Unique determination of fractional order in the TFDE using one observation

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Contents

Introduction

- Motivation: To study inverse coefficient problems in PDEs by using limited measurable data.
- For inverse fractional order problems in TFDEs, Can we determine the fractional orders by finite observations of the solution?
- This is regarded as an open problem by Yamamoto M and Li ZY.
- For example, the TFDE is given as

$$\partial_t^\alpha u + Au = 0, \tag{1.1}$$

  where A denotes elliptic operator, and ∂_t^α denotes Caputo's derivative of the order $\alpha \in (0, 1)$.
- To determine the order only using the measurement at a single space-time point–This is the topic of this talk.

Introduction

<u>反问题</u>：基于数学、物理模型,从可以测量到的信息中提取不可测量的信息.

- 计算机层析成像（CT技术）；
- 核磁共振（MRI技术）；
- 生命科学（离子通道识别，蛋白质结构测定）；
- 地球内部结构探测（地质勘察、石油勘探）；
- 气候气象科学（数值天气预报）；
- 装备制造业（工业设计）,等.

Introduction

从实际生活看:

- “盲人听鼓”;
- 管中窥豹;
- 看病就医;
- 警察破案,侦探推理;
- 挑选西瓜–有经验的人通过拍打瓜皮发出的声音,就可知道瓜瓤长得怎样.这是因为当物体的材料确定后,它的音调高低与其形状密切相关.

Introduction

从数学上看:

- 由$ax = b$, 求$x = a^{-1}b$;
- 由$Ax = b$, 求$x = A^{-1}b$;
- 由数据(x, y), 求经验公式$y = f(x)$;
- 若给定微分方程模型$\mathcal{L}(D)u = f$及适当的初边值条件,可形成定解问题（称为正问题）;
- 若模型系数$D = D(x)$未知（难以测量或不可测量）,则需要确定参数D并求解u.
- 这时,额外增加关于解的部分信息(可测量数据),就形成所谓的数学物理反问题.

Introduction

Inverse problems related with TFDE:

- IHCP with order $\frac{1}{2}$ Murio [CMA 2007]
- Backward problems Liu-Yamamoto [AA 2010]; Wang-Wei-Zhou [AMM 2013], Wei-Wang [ESAIM 2014]; Ren-Xu-Lu [JIIP 2014]; Sun-Li-Jia [AAMM 2017]
- Unique continuation Zhang-Xu [IP 2011], Xu-Cheng-Yamamoto [AA 2011], Yamamoto-Zhang [IP 2012]; Cheng-Lin-Nakamura [JDE 2013]; Jiang-Li-Yamamoto [IP 2016]; Lin-Nakamura [Commun PDE 2016], Lin-Nakamura [Math Annalen 2018]; Li-Liu-Yamamoto [IPI 2022]

Introduction

- Inverse coefficient problems Jin-Rundell [IP 2012]; Miller-Yamamoto [IP 2013]; Sun-Wei [ANM 2017]; Sun-Yan-Wei [JCAM 2019]; Li-Luchko-Yamamoto [CMA 2016]; Rundell-Zhang [IP 2017]; Kian-Oksanen-Soccorsi-Yamamoto [JDE 2018]; Li-Fujishiro-Li [JCAM 2020]
- Inverse source problems, for linear source Wei-Chen-Sun [IPSE 2010]; Chi-Li [CMA 2011]; Wang-Zhou-Wei [ANM 2013], Wei-Li-Li [IP 2016], Wei-Sun-Li [AML 2017]; Liu-Rundell-Yamamoto [FCAA 2016]; Sun-Liu [IP 2020]. For nonlinear source Luchko-Rundell-Yamamoto [IP 2013]

Simultaneous inverse problems

- Cheng-Nakagawa-Yamamoto-Yamazaki [IP 2009], Bondarenko-Ivaschenko [JIIP 2009], Li-Zhang-Jia-Yamamoto [IP 2013] for diffusion coefficient and fractional order in time-FDE;
- Li-Yamamoto [AA 2015], Li-Imanuvilov-Yamamoto [IP 2016] for fractional orders and model coefficients in multi-term time-FDE;
- Ruan-Zhang-Wang [AMC 2018] for fractional order and space-source in TFDE;
- Jing-Peng [AML 2020] for fractional-order, potential and Robin coefficient in TFDE...

- Machine Learning, Bayesian Inversion

- Xu-Yang-Sun [Sci China Math 2017];
- E Wei-Nan [ArXiv 2020];
- Bao-Ye-Zang-Zhou [IP 2020];

- Zhang-Jia-Yan [IP 2018] for source and fractional orders in space-time FDE; Fan-Jiang-Chen [CMA 2016] for multiple parameters in time-FDE;
- Li-Schwab-Antholzer-Haltmeier [IP 2020]; Kamyab-Azimifar-Sabzi [PeerJ Computer Sci 2022]...

Introduction

- Inverse fractional order problems
- Cheng J et al. [IP, 2009], Tatar and Ulusoy [EJDE, 2013], Li GS et al. [IP 2013], Li ZY and Yamamoto [AA 2015], Chen-Liu-Jiang [SIAM J Math Anal 2016];
- Li-Cheng-Li [J Math Phys 2019]; Zheng-Cheng-Wang [IP 2019];
- Li-Liu-Yamamoto [2019]...

Introduction

Recently see

- Li ZY et al. [JCAM 2020]
- Jin and Kian [arXiv 2021]
- Sun LL et al. [IP 2021]
- Yamamoto [IP 2021]
- Kian-Li-Liu-Yamamoto [Math Ann 2021]...
- Alimor and Ashurov [JIIP 2020, arXiv 2021]
- Ashurov and Umarov [FCAA 2020, FCAA 2022, FCAA 2023]

Introduction

- It is noted that in the existing work on inverse fractional order problems, most of them were studied by using
- Subdomain measurements, or one-point measurements at $t \in (0, T)$;
- Subboundary data also at $t \in (0, T)$ for arbitrary given $T > 0$.
- For equation (1.1), we are to determine the fractional order uniquely only using the measurement at one space-time point.

Introduction

- It is difficult to give an answer in theory for the above question, but the situation could be changed if having suitable conditions.
- By the eigenfunction expansion method, the solution of the forward problem is expressed by the Mittag-Leffler function, and the inverse problem is transformed to a nonlinear algebraic equation.
- By choosing the initial values and the measured time with suitable model parameters, the nonlinear equation can be solved uniquely by the monotonicity of the nonlinear function on the fractional orders.
- The key points are the Mittag-Leffler function, the Gamma function and their properties.

Mittag-Leffler function and its property

- The Gamma function:

$$\Gamma(z) = \int_0^\infty x^z e^{-x}\, dx, \tag{2.1}$$

where $\Re(z) > 0$. There hold the formula $\Gamma(z+1) = z\Gamma(z)$, and the Stirling's approximate formula:

$$\Gamma(z+1) \sim \sqrt{2\pi z}\, e^{-z}\, z^z,\ z \to +\infty, \tag{2.2}$$

and the derivative's formula:

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+z}), \tag{2.3}$$

where $\gamma = 0.5772157\ldots$ is the Euler constant.

Mittag-Leffler function and its property

- On the Gamma function, we have the following assertion.

  Lemma 1 For the parameter $\alpha \in (0,1)$, there holds

$$\lim_{j\to\infty} \frac{\Gamma(\alpha j)}{\Gamma(\alpha j + \alpha)} = 0. \tag{2.4}$$

- Following Mittag-Leffler's classical definition, the one-parametric Mittag-Leffler function:

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \ z \in \mathcal{C}, \ \alpha > 0. \tag{2.5}$$

Mittag-Leffler function and its property

- Obviously there is $E_1(z) = e^z$ as $\alpha = 1$, i.e., $E_\alpha(z)$ is a generalization of e^z. In general there holds basic estimate:

$$|E_\alpha(z)| \leq \frac{c}{1 + |z|}, \tag{2.6}$$

  where $\alpha > 0$, and $c > 0$ is a constant.

- For researches on Mittag-Leffler functions, see the monograph by Gorenflo R, Kilbas A A, Mainardi F, et al. (2020).

Mittag-Leffler function and its property

- **Lemma 2** For $\alpha \in (0,1)$ and $j = 1, 2, \cdots$, denote

$$\gamma_j = \gamma + \frac{1}{\alpha j + 1} - \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n + \alpha j + 1}\right). \tag{2.7}$$

- There hold

$$\lim_{j \to \infty} \frac{\gamma_{j+1}}{\gamma_j} = 1, \tag{2.8}$$

and $\gamma_j < 0$ for almost $j > 1$, and $\lim_{j\to\infty} \gamma_j = O(\ln(j))$.

- The series γ_j and $\frac{\gamma_{j+1}}{\gamma_j}$, $j = 1, 2, \cdots$, with $\alpha \in (0, 1)$ are plotted in Figure 1, respectively.

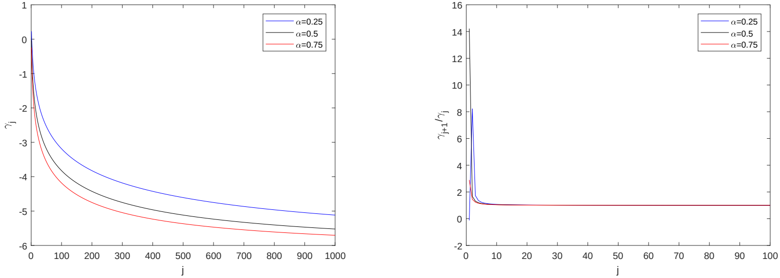

Figure 1. The pictures of γ_j and γ_{j+1}/γ_j

Mittag-Leffler function and its property

- **Theorem 1** Let $g(\alpha) = E_\alpha(-ct^\alpha)$ for $\alpha \in (0,1)$. The function $g(\alpha)$ is differentiable on $\alpha \in (0,1)$, and there holds

$$g'(\alpha) = \sum_{j=1}^{\infty} (-1)^j \, c^j \, j \, t^{\alpha j} \frac{\ln(t) + \gamma_j}{\Gamma(\alpha j + 1)}, \tag{2.9}$$

  and $g'(\alpha) < 0$ if $t > 0$ is suitably large and $c > 0$ is small, where γ_j is given by (2.7).

- **Proof idea** 1) The convergence of the series (2.9);
  2) The sign of the finite sum.

- The pictures of $g'(\alpha)$ with $c = 0.01$ and $c = 0.1$ and $t = 20, 50, 100, 200$ are plotted in Figure 2, respectively.

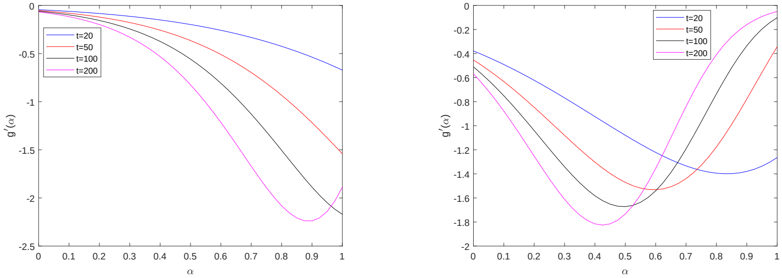

Figure 2. The pictures of $g'(\alpha)$

3.1 The forward problem

- Consider the homogeneous TFDE

$$\partial_t^\alpha u - D\Delta u = 0, (x,t) \in \Omega_T, \tag{3.1}$$

where $\Omega_T = \Omega \times (0,T)$ and $\Omega \subset \mathbf{R}^d$ ($d = 1,2,3$) is an open bounded domain with smooth boundary $\partial\Omega$.

- For the equation (3.1), given the homogeneous boundary condition

$$u|_{\partial\Omega} = 0, \ x \in \partial\Omega, 0 < t \leq T, \tag{3.2}$$

and the initial distribution

$$u(x,0) = f(x), \ x \in \Omega, \tag{3.3}$$

where $f(x) \in L^2(\Omega)$.

The inverse fractional order problem

- By using the eigenfunction expansion method, there exists a unique solution:

$$u(x,t) \in C([0,T], L^2(\Omega)) \cap C((0,T], H^2(\Omega) \cap H_0^1(\Omega)), \tag{3.4}$$

which can be expressed by the Mittag-Lellfer function given as

$$u(x,t) = \sum_{n=1}^{\infty} f_n \, E_\alpha(-D\lambda_n t^\alpha) \, \varphi_n(x), \tag{3.5}$$

- where $f_n = (f, \varphi_n)$, and λ_n and $\varphi_n(x)$ compose the eigensystem of the Laplace operator, which satisfy the condition $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and $\Delta\varphi_n = \lambda_n\varphi_n$ for $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$, and $E_\alpha(\cdot)$ is given by (2.5).

The inverse fractional order problem

- The asymptotic property of the eigenfunctions
- Suppose that the eigenfunctions are normalized by L^2 norm, i.e., $\|\varphi_n(x)\|_{L^2} = 1$. By the property of $\lambda_n \sim n^{\frac{2}{d}}$, where d is dimension of the space domain, there holds for some $\mu \in (0, 1)$:

$$\|\varphi_n\|_{H^{2\mu}} \sim \lambda_n^\mu \|\varphi_n\|_{L^2} = \lambda_n^\mu \sim n^{\frac{2\mu}{d}}. \tag{3.6}$$

- By Sobolev embedding theorem, there is $H^{2\mu} \subset L^\infty$, and then we get for some $C > 0$

$$\|\varphi_n\|_{L^\infty} \leq C\|\varphi_n\|_{H^{2\mu}} \sim n^{\frac{2\mu}{d}}, \tag{3.7}$$

which shows that the eigenfunctions can grow with their numbers.

The inverse fractional order problem

3.2 The inverse problem

- Let $x_0 \in \Omega$ be fixed, and we have the measured data at a time $t_1 > 0$ given as
$$u(x_0, t_1) := m. \tag{3.8}$$
- The inverse problem is to identify the order $\alpha \in (0, 1)$ by (3.8) based on the forward problem (3.1)-(3.3).
- Noting the solution's expression (3.5), we get a nonlinear algebraic equation:
$$\sum_{n=1}^{\infty} f_n \, E_\alpha(-D\lambda_n t_1^\alpha) \, \varphi_n(x_0) = m. \tag{3.9}$$
- As a result the inverse order problem is transformed to solving of the nonlinear equation (3.9).

The inverse fractional order problem

3.3 Unique solvability

- By (3.9) we set

$$F(\alpha) := \sum_{n=1}^{\infty} f_n E_\alpha(-D\lambda_n t_1^\alpha)\, \varphi_n(x_0), \tag{3.10}$$

  where $x_0 \in \Omega$ is the measured point and $t_1 > 0$ is the measured time, and we need to solve the equation $F(\alpha) = m$.
- We are to show $F(\alpha)$ is monotonic on $\alpha \in (0, 1)$ under suitable conditions for the initial function, the diffusion coefficient and the measured point and time.

The inverse fractional order problem

- By the extremum principle of the time fractional diffusion equation, the solution of the forward problem takes nonnegative values if the initial function $f(x)$ is nonnegative for $x \in \Omega$.
- However, it is not sufficient to guarantee the monotonicity of the function $F(\alpha)$, we need more strong conditions for the initial value function as well as the measured point and the model parameters.
- We give the uniqueness result.

The inverse fractional order problem

- **Theorem 2** Denote $I = (0, 1)$. Assume that
- $D > 0$ is small enough and the measured time $t_1 > 0$ is suitably large;
- There exists a positive integer $N > 1$ and a measured point $x_0 \in \Omega$ such that $f_n \geq 0$ and $\varphi_n(x_0) \geq 0$, $n = 1, 2, \cdots, N$, and for some $n_0 \in \{1, 2, \cdots, N\}$, $f_{n_0} > 0$ and $\varphi_{n_0}(x_0) > 0$.
- Then the nonlinear function $F(\alpha)$ is strictly monotonic decreasing on $\alpha \in I$, and the inverse order problem is of uniqueness.

The inverse fractional order problem

- **Proof idea**

1) Convergence of $\sum_{n=1}^{\infty} f_n \varphi_n(x_0)\ E_\alpha(-D\lambda_n t^\alpha)$;

2) Assume that $\alpha_1, \alpha_2 \in I$ and $\alpha_1 < \alpha_2$, there holds

$$F(\alpha_1) - F(\alpha_2) = \sum_{n=1}^{\infty} f_n \varphi_n(x_0)\ [E_{\alpha_1}(-D\lambda_n t_1^{\alpha_1}) - E_{\alpha_2}(-D\lambda_n t_1^{\alpha_2})],$$

and rewrite as

$$\begin{aligned} F(\alpha_1) - F(\alpha_2) &= \sum_{n=1}^{N} f_n \varphi_n(x_0)\ [E_{\alpha_1}(-D\lambda_n t_1^{\alpha_1}) - E_{\alpha_2}(-D\lambda_n t_1^{\alpha_2})] \\ &+ \varepsilon_0 := G(\alpha_1, \alpha_2) + \varepsilon_0; \end{aligned}$$

3) $G(\alpha_1, \alpha_2) > 0 \Longrightarrow F(\alpha_1) - F(\alpha_2) > 0.$

3.4 Numerical experiments

- Based on the series expression of $F(\alpha)$ by (3.10), it is transformed to solve a disturbed equation from numerics, here the disturbed equation is given as

$$F_N(\alpha) = m^\delta, \tag{3.11}$$

where

$$F_N(\alpha) = \sum_{n=1}^{N} f_n \varphi_n(x_0)\, E_\alpha(-D\lambda_n t_1^\alpha), \tag{3.12}$$

and d^δ is the noisy data which is expressed by

$$m^\delta = m(1 + \delta\theta), \tag{3.13}$$

where $\delta > 0$ is noise level, and $\theta \in [-1, 1]$ is random number.

The inverse fractional order problem

- **Example 1.** In this example we set $D = 0.1$, and the initial function $f(x) = \frac{1}{2}\sin(2x)$ for $x \in [0, \pi]$.
- We choose $x_0 = \frac{\pi}{4}$ as the measured point, and $t_1 = 10$ as the measured time, i.e., the additional measurement is $m = u(\frac{\pi}{4}, 10)$.
- Noting $f_n = \begin{cases} \frac{1}{2}, & n = 2, \\ 0, & n \neq 2, \end{cases}$ and $\lambda_2 = 4$, and $\sin(2x_0) = 1$, we have by (3.12)

$$F(\alpha) = \frac{1}{2} E_\alpha\left(-\frac{2^{1+\alpha}}{5}\right), \ 0 < \alpha < 1. \tag{3.14}$$

The inverse fractional order problem

- Let the exact order be $\alpha^{exa} = 0.25$, by which we get the additional data $m = 0.275482$ by solving the forward problem.
- To see the unique solvability, we plot the functions $F(\alpha)$ and d on $\alpha \in [0, 1]$ in Fig.3, where $F(0)$ and $F(1)$ are defined by

$$F(0) = \frac{1}{2}\sum_{k=0}^{\infty}(-\frac{2}{5})^k, \ F(1) = \frac{1}{2}e^{-4}.$$

- It can be seen clearly that the function $F(\alpha)$ is strictly monotone on $\alpha \in [0, 1]$, and the inverse order problem is of uniqueness.

The inverse fractional order problem

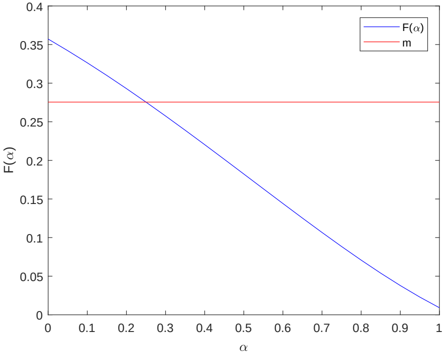

Fig.3 The functions $F(\alpha)$ and m in Ex.1.

The inverse fractional order problem

- **Example 2.** In this example, we take the initial function be

$$f(x) = \sin(x) + 2\sin(2x) + \frac{1}{4}\sin(4x), \ x \in [0, \pi], \tag{3.15}$$

- The solution of the forward problem is given by

$$\begin{aligned} u(\alpha)(x,t) &= E_\alpha(-Dt^\alpha)\sin(x) + 2E_\alpha(-4Dt^\alpha)\sin(2x) \\ &+ \tfrac{1}{4}E_\alpha(-16Dt^\alpha)\sin(4x). \end{aligned} \tag{3.16}$$

- Let the exact fractional order be $\alpha^{exa} = 0.75$.
- Let $D = 0.01$, $t_1 = 100$, and choose the measured point $x_0 = \frac{3}{8}\pi$, and the additional data $m = 1.1376$. The functions $F(\alpha)$ and m are plotted in Fig.4.

The inverse fractional order problem

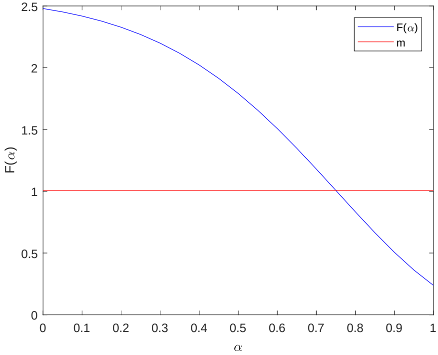

Fig.4 The functions $F(\alpha)$ and m in Ex.2.

The inverse fractional order problem

- **Example 3.** In this example, let the initial function be

$$f(x) = 1 + x,\ x \in [0, \pi], \tag{3.17}$$

and the expansive coefficients in the eigenfunction space are computed by

$$f_n = \frac{2}{\pi}\int_0^{\pi}(1+x)\sin(nx)dx = \begin{cases} \frac{4}{n\pi} + \frac{2}{n}, & n = 1, 3, 5, \cdots, \\ -\frac{2}{n}, & n = 2, 4, 6, \cdots. \end{cases} \tag{3.18}$$

The inverse fractional order problem

- Let the exact fractional order be $\alpha^{exa} = 0.45$, and the measured point be $x_0 = \frac{\pi}{N+1}$ such that $\sin(nx_0) > 0$ for $n = 1, 2, \cdots, N$.
- On the concrete computations, we choose $N = 30$, $D = 1e - 4$ and $t_1 = 50$, and the additional data $m = 1.17203$.
- The functions $F(\alpha)$ and d are plotted in Fig.5.

The inverse fractional order problem

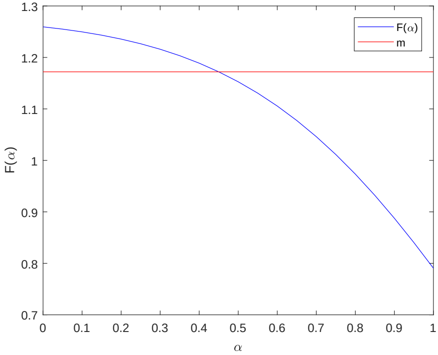

Fig.5 The functions $F(\alpha)$ and d in Ex.3.

The inverse fractional order problem

Reference

Gongsheng Li, Zhen Wang, Xianzheng Jia, Yi Zhang

An inverse problem of determining the fractional order in the TFDE using the measurement at one space-time point. *Fractional Calculus and Applied Analysis*, 2023, 26(4): 1770-1785.

谢　谢！