Weak Galerkin Spectral Element Methods for Elliptic Eigenvalue Problems: Lower Bound Approximation and Superconvergence

李会元
中国科学院软件研究所 (ISCAS)
潘佳佳
上海理工大学 (USST)
Email: huiyuan@iscas.ac.cn

Numerical Methods for Spectral Problems: Theory and Applications 2024

Backgrounds

History

- h-version: abundant literature
  - first proposed by Junping Wang and Xiu Ye in 2011
    - in primal formulation, without stabilizer, regular triangular or tetrahedral meshes
  - applied for various PDEs
    - Helmholtz equations [12], Maxwell equations [14], Stokes equations [19], Navier-Stokes equations [8], biharmonic equations [11], parabolic equations [6], etc..
  - typical polynomial space triplets
    - $(\mathbb{P}_n(K), \mathbb{P}_{n+1}(e), \mathbb{P}_{n+1}(K)^d)$ [18, 5, 3, 29, 1]
    - $(\mathbb{P}_n(K), \mathbb{P}_n(e), \mathbb{P}_j(K)^d)$ with $j > n$ [22, 17, 2]
    - $(\mathbb{P}_n(K), \mathbb{P}_{n-1}(e), \mathbb{P}_{n-1}(K)^d)$ [21, 27, 4, 7, 13]
    - $(\mathbb{P}_n(K), \mathbb{P}_n(e), \mathbb{P}_{n-1}(K)^d)$ [15, 9, 10]
- p- and hp-version: little attention
  - hp hybridizable method (Zhimin Zhang etc)
  - weak Galerkin spectral element method (WGSEM) [16].
- flexible choices of approximation spaces

Objections

- space triplets for well-posedness
- properties of well-posed WGSEM for PDE eigenvalues
  - lower and upper bound approximation
  - super convergence

Lower bound and super convergence

- lower bound approximation of weak Galerkin finite element methods for eigenvalue problems with the specific space triplet $(\mathbb{P}_n(K), \mathbb{P}_{n-1}(e), \mathbb{P}_{n-1}(K)^2)$ has been studied [21, 27, 28, 26, 4],
- super-convergence for PDE source problems with certain space triplets [18, 7, 20, 3, 23, 24, 29, 17, 25]

Spectral weak gradients

Definition

Let $\mathcal{H}_N(K)$ be a finite space satisfying that $\mathcal{H}_N(K) \subset H^1(K)^2$. The discrete gradient of any weak function $v \in W(K)$ is defined as the unique $\nabla_N v \in \mathcal{H}_N(K)$ satisfying

$$(\nabla_N v, \boldsymbol{q})_K = -(v_0, \nabla \cdot \boldsymbol{q})_K + (v_b, \boldsymbol{q} \cdot \boldsymbol{n})_{\partial K}, \quad \boldsymbol{q} \in \mathcal{H}_N(K). \tag{1}$$

- assume that $v_0 \in H^1(K)$, then
$$(\nabla_N v, \mathbf{q})_K = (\nabla v_0, \mathbf{q})_K + (v_b - v_0, \mathbf{q} \cdot \boldsymbol{n})_{\partial K}, \quad \mathbf{q} \in \mathcal{H}_N(K). \tag{2}$$
- define L^2-orthogonal projection $\Pi_N : L^2(K)^2 \to \mathcal{H}_N(K)$ such that
$$(\Pi_N \mathbf{p} - \mathbf{p}, \mathbf{q})_K = 0, \quad \forall \mathbf{q} \in \mathcal{H}_N(K). \tag{3}$$
- suppose $\{\psi_i\}_{i=1}^{N_v}$ be the L^2-orthonormal basis of $\mathcal{H}_N(K)$ and denote
$$\hat{f}_i = (v_b - v_0, \psi_i \cdot \boldsymbol{n})_{\partial K}, \qquad 1 \le i \le N_v.$$
- it then follows that
$$\nabla_N v = \Pi_N(\nabla v_0) + \sum_{i=1}^{N_v} \hat{f}_i \psi_i.$$

Elements with general SM characteristics

- let $\hat{K} \in \mathbb{R}^2$ be a reference polygon whose boundary $\partial\hat{K}$ consists of several edges of $\hat{K}$, K be a physical element such that there is a one-to-one onto mapping $F_K : \bar{\hat{K}} \to \bar{K}$.
- $\hat{K}$= $\hat{T}$ or $\hat{Q}$, the reference triangle or the reference square is of greatest interest.

- For K being an arbitrary triangle with three vertices $(x_i, y_i)^{\mathrm{t}}$, $i = 1, 2, 3$, there is a one-to-one affine mapping $\Phi_K : \bar{\hat{T}} \to \bar{K}$ such that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi_K \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \hat{x} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \hat{y} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + (1 - \hat{x} - \hat{y}) \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}. \tag{4}$$

- For K being a quadrilateral with four vertices $(x_i, y_i)^{\mathrm{t}}$, $i = 1, 2, 3, 4$, there is a one-to-one bilinear mapping $\Phi_K : \bar{\hat{Q}} \to \bar{K}$ such that

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} = \Phi_K \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} &= \frac{(1-\hat{x})(1-\hat{y})}{4} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \frac{(1+\hat{x})(1-\hat{y})}{4} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ &+ \frac{(1+\hat{x})(1+\hat{y})}{4} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} + \frac{(1-\hat{x})(1+\hat{y})}{4} \begin{pmatrix} x_4 \\ y_4 \end{pmatrix}. \end{aligned} \tag{5}$$

Spectral gradients on the reference domain $\hat{K}$

- polynomial spaces for weak gradients on $\hat{K}$:

$$\mathcal{H}_N(\hat{K}) = \mathbb{P}_N(\hat{T})^2 \quad \text{if} \quad \hat{K} = \hat{T},$$
$$\mathcal{H}_N(\hat{K}) = \mathbb{Q}_N(\hat{Q})^2 \quad (\text{resp. } \mathbb{Q}_{N,N-1}(\hat{Q}) \times \mathbb{Q}_{N-1,N}(\hat{Q})) \text{ if } \quad \hat{K} = \hat{Q}.$$

- polynomial space for weak functions on $\hat{K}$:

$$W_{n,m}(\hat{K}) = \{v = \{v_0, v_b\} : v_0 \in \mathcal{P}_n(\hat{K}), v_b \in X_m(\partial\hat{K})\},$$

where

$$\mathcal{P}_n(\hat{K}) = \mathbb{P}_n(\hat{T}) \quad \text{if} \quad \hat{K} = \hat{T}, \qquad \text{and} \quad \mathcal{P}_n(\hat{K}) = \mathbb{Q}_n(\hat{Q}) \quad \text{if} \quad \hat{K} = \hat{Q},$$
$$X_m(\partial\hat{K}) = \{v : v \in \mathbb{P}_m(\hat{e}) \text{ for any edge } \hat{e} \text{ of } \hat{K}\}.$$

Spectral gradient on a triangle or a convex quadrilateral K

spectral approximation space of weak functions

$$W_{n,m}(K) = \{\{v_0, v_b\} : v_0 \in \mathcal{P}_n(K), v_b \in X_m(\partial K)\},$$

where

$$\mathcal{P}_n(K) = \{\hat{w} \circ \Phi_K^{-1} : \hat{w} \in \mathcal{P}_n(\hat{K})\},$$
$$X_m(\partial K) = \{v : v \in \mathbb{P}_m(e) \text{ for any edge } e \text{ of } K\}.$$

approximation space of spectral weak gradients

- mapped polynomial space

$$\mathcal{H}_N(K) = \mathcal{H}_N^I(K) := \{\hat{\mathbf{q}} \circ \Phi_K^{-1} \text{ for } \hat{\mathbf{q}} \in \mathcal{H}_N(\hat{K})\}. \quad \text{(I)}$$

- rational space defined by Piola transform (in a framework of de Rham complex)

$$\mathcal{H}_N(K) = \mathcal{H}_N^{II}(K) := \{\boldsymbol{q} : \boldsymbol{q} = J_K^{-1} F_K \hat{\mathbf{q}} \circ \Phi_K^{-1} \text{ for } \hat{\mathbf{q}} \in \mathcal{H}_N(\hat{K})\}. \quad \text{(II)}$$

- $\mathcal{H}_N^{II}(Q) \subseteq \{(J_K^{-1}\hat{v}) \circ \Phi_K^{-1} : \hat{v} \in \mathbb{Q}_N(\hat{Q})^2\}$ for $\mathcal{H}_N(\hat{Q}) = \mathbb{Q}_{N,N-1}(\hat{Q}) \times \mathbb{Q}_{N-1,N}(\hat{Q})$.

Spectral gradients

Properties for spectral gradients in $\mathcal{H}_N^{II}(K)$

- $(\nabla_N u, \boldsymbol{v})_K = (\hat{\nabla}_N \hat{u}, \hat{\boldsymbol{v}})_{\hat{K}}$ for $\boldsymbol{v} \in \mathcal{H}_N^{II}(K)$
- $\nabla_N v = \Pi_N(F_K^{-\mathrm{t}} \hat{\nabla}_N \hat{v} \circ \phi_K^{-1})$
- $\nabla_N v = F_K^{-\mathrm{t}} \hat{\nabla}_N \hat{v} \circ \phi_K^{-1}$ and $\mathcal{H}_N^{II}(K) = \mathcal{P}_N(K)^2 = \mathcal{H}_N^{I}(K)$
  for any triangle or parallelogram K with $\mathcal{H}_N(\hat{K}) = \mathcal{P}_N(\hat{K})^2$.

Lemma (Nullity of spectral gradient on K)

Suppose $v \in W_{n,m}(K)$ with $\nabla_N v = 0$ on K. Then v_0 is constant on K and $v_b = v_0|_{\partial K}$ if and only if

(i) $n \le N-1$ *and* $m \le N$ *for* $K = T$ *and* $\mathcal{H}_N(\hat{K}) = \mathbb{P}_N(\hat{T})^2$;

(ii) $n \le N-1$ *and* $m \le N$ *for* $K = Q$ *and* $\mathcal{H}_N(\hat{K}) = \mathbb{Q}_N(\hat{Q})^2$;

(iii) $n \le N-1$ *and* $m \le N-1$ *for* $K = Q$ *and* $\mathcal{H}_N(\hat{K}) = \mathbb{Q}_{N,N-1}(\hat{Q}) \times \mathbb{Q}_{N-1,N}(\hat{Q})$.

- hold both for $\mathcal{H}_N^{I}(K)$ and$\mathcal{H}_N^{II}(K)$.

Problem

- model elliptic eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{6}$$

where Ω is a polygonal domain in $\mathbb{R}^2$.

- The classical variational form: to find $u \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$a(u, v) := (\nabla u, \nabla v) = \lambda(u, v), \quad v \in H_0^1(\Omega), \tag{7}$$

Meshes for brevity

Meshes

$\mathcal{T}_h = \{T_k\}_{k=1}^{N_h}$: triangular and/or (convex) quadrilateral elements.

Assumptions

A1. $\mathcal{T}_h$ is shape regular in the sense that the condition number of the Jacobian is bounded for all elements,

$$\|F_K(\hat{x},\hat{y})\| \, \|F_K^{-1}(\hat{x},\hat{y})\| \leq C, \quad \forall K \in \mathcal{T}_h, \ (\hat{x},\hat{y}) \in \hat{K}.$$

- $\mathcal{E}_h$: the set of all edges in $\mathcal{T}_h$
- $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$: the set of all interior edges.

Approximation scheme

Approximation spaces

- The weak Galerkin-spectral element approximation space on $\mathcal{T}_h$,

$$
\begin{aligned}
V_\delta &:= \big\{ v = \{v_0, v_b\} : \{v_0, v_b\}|_K \in W_{n,m}(K) \text{ for all } K \in \mathcal{T}_h \big\}, \\
V_\delta^0 &:= \{ v : v \in V_\delta,\ v_b|_{\partial K \cap \partial \Omega} = 0 \text{ for all } K \in \mathcal{T}_h \},
\end{aligned} \tag{8}
$$

where $\delta = \delta(h, n, m, N)$.

bilinear forms on V_δ^0

$$
\begin{aligned}
s(v, w) &:= \rho \sum_{K \in \mathcal{T}_h} h_K^{\varepsilon - 1} n^2 (v_0 - v_b, w_0 - w_b)_{\partial K}, \quad \rho \geq 0,\ 0 \leq \varepsilon < 1, \\
a_\delta(v, w) &:= \sum_{K \in \mathcal{T}_h} (\nabla_N v, \nabla_N w)_T + s(v, w), \qquad v, w \in V_\delta^0.
\end{aligned}
$$

Remark

$$
\|u\|_{\partial K} \leq C h^{-1/2} n \|u\|_K, \quad u \in \mathbb{P}_n(K).
$$

Approximation scheme and well-posedness constraints

Weak Galerkin spectral element approximation scheme

to find $\lambda_\delta \in \mathbb{R}$ and $u_\delta = \{u_{\delta,0}, u_{\delta,b}\} \in V_\delta^0$ such that

$$a_\delta(u_\delta, v) = \lambda_\delta(u_{\delta,0}, v_0), \quad v \in V_\delta^0, \tag{9}$$

Well-posedness constraints on (n, m, N) for $\rho = 0$

(i) $n \leq N-1$ and $m \leq N$ for any triangular/quadrilateral mesh $\mathcal{T}_h$ with $\mathcal{H}_N(K) = \mathcal{P}_N(K)^2$;

(ii) $n \leq N-1$ and $m \leq N-1$ for any quadrilateral mesh $\mathcal{T}_h$ with $\mathcal{H}_N(K) = \{\boldsymbol{q} = J_K^{-1}F_K\hat{\mathbf{q}} \circ \Phi_K^{-1} : \hat{\mathbf{q}} \in \mathbb{Q}_{N,N-1}(\hat{Q}) \times \mathbb{Q}_{N-1,N}(\hat{Q})\}$.

Well-posedness constraints on (n, m, N) for $\rho > 0$

(i) $n \leq N+1$ for any triangular mesh $\mathcal{T}_h$ with $\mathcal{H}_N(K) = \mathcal{P}_N(K)^2$;

(ii) $n \leq N$ for any quadrilateral mesh $\mathcal{T}_h$ with $\mathcal{H}_N(K) = \mathcal{P}_N(K)^2$;

(iii) $n \leq N-1$ for any quadrilateral mesh $\mathcal{T}_h$ with $\mathcal{H}_N(K) = \{\boldsymbol{q} = J_K^{-1}F_K\hat{\mathbf{q}} \circ \Phi_K^{-1} : \hat{\mathbf{q}} \in \mathbb{Q}_{N,N-1}(\hat{Q}) \times \mathbb{Q}_{N-1,N}(\hat{Q})\}$.

A convergence theorem for the polynomial degree triplet (N, N, N)

Theorem ($\epsilon = 0,\ \rho > 0$)

- *Suppose that* $u_k \in H^s(\Omega)$ *for any* $u_k \in E(\lambda_k)$*, then for* $j = k, k+1, \cdots, k+q-1$,

$$|\lambda_k - \lambda_{j,\delta}| \le Ch^{2\mu-2} N^{3-2s} \sup_{u_k \in E(\lambda_k)} \|u_k\|_s^2 \quad \textit{with} \quad \mu = \min\{N+1, s\}. \tag{10}$$

- *Let* $u_{j,\delta}$ *be an eigenfunction corresponding to* $\lambda_{j,\delta}$ *for* $j = k, k+1, \cdots, k+q-1$ *with* $\|u_{j,\delta}\|_V = 1$*, then*

$$\inf_{u \in E(\lambda_k)} \|u - u_{j,\delta}\|_V \le Ch^{\mu-1} N^{3/2-s} \sup_{u_k \in E(\lambda_k)} \|u_k\|_s. \tag{11}$$

- Let u_k be a eigenfunction corresponding to λ_k with $\|u_k\|_V = 1$, then there exist a function $v_\delta \in \mathrm{span}\{u_{k,\delta}, \cdots, u_{k+q-1,\delta}\}$ with $\|v_\delta\|_V = 1$ such that

$$\|u_k - v_\delta\|_V \le Ch^{\mu-1} N^{3/2-s} \sup_{u_k \in E(\lambda_k)} \|u_k\|_s. \tag{12}$$

Convergence order baseline

Baseline order of convergence for eigenvalues

$$\mathcal{O}(h^{2\min(n,m,N+1,s)-2\varepsilon}) \text{ if } \rho > 0, 0 \leq \varepsilon < 1, \text{ or } \mathcal{O}(h^{2\min(n,m,N+1,s)}) \text{ if } \rho = 0, \tag{13}$$

Qualitative analysis for rules of thumbs

main objects

- lower bound approximation of eigenvalues
- superconvergence of numerical eigenvalues

main settings

- $\mathcal{H}_N(K) = \mathcal{P}_N(K)^2$, full polynomial space of degree $\leq N$, without Piola transforms
- well-posedness constraints on (n, m, N) for $\rho = 0$
  (i) $n \leq N - 1$ and $m \leq N$ for any triangular/quadrilateral mesh
- well-posedness constraints on (n, m, N) for $\rho > 0$
  (i) $n \leq N + 1$ for any triangular mesh
  (ii) $n \leq N$ for any quadrilateral mesh

lower/upper approximation

- $(\lambda, u) \in \mathbb{R}^+ \times H_0^1(\Omega)$ with $\|u\| = 1$, and $\left(\lambda_\delta, u_\delta\right) \in \mathbb{R}^+ \times V_\delta^0$ with $\|u_{\delta,0}\|_\Omega = 1$

for any $v = \{v_0, v_b\} \in V_\delta^0$ with arbitrary polynomial degrees n and m

$$
\begin{aligned}
\lambda - \lambda_\delta = & \sum_{K \in \mathcal{T}_h} \|\nabla u - \nabla_N u_\delta\|_K^2 + s_\rho(u_\delta - v, u_\delta - v) + 2 \sum_{K \in \mathcal{T}_h} (\nabla u - \nabla_N v, \nabla_N u_\delta)_K \\
& - \lambda_\delta \sum_{K \in \mathcal{T}_h} \|u_{\delta,0} - v_0\|_K^2 - \lambda_\delta \sum_{K \in \mathcal{T}_h} \left(\|u_{\delta,0}\|_K^2 - \|v_0\|_K^2\right) - s_\rho(v, v) \\
& := I_1 + I_2 + I_3 - \lambda_\delta I_4 - \lambda_\delta I_5 - I_6
\end{aligned}
$$

- taking $v\big|_K = \{\Pi_n^0(u\big|_K), \Pi_m^b(u\big|_{\partial K})\}$, Π_n^0, Π_m^b are the L^2-orthogonal projections onto $\mathcal{P}_n(K)$, $X_m(\partial K)$
- $I_3 = 0$ if $N - 1 \le n \le N + 1$ and $m \ge N$ on triangular meshes

lower bound approximation on triangular meshes:

- $n = N, N + 1$, $m \ge N$, $0 \le \varepsilon < 1$ in the case of a small $\rho > 0$

<table>
<tr><th colspan="3">Lower bound approximation</th></tr>
<tr><th>Mesh type</th><th>(n, m, N)</th><th>Remarks</th></tr>
<tr><td>quadrilateral</td><td>$n = N,\ m \geq N$</td><td rowspan="2">h- and/or p-version methods with appropriate $\rho > 0$</td></tr>
<tr><td rowspan="2">triangular</td><td>$n = N,\ N+1,\ m \geq N$</td></tr>
<tr><td>$n = N+1,\ m = N$</td><td>h-version method with $\rho = 1$ [21, 27, 28, 26]</td></tr>
</table>

superconvergence

- Π_N are the L^2-orthogonal projections onto $\mathcal{H}_N(K)$
- $(\Pi_\delta u)\big|_K = \{\Pi_n^0(u\big|_K), \Pi_m^b(u\big|_{\partial K})\}$

if $n \geq N-1$ and $m \geq N$ for K being a triangular element

$$\nabla_N((\Pi_\delta u)\big|_K) = \Pi_N(\nabla u\big|_K), \quad \forall u \in H^1(\Omega).$$

$$\begin{aligned}(\nabla_N(\Pi_\delta u), \boldsymbol{q})_K &= -(\Pi_n^0 u, \nabla\cdot\boldsymbol{q})_K + (\Pi_m^b u, \boldsymbol{q}\cdot\boldsymbol{n})_{\partial K} = -(u, \nabla\cdot\boldsymbol{q})_K + (u, \boldsymbol{q}\cdot\boldsymbol{n})_{\partial K} \\ &= (\nabla u, \boldsymbol{q})_K = \big(\Pi_N(\nabla u), \boldsymbol{q}\big)_K\end{aligned}$$

if $n \geq N-1$ and $m \geq N-1$ for any quadrilateral K

$$\nabla_N^*((\Pi_\delta u)\big|_K) = \Pi_N(\nabla u\big|_K), \quad \forall u \in H^1(K).$$

super-convergence: higher than the convergence order baseline $\mathcal{O}(h^{2n-2\varepsilon})$

- $n = N-1,\ m = N$ in the case of $\rho = 0$
- or $n = N-1,\ m \geq N+1$, or $n = N,\ m \geq N$ in the case of $0 < \rho \ll 1$

Superconvergence		
Mesh type	$n,\ m$	Remarks
quadrilateral	$n = N,\ m \geq N$	h-version (resp. p-version) methods with appropriate penalty terms for eigenfunctions smooth enough (resp. with limited regularity)
	$n = N-1,\ m \geq N+1$	h-version methods with appropriate penalty terms for eigenfunctions smooth enough on rectangular meshes
triangular	$n = N,\ m \geq N$ or $n = N-1,\ m \geq N+1$	h-version methods with appropriate penalty term
	$n = N-1, m = N$	h-version method without any penalty terms

Remark

Using Piola transforms under the de Rham complex for more superconvergence cases

Case ((N, N, N) and $\rho > 0$ with triangular or quadrilateral meshes)

h-version:

$\rho = 0.01$: *lower bound; superconvergence*

$\rho = 1$: *upper bound; no superconvergence (baseline: $\mathcal{O}(h^{2N})$)*

p-version: difficult to observe owing to the exponential orders of convergence

take $N = 1$ for example

h	0.088		0.044		0.022		0.011		0.006	
	error	order	error	order	error	order	error	order	error	order
λ_1	0.306	–	0.005	5.87	2.690e-4	4.28	1.467e-5	4.20	5.334e-7	4.78
λ_2	24.96	–	0.149	7.39	0.005	4.80	2.892e-4	4.20	1.493e-5	4.28
λ_3	54.50	–	1.226	5.47	0.021	5.88	0.001	4.28	5.867e-5	4.20
λ_4	74.22	–	6.697	3.47	0.044	7.26	0.002	4.34	1.196e-4	4.17

Table: $\lambda_i - \lambda_{i,\delta}, i = 1, 2, 3, 4$ versus h for $\rho = 0.01$ with $N = 1$ and $\varepsilon = 0$ for triangular meshes.

h	0.088		0.044		0.022		0.011		0.006	
	error	order	error	order	error	order	error	order	error	order
λ_1	-0.011	–	-0.003	1.94	-7.333e-4	1.99	-1.838e-4	2.00	-4.597e-5	2.00
λ_2	-0.044	–	-0.013	1.81	-0.003	1.96	-8.254e-4	1.99	-2.067e-4	2.00
λ_3	-0.149	–	-0.045	1.74	-0.012	1.94	-0.003	1.99	-7.350e-4	2.00
λ_4	-0.180	–	-0.060	1.58	-0.016	1.91	-0.004	1.98	-0.001	1.99

Table: $\lambda_i - \lambda_{i,\delta}, i = 1, 2, 3, 4$ versus h for $\rho = 1$ with $N = 1$ and $\varepsilon = 0$ for triangular meshes.

h	0.120		0.063		0.031		0.016		0.008	
	error	order	error	order	error	order	error	order	error	order
λ_1	6.212	—	0.003	12.0	1.258e-4	4.36	6.380e-6	4.35	1.305e-7	5.72
λ_2	35.78	—	0.334	7.21	0.005	6.15	2.393e-4	4.34	1.223e-5	4.37
λ_3	64.96	—	29.00	1.24	0.010	11.6	4.905e-4	4.33	2.501e-5	4.37
λ_4	84.70	—	47.38	0.90	0.063	9.54	0.003	4.54	1.543e-4	4.28

Table: $\lambda_i - \lambda_{i,\delta}, i = 1, 2, 3, 4$ versus h for $\rho = 0.01$ with $N = 1$, $\varepsilon = 0$ for quadrilateral meshes .

h	0.127		0.063		0.031		0.016		0.008	
	error	order	error	order	error	order	error	order	error	order
λ_1	-0.007	—	-0.002	1.85	-4.877e-4	1.97	-1.218e-4	2.05	-3.054e-5	2.01
λ_2	-0.039	—	-0.014	1.42	-0.004	1.91	-9.518e-4	2.03	-2.392e-4	2.01
λ_3	-0.082	—	-0.029	1.48	-0.008	1.89	-0.002	2.03	-4.878e-4	2.01
λ_4	-0.017	—	-0.057	-1.75	-0.017	1.73	-0.004	2.00	-0.001	2.00

Table: $\lambda_i - \lambda_{i,\delta}, i = 1, 2, 3, 4$ versus h for $\rho = 1$ with $N = 1$ and $\varepsilon = 0$ for quadrilateral meshes.

Case $((N-1, N+1, N)$ and $\rho > 0$ with triangular or rectangular meshes)

h-version:

$\rho = 0.0001$: *upper bound; superconvergence*

$\rho = 1$: *upper bound; no superconvergence (baseline: $\mathcal{O}(h^{2N-2})$)*

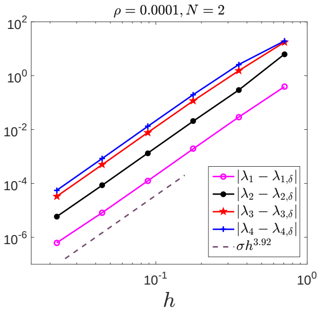
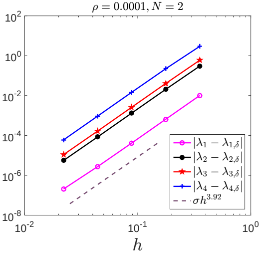


Figure: $|\lambda_i - \lambda_{i,\delta}|, i = 1, 2, 3, 4$ versus h with $\rho = 0.0001$, $\varepsilon = 0$ for $N = 2$ (left) and $N = 3$ (right) on triangular meshes.



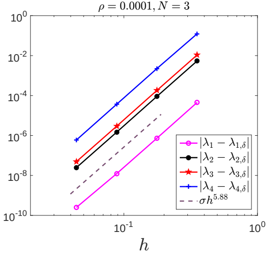


Figure: $|\lambda_i - \lambda_{i,\delta}|, i = 1, 2, 3, 4$ versus h with $\rho = 0.0001$, $\varepsilon = 0$ on rectangular meshes for $N = 2$ (left) and $N = 3$ (right).

Case ($(N-1, N-1, N)$ and $\rho = 0$ with triangular or quadrilateral meshes)

h-version: upper bound; no superconvergence

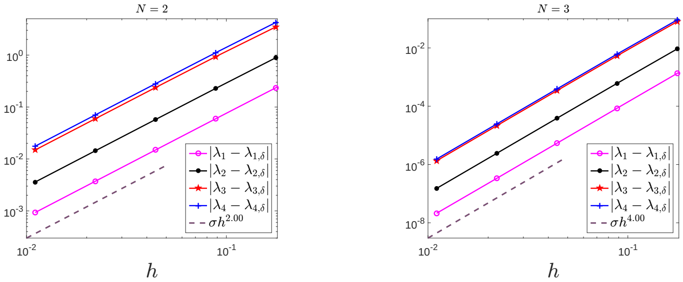
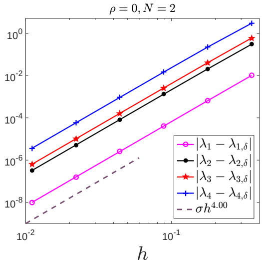


Figure: $|\lambda_i - \lambda_{i,\delta}|, i = 1, 2, 3, 4$ versus h in log-log scale for $\rho = 0$ with $N = 2$ (left) and $N = 3$ (right) for triangular meshes.

Case $((N-1, N-1, N)$ and $\rho = 0$ with rectangular meshes)

h-version: upper bound; superconvergence $(\mathcal{O}(h^{2N})$ vs. baseline $\mathcal{O}(h^{2N-2}))$



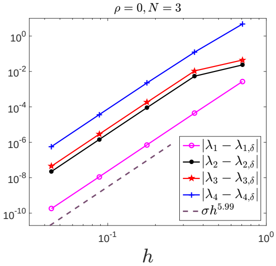


Figure: $|\lambda_i - \lambda_{i,\delta}|, i = 1, 2, 3, 4$ versus h with $n = N-1$, $m = N-1$, $\rho = 0$ for $N = 2$ (left) and $N = 3$ (right) on rectangular meshes with $\mathcal{H}_N(\hat{K}) = \mathbb{Q}_N(\hat{Q})^2$.

Case ($(N-1, N, N)$ and $\rho = 0$ with triangular or rectangular meshes)

h-version: upper bound; superconvergence ($\mathcal{O}(h^{2N})$ vs. $\mathcal{O}(h^{2N-2})$)

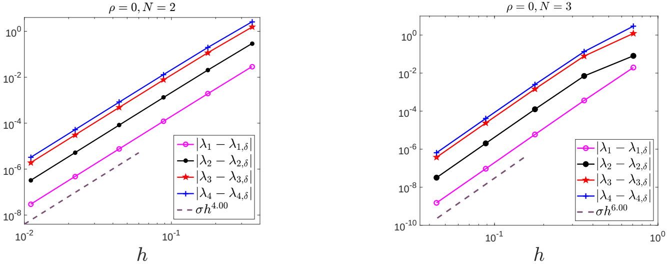


Figure: $|\lambda_i - \lambda_{i,\delta}|$, $i = 1, 2, 3, 4$, versus h with $n = N-1$, $m = N$, $\rho = 0$ for $N = 2$ (left) and $N = 3$ (right) on triangular meshes of the square.

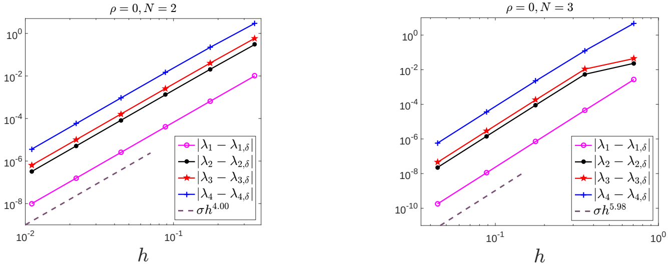


Figure: $|\lambda_i - \lambda_{i,\delta}|, i = 1, 2, 3, 4$ versus h with $n = N - 1$, $m = N$, $\rho = 0$ for $N = 2$ (left) and $N = 3$ (right) on rectangular meshes with $\mathcal{H}_N(\hat{K}) = \mathbb{Q}_N(\hat{Q})^2$.

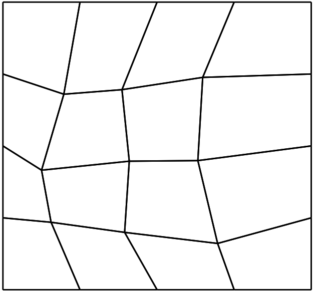
			well-posedness	lower bound	superconvergence
$\rho = 0$	T	$\mathbb{P}_N^2$	$n \leq N-1, m \leq N$		$n = N-1, m = N$
	Q	$\mathbb{Q}_N^2$	$n \leq N-1, m \leq N$		
$\rho > 0$	T	$\mathbb{P}_N^2$	$n \leq N+1$	$n = N, m \geq N$ $n = N+1, m \geq N$	$n = N, m \geq N$ $n = N-1, m \geq N+1$
	Q	$\mathbb{Q}_N^2$	$n \leq N$	$n = N, m \geq N$	$n = N, m \geq N$
	R	$\mathbb{Q}_N^2$	$n \leq N$	$n = N, m \geq N$	$n = N-1, m \geq N+1$

Single ridge rectangular waveguide $\Omega = [-1,1]^2 \setminus \left([-\frac{1}{2},\frac{1}{2}] \times [0,1]\right)$

- u_1: singularity of type $r^{2/3}$
- $\mathcal{O}(h^{4/3})$ in general for h-version
- $\mathcal{O}(N^{-8/3})$ for p-version

$$\lambda_1 = 12.053240106029265988, \quad \lambda_2 = 18.79637554640384564,$$
$$\lambda_3 = 30.157720368619479245, \quad \lambda_4 = 39.626151901149341938,$$

- the eighth eigenvalue is explicitly formulated as $\lambda_8 = 5\pi^2$.



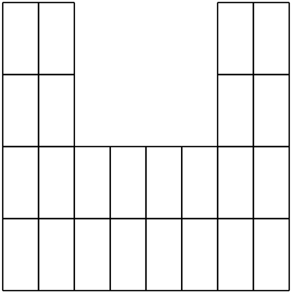


Figure: Left: quadrilateral meshes of the square $\Omega = [0,1]^2$ with mesh size $h = 0.475$; Right: rectangular meshes of the single ridge rectangular waveguide $\Omega = [-1,1]^2 \setminus \left([-\frac{1}{2},\frac{1}{2}] \times [0,1]\right)$ with mesh size $h = 0.559$.

h	0.070		0.035		0.017		0.009		0.004	
		order		order		order		order		order
$\lambda_1 - \lambda_{1,\delta}$	0.075	–	0.006	3.73	0.002	1.74	6.443e-4	1.40	2.532e-4	1.35
$\lambda_2 - \lambda_{2,\delta}$	2.683	–	0.012	7.81	0.004	1.49	0.002	1.36	6.564e-4	1.34
$\lambda_3 - \lambda_{3,\delta}$	14.02	–	0.020	9.44	0.006	1.69	0.002	1.38	9.511e-4	1.34
$\lambda_4 - \lambda_{4,\delta}$	23.49	–	0.185	6.99	0.013	3.87	0.004	1.80	0.001	1.40
$\lambda_8 - \lambda_{8,\delta}$	33.20	–	0.019	$\boxed{10.74}$	8.654e-4	$\boxed{4.49}$	4.887e-5	$\boxed{4.15}$	2.589e-6	$\boxed{4.24}$

h	0.280		0.140		0.070		0.035		0.017	
		order		order		order		order		order
$\lambda_1 - \lambda_{1,\delta}$	3.898	–	0.006	9.30	0.002	1.41	9.121e-4	1.35	3.597e-4	1.34
$\lambda_2 - \lambda_{2,\delta}$	10.64	–	0.015	9.43	0.006	1.36	0.002	1.34	9.317e-4	1.34
$\lambda_3 - \lambda_{3,\delta}$	22.00	–	4.939	2.16	0.009	9.18	0.003	1.33	0.001	1.33
$\lambda_4 - \lambda_{4,\delta}$	31.47	–	14.40	1.13	0.013	10.14	0.005	1.39	0.002	1.33
$\lambda_8 - \lambda_{8,\delta}$	35.47	–	24.12	$\boxed{0.56}$	2.453e-5	$\boxed{19.9}$	3.108e-7	$\boxed{6.30}$	4.191e-9	$\boxed{6.21}$

Table: Lower bound approximation and convergence rates of $\lambda_{i,\delta}, i = 1, 2, 3, 4, 8$, in h-version methods with $n = N$, $m = N$, $\rho = 0.01$, $\varepsilon = 0$ on rectangular meshes of the single ridge rectangular waveguide. Top: $N = 1$; Bottom: $N = 2$.

N	4		8		16		32		48	
		order		order		order		order		order
$\lambda_1 - \lambda_{1,\delta}$	0.008	–	0.002	2.45	2.731e-4	2.50	4.642e-5	2.56	1.653e-5	2.55
$\lambda_2 - \lambda_{2,\delta}$	6.131	–	0.004	10.6	7.071e-4	2.49	1.205e-4	2.55	4.296e-5	2.54
$\lambda_3 - \lambda_{3,\delta}$	17.49	–	0.006	11.6	0.001	2.48	1.748e-4	2.54	6.242e-5	2.54
$\lambda_4 - \lambda_{4,\delta}$	20.85	–	0.008	11.3	0.001	2.48	2.503e-4	2.55	8.934e-5	2.54

N	4	5	6	7	8
$\lambda_8 - \lambda_{8,\delta}$	26.38	9.747	6.485	2.970e-12	1.563e-13

Table: Lower bound approximation of $\lambda_{i,\delta}, i = 1, 2, 3, 4, 8$, and convergence rates in p-version methods with $n = N$, $m = N$, $\rho = 0.01$, $\varepsilon = 0$ on rectangular meshes with $h = 0.559$ of the single ridge rectangular waveguide.

Case ((N, N, N) via Piola transform and $\rho > 0$ with quadrilateral meshes)

h-version: no superconvergence

$\rho = 0.1$: *lower bound;*

$\rho = 1$: *upper bound;*

h	0.121		0.063		0.032		0.016		0.008	
	error	order	error	order	error	order	error	order	error	order
λ_1	0.009	—	0.002	2.47	4.283e-4	2.12	1.045e-4	2.03	2.620e-5	2.01
λ_2	0.186	—	0.019	3.43	0.003	2.55	7.316e-4	2.19	1.762e-4	2.07
λ_3	0.407	—	0.043	3.38	0.008	2.54	0.002	2.12	4.207e-4	2.07
λ_4	2.644	—	0.135	4.48	0.017	3.07	0.003	2.43	7.077e-4	2.13

Table: $\lambda_i - \lambda_{i,\delta}, i = 1, 2, 3, 4$ versus h for $\rho = 0.1$ with $N = 1$ and $\varepsilon = 0$ for quadrilateral meshes.

h	0.240		0.124		0.063		0.031		0.016	
	error	order	error	order	error	order	error	order	error	order
λ_1	-0.042	—	-0.012	1.92	-3.029e-3	2.00	-7.426e-4	2.03	-1.842e-4	2.07
λ_2	-0.150	—	-0.089	0.79	-0.024	1.93	-5.996e-3	1.98	-1.506e-3	2.05
λ_3	-0.341	—	-0.175	1.01	-0.047	1.93	-0.012	2.00	-2.947e-3	2.05
λ_4	0.273	—	-0.360	-0.42	-0.108	1.75	-0.028	1.93	-7.231e-3	2.04

Table: $\lambda_i - \lambda_{i,\delta}, i = 1, 2, 3, 4$ versus h for $\rho = 1$ with $N = 1$ and $\varepsilon = 0$ for quadrilateral meshes.

Case ($(N-1, N-1, N)$ via Piola transform and $\rho = 0$ with quadrilateral meshes)

h-version: *upper bound; no superconvergence*

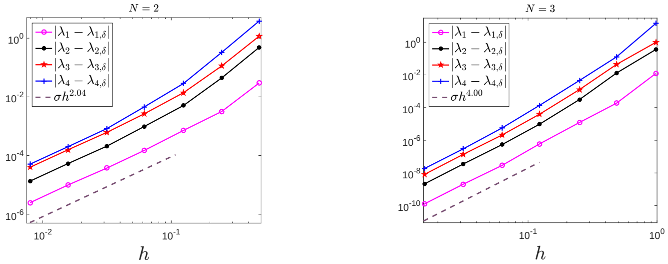


Figure: $|\lambda_i - \lambda_{i,\delta}|, i = 1, 2, 3, 4$ versus h in log-log scale for $\rho = 0$ with $N = 2$ (left) and $N = 3$ (right) on quadrilateral meshes.

Case $((N-1, N-1, N)$ using Piola transform and $\mathcal{H}_N(\hat{K}) = \mathbb{Q}_{N,N-1}(\hat{Q}) \times \mathbb{Q}_{N-1,N}(\hat{Q})$, $\rho = 0$ with quadrilateral meshes (De Rham complex constrained))

h-version: upper bound; superconvergence

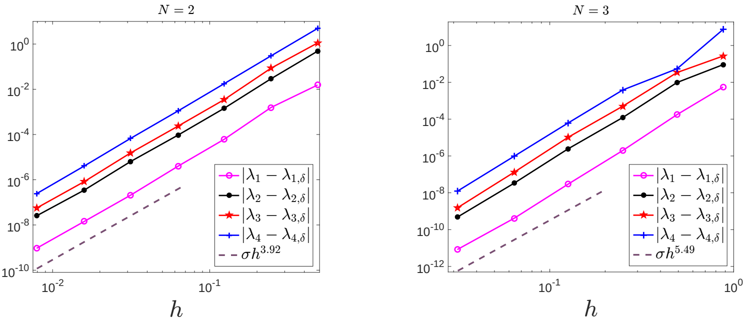


Figure: $|\lambda_i - \lambda_{i,\delta}|, i = 1, 2, 3, 4$ versus h in log-log scale for $\rho = 0$ with $N = 2$ (left) and $N = 3$ (right) for quadrilateral meshes with $\mathcal{H}_N(\hat{K}) = \mathbb{Q}_{N,N-1}(\hat{Q}) \times \mathbb{Q}_{N-1,N}(\hat{Q})$.

A. Al-Taweel and X. Wang.
The lowest-order stabilizer free weak Galerkin finite element method.
Applied Numerical Mathematics, 157:434–445, 2020.

A. Al-Taweel and X. Wang.
A note on the optimal degree of the weak gradient of the stabilizer free weak Galerkin finite element method.
Applied Numerical Mathematics, 150:444–451, 2020.

A. Al-Taweel, X. Wang, X. Ye, and S. Zhang.
A stabilizer free weak Galerkin element method with supercloseness of order two.
Numerical Methods for Partial Differential Equations, 37:1012–1029, 2021.

C. Carstensen, Q. Zhai, and R. Zhang.
A skeletal finite element method can compute lower eigenvalue bounds.
SIAM J. Numer. Anal., 58(1):109–124, 2020.

Y. Feng, Z. Guan, H. Xie, and C. Zhou.
Augmented subspace scheme for eigenvalue problem by weak Galerkin finite element method.
arXiv preprint arXiv: 2401.04063, 2024.

F. Gao and L. Mu.
On L^2 error estimate for weak Galerkin finite element methods for parabolic problems.
J. Comput. Math., 32(2):195–204, 2014.

A. Harris and S. Harris.
Superconvergence of weak Galerkin finite element approximation for second order elliptic problems by L^2-projections.
Applied Mathematics and Computation, 227:610–621, 2014.

X. Liu, J. Li, and Z. Chen.
A weak Galerkin finite element method for the Navier-Stokes equations.
Journal of Computational and Applied Mathematics, 333:442–457, 2018.

Y. Liu, G. Wang, M. Wu, and Y. Nie.
A recovery-based a posteriori error estimator of the weak Galerkin finite element method for elliptic problems.
Journal of Computational and Applied Mathematics, 406:113926, 2022.

L. Mu.
Weak Galerkin based a posteriori error estimates for second order elliptic interface problems on polygonal meshes.
Journal of Computational and Applied Mathematics, 361:413–425, 2019.

L. Mu, J. Wang, Y. Wang, and X. Ye.
A weak Galerkin mixed finite element method for biharmonic equations.
Numerical Solution of Partial Differential Equations: Theory, Algorithms, and Their Applications, pages 247–277, 2013.

L. Mu, J. Wang, and X. Ye.
A new weak Galerkin finite element method for the Helmholtz equation.
IMA Journal of Numerical Analysis, 35(3):1228–1255, 2014.

L. Mu, J. Wang, and X. Ye.
A weak Galerkin finite element method with polynomial reduction.
Journal of Computational and Applied Mathematics, 285:45–58, 2015.

L. Mu, J. Wang, X. Ye, and S. Zhang.
A weak Galerkin finite element method for the Maxwell equations.
Journal of Scientific Computing, 65(1):363–386, 2015.

L. Mu, X. Ye, and J. Wang.
Weak Galerkin finite element methods on polytopal meshes.

Int. J. Numer. Anal. Model., 12:31–53, 2015.

J. Pan and H. Li.
A penalized weak Galerkin spectral element method for second order elliptic equations.
Journal of Computational and Applied Mathematics, 386:113–228, 2021.

J. Wang, X. Wang, X. Ye, S. Zhang, and P. Zhu.
Two-order superconvergence for a weak Galerkin method on rectangular and cuboid grids.
Numerical Methods for Partial Differential Equations, 39(1):744–758, 2023.

J. Wang and X. Ye.
A weak Galerkin finite element method for second-order elliptic problems.
Journal of Computational and Applied Mathematics, 241:103–115, 2013.

J. Wang and X. Ye.
A weak Galerkin finite element method for the Stokes equations.
Advances in Computational Mathematics, 42(1):155–174, 2016.

R. Wang, R. Zhang, X. Zhang, and Z. Zhang.
Supercloseness analysis and polynomial preserving recovery for a class of weak Galerkin methods.
Numerical Methods for Partial Differential Equations, 34(1):317–335, 2018.

H. Xie, Q. Zhai, and R. Zhang.
The weak Galerkin method for eigenvalue problems.
arXiv preprint arXiv:1508.05304, 2015.

X. Ye and S. Zhang.
A stabilizer-free weak Galerkin finite element method on polytopal meshes.
Journal of Computational and Applied Mathematics, 371:112699, 2020.

X. Ye and S. Zhang.
A stabilizer free weak Galerkin finite element method on polytopal mesh: Part II.
Journal of Computational and Applied Mathematics, 394:113525, 2021.

X. Ye and S. Zhang.
A stabilizer free weak Galerkin finite element method on polytopal mesh: Part III.
Journal of Computational and Applied Mathematics, 394:113538, 2021.

X. Ye and S. Zhang.
Order two superconvergence of the CDG finite elements on triangular and tetrahedral meshes.
CSIAM Transactions on Applied Mathematics, 4(2):256–274, 2023.

Q. Zhai, H. Xie, R. Zhang, and Z. Zhang.
Acceleration of weak Galerkin methods for the Laplacian eigenvalue problem.
Journal of Scientific Computing, 79:914–934, 2019.

Q. Zhai, H. Xie, R. Zhang, and Z. Zhang.
The weak Galerkin method for elliptic eigenvalue problems.
Commun. Comput. Phys., 26(1):160–191, 2019.

Q. Zhai and R. Zhang.
Lower and upper bounds of Laplacian eigenvalue problem by weak Galerkin method on triangular meshes.
Discrete and Continuous Dynamical Systems Series B, 24(1):403–413, 2019.

P. Zhu and S. Xie.
Superconvergent weak Galerkin methods for non-self adjoint and indefinite elliptic problems.
Applied Numerical Mathematics, 172:300–314, 2022.

Conclusion

- WG is studied in the framework of SEM in the two-dimensional settings
  - flexible choices of approximation spaces
  - lower bound approximation
  - super convergence
- an alternative high-order approach for solving PDEs

Thank you!