Hp-error estimates of fractional collocation methods for weakly singular Volterra integro-differential equations

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Introduction

Volterra integro-differential equations (VIDEs) with weakly singular kernels

$$\begin{cases} u'(t) = g(t) + a(t)u(t) + \int_0^t (t-s)^{\mu-1} K(t,s)u(s)ds, \ \ t \in I := [0,T] \\ u(0) = u_0, \quad \mu \in (0,1). \end{cases} \tag{1}$$

Proposition 1.1 (Brunner, 2004)

Assume that $a, g \in C^m(I)$ *and* $K \in C^m(D)$, $D = \{(t,s) : 0 \leq s \leq t \leq T\}$, $(m \geq 1)$, *with* $K(t,t) \neq 0$ *on* I. $\mu \in (0,1)$. *Then the unique solution* u *of equation* (1) *lies in* $C^1(I) \cap C^{m+1}(0,T]$. *The solution can be written in the form*

$$u(t) = \sum_{(j,k)_{\mu,m}} \alpha_{j,k}(\mu) t^{j+k(1+\mu)} + Y_{m+1}(t,\mu) \quad \text{for} \quad t \in I,$$

where $(j,k)_{\mu,m} := \{(j,k) : j,k \in \mathbb{N}_0, \ j + k(1+\mu) < m+1\}$, *the constants* $\alpha_{j,k}(\mu)$ *depend on* μ, *and* $Y_{m+1} \in C^{m+1}(I)$.

Introduction

h-version numerical methods: convergence is achieved on successively refined time steps

- Collocation methods on uniform mesh: m(≥ 2) points
  - Global order: $1+\mu$
- Collocation on graded meshes
  - H. Brunner, IMAJNA 1986: $r = m/\mu$
  - T. Tang, IMAJNA 1993: $r = m/(1+\mu)$
  - T. Tang, Numer Math 1992: Superconvergence
- Geometric meshes: Q. Hu, IMAJNA, 1998
- Nonpolynomial collocation
  - H, Brunner, SINUM 1983
  - Q. Hu, SINUM 1996
- Smoothing transformation: T. Diogo et al, 2017
- DG: J. Ma 2009; Mustapha, Math Comp, 2013
- RK, LMM, BVM: Lubich, Math Comp, 1983; D. Xu SINUM 2007; C. Zhang $\cdots$

Introduction

p-version numerical methods: convergence is achieved by successively increasing the approximation order

- Spectral method for VIEs: T. Tang, X. Xu, J. Cheng, 2008
- Spectral Petrov-Galerkin methods for VIDEs (smooth kernel): Z. Xie, X. Tao et al, 2011
- Spectral collocation methods for VIDEs (several classes): Y. Huang, Y. Chen, Y. Yang et al, 2012-2020
- Spectral Galerkin method based on nonpolynomial (λ-polynomials) basis functions: D. Hou, C. Xu, 2017
- ......

Introduction

hp-version numerical methods: flexible choice of locally varying time-steps and approximation orders

- Discontinuous Galerkin method: H. Brunner, D. Schotzau, SINUM 2006
- Petrov-Galerkin method: L. Yi, B. Guo, SINUM 2015
- Collocation method: Z. Wang, Y. Guo et al, Math Comp 2017, JSC 2017, NMTMA 2019
- All these methods are based on piecewise polynomial approximation. The start singularity are resolved by using graded meshes or geometric meshes.

Aim of our work

- Multi-domain fractional spectral collocation method
  - exponential convergence over arbitrary meshes
  - high order convergence (h-version) on uniform mesh
- Error analysis of *hp*-version

Contents

Consider nonlinear Volterra integro-differential equations (VIDEs)

$$u'(t) = f(t, u(t)) + \int_0^t (t-s)^{\mu-1} K(t, s, u(s)) ds, \quad (2)$$

with $u(0) = u_0$, $\mu \in (0,1)$.

- The partition $\mathscr{T}_M$ of the interval I:

  $$\mathscr{T}_M := \{t_i : 0 = t_0 < t_1 < \cdots < t_M = T\},$$

  Define $\sigma_i = (t_{i-1}, t_i]$, $h_i = t_i - t_{i-1}$. Let $\lambda \in (0,1]$. Define $\sigma_{i,\lambda} = (t_{i-1}^\lambda, t_i^\lambda]$, $h_{i,\lambda} = t_i^\lambda - t_{i-1}^\lambda$.
- The approximate space:

  $$S_\lambda(\mathscr{T}_M) := \{v : v|_{\sigma_i} \in P_{N_i}^\lambda, i = 1, \cdots, M\}, \quad (3)$$

  where $P_{N_i}^\lambda := span\{1, t^\lambda, \cdots, t^{N_i\lambda}\}$ with $\{N_i\}$ being given sequence of natural number.
- *Müntz* theorem shows that P_∞^λ is a dense subset of $C(I)$.

Notation

- Let $\rho(s) := s^{1/\lambda}$. Collocation points $\{t_{i,k}^C\}_{k=0}^{N_i} \in \sigma_i = (t_{i-1}, t_i]$

$$X_i := \{t_{i,k}^C : t_{i,k}^C = \rho(\xi_{i,k}^C), \quad k = 0, \cdots N_i\},$$

  where $\xi_{i,k}^C$ are the shifted Chebyshev-Gauss points on the interval $\sigma_{i,\lambda} = (t_{i-1}^\lambda, t_i^\lambda]$.
- The interpolation basis functions on σ_i:

$$L_{i,k}^{\lambda,C}(t) = \prod_{j=0,j\neq k}^{N_i} \frac{t^\lambda - (t_{i,j}^C)^\lambda}{(t_{i,k}^C)^\lambda - (t_{i,j}^C)^\lambda}.$$

- The interpolation operator $I_{N_i,i}^{\lambda,C} : C(\sigma_i) \to P_{N_i}^\lambda(\sigma_i)$

$$(I_{N_i,i}^{\lambda,C} v)(t) := \sum_{k=0}^{N_i} L_{i,k}^{\lambda,C}(t) v(t_{i,k}^C), \quad t \in \sigma_i, \quad i = 1, \cdots, M.$$

Notation

- Let $\{\xi_{i,k}^L\}_{k=1}^{N_i+1}$ be the shifted Legendre-Gauss-Lobatto points on the interval $[t_{i-1}^\lambda, t_i^\lambda]$.
- Let $t_{i,k}^L = \rho(\xi_{i,k}^L) = (\xi_{i,k}^L)^{1/\lambda} \in [t_{i-1}, t_i]$. The interpolation basis functions on $[t_{i-1}, t_i]$:

$$L_{i,k}^{\lambda,L}(t) = \prod_{j=0, j\neq k}^{N_i+1} \frac{t^\lambda - (t_{i,j}^L)^\lambda}{(t_{i,k}^L)^\lambda - (t_{i,j}^L)^\lambda}.$$

- The interpolation operator $I_{N_i+1,i}^{\lambda,L} : C([t_{i-1}, t_i]) \to P_{N_i+1}^\lambda([t_{i-1}, t_i])$

$$(I_{N_i+1,i}^{\lambda,L} v)(t) := \sum_{k=0}^{N_i+1} L_{i,k}^{\lambda,L}(t) v(t_{i,k}^L), \quad t \in [t_{i-1}, t_i], \quad i = 1, \cdots, M.$$

Fractional collocation method for (2): find a function $V \in S_\lambda(\mathscr{T}_M)$ and U such that for $i = 1, 2, \cdots, M$,

$$V(t_{i,k}^C) = f(t_{i,k}^C, U_i(t_{i,k}^C)) + \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} (t_{i,k}^C - s)^{\mu-1} I_{N_j+1,j}^{\lambda,L} K(t_{i,k}^C, s, U_j(s)) ds$$
$$+ \int_{t_{i-1}}^{t_{i,k}^C} (t_{i,k}^C - s)^{\mu-1} I_{N_i+1,i}^{\lambda,L} K(t_{i,k}^C, s, U_i(s)) ds, \quad k = 0, \cdots, N_i, \quad (4)$$

$$U_i(t) = U_{i-1}(t_{i-1}) + \int_{t_{i-1}}^{t} V_i(s) ds, \quad t \in [t_{i-1}, t_i], \quad (5)$$

with $U_0(t_0) = u_0$, and the numerical solution is given by the continuous function $U(t)$ which is piecewise defined by (5).

- Let $Y_{i,l} = V_i(t_{i,l}^C)$ and $\beta_{i,l}(t) = \int_{t_{i-1}}^{t} L_{i,l}^{\lambda,C}(s)ds$, then

$$V_i(t) = \sum_{l=0}^{N_i} L_{i,l}^{\lambda,C}(t) Y_{i,l}, \quad U_i(t) = U_{i-1}(t_{i-1}) + \sum_{l=0}^{N_i} \beta_{i,l}(t) Y_{i,l}.$$

- Let

$$\phi_{k,p}^{i,j} = \begin{cases} \displaystyle\int_{t_{j-1}}^{t_j} (t_{i,k}^C - s)^{\mu-1} L_{j,p}^{\lambda,L}(s)ds, & 1 \le j \le i-1, \\ \displaystyle\int_{t_{i-1}}^{t_{i,k}^C} (t_{i,k}^C - s)^{\mu-1} L_{i,p}^{\lambda,L}(s)ds, & j = i. \end{cases}$$

Rewritten the numerical scheme: for $k = 0, \cdots, N_i$,

$$
\begin{aligned}
Y_{i,k} = f &\left(t^C_{i,k}, U_{i-1}(t_{i-1}) + \sum_{l=0}^{N_i} \beta_{i,l}(t^C_{i,k}) Y_{i,l} \right) \\
&+ \sum_{j=1}^{i-1} \sum_{p=0}^{N_j+1} K\left(t^C_{i,k}, t^L_{j,p}, U_j(t^L_{j,p})\right) \phi^{i,j}_{k,p} \\
&+ \sum_{p=0}^{N_i+1} K\left(t^C_{i,k}, t^L_{i,p}, U_{i-1}(t_{i-1}) + \sum_{l=0}^{N_i} \beta_{i,l}(t^L_{i,p}) Y_{i,l}\right) \phi^{i,i}_{k,p}, \\
U_i(t_i) = U_{i-1}&(t_{i-1}) + \sum_{l=0}^{N_i} \beta_{i,l}(t_i) Y_{i,l},
\end{aligned}
$$

which is a nonlinear system about $\{Y_{i,k}\}_{k=0}^{N_i}$ and can be solved by an iterative process.

Contents

Some Lemmas

Lemma 3.1 (Wang et al, Math Comp, 2017)

Let $\hat{I}^{L}_{N_i+1,i}$ be the shifted Legendre-Gauss-Lobatto interpolation operator. For $f \in H^m(\hat{\sigma}_i)$ with $1 \le m \le N_i + 2$,

$$\|f - \hat{I}^{L}_{N_i+1,i} f\|_{H^k(\hat{\sigma}_i)} \le c h_{i,\lambda}^{m-k} (N_i + 1)^{k-m} \|\partial_t^m f\|_{L^2(\hat{\sigma}_i)}, \quad k = 0, 1.$$

Lemma 3.2

Let $\hat{I}^{C}_{N_i+1,i}$ be the shifted Chebyshev-Gauss interpolation operator. For $f \in H^m(\hat{\sigma}_i)$ with $1 \le m \le N_i + 1$,

$$\|f - \hat{I}^{C}_{N_i,i} f\|_{L^2(\hat{\sigma}_i)} \le c h_{i,\lambda}^{m} N_i^{-m} \|\partial_t^m f\|_{L^2(\hat{\sigma}_i)}.$$

Lemma 3.3

Let $0 < \nu < 1$ *and* $\kappa \geq t_{i,k}^{C}$ *be any given constants. For any* $u \in H^1(0, \xi_{i,k}^{C})$ *with* $u(0) = 0$*, we have*

$$\int_0^{\xi_{i,k}^{C}} u^2(y)(\kappa - y^{1/\lambda})^{-\nu} y^{1/\lambda - 1} dy \leq \frac{2\lambda \kappa^{1-\nu}}{1-\nu} \|u\|_{L^2(0,\xi_{i,k}^{C})} \|u'\|_{L^2(0,\xi_{i,k}^{C})}.$$

Assumptions

- Assume that there exist non-negative constants L_i, $i = 1, \ldots, 5$, such that for all $s, t \in [0, T]$, $w_1, w_2 \in \mathbb{R}$,

$$|f(t, w_1) - f(t, w_2)| \leq L_1 |w_1 - w_2|, \tag{6a}$$
$$|K(t, s, w_1) - K(t, s, w_2)| \leq L_2 |w_1 - w_2|, \tag{6b}$$
$$|K_2(t, s, w_1) - K_2(t, s, w_2)| \leq L_3 |w_1 - w_2|, \tag{6c}$$
$$|K_3(t, s, w_1) - K_3(t, s, w_2)| \leq L_4 |w_1 - w_2|, \tag{6d}$$
$$|K_3(t, s, w_1)| \leq L_5, \tag{6e}$$

  where $K_2(t, s, w) = \partial_s K(t, s, w)$ and $K_3(t, s, w) = \partial_w K(t, s, w)$.
- Example 1: $K(t, s, w) = k(t, s) w$
- Example 2: $K(t, s, w) = k(t, s) \sin w$

Convergence results

- Define $\mathbb{K}(\hat{w})(s) := \hat{K}(t,s,\hat{w}(s)) = K(t,\rho(s),w(\rho(s)))$.
- Let $e(t) = u(t) - U(t)$, $e'(t) = u'(t) - V(t)$.

Theorem 3.1

Let u be the exact solution of equation (2) and U be the solution of numerical scheme (4)-(5). Assume that f and K are continuous and satisfy the conditions in (6), $u \in H^1_{0,\lambda-1}(I)$, $u(t^{1/\lambda})|_{\sigma_{i,\lambda}} \in H^{m_i+1}(\sigma_{i,\lambda})$, $u'(t^{1/\lambda})|_{\sigma_{i,\lambda}} \in H^{m_i}(\sigma_{i,\lambda})$ and $\mathbb{K} : H^{m_i+1}(\sigma_{i,\lambda}) \to H^{m_i+1}(\sigma_{i,\lambda})$ with $1 \le m_i \le N_i + 1$. Then we have

$$\|e\|^2_{H^1_{0,\lambda-1}(I)} \le \exp\left(cT^{2-\lambda+2\mu}\right) \sum_{i=1}^{M} h_{i,\lambda}^{2m_i} N_i^{-2m_i} \cdot \left(|u'(t^{1/\lambda})|^2_{H^{m_i}(\sigma_{i,\lambda})} + \left\| \partial_s^{m_i+1} K\left(t, s^{1/\lambda}, u_i(s^{1/\lambda})\right) \right\|^2_{L^\infty(I;L^2(\sigma_{i,\lambda}))} \right) \tag{7}$$

Convergence results

Special case: $N_i = N$,

$$\mathscr{T}_M = \left\{ t_i = \left(\frac{i}{M}\right)^q T, \quad i = 0, 1, \cdots, M \right\}, \quad q \geq 1. \tag{8}$$

Theorem 3.2

Let $N_i = N$ and $\mathscr{T}_M$ be the graded mesh defined by (8). Let u be the exact solution of equation (2) and U be the solution of numerical scheme (4)-(5). Assume that f and K are continuous and satisfy the conditions in (6), $u \in H^1_{0,\lambda-1}(I)$, $u(t^{1/\lambda}) \in H^{m+1}(I)$, $u'(t^{1/\lambda})| \in H^m(I)$ and $\mathbb{K} : H^{m+1}(\hat{\sigma}_i) \to H^{m+1}(\hat{\sigma}_i)$ with $1 \leq m \leq N+1$. Then we have

$$\|e\|^2_{H^1_{0,\lambda-1}(I)} \leq \exp\left(cT^{2-\lambda+2\mu}\right) M^{-2\min\{q\lambda m, m\}} N^{-2m} \cdot \left(|u'(t^{1/\lambda})|^2_{H^m(I)} + \sum_{i=1}^{M} \left\| \partial_s^{m+1} K\left(t, s^{1/\lambda}, u_i(s^{1/\lambda})\right) \right\|^2_{L^\infty(I; L^2(\hat{\sigma}_i))} \right). \tag{9}$$

Contents

Numerical experiments

Example 1

Consider the linear VIDE

$$\begin{cases} u'(t) = f(t) - u(t) - \int_0^t (t-s)^{\mu-1} e^{s^\mu} u(s) ds, \quad t \in [0,1], \\ u(0) = 0. \end{cases} \tag{10}$$

where $f(t) = (1 - \mu t^\mu + t)e^{-t^\mu} + B(\mu, 2)t^{1+\mu}$. *The exact solution is* $u(t) = te^{-t^\mu}$.

- The exact solution u has a weak singularity at $t = 0$ for $\mu \in (0,1)$. Set $\mu = 1/2$ in this example.
- $u(t^{1/\lambda})$ and $u'(t^{1/\lambda})$ are analytic if $\lambda = 1/2$.

Numerical experiments

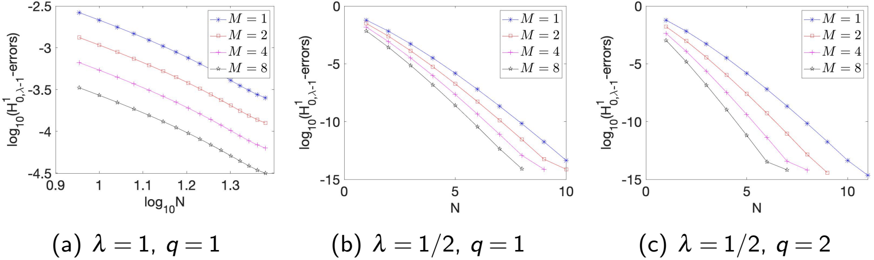


Figure 1: The p-version convergence in the $H^1_{0,\lambda-1}$ norm for Example 1

Table 1: The h-version convergence in the $H^1_{0,\lambda-1}$-norm for Example 1.

M	$\lambda=1(q=1)$				$\lambda=1/2(q=2)$			
	$N=2$		$N=3$		$N=2$		$N=3$	
	error	rate	error	rate	error	rate	error	rate
64	1.18E-03	1.00	5.00E-04	1.00	3.00E-08	3.00	3.58E-11	4.00
128	5.91E-04	1.00	2.50E-04	1.00	3.75E-09	3.00	2.24E-12	4.00
256	2.96E-04	1.00	1.25E-04	1.00	4.69E-10	3.00	1.40E-13	4.00

M	$\lambda=1/2(q=1)$							
	$N=1$		$N=2$		$N=3$		$N=4$	
	error	rate	error	rate	error	rate	error	rate
64	6.10E-04	1.20	8.06E-06	1.70	7.73E-08	2.20	5.64E-10	2.71
128	2.63E-04	1.21	2.45E-06	1.72	1.67E-08	2.21	8.56E-11	2.72
256	1.13E-04	1.22	7.40E-07	1.73	3.57E-09	2.22	1.29E-11	2.73

Numerical experiments

Example 2

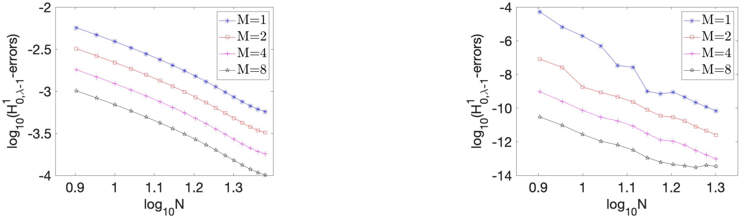
Consider the linear VIDE

$$\begin{cases} u'(t) = f(t) - u(t) - \int_0^t (t-s)^{\mu-1} e^s u(s) ds, \quad t \in [0,1], \\ u(0) = 0. \end{cases} \tag{11}$$

with exact solution $u(t) = (t^{1+\nu_1} + t^{1+\nu_2})e^{-t}$.

- $u(t^{1/\lambda})$ has only finite regularity for general ν_1 and ν_2.
- Set $\mu = 1/3$, $\nu_1 = 1/3$ and $\nu_2 = \sqrt{2}$. Take $\lambda = 1$ and $1/3$, respectively.

Numerical experiments



(a) $\lambda = 1,\ q = 1$ (b) $\lambda = 1/3,\ q = 3$

Figure 2: The p-version convergence in the $H^1_{0,\lambda-1}$ norm of Example 2

Table 2: The h-version convergence in the $H^1_{0,\lambda-1}$-norm of Example 2.

M	$\lambda = 1 (q = 1)$				$\lambda = 1/3 (q = 3)$			
	$N = 2$		$N = 3$		$N = 2$		$N = 3$	
	error	rate	error	rate	error	rate	error	rate
64	1.21E-03	0.83	7.38E-04	0.83	3.14E-07	3.00	2.66E-09	4.00
128	6.77E-04	0.83	4.14E-04	0.83	3.92E-08	3.00	1.66E-10	4.00
256	3.80E-04	0.83	2.32E-04	0.83	4.91E-09	3.00	1.04E-11	4.00

Numerical experiments

Example 3

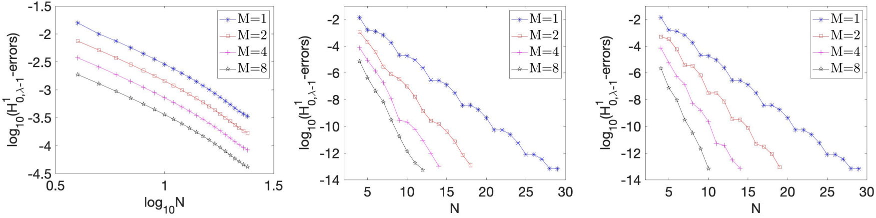
Consider the nonlinear VIDE

$$\begin{cases} u'(t) = 1 - u(t) + \int_0^t (t-s)^{\mu-1}(1+\sin(u))ds, \quad t \in [0,1], \\ u(0) = 0, \end{cases} \tag{12}$$

with $\mu = 1/2$.

- Nonlinear equation which satisfies the conditions in (6).
- The exact solution is unknown. Take a very fine collocation solution ($M = 1$, $N = 33$, $\lambda = 1/2$) as 'exact' solution.

Numerical experiments



(a) $\lambda = 1,\ q = 1$ (b) $\lambda = 1/2,\ q = 1$ (c) $\lambda = 1/2,\ q = 2$

Figure 3: The p-version convergence in the $H^1_{0,\lambda-1}$ norm of Example 3

Numerical experiments

Table 3: The h-version convergence in the $H^1_{0,\lambda-1}$-norm for Example 3.

M	$\lambda = 1(q=1)$				$\lambda = 1/2(q=2)$			
	$N=2$		$N=3$		$N=2$		$N=3$	
	error	rate	error	rate	error	rate	error	rate
64	6.68E-04	1.00	3.70E-04	1.00	8.91E-07	3.00	6.79E-09	4.00
128	3.34E-04	1.00	1.85E-04	1.00	1.11E-07	3.00	4.25E-10	4.00
256	1.67E-04	1.00	9.24E-05	1.00	1.39E-08	3.00	2.66E-11	4.00

M	$\lambda = 1/2(q=1)$					
	$N=1$		$N=3$		$N=5$	
	error	rate	error	rate	error	rate
64	6.23E-04	1.17	1.23E-06	2.17	3.23E-10	3.19
128	2.70E-04	1.21	2.68E-07	2.20	3.66E-11	3.14
256	1.15E-04	1.23	5.77E-08	2.22	4.24E-12	3.11

Numerical experiments

Example 4

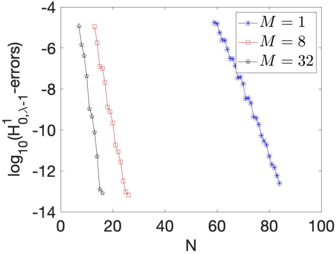
Consider the nonlinear VIDE

$$
\begin{cases}
u'(t) = f(t) - u(t) + \int_0^t (t-s)^{-\frac{1}{2}} u^2(s)ds, \quad t \in [0,1], \\
u(0) = 0,
\end{cases}
\tag{13}
$$

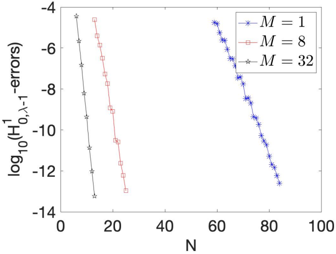
with exact solution $u(t) = t^{\frac{1}{2}} \sin(bt)$.

- Nonlinear equation which does not satisfy the conditions in (6).
- The exact solution has not only weak singularity at $t = 0$, but also highly oscillatory behaviour for large b. Take $b = 35$ in this example.
- $u(t^{1/\lambda})$ and $u'(t^{1/\lambda})$ are analytic if $\lambda = 1/2$.

Numerical experiments



(a) $\lambda = 1/2,\ q = 1$



(b) $\lambda = 1/2,\ q = 2$

Figure 4: The errors of the fractional collocation method with different M and N for Example 4

Table 4: The errors of the fractional collocation method with $q = 2$ and different M and N for Example 4.

M	512	128	32	13	8	1
N	4	6	8	14	19	73
error	5.31E-09	2.20E-09	6.23E-09	3.43E-09	1.21E-09	2.06E-09
time (in seconds)	199	36	9.6	8.9	10.7	69

Contents

Concluding remarks

- A fractional collocation method for solving second kind VIDEs with weakly singular kernels.
- A rigorous error analysis of *hp*-version.
- The *p*- and *hp*-versions can attain exponential convergence rates if $u(t^{1/\lambda})$ and $u'(t^{1/\lambda})$ are sufficiently smooth, even though $u(t)$ has low regularity.
- The *h*-version has no order barrier for uniform meshes, and optimal algebraic convergence rates of the *h*-version can be achieved by using graded meshes.

Thanks for your attention!