A Fast Fourier-Galerkin Method for Solving Boundary Integral Equations on Torus-Shaped Surfaces

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Numerical Methods for Spectral Problems: Theory and Applications 2024
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Aug 6, 2024

Joined work with Yiying Fang

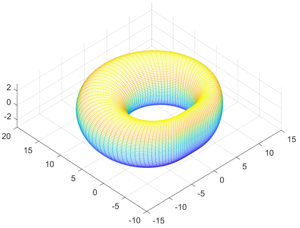
Outline

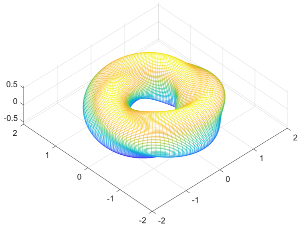
Background

Consider the following problem: A BIE derived from the interior Dirichlet problem for the Laplace equation

$$\begin{aligned} \Delta u &= 0, \quad && \text{in } \Omega, \\ u &= h, \quad && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^3$ represents a bounded domain with a smooth boundary $\partial\Omega$, $\partial\Omega$ is C^∞ diffeomorphism mapped onto a torus, h is a given sufficiently smooth function on $\partial\Omega$. The tours-shaped domain frequently appears in the study of magnetohydrodynamic equilibria and instabilities.





Background

Computational methods

- Finite difference method (FDM)
- Finite element method (FEM)
- Finite volume method (FVM)
- Boundary integral equation (BIE) method

Boundary integral equation methods are applied in various fields:

- Acoustic and electromagnetic wave scattering [Bao 2019; Lai 2019]
- Multi-medium elasticity [Yang 2015]
- Steady and transient heat conduction problems [Jiang 2020]
- Fluid-structure interactions [van Opstal 2015]
- Navier-Stokes equations [Klinteberg 2020] and multiphase flows [Wei 2020]

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Advantages of BIE

- Reduces the problem dimension, eliminating the need for volume mesh generation
- Far-field boundary conditions can be handled more easily
- Ill-conditioning can be avoided
- Achieves high convergence rates with smooth domain boundaries and boundary conditions

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Main Disadvantages of BIE

- The coefficient matrix is dense, scaling as $\mathcal{O}(N^2)$, where $N \sim h^{-1}$ in 2D, and $N \sim h^{-2}$ in 3D
- Requires significant computational resources to generate the coefficient matrix
- Involves high dimensional singular integrations

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Fast Methods for Solving BIEs

- **Fast multipole methods** [Rokhlin and Greengard 1985, 1987]. Complexity $\mathcal{O}(150Np^2)$ for 3D
- **Wavelet methods** [Dahmen, Prossdorf and Schneider 1993; Petersdorff and Schwab 1996, 1997; Micchelli, Xu and Zhao 1997; Chen, Micchelli and Xu 2002; Harbrecht and Schneider 2002, 2006, 2009]. Complexity $\mathcal{O}(N\log^2 N)$ for 3D
- **Nyström methods** [Ying, Biros and Zorin 2006; Kong, Bremer and Rokhlin 2011; Bremer 2012; Fermo and Laurita 2015; Young, Hao and Martinsson 2012]. Complexity $\mathcal{O}(N^2)$ and errors $\mathcal{O}(N^{-\frac{p-1}{2}})$ for 3D
- **Nyström-Fourier methods on axially symmetric surfaces** [Helsing and Karlsson 2014; O'Neil and Cerfon 2018; Lai and O'Neil 2019]. Complexity $\mathcal{O}(N^{3/2})$ and errors $\mathcal{O}(N^{-\frac{p-1}{2}})$ for 3D
- **Fast spectral methods** [Cai and Xu 2008; Jiang, Wang and Xu 2014, 2018; Guang, Jiang and Xu 2020; Jiang and Wang 2021]. A lack of 3D fast spectral algorithms

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2. Fourier-Galerkin Method and a key observation
3. Fast Fourier-Galerkin Method for Solving BIEs
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The solution u of (1) can be represented as (ref. [Kress 2013])

$$u(\mathbf{x}) = \int_{\partial\Omega} \varrho(\mathbf{y}) \frac{\partial}{\partial \boldsymbol{\nu}_{\mathbf{y}}} \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} \right) ds(\mathbf{y}), \quad \mathbf{x} \in \Omega,$$

where $\boldsymbol{\nu}_{\mathbf{y}}$ is the outward unit normal vector, and ϱ is determined by the BIE

$$\varrho(\mathbf{x}) - \frac{1}{2\pi} \int_{\partial\Omega} \varrho(\mathbf{y}) \frac{\partial}{\partial \boldsymbol{\nu}_{\mathbf{y}}} \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} \right) ds(\mathbf{y}) = -\frac{1}{2\pi} h(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega.$$

Substituting the parametric equation Γ of $\partial\Omega$ into (9), we reformulate the boundary integral equation as follows,

$$(\mathcal{I} - \mathcal{K})\rho = g, \tag{2}$$

where $\mathcal{K}$ is the integral operator defined by

$$(\mathcal{K}\rho)(\boldsymbol{\theta}) := \int_{I_{2\pi}^2} K(\boldsymbol{\theta}, \boldsymbol{\eta}) \rho(\boldsymbol{\eta}) d\boldsymbol{\eta},$$

with

$$K(\boldsymbol{\theta}, \boldsymbol{\eta}) := -\frac{1}{2\pi} \frac{(\Gamma(\boldsymbol{\theta}) - \Gamma(\boldsymbol{\eta})) \cdot \left(\frac{\partial \Gamma}{\partial \eta_0} \times \frac{\partial \Gamma}{\partial \eta_1} \right)(\boldsymbol{\eta})}{|\Gamma(\boldsymbol{\theta}) - \Gamma(\boldsymbol{\eta})|^3}. \tag{3}$$

Define

- $e_k(\theta) := \frac{1}{\sqrt{2\pi}} e^{ik\theta}$ for $\theta \in I_{2\pi} := [0, 2\pi]$
- $e_{\mathbf{k}}(\boldsymbol{\theta}) := e_{k_0}(\theta_0) e_{k_1}(\theta_1)$ for $\boldsymbol{\theta} := [\theta_0, \theta_1] \in I_{2\pi}^2$ and $\mathbf{k} := [k_0, k_1] \in \mathbb{Z}^2$
- $\mathbf{K}_N := \left[K_{\mathbf{k},\mathbf{l}} : \mathbf{k}, \mathbf{l} \in \mathbb{Z}^2_{-n,n} \right]$, where $K_{\mathbf{k},\mathbf{l}} := \langle \mathcal{K} e_{\mathbf{l}}, e_{\mathbf{k}} \rangle$, $\mathbb{Z}_{-n,n} := \{-n, \ldots, -1, 0, 1, \ldots, n\}$ and $N = (2n+1)^2$
- $\mathbf{g}_N := \left[\langle g, e_{\mathbf{k}} \rangle : \mathbf{k} \in \mathbb{Z}^2_{-n,n} \right]$

Based on the standard Fourier-Galerkin method, the linear equations of (2) can be obtained

$$(\mathbf{I}_N - \mathbf{K}_N)\boldsymbol{\rho}_N = \mathbf{g}_N, \tag{4}$$

Define

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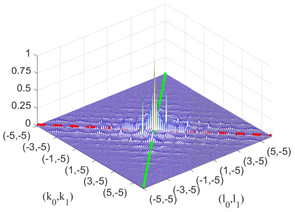
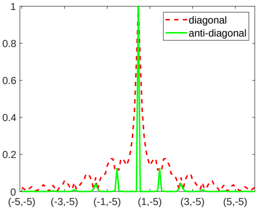
where $\mathbf{I}_N$ is the identity matrix, $\mathbf{K}_N$ is dense.



(a) $\mathbf{K}_N$, $\mathcal{O}(N^2)$



(b) Decay Pattern

Singularity of kernel function

$$K(\boldsymbol{\theta},\boldsymbol{\eta}) = -\frac{1}{2\pi}\frac{(\Gamma(\boldsymbol{\theta})-\Gamma(\boldsymbol{\eta}))\cdot\left(\frac{\partial\Gamma}{\partial\eta_0}\times\frac{\partial\Gamma}{\partial\eta_1}\right)(\boldsymbol{\eta})}{|\Gamma(\boldsymbol{\theta})-\Gamma(\boldsymbol{\eta})|^3}.$$

Taylor's formula

$$\Gamma(\boldsymbol{\theta})-\Gamma(\boldsymbol{\eta}) = \left(\frac{\partial\Gamma}{\partial\eta_0}(\boldsymbol{\eta}),\frac{\partial\Gamma}{\partial\eta_1}(\boldsymbol{\eta})\right)(\boldsymbol{\theta}-\boldsymbol{\eta})+\mathcal{O}(|\boldsymbol{\theta}-\boldsymbol{\eta}|^2),$$

shows

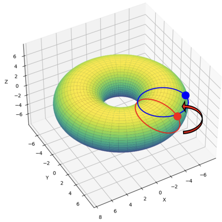
$$K(\boldsymbol{\theta},\boldsymbol{\eta}) = \mathcal{O}\left(|\Gamma(\boldsymbol{\theta})-\Gamma(\boldsymbol{\eta})|^{-1}\right).$$

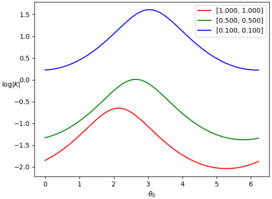
This means $K\in L^1(I_{2\pi}^4)$ and $K\notin L^2(I_{2\pi}^4)$.

Regularity of kernel function

$$K(\boldsymbol{\theta},\boldsymbol{\eta}) = -\frac{1}{2\pi}\frac{(\Gamma(\boldsymbol{\theta})-\Gamma(\boldsymbol{\eta}))\cdot\left(\frac{\partial\Gamma}{\partial\eta_0}\times\frac{\partial\Gamma}{\partial\eta_1}\right)(\boldsymbol{\eta})}{|\Gamma(\boldsymbol{\theta})-\Gamma(\boldsymbol{\eta})|^3}.$$

With fixed $\boldsymbol{\theta}-\boldsymbol{\eta}$ and θ_1, let θ_0 change from 0 to 2π:





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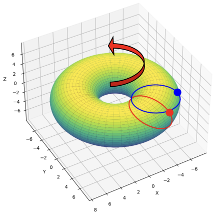

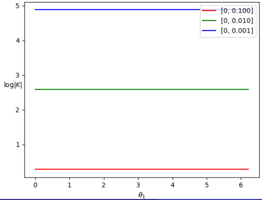

An observation

From the above example,

- consider function $G(\boldsymbol{\theta}, \boldsymbol{\vartheta}) := K(\boldsymbol{\theta}, \boldsymbol{\theta} + \boldsymbol{\vartheta})$ (there holds $\hat{G}_{\mathbf{k},\mathbf{l}} = \hat{K}_{\mathbf{k}-\mathbf{l},\mathbf{l}}$)

$$G(\boldsymbol{\theta}, \boldsymbol{\vartheta}) = -\frac{1}{2\pi} \frac{(\Gamma(\boldsymbol{\theta}) - \Gamma(\boldsymbol{\theta}+\boldsymbol{\vartheta})) \cdot \left(\frac{\partial \Gamma}{\partial \theta_0} \times \frac{\partial \Gamma}{\partial \theta_1}\right)(\boldsymbol{\theta}+\boldsymbol{\vartheta})}{d^{3/2}(\boldsymbol{\theta}, \boldsymbol{\vartheta})},$$

  where $d(\boldsymbol{\theta}, \boldsymbol{\vartheta}) := |\Gamma(\boldsymbol{\theta}) - \Gamma(\boldsymbol{\theta}+\boldsymbol{\vartheta})|_2^2$.

- when Γ is holomorphic in a domain of $\mathbb{C}^2$, for given $\boldsymbol{\vartheta}$ with $\zeta(\boldsymbol{\vartheta}) \neq 0$, function $G(\cdot, \boldsymbol{\vartheta})$ is also holomorphic in a domain within $\mathbb{C}^2$, where $\zeta(\boldsymbol{\theta}) := \sqrt{(\min\{\theta_0, 2\pi - \theta_0\})^2 + (\min\{\theta_1, 2\pi - \theta_1\})^2}$.

Question: *how does this analytic extension domain of function $G(\cdot, \boldsymbol{\vartheta})$ vary with changes in $\boldsymbol{\vartheta}$?* (Find the zero free domain of $d(\cdot, \boldsymbol{\vartheta})$ in $\mathbb{C}^2$.)

Assumption 1

(A1) There exists a constant $C_0 > 0$ such that for all $\boldsymbol{\theta}, \boldsymbol{\eta} \in I_{2\pi}^2$,

$$\|\Gamma(\boldsymbol{\theta}) - \Gamma(\boldsymbol{\eta})\|_2 \geqslant C_0 \zeta(\boldsymbol{\theta} - \boldsymbol{\eta}),$$

where $\zeta(\boldsymbol{\theta}) := \sqrt{(\min\{\theta_0, 2\pi - \theta_0\})^2 + (\min\{\theta_1, 2\pi - \theta_1\})^2}$.

Assumption 2

(A2) There exists a constant $R_0 > 0$ such that the periodic extension of Γ can be analytic extended to $\overline{\mathbb{W}(R_0)^2}$, where $\mathbb{W}(R_0) := \{z = x + iy : -R_0 < y < R_0\}$.

Lemma

If assumptions **(A1)** *and* **(A2)** *hold, then there is $b^* > 0$ and $C_1 > 0$ such that for all $\mathbf{z} \in \overline{\mathbb{W}(b^*)^2}$ and $\boldsymbol{\vartheta} \in I_{2\pi}^2$,*

$$|d(\mathbf{z}, \boldsymbol{\vartheta})| \geqslant C_1 (\zeta(\boldsymbol{\vartheta}))^2.$$

$$G(\boldsymbol{\theta},\boldsymbol{\vartheta}) = -\frac{1}{2\pi}\frac{(\Gamma(\boldsymbol{\theta})-\Gamma(\boldsymbol{\theta}+\boldsymbol{\vartheta}))\cdot\left(\frac{\partial\Gamma}{\partial\theta_0}\times\frac{\partial\Gamma}{\partial\theta_1}\right)(\boldsymbol{\theta}+\boldsymbol{\vartheta})}{d^{3/2}(\boldsymbol{\theta},\boldsymbol{\vartheta})}.$$

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Remark: Here, $0 < b^* < \min\left\{\frac{C_0^2}{\sqrt{2}M_0M_1}, \frac{R_0}{2}\right\}$, M_0 and M_1 are the gredient and Hessian matrix of $d(\cdot,\vartheta)$.

We say $G(\cdot,\boldsymbol{\vartheta})$ can be uniformly analytic extended to $\mathbb{W}(b^*)^2$ with respect to $\boldsymbol{\theta}$, in the following sense:

if there is a function $\tilde{G}$ defined on $\mathbb{W}(b^*)^2 \otimes I_{2\pi}^2$ such that for all $\boldsymbol{\vartheta} \in I_{2\pi}^2$ with $\zeta(\boldsymbol{\vartheta}) \neq 0$, $\tilde{G}(\cdot,\boldsymbol{\vartheta})$ is holomorphic on $\mathbb{W}(b^*)^2$ and satisfies $\tilde{G}(\cdot,\boldsymbol{\vartheta}) = G(\cdot,\boldsymbol{\vartheta})$ on $I_{2\pi}^2$.

Lemma

If assumptions **(A1)** *and* **(A2)** *hold, then there exists a positive constant* c *such that for all* $\mathbf{z} \in \overline{\mathbb{W}(b^*)^2}$ *and* $\boldsymbol{\vartheta} \in l^2_{2\pi}$,

$$\left|(\Gamma(\mathbf{z}) - \Gamma(\mathbf{z}+\boldsymbol{\vartheta})) \cdot \left(\frac{\partial \Gamma}{\partial z_0} \times \frac{\partial \Gamma}{\partial z_1}\right)(\mathbf{z}+\boldsymbol{\vartheta})\right| \leqslant c(\zeta(\boldsymbol{\vartheta}))^2. \tag{5}$$

a

Proposition

If assumptions **(A1)** and **(A2)** hold, then $G(\cdot, \boldsymbol{\vartheta})$ can be uniformly analytic extended to $\overline{\mathbb{W}(b^*)^2}$ with respect to $\boldsymbol{\theta}$, and there exists a positive constant c such that for all $\mathbf{z} \in \overline{\mathbb{W}(b^*)^2}$ and $\boldsymbol{\vartheta} \in l^2_{2\pi}$,

$$|G(\mathbf{z}, \boldsymbol{\vartheta})| \leqslant c(\zeta(\boldsymbol{\vartheta}))^{-1}. \tag{6}$$

[a] V. Scheidemann, Introduction to Complex Analysis in Several Variables, Birkhäuser Cham, 2023.

By conducting some computation,

$$\left|(\widehat{G(\cdot,\vartheta)})_{\mathbf{k}}\right| \leqslant S_{b^*,G}(\vartheta)e^{-b^*\|\mathbf{k}\|_1},$$

where $S_{b^*,G}(\vartheta) := 2\pi \sup\{|G(\mathbf{z},\vartheta)| : \mathbf{z} = (z_0, z_1) \in \mathbb{C}^2, \text{ and } |\mathrm{Im}(z_0)| = |\mathrm{Im}(z_1)| = b^*\}$.

Theorem

If assumptions **(A1)** *and* **(A2)** *hold, then there exists $M > 0$ such that for all $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^2$, there holds* $\left|\hat{G}_{\mathbf{k},\mathbf{l}}\right| \leqslant \frac{M}{2\pi}e^{-b^*\|\mathbf{k}\|_1}$.

[b]

Corollary

If assumptions **(A1)** *and* **(A2)** *hold, then there exists a constant $M > 0$ such that for all $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^2$,*

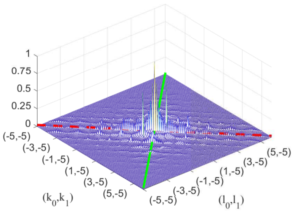
$$|K_{\mathbf{k},\mathbf{l}}| \leqslant \frac{M}{2\pi}e^{-b^*\|\mathbf{l}-\mathbf{k}\|_1}.$$

[b] V. Scheidemann, Introduction to Complex Analysis in Several Variables, Birkhäuser

Summary of this section

- $G(\cdot, \vartheta)$ can be uniformly analytic extended to $\mathbb{W}(b^*)^2$ with respect to $\boldsymbol{\theta}$. (Remark: this to be an inherent property of the Green's function itself, suggesting that a similar conclusion may also be applicable to general surfaces.)
- $\mathbf{K}_N$ is dense and can be compressed, since

$$|K_{\mathbf{k},\mathbf{l}}| \leqslant \frac{M}{2\pi} e^{-b^* \|\mathbf{l}-\mathbf{k}\|_1}$$



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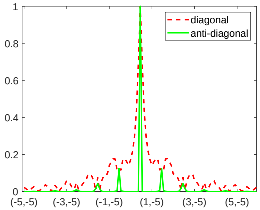

(b) Decay Pattern

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1. Background
2. Fourier-Galerkin Method and a key observation
3. Fast Fourier-Galerkin Method for Solving BIEs
4. Numerical Experiments

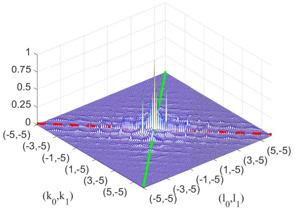
Fast Fourier-Galerkin Method for Solving BIEs

Let $\mathbb{L}_N(q) := \left\{(\mathbf{k},\mathbf{l}) : \mathbf{k},\mathbf{l} \in \mathbb{Z}^2_{-n,n} \text{ and } \|\mathbf{k}-\mathbf{l}\|_1 \leqslant q \ln N\right\}$, for $n \in \mathbb{N}$ and $q > 0$, where $N := (2n+1)^2$. For each pair $(\mathbf{k},\mathbf{l}) \in \mathbb{Z}^4_{-n,n}$, define

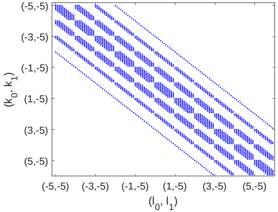
$$\widetilde{K}_{\mathbf{k},\mathbf{l}} := \begin{cases} K_{\mathbf{k},\mathbf{l}}, & (\mathbf{k},\mathbf{l}) \in \mathbb{L}_N(q), \\ 0, & \text{otherwise.} \end{cases}$$

The fast Fourier-Galerkin method which is to find $\tilde{\boldsymbol{\rho}}_N$ that satisfies

$$(\mathbf{I}_N - \widetilde{\mathbf{K}}_N)\tilde{\boldsymbol{\rho}}_N = \mathbf{g}_N. \tag{7}$$



(a) $\mathbf{K}_N$, $\mathcal{O}(N^2)$



(b) $\mathbb{L}_N(q)$, $\mathcal{O}(N \ln^2 N)$

Main results

Theorem

Let $p \in \mathbb{N}$ and assumptions **(A1)-(A2)** *hold. If the operator $\mathcal{I} - \mathcal{K}$ is injective from $L_2(I_{2\pi}^2)$ to $L_2(I_{2\pi}^2)$, the right hand side g is in $H_p(I_{2\pi}^2)$, and $q > 0$ with $qb^* \geqslant p/2$, then there exists a positive constant c such that for sufficiently large n and any $\varphi \in H_p(I_{2\pi}^2)$,*

$$\|(\mathcal{I} - \widetilde{\mathcal{K}}_n)\mathcal{P}_n\varphi\| \geqslant c\|\varphi\|,$$

and

$$\|\rho - \tilde{\rho}_n\| \leqslant cN^{-p/2} \ln N \|\rho\|_{H_p},$$

where $\widetilde{\mathcal{K}}_n$ and $\tilde{\rho}_n$ depend on q.

Main results

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where $\widetilde{\mathcal{K}}_n$ and $\tilde{\rho}_n$ depend on q.

	Classical Fourier-Galerkin	Fast Fourier-Galerkin
Sparsity	$\mathcal{O}(N^2)$	$\mathcal{O}(N \ln N)$
Convergency	Optimal	Optimal
Satbility	Yes	Yes

Outline

Example 1.Consider solving BIE (9) with $h(\mathbf{x}) = 2|\mathbf{x} - \Gamma(\pi,\pi)|^2 \ln|\mathbf{x} - \Gamma(\pi,\pi)|$ on the bagel-shaped surface $\partial\Omega$, represented by

$$\Gamma = \begin{bmatrix} (5+3\cos\theta_0)(1+0.5\sin\theta_1)\cos\theta_1 \\ (5+3\cos\theta_0)(1+0.5\sin\theta_1)\sin\theta_1 \\ 3\sin\theta_0 \end{bmatrix}$$

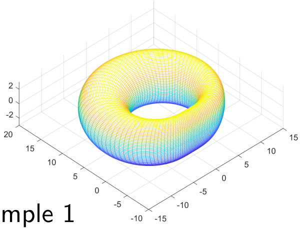

Table 1: Numerical results for Example 1

n	non-truncated			truncated			
	e_n	C.O.	Cond	$\tilde{e}_n$	C.O.	Cond	C.R.
15	8.8193e-06	-	3.9851	1.0644e-05	-	3.9851	0.3443
25	3.1172e-06	2.04	3.9851	3.2692e-06	2.31	3.9851	0.2026
35	1.4445e-06	2.29	3.9851	1.5206e-06	2.27	3.9851	0.1243
45	7.0882e-07	2.83	3.9851	7.6028e-07	2.76	3.9851	0.0868

Numerical Experiments

Example 2.Consider solving BIE (9) with $h(\mathbf{x}) = 2|x_0 - \pi|^2 \ln|x_0 - \pi|$ on the cruller surface $\partial\Omega$, represented by

$$\Gamma = \begin{bmatrix} (1+\omega(\boldsymbol{\theta})\cos\theta_0)\cos\theta_1 \\ (1+\omega(\boldsymbol{\theta})\cos\theta_0)\sin\theta_1 \\ \omega(\boldsymbol{\theta})\sin\theta_0 \end{bmatrix}$$

where $\omega(\boldsymbol{\theta}) := \frac{1}{2} + 0.065\cos(3\theta_0 + 3\theta_1)$

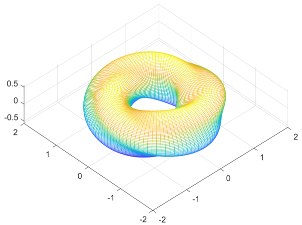

Table 2: Numerical results for Example 2

n	non-truncated			truncated			
	e_n	C.O.	Cond	$\tilde{e}_n$	C.O.	Cond	C.R.
15	2.1270e-03	-	3.6919	2.7481e-03	-	3.6923	0.3443
25	7.2865e-04	2.10	3.6919	7.9016e-04	2.44	3.6920	0.2026
35	3.4019e-04	2.26	3.6919	4.0777e-04	1.97	3.6919	0.1243
45	1.7482e-04	2.65	3.6919	2.3748e-04	2.15	3.6919	0.0868

Thank you!