第八届谱方法及其应用学术研讨会

# Fast and stable augmented Levin methods for highly oscillatory and singular integrals


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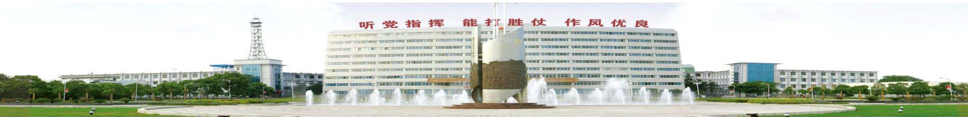

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## Highly oscillatory integrals

We are considering the computation of oscillatory integrals

$$I_{\omega}^{[a,b]}[f,g] := \int_a^b e^{iwg(x)} f(x)\,dx,$$

- $\omega \gg 1$ ;
- $f$ may be singular in $[a, b]$;
- $g$ are suitably smooth functions on $[a, b]$;
- It occurs in a wide range of practical problems and applications, e.g., nonlinear optics, electromagnetics and acoustic scattering.

# Highly oscillatory integrals

Many effective methods have been proposed for oscillatory integrals[1]:

- Asymptotic methods
- Filon-type methods
- Levin methods
- the generalized quadrature rule
- numerical steepest-descent methods

[1] Alfredo Deano, Daan Huybrechs, and Arieh Iserles. *Computing highly oscillatory integrals*. Philadelphia: SIAM, 2018.

# Levin method

Levin method is to find a function $p$ such that

$$\Big(p(x)\,e^{i\omega g(x)}\Big)' = f(x)\,e^{i\omega g(x)},$$

$$\text{i.e. } \mathcal{L}[p](x) \equiv p'(x) + i\omega g'(x)p(x) = f(x), \quad \text{(Levin-ODE)}$$

Thus, $I_\omega^{[a,b]}[f,g] = \int_a^b f(x)\,e^{i\omega g(x)}\,dx = p(b)\,e^{i\omega g(b)} - p(a)\,e^{i\omega g(a)}$.

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- When $g'(x) \neq 0$ and $f$ is smooth, the Levin-ODE problem is solved directly by collocation methods.

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- When $g'(x) \neq 0$ and $f$ is smooth, the Levin-ODE problem is solved directly by collocation methods.
- When $g'(x) \neq 0$ and $f$ possesses explicit algebraic and/or logarithmic singularities, the Levin-ODE problem can be solved with the help of singularity separation and superposition principle[2].

[2] Y. Wang and S. Xiang. *Levin methods for highly oscillatory integrals with singularities.* Science China Mathematics, 2022: 65(3).

# Levin method

Assume $g(a) = 0$, otherwise set $\hat{g}(x) = g(x) - g(a)$. Find a particular solution

$$p(x) = [q(x) + c_0(1 - e^{iwg(x)})]g^{\alpha}(x) + h(x), \text{for } f(x) = x^{\alpha} s(x)$$

or

$$\begin{aligned} p(x) = & [q(x) + c_0(1 - e^{iwg(x)})]g^{\alpha}(x) \log g(x) \\ & + [\ell(x) + d_0(1 - e^{iwg(x)})]g^{\alpha}(x) + h(x), \end{aligned} \text{ for } f(x) = x^{\alpha} \log x s(x)$$

where $s$ is a smooth function, $(c_0, q(x)), (d_0, \ell(x))$ can be computed from the ode of the type

$$iwg'(x)c_0 + g(x)q'(x) + [1 + \alpha + iwg(x)]g'(x)q(x) = f_1(x).$$

and $h(x)$ can be obtained explicitly in terms of special functions.

## Levin method

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$$\mathcal{L}[p](x) \equiv p'(x) + 2i\omega x p(x) = 1 + x,$$

we see that

$$p'(0) = f(0) = 1, \; p''(0) = 1 - 2i\omega p(0), \; p^{(3)}(0) = -4i\omega,$$

and

$$p^{(4)}(0) = -6i\omega - 12\omega^2 p(0), \; p^{(k+1)}(0) = -2ki\omega p^{(k-1)}(0), \quad k = 4, 5, \ldots.$$

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$$p^{(4)}(0) = -6i\omega - 12\omega^2 p(0),\ p^{(k+1)}(0) = -2ki\omega p^{(k-1)}(0), \quad k = 4, 5, \ldots.$$

In fact, the solution $p(x) = \frac{x}{2i\omega} + \left( C + \frac{\sqrt{\pi}\mathrm{Erf}(\sqrt{-i\omega}x)}{2\sqrt{-i\omega}} \right) e^{-i\omega x^2}$, is highly oscillatory, where Erf is the error function.

# Our goal

Our goal is to develop a new fast and stable Levin method for highly oscillatory integrals to address[3]:

$$I_\omega[f, g] := \int_0^a e^{iwg(x)} f(x)\,dx,$$

- the large absolute value of $\omega$ (leading to high oscillations),
- the singularity of $f$ (especially $x^\alpha$ or $x^\alpha \log(x)$).
- the stationary point of $g$ $(g(0) = g'(0) = g''(0) = \cdots = g^{(r-1)}(0) = 0, g^{(r)}(x) \neq 0)$.

[3] Y. Wang and S. Xiang. *Fast and stable augmented Levin methods for highly oscillatory and singular integrals.* Mathematics of Computation, 2022: 91(336).

# Contents

# Key idea

Consider $I = \int_0^1 (1+x) e^{i\omega x^2}\,dx$ again. To avoid the high oscillation of the solution $\mathcal{L}[p](x) \equiv p'(x) + 2i\omega x p(x) = 1 + x$, we augment with a free parameter $c_0$ such that

$$i\omega c_0 + q'(x) + 2i\omega x q(x) = 1 + x.$$

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If $q(x) = a_0 + a_1 x$, it follows that $c_0 = \frac{1}{i\omega}$, $a_0 = \frac{1}{2i\omega}$, and $a_1 = 0$, and thus, $q(x) = \frac{1}{2i\omega}$ is not oscillatory.

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However, the solution $q$ is not a solution of the Levin-ODE. We need to consider an extra differential equation

$$-i\omega c_0 + h'(x) + 2i\omega x h(x) = 0,$$

$$h(x) = e^{-i\omega x^2} \int_0^x e^{i\omega t^2}\, dt = \frac{(-1)^{1/4} e^{-i\omega x^2} \sqrt{\pi}\,\mathrm{Erf}\left((-1)^{3/4}\sqrt{\omega} x\right)}{2\sqrt{\omega}}.$$

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It is clear that $p = q + h$ is a particular solution of the Levin-ODE.

# Key idea

For $\int_0^a e^{iwg(x)} f(x)\,dx$, find a function $p(x) = q(x) + h(x)$ such that

$$\begin{aligned} \sum_{j=0}^{r-2} i\omega\, c_j \beta(x) \sqrt[r]{g}^{j}(x) + q'(x) + i\omega\, g'(x)\, q(x) &= f(x) \quad \text{(basic ODE)}, \\ -\sum_{j=0}^{r-2} i\omega\, c_j \beta(x) \sqrt[r]{g}^{j}(x) + h'(x) + i\omega\, g'(x)\, h(x) &= 0, \end{aligned}$$

where

$$\sqrt[r]{g}(x) := \begin{cases} (g^{(r)}(0))^{1/r} x \left( \frac{g(x)}{g^{(r)}(0) x^r} \right)^{1/r}, & x \neq 0, \\ 0, & x = 0. \end{cases}, \quad \beta(x) := \frac{d}{dx} \left( \sqrt[r]{g}(x) \right) \tag{1}$$

## Key idea

For $\int_0^a e^{iwg(x)} x^\alpha f(x)\, dx$, find a function

$$p(x) = \left( q(x) \sqrt[r]{g}(x) + q_0 \left( 1 - e^{-i\omega g(x)} \right) \right) \sqrt[r]{g}^{\alpha}(x) + h(x),$$

such that (**augmented Levin-ODE**)

$$\sum_{j=0}^{r-1} c_j i\omega \beta(x) \sqrt[r]{g}^{j}(x) + \sqrt[r]{g}(x) q'(x) + [1 + \alpha + i\omega r g(x)]\, \beta(x) q(x)$$
$$= f_1(x) \quad \text{(basic ODE)},$$

$$-\sum_{j=0}^{r-2} c_j i\omega \beta(x) \sqrt[r]{g}^{j+\alpha}(x) + h'(x) + i\omega g'(x) h(x) + \alpha \beta(x) q_0 \frac{1 - e^{-i\omega g(x)}}{\sqrt[r]{g}^{1-\alpha}(x)} = 0,$$

where $c_{r-1} = r q_0$.

# Key idea

For $\int_0^a e^{iwg(x)} x^\alpha \log x f(x)\,dx$, find a function

$$p(x) = \left( q(x)\sqrt[r]{g}(x) + q_0 \left(1 - e^{-i\omega g(x)}\right)\right) \sqrt[r]{g}^{\alpha}(x) \log\left(\sqrt[r]{g}(x)\right) + \left( \ell(x)\sqrt[r]{g}(x) + \ell_0 \left(1 - e^{-i\omega g(x)}\right)\right) \sqrt[r]{g}^{\alpha}(x) + h(x).$$

such that (**augmented Levin-ODE system**)

$$\sum_{j=0}^{r-1} c_j i\omega \beta(x) \sqrt[r]{g}^{j}(x) + \sqrt[r]{g}(x) q'(x) + \left[1 + \alpha + i\omega r g(x)\right] \beta(x) q(x) = f_1(x),$$

$$\sum_{j=0}^{r-1} d_j i\omega \beta(x) \sqrt[r]{g}^{j}(x) + \sqrt[r]{g}(x) \ell'(x) + \left[1 + \alpha + i\omega r g(x)\right] \beta(x) \ell(x) = -\beta(x) q($$

# Solution of Basic ODE

$$i\omega\beta(x)\sum_{j=0}^{r-2} c_j \sqrt[r]{g}^{j}(x) + q'(x) + i\omega g'(x)q(x) = f(x),$$

$$i\omega\beta(x)\sum_{j=0}^{r-1} c_j \sqrt[r]{g}^{j}(x) + \sqrt[r]{g}(x)q'(x) + [1 + \alpha + i\omega r g(x)]\beta(x)q(x) = f(x).$$

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$$\Phi^{[k]}(x) := \frac{1}{i\omega}\left(f(x) - \mathcal{D}q^{[k-1]}(x)\right), \ q^{[0]} \equiv 0$$

$$c_j^{[k]} := \frac{1}{j!}\mathcal{D}^j\left[\frac{\Phi^{[k]}\circ\eta}{\beta\circ\eta}\right](0), \ j = 0, 1, \ldots, r-2,$$

$$q^{[k]}(x) := \frac{1}{r\beta(x)\sqrt[r]{g}^{r-1}(x)}\left(\Phi^{[k]}(x) - \sum_{j=0}^{r-2} c_j^{[k]}\beta(x)\sqrt[r]{g}^j(x)\right),$$

# Solution of Basic ODE

Theorem 1
*Given* $n \in \mathbb{N}$, *if the coefficients* $c_j = c_j^{[n+1]}(j = 0, \dots, r-2)$, *where* $c_j^{[n+1]}$ *are obtained by iteration, then*

$$\max_{j=0,\dots,r-2} |c_j| \leq C\omega^{-1}, \tag{2}$$

*and the solution* $q \in C^n[0,1]$ *of ODEs subject to the condition* $q(0) = q^{[n+1]}(0)$ *satisfies*

$$\|\mathcal{D}^m q\|_{\infty,[0,1]} \leq C\omega^{-1},\ m = 0, 1, \dots, n, \tag{3}$$

*where* $C$ *is a constant independent of* $\omega$.

# Contents

We solve the basic ODEs by the sparse spectral method proposed by Olver and Townsend[4].

Let $\mathcal{D}$ and $\mathcal{S}$ represent the differential and conversion operators, respectively,

$$\mathcal{D} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & 3 & \\ & & & \ddots & \ddots \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & & & \\ & \frac{1}{2} & 0 & -\frac{1}{2} & & \\ & & \frac{1}{2} & 0 & -\frac{1}{2} & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

[4] S. Olver and A. Townsend. "A fast and well-conditioned spectral method". In: *SIAM Review* 3 (2013), pp. 462–489.

Given $g(x) = \sum_{j=0}^{\infty} g_j T_j^*(x)$, $T_j^*(x) = T_j(2x-1)$, define two multiplication operators $\mathcal{M}_1$ and $\mathcal{M}_2$

$$\mathcal{M}_0[g] = \frac{1}{2}\left[\begin{pmatrix} 2g_0 & g_1 & g_2 & g_3 & \cdots \\ g_1 & 2g_0 & g_1 & g_2 & \ddots \\ g_2 & g_1 & 2g_0 & g_1 & \ddots \\ g_3 & g_2 & g_1 & 2g_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ g_1 & g_2 & g_3 & g_4 & \\ g_2 & g_3 & g_4 & g_5 & \\ g_3 & g_4 & g_5 & g_6 & \\ \vdots & & & & \end{pmatrix}\right],$$

$$\mathcal{M}_1[g] = \frac{1}{2}\left[\begin{pmatrix} 2g_0 & g_1 & g_2 & g_3 & \cdots \\ g_1 & 2g_0 & g_1 & g_2 & \ddots \\ g_2 & g_1 & 2g_0 & g_1 & \ddots \\ g_3 & g_2 & g_1 & 2g_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} - \begin{pmatrix} g_2 & g_3 & g_4 & g_5 & \cdots \\ g_3 & g_4 & g_5 & g_6 & \\ g_4 & g_5 & g_6 & g_7 & \\ g_5 & g_6 & g_7 & g_8 & \\ \vdots & & & & \end{pmatrix}\right].$$

Given a positive integer $n$, we let $\mathcal{P}_n$ denote an $n \times \infty$ projection operator defined by

$$\mathcal{P}_n = (\mathcal{I}_n, \mathbf{0}), \tag{4}$$

where $\mathcal{I}_n$ is the $n \times n$ identity matrix.

$$i\omega\beta(x)\sum_{j=0}^{r-2} c_j \sqrt[r]{g}^j(x) + q'(x) + i\omega g'(x) q(x) = f(x),$$

$$\sum_{j=0}^{r-2} c_j \mathcal{P}_n \mathcal{S} \mathbf{g}^j + \mathcal{P}_n \left[\frac{2}{i\omega}\mathcal{D} + \mathcal{S}\mathcal{M}_0[g']\right] \mathcal{P}_{n-r+1}^\top \mathcal{P}_{n-r+1} \mathbf{q} = \frac{1}{i\omega}\mathcal{P}_n \mathcal{S}\mathbf{f},$$

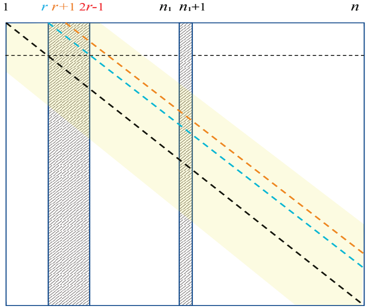


Figure 1: Diagram for $\bar{\mathbf{A}}_n$

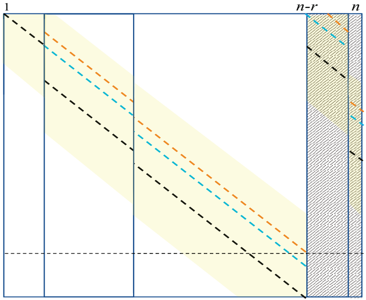


Figure 2: Diagram for $\bar{\bar{\mathbf{A}}}_n$

$$\bar{\mathbf{A}}_n \bar{\mathbf{q}}_n = \mathbf{f}_n, \quad \bar{\bar{\mathbf{A}}}_n = \begin{pmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{pmatrix}.$$

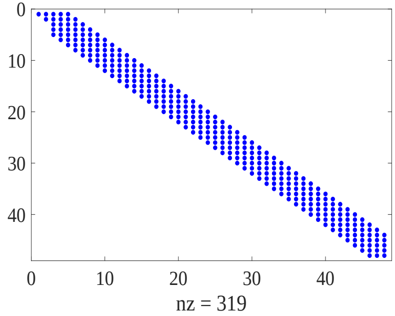


Figure 3: Sparsity

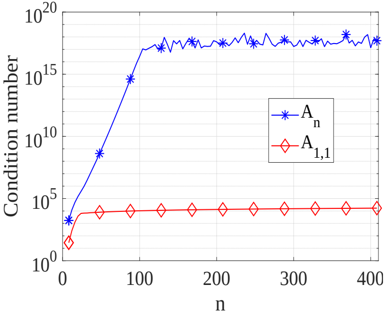


Figure 4: Condition numbers

# Error analysis

## Theorem 2

*Assume that the numerical solutions of ODEs* (2) *and* (2) *satisfy* $\|\hat{q}\|_{\omega,[0,1]} = O(\omega^{-1})$, $\|\hat{q}'\|_{\omega,[0,1]} = O(\omega^{-1})$, *and* $\hat{c}_j = O(\omega^{-1})$ *of ODEs independent of* $n$. $Q_1$ *and* $Q_2$ *are then of spectral (super-algebraic) convergence, i.e., for each fixed positive integer* $k$,

$$|I_1 - Q_1| \le \frac{C|\omega|^{-\min\{(1+\alpha)/r,1\}}}{(n-3)(n-4)\ldots(n-k)},$$

*and*

$$|I_2 - Q_2| \le \frac{C\delta_{\alpha,r}(\omega)|\omega|^{-\min\{(1+\alpha)/r,1\}}}{(n-3)(n-4)\ldots(n-k)},$$

*for* $n \ge k+2 > 4$ *and a positive constant* $C$ *independent of* $n$ *and* $\omega$.

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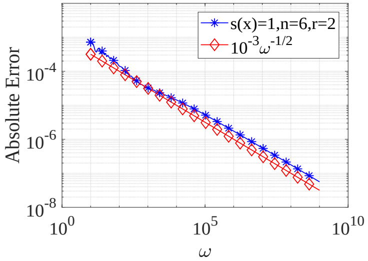


Figure 5: $s(x) = 1$

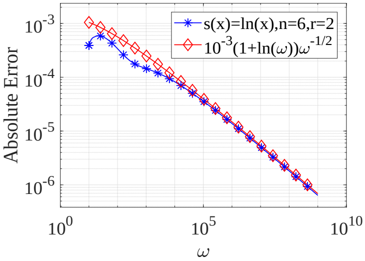


Figure 6: $s(x) = \ln x$

Absolute errors of the new Levin methods with $n = 6$ for $\int_0^1 s(x) e^{x^2} e^{iwx^2} dx$ with $w \in [10, 10^9]$

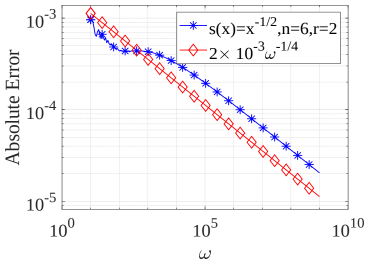


Figure 7: $s(x) = x^{-\frac{1}{2}}$

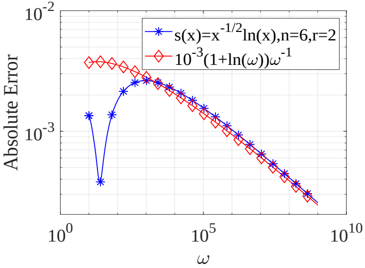


Figure 8: $s(x) = x^{-\frac{1}{2}} \ln x$

Absolute errors of the new Levin methods with $n = 6$ for $\int_0^1 s(x) e^{x^2} e^{iwx^2} dx$ with $w \in [10, 10^9]$

Table 1: Absolute errors of the new Levin methods for $\int_0^1 s(x)\, e^{x^2}\, e^{i\omega x^2}\, dx$ with $\omega = 10$

| n | $s(x) = 1$ | $x^{-1/2}$ | $x^{1/2}$ | $\ln x$ | $x^{-1/2} \ln x$ | $x^{1/2} \ln x$ |
|---|---|---|---|---|---|---|
| 8 | $1.20e-05$ | $1.35e-05$ | $1.00e-05$ | $5.73e-06$ | $2.01e-05$ | $3.79e-06$ |
| 10 | $6.68e-08$ | $7.66e-08$ | $6.53e-08$ | $1.42e-08$ | $3.14e-07$ | $2.31e-08$ |
| 12 | $2.22e-10$ | $2.32e-10$ | $1.93e-10$ | $3.60e-10$ | $3.26e-09$ | $1.09e-10$ |
| 14 | $1.31e-12$ | $9.64e-13$ | $9.17e-13$ | $2.24e-12$ | $3.37e-11$ | $2.43e-13$ |
| 16 | $5.51e-15$ | $2.80e-15$ | $3.31e-15$ | $1.80e-14$ | $2.93e-13$ | $4.16e-16$ |
| 18 | $1.89e-16$ | $4.58e-16$ | $9.44e-17$ | $4.58e-16$ | $4.97e-16$ | $8.33e-17$ |

Table 2: Absolute errors of the new Levin methods for $\int_0^1 s(x)\, e^{x^2} e^{i\omega x^2}\, dx$ with $\omega = 10^5$

| n | $s(x) = 1$ | $x^{-1/2}$ | $x^{1/2}$ | $\ln x$ | $x^{-1/2} \ln x$ | $x^{1/2} \ln x$ |
|---|---|---|---|---|---|---|
| 8 | $1.17e-07$ | $4.49e-06$ | $4.17e-09$ | $8.03e-07$ | $3.60e-05$ | $2.80e-08$ |
| 10 | $1.92e-09$ | $7.59e-08$ | $6.70e-11$ | $1.33e-08$ | $6.14e-07$ | $4.52e-10$ |
| 12 | $2.47e-11$ | $1.01e-09$ | $8.51e-13$ | $1.72e-10$ | $8.26e-09$ | $5.72e-12$ |
| 14 | $2.61e-13$ | $1.11e-11$ | $9.14e-15$ | $1.82e-12$ | $9.14e-11$ | $6.03e-14$ |
| 16 | $2.38e-15$ | $1.02e-13$ | $8.88e-17$ | $1.64e-14$ | $8.53e-13$ | $5.61e-16$ |
| 18 | $1.85e-17$ | $7.95e-16$ | $7.89e-19$ | $1.20e-16$ | $6.80e-15$ | $4.68e-18$ |

Table 3: Absolute errors of the new Levin methods for $\int_0^1 s(x)e^{x^2}e^{i\omega x^2}\,dx$ with $\omega = 10^5$

| n | $s(x)=1$ | $x^{-1/2}$ | $x^{1/2}$ | $\ln x$ | $x^{-1/2}\ln x$ | $x^{1/2}\ln x$ |
|---|---|---|---|---|---|---|
| 128 | $1.94e-18$ | $6.55e-17$ | $1.91e-19$ | $1.01e-17$ | $4.48e-16$ | $6.35e-19$ |
| 256 | $2.17e-18$ | $1.96e-17$ | $1.59e-19$ | $1.55e-17$ | $1.78e-16$ | $1.63e-19$ |
| 512 | $4.09e-18$ | $9.71e-17$ | $2.85e-19$ | $3.23e-17$ | $1.38e-15$ | $7.67e-19$ |
| 1024 | $4.34e-19$ | $3.10e-17$ | $1.32e-19$ | $3.88e-18$ | $1.57e-16$ | $5.42e-19$ |
| 2048 | $3.13e-18$ | $1.77e-16$ | $4.31e-19$ | $3.23e-17$ | $2.56e-15$ | $8.20e-19$ |
| 4096 | $9.70e-19$ | $7.47e-17$ | $1.52e-19$ | $1.48e-17$ | $1.52e-15$ | $6.06e-19$ |

# Contents

Augmented Levin method:

- Augmentation of free parameters to address the stationary point,
- Separation of singularities to address the singularities,
- Levin idea to address the high oscillation,
- Ultraspherical spectral method to address the fast computation,
- TSVD to address the stable computation.

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Future work:
Computing highly oscillatory integrals with Hankel functions and turning points:

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} + \begin{pmatrix} -\frac{1}{x} & w \\ -w & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

# Thank you for your attention!