Numerical Approximation of Optimal Convergence for Fractional Elliptic Equations with Additive Noise [1]

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[1] This talk is based on the joint work with Professor Zhongqiang Zhang, Worcester Polytechnic Institute, USA

Consider the nonlocal stochastic boundary value problem

$$(-\Delta)_\Omega^{\alpha/2} u + \mu u = \mathfrak{N}(x), \quad x \in \Omega = (-1, 1). \tag{1}$$

Here $u = 0$ if $x \in \partial\Omega$ and $\mu \geq 0$. The regional Laplacian $(-\Delta)_\Omega^{\alpha/2}$ with $\alpha \in (1, 2)$

$$(-\Delta)_\Omega^{\alpha/2} u(x) = c_{1,\alpha} \int_\Omega \frac{u(x) - u(y)}{|x - y|^{1+\alpha}}\, dy, \quad c_{1,\alpha} = \frac{2^\alpha \Gamma(\frac{\alpha+1}{2})}{\pi^{1/2} |\Gamma(-\alpha/2)|}.$$

- Long range interaction or anomalous diffusion
- Incomplete information, lack of knowledge or uncertainty for the inputs

The connection between nonlocal operators

For the target function u vanishing outside of the domain,

$$(-\Delta)^{\alpha/2}u(x) = (-\Delta)_{\Omega}^{\alpha/2}u(x) + \rho_{\Omega}(x)u(x), \quad x \in \Omega \tag{2}$$

where

$$\rho_{\Omega}(x) = c_{1,\alpha}\int_{\Omega^c} \frac{1}{|x-y|^{d+\alpha}}\,dy, \quad (-\Delta)^{\alpha/2}u(x) := c_{1,\alpha}\int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+\alpha}}\,dy, \tag{3}$$

Lemma (integration-by-parts formula)

Assume that u, v vanish outside of $\Omega \subseteq \mathbb{R}$ almost everywhere. Then it holds that

$$\begin{aligned} &\int_{\Omega} v(-\Delta)^{\alpha/2}u(x)\,dx \\ = \quad &\frac{c_{1,\alpha}}{2}\iint_{\Omega\otimes\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+\alpha}}\,dy\,dx + \int_{\Omega} u(x)v(x)\rho_{\Omega}(x)\,dx, \end{aligned}$$

when all the integrals are well-defined.

Lemma

For the n-th order Jacobi polynomial $P_n^{\alpha/2}(x)$, it holds that

$$\begin{aligned}(-\Delta)^{\alpha/2}[\omega^{\alpha/2}P_n^{\alpha/2}(x)] &= \lambda_n^\alpha P_n^{\alpha/2}(x), &(4)\\ \lambda_n^\alpha &= \frac{\Gamma(\alpha+n+1)}{n!} \approx n^\alpha. &(5)\end{aligned}$$

The Jacobi polynomials $P_n^\gamma(x)$ are mutually orthogonal as

$$\int_{-1}^{1}(1-x^2)^\gamma P_m^\gamma(x)P_n^\gamma(x)\,dx = h_n^\gamma \delta_{nm}, \quad \gamma > -1. \tag{6}$$

Here δ_{nm} is equal to 1 if $n = m$ and zero otherwise and

$$h_n^\gamma = \|P_n^\gamma\|_{\omega^\gamma}^2 = \frac{2^{2\gamma+1}(\Gamma(n+\gamma+1))^2}{(2n+2\gamma+1)\Gamma(n+2\gamma+1)\Gamma(n+1)}.$$

Fractional elliptic with additive Gaussian noise

For the convenience of the discussion, we focus on the model problem

$$(-\Delta)^{\alpha/2}u + \mu u = \mathfrak{N}(x), \quad x \in I = (-1, 1). \tag{7}$$

Here $u = 0$ if $x \in I^c$. The integral fractional Laplacian $(-\Delta)^{\alpha/2}$ with $\alpha \in (0, 2)$.

$$(-\Delta)^{\alpha/2}u(x) = c_{1,\alpha} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+\alpha}}\, dy, \quad c_{1,\alpha} = \frac{2^{\alpha}\Gamma(\frac{\alpha+1}{2})}{\pi^{1/2}|\Gamma(-\alpha/2)|}.$$

♢ Elliptic problems with additive noise $\mathfrak{N}(x)$

- white noise $\dot{W}(x)$, Zhimin Zhang et al 1998, Qiang Du 2002, George Karniadakis et al. 2016
- fractional Brownian motion $\dot{B}^H(x)$, Yanzhao Cao, Jialin Hong 2018
- $1/f^{\beta}$ noise, Max Gunzburger et al. 2011, 2015

♢ In 1D, spectral method for spatial discretization in physical domain: Mao and Shen 2016...

Goal: present a unified framework for numerically solving such equations and analyzing the convergence.

◇ Main challenges to compute the fractional Laplacian

- nonlocal and thus high storage cost
- weakly singular solutions; boundary singularity, low-order convergence
- the resulting dense matrix need fast solver
- In particular, hyper-singular integral form difficult to evaluate or discretize the fractional Laplacian

◇ Approximation of noise

- stationary and non-stationary noises.
- white and color noises.
- regular computation domain

We use spectral methods to discretize both nonlocal operators in physical domain and noises in random space

White noise and colored noise

- In many situations, it is assumed that a stochastic process $v(t)$ satisfies the following properties
  - 1. The expectation of $v(t)$ is zero for all t, i.e., $E[v(t)] = 0$.
  - 2. The covariance (two-point correlation) function of $v(t)$ is more or less known. That is,

$$Cov[(v(t), v(s))] = E[(v(t) - E[v(t)])(v(s) - E[v(s)])].$$

- When the covariance function is proportional to the Dirac function $\delta(t - s)$, the process $v(t)$ is called uncorrelated, which is usually referred to as white noise.
- Otherwise, it is correlated and is referred to as color noise.
- The white noise can be intuitively described as a stochastic process, which has independent values at each time instance and has an infinite variance.

Approximation of white noise by spectral expansion

Gaussian process with mean zero and covariance $\delta(x-y)$.

$$\frac{\partial^d}{\partial x_1\cdots\partial x_d}W = \sum_{|\alpha|<\infty,\ \alpha\in\mathbb{N}^d} e_\alpha(x)\xi_\alpha.$$

Here $\xi_\alpha \sim \mathcal{N}(0,1)$'s are i.i.d and e_αs form a complete orthonormal basis.

- Truncation (Wong-Zakai approximation)

$$\frac{\partial^d}{\partial x_1\cdots\partial x_d}W_{\mathrm{n}} = \sum_{|\alpha|\le \mathrm{n},\ \alpha\in\mathbb{N}^d} e_\alpha(x)\xi_\alpha.$$

  - (special case, using the basis of piecewise constants) piecewise linear approximation (polygonal approximation)

$$W_h(x) = W(x_i) + (W(x_{i+1}) - W(x_i))\frac{x-x_i}{x_{i+1}-x_i}, \quad x\in[x_i, x_{i+1}).$$

$1/f^\beta$ noise

$$\dot{W}^\beta(x) = \sum_{k=1}^{\infty} k^{-\frac{\beta}{2}} e_k(x) \xi_k.$$

Here $e_k(x) = \sin(k\pi \frac{x+1}{2})$, $x \in (-1, 1)$.

When $0 \leq \beta \leq 2$, the noise is called $1/f^\beta$ noise.

- (White noise) $\beta = 0$
- (Pink noise) $\beta = 1$
- (Brownian noise) $\beta = 2$

$\diamondsuit$ Truncating the infinite series leads to "good " approximation of the noise.

Colored noises

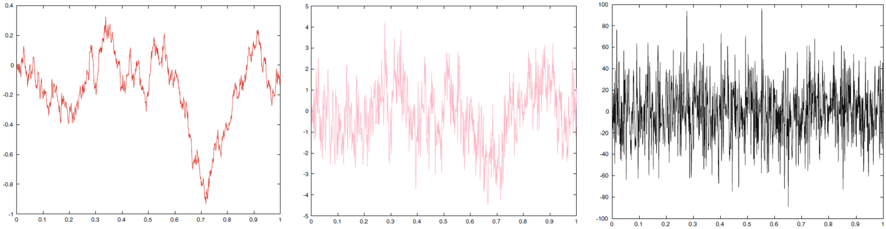


Figure: Sample paths of different Gaussian $1/f^{\beta}$ processes: Brownian motion, $\beta = 2$ (left); Pink noise, $\beta = 1$ (middle); White noise, $\beta = 0$ (right)

A fBm of Hurst index $H \in (0,1)$ denoted by $W^H(t)$(or $B^H(t)$), $t \geq 0$, is a centered continuous Gaussian process, describing the correlated random fluctuations, which satisfies:

$$\mathbb{E}[W^H(t)] = 0;$$

$$\mathrm{Cov}(s,t) := \mathbb{E}[W^H(t)W^H(s)] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

- **self-similarity:**
  $\mathrm{Cov}(\alpha t, \alpha s) = \alpha^{2H}\mathrm{Cov}(t,s)$ for $\alpha > 0,\ t,\ s \geq 0$;
- **stationary increments:**
  $W^H(t) - W^H(s) \sim W^H(t-s)$ for $t > s > 0$;
- **Hölder continuous property**
  $|W^H(t) - W^H(s)| \leq M|t-s|^\gamma$, $\gamma < H$, a.s.,
  where $M > 0$ is a constant, $t, s > 0$.

[A. N. Kolmogorov (1940); B. B. Mandelbrot, J. W. Van Ness, SIAM Review (1968)]

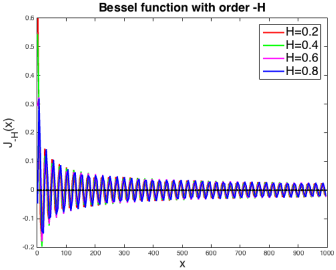
A representation of the fBm

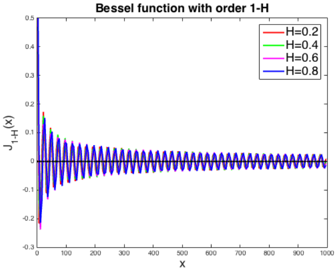
$$W^H = c_H \left(\sum_{k=1}^{\infty} \frac{\sin(\alpha_k x)}{\alpha_k^{1+H} J_{1-H}(\alpha_k)} \xi_k + \sum_{k=1}^{\infty} \frac{\cos(\beta_k x)}{\beta_k^{1+H} J_{-H}(\beta_k)} \zeta_k \right), \quad x \in [0,1], \qquad (8)$$

- $c_H = \sqrt{2/\pi}\,\Gamma^{1/2}(1+2H)\sin^{1/2}(\pi H)$
- α_k's, β_k's are the positive zeros of the Bessel function J_{-H} and J_{1-H}
- ξ_k's and ζ_k's are mutually independent standard Gaussian random variables.

[Dzhaparidze-Zanten'04, Probab. Theory Relat. Fields]

Bessel function





$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-\alpha^2)y=0;$$

$$J_\alpha(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}.$$

Lemma

For the Bessel function J_ν, where $\nu > -1$, we have

$$\begin{aligned} J^2_{1+\nu}(z) + J^2_\nu(z) &\approx \frac{2}{\pi z}, \text{ for large } |z|. \\ 2\nu J_\nu(x) &= xJ_{\nu+1}(x) + xJ_{\nu-1}(x). \end{aligned}$$

Let z_k be the positive zeros of the Bessel function J_ν. When k is large, we have

$$z_k = k\pi + \frac{\pi}{2}(\nu - \frac{1}{2}) - \frac{4\nu^2 - 1}{8(k\pi + \frac{\pi}{2}(\nu - \frac{1}{2}))} + O(\frac{1}{k^3}).$$

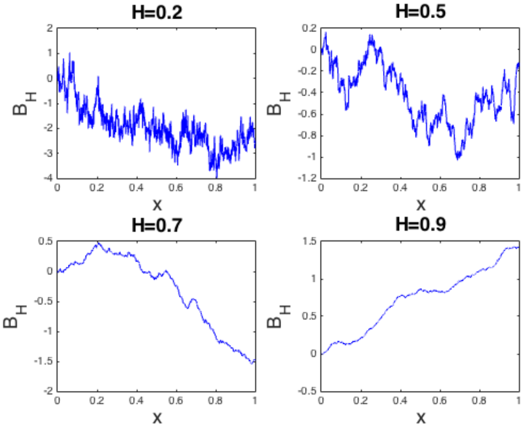
Lemma

For positive real numbers ν and z, it holds that when z is large enough,

$$|J_\nu(z)| \leq C_l(\frac{z}{\nu})^l z^{-1/2}, \tag{9}$$

where C_l is a constant depending on the positive number l.

Simulation of the fBm



Pros and cons of the spectral methods

• (pros) The expansion only depends on the domain and the underlying process but not on the mesh.

- free parameters
- often achieve optimal convergence

• (cons) More work, not necessary easy to obtain

- need to recompute when domain changes
- pre-simulated sampling points
- need further discretization, balancing errors to obtain optimal estimates.

Difficulties and issues

- The equation (7) is not diagonalizable
  - the integral formulation with Green function is unclear
  - eigenfunctions on bounded domains not explicitly known

◇ Use the spectral approximation for the integral fractional Laplacian operator. full regularity and convergence analysis (Hao et al 2020) when the RHS is smooth.

◇ Need further analysis when RHS is rough noise

A problem is diagonalizable if the leading-order operator and the noise have same eigenfunctions in physical space.

Theorem (Hao et al 2020)

When $\mu > 0$ *and* $g \in H^r$ *with* $r \geq 0$, *then* $u \in H^{\min(1/2+\alpha/2-\epsilon, \alpha+r)}$ *with* $\epsilon > 0$ *arbitrarily small, i.e., there exists a positive constant* c *such that* $\|u\|_{H^{\min\{1/2+\alpha/2-\varepsilon, \alpha+r\}}} \leq c\|g\|_{H^r}$.

Define the operator $\mathcal{T}$ such that $\mathcal{T}g = u$. From Theorem 5,

$$\|\mathcal{T}g\|_{H^{\min\{1/2+\alpha/2-\epsilon, \alpha+r\}}} \leq c\|g\|_{H^r}, \quad r \geq -\alpha/2. \tag{10}$$

The best smoothness index in non-weighted Sobolev spaces is

$$1/2 + \alpha/2 - \epsilon < 3/2!$$

♢ Introduce the weighted Sobolev spaces

$$B^s_\gamma = \{u : u = \sum_{n=0}^{\infty} u_n^\gamma P_n^\gamma, \text{where} \sum_{n=0}^{\infty} (u_n^\gamma)^2 h_n^\gamma (1+n^2)^s < \infty\}. \tag{11}$$

Theorem

When $\mu > 0$ and $g \in B^r_{\alpha/2}$ with $r \geq -\alpha/2$, then we have $\omega^{-\alpha/2} u \in B^{\min(5\alpha/2+1-\epsilon,\alpha+r)}_{\alpha/2}$ with $\epsilon > 0$ arbitrarily small, i.e., there exists a constant c such that $\|\omega^{-\alpha/2} u\|_{B^{\min\{5\alpha/2+1-\epsilon,\alpha+r\}}_{\alpha/2}} \leq c\|g\|_{B^r_{\alpha/2}}$.

- Spectral methods $u_{M,N} = (1-x^2)^{\alpha/2} \sum_{m=0}^{N} u^m_{M,N} P^{\alpha/2}_m(x)$

$$((-\Delta)^{\alpha/2} u_{M,N}, v_N) + (u_{M,N}, v_N) = (\dot{W}^H_M, v_N), \quad \forall v_N \in U_N,$$

where $U_N = (1-x^2)^{\alpha/2}\mathbb{P}_N$ and

$$\dot{W}^H_M(x) = c_H \sum_{k=1}^{M} \frac{\cos(\alpha_k \frac{x+1}{2})}{\alpha^H_k J_{1-H}(\alpha_k)} \xi_k + c_H \sum_{k=1}^{M} \frac{\sin(\beta_k \frac{x+1}{2})}{\beta^H_k J_{-H}(\beta_k)} \zeta_k.$$

Theorem

Let $\alpha \in (0,2]$. For $H \in (0,1)$, $M = N$,

$$\mathbb{E}[\|u - u_{M,N}\|^2_{L^2_{-\alpha/2}}] \leq CN^{-2(\alpha+H-1-\epsilon)},$$

and for $H \in (1-\alpha/2, 1)$, $M = N$,

$$\mathbb{E}[\|u - u_{M,N}\|^2_{H^{\alpha/2}}] \leq CN^{-2(\alpha/2+H-1-\epsilon)}.$$

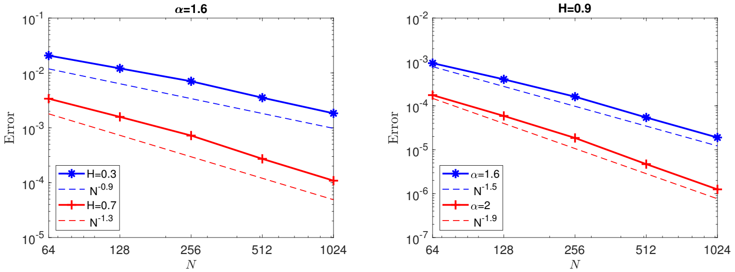


Figure: Convergence orders and errors of the spectral Galerkin method for the equation $(-\Delta)^{\alpha/2}u + u = \dot{W}^H(x)$. The theoretical mean-square convergence order is $\alpha + H - 1 - \epsilon$ in L^2 -norm.

Basic observations

The weak formulation is to find $u \in H_0^{\alpha/2}$, for any $g \in H^{-\alpha/2}$, such that

$$a(u,v) := ((-\Delta)^{\alpha/2}u, v) + \mu(u,v) = (g,v), \quad \forall v \in H_0^{\alpha/2}, \tag{12}$$

Lemma (e.g. in [?])

For any $v \in H_0^{\alpha/2}$ with $1 < \alpha \leq 2$, it holds that

$$\|\omega^{-\alpha/2} v\|_{B^{\alpha/2}_{\alpha/2}} \approx ((-\Delta)^{\alpha/2}v, v) \approx |v|^2_{H^{\alpha/2}}. \tag{13}$$

• Regularity of $1/f^\beta$ noise

$$\mathbb{E}[\left\|\dot{W}^\beta\right\|^2_{H^t}] \simeq \sum_{k=1}^{\infty} k^{2t-\beta} < \infty, \quad \text{when } t < \frac{\beta}{2} - \frac{1}{2}. \tag{14}$$

In standard Sobolev spaces H^t are known :

- white noise, $\beta = 0$, and $t < -\frac{1}{2}$,
- pink noise (flicker noise), $\beta = 1$, and $t < 0$,
- Brownian noise, $\beta = 2$, $t < \frac{1}{2}$.

For deterministic problems.

Theorem ([?])

Suppose that u and u_N satisfy the problems (15) and (12), respectively. Suppose that $g \in B^r_{\alpha/2}$ with $r \geq -\alpha/2$ and $\epsilon > 0$ arbitrarily small, we have the following error estimates:

$$\|u - u_N\|_{L^2_{-\alpha/2}} + N^{-\alpha/2}\|u - u_N\|_{H^{\alpha/2}} \leq CN^{-\min\{\alpha+r, 5\alpha/2+1-\epsilon\}}\|g\|_{B^r_{\alpha/2}}.$$

Let $u_N = \mathcal{T}_N g$ and u_N is the solution to (12). Then

$$\|\mathcal{T} g - \mathcal{T}_N g\|_{L^2_{-\alpha/2}} + N^{-\alpha/2}\|\mathcal{T} g - \mathcal{T}_N g\|_{H^{\alpha/2}} \quad \leq \quad CN^{-\min\{\alpha+r, 5\alpha/2+1-\epsilon\}}\|g\|_{B^r_{\alpha/2}}.$$

• Can be readily applied to the noise when the noise is not so rough.

• Not good for $\alpha = 2$ while $H = 1/2$ (white noise) as the white noise belongs to $H^{-1/2-}$. Need further work

• Recall that the weak formulation is to find $u \in H_0^{\alpha/2}$, for any $g \in H^{-\alpha/2}$, such that

$$a(u,v) := ((-\Delta)^{\alpha/2}u, v) + \mu(u,v) = (g,v), \quad \forall v \in H_0^{\alpha/2}.$$

• Ultra-weak formulation of the original problem, that is, to find $u \in L^2_{-\alpha/2}$ such that, for any $g \in B^{-\alpha}_{\alpha/2}$,

$$b(u,v) := (u, (-\Delta)^{\alpha/2}(v\omega^{\alpha/2})) + \mu(u,v)_{\omega^{\alpha/2}} = (g,v)_{\omega^{\alpha/2}}, \quad \forall v \in B^{\alpha}_{\alpha/2}.$$

Theorem

When $\mu \geq 0$ and $g \in B^{-\alpha}_{\alpha/2}$, there exists an unique solution $u \in L^2_{-\alpha/2}$ such that

$$\|u\|_{L^2_{-\alpha/2}} \leq c\|g\|_{B^{-\alpha}_{\alpha/2}}. \tag{15}$$

Key steps

Step 1.

$$b(u,v) := (u, (-\Delta)^{\alpha/2}(v\omega^{\alpha/2})) + \mu(u,v)_{\omega^{\alpha/2}} = (g,v)_{\omega^{\alpha/2}}, \quad \forall v \in B^{\alpha}_{\alpha/2}.$$

$$b(u,v) \leq (1+\mu)\|u\|_{L^2_{-\alpha/2}} \|v\|_{B^{\alpha}_{\alpha/2}}.$$

- Step 2

$$b(\phi, v) = (\phi,\, (-\Delta)^{\alpha/2}(\omega^{\alpha/2}v)\,) + \mu(\phi, v)_{\omega^{\alpha/2}} = \tilde{a}(v,\phi) = \|\phi\|^2_{L^2_{-\alpha/2}}.$$

where $v \in\in B^{\alpha}_{\alpha/2}$ (existence and stability follows from previous regularity) satisfies

$$\tilde{a}(v,\phi) := ((-\Delta)^{\alpha/2}(v\omega^{\alpha/2}), \phi) + \mu(v\omega^{\alpha/2}), \phi), \forall \phi \in L^2_{-\alpha/2}.$$

- (From previous regularity) There exists an unique $v \in B^{\alpha}_{\alpha/2}$ such that

$$\tilde{a}(v,w) = (\omega^{-\alpha/2}\phi, w) = (\phi, w)_{-\alpha/2}, \quad \forall w \in L^2_{-\alpha/2}, \tag{16}$$

where $\|v\|_{B^{\alpha}_{\alpha/2}} \leq c\|\omega^{-\alpha/2}\phi\|_{L^2_{\alpha/2}} = c\|\phi\|_{L^2_{-\alpha/2}}$.

$$\sup_{0\neq v\in B^{\alpha}_{\alpha/2}} \frac{b(\phi,v)}{\|v\|_{B^{\alpha}_{\alpha/2}}} = \sup_{0\neq v\in B^{\alpha}_{\alpha/2}} \frac{\|\phi\|^2_{L^2_{-\alpha/2}}}{\|v\|_{B^{\alpha}_{\alpha/2}}} \geq \frac{1}{c}\|\phi\|_{L^2_{-\alpha/2}}, \quad \forall\, 0\neq\phi\in L^2_{-\alpha/2}. \tag{17}$$

• Step 3. For $v \in B^{\alpha}_{\alpha/2}$, take $\omega^{-\alpha/2}\phi = (-\Delta)^{\alpha/2}(\omega^{\alpha/2}v) + \mu(\omega^{\alpha/2}v)$. Then the following transposed inf-sup condition holds,

$$\sup_{0\neq\phi\in L^2_{-\alpha/2}} b(\phi,v) > 0, \quad \forall\, 0\neq v\in B^{\alpha}_{\alpha/2}. \tag{18}$$

Theorem

With $\mu \geq 0$, if $g \in B^r_{\alpha/2}$ with $r \geq -\alpha$, then for any $\epsilon > 0$, we have the following stability estimate

$$\|\omega^{-\alpha/2} u\|_{B_{\alpha/2}^{\min\{\alpha+r, 5\alpha/2+1-\epsilon\}}} \leq c\|g\|_{B^r_{\alpha/2}}. \tag{19}$$

Theorem

Let $\alpha \in (0, 2]$. For $H \in (0, 1)$, $M = N$,

$$\mathbb{E}[\|u - u_{M,N}\|^2_{L^2_{-\alpha/2}}] \leq CN^{-2(\alpha+H-1-\epsilon)},$$

and for $H \in (1-\alpha/2, 1)$, $M = N$,

$$\mathbb{E}[\|u - u_{M,N}\|^2_{H^{\alpha/2}}] \leq CN^{-2(\alpha/2+H-1-\epsilon)}.$$

$1/f^\beta$ noise

$$\dot{W}^\beta(x) = \sum_{k=1}^{\infty} k^{-\frac{\beta}{2}} e_k(x)\xi_k. \tag{20}$$

where $\{e_k(x)\}$ can be any orthonormal basis in $L^2(I)$ and ξ_k's are i.i.d. standard normal.

Lemma

For any small number $\epsilon > 0$ and the truncation

$$\dot{W}_M^\beta(x) = \sum_{k=1}^{M} k^{-\frac{\beta}{2}} e_k(x)\xi_k,$$

$$\begin{aligned} \mathbb{E}[\|\dot{W}_M^\beta\|^2_{B_{\alpha/2}^{\frac{\beta-1}{2}-\epsilon}}] &< \infty, \\ \mathbb{E}[\|\dot{W}_M^\beta\|^2_{B_{\alpha/2}^{s}}] &\leq CM^{2s-\beta+1+2\epsilon}, \quad s \geq \frac{\beta-1}{2}. \end{aligned}$$

Key steps in the proof

$$\dot{W}^{\beta}(x) = \sum_{k=1}^{\infty} k^{-\beta/2} e_k(x)\xi_k = \sum_{n=0}^{\infty} a_n^{\alpha/2}(\beta) P_n^{\alpha/2}(x),$$

$$a_n^{\alpha/2}(\beta) = \frac{1}{h_n^{\alpha/2}} \int_{-1}^{1} \dot{W}^{\beta}(x)(1-x^2)^{\alpha/2} P_n^{\alpha/2}(x)dx = \sum_{k=1}^{\infty} k^{-\beta/2} a_{n,k}^{\alpha/2} \xi_k.$$

$$a_{n,k}^{\alpha/2} = \frac{1}{h_n^{\alpha/2}} \int_{-1}^{1} e_k(x)(1-x^2)^{\alpha/2} P_n^{\alpha/2}(x)dx$$

$$\left| a_{n,k}^{\alpha/2} \right| \leq C n^{\alpha/2+1-l} k^{l-\alpha/2-1}.$$

Theorem

When $\beta \geq 1 - 2\alpha$ *and* $2s > \beta - 1$*,*

$$\mathbb{E}[\|u - u_{M,N}\|^2_{L^2_{-\alpha/2}}] \leq C\epsilon_0^{-1} N^{-(2\alpha+2s)} M^{2s-\beta+1-2\epsilon} + CN^{-2(\alpha+\frac{\beta-1}{2}-\epsilon)}. \quad (21)$$

Moreover, if $\beta > 1 - \alpha$*, we have the following error estimates:*

$$\mathbb{E}[\|u - u_{M,N}\|^2_{L^2_{-\alpha/2}}] \leq C\epsilon_0^{-1} M^{-2(\alpha+\frac{\beta-1}{2}-\epsilon_0)} + CN^{-2(\alpha+\frac{\beta-1}{2}-\epsilon)}, \quad (22)$$

$$\mathbb{E}[\|u - u_{M,N}\|^2_{H^{\alpha/2}}] \leq C\epsilon_0^{-1} M^{-2(\frac{\alpha}{2}+\frac{\beta-1}{2}-\epsilon_0)} + CN^{-2(\frac{\alpha}{2}+\frac{\beta-1}{2}-\epsilon)}. \quad (23)$$

Here ϵ *and* ϵ_0 *can be any positive numbers. Here* ϵ *and* ϵ_0 *can be any positive numbers. Taking* $M = N$*, we obtain the optimal convergence order of the spectral method is*

$$\alpha + \frac{\beta - 1}{2} - \epsilon.$$

Numerical results: $\beta = 0$

Table: Strong convergence order for $\beta = 0$ (white noise): $\alpha - \frac{1}{2}$.

α	1.2		1.4		1.8		2	
16	3.97e-04		1.75e-04		2.97e-05		1.16e-05	
32	2.58e-04	0.62	1.02e-04	0.77	1.34e-05	1.15	4.30e-06	1.43
64	1.64e-04	0.65	5.90e-05	0.79	6.15e-06	1.13	1.57e-06	1.46
128	1.02e-04	0.69	3.35e-05	0.82	2.84e-06	1.11	5.62e-07	1.48
256	6.04e-05	0.75	1.83e-05	0.87	1.31e-06	1.12	2.00e-07	1.49
		0.7		0.9		1.3		1.5

Taking $M = N$ leads to the order of $\alpha + \frac{\beta-1}{2}-$.

Numerical results: $\beta = 1$

Table: Strong convergence order for $\beta = 1$ (pink noise): α.

α	1.0		1.1		1.9		2	
8	9.53e-02		7.31e-02		6.79e-03		5.01e-03	
16	5.28e-02	0.85	3.81e-02	0.94	2.07e-03	1.71	1.42e-03	1.82
32	2.81e-02	0.91	1.90e-02	1.00	6.00e-04	1.78	3.76e-04	1.91
64	1.47e-02	0.94	9.37e-03	1.02	1.74e-04	1.79	9.80e-05	1.94
128	7.54e-03	0.96	4.54e-03	1.04	5.06e-05	1.78	2.50e-05	1.97
256	3.81e-03	0.98	2.17e-03	1.07	1.51e-05	1.74	6.31e-06	1.99
		1.0		1.1		1.9		2

Taking $M = N$ leads to the order of $\alpha + \frac{\beta-1}{2}-$.

Numerical results: $\beta = 2$

Table: Strong convergence order for $\beta = 2$ (Brownian noise): $\alpha + 1/2$.

α	1.0		1.1		1.9		2	
8	2.83e-02		2.16e-02		2.05e-03		1.52e-03	
16	1.16e-02	1.29	8.28e-03	1.39	4.58e-04	2.16	3.17e-04	2.26
32	4.42e-03	1.39	2.96e-03	1.48	9.50e-05	2.27	6.05e-05	2.39
64	1.65e-03	1.42	1.04e-03	1.51	1.94e-05	2.29	1.13e-05	2.43
128	6.04e-04	1.45	3.58e-04	1.54	3.96e-06	2.29	2.04e-06	2.46
256	2.19e-04	1.46	1.22e-04	1.55	8.25e-07	2.26	3.65e-07	2.48
		1.5		1.6		2.4		2.5

Taking $M = N$ leads to the order of $\alpha + \frac{\beta-1}{2}-$.

Summary

• Spectral methods for simulation of noises: fractional Brownian motion and $1/f^{\beta}$ noise.

• Elliptic with additive fractional Gaussian noise: spectral methods lead to optimal convergence

• The convergence analysis framework can be readily extended to other nonlocal problems such as mixed diffusion, nonsymmetrical operators

To do

• Multiplicative noise;

• Weak convergence in moments and in probability

• Extension to general domains

Partially relevant references on spectral methods

- **Zhaopeng Hao**, Zhongqiang Zhang, SINUM, 2020
- **Zhaopeng Hao**, Zhangqiang Zhang, SIAM/ASA UQ, 2021
- **Zhaopeng Hao**, Huiyuan Li, Zhimin Zhang, Zhongqiang Zhang, Math. Comput. 2021,
- **Zhaopeng Hao**, Zhangqiang Zhang, ANM, 2021
- **Zhaopeng Hao**, Calcolo, 2023

Thanks for your attention !