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Chebyshev Approximations of Functions with Endpoint Singularities

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Motivations

- We consider the Poisson equation:

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

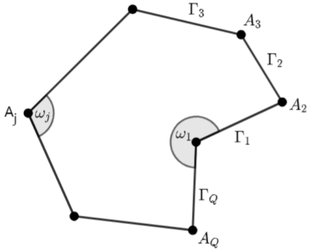

Figure 1: Polygonal region Ω with boundary vertices $A_j, j = 1, 2, \ldots, Q$ and interior angles ω_j.

Motivations

Melenk [1]

$$V_{sj}\left(r_j,\theta_j\right) = \begin{cases} r_j^{s\pi/\omega_j} \sin\left(\frac{s\pi}{\omega_j}\theta_j\right) & s\pi/\omega_j \notin \mathbb{N}, \\ r_j^{s\pi/\omega_j}\left(\ln r_j \sin\left(\frac{s\pi}{\omega_j}\theta_j\right) + \theta_j \cos\left(\frac{s\pi}{\omega_j}\theta_j\right)\right) & s\pi/\omega_j \in \mathbb{N}. \end{cases}$$

If $f \in H^k(\Omega)$, then the solution u can be decomposed as

$$u = \sum_{j=1}^{Q} \sum_{\substack{s\in\mathbb{N} \\ s\pi/\omega_j<k}} a_{sj}(f) V_{sj}\left(r_j,\theta_j\right) + u_0,$$

for some $a_{sj}(f) \in \mathbb{R}$ and $u_0 \in H^{k+1}(\Omega)$.

[1] Melenk, J.: *hp*-Finite Element Methods for Singular Perturbations. Springer-Verlag, Berlin (2002).

Motivations

- Consider the approximation of

$$u(x) = (1+x)^{\alpha}\varphi(x), \quad x \in (-1,1).$$

- Theory

$$\|u - \pi_N^C u\|_{L^2_\omega(\Omega)} \le cN^{-m}\|u^{(m)}\|_{\omega^{m-1/2,m-1/2}}, \quad m < 2\alpha + 1/2,$$

$$\|u^{(m)}\|^2_{\omega^{m-1/2,m-1/2}} = \int_{-1}^{1} |u^{(m)}|^2 (1-x^2)^{m-1/2} \mathrm{d}x.$$

Numerical

$$\|u - \pi_N^C u\|_{L^2_\omega(\Omega)} = \mathcal{O}(N^{-2\alpha-1/2}).$$

Goals

On the space $\mathcal{B}^\sigma$ for general singular function u

$$\|\Pi_N u - u\|_S \leq cN^{-\sigma}|u|_{\mathcal{B}^\sigma}, \quad \sigma \geq 0,$$

where Π_N is a operator (Projection, Numerical solution, etc).

1. Contain the classes of functions as broad as possible
2. Can best characterize their endpoint regularity leading to optimal convergence order
3. The positive constant c as sharp as possible

In this talk

- **Goal**

  1. Provide a natural means to introduce the fractional spaces to characterise fractional regularity of endpoint singular functions
  2. Derive optimal estimates for polynomial approximation in L^∞- and L^2- Sobolev norms.

- **Tools**

  1. Generalised Gegenbauer functions of fractional degree (GGF-Fs)
  2. Integration by parts with fractional integration by parts
  3. Uniform upper bounds of GGF-Fs

From Gegenbauer polynomials to GGF-Fs

Definition: for real $\lambda > -1/2$ and real $\nu \geq 0$, the **right GGF-F of degree ν** is defined by the Hypergeometric function as: for $x \in (-1, 1)$,

$$ {}^{r}G_{\nu}^{(\lambda)}(x) = {}_2F_1\Big(-\nu, \nu + 2\lambda; \lambda + \frac{1}{2}; \frac{1-x}{2}\Big), $$

while the **left GGF of degree ν** is defined by

$$ {}^{l}G_{\nu}^{(\lambda)}(x) := (-1)^{[\nu]}\, {}^{r}G_{\nu}^{(\lambda)}(-x), $$

where $[\nu]$ is the largest integer $\leq \nu$.

Recall the hypergeometric function:

$$ {}_2F_1(a, b; c; x) = 1 + \sum_{j=1}^{\infty} \frac{a(a+1)\cdots(a+j-1)}{1 \cdot 2 \cdots j} \frac{b(b+1)\cdots(b+j-1)}{c(c+1)\cdots(c+j-1)} x^j. $$

From Gegenbauer polynomials to GGF-Fs

$$ {}^{r}G_n^{(\lambda)}(x) = {}^{l}G_n^{(\lambda)}(x) = G_n^{(\lambda)}(x), \quad n \geq 0; \quad {}^{r}G_\nu^{(\lambda)}(1) = 1, \quad G_n^{(\lambda)}(-1) = (-1)^n. $$

(i) If $-1/2 < \lambda < 1/2$, then

$$ {}^{r}G_\nu^{(\lambda)}(-1) = \frac{\cos((\nu+\lambda)\pi)}{\cos(\lambda\pi)}. $$

(ii) If $\lambda = 1/2$ and $\nu \notin \mathbb{N}_0$, then

$$ \lim_{x \to -1^+} \frac{{}^{r}G_\nu^{(\lambda)}(x)}{\ln(1+x)} = \frac{\sin(\nu\pi)}{\pi}. $$

(iii) If $\lambda > 1/2$ and $\nu \notin \mathbb{N}_0$, then

$$ \lim_{x \to -1^+} \left(\frac{1+x}{2}\right)^{\lambda-1/2} {}^{r}G_\nu^{(\lambda)}(x) = -\frac{\sin(\nu\pi)}{\pi} \frac{\Gamma(\lambda-1/2)\Gamma(\lambda+1/2)\Gamma(\nu+1)}{\Gamma(\nu+2\lambda)}. $$

Fractional integration by parts

Lemma **(Liu-Wang-Li.'19 MCOM; and Liu-Wang-Wu.'21 AICM)** Let $\rho \geq 0$, $f(x) \in L^1(\Omega)$ and $g(x) \in \mathrm{AC}(\bar{\Omega})$.

(i) If $I^{\rho}_{b-} f(x) \in \mathrm{BV}(\bar{\Omega})$, then

$$\int_a^b f(x)\, I^{\rho}_{a+} g'(x)\, \mathrm{d}x = \{g(x)\, I^{\rho}_{b-} f(x)\}\big|_{a^+}^{b^-} - \int_a^b g(x)\, \mathrm{d}\{I^{\rho}_{b-} f(x)\}.$$

(ii) If $I^{\rho}_{a+} f(x) \in \mathrm{BV}(\bar{\Omega})$, then

$$\int_a^b f(x)\, I^{\rho}_{b-} g'(x)\, \mathrm{d}x = \{g(x)\, I^{\rho}_{a+} f(x)\}\big|_{a^+}^{b^-} - \int_a^b g(x)\, \mathrm{d}\{I^{\rho}_{a+} f(x)\}.$$

Recall the Riemann-Liouville (RL) fractional integrals/derivatives

$$(I^s_{a+} u)(x) = \frac{1}{\Gamma(s)} \int_a^x \frac{u(y)}{(x-y)^{1-s}} \mathrm{d}y;$$

$$(I^s_{b-} u)(x) = \frac{1}{\Gamma(s)} \int_x^b \frac{u(y)}{(y-x)^{1-s}} \mathrm{d}y, \quad x \in \Omega.$$

$$(\mathcal{D}^s_{a+} u)(x) = \mathcal{D}^k\{\, I^{k-s}_{a+}\, u\}(x); \quad (\mathcal{D}^s_{b-} u)(x) = (-1)^k \mathcal{D}^k\{\, I^{k-s}_{b-}\, u\}(x).$$

Fractional integral/derivative formulas

Theorem **(Liu-Wang-Li.'19 MCOM)**: For real $\nu \geq s > 0$ and real $\lambda > -1/2$, the GGF-Fs on $(-1, 1)$ satisfy the Riemann-Liouville fractional integral formulas:

$$I^s_{1-}\{\omega_\lambda(x)\, {}^rG^{(\lambda)}_\nu(x)\} = h^{(-s)}_\lambda\, \omega_{\lambda+s}(x)\, {}^rG^{(\lambda+s)}_{\nu-s}(x),$$

$$I^s_{-1+}\{\omega_\lambda(x)\, {}^lG^{(\lambda)}_\nu(x)\} = (-1)^{[\nu]+[\nu-s]}\, h^{(-s)}_\lambda\, \omega_{\lambda+s}(x)\, {}^lG^{(\lambda+s)}_{\nu-s}(x).$$

For real $\lambda > s - 1/2$ and real $\nu \geq 0$, the GGF-Fs on $(-1, 1)$ satisfy the Riemann-Liouville fractional derivative formulas:

$$\mathcal{D}^s_{1-}\{\omega_\lambda(x)\, {}^rG^{(\lambda)}_\nu(x)\} = h^{(s)}_\lambda\, \omega_{\lambda-s}(x)\, {}^rG^{(\lambda-s)}_{\nu+s}(x),$$

$$\mathcal{D}^s_{-1+}\{\omega_\lambda(x)\, {}^lG^{(\lambda)}_\nu(x)\} = (-1)^{[\nu]+[\nu+s]}\, h^{(s)}_\lambda\, \omega_{\lambda-s}(x)\, {}^lG^{(\lambda-s)}_{\nu+s}(x).$$

In the above, we denote

$$\omega_\alpha(x) = (1 - x^2)^{\alpha-\frac{1}{2}}, \quad h^{(\beta)}_\lambda = \frac{2^\beta\, \Gamma(\lambda + 1/2)}{\Gamma(\lambda - \beta + 1/2)}.$$

Uniform upper bounds

Theorem **(Liu-Wang-Li.'19 MCOM; Liu-Wang.'20 JAT)** For real $\lambda \geq 1$ and $\nu \geq 0$, we have the bound

$$\max_{|x|\leq 1} \left\{\omega_\lambda(x)\left|{}^{r}G_{\nu}^{(\lambda)}(x)\right|,\ \omega_\lambda(x)\left|{}^{l}G_{\nu}^{(\lambda)}(x)\right|\right\} \leq \frac{\Gamma(\lambda+1/2)\Gamma((\nu+1)/2)}{\sqrt{\pi}\Gamma((\nu+1)/2+\lambda)},$$

where $\omega_\lambda(x) = (1-x^2)^{\lambda-1/2}$.

Chebyshev Projections

For any $u \in L^2_\omega(\Omega)$, we expand it in Chebyshev series and denote the partial sum by

$$u(x) = {\sum_{n=0}^{\infty}}' \, \hat{u}_n^C \, T_n(x), \quad \pi_N^C u(x) = {\sum_{n=0}^{N}}' \, \hat{u}_n^C \, T_n(x),$$

where the prime denotes a sum whose first term is halved, and

$$\hat{u}_n^C = \frac{2}{\pi} \int_{-1}^{1} u(x) \frac{T_n(x)}{\sqrt{1-x^2}} \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} u(\cos\theta) \cos(n\theta) \mathrm{d}\theta.$$

L^∞ error

$$\|u - \pi_N^C u\|_{L^\infty} \leq \sum_{n=N+1}^{\infty} | \, \hat{u}_n^C \, T_n(x)| \leq \sum_{n=N+1}^{\infty} |\hat{u}_n^C|.$$

L^2_ω error

$$\|u - \pi_N^C u\|^2_{L^2_\omega(\Omega)} = \| \sum_{n=N+1}^{\infty} \hat{u}_n^C \, T_n(x)\|^2_{L^2_\omega(\Omega)} = \frac{\pi}{2} \sum_{n=N+1}^{\infty} |\hat{u}_n^C|^2.$$

Main results

Theorem (Liu-Wang-Li.'19 MCOM): Given $\theta \in (-1,1)$, if $u \in \mathbb{W}_{\theta}^{m+s}(\Omega)$ with $s \in (0,1)$ and integer $m \geq 0$, then for $n \geq m+s > 1/2$, we have

$$\begin{aligned}\hat{u}_n^C = -C_{m,s}\Big\{ & \int_{-1}^{\theta} {}^{l}G_{n-m-s}^{(m+s)}(x)\,\omega_{m+s}(x)\,\mathrm{d}\big\{{}_{x}I_{\theta}^{1-s}u^{(m)}(x)\big\} \\ & + \big\{{}_{x}I_{\theta}^{1-s}u^{(m)}(x)\,{}^{l}G_{n-m-s}^{(m+s)}(x)\,\omega_{m+s}(x)\big\}\big|_{x=\theta-} \\ & - \int_{\theta}^{1} {}^{r}G_{n-m-s}^{(m+s)}(x)\,\omega_{m+s}(x)\,\mathrm{d}\big\{{}_{\theta}I_{x}^{1-s}u^{(m)}(x)\big\} \\ & - \big\{{}_{\theta}I_{x}^{1-s}u^{(m)}(x)\,{}^{r}G_{n-m-s}^{(m+s)}(x)\,\omega_{m+s}(x)\big\}\big|_{x=\theta+}\Big\},\end{aligned}$$

where ${}^{l}G_{\nu}^{(\lambda)}, {}^{r}G_{\nu}^{(\lambda)}$ are GGFs of fractional degree ν, and

$$\omega_\lambda(x) = (1-x^2)^{\lambda-1/2}, \quad C_{m,s} := \frac{1}{\sqrt{\pi}\,2^{m+s-1}\Gamma(m+s+1/2)}.$$

Derivation: fractional integration by part

By **integration by parts**: for $u \in W^{m,1}(\Omega)$,

$$\hat{u}_n^C = \frac{2}{\pi}\int_{-1}^{1} u(x)\left\{T_n(x)(1-x^2)^{-1/2}\right\}\mathrm{d}x = -\frac{2}{\pi}\int_{-1}^{1} u(x)\left\{G_{n-1}^{(1)}(x)\omega_1(x)\right\}'\mathrm{d}x$$
$$= \cdots = \frac{1}{(2m-1)!!}\,\frac{2}{\pi}\int_{-1}^{1} u^{(m)}(x)\,G_{n-m}^{(m)}(x)\omega_m(x)\,\mathrm{d}x.$$

Proceed with moving a fractional step by fractional integral formulas of GGFs and **fractional integration by parts (traces theorem of BV functions)**: If $I_{\theta-}^{1-s}f(x) \in \mathrm{BV}(\bar{\Omega}_\theta^-)$,

$$\int_{-1}^{\theta} f(x)\,I_{-1+}^{1-s}g'(x)\,dx = \int_{-1}^{\theta} g'(x)\,I_{\theta-}^{1-s}f(x)\,\mathrm{d}x$$
$$= \left\{g(x)\,I_{\theta-}^{1-s}f(x)\right\}\Big|_{x=\theta-} - \int_{-1}^{\theta} g(x)\,\mathrm{d}\left\{I_{\theta-}^{1-s}f(x)\right\}, \quad \theta \in (-1,1),$$

where $f(x) = u^{(m)}(x)$, and $g(x) = \omega_{m+s}(x)\,{}^{l}G_{n-m-s}^{(m+s)}(x) \in C^\infty(\Omega_\theta^-)$.

Fractional space

For $s \in (0,1)$ and $m \in \mathbb{N}_0$, define the fractional space

$$\mathbb{W}_\theta^{m+s}(\Omega) := \Big\{u \in L^1(\Omega) \,:\, u, u', \cdots, u^{(m-1)} \in \mathrm{AC}(\Omega) \;\text{ and }\; {}_xI_\theta^{1-s}u^{(m)} \in \mathrm{BV}(\Omega_\theta^-),\;\; {}_\theta I_x^{1-s}u^{(m)} \in \mathrm{BV}(\Omega_\theta^+)\Big\},$$

where $\Omega_\theta^- := (-1,\theta)$ and $\Omega_\theta^+ := (\theta,1)$ with $\theta \in (-1,1)$. Equipped with the norm:

$$\|u\|_{\mathbb{W}_\theta^{m+s}(\Omega)} = \sum_{k=0}^{m} \|u^{(k)}\|_{L^1(\Omega)} + U_\theta^{m,s},$$

the semi-norm is defined by

$$U_\theta^{m,s} := \int_{-1}^{\theta} \big|\mathrm{d}\big\{{}_xI_\theta^{1-s}u^{(m)}\big\}(x)\big| + \int_{\theta}^{1} \big|\mathrm{d}\big\{{}_\theta I_x^{1-s}u^{(m)}\big\}(x)\big| + \Big|\big\{{}_xI_\theta^{1-s}u^{(m)}\big\}(\theta-)\Big| + \Big|\big\{{}_\theta I_x^{1-s}u^{(m)}\big\}(\theta+)\Big|.$$

Main results: L^∞- and L^2_ω-estimates

Theorem (Liu-Wang-Li.'19 MCOM): Given $\theta \in (-1,1)$, if $u \in \mathbb{W}_\theta^{m+s}(\Omega)$ with $s \in (0,1)$ and integer $m \geq 1$, then for $n \geq m+s$, we have

$$|\hat{u}_n^C| \leq \frac{1}{2^{m+s-1}\pi} \frac{\Gamma((n-m-s+1)/2)}{\Gamma((n+m+s+1)/2)} U_\theta^{m,s},$$

and

$$\|u - \pi_N^C u\|_{L^\infty(\Omega)} \leq \frac{U_\theta^{m,s}}{2^{m+s-2}(m+s-1)\pi} \frac{\Gamma((N-m-s)/2+1)}{\Gamma((N+m+s)/2)},$$

$$\|u - \pi_N^C u\|_{L^2_\omega(\Omega)} \leq \left\{ \frac{2^3}{(2m+2s-1)\pi} \frac{\Gamma(N-m-s+1)}{\Gamma(N+m+s)} \right\}^{1/2} U_\theta^{m,s}.$$

Recall that: for $a < b$,

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b} + \frac{1}{2}(a-b)(a+b-1)z^{a-b-1} + O(z^{a-b-2}), \quad z \gg 1.$$

Approximation of typical singular functions

- **Type-I**: $u(x) = |x-\theta|^\alpha g(x)$ with $\alpha > -1/2, \theta \in (-1,1)$ and g being smooth. Then $u \in \mathbb{W}_\theta^{\alpha+1}(\Omega)$, and have the optimal order:

$$|\hat{u}_n^C| = \mathcal{O}(n^{-\alpha-1}), \;\; \|u - \pi_N^C u\|_\infty = \mathcal{O}(N^{-\alpha}),$$

$$\|u - \pi_N^C u\|_{L^2_\omega(\Omega)} = \mathcal{O}(N^{-\alpha-1/2}).$$

Table 1: Order of $\|u - \pi_N^C u\|_{L^2_\omega}$ with $u = |x|^\alpha$ (Expect: $\mathcal{O}(N^{-\alpha-1/2})$)

N	$\alpha = 0.1$	Order	$\alpha = 1.2$	Order	$\alpha = 2.6$	Order
2^5	1.88e-2	–	1.68e-3	–	2.68e-5	–
2^6	1.25e-2	0.59	5.31e-4	1.66	3.27e-6	3.03
2^7	8.29e-3	0.59	1.66e-4	1.68	3.91e-7	3.07
2^8	5.48e-3	0.60	5.13e-5	1.69	4.61e-8	3.08
2^9	3.61e-3	0.60	1.58e-5	1.70	5.41e-9	3.09
2^{10}	2.37e-3	0.61	4.88e-6	1.70	6.33e-10	3.10

Endpoint singularities

- **Type-II**: $u(x) = (1+x)^\alpha g(x) \in \mathbb{W}^{\alpha+1}_{-1}(\Omega)$ with $\alpha > -1/2, \theta \in (-1,1)$ and g being smooth.

Table 2: Order of $|\hat{u}_n^C|$ with $u = (1+x)^\alpha \sin x$.

n	$\alpha = 0.1$	order	$\alpha = 1.2$	order	$\alpha = 2.3$	order
2^3	1.18e-02	–	3.79e-04	–	6.06e-05	–
2^4	5.10e-03	1.21	3.37e-05	3.49	1.05e-06	5.85
2^5	2.21e-03	1.21	3.14e-06	3.43	2.06e-08	5.67
2^6	9.59e-04	1.21	2.96e-07	3.41	4.18e-10	5.62
2^7	4.14e-04	1.21	2.80e-08	3.40	8.59e-12	5.60
2^8	1.77e-04	1.23	2.65e-09	3.40	1.77e-13	5.60
2^9	7.35e-05	1.27	2.51e-10	3.40	3.65e-15	5.60

Numerical results show

$$|\hat{u}_n^C| = \mathcal{O}(n^{-2\alpha-1})$$

Main results: endpoint singularities

(Liu-Wang-Li.'19 MCOM): $\theta \to -1$

$$\hat{u}_n^C = \frac{1}{\sqrt{\pi}\, 2^{\mu-1}\Gamma(\mu+1/2)} \Big\{ \int_{-1}^{1} {}^r\mathcal{G}_n^{(\mu)}(x) v'(x)\mathrm{d}x + \lim_{x\to -1^+} \big\{ {}^r\mathcal{G}_n^{(\mu)}(x) v(x) \big\} \Big\}.$$

where $v(x) = I_{-1+}^{1-s} u^{(m)}(x)$, ${}^r\mathcal{G}_n^{(\mu)}(x) := \big(1-x^2\big)^{\mu-1/2}\, {}^rG_{n-\mu}^{(\mu)}(x)$.
By **integration by parts**:

$$\begin{aligned}
\int_{-1}^{1} {}^r\mathcal{G}_n^{(\mu)}(x) v'(x)\mathrm{d}x &= -\frac{1}{2\mu+1} \int_{-1}^{1} \big\{ {}^r\mathcal{G}_n^{(\mu+1)}(x) \big\}' v'(x)\mathrm{d}x \\
&= \frac{1}{2\mu+1} \int_{-1}^{1} {}^r\mathcal{G}_n^{(\mu+1)}(x) v''(x)\mathrm{d}x - \frac{1}{2\mu+1} \big\{ {}^r\mathcal{G}_n^{(\mu+1)}(x) v'(x) \big\}\Big|_{x=-1+} \\
&= \cdots = \frac{\Gamma(\mu+1/2)}{2^k\Gamma(\mu+k+1/2)} \int_{-1}^{1} {}^r\mathcal{G}_n^{(\mu+k)}(x)\, \mathrm{d}v^{(k)}(x) \\
&\quad + \sum_{j=1}^{k} \frac{\Gamma(\mu+1/2)}{2^j\Gamma(\mu+j+1/2)} \big\{ {}^r\mathcal{G}_n^{(\mu+j)}(x) v^{(j)}(x) \big\}\Big|_{x=-1+}.
\end{aligned}$$

Main results: endpoint singularities

Theorem For some $m, k \in \mathbb{N}_0$, $s \in (0,1)$ and $\mu := m + s$, assume that $u \in \mathbb{W}^{\mu}_{-1+}(\Omega)$, $v \in \mathbb{W}^{k+1}(\Omega)$ with $v(x) = I^{1-s}_{-1+} u^{(m)}(x)$. Then for $\mu > 1/2$ and $n \geq \mu + k + 1$, we have

$$\hat{u}_n^C = \frac{1}{\sqrt{\pi}\, 2^{\mu+k-1}\Gamma(\mu + k + 1/2)} \int_{-1}^{1} {}^r\mathcal{G}_n^{(\mu+k)}(x)\, \mathrm{d}v^{(k)}(x) + \sum_{j=0}^{k} (-1)^{n+j} \widehat{C}_{n,\mu+j} \sin(\mu\pi) v^{(j)}(-1+),$$

where ${}^r\mathcal{G}_n^{(\mu)}(x) := \left(1 - x^2\right)^{\mu-1/2}\, {}^rG^{(\mu)}_{n-\mu}(x)$, and

$$\widehat{C}_{n,\rho} := \frac{2^{\rho}\Gamma(\rho - 1/2)}{\pi^{3/2}} \frac{\Gamma(n - \rho + 1)}{\Gamma(n + \rho)}.$$

Moreover, we have the following bound:

$$|\hat{u}_n^C| \leq \left\{ \frac{1}{2^{\mu+k-1}\pi} \frac{\Gamma((n - \mu - k + 1)/2)}{\Gamma((n + \mu + k + 1)/2)} + \frac{2^{\mu}\Gamma(\mu - 1/2)}{\pi^{3/2}} \frac{n\Gamma(n - \mu + 1)}{(n - \mu - k)\Gamma(n + \mu)} \right\} U^{\mu,k}_{-1+}.$$

Fractional space: endpoint singularities

We say u is of $\mathbb{W}^k(\Omega)$ if $u \in \mathrm{AC}^{k-1}(\bar{\Omega})$ and $u^{(k-1)} \in \mathrm{BV}(\bar{\Omega})$ with integral $k \geq 1$. Also, we denote $\mathbb{W}^0(\Omega) := L^1(\Omega)$. Equipped with the norm:

$$\|u\|_{\mathbb{W}^k(\bar{\Omega})} := \|u\|_{\mathrm{AC}^{k-1}(\bar{\Omega})} + V_{\bar{\Omega}}[u], \quad k \geq 1; \quad \|u\|_{\mathbb{W}^0(\bar{\Omega})} = \|u\|_{L^1(\Omega)}.$$

To deal with endpoint singularities, we define the fractional spaces as follows: for $m \in \mathbb{N}_0$ and $s \in (0,1)$, $\mu := m + s$

$$\mathbb{W}^{\mu}_{a+}(\Omega) : = \left\{u \in \mathrm{AC}^m(\bar{\Omega}),\ I^{1-s}_{a+} u^{(m)} \in \mathrm{BV}(\bar{\Omega})\right\},$$
$$\mathbb{W}^{\mu}_{b-}(\Omega) : = \left\{u \in \mathrm{AC}^m(\bar{\Omega}),\ I^{1-s}_{b-} u^{(m)} \in \mathrm{BV}(\bar{\Omega})\right\}.$$

For some $m, k \in \mathbb{N}_0$, $s \in (0,1)$ and $\mu := m + s$, assume that $u \in \mathbb{W}^{\mu}_{a+}(\Omega)$, $v \in \mathbb{W}^{k+1}(\Omega)$ with $v(x) = I^{1-s}_{a+} u^{(m)}(x)$. Also, we say u is of $\mathbb{W}^{\mu,k}_{a+}(\Omega)$. Accordingly, we denote

$$U^{\mu,k}_{a+} := V_{\bar{\Omega}}\left[v^{(k)}\right] + |\sin(s\pi)| \sum_{j=0}^{k} \left|v^{(j)}(a+)\right|,$$

and

$$U^{\mu,k}_{b-} := V_{\bar{\Omega}}\left[v^{(k)}\right] + |\sin(s\pi)| \sum_{j=0}^{k} \left|v^{(j)}(b-)\right|.$$

Fractional space: endpoint singularities

For $k \in \mathbb{N}_0$, $\mathrm{AC}^{k+1}(\bar{\Omega}) \subseteq \mathbb{W}^{k+1}(\Omega) \subseteq \mathrm{AC}^{k}(\bar{\Omega}) \subseteq \mathbb{W}^{k}(\Omega)$.
If $k = 0$, we have that $\mathrm{AC}^{0}(\bar{\Omega}) = \mathbb{W}^{0}(\Omega) = L^1(\Omega)$.
If $k = 1$, we have that $\mathrm{AC}^{1}(\bar{\Omega}) = \mathrm{AC}(\bar{\Omega})$ and $\mathbb{W}^{1}(\Omega) = \mathrm{BV}(\bar{\Omega})$.
If $k \to \infty$, we have that $\mathrm{AC}^{\infty}(\bar{\Omega}) = \mathbb{W}^{\infty}(\Omega) = C^{\infty}(\Omega)$.
[2]If $u \in \mathrm{BV}(\bar{\Omega})$ and $\rho > 0$, then $I^{\rho}_{a+}u(x) \in \mathrm{BV}(\bar{\Omega})$ and $I^{\rho}_{b-}u(x) \in \mathrm{BV}(\bar{\Omega})$.
For $0 < \mu_1 \leq \mu_2$, we have

$$\mathbb{W}^{\mu_1}_{a+}(\Omega) \supseteq \mathbb{W}^{\mu_2}_{a+}(\Omega), \quad \mathbb{W}^{\mu_1}_{b-}(\Omega) \supseteq \mathbb{W}^{\mu_2}_{b-}(\Omega).$$

If $k = 0$, $\mathbb{W}^{\mu,0}_{a+}(\Omega) = \mathbb{W}^{\mu}_{a+}(\Omega)$ and $\mathbb{W}^{\mu,0}_{b-}(\Omega) = \mathbb{W}^{\mu}_{b-}(\Omega)$.
If $k_1, k_2 \in \mathbb{N}_0$ and $k_1 \leq k_2$, we have $\mathbb{W}^{\mu}_{a+}(\Omega) \supseteq \mathbb{W}^{\mu,k_1}_{a+}(\Omega) \supseteq \mathbb{W}^{\mu,k_2}_{a+}(\Omega)$ and $\mathbb{W}^{\mu}_{b-}(\Omega) \supseteq \mathbb{W}^{\mu,k_1}_{b-}(\Omega) \supseteq \mathbb{W}^{\mu,k_2}_{b-}(\Omega)$.

[2] Liang, Y.: Box dimensions of Riemann-Liouville fractional integrals of continuous functions of bounded variation. Nonlinear Anal. 72(11), 4304–4306 (2010).

Main results: L^∞- and L^2_ω-estimates

Theorem For some $m, k \in \mathbb{N}_0$, $s \in (0,1)$ and $\mu := m + s$, assume that $u \in \mathbb{W}^{\mu}_{-1+}(\Omega)$, $v \in \mathbb{W}^{k+1}(\Omega)$ with $v(x) = I^{1-s}_{-1+} u^{(m)}(x)$. Then we have that for $1 < \mu \le \mu + k \le N$,

$$\|u - \pi^C_N u\|_{L^\infty(\Omega)} \le \Big\{ \frac{1}{2^{\mu+k-2}(\mu+k-1)\pi} \frac{\Gamma((N-\mu-k)/2+1)}{\Gamma((N+\mu+k)/2)} + \frac{N+1}{N-\mu-k+1} \frac{2^{\mu-1}\Gamma(\mu-1/2)}{\pi^{3/2}(\mu-1)} \frac{\Gamma(N-\mu+2)}{\Gamma(N+\mu)} \Big\} U^{\mu,k}_{-1+},$$

and

$$\big\|u - \pi^C_N u\big\|_{L^2_\omega(\Omega)} \le \Big\{ \frac{4}{(2\mu+2k-1)\pi} \frac{\Gamma(N-\mu-k+1)}{\Gamma(N+\mu+k)} + \Big(\frac{N+1}{N-\mu-k+1}\Big)^2 \frac{2^{6\mu-4}\Gamma^2(\mu-1/2)}{\pi^2(\mu-1)(4N+3)} \times \frac{\Gamma(2N-2\mu+3)}{\Gamma(2N+2\mu-1)} \Big\}^{1/2} U^{\mu,k}_{-1+}.$$

Approximation of typical singular functions

- **Type-II**: $u(x) = (x+1)^\alpha g(x)$ with $\alpha \geq 0$. Then we have $u \in \mathbb{W}_{-1}^{\alpha+1}(\Omega)$, and $I_{-1+}^{1-s} u^{(m)}(x) \in C^\infty(\Omega), \alpha + 1 = \mu$

$$|\hat{u}_n^C| = \mathcal{O}(n^{-2\alpha-1}), \quad \|u - \pi_N^C u\|_\infty = \mathcal{O}(N^{-2\alpha}),$$
$$\|u - \pi_N^C u\|_{L^2_\omega(\Omega)} = \mathcal{O}(N^{-2\alpha-1/2}).$$

Table 3: Order of $\|u - \pi_N^C u\|_{L^\infty}$ with $u = (1+x)^\alpha \sin x$.

N	$\alpha = 0.1$	order	$\alpha = 1.2$	order	$\alpha = 2.3$	order
2^3	4.63e-01	–	1.04e-03	–	7.27e-05	–
2^4	4.05e-01	0.19	2.06e-04	2.34	3.08e-06	4.56
2^5	3.54e-01	0.20	4.02e-05	2.36	1.32e-07	4.54
2^6	3.09e-01	0.20	7.74e-06	2.38	5.60e-09	4.56
2^7	2.69e-01	0.20	1.48e-06	2.39	2.35e-10	4.58
2^8	2.35e-01	0.19	2.81e-07	2.39	9.76e-12	4.59
2^9	2.07e-01	0.19	5.35e-08	2.40	4.04e-13	4.59
Pred.		0.20		2.40		4.60

Approximation of typical singular functions

Table 4: Order of $\|u - \pi_N^C u\|_{L^2_\omega}$ with $u = (1+x)^\alpha \sin x$.

N	$\alpha = 0.1$	order	$\alpha = 1.2$	order	$\alpha = 2.3$	order
2^3	3.36e-02	–	4.49e-04	–	4.58e-05	–
2^4	2.11e-02	0.68	6.35e-05	2.82	1.38e-06	5.06
2^5	1.31e-02	0.69	8.81e-06	2.85	4.19e-08	5.04
2^6	8.04e-03	0.70	1.20e-06	2.87	1.26e-09	5.06
2^7	4.95e-03	0.70	1.63e-07	2.88	3.74e-11	5.07
2^8	3.09e-03	0.68	2.19e-08	2.89	1.10e-12	5.09
2^9	2.04e-03	0.60	2.95e-09	2.90	3.22e-14	5.09
Pred.		0.70		2.90		5.10

Sharpness of the bounds

Consider $u(x) = (1+x)^\alpha$ on $\bar{\Omega} = [-1, 1]$ with $\alpha > 0$ and integer $N > \alpha - 1$, then we have

$$\|u - \pi_N^C u\|_{L^\infty(\Omega)} = \widetilde{M}_\infty^\alpha,$$

where

$$\widetilde{M}_\infty^\alpha := C_\alpha \frac{\Gamma(N-\alpha+1)}{\Gamma(N+\alpha+1)}, \quad C_\alpha := \frac{|\sin(\pi\alpha)|\Gamma(2\alpha)}{2^{\alpha-1}\pi}.$$

$$\|u - \pi_N^C u\|_{L^2_\omega(\Omega)} = \widetilde{M}_2^\alpha = \Big(\frac{\widetilde{C}_\alpha}{N+\alpha+1} \frac{\Gamma(2N-2\alpha+1)}{\Gamma(2N+2\alpha+1)}\Big)^{1/2},$$

where

$$\widetilde{C}_\alpha := \frac{2^{2\alpha+2}\alpha \sin^2(\pi\alpha)\Gamma^2(2\alpha)}{\pi}.$$

Sharpness of the bounds

Consequently, for $u(x) = (1+x)^\alpha$ on $\bar{\Omega} = [-1, 1]$ with $\alpha > 0$ and integer $N > \alpha - 1$, we have $\sigma = \alpha + 1$, $k = [\sigma]$,

$$U^{\mu,k}_{-1+} = |\sin(\pi\alpha)|\Gamma(\alpha+1),$$

thus, we obtain the right bound

$$M^{\alpha,k}_{\infty} = \left\{ \frac{1}{2^{\alpha+k-1}(\alpha+k)\pi} \frac{\Gamma((N-\alpha-k+1)/2)}{\Gamma((N+\alpha+k+1)/2)} + \frac{N+1}{N-\alpha-k} \frac{2^\alpha \Gamma(\alpha+1/2)}{\pi^{3/2}\alpha} \frac{\Gamma(N-\alpha+1)}{\Gamma(N+\alpha+1)} \right\} |\sin(\pi\alpha)|\Gamma(\alpha+1).$$

We get

$$R_{L^\infty} := \lim_{N\to\infty} \frac{\widetilde{M}^{\alpha}_{\infty}}{M^{\alpha,k}_{\infty}} = 1.$$

Sharpness of the bounds

Consequently, for $u(x) = (1+x)^\alpha$ on $\bar{\Omega} = [-1, 1]$ with $\alpha > 0$ and integer $N > \alpha - 1$, we have $\sigma = \alpha + 1$, $k = [\sigma]$,

$$U^{\mu,k}_{-1+} = |\sin(\pi\alpha)|\Gamma(\alpha+1),$$

thus, we obtain the right bound

$$M^{\alpha,k}_2 = \left\{ \frac{4}{(2\alpha+2k+1)\pi} \frac{\Gamma(N-\alpha-k)}{\Gamma(N+\alpha+k+1)} \right.$$
$$\left. + \left(\frac{N+1}{N-\alpha-k} \right)^2 \frac{2^{6\alpha+2}\Gamma^2(\alpha+1/2)}{\pi^2\alpha(4N+3)} \frac{\Gamma(2N-2\alpha+1)}{\Gamma(2N+2\alpha+1)} \right\}^{1/2} |\sin(\pi\alpha)|\Gamma(\alpha+1).$$

We get

$$R_{L^2_\omega} := \lim_{N\to\infty} \frac{\widetilde{M}^\alpha_2}{M^{\alpha,k}_2} = 1.$$

Remark

Applying the similar technique of the separation of singularities in Tuan and Elliott [3], the above results can be extended to the general functions with interior and endpoint singularities for

$$f(x) = g(x) \prod_{i=1}^{m} |x - a_i|^{\beta_i} = \sum_{i=1}^{m} g_i(x) |x - a_i|^{\beta_i},$$

where $-1 \leq a_1 < a_2 < \cdots < a_m \leq 1$, $g(x),\ g_i(x) \in C^\infty[-1,1]$, and $\beta_i \geq 0$.
The above results can be extended to Legendre, Gegenbauer and Jacobi polynomial approximations.

[3] P.D. Tuan and D. Elliott. Coefficients in series expansions for certain classes of functions. Math. Comp., 26:213–232, 1972.

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Thanks for your attention!