Analysis and Hermite spectral approximation of diffusive-viscous wave equations in unbounded domains arising in geophysics

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Outline

Outline

exploration seismology

Modeling of seismic wave

- Acoustic wave theory
- Elastic wave theory
- Biot's theory (Biot 1956, 1962)
- Diffusive-viscous wave theory (Knorneev, 2004)

Existing research works on DVWEs

- Interlayer-flow model using finite difference method (Quintal et al. 2007)
- Seismic data-driven geological model (Chen et al. 2013)
- Finite difference method (Zhao et al. 2014, 2018)
- Finite volume method (Mensah et al. 2019)
- Well-posedness of a general initial-boundary value problem (Han et al. 2020)
- Finite element method (Han et al. 2021, Zhao et al. 2022)
- Local discontinuous Galerkin method (Ling et al. 2023)

Motivation

- A main aspect for the DVWEs is the non-reflection boundary conditions.
- Numerical simulations performed in bounded domains may generate artificial reflections due to the truncation of the model.
- The perfectly matched layer boundary condition can be used to absorb the artificial reflections but the implementation is more complex and more expensive.
- The Hermite spectral method is a good choice for unbounded domain problems and it enjoys high accuracy if the solution is smooth enough.

Our work

① Consider the DVWE without the truncation of the domain

② Establish the existence and uniqueness of the weak solution

③ Discuss the regularity of the solutions in terms of the initial conditions and the source term

④ Develop an efficient Hermite spectral Galerkin method and build its stability and error estimate

Outline

The governing equation

The diffusive-viscous wave equation in unbounded domain is described as follows:

$$\partial_t^2 u + \alpha \partial_t u - \partial_t \mathrm{div}(\beta \nabla u) - \mathrm{div}(\gamma^2 \nabla u) = f, \quad \boldsymbol{x} \in \mathbb{R}^d, \quad t > 0 \tag{1}$$

subjecting to following initial conditions

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \partial_t u(\boldsymbol{x}, 0) = w_0(\boldsymbol{x}), \tag{2}$$

where d is the dimension in space, $u = u(\boldsymbol{x}, t)$ is the wave field, $\alpha = \alpha(\boldsymbol{x})$ and $\beta = \beta(\boldsymbol{x})$ are the diffusive and viscous attenuation parameters respectively, $\gamma = \gamma(\boldsymbol{x})$ is the wave propagation speed in the non-dispersive medium, $f = f(\boldsymbol{x}, t)$ is the source.

Weak formulation

Let $H^1(\mathbb{R}^d)$ be the usual Sobolev space. By using the integration by parts, we have the weak form of the problem (1):

For a.e. $t \in (0, T)$, find $u(t),\ \partial_t u(t) \in H^1(\mathbb{R}^d), \partial_t^2 u(t) \in H^{-1}(\mathbb{R}^d)$, such that

$$A(u, v) = (f, v), \quad \forall v \in H^1(\mathbb{R}^d) \tag{3}$$

with $u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}),\ \partial_t u(\boldsymbol{x}, 0) = w_0(\boldsymbol{x})$, where

$$A(u, v) := (\partial_t^2 u, v) + (\alpha \partial_t u, v) + (\beta \partial_t \nabla u, \nabla v) + (\gamma^2 \nabla u, \nabla v).$$

Well-posedness of the weak solution

Theorem (Ling & Mao, JSC, 2023)

Assume $u_0 \in H^1(\mathbb{R}^d),\ w_0 \in L^2(\mathbb{R}^d),\ f \in L^2(0,T;H^{-1}(\mathbb{R}^d))$, *then the weak problem* (3) *admits a unique solution* $u \in L^\infty(0,T;H^1(\mathbb{R}^d))$, $\partial_t u \in L^\infty(0,T;L^2(\mathbb{R}^d)) \cap L^2(0,T;H^1(\mathbb{R}^d))$, *and* $\partial_t^2 u \in L^2(0,T;H^{-1}(\mathbb{R}^d))$ *satisfying*

$$\begin{aligned}\|\partial_t u\|_{L^\infty(0,T;L^2(\mathbb{R}^d))} &+ \|\partial_t u\|_{L^2(0,T;H^1(\mathbb{R}^d))} \\ &+ \|u\|_{L^\infty(0,T;H^1(\mathbb{R}^d))} \le C(u_0, w_0, f).\end{aligned}$$

Regularity of the weak solution

Theorem (Ling & Mao, JSC, 2023)

Assume α, β, γ *are constants, let* u *be the solution of* (3), *if* $u_0 \in H^{k+2}(\mathbb{R}^d),\ w_0 \in H^{k+2}(\mathbb{R}^d), f \in H^1(0,T;H^k(\mathbb{R}^d)), k \geq 0$, *then*

$$
\begin{aligned}
&\|\partial_t^2 u\|_{L^\infty(0,T;H^k(\mathbb{R}^d))} + \|\partial_t u\|_{L^\infty(0,T;H^{k+1}(\mathbb{R}^d))} + \|u\|_{L^\infty(0,T;H^{k+2}(\mathbb{R}^d))} \\
&+ \|\partial_t^2 u\|_{L^2(0,T;H^{k+1}(\mathbb{R}^d))} + \|\partial_t u\|_{L^2(0,T;H^{k+2}(\mathbb{R}^d))} \\
&\leq C \left(\|f\|_{H^1(0,T;H^k(\mathbb{R}^d))} + \|w_0\|_{H^{k+2}(\mathbb{R}^d)} + \|u_0\|_{H^{k+2}(\mathbb{R}^d)} \right).
\end{aligned}
$$

Outline

Hermite polynomials

The orthonormal Hermite polynomials $\{H_n(x)\}$ are defined in $\mathbb{R}$ by the three-term recurrence relation:

$$H_0(x) = \pi^{-1/4}, \quad H_1(x) = \sqrt{2}\pi^{-1/4}x,$$
$$H_{n+1}(x) = x\sqrt{\frac{2}{n+1}}H_n(x) - \sqrt{\frac{n}{n+1}}H_{n-1}(x), \ n \geq 1.$$

Weighted orthogonality:

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)\omega(x)dx = \delta_{mn}, \qquad \omega(x) = \mathrm{e}^{-x^2}.$$

Denote $P_N(x)$ the space of the polynomials of degree at most N, i.e.

$$P_N(x) = \mathrm{span}\Big\{H_0(x), H_1(x), \cdots, H_N(x)\Big\}.$$

Hermite functions

Let us introduce the Hermite orthogonal functions

$$\phi_j(x) = \mathrm{e}^{-x^2/2} H_j(x),\ j = 0, 1, \ldots,$$

which form an orthogonal basis in $L^2(\mathbb{R})$, i.e.,

$$\int_{-\infty}^{\infty} \phi_m(x)\phi_n(x)dx = \delta_{mn}.$$

Let

$$\mathcal{P}_N(x) = \left\{\mathrm{e}^{-\frac{x^2}{2}} v \mid v \in P_N(x)\right\} = \mathsf{span}\left\{\phi_0(x), \phi_1(x), \cdots, \phi_N(x)\right\},$$

and denote V_N the d dimension tensor product of $\mathcal{P}_N$.

Hermite spectral Galerkin scheme

The Hermite spectral Galerkin approximation to (3) is to find $u_N(t)$, $\partial_t u_N(t) \in V_N$, such that for any $v \in V_N$

$$A(u_N, v) = (f, v), \quad \forall v \in V_N, \tag{4}$$

where $u_N(\boldsymbol{x}, 0) = \hat{\mathbf{\Pi}}_N u_0, \partial_t u_N(\boldsymbol{x}, 0) = \hat{\mathbf{\Pi}}_N w_0$ and V_N is the d dimension tensor product of $\mathcal{P}_N$, $\hat{\mathbf{\Pi}}_N$ is the L^2 projection defined as follows:

Projection

Recall the orthogonal projection $\mathbf{\Pi_N} : L^2_{\boldsymbol{\omega}}(\mathbb{R}^d) \to P_N^d$,

$$\int_{\mathbb{R}^d} (\mathbf{\Pi}_N u - u) v_N \boldsymbol{\omega}(\boldsymbol{x}) dx = 0, \quad \forall\, v_N \in P_N^d.$$

Note that for any $u \in L^2(\mathbb{R}^d)$, we have $u\boldsymbol{\omega}^{-1/2} \in L^2_{\boldsymbol{\omega}}(\mathbb{R}^d)$. Define

$$\hat{\mathbf{\Pi}}_N u := \boldsymbol{\omega}^{1/2} \mathbf{\Pi}_N (u\boldsymbol{\omega}^{-1/2}) \in V_N.$$

Then for $u \in L^2(\mathbb{R}^d)$, we derive immediately the following

$$\int_{\mathbb{R}^d} (\hat{\mathbf{\Pi}}_N u - u) v_N d\boldsymbol{x} = 0, \quad \forall\, v_N \in V_N.$$

Approximation theory

Define the operator $\hat{\partial}_{x_j} = \partial_{x_j} + x_j$, $\hat{\partial}_{\boldsymbol{x}} := \prod_{j=1}^{d} \hat{\partial}_{x_j}$, $\hat{\boldsymbol{\partial}}_{\boldsymbol{x}}^{\boldsymbol{k}} := \prod_{j=1}^{d} \hat{\partial}_{x_j}^{k_j}$ and weighted Sobelev spaces:

$$\hat{B}^m(\mathbb{R}^d) := \{u : \hat{\partial}_{\boldsymbol{x}}^{\boldsymbol{k}} u \in L^2(\mathbb{R}^d),\ 0 \le |\boldsymbol{k}|_1 \le m\},\ \forall m \in \mathbb{N},$$

$$\|u\|_{\hat{B}^m(\mathbb{R}^d)} = \Big(\sum_{0 \le |\boldsymbol{k}|_1 \le m} \|\hat{\partial}_{\boldsymbol{x}}^{\boldsymbol{k}} u\|^2 \Big)^{\frac{1}{2}}, \quad |u|_{\hat{B}^m(\mathbb{R}^d)} = \Big(\sum_{j=1}^{d} \|\hat{\partial}_{x_j}^{m} u\|^2 \Big)^{\frac{1}{2}}.$$

Lemma

For any $u \in \hat{B}^m(\mathbb{R}^d)$ with $m \ge 1$, we have

$$\|\hat{\boldsymbol{\Pi}}_N u - u\|_{H^\mu(\mathbb{R}^d)} \le C N^{(\mu-m)/2} |u|_{\hat{B}^m(\mathbb{R}^d)}, \quad 0 \le \mu \le m.$$

Error estimate

Let us denote the errors as follows:

$$e_N = u - u_N = \xi_N + \eta_N, \quad \xi_N = \widehat{\mathbf{\Pi}}_N u - u_N, \quad \eta_N = u - \widehat{\mathbf{\Pi}}_N u.$$

Triangle inequality:

$$\|e_N\| = \|\xi_N + \eta_N\| \le \|\xi_N\| + \|\eta_N\|.$$

Error equation:

$$A(u - u_N, v) = 0, \quad \forall v \in V_N \quad \Longrightarrow \quad A(\xi_N, v) = -A(\eta_N, v).$$

Theorem (Ling & Mao, JSC, 2023)

Let u and u_N be the solutions of the weak problem (3) *and the Hermite spectral Galerkin scheme* (4), *respectively. Assume* $u_0 \in \hat{B}_\mu(\mathbb{R}^d),\ w_0 \in \hat{B}_\nu(\mathbb{R}^d), \partial_t^2 u(t) \in L^2(0,T;\hat{B}_p(\mathbb{R}^d))$, *and*

$$u(t) \in L^2(0,T;\hat{B}_r(\mathbb{R}^d)) \cap L^\infty(0,T;\hat{B}_s(\mathbb{R}^d)),$$
$$\partial_t u(t) \in L^2(0,T;\hat{B}_q(\mathbb{R}^d)) \cap L^\infty(0,T;\hat{B}_\tau(\mathbb{R}^d)),$$

with $t \geq 0, q, s > 1, p, \mu, \nu, \tau > 0$, *then there holds*

$$\begin{aligned}\|\partial_t e_N\| + \|e_N\|_{H^1(\mathbb{R}^d)} \lesssim\ & N^{-\frac{p}{2}}|\partial_t^2 u|_{L^2(0,T;\hat{B}_p(\mathbb{R}^d))} + N^{\frac{1-q}{2}}|\partial_t u|_{L^2(0,T;\hat{B}_q(\mathbb{R}^d))} \\ & + N^{-\frac{\tau}{2}}|\partial_t u|_{L^\infty(0,T;\hat{B}_\tau(\mathbb{R}^d))} + N^{\frac{1-r}{2}}|u|_{L^2(0,T;\hat{B}_r(\mathbb{R}^d))} \\ & + N^{\frac{1-s}{2}}|u|_{L^\infty(0,T;\hat{B}_s(\mathbb{R}^d))} + N^{-\frac{\mu}{2}}|u_0|_{\hat{B}_\mu(\mathbb{R}^d)} + N^{-\frac{\nu}{2}}|w_0|_{\hat{B}_\nu(\mathbb{R}^d)}.\end{aligned}$$

Recall that if

$u_0 \in H^{k+2}(\mathbb{R}^d),\ w_0 \in H^{k+2}(\mathbb{R}^d), f \in H^1(0,T; H^k(\mathbb{R}^d)), k \geq 0$, then

$$\begin{aligned} &\|\partial_t^2 u\|_{L^\infty(0,T;H^k(\mathbb{R}^d))} + \|\partial_t u\|_{L^\infty(0,T;H^{k+1}(\mathbb{R}^d))} + \|u\|_{L^\infty(0,T;H^{k+2}(\mathbb{R}^d))} \\ &+ \|\partial_t^2 u\|_{L^2(0,T;H^{k+1}(\mathbb{R}^d))} + \|\partial_t u\|_{L^2(0,T;H^{k+2}(\mathbb{R}^d))} \\ \leq\ & C\left(\|f\|_{H^1(0,T;H^k(\mathbb{R}^d))} + \|w_0\|_{H^{k+2}(\mathbb{R}^d)} + \|u_0\|_{H^{k+2}(\mathbb{R}^d)}\right). \end{aligned}$$

Theorem (Ling & Mao, JSC, 2023)

Let u and u_N be the solutions of the weak problem (3) *and the Hermite spectral Galerkin scheme* (4)*, respectively. If $u, \partial_t u, \partial_t^2 u$ decay sufficiently fast at infinity, α, β, γ are sufficiently smooth, and $f \in H^1(0,T; H^k(\mathbb{R}^d)),\ u_0 \in H^{k+2}(\mathbb{R}^d),\ w_0 \in H^{k+2}(\mathbb{R}^d)$, then the error estimate is reduced as*

$$\|\partial_t e_N\| + \|e_N\|_{H^1(\mathbb{R}^d)} \leq C N^{-\frac{k+1}{2}},$$

where C is independent of N and just depends on α, β, γ as well as $\|f\|_{H^1(0,T;H^k(\mathbb{R}^d))}$, $\|u_0\|_{H^{k+2}(\mathbb{R}^d)}$, $\|w_0\|_{H^{k+2}(\mathbb{R}^d)}$.

Time discretization

Let us recall the Hermite spectral Galerkin method: find $u_N(t)$, $\partial_t u_N(t) \in V_N$, such that for any $v \in V_N$

$$(\partial_t^2 u_N, v) + (\alpha \partial_t u_N, v) + (\beta \partial_t \nabla u_N, \nabla v) + (\gamma^2 \nabla u_N, \nabla v) = (f, v).$$

Time discretization

Let us recall the Hermite spectral Galerkin method: find $u_N(t)$, $\partial_t u_N(t) \in V_N$, such that for any $v \in V_N$

$$(\partial_t^2 u_N, v) + (\alpha \partial_t u_N, v) + (\beta \partial_t \nabla u_N, \nabla v) + (\gamma^2 \nabla u_N, \nabla v) = (f, v).$$

By introducing an auxiliary variable $w_N = \partial_t u_N$, we rewrite it as the following ODE system:

$$\begin{cases} (\partial_t u_N, v_1) = (w_N, v_1), \\ (\partial_t w_N, v_2) + (\alpha w_N, v_2) + (\beta \nabla w_N, \nabla v_2) + (\gamma^2 \nabla u_N, \nabla v_2) = (f, v_2). \end{cases}$$

Then the explicit or implicit time discretization methods can be used, e.g. RK3, Crank-Nicolson, BDFs.

Fully-discrete scheme

Take the BDF1 as an example: let $\delta_\tau u_N^{k+1} = \frac{1}{\tau}\left(u_N^{k+1} - u_N^k\right)$

$$\left(\delta_\tau u_N^{k+1}, v_1\right) = (w_N^{k+1}, v_1),$$
$$\left(\delta_\tau w_N^{k+1}, v_2\right) + (\alpha w_N^{k+1}, v_2) + (\beta \nabla w_N^{k+1}, \nabla v_2) + (\gamma^2 \nabla u_N^{k+1}, \nabla v_2) = (f^{k+1}, v_2).$$

The key is to solve the second equation, which is equivalent to the following problem:

$$\begin{cases} \text{Find } u_N \in V_N \text{ such that} \\ (a u_N, v) + (b \nabla u_N, \nabla v) = (f, v), \quad \forall v \in V_N. \end{cases} \tag{5}$$

Implementation in 1D case

1D case:

$$V_N = \text{span}\{\varphi_i(x) : i = 0, 1, \cdots, N\}.$$

We denote

$$u_N = \sum_{i=0}^{N} \widehat{u}_i \varphi_i(x), \quad \boldsymbol{u} = \left(\widehat{u}_0, \widehat{u}_1, \cdots, \widehat{u}_N\right)^T,$$

$$f_i = \int_{\mathbb{R}} f(x)\varphi_i(x)dx, \quad \boldsymbol{f} = \left(f_0, f_1, \cdots, f_N\right)^T,$$

$$m_{ij} = \int_{\mathbb{R}} a(x)\varphi_i(x)\varphi_j(x)dx, \quad M = (m_{ij})_{i,j=0,1,\cdots,N},$$

$$s_{ij} = \int_{\mathbb{R}} b(x)\varphi_i'(x)\varphi_j'(x)dx, \quad S = (s_{ij})_{i,j=0,1,\cdots,N}.$$

Then (5) yields to the following linear system:

$$(M + S)\boldsymbol{u} = \boldsymbol{f}.$$

Implementation in 2D case

If $a(x,y)$ and $b(x,y)$ are constants, i.e.

$$a(x,y) \equiv a, \quad b(x,y) \equiv b,$$

then we denote

$$\begin{aligned}
U &= (\widehat{u}_{nm})_{n,m=0,1,\cdots,N}, & F &= (f_{ij})_{i,j=0,1,\cdots,N}, \\
m_{ij} &= \int_{\mathbb{R}} \varphi_i(x)\varphi_j(x)dx, & M &= (m_{ij})_{i,j=0,1,\cdots,N}, \\
s_{ij} &= \int_{\mathbb{R}} \varphi_i'(x)\varphi_j'(x)dx, & S &= (s_{ij})_{i,j=0,1,\cdots,N}.
\end{aligned}$$

And we can get the following equivalent linear system

$$aMUM^T + b\big(SUM^T + MUS^T\big) = F. \tag{6}$$

Matrix diagonalization method

Consider the generalized eigenvalue problem

$$Mx = \lambda Sx. \tag{7}$$

Let Λ be the diagonal matrix whose diagonal entries $\{\lambda_k\}$ are the eigenvalues of (7), and E be the matrix whose columns are the corresponding eigenvectors of (7), that is to say,

$$ME = SE\Lambda.$$

Then we set $U = EWE^T$ in (6) and can obtain

$$aSE\Lambda W\Lambda^T E^T S^T + b\big(SEW\Lambda^T E^T S^T + SE\Lambda W E^T S^T\big) = F. \tag{8}$$

Matrix diagonalization method

Multiplying the left (resp. right) of (8) by $(SE)^{-1}$ (resp. $(SE)^{-T}$) gives

$$a\Lambda W\Lambda^T + b\big(W\Lambda^T + \Lambda W\big) = (SE)^{-1}F(SE)^{-T} := G.$$

It is equivalent to

$$\big(a\lambda_i\lambda_j + b(\lambda_j + \lambda_i)\big)w_{ij} = g_{ij}, \qquad i,j = 0,1,\cdots,N, \tag{9}$$

where w_{ij} and g_{ij} are the entries of W and G respectively.

Efficient implementation

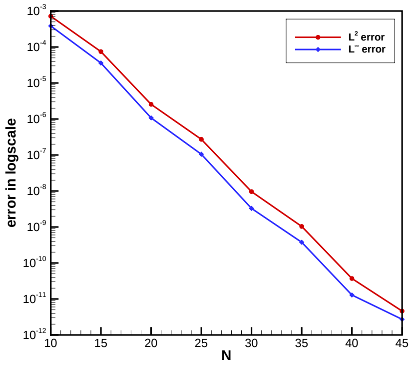
An efficient implementation for solving (6) can be summarized into the following steps:

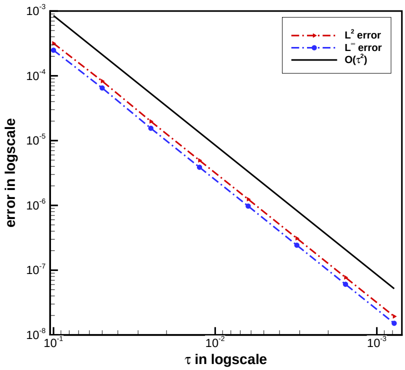
1. Pre-processing: compute the generalized eigenvalue problem (7) to obtain the eigenvalues Λ and eigenvectors E and compute $(SE)^{-1}$;
2. Compute $G = (SE)^{-1}F(SE)^{-T}$;
3. Obtain W from (9);
4. Set $U = EWE^T$.

Outline

2D DVWEs with $\alpha = \beta = \gamma = 1$:

$$u(x,y,0) = \mathrm{e}^{-(x^2+y^2)}, \quad u_t(x,y,0) = -\mathrm{e}^{-(x^2+y^2)}, \quad f = 0.$$



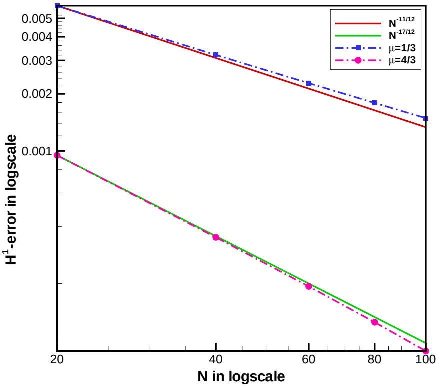


Regularity tests

We set $\alpha = \beta = \gamma = 1, u = u_t = 0$ and the source function is

$$u(x,0) = u_t(x,0) = 0, \quad f(x,t) = x^{\mu} \mathrm{e}^{-x^2} \cos(t).$$

When $\mu = 1/3$ and $\mu = 4/3$, the expected convergence rates of $\|e\|_{H^1(\mathbb{R})}$ are almost $11/12$ and $17/12$ respectively, according to (5).



Wave propagation within homogeneous medium

We set the parameters as $\alpha = 1, \beta = 0.01$ and $\gamma = 20$. A Ricker wavelet with dominant frequency of 15Hz located at $(x_0, y_0) = (10, 10)$ is used to generate the vibration. The source function is taken as follows

$$f(x,y,t) = g(x,y)h(t), \qquad g(x,y) = \mathrm{e}^{-[(x-x_0)^2+(y-y_0)^2]}, \tag{10}$$

and

$$h(t) = \left[1 - 2(\pi f_0(t-t_0))^2\right]\mathrm{e}^{-(\pi f_0(t-t_0))^2} \tag{11}$$

with the dominant frequency $f_0 = 15$ and the time delay $t_0 = 0.05$.

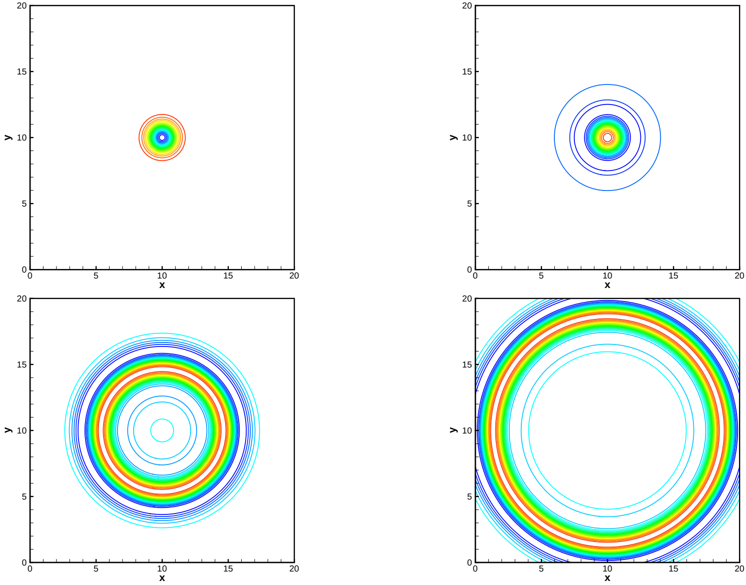


Figure 1: Contours of the numerical approximations at $T = 0.005, 0.1, 0.3, 0.5$ with $N = 200$.

Wave propagation within heterogeneous media

Now we consider the wave propagation within two different media. The parameters are set as

$$(\alpha, \beta, \gamma) = \begin{cases} (1.0, 0.02, 15.6), & \text{if } y \leq 16.5, \\ (2.5, 0.05, 20.4), & \text{if } y > 16.5. \end{cases}$$

The source function is defined as that in (10) and (11) with $(x_0, y_0) = (15, 15)$, $f_0 = 20$ and $t_0 = 0.05$.

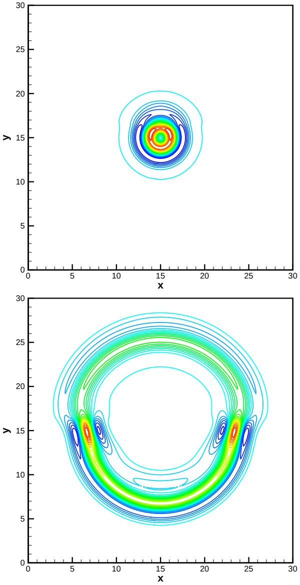


Figure 2: Contours of the numerical approximations at $T = 0.15, 0.4, 0.6, 0.8$ with $N = 300$.

Outline

Summary

① The diffusive-viscous wave model in unbounded domains is considered.

② We analyzed the existence, uniqueness and regularity of the weak solution.

③ We further developed a high accuracy Hermite spectral Galerkin method and then derived the error estimate.

④ Several numerical examples are provided to demonstrate the effectiveness of the present algorithm.

Thanks for your attention!

Q & A