Efficient spectral collocation method for tempered fractional differential equation

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Motiviation

- Why "tempered" ? The fractional sub-diffusive and/or superdiffusive models, based on the continuous time random walk (CTRW) model with the power law waiting time distribution **having divergent first moment** and/or the power law jump length distribution **having divergent second moment**.
- Efficient numerical method are required to solve the tempered equation, then to extend and dig out the potential applications of the tempered dynamics.
- Mathematically, the common-used fractional calculus is the special case of the tempered fractional calculus with the parameter $\kappa = 0$.

Related works

- (2015) F. Sabzikar, M.M. Meerschaert, J.H. Chen, Tempered fractional calculus, J. Comput. Phys. ;...
- FDM/FEM: Li and Deng(Adv. Comput. Math. 2016); Wang and Li(Appl. Math. Lett. 2022); Deng and Zhang(NMPDEs. 2018); Chen and Deng(Appl. Math. Lett. 2017);Ding and Li(J. Sci. Comput. 2019); Guo et al. (SIAM J. Sci. Comput. 2019); Cao et al.(J. Sci. Comput. 2020);...
- LDG: Sun et al.(Appl. Math. Comput. 2020); Safari et al.(J. Sci. Comput. 2022);
- Multigrid: Bu and Oosterlee(Fractal Fract. 2021);...
- Spectral method: M. Zayernourt, M. Ainsworkth, G.E. Karniadakis (SIAM J. Sci. Comput. 2015); Hanert and Piret(SIAM J. Sci. Comput. 2015); Chen et al.(J. Sci. Comput. 2018); Huang et al.(J. Sci. Comput. 2018)...
- Others ...

Aims

we aim to develop a high accuracy spectral collocation method which uses the tempered fractional Jacobi functions (TFJFs) as the basis functions for solving TFD equations. The main contributions of this work are as follows:

- † We define the TFJFs and derive the approximation results of orthogonal projection and interpolation based on the TFJFs.
- † We derive the differentiation matrix of the tempered fractional Caputo derivative and give a fast and stable evaluation method based on recurrence relationship.
- † We demonstrate the effectiveness of the proposed spectral collocation method for the initial or boundary value problems, i.e., the fractional Helmholtz equation, and the fractional Burgers equation.

定义 (left and right tempered fractional integrals)

For $\kappa \geq 0$, the left and right tempered fractional integrals of function $u(t)$ on (a,b) of order $\mu > 0$ are defined, respectively, by

$$_{a}\mathrm{I}_{t}^{\mu,\kappa}u(t) := \frac{1}{\Gamma(\mu)}\int_{a}^{t}\frac{e^{-\kappa(t-s)}u(s)}{(t-s)^{1-\mu}}\mathrm{d}s,$$

and

$$_{t}\mathrm{I}_{b}^{\mu,\kappa}u(t) := \frac{1}{\Gamma(\mu)}\int_{t}^{b}\frac{e^{-\kappa(s-t)}u(s)}{(s-t)^{1-\mu}}\mathrm{d}s,$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

定义 (tempered R-L, Caputo fractional derivatives)

For $\kappa \geq 0$, the left and right tempered Riemman-Liouville, Caputo fractional derivatives of function $u(t)$ on (a, b) of order $\mu > 0$ are defined, respectively, by

$$
\begin{aligned}
{}_a^R\mathrm{D}_t^{\mu,\kappa} u(t) :&= \frac{e^{-\kappa t}}{\Gamma(n-\mu)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{e^{\kappa s} u(s)}{(t-s)^{\mu-n+1}} \mathrm{d}s, \\
{}_t^R\mathrm{D}_b^{\mu,\kappa} u(t) :&= \frac{e^{\kappa t}}{\Gamma(n-\mu)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_t^b \frac{e^{-\kappa s} u(s)}{(s-t)^{\mu-n+1}} \mathrm{d}s, \\
{}_a^C\mathrm{D}_t^{\mu,\kappa} u(t) :&= \frac{e^{-\kappa t}}{\Gamma(n-\mu)} \int_a^t \frac{(e^{\kappa s} u(s))^{(n)}}{(t-s)^{\mu-n+1}} \mathrm{d}s, \\
{}_t^C\mathrm{D}_b^{\mu,\kappa} u(t) :&= \frac{(-1)^n e^{\kappa t}}{\Gamma(n-\mu)} \int_t^b \frac{(e^{-\kappa s} u(s))^{(n)}}{(s-t)^{\mu-n+1}} \mathrm{d}s
\end{aligned}
$$

where $n-1 < \mu \leq n$.

If $\kappa \equiv 0$, then the left and right tempered fractional calculus reduce to the left and right (non-tempered) fractional calculus, denoted by ${}_a\mathrm{I}_t^{\mu}$, ${}_t\mathrm{I}_b^{\mu}$, ${}_a^R\mathrm{D}_t^{\mu}$, ${}_t^R\mathrm{D}_b^{\mu}$, ${}_a^C\mathrm{D}_t^{\mu}$ and ${}_t^C\mathrm{D}_b^{\mu}$, respectively. Then we have

$$
\begin{aligned}
{}_a\mathrm{I}_t^{\mu,\kappa}u(t) &= e^{-\kappa t}\,{}_a\mathrm{I}_t^{\mu}(e^{\kappa t}u(t)), & {}_t\mathrm{I}_b^{\mu,\kappa}u(t) &= e^{\kappa t}\,{}_t\mathrm{I}_b^{\mu}(e^{-\kappa t}u(t)),\\
{}_a^R\mathrm{D}_t^{\mu,\kappa}u(t) &= e^{-\kappa t}\,{}_a^R\mathrm{D}_t^{\mu}(e^{\kappa t}u(t)), & {}_t^R\mathrm{D}_b^{\mu,\kappa}u(t) &= e^{\kappa t}\,{}_t^R\mathrm{D}_b^{\mu}(e^{-\kappa t}u(t)),\\
{}_a^C\mathrm{D}_t^{\mu,\kappa}u(t) &= e^{-\kappa t}\,{}_a^C\mathrm{D}_t^{\mu}(e^{\kappa t}u(t)), & {}_t^C\mathrm{D}_b^{\mu,\kappa}u(t) &= e^{\kappa t}\,{}_t^C\mathrm{D}_b^{\mu}(e^{-\kappa t}u(t)).
\end{aligned}
$$

Similar to the corresponding non-tempered case, for $0 < \mu < 1$ and $u(t)$ be absolutely continuous in $[a, b]$, the relations between Riemann-Liouville and Caputo derivatives are

$$
{}_a^R\mathrm{D}_t^{\mu,\kappa}u(t) = {}_a^C\mathrm{D}_t^{\mu,\kappa}u(t) + \frac{1}{\Gamma(1-\mu)}e^{-\kappa t}(t-a)^{-\mu}u(a), \tag{1}
$$

and

$$
{}_t^R\mathrm{D}_b^{\mu,\kappa}u(t) = {}_t^C\mathrm{D}_b^{\mu,\kappa}u(t) + \frac{1}{\Gamma(1-\mu)}e^{-\kappa t}(b-t)^{-\mu}u(b). \tag{2}
$$

Denote $\mathbb{P}_N(I)$ the space of all algebraic polynomials with degree at most N defined on $I =: (-1,1)$. Let $P_n^{\alpha,\beta}(s)$ $(\alpha,\beta > -1), s \in I$ be the Jacobi orthogonal polynomials satisfying:

$$\int_{-1}^{1} P_n^{\alpha,\beta}(s) P_m^{\alpha,\beta}(s)\omega^{\alpha,\beta}(s)\mathrm{d}s = \gamma_n^{\alpha,\beta}\delta_{mn}, \tag{3}$$

定义 (Tempered Fractional Jacobi Function)

Define the TFJFs by

$$J_{n,l}^{\alpha,\beta,\delta,\kappa}(s) =: e^{-\kappa s}(1+s)^{\delta} P_n^{\alpha,\beta}(s), \quad J_{n,r}^{\alpha,\beta,\delta,\kappa}(s) =: e^{\kappa s}(1-s)^{\delta} P_n^{\alpha,\beta}(s),$$

for $s \in I, n = 0, 1, \cdots$.

The tempered fractional derivatives of the TFJFs can also be represented by the same class of functions.

定理

For $n \geq 0$, there holds

$$
{}_{-1}^{R}\mathrm{D}_{s}^{\mu,\kappa}\left(J_{n,l}^{\alpha,\beta,\beta,\kappa}(s)\right) = \lambda_{n}^{\beta,\mu} J_{n,l}^{\alpha+\mu,\beta-\mu,\beta-\mu,\kappa}(s), \quad \alpha \in \mathbb{R},\ \beta > \mu - 1.
$$

$$
{}_{s}^{R}\mathrm{D}_{1}^{\mu,\kappa}\left(J_{n,r}^{\alpha,\beta,\alpha,\kappa}(s)\right) = \lambda_{n}^{\alpha,\mu} J_{n,r}^{\alpha-\mu,\beta+\mu,\alpha-\mu,\kappa}(s), \quad \alpha > \mu - 1,\ \beta \in \mathbb{R}.
$$

where

$$
\lambda_{n}^{\beta,\mu} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta-\mu+1)}.
$$

Discrete subspace

We introduce the $(N+1)$-dimensional spaces of the TFJFs as:

$$\mathbb{F}_{N,l}^{\delta,\kappa} := e^{-\kappa s}(1+s)^{\delta}\mathbb{P}_N = \text{span}\left\{J_{0,l}^{\alpha,\beta,\delta,\kappa}, J_{1,l}^{\alpha,\beta,\delta,\kappa}, \cdots, J_{N,l}^{\alpha,\beta,\delta,\kappa}\right\},$$

and

$$\mathbb{F}_{N,r}^{\delta,\kappa} := e^{\kappa s}(1-s)^{\delta}\mathbb{P}_N = \text{span}\left\{J_{0,r}^{\alpha,\beta,\delta,\kappa}, J_{1,r}^{\alpha,\beta,\delta,\kappa}, \cdots, J_{N,r}^{\alpha,\beta,\delta,\kappa}\right\}.$$

Projections

Let $\alpha, \beta > -1$. Denote $\mathbf{P}_{N,l}^{\delta,\kappa} : L^2_{\varpi_l^{\alpha,\beta,\delta,\kappa}}(I) \to \mathbb{F}_{N,l}^{\delta,\kappa}$ as the orthogonal projection such that

$$(\mathbf{P}_{N,l}^{\delta,\kappa} u - u, v)_{\varpi_l^{\alpha,\beta,\delta,\kappa}} = 0 \quad \forall v \in \mathbb{F}_{N,l}^{\delta,\kappa}, \tag{4}$$

and $\mathbf{P}_{N,r}^{\delta,\kappa} : L^2_{\varpi_r^{\alpha,\beta,\delta,\kappa}}(I) \to \mathbb{F}_{N,r}^{\delta,\kappa}$ as the orthogonal projection such that

$$(\mathbf{P}_{N,r}^{\delta,\kappa} u - u, v)_{\varpi_r^{\alpha,\beta,\delta,\kappa}} = 0 \quad \forall v \in \mathbb{F}_{N,r}^{\delta,\kappa}. \tag{5}$$

Error of the projections

For describe the projection error, we define

$$\check{B}^{\nu}_{\alpha,\beta,\delta,l}(I) := \left\{ u \in L^2_{\varpi_l^{\alpha,\beta,\delta,\kappa}}(I) : {}^{R}_{-1}\mathrm{D}^{\mu,\kappa}_{s} u \in L^2_{\varpi_l^{\alpha+\mu,\beta-\mu,\delta-\mu,\kappa}}(I), \quad 0 \le \mu \le \nu \right\},$$

$$\check{B}^{\nu}_{\alpha,\beta,\delta,r}(I) := \left\{ u \in L^2_{\varpi_r^{\alpha,\beta,\delta,\kappa}}(I) : {}^{R}_{s}\mathrm{D}^{\mu,\kappa}_{1} u \in L^2_{\varpi_r^{\alpha-\mu,\beta+\mu,\delta-\mu,\kappa}}(I), \quad 0 \le \mu \le \nu \right\}.$$

定理 (Error estimate of the projections)

Let $\alpha > -1, \beta > \nu - 1, 0 \le \mu \le \nu$. For any $u \in \check{B}^{\nu}_{\alpha,\beta,\beta,l}(I)$,

$$\|{}^{R}_{-1}\mathrm{D}^{\mu,\kappa}_{s}(\mathbf{P}^{\beta,\kappa}_{N,l}u - u)\|_{\varpi_l^{\alpha+\mu,\beta-\mu,\beta-\mu,\kappa}} \le cN^{\mu-\nu}\|{}^{R}_{-1}\mathrm{D}^{\nu,\kappa}_{s}u\|_{\varpi_l^{\alpha+\nu,\beta-\nu,\beta-\nu,\kappa}}.$$

and for $\alpha > \nu - 1, \beta > -1, u \in \check{B}^{\nu}_{\alpha,\beta,\alpha,r}(I)$,

$$\|{}^{R}_{s}\mathrm{D}^{\mu,\kappa}_{1}(\mathbf{P}^{\alpha,\kappa}_{N,r}u - u)\|_{\varpi_r^{\alpha-\mu,\beta+\mu,\alpha-\mu,\kappa}} \le cN^{\mu-\nu}\|{}^{R}_{s}\mathrm{D}^{\nu,\kappa}_{1}u\|_{\varpi_r^{\alpha-\nu,\beta+\nu,\alpha-\nu,\kappa}}.$$

Interpolations

Denote $\{\xi_i\}_{i=0}^N$ as the Jacobi-Gauss-Lobatto points(JGL) in $\bar{I}$.
For a function $u(s) \in C(I)$ which satisfies that $e^{\kappa s}(1+s)^{-\delta}u(s)$ being continuous on $\bar{I}$ for some $\delta > -1$, we define the interpolation operator $\Pi_{N,l}^{\delta,\kappa} : C(I) \to \mathbb{F}_{N,l}^{\delta,\kappa}$ at given nodes $\{\xi_i\}_{i=0}^N$ as

$$\Pi_{N,l}^{\delta,\kappa}u(s) = u(s), \quad s = \xi_i, \ i = 0, 1, \cdots, N. \tag{6}$$

Similarly, a function $u(s) \in C(I)$ which satisfies that $e^{-\kappa s}(1-s)^{-\delta}u(s)$ being continuous on $\bar{I}$ for some $\delta > -1$, we define the interpolation operator $\Pi_{N,r}^{\delta,\kappa} : C(I) \to \mathbb{F}_{N,r}^{\delta,\kappa}$ at the same nodes $\{\xi_i\}_{i=0}^N$ as

$$\Pi_{N,r}^{\delta,\kappa}u(s) = u(s), \quad s = \xi_i, \ i = 0, 1, \cdots, N. \tag{7}$$

Error of the interpolations

定理 (Error estimate of the interpolations)

For $\alpha, \beta > \nu - 1$ and $0 \leq \mu \leq \nu$. If $u \in \check{B}^{\nu}_{\alpha,\beta,\beta,l}(I)$, then

$$\|{}^{R}_{-1}\mathrm{D}^{\mu,\kappa}_{s}(\Pi^{\beta,\kappa}_{N,l}u - u)\|_{\varpi_l^{\alpha+\mu,\beta-\mu,\beta-\mu,\kappa}} \leq cN^{\mu-\nu}\|{}^{R}_{-1}\mathrm{D}^{\nu,\kappa}_{s}u\|_{\varpi_l^{\alpha+\nu,\beta-\nu,\beta-\nu,\kappa}},$$

and if $u \in \check{B}^{\nu}_{\alpha,\beta,\alpha,r}(I)$, then

$$\|{}^{R}_{s}\mathrm{D}^{\mu,\kappa}_{1}(\Pi^{\alpha,\kappa}_{N,r}u - u)\|_{\varpi_r^{\alpha-\mu,\beta+\mu,\alpha-\mu,\kappa}} \leq cN^{\mu-\nu}\|{}^{R}_{s}\mathrm{D}^{\nu,\kappa}_{1}u\|_{\varpi_r^{\alpha-\nu,\beta+\nu,\alpha-\nu,\kappa}}.$$

The tempered fractional Lagrange interpolate basis

Let $\{L_i(s)\}_{i=0}^N$ be the Lagrange interpolation basis functions with respect to the nodes $\{\xi_i\}_{i=0}^N$. The tempered fractional Lagrange interpolation basis functions are defined by

$$F_{i,l}^{\delta,\kappa}(s) := e^{-\kappa(s-\xi_i)}\left(\frac{1+s}{1+\xi_i}\right)^{\delta} L_i(s), \quad i = 0, 1, 2, \cdots, N, \tag{8}$$

which satisfies $F_{i,l}^{\delta,\kappa}(\xi_k) = \delta_{ik}$.

The tempered fractional Lagrange interpolation basis functions are defined by

$$F_{i,r}^{\delta,\kappa}(s) := e^{\kappa(s-\xi_i)}\left(\frac{1-s}{1-\xi_i}\right)^{\delta} L_i(s), \quad i = 0, 1, 2, \cdots, N, \tag{9}$$

which satisfies $F_{i,r}^{\delta,\kappa}(\xi_k) = \delta_{ik}$.

Differentiation matrix(DM)

The differentiation matrix of left tempered Caputo derivative is:

$$\left[\mathbf{D}^{\mu,\kappa}_{s,l}\right]_{(N+1)\times(N+1)} := \left({}^{C}_{-1}\mathrm{D}^{\mu,\kappa}_{s} F^{\delta,\kappa}_{i,l}(\xi_k)\right)^{N}_{k,i=0}.$$

$${}^{C}_{-1}\mathrm{D}^{\mu,\kappa}_{s} F^{\delta,\kappa}_{i,l}(s) = \frac{e^{\kappa\xi_i}}{(1+\xi_i)^{\delta}} \sum_{j=0}^{N} l_{ij}\, {}^{C}_{-1}\mathrm{D}^{\mu,\kappa}_{s} J^{\alpha,\beta,\delta,\kappa}_{i,l}(s). \qquad (10)$$

where $l_{ij} = \frac{P^{\alpha,\beta}_{j}(\xi_i)\omega_i}{\gamma^{\alpha,\beta}_{j}} (j = 0, 1, \cdots, N-1), \quad l_{iN} = \frac{P^{\alpha,\beta}_{N}(\xi_i)\omega_i}{(2+\frac{\alpha+\beta+1}{N})\gamma^{\alpha,\beta}_{N}}.$

The differentiation matrix of right tempered Caputo derivative is:

$$\left[\mathbf{D}^{\mu,\kappa}_{s,r}\right]_{(N+1)\times(N+1)} := \left({}^{C}_{s}\mathrm{D}^{\mu,\kappa}_{1} F^{\delta,\kappa}_{i,r}(\xi_k)\right)^{N}_{k,i=0}.$$

We have the similar formula as (10) for right case.

Fast evaluation of the DM

Denote

$$S_{n,\alpha,\beta,l}^{\delta,\kappa,\mu}(s) := {}_{-1}\mathrm{I}_s^{\mu,\kappa}\left(J_{n,l}^{\alpha,\beta,\delta,\kappa}(s)\right) = \frac{1}{\Gamma(\mu)}\int_{-1}^{s}(s-t)^{\mu-1}e^{-\kappa(s-t)}J_{n,l}^{\alpha,\beta,\delta,\kappa}(t)\mathrm{d}t. \tag{11}$$

定理 (Recurrence relationship)

Let $\alpha,\beta,\delta>-1$ *and* $\mu>0, s\in I$. *Then* $S_{n,l} := S_{n,\alpha,\beta,l}^{\delta,\kappa,\mu}(s)$ *and* $\widehat{S}_{n,l} := \mu S_{n,\alpha,\beta,l}^{\delta,\kappa,\mu+1}(s)$ *satisfy*

$$S_{n+1,l} = (A_n^{\alpha,\beta}s - B_n^{\alpha,\beta})S_{n,l} - C_n^{\alpha,\beta}S_{n-1,l} - A_n^{\alpha,\beta}\widehat{S}_{n,l},\ \ n\geq 1$$

$$\widehat{S}_{n+1,l} = \widetilde{A}_n\widehat{S}_{n,l} - \widetilde{B}_n\widehat{S}_{n-1,l} + (1+s)\left(\widetilde{a}_nS_{n-1,l} + \widetilde{b}_nS_{n,l} + \widetilde{c}_nS_{n+1,l}\right),\ n\geq 1$$

with the starting terms $S_{0,l}, \widehat{S}_{0,l}, S_{1,l}, \widehat{S}_{1,l}$.

Condition number of DM

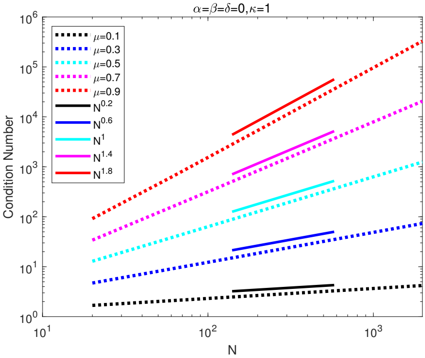

图 1: Condition number of DMLTCD with $0 < \mu < 1, \alpha = \beta = 0, \delta = 0, \kappa = 1$.

Application to Fractional ordinary differential equation

Let $0 < \mu < 1$. We apply the spectral collocation method to the initial value problem of nonlinear fractional ordinary differential equation

$$\begin{cases} {}_0^C\mathrm{D}_x^{\mu,\kappa} u(x) = f(x,u), \quad 0 < x \le T, \\ u(0) = u_0. \end{cases} \tag{12}$$

The spectral collocation method based on the TFJFs for (12) is to find $u_N \in \mathbb{F}_{N,l}^{\delta,\kappa}$ such that for $x = x_j^{\alpha,\beta} = \frac{T(1+\xi_j)}{2}, \quad j = 1, 2, \cdots, N,$

$${}_0^C\mathrm{D}_x^{\mu,\widetilde{\kappa}} u_N(x) = f(x, u_N(x)), \tag{13}$$

and

$$u_N(0) = u_0. \tag{14}$$

Application to Fractional ordinary differential equation

The above two equations lead to the following system

$$\left(\frac{2}{T}\right)^{\mu} \overline{\mathbf{D}}_{s,l}^{\mu,\widetilde{\kappa}} \mathbf{u} = \mathbf{f}(\mathbf{u}), \tag{15}$$

Consider (12) and $f(x,u) = g(x) - u^2$, where $g(x)$ is determined by

$$g(x) = u^2 + {}_0^C\mathrm{D}_x^{\mu,\kappa} u(x).$$

In this example, the Newton method is applied to solving the nonlinear system (15), which takes the following form:

$$\mathbf{u}^{n+1} = \mathbf{u}^n + (\mathbf{D} + 2\mathrm{diag}(\mathbf{u}^n))^{-1} (\mathbf{D}\mathbf{u}^n + \mathbf{u}^n .* \mathbf{u}^n - \mathbf{g}), \quad n = 0, 1, \cdots,$$

where $\mathbf{g} = [g(x_1^{\alpha,\beta}), g(x_2^{\alpha,\beta}), \cdots, g(x_N^{\alpha,\beta})]^T$ and $\mathbf{D} = (\frac{2}{T})^{\mu} \overline{\mathbf{D}}_{s,l}^{\mu,\widetilde{\kappa}}$.

Numerical results to fractional ordinary differential equation

C1. $u(x) = e^{-\kappa x}(x^2 - x)$ with $T = 5$.

C2. $u(x) = e^{-\kappa x}\left(x^8 - 3x^{4+\mu/2} + \frac{9}{4}x^{\mu}\right)$ with $T = 1$.

表 1: The numerical errors E_0 with C2 solution $(\alpha = 0, \beta = 0, \kappa = 1)$

$N(1/h)$	$\mu = 0.2$[5]	$\mu = 0.2$	$\mu = 0.5$[5]	$\mu = 0.5$	$\mu = 0.9$[5]	$\mu = 0.9$
10	5.81e-1	1.09e-07	2.44e-2	8.41e-07	7.77e-3	1.40e-06
20	2.00e-1	5.17e-10	6.38e-3	2.85e-09	1.44e-3	3.32e-09
40	6.50e-2	3.87e-13	2.82e-3	7.41e-12	6.00e-4	7.58e-12
80	8.35e-3	1.23e-13	3.84e-4	3.65e-13	1.52e-4	1.14e-13
160	4.72e-4	1.82e-11	2.23e-5	1.52e-12	1.54e-5	1.71e-13

Application to Helmholtz equation

Let $1 < \mu < 2$. We apply the spectral collocation method to the following fractional Helmholtz equation

$$\begin{cases} \lambda^2 u(x) - {}_a^C\mathrm{D}_x^{\mu,\kappa} u(x) = f(x), & a < x < b, \\ u(a) = u_a, \quad u(b) = u_b. \end{cases} \tag{16}$$

The spectral collocation method based on the TFJFs for (16) is to find $u_N \in \mathbb{F}_N^{\delta,\kappa}$ such that

$$\lambda^2 u_N(x) - {}_a^C\mathrm{D}_x^{\mu,\kappa} u_N(x) = f(x), \quad x = x_j^{\alpha,\beta}, \quad j = 1, 2, \cdots, N-1, \tag{17}$$

and

$$u_N(a) = u_a, \quad u_N(b) = u_b. \tag{18}$$

Numerical errors for fractional Helmholtz equation

The source term $f(x)$ is chosen such that the problem satisfies one of the following two cases:

C1. Smooth solution: $u(x) = e^{-\kappa x} \sin(\pi x)$.

C2. Solution with low regularity: $u(x) = e^{-\kappa x} \left[2^{\sigma+1} x^{\sigma} - x^{2\sigma+1}\right]$.

表 2: The numerical errors E_0 with C1 solution $(\alpha = \beta = 0, \delta = 0, \kappa = 1, \lambda = 0)$

N	$\mu = 1.1$	$\mu = 1.3$	$\mu = 1.5$	$\mu = 1.7$	$\mu = 1.9$	$\mu = 1.99$
4	1.781e-01	1.191e-01	9.844e-02	1.091e-01	9.028e-02	4.519e-02
8	5.110e-04	3.488e-04	2.818e-04	2.931e-04	2.610e-04	4.833e-05
12	2.173e-07	1.373e-07	1.016e-07	1.448e-07	1.699e-07	3.235e-08
16	2.618e-11	1.550e-11	1.160e-11	1.732e-11	2.501e-11	5.182e-12
20	1.200e-14	1.299e-14	1.241e-14	1.474e-14	1.302e-14	1.258e-14

Numerical errors for fractional Helmholtz equation

表 3: The numerical errors E_0 with C1 solution $(\mu = 1.4, \delta = 0, \kappa = 5, \lambda = 2$

N	$\alpha=\beta=0$	$\alpha=\beta=-.5$	$\alpha=-\beta=.5$	$(\alpha,\beta)=(-.3,.8)$	$\alpha=\beta=1$
4	2.282e-02	2.901e-02	1.322e-02	2.338e-02	1.605e-02
8	9.931e-05	5.828e-05	2.701e-05	3.397e-04	1.821e-04
12	3.315e-08	2.504e-08	1.210e-08	1.889e-07	1.075e-07
16	3.008e-12	2.379e-12	1.493e-12	2.353e-11	1.419e-11
20	1.174e-14	2.498e-15	1.443e-15	1.282e-14	2.776e-14

Numerical errors for fractional Helmholtz equation

表 4: The numerical errors E_0 with C1 solution $(\mu = 1.6, \alpha = \beta = 0, \delta = 0, \lambda = 1000)$

N	$\kappa = 1$	$\kappa = 2$	$\kappa = 3$	$\kappa = 5$	$\kappa = 8$	$\kappa = 10$
4	2.131e-06	1.509e-06	1.068e-06	5.353e-07	1.900e-07	9.521e-08
8	3.355e-08	3.035e-08	2.745e-08	2.247e-08	1.663e-08	1.361e-08
12	3.806e-11	3.632e-11	3.467e-11	3.158e-11	2.745e-11	2.500e-11
16	9.076e-15	8.868e-15	8.618e-15	8.188e-15	7.522e-15	7.119e-15
20	2.220e-16	1.110e-16	1.110e-16	5.551e-17	2.082e-17	2.776e-17

Numerical errors for fractional Helmholtz equation

表 5: The numerical errors E_0 and the order of convergence *Ord* with C2 solution $(\sigma = 1.4, \alpha = \beta = 0, \delta = 0, \kappa = 1, \lambda = 1)$

	$\mu = 1.1$		$\mu = 1.3$		$\mu = 1.5$		$\mu = 1.9$	
N	E_0	*Ord*	E_0	*Ord*	E_0	*Ord*	E_0	*Ord*
8	3.74e-2	-	4.30e-2	-	4.54e-2	-	2.32e-2	-
16	6.84e-3	2.45	8.42e-3	2.35	1.33e-2	1.77	1.35e-2	0.78
32	1.07e-3	2.67	1.88e-3	2.16	3.97e-3	1.75	7.14e-3	0.92
64	1.61e-4	2.74	4.12e-4	2.19	1.15e-3	1.79	3.63e-3	0.97
128	2.48e-5	2.69	8.96e-5	2.20	3.33e-4	1.79	1.83e-3	0.99
256	3.95e-6	2.65	1.95e-5	2.20	9.59e-5	1.80	9.17e-4	1.00
512	6.37e-7	2.63	4.25e-6	2.20	2.76e-5	1.80	4.59e-4	1.00

Numerical errors for fractional Helmholtz equation

表 6: The numerical errors E_0 and the order of convergence *Ord* with C2 solution $(\sigma = 1.4, \alpha = \beta = 0, \delta = 0.4, \kappa = 1, \lambda = 1)$

	$\mu = 1.1$		$\mu = 1.3$		$\mu = 1.5$		$\mu = 1.9$	
N	E_0	*Ord*	E_0	*Ord*	E_0	*Ord*	E_0	*Ord*
8	4.53e-05	-	1.45e-05	-	9.69e-06	-	5.74e-06	-
16	5.36e-07	6.40	1.07e-07	7.09	5.75e-08	7.40	3.73e-08	7.27
32	1.06e-08	5.66	7.20e-10	7.21	3.76e-10	7.26	2.17e-10	7.43
64	1.07e-10	6.63	4.43e-12	7.34	2.13e-12	7.47	1.18e-12	7.52
128	7.94e-14	10.40	6.14e-14	6.17	5.21e-14	5.35	4.95e-14	4.58

Application to Fractional Burgers equation

Consider the fractional Burgers equation (FBE),

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = \epsilon\, {}_a^C\mathrm{D}_x^{\mu,\kappa} u(x,t), \tag{19}$$

subject to $u(x,0) = u_0(x)$.
For time advancing, we employ Crank-Nicolson/leapfrog scheme. Then, the full discretization scheme reads as:

$$\begin{cases} (\mathbb{I} - \epsilon\tau\widetilde{\mathbf{D}})\mathbf{u}^{n+1} = (\mathbb{I} + \epsilon\tau\widetilde{\mathbf{D}})\mathbf{u}^{n-1} - 2\tau(\mathrm{diag}(\mathbf{u}^n)\mathbf{D})\mathbf{u}^n, \quad n \geq 1, \\ \mathbf{u}^1 = (\mathbb{I} + \epsilon\tau\widetilde{\mathbf{D}})\mathbf{u}^0 - \tau(\mathrm{diag}(\mathbf{u}^0)\mathbf{D})\mathbf{u}^0, \\ \mathbf{u}^0 = u_0(\mathbf{x}), \end{cases} \tag{20}$$

where $\mathbf{D}$ is the first-order differentiation matrix and $\widetilde{\mathbf{D}} = (\frac{2}{b-a})^{\mu}\widetilde{\mathbf{D}}_{s,l}^{\mu,\kappa}$.
Consider (19) with the initial profile as one of the belows:

C1. $u_0(x) = \sin(\pi x), \quad x \in [-1, 1]$.

C2. $u_0(x) = e^{-4x^2}, \quad x \in [-6, 6]$.

In the example, we always take $\alpha = \beta = 0, \tau = 10^{-3}$.

Numerical solution to fractional Burgers equation

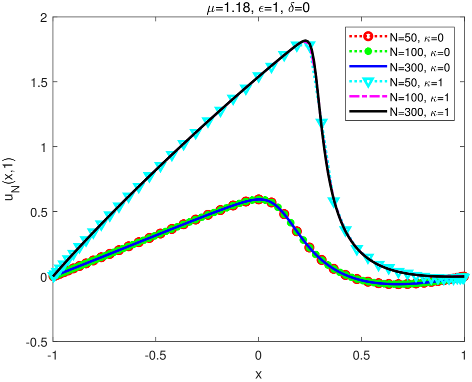

图 2: Numerical solutions at $t = 1$ of $u_0(x) = \sin(\pi x)$ with $\mu = 1.18, \epsilon = 1, \alpha = \beta = 0, \tau = 10^{-3}$.

Numerical solution to fractional Burgers equation

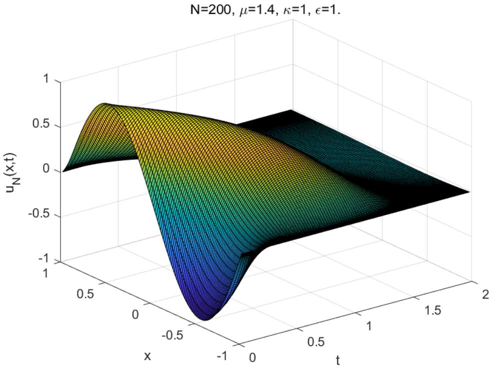

图 3: Numerical solutions of $u_0(x) = \sin(\pi x)$ with $\mu = 1.4, \epsilon = 1, N = 200, \alpha = \beta = 0, \tau = 10^{-3}$.

Numerical solution to fractional Burgers equation

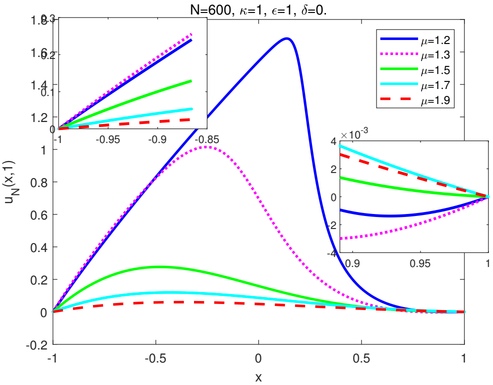

图 4: Numerical solutions at $t = 1$ of $u_0(x) = \sin(\pi x)$ with $N = 600, \epsilon = 1, \alpha = \beta = 0, \tau = 10^{-3}$.

Numerical solution to fractional Burgers equation

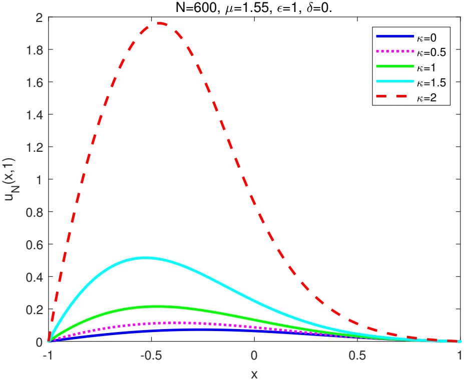

图 5: Numerical solutions at $t = 1$ of $u_0(x) = \sin(\pi x)$ with $\epsilon = 1, N = 600, \alpha = \beta = 0, \tau = 10^{-3}$.

Numerical solution to fractional Burgers equation

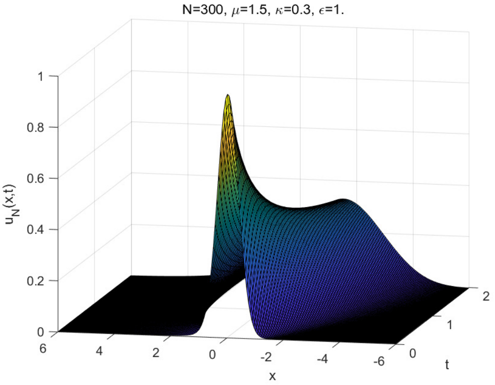

图 6: Numerical solutions of $u_0(x) = e^{-4x^2}$ with $N = 300, \mu = 1.5, \epsilon = 1, \alpha = \beta = 0, \tau = 10^{-3}$.

Numerical solution to fractional Burgers equation

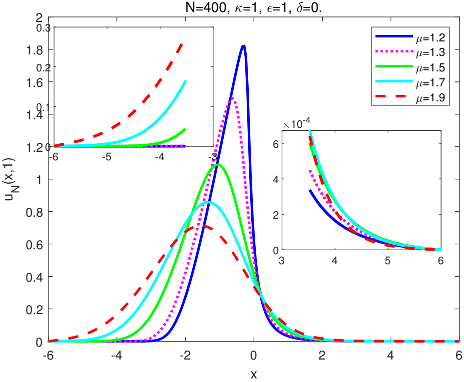

图 7: Numerical solutions at $t=1$ of $u_0(x)=e^{-4x^2}$ with $N=400, \kappa=1, \epsilon=1, \alpha=\beta=0, \tau=10^{-3}$.

Numerical solution to fractional Burgers equation

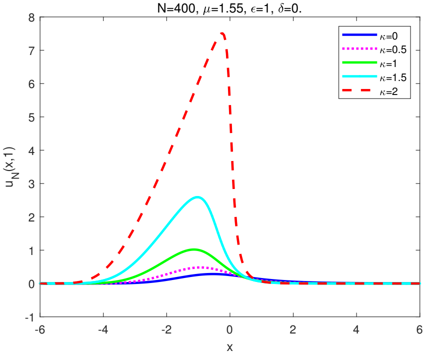

图 8: Numerical solutions at $t = 1$ of $u_0(x) = e^{-4x^2}$ with $N = 400, \mu = 1.55, \epsilon = 1, \alpha = \beta = 0, \tau = 10^{-3}$.

Summary

We present a spectral collocation method using the TFJFs as basis functions and obtain an efficient algorithm to solve tempered fractional differential equations. The key in implementing is to stably evaluate the collocation differentiation matrix by utilizing a recurrence relation.

- Advantages
  - High-accuracy, Fast
  - No difficulty for nonlinear problem, variable-order case
- Disadvantage
  - disable to deal with two-sided tempered fractional differential equation
  - hard to derive the error estimate.

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Thanks you!