B_1 \\
& + \left(\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} \right) \frac{\partial B_1}{\partial x_1} - \left(\frac{\partial^2 B_1}{\partial x_2 \partial x_3} - \frac{\partial^2 B_3}{\partial x_1 \partial x_2} \right) B_3 - \left(\frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1} \right) \frac{\partial B_3}{\partial x_2} \\
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Let $\mathbf{B} = (B_1, B_2, B_3)$ and $\nabla \times \left((\nabla \times \mathbf{B}) \times \mathbf{B} \right) = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)^{\mathrm{T}}$. We have

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\mathcal{H}_2 =& \left(\frac{\partial^2 B_1}{\partial x_3^2} - \frac{\partial^2 B_3}{\partial x_3 \partial x_1} \right) B_3 + \left(\frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1} \right) \frac{\partial B_3}{\partial x_3} - \left(\frac{\partial^2 B_2}{\partial x_3 \partial x_1} - \frac{\partial^2 B_1}{\partial x_2 \partial x_3} \right) B_2 - \left(\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} \right) \frac{\partial B_2}{\partial x_3} \\
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&+ \left(\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} \right) \frac{\partial B_1}{\partial x_1} - \left(\frac{\partial^2 B_1}{\partial x_2 \partial x_3} - \frac{\partial^2 B_3}{\partial x_1 \partial x_2} \right) B_3 - \left(\frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1} \right) \frac{\partial B_3}{\partial x_2} \\
&+ \left(\frac{\partial^2 B_2}{\partial x_1 \partial x_2} - \frac{\partial^2 B_1}{\partial x_2^2} \right) B_2 + \left(\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} \right) \frac{\partial B_2}{\partial x_2}.
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Enlightening Research

- How to decouple $(\mathbf{u}, p)$?
  - Standard projection method [*Chorin, Math. Comp., 1969; Temam, Arch. Rat. Mech. Anal., 1969*]
  - Rotational form of pressure-correction scheme [*Timmermans-Minev-Van De Vosse, J. Numer. Methods Fluids, 1996*]
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Based on the above findings, we shall design the numerical scheme for Hall-MHD system with the following properties:

- Decouple of $\mathbf{u}$ and p
- Completely linearized
- Unconditionally energy stable
- Adaptive time step size

We introduce a non-zero constant scalar auxiliary variable C

$$r(t) = C \quad \text{with} \quad C \neq 0.$$

Rewrite the Hall-MHD system (1.1)-(1.3) as

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla p + \frac{r(t)}{C}\Big((\mathbf{u}\cdot\nabla)\mathbf{u} - (\nabla \times \mathbf{B}) \times \mathbf{B}\Big) = \mathbf{0}, \tag{2.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \eta \nabla \times (\nabla \times \mathbf{B}) + \frac{r(t)}{C}\Big(\nabla \times (\mathbf{B} \times \mathbf{u}) + \lambda \nabla \times \big((\nabla \times \mathbf{B}) \times \mathbf{B}\big)\Big) = \mathbf{0}, \tag{2.2}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{B} = 0, \tag{2.3}$$

$$\begin{aligned}\frac{dr(t)}{dt} = \frac{r(t)}{C}\int_\Omega &(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} - (\nabla \times \mathbf{B}) \times \mathbf{B} \cdot \mathbf{u} + (\mathbf{B} \times \mathbf{u}) \cdot (\nabla \times \mathbf{B}) \\ &+ \lambda(\nabla \times \mathbf{B}) \times \mathbf{B} \cdot (\nabla \times \mathbf{B}) d\mathbf{x},\end{aligned} \tag{2.4}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0(\mathbf{x}), \quad r(0) = C, \tag{2.5}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{n} \times \mathbf{B}|_{\partial\Omega} = \mathbf{0}, \quad \nabla \cdot \mathbf{B}|_{\partial\Omega} = 0. \tag{2.6}$$

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$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{B}(\mathbf{x},0) = \mathbf{B}_0(\mathbf{x}), \quad r(0) = C, \tag{2.5}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{n}\times\mathbf{B}|_{\partial\Omega} = \mathbf{0}, \quad \nabla\cdot\mathbf{B}|_{\partial\Omega} = 0. \tag{2.6}$$

Theorem 2.1

The reformulated equivalent Hall-MHD system (2.1)-(2.6) *admits the following energy dissipation law*

$$\frac{d}{dt}\widetilde{E}(\mathbf{u},\mathbf{B},r) = -\nu\|\nabla\mathbf{u}\|^2 - \eta\|\nabla\times\mathbf{B}\|^2, \tag{2.7}$$

where

$$\widetilde{E}(\mathbf{u},\mathbf{B},r) = \frac{1}{2}\left(\|\mathbf{u}\|^2 + \|\mathbf{B}\|^2 + 2r(t)\right). \tag{2.8}$$

Define some function spaces as follows

$$
\begin{aligned}
&H^1(\Omega) = \left\{v \in L^2(\Omega) : \nabla v \in L^2(\Omega)^3\right\},\\
&H_0^1(\Omega) = \left\{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\right\},\\
&L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q dx = 0\right\},\\
&\mathbf{H}^1(\Omega) = H^1(\Omega)^3, \mathbf{H}_0^1(\Omega) = H_0^1(\Omega)^3,\\
&\mathbf{H}(\mathrm{curl};\Omega) = \left\{\mathbf{s} \in L^2(\Omega)^3 : \nabla \times \mathbf{s} \in L^2(\Omega)^3\right\},\\
&\mathbf{H}_0(\mathrm{curl};\Omega) = \{\mathbf{s} \in \mathbf{H}(\mathrm{curl};\Omega) : \mathbf{s} \times \mathbf{n}|_{\partial\Omega} = \mathbf{0}, \nabla \cdot \mathbf{s}|_{\partial\Omega} = \mathbf{0}\}.
\end{aligned}
$$

The variational formulation reads: Find $\mathbf{u}(t) \in \mathbf{H}_0^1(\Omega)$, $p(t) \in L_0^2(\Omega)$, $\mathbf{B}(t) \in \mathbf{H}_0(\mathrm{curl};\Omega)$, and $r(t) \in \mathbb{R}$ such that

$$\left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\right) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + \frac{r(t)}{C}\Big(((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}) - ((\nabla \times \mathbf{B}) \times \mathbf{B}, \mathbf{v})\Big) = 0, \tag{3.1}$$

$$\left(\frac{\partial \mathbf{B}}{\partial t}, \mathbf{s}\right) + \eta(\nabla \times \mathbf{B}, \nabla \times \mathbf{s}) + \frac{r(t)}{C}\Big((\mathbf{B} \times \mathbf{u}, \nabla \times \mathbf{s}) + \lambda\big((\nabla \times \mathbf{B}) \times \mathbf{B}, \nabla \times \mathbf{s}\big)\Big) = 0, \tag{3.2}$$

$$\frac{dr(t)}{dt} = \frac{r(t)}{C}\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u} - (\nabla \times \mathbf{B}) \times \mathbf{B} \cdot \mathbf{u} + (\mathbf{B} \times \mathbf{u}) \cdot (\nabla \times \mathbf{B}) + \lambda(\nabla \times \mathbf{B}) \times \mathbf{B} \cdot (\nabla \times \mathbf{B}) d\mathbf{x}, \tag{3.3}$$

$$(\nabla \cdot \mathbf{u}, q) = 0, \tag{3.4}$$

for $\forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \forall \mathbf{s} \in \mathbf{H}_0(\mathrm{curl};\Omega)$, and $\forall q \in L_0^2(\Omega)$.

Denote

$$\tau_n = t_{n+1} - t_n, \qquad \zeta_{n+1} = \frac{\tau_{n+1}}{\tau_n}$$

For the given sequences $\{h^n\}$ and $\left\{\widehat{h}^n\right\}$, we define

$$h^{n+\frac{1}{2}} = \frac{h^{n+1} + h^n}{2}, \quad \overline{h}^{n+\frac{1}{2}} = \left(1 + \frac{\zeta_n}{2}\right) h^n - \frac{\zeta_n}{2} h^{n-1}, \quad \widehat{h}^{n+\frac{1}{2}} = \frac{\widehat{h}^{n+1} + h^n}{2}.$$

Define the following approximation spaces

$$\begin{aligned}
\mathbf{X}_N &= [P_N(I_{x_1}) \otimes P_N(I_{x_2}) \otimes P_N(I_{x_3})]^3 \cap \mathbf{H}_0^1(\Omega), \\
\mathbf{S}_N &= [P_N(I_{x_1}) \otimes P_N(I_{x_2}) \otimes P_N(I_{x_3})]^3 \cap \mathbf{H}_0(\mathrm{curl}; \Omega), \\
M_N &= [P_{N-2}(I_{x_1}) \otimes P_{N-2}(I_{x_2}) \otimes P_{N-2}(I_{x_3})] \cap L_0^2(\Omega).
\end{aligned} \tag{3.5}$$

Denote

$$\tau_n = t_{n+1} - t_n, \qquad \zeta_{n+1} = \frac{\tau_{n+1}}{\tau_n}$$

For the given sequences $\{h^n\}$ and $\left\{\widehat{h}^n\right\}$, we define

$$h^{n+\frac{1}{2}} = \frac{h^{n+1} + h^n}{2}, \quad \overline{h}^{n+\frac{1}{2}} = \left(1 + \frac{\zeta_n}{2}\right) h^n - \frac{\zeta_n}{2} h^{n-1}, \quad \widehat{h}^{n+\frac{1}{2}} = \frac{\widehat{h}^{n+1} + h^n}{2}.$$

Define the following approximation spaces

$$\begin{aligned} \mathbf{X}_N &= \left[P_N(I_{x_1}) \otimes P_N(I_{x_2}) \otimes P_N(I_{x_3})\right]^3 \cap \mathbf{H}_0^1(\Omega), \\ \mathbf{S}_N &= \left[P_N(I_{x_1}) \otimes P_N(I_{x_2}) \otimes P_N(I_{x_3})\right]^3 \cap \mathbf{H}_0(\mathrm{curl}; \Omega), \\ M_N &= \left[P_{N-2}(I_{x_1}) \otimes P_{N-2}(I_{x_2}) \otimes P_{N-2}(I_{x_3})\right] \cap L_0^2(\Omega). \end{aligned} \tag{3.5}$$

The fully-discrete scheme reads: Find $\mathbf{u}_N^{n+1}$ and $\widehat{\mathbf{u}}_N^{n+1} \in \mathbf{X}_N$, $p_N^{n+1} \in M_N$, $\mathbf{B}_N^{n+1} \in \mathbf{S}_N$, and $r_N^{n+1} \in \mathbb{R}$ such that

$$\left(\frac{\widehat{\mathbf{u}}_N^{n+1} - \mathbf{u}_N^n}{\tau_n}, \mathbf{v}_N\right) + \nu\left(\nabla \widehat{\mathbf{u}}_N^{n+\frac{1}{2}}, \nabla \mathbf{v}_N\right) - (p_N^n, \nabla \cdot \mathbf{v}_N)$$
$$+ \overline{\rho}_N^{n+\frac{1}{2}}\left(\left(\left(\overline{\mathbf{u}}_N^{n+\frac{1}{2}} \cdot \nabla\right)\overline{\mathbf{u}}_N^{n+\frac{1}{2}}, \mathbf{v}_N\right) - \left(\left(\nabla \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}}\right) \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}}, \mathbf{v}_N\right)\right) = 0, \tag{3.6}$$

$$\left(\frac{\mathbf{B}_N^{n+1} - \mathbf{B}_N^n}{\tau_n}, \mathbf{s}_N\right) + \eta\left(\nabla \times \mathbf{B}_N^{n+\frac{1}{2}}, \nabla \times \mathbf{s}_N\right) + \overline{\rho}_N^{n+\frac{1}{2}}\left(\left(\overline{\mathbf{B}}_N^{n+\frac{1}{2}} \times \overline{\mathbf{u}}_N^{n+\frac{1}{2}}, \nabla \times \mathbf{s}_N\right)\right.$$
$$\left.+ \lambda\left(\left(\nabla \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}}\right) \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}}, \nabla \times \mathbf{s}_N\right)\right) = 0, \tag{3.7}$$

$$\frac{r_N^{n+1} - r_N^n}{\tau_n} = \overline{\rho}_N^{n+\frac{1}{2}} \int_\Omega \left(\overline{\mathbf{u}}_N^{n+\frac{1}{2}} \cdot \nabla\right)\overline{\mathbf{u}}_N^{n+\frac{1}{2}} \cdot \widehat{\mathbf{u}}_N^{n+\frac{1}{2}} - \left(\nabla \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}}\right) \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}} \cdot \widehat{\mathbf{u}}_N^{n+\frac{1}{2}}$$
$$+ \left(\overline{\mathbf{B}}_N^{n+\frac{1}{2}} \times \overline{\mathbf{u}}_N^{n+\frac{1}{2}}\right) \cdot \left(\nabla \times \mathbf{B}_N^{n+\frac{1}{2}}\right) + \lambda\left(\nabla \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}}\right) \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}} \cdot \left(\nabla \times \mathbf{B}_N^{n+\frac{1}{2}}\right) d\mathbf{x}, \tag{3.8}$$

$$\left(\frac{\mathbf{u}_N^{n+1} - \widehat{\mathbf{u}}_N^{n+1}}{\tau_n}, \mathbf{l}_N\right) - \frac{1}{2}\left(p_N^{n+1} - p_N^{n}, \nabla \cdot \mathbf{l}_N\right) = 0, \tag{3.9}$$

$$\left(\nabla \cdot \mathbf{u}_N^{n+1}, q_N\right) = 0, \tag{3.10}$$

for $\forall \mathbf{v}_N$ and $\mathbf{l}_N \in \mathbf{X}_N, \forall \mathbf{s}_N \in \mathbf{S}_N$, and $\forall q_N \in M_N$. Here, we define $\overline{\rho}_N^{n+\frac{1}{2}} = \dfrac{\overline{r}_N^{n+\frac{1}{2}}}{C}$.

The first step solutions $\mathbf{u}_N^1$, $\widehat{\mathbf{u}}_N^1, p_N^1$, $\mathbf{B}_N^1$, and r_N^1 which can be computed by the first-order scheme.

$$\left(\frac{\mathbf{u}_N^{n+1}-\widehat{\mathbf{u}}_N^{n+1}}{\tau_n},\mathbf{l}_N\right)-\frac{1}{2}\left(p_N^{n+1}-p_N^n,\nabla\cdot\mathbf{l}_N\right)=0, \tag{3.9}$$

$$\left(\nabla\cdot\mathbf{u}_N^{n+1},q_N\right)=0, \tag{3.10}$$

for $\forall\mathbf{v}_N$ and $\mathbf{l}_N\in\mathbf{X}_N,\forall\mathbf{s}_N\in\mathbf{S}_N$, and $\forall q_N\in M_N$. Here, we define $\overline{\rho}_N^{n+\frac{1}{2}}=\dfrac{\overline{r}_N^{n+\frac{1}{2}}}{C}$. The first step solutions $\mathbf{u}_N^1$, $\widehat{\mathbf{u}}_N^1,p_N^1$, $\mathbf{B}_N^1$, and r_N^1 which can be computed by the first-order scheme.

Theorem 3.1

The fully-discrete scheme is unconditionally stable in the sense that

$$E_*^{n+1} = E_*^n - \nu\tau_n \left\| \nabla \widehat{\mathbf{u}}_N^{n+\frac{1}{2}} \right\|^2 - \eta\tau_n \left\| \nabla \times \mathbf{B}_N^{n+\frac{1}{2}} \right\|^2, \tag{3.11}$$

where

$$E_*^{n+1} = \frac{1}{2}\left(\|\mathbf{u}_N^{n+1}\|^2 + \|\mathbf{B}_N^{n+1}\|^2 + 2r_N^{n+1} + \frac{\tau_n^2}{4}\|\nabla_N p_N^{n+1}\|^2 \right). \tag{3.12}$$

Theorem 3.2

For the fully-discrete scheme (3.6)-(3.10)*, it holds that*

$$\|\nabla \cdot \mathbf{B}_N^{n+1}\|^2 \le \|\nabla \cdot \mathbf{B}_N^{0}\|^2, \quad n = 0, 1, \cdots . \tag{3.13}$$

Theorem 3.2 reveals that, if the initial condition is divergence-free, the magnetic field $\mathbf{B}_N^{n+1}$ will satisfy the divergence-free constraint.

Theorem 3.2

For the fully-discrete scheme (3.6)-(3.10), *it holds that*

$$\|\nabla \cdot \mathbf{B}_N^{n+1}\|^2 \leq \|\nabla \cdot \mathbf{B}_N^0\|^2, \quad n = 0, 1, \cdots . \tag{3.13}$$

Theorem 3.2 reveals that, if the initial condition is divergence-free, the magnetic field $\mathbf{B}_N^{n+1}$ will satisfy the divergence-free constraint.

Algorithm. Adaptive time-stepping strategy

Give: the values of $\mathbf{u}_N^k$, $\mathbf{B}_N^k$, and r_N^k $(k = n-1, n)$; the minimum and maximin time-stepping τ_{min} and τ_{max}; the current time-stepping τ_{n-1}; the parameter $\sigma > 0$.

1): Compute $e_{n-1} = \frac{E_*^{n-1} - E_*^n}{\tau_{n-1}}$.

2): **if** $e_{n-1} = 0$

Set $\tau_n = \tau_{max}$.

Compute $\mathbf{u}_N^{n+1}$, $\mathbf{B}_N^{n+1}$, and r_N^{n+1} by the scheme.

3): **else**

Update the time-stepping $\tau_n = \min\left\{\tau_{max}, \max\left\{\frac{\tau_{n-1}}{\sigma e_{n-1}}, \tau_{min}\right\}\right\}$.

Compute $\mathbf{u}_N^{n+1}$, $\mathbf{B}_N^{n+1}$, and r_N^{n+1} by by the scheme.

4): **endif**

Let $L_{m_s}(x_s)$ be the m_s-th $(s = 1, 2, 3)$ Legendre polynomials defined over the domains $I_{x_s} = [-1, 1]$.
Define the functions

$$\Phi_{m_s}(x_s) = L_{m_s}(x_s) - L_{m_s+2}(x_s), \tag{4.1}$$

$$\Psi_{m_s}(x_s) = -\frac{m_s+3}{m_s+1}L_{m_s}(x_s) + \frac{m_s}{m_s+2}L_{m_s+2}(x_s), \tag{4.2}$$

which satisfy

$$\Phi_{m_s}(\pm 1) = 0, \qquad \left.\frac{d\Psi_{m_s}(x_s)}{dx_s}\right|_{x_s=\pm 1} = 0.$$

Define the following function spaces

$$
\begin{aligned}
&X_N^{x_s} = \text{span}\left\{\Phi_{m_1}(x_1)\Phi_{m_2}(x_2)\Phi_{m_3}(x_3):\ m_1, m_2, m_3 = 0, 1, \ldots, N-2\right\}, \quad s = 1, 2, 3,\\
&S_N^{x_1} = \text{span}\left\{\Psi_{m_1}(x_1)\Phi_{m_2}(x_2)\Phi_{m_3}(x_3):\ m_1, m_2, m_3 = 0, 1, \ldots, N-2\right\},\\
&S_N^{x_2} = \text{span}\left\{\Phi_{m_1}(x_1)\Psi_{m_2}(x_2)\Phi_{m_3}(x_3):\ m_1, m_2, m_3 = 0, 1, \ldots, N-2\right\},\\
&S_N^{x_3} = \text{span}\left\{\Phi_{m_1}(x_1)\Phi_{m_2}(x_2)\Psi_{m_3}(x_3):\ m_1, m_2, m_3 = 0, 1, \ldots, N-2\right\},\\
&\mathbf{X}_N = X_N^{x_1} \times X_N^{x_2} \times X_N^{x_3}, \quad \mathbf{S}_N = S_N^{x_1} \times S_N^{x_2} \times S_N^{x_3},\\
&M_N = \left\{\sum_{m_1=0}^{N-2}\sum_{m_2=0}^{N-2}\sum_{m_3=0}^{N-2} l_{m_1m_2m_3} L_{m_1}(x_1)L_{m_2}(x_2)L_{m_3}(x_3):\ l_{m_1m_2m_3} \in \mathbb{R}, l_{000} = 0\right\}.
\end{aligned}
\tag{4.3}
$$

Introduce the following matrices

$$(\mathcal{A}_1^{x_s})_{ij} = (\Phi_j(x_s), \Phi_i(x_s)) = \begin{cases} \frac{2}{2j+1} + \frac{2}{2j+5}, & i = j, \\ -\frac{2}{2j+5}, & i = j+2, \\ -\frac{2}{2j+1}, & i = j-2, \\ 0, & \text{otherwise}, \end{cases}$$

$$(\mathcal{A}_2^{x_s})_{ij} = (\Psi_j(x_s), \Psi_i(x_s)) = \begin{cases} \frac{2(j+3)^2}{(j+1)^2(2j+1)} + \frac{2j^2}{(j+2)^2(2j+5)}, & i = j, \\ -\frac{2j(j+5)}{(j+2)(j+3)(2j+5)}, & i = j+2, \\ -\frac{2(j-2)(j+3)}{j(j+1)(2j+1)}, & i = j-2, \\ 0, & \text{otherwise}, \end{cases}$$

$$(\mathcal{B}^{x_s})_{ij} = \left(\frac{d\Phi_j(x_s)}{dx_s}, \frac{d\Phi_i(x_s)}{dx_s} \right) = \begin{cases} 4j+6, & i = j, \\ 0, & \text{otherwise}, \end{cases}$$

$$
(\mathcal{C}_1^{x_s})_{ij} = (L_j(x_s), \Phi_i(x_s)) = \begin{cases} \frac{2}{2j+1}, & i = j, \\ -\frac{2}{2j+1}, & i = j-2, \\ 0, & \text{otherwise}, \end{cases} \tag{4.4}
$$

$$
(\mathcal{C}_2^{x_s})_{ij} = (L_j(x_s), \Psi_i(x_s)) = \begin{cases} -\frac{2(j+3)}{(2j+1)(j+1)}, & i = j \\ \frac{2j-4}{j(2j+1)}, & i = j-2, \\ 0, & \text{otherwise}, \end{cases}
$$

$$
(\mathcal{D}^{x_s})_{ij} = \left(L_j(x_s), \frac{d\Phi_i(x_s)}{dx_s}\right) = \begin{cases} -2, & i = j-1, \\ 0, & \text{otherwise}, \end{cases}
$$

$$
(\mathcal{E}^{x_s})_{ij} = \left(\frac{d\Phi_j(x_s)}{dx_s}, \Psi_i(x_s) \right) = \begin{cases} \frac{2(j+4)}{(j+2)}, & i = j+1, \\ -\frac{2(j-1)}{(j+1)}, & i = j-1, \\ 0, & \text{otherwise}, \end{cases}
$$

$$
(\mathcal{F}^{x_s})_{ij} = (L_j(x_s), L_i(x_s)) = \begin{cases} \frac{2}{2j+1}, & i = j, \\ 0, & \text{otherwise}, \end{cases}
$$

For the $N \times N$ matrices $\mathcal{M}_s = \left(m_{ij}^{(s)}\right)_{N \times N}$ ($s = 1, 2$, and 3) and $N \times N \times N$ matrix $\mathrm{W} = (w_{ijk})_{N \times N \times N}$, the matrix multiplication, denoted by $\mathcal{M}_1 \cdot \underset{\dot{\mathcal{M}}_3}{\mathrm{W}} \cdot \mathcal{M}_2$, creates a $N \times N \times N$ matrix:

$$\left(\mathcal{M}_1 \cdot \underset{\dot{\mathcal{M}}_3}{\mathrm{W}} \cdot \mathcal{M}_2\right)_{ijk} = \sum_{p1=1}^{N} \sum_{p2=1}^{N} \sum_{p3=1}^{N} m_{ip_1}^{(1)} w_{p_1 p_2 p_3} m_{jp_2}^{(2)} m_{kp_3}^{(3)}. \tag{4.5}$$

For $s = 1, 2$, and 3, we denote

$$u_{N,s}^{n+1} = \sum_{m_1,m_2,m_3=0}^{N-2} \widetilde{u}_{s,m_1m_2m_3}^{n+1} \Phi_{m_1}(x_1)\Phi_{m_2}(x_2)\Phi_{m_3}(x_3),$$

$$\widehat{u}_{N,s}^{n+1} = \sum_{m_1,m_2,m_3=0}^{N-2} \widetilde{\widehat{u}}_{s,m_1m_2m_3}^{n+1} \Phi_{m_1}(x_1)\Phi_{m_2}(x_2)\Phi_{m_3}(x_3),$$

$$B_{N,1}^{n+1} = \sum_{m_1,m_2,m_3=0}^{N-2} \widetilde{B}_{1,m_1m_2m_3}^{n+1} \Psi_{m_1}(x_1)\Phi_{m_2}(x_2)\Phi_{m_3}(x_3),$$

$$B_{N,2}^{n+1} = \sum_{m_1,m_2,m_3=0}^{N-2} \widetilde{B}_{2,m_1m_2m_3}^{n+1} \Phi_{m_1}(x_1)\Psi_{m_2}(x_2)\Phi_{m_3}(x_3),$$

$$B_{N,3}^{n+1} = \sum_{m_1,m_2,m_3=0}^{N-2} \widetilde{B}_{3,m_1m_2m_3}^{n+1} \Phi_{m_1}(x_1)\Phi_{m_2}(x_2)\Psi_{m_3}(x_3),$$

$$p_N^{n+1} = \sum_{m_1,m_2,m_3=0}^{N-2} \widetilde{p}_{m_1m_2m_3}^{n+1} L_{m_1}(x_1)L_{m_2}(x_2)L_{m_3}(x_3) \quad \text{with} \quad \widetilde{p}_{000}^{n+1} = 0,$$

Step 1: Computation of $\widehat{\mathrm{u}}_N^{n+1}$ and B_N^{n+1}.

The algebra systems for the unknown matrices $\widetilde{\widehat{\mathrm{U}}}_s^{n+1} = (\widetilde{\widehat{u}}_{s,m_1m_2m_3}^{n+1})_{m_1,m_2,m_3=0}^{N-2}$ and $\widetilde{\mathrm{B}}_s^{n+1} = (\widetilde{B}_{s,m_1m_2m_3}^{n+1})_{m_1,m_2,m_3=0}^{N-2}$ are

$$\mathcal{A}_1^{x1}\cdot\underset{\dot{\mathcal{A}}_1^{x3}}{\widetilde{\widehat{\mathrm{U}}}_1^{n+1}}\cdot\mathcal{A}_1^{x2}+\frac{\nu\tau_n}{2}\left(\mathcal{B}^{x1}\cdot\underset{\dot{\mathcal{A}}_1^{x3}}{\widetilde{\widehat{\mathrm{U}}}_1^{n+1}}\cdot\mathcal{A}_1^{x2}+\mathcal{A}_1^{x1}\cdot\underset{\dot{\mathcal{A}}_1^{x3}}{\widetilde{\widehat{\mathrm{U}}}_1^{n+1}}\cdot\mathcal{B}^{x2}+\mathcal{A}_1^{x1}\cdot\underset{\dot{\mathcal{B}}^{x3}}{\widetilde{\widehat{\mathrm{U}}}_1^{n+1}}\cdot\mathcal{A}_1^{x2}\right)=-\tau_n\widetilde{\mathrm{Q}}_1^n, \tag{4.6}$$

$$\mathcal{A}_1^{x1}\cdot\underset{\dot{\mathcal{A}}_1^{x3}}{\widetilde{\widehat{\mathrm{U}}}_2^{n+1}}\cdot\mathcal{A}_1^{x2}+\frac{\nu\tau_n}{2}\left(\mathcal{B}^{x1}\cdot\underset{\dot{\mathcal{A}}_1^{x3}}{\widetilde{\widehat{\mathrm{U}}}_2^{n+1}}\cdot\mathcal{A}_1^{x2}+\mathcal{A}_1^{x1}\cdot\underset{\dot{\mathcal{A}}_1^{x3}}{\widetilde{\widehat{\mathrm{U}}}_2^{n+1}}\cdot\mathcal{B}^{x2}+\mathcal{A}_1^{x1}\cdot\underset{\dot{\mathcal{B}}^{x3}}{\widetilde{\widehat{\mathrm{U}}}_2^{n+1}}\cdot\mathcal{A}_1^{x2}\right)=-\tau_n\widetilde{\mathrm{Q}}_2^n, \tag{4.7}$$

$$\mathcal{A}_1^{x1}\cdot\underset{\dot{\mathcal{A}}_1^{x3}}{\widetilde{\widehat{\mathrm{U}}}_3^{n+1}}\cdot\mathcal{A}_1^{x2}+\frac{\nu\tau_n}{2}\left(\mathcal{B}^{x1}\cdot\underset{\dot{\mathcal{A}}_1^{x3}}{\widetilde{\widehat{\mathrm{U}}}_3^{n+1}}\cdot\mathcal{A}_1^{x2}+\mathcal{A}_1^{x1}\cdot\underset{\dot{\mathcal{A}}_1^{x3}}{\widetilde{\widehat{\mathrm{U}}}_3^{n+1}}\cdot\mathcal{B}^{x2}+\mathcal{A}_1^{x1}\cdot\underset{\dot{\mathcal{B}}^{x3}}{\widetilde{\widehat{\mathrm{U}}}_3^{n+1}}\cdot\mathcal{A}_1^{x2}\right)=-\tau_n\widetilde{\mathrm{Q}}_3^n. \tag{4.8}$$

Step 1: Computation of $\widehat{\mathrm{u}}_N^{n+1}$ and B_N^{n+1}.

The algebra systems for the unknown matrices $\widetilde{\widehat{\mathrm{U}}}_s^{n+1} = (\widetilde{\widehat{u}}_{s,m_1m_2m_3}^{n+1})_{m_1,m_2,m_3=0}^{N-2}$ and $\widetilde{\mathrm{B}}_s^{n+1} = (\widetilde{B}_{s,m_1m_2m_3}^{n+1})_{m_1,m_2,m_3=0}^{N-2}$ are

$$\mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{x3}}{\underset{\cdot}{\widetilde{\widehat{\mathrm{U}}}_1^{n+1}}} \cdot \mathcal{A}_1^{x2} + \frac{\nu\tau_n}{2}\left(\mathcal{B}^{x1} \cdot \underset{\mathcal{A}_1^{x3}}{\underset{\cdot}{\widetilde{\widehat{\mathrm{U}}}_1^{n+1}}} \cdot \mathcal{A}_1^{x2} + \mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{x3}}{\underset{\cdot}{\widetilde{\widehat{\mathrm{U}}}_1^{n+1}}} \cdot \mathcal{B}^{x2} + \mathcal{A}_1^{x1} \cdot \underset{\mathcal{B}^{x3}}{\underset{\cdot}{\widetilde{\widehat{\mathrm{U}}}_1^{n+1}}} \cdot \mathcal{A}_1^{x2}\right) = -\tau_n \widetilde{\mathrm{Q}}_1^n, \tag{4.6}$$

$$\mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{x3}}{\underset{\cdot}{\widetilde{\widehat{\mathrm{U}}}_2^{n+1}}} \cdot \mathcal{A}_1^{x2} + \frac{\nu\tau_n}{2}\left(\mathcal{B}^{x1} \cdot \underset{\mathcal{A}_1^{x3}}{\underset{\cdot}{\widetilde{\widehat{\mathrm{U}}}_2^{n+1}}} \cdot \mathcal{A}_1^{x2} + \mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{x3}}{\underset{\cdot}{\widetilde{\widehat{\mathrm{U}}}_2^{n+1}}} \cdot \mathcal{B}^{x2} + \mathcal{A}_1^{x1} \cdot \underset{\mathcal{B}^{x3}}{\underset{\cdot}{\widetilde{\widehat{\mathrm{U}}}_2^{n+1}}} \cdot \mathcal{A}_1^{x2}\right) = -\tau_n \widetilde{\mathrm{Q}}_2^n, \tag{4.7}$$

$$\mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{x3}}{\underset{\cdot}{\widetilde{\widehat{\mathrm{U}}}_3^{n+1}}} \cdot \mathcal{A}_1^{x2} + \frac{\nu\tau_n}{2}\left(\mathcal{B}^{x1} \cdot \underset{\mathcal{A}_1^{x3}}{\underset{\cdot}{\widetilde{\widehat{\mathrm{U}}}_3^{n+1}}} \cdot \mathcal{A}_1^{x2} + \mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{x3}}{\underset{\cdot}{\widetilde{\widehat{\mathrm{U}}}_3^{n+1}}} \cdot \mathcal{B}^{x2} + \mathcal{A}_1^{x1} \cdot \underset{\mathcal{B}^{x3}}{\underset{\cdot}{\widetilde{\widehat{\mathrm{U}}}_3^{n+1}}} \cdot \mathcal{A}_1^{x2}\right) = -\tau_n \widetilde{\mathrm{Q}}_3^n. \tag{4.8}$$

and

$$\mathcal{A}_2^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\widetilde{\mathrm{B}}_1^{n+1}} \cdot \mathcal{A}_1^{x2} + \frac{\eta\tau_n}{2}\Big(\mathcal{A}_2^{x1} \cdot \underset{\mathcal{B}^{\dot{x}3}}{\widetilde{\mathrm{B}}_1^{n+1}} \cdot \mathcal{A}_1^{x2} - \mathcal{E}^{x1} \cdot \underset{(\mathcal{E}^{\dot{x}3})^{\mathsf{T}}}{\widetilde{\mathrm{B}}_3^{n+1}} \cdot \mathcal{A}_1^{x2} - \mathcal{E}^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\widetilde{\mathrm{B}}_2^{n+1}} \cdot \mathcal{E}^{x2} \tag{4.9}$$

$$+ \mathcal{A}_2^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\widetilde{\mathrm{B}}_1^{n+1}} \cdot \mathcal{B}^{x2}\Big) = -\tau_n \widetilde{\mathrm{Z}}_1^n$$

$$\mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\widetilde{\mathrm{B}}_2^{n+1}} \cdot \mathcal{A}_2^{x2} + \frac{\eta\tau_n}{2}\Big(-\mathcal{A}_1^{x1} \cdot \underset{(\mathcal{E}^{\dot{x}3})^{\mathsf{T}}}{\widetilde{\mathrm{B}}_3^{n+1}} \cdot \mathcal{E}^{x2} + \mathcal{A}_1^{x1} \cdot \underset{\mathcal{B}_1^{\dot{x}3}}{\widetilde{\mathrm{B}}_2^{n+1}} \cdot \mathcal{A}_2^{x2} - \mathcal{B}^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\widetilde{\mathrm{B}}_2^{n+1}} \cdot \mathcal{A}_1^{x2} \tag{4.10}$$

$$+ (\mathcal{E}^{x1})^{\mathsf{T}} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\widetilde{\mathrm{B}}_1^{n+1}} \cdot \mathcal{E}^{x2}\Big) = -\tau_n \widetilde{\mathrm{Z}}_2^n,$$

$$\mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_2^{\dot{x}3}}{\widetilde{\mathrm{B}}_3^{n+1}} \cdot \mathcal{A}_1^{x2} + \frac{\eta\tau_n}{2}\left(\mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_2^{\dot{x}3}}{\widetilde{\mathrm{B}}_3^{n+1}} \cdot \mathcal{B}^{x2} - \mathcal{A}_1^{x1} \cdot \underset{\mathcal{E}^{\dot{x}3}}{\widetilde{\mathrm{B}}_2^{n+1}} \cdot \mathcal{E}^{x2} - (\mathcal{E}^{x1})^{\mathsf{T}} \cdot \underset{\mathcal{E}^{\dot{x}3}}{\widetilde{\mathrm{B}}_1^{n+1}} \cdot \mathcal{A}_1^{x2} + \mathcal{B}^{x1} \cdot \underset{\mathcal{A}_2^{\dot{x}3}}{\widetilde{\mathrm{B}}_3^{n+1}} \cdot \mathcal{A}_1^{x2}\right) \tag{4.11}$$

$$= -\tau_n \widetilde{\mathrm{Z}}_3^n.$$

Step 2: Computation of r_N^{n+1}.

After obtaining $\widehat{\mathbf{u}}_N^{n+1}$ and $\mathbf{B}_N^{n+1}$ in Step 1, we have

$$r_N^{n+1} = r_N^n + \tau_n \overline{\rho}_N^{n+1}(\Xi + \Lambda), \tag{4.12}$$

where the expressions of Ξ and Λ are as follows

$$\Xi = \left(\left(\left(\overline{\mathbf{u}}_N^{n+\frac{1}{2}} \cdot \nabla \right) \overline{\mathbf{u}}_N^{n+\frac{1}{2}}, \widehat{\mathbf{u}}_N^{n+1} \right) - \left(\left(\nabla \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}} \right) \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}}, \widehat{\mathbf{u}}_N^{n+1} \right) \right),$$

$$\Lambda = \left(\left(\overline{\mathbf{B}}_N^{n+\frac{1}{2}} \times \overline{\mathbf{u}}_N^{n+\frac{1}{2}}, \nabla \times \widehat{\mathbf{B}}_N^{n+1} \right) + \lambda \left(\left(\nabla \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}} \right) \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}}, \nabla \times \widehat{\mathbf{B}}_N^{n+1} \right) \right).$$

Step 2: Computation of r_N^{n+1}.

After obtaining $\widehat{\mathbf{u}}_N^{n+1}$ and $\mathbf{B}_N^{n+1}$ in Step 1, we have

$$r_N^{n+1} = r_N^n + \tau_n \overline{\rho}_N^{n+1}(\Xi + \Lambda), \tag{4.12}$$

where the expressions of Ξ and Λ are as follows

$$\begin{aligned}
\Xi &= \left(\left(\left(\overline{\mathbf{u}}_N^{n+\frac{1}{2}} \cdot \nabla \right) \overline{\mathbf{u}}_N^{n+\frac{1}{2}}, \widehat{\mathbf{u}}_N^{n+1} \right) - \left(\left(\nabla \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}} \right) \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}}, \widehat{\mathbf{u}}_N^{n+1} \right) \right), \\
\Lambda &= \left(\left(\overline{\mathbf{B}}_N^{n+\frac{1}{2}} \times \overline{\mathbf{u}}_N^{n+\frac{1}{2}}, \nabla \times \widehat{\mathbf{B}}_N^{n+1} \right) + \lambda \left(\left(\nabla \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}} \right) \times \overline{\mathbf{B}}_N^{n+\frac{1}{2}}, \nabla \times \widehat{\mathbf{B}}_N^{n+1} \right) \right).
\end{aligned}$$

Step 3: Computation of $\mathbf{u}_N^{n+1}$ and p_N^{n+1}.

We obtain the following matrix equations for the unknowns $\widetilde{\mathrm{U}}_s^{n+1}$ $(s = 1, 2,$ and $3)$ and $\widetilde{\mathrm{P}}^{n+1}$

$$\mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\widetilde{\mathrm{U}}_1^{n+1}} \cdot \mathcal{A}_1^{x2} - \frac{\tau n}{2} \mathcal{D}^{x1} \cdot \underset{\mathcal{C}_1^{\dot{x}3}}{\widetilde{\mathrm{P}}^{n+1}} \cdot \mathcal{C}_1^{x2} = \mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\left(\widetilde{\mathrm{U}}_{A1}^{n+1} + \rho_N^{n+\frac{1}{2}} \widetilde{\mathrm{U}}_{B1}^{n+1} \right)} \cdot \mathcal{A}_1^{x2} - \frac{\tau n}{2} \mathcal{D}^{x1} \cdot \underset{\mathcal{C}_1^{\dot{x}3}}{\widetilde{\mathrm{P}}^{n}} \cdot \mathcal{C}_1^{x2}, \tag{4.13}$$

$$\mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\widetilde{\mathrm{U}}_2^{n+1}} \cdot \mathcal{A}_1^{x2} - \frac{\tau n}{2} \mathcal{C}^{x1} \cdot \underset{\mathcal{C}_1^{\dot{x}3}}{\widetilde{\mathrm{P}}^{n+1}} \cdot \mathcal{D}_1^{x2} = \mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\left(\widetilde{\mathrm{U}}_{A2}^{n+1} + \rho_N^{n+\frac{1}{2}} \widetilde{\mathrm{U}}_{B2}^{n+1} \right)} \cdot \mathcal{A}_1^{x2} - \frac{\tau n}{2} \mathcal{C}^{x1} \cdot \underset{\mathcal{C}_1^{\dot{x}3}}{\widetilde{\mathrm{P}}^{n}} \cdot \mathcal{D}_1^{x2}, \tag{4.14}$$

$$\mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\widetilde{\mathrm{U}}_3^{n+1}} \cdot \mathcal{A}_1^{x2} - \frac{\tau n}{2} \mathcal{C}^{x1} \cdot \underset{\mathcal{D}_1^{\dot{x}3}}{\widetilde{\mathrm{P}}^{n+1}} \cdot \mathcal{C}_1^{x2} = \mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\left(\widetilde{\mathrm{U}}_{A1}^{n+1} + \rho_N^{n+\frac{1}{2}} \widetilde{\mathrm{U}}_{B1}^{n+1} \right)} \cdot \mathcal{A}_1^{x2} - \frac{\tau n}{2} \mathcal{C}^{x1} \cdot \underset{\mathcal{D}_1^{\dot{x}3}}{\widetilde{\mathrm{P}}^{n}} \cdot \mathcal{C}_1^{x2}, \tag{4.15}$$

$$\mathcal{D}^{x1} \cdot \underset{\mathcal{C}_1^{\dot{x}3}}{\widetilde{\mathrm{U}}_1^{n+1}} \cdot \mathcal{C}_1^{x2} + \mathcal{C}^{x1} \cdot \underset{\mathcal{C}_1^{\dot{x}3}}{\widetilde{\mathrm{U}}_2^{n+1}} \cdot \mathcal{D}^{x2} + \mathcal{D}^{x1} \cdot \underset{\mathcal{D}^{\dot{x}3}}{\widetilde{\mathrm{U}}_3^{n+1}} \cdot \mathcal{C}_1^{x2} = 0. \tag{4.16}$$

Step 3: Computation of $\mathbf{u}_N^{n+1}$ and p_N^{n+1}.

We obtain the following matrix equations for the unknowns $\widetilde{\mathrm{U}}_s^{n+1}$ ($s = 1, 2$, and 3) and $\widetilde{\mathrm{P}}^{n+1}$

$$\mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\widetilde{\mathrm{U}}_1^{n+1}} \cdot \mathcal{A}_1^{x2} - \frac{\tau_n}{2}\mathcal{D}^{x1} \cdot \underset{\mathcal{C}_1^{\dot{x}3}}{\widetilde{\mathrm{P}}^{n+1}} \cdot \mathcal{C}_1^{x2} = \mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\left(\widetilde{\mathrm{U}}_{A1}^{n+1} + \rho_N^{n+\frac{1}{2}}\widetilde{\mathrm{U}}_{B1}^{n+1}\right)} \cdot \mathcal{A}_1^{x2} - \frac{\tau_n}{2}\mathcal{D}^{x1} \cdot \underset{\mathcal{C}_1^{\dot{x}3}}{\widetilde{\mathrm{P}}^{n}} \cdot \mathcal{C}_1^{x2}, \tag{4.13}$$

$$\mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\widetilde{\mathrm{U}}_2^{n+1}} \cdot \mathcal{A}_1^{x2} - \frac{\tau_n}{2}\mathcal{C}^{x1} \cdot \underset{\mathcal{C}_1^{\dot{x}3}}{\widetilde{\mathrm{P}}^{n+1}} \cdot \mathcal{D}_1^{x2} = \mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\left(\widetilde{\mathrm{U}}_{A2}^{n+1} + \rho_N^{n+\frac{1}{2}}\widetilde{\mathrm{U}}_{B2}^{n+1}\right)} \cdot \mathcal{A}_1^{x2} - \frac{\tau_n}{2}\mathcal{C}^{x1} \cdot \underset{\mathcal{C}_1^{\dot{x}3}}{\widetilde{\mathrm{P}}^{n}} \cdot \mathcal{D}_1^{x2}, \tag{4.14}$$

$$\mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\widetilde{\mathrm{U}}_3^{n+1}} \cdot \mathcal{A}_1^{x2} - \frac{\tau_n}{2}\mathcal{C}^{x1} \cdot \underset{\mathcal{D}_1^{\dot{x}3}}{\widetilde{\mathrm{P}}^{n+1}} \cdot \mathcal{C}_1^{x2} = \mathcal{A}_1^{x1} \cdot \underset{\mathcal{A}_1^{\dot{x}3}}{\left(\widetilde{\mathrm{U}}_{A1}^{n+1} + \rho_N^{n+\frac{1}{2}}\widetilde{\mathrm{U}}_{B1}^{n+1}\right)} \cdot \mathcal{A}_1^{x2} - \frac{\tau_n}{2}\mathcal{C}^{x1} \cdot \underset{\mathcal{D}_1^{\dot{x}3}}{\widetilde{\mathrm{P}}^{n}} \cdot \mathcal{C}_1^{x2}, \tag{4.15}$$

$$\mathcal{D}^{x1} \cdot \underset{\mathcal{C}_1^{\dot{x}3}}{\widetilde{\mathrm{U}}_1^{n+1}} \cdot \mathcal{C}_1^{x2} + \mathcal{C}^{x1} \cdot \underset{\mathcal{C}_1^{\dot{x}3}}{\widetilde{\mathrm{U}}_2^{n+1}} \cdot \mathcal{D}^{x2} + \mathcal{D}^{x1} \cdot \underset{\mathcal{D}^{\dot{x}3}}{\widetilde{\mathrm{U}}_3^{n+1}} \cdot \mathcal{C}_1^{x2} = 0. \tag{4.16}$$

Accuracy tests

Consider a problem with a known analytical solution as follows

$$\mathbf{u}(\mathbf{x},t) = t^4 k \begin{pmatrix} \sin^2(\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3) \\ \sin(2\pi x_1)\sin^2(\pi x_2)\sin(2\pi x_3) \\ -2\sin(2\pi x_1)\sin(2\pi x_2)\sin^2(\pi x_3) \end{pmatrix},$$

$$\mathbf{B}(\mathbf{x},t) = t^4 k \begin{pmatrix} \cos(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3) \\ \sin(2\pi x_1)\cos(2\pi x_2)\sin(2\pi x_3) \\ -2\sin(2\pi x_1)\sin(2\pi x_2)\cos(2\pi x_3) \end{pmatrix},$$

$$p(\mathbf{x},t) = t^4 k \left(\sin^2(2\pi x_1)\sin^2(2\pi x_2)\sin^2(2\pi x_3) - \frac{1}{8} \right),$$

where $k = 0.1$. In addition, we set $\nu = \eta = \lambda = C = 1$. Moreover, we have $\nabla p|_{\partial\Omega} = \mathbf{0}$.

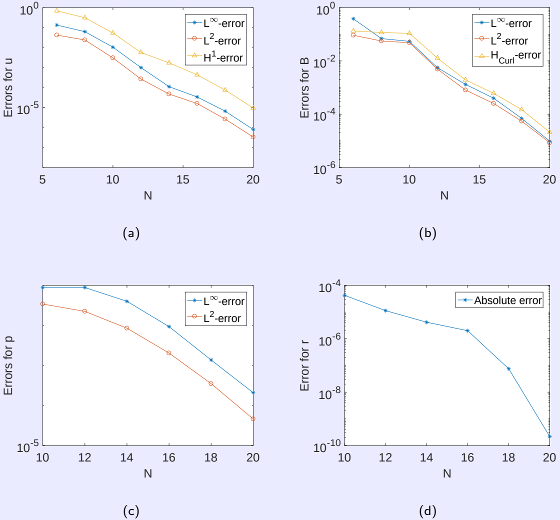


Figure 5.1: Spatial discretization errors v.s. N with $\tau = 1e-3$ at $T = 1$.

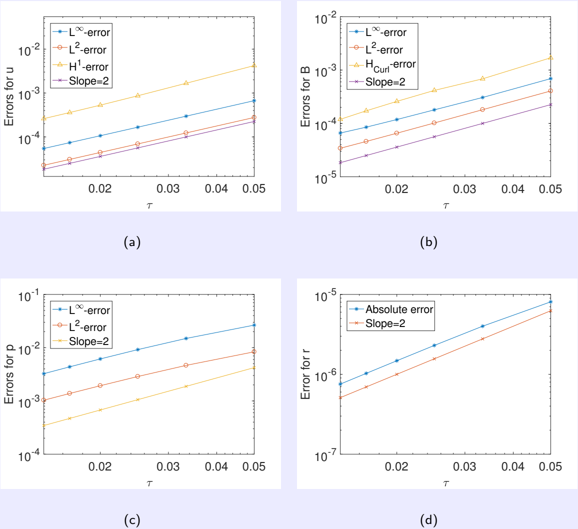


Figure 5.2: Temporal discretization errors v.s. τ with $N = 20$ at $T = 1$.

Adaptive time-stepping strategy

In this example, the initial conditions are as follows

$$\mathbf{u}(\mathbf{x},0) = k\begin{pmatrix} \cos^2\left(\frac{\pi}{2}x_1\right)\sin(\pi x_2)\sin(\pi x_3) \\ \sin(\pi x_1)\cos^2\left(\frac{\pi}{2}x_2\right)\sin(\pi x_3) \\ -2\sin(\pi x_1)\sin(\pi x_2)\cos^2\left(\frac{\pi}{2}x_3\right) \end{pmatrix}, \tag{5.1}$$

$$\mathbf{B}(\mathbf{x},0) = k\begin{pmatrix} \left(\frac{x_1^3}{3} - x_1\right)(x_2^2-1)(x_3^2-1) \\ (x_1^2-1)\left(\frac{x_2^3}{3} - x_2\right)(x_3^2-1) \\ -2(x_1^2-1)(x_2^2-1)\left(\frac{x_3^3}{3} - x_3\right) \end{pmatrix}, \tag{5.2}$$

$$p(\mathbf{x},0) = 0. \tag{5.3}$$

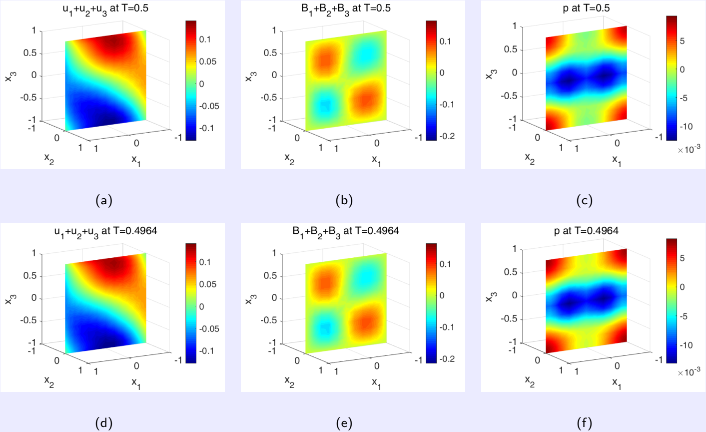
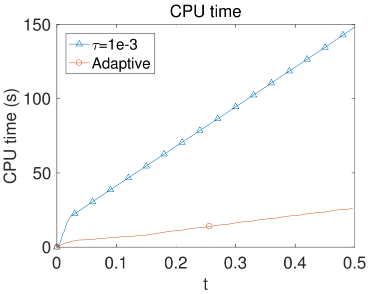
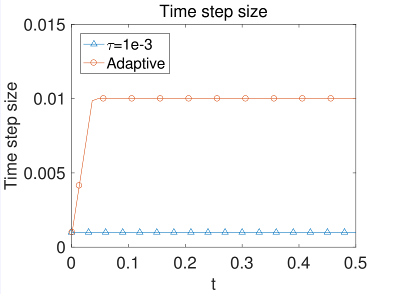


Figure 5.3: Numerical comparisons between small uniform time-stepping (the first row) and adaptive time-stepping (the second row) with $N = 14$ at $T = 0.5$.



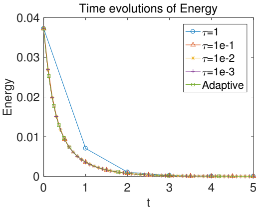
(a)



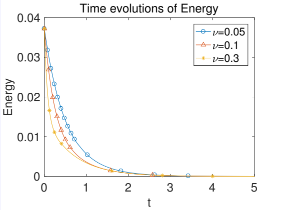
(b)

Figure 5.4: CPU time (left) and time step size (right). The parameters are the same as that in Fig. 5.3.

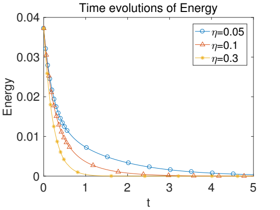
Unconditional energy stability



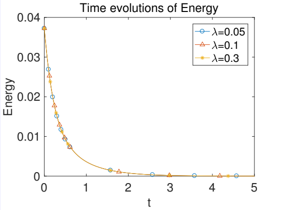
(a)



(b)



(c)



(d)

Figure 5.5: Time evolutions of L^2-energy for different time step sizes (plot(a)), for

Magnetic O- and X-points in Hall-MHD regime

In this simulation, the initial conditions are

$$\mathbf{u}(\mathbf{x},0) = \mathbf{0}, \tag{5.4}$$

$$\mathbf{B}(\mathbf{x},0) = \begin{pmatrix} 0 \\ 0 \\ -2.5\sin(\pi x_1) \end{pmatrix}, \tag{5.5}$$

$$p(\mathbf{x},0) = 0. \tag{5.6}$$

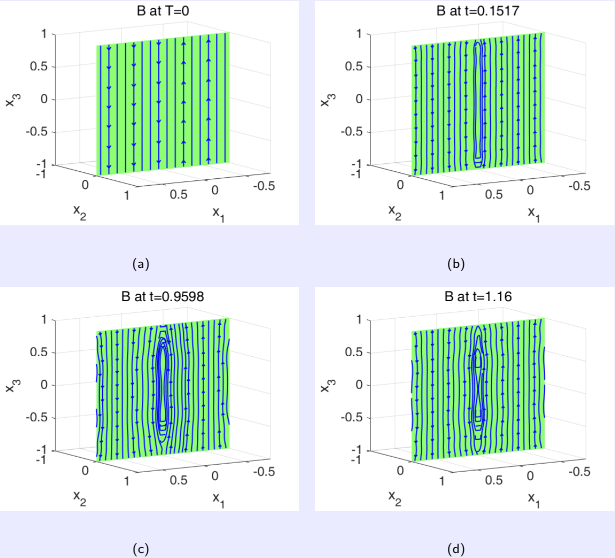


Figure 5.6: Time evolutions of magnetic field lines on the plane $x_2 = 0$.

These numerical results coincide very well with the numerical simulations in [1], qualitatively.

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[1] K. Bora, R. Bhattacharyya, P. K. Smolarkiewicz, Evolution of three-dimensional coherent structures in Hall magnetohydrodynamics, The Astrophysical Journal 906 (2021) 102.

Recent works on spectral methods for PDEs:

- **Shimin Guo**, Liquan Mei, Wenjing Yan, Ying Li, Mass-, energy-, and momentum-preserving spectral scheme for Klein-Gordon-Schrödinger system on infinite domains, **SIAM Journal on Scientific Computing** 45 (2023) B200-B230.
- **Shimin Guo**, Liquan Mei, Can Li, Wenjing Yan, Jinghuai Gao, IMEX Hermite-Galerkin spectral schemes with adaptive time stepping for the coupled nonlocal Gordon-type systems in multiple dimensions, **SIAM Journal on Scientific Computing** 43 (2021) B1133-B1163.
- **Shimin Guo**, Can Li, Xiaoli Li, Liquan Mei, Energy-conserving and time-stepping-varying ESAV-Hermite-Galerkin spectral scheme for nonlocal Klein-Gordon-Schrödinger system with fractional Laplacian in unbounded domains, **Journal of Computational Physics** 458 (2022) 111096.
- **Shimin Guo**, Wenjing Yan, Can Li, Liquan Mei, Dissipation-preserving rational spectral-Galerkin method for strongly damped nonlinear wave system involving mixed fractional Laplacians in unbounded domains, **Journal of Scientific Computing** 93 (2022) 53.
- **Shimin Guo**, Liquan Mei, Can Li, Zhengqiang Zhang, Ying Li, Semi-implicit Hermite-Galerkin spectral method for distributed-order fractional-in-space nonlinear reaction-diffusion equations in multidimensional unbounded domains, **Journal of Scientific Computing** 85 (2020) 15.
- ......

Thanks a lot for your attention!