On the eigen-spectra of Riesz derivative operator

Changpin LI

Shanghai University

Joint work with Drs Min CAI and Caiyu JIAO

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Section 1

Riesz derivative on $\mathbb{R}^n$

Riesz derivative defined on $\mathbb{R}^n$

Definition 1 [Samko, Kilbas, Marichev, 1993]

For suitably smooth function $u(\mathbf{x})$ defined on $\mathbb{R}^n$, Riesz derivative is given by

$$
{}_{RZ}\mathrm{D}_{\mathbf{x}}^{\alpha} u(\mathbf{x}) = \frac{1}{d_{n,l}(\alpha)} \int_{\mathbb{R}^n} \frac{(\Delta_{\mathbf{y}}^{l} u)(\mathbf{x})}{|\mathbf{y}|^{n+\alpha}} \mathrm{d}\mathbf{y}, \ 0 < \alpha < l.
$$

Here l can be arbitrary integer satisfying $l > \alpha$, and $\left(\Delta_{\mathbf{y}}^{l} u\right)(\mathbf{x})$ denotes the centred differences

$$
\left(\Delta_{\mathbf{h}}^{l} u\right)(\mathbf{x}) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} u\left[\mathbf{x} + \left(\frac{l}{2} - k\right)\mathbf{h}\right]
$$

(with a vector step $\mathbf{h}$ and center $\mathbf{x}$) or non-centered differences

$$
\left(\Delta_{\mathbf{h}}^{l} u\right)(\mathbf{x}) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} u(\mathbf{x} - k\mathbf{h}).
$$

Definition 1 (continued)

The normalizing constant $d_{n,l}(\alpha)$ is the analytic function of the parameter α given by the relation

$$d_{n,l}(\alpha) = \beta_n(\alpha) \frac{A_l(\alpha)}{\sin\left(\frac{\pi\alpha}{2}\right)},$$

with

$$\beta_n(\alpha) = \frac{\pi^{1+\frac{n}{2}}}{2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)},$$

and

$$A_l(\alpha) = \begin{cases} \sum_{k=0}^{l} (-1)^{k-1} \binom{l}{k} k^\alpha, & \text{in the case of non-centred difference,} \\ 2 \sum_{k=0}^{[\frac{l}{2}]} (-1)^{k-1} \binom{l}{k} \left(\frac{l}{2} - k\right)^\alpha, & \text{in the case of centred difference,} \end{cases}$$

except for the case of a centred difference with an odd integer l, when $d_{n,l}(\alpha) = 0$.

Remark 1

In the case of centred difference with $l = 2$ and $\mathbf{h} = \mathbf{y}$, a simple calculation yields that for $0 < \alpha < 2$,

$$
{}_{RZ}\mathrm{D}_{\mathbf{x}}^{\alpha} u(\mathbf{x}) = -\frac{2^{\alpha-2}\alpha\Gamma\left(\frac{n+\alpha}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \frac{u(\mathbf{x}+\mathbf{y}) - 2u(\mathbf{x}) + u(\mathbf{x}-\mathbf{y})}{|\mathbf{y}|^{n+\alpha}} \mathrm{d}\mathbf{y}. \tag{1.1}
$$

Schwartz space

Definition 2 [Stein, Shakarchi, 2003]

The Schwartz space $\mathcal{S}(\mathbb{R})$ consists of a set of all indefinitely differentiable functions $u(x)$ such that $u(x)$ and all its derivatives $u'(x), u''(x), \ldots, u^{(l)}(x), \ldots$ are rapidly decreasing, in the sense that

$$\sup_{x\in\mathbb{R}} |x|^k |u^{(l)}(x)| < \infty \text{ for any } k,\ l \in \mathbb{Z}^+.$$

The Schwartz space on $\mathbb{R}^n$ is the function space

$$\mathcal{S}(\mathbb{R}^n) = \{u(\mathbf{x}) \in C^\infty(\mathbb{R}^n) : \|u(\mathbf{x})\|_{\mathbf{s},\mathbf{k}} < \infty\},$$

where $\mathbf{s}, \mathbf{k}$ are multi-indices and

$$\|u(\mathbf{x})\|_{\mathbf{s},\mathbf{k}} = \sup_{\mathbf{x}\in\mathbb{R}^n} \left| \mathbf{x}^{\mathbf{s}} \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\mathbf{k}} u(\mathbf{x}) \right|.$$

Fractional Laplacian

Definition 3

For any $u(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0,1)$, the fractional Laplacian $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(\mathbf{x}) = C(n,s)\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+2s}}\mathrm{d}\mathbf{y}$$

Here $\mathrm{P.V.}$ is a commonly used abbreviation for "in the principle value sense" and $C(n,s)$ is a dimensional constant that depends on n and s, precisely given by

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-\cos\zeta_1}{|\zeta|^{n+2s}}\mathrm{d}\zeta\right)^{-1}$$

with ζ_1 being the first component of $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{R}^n$.

Remark 2 [Nezza, Palatucci, Valdinoci, 2011]

Let $(-\Delta)^s$ with $0 < s < 1$ be the fractional Laplacian. Then for any $u(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^n)$, it holds for $\mathbf{x} \in \mathbb{R}^n$ that

$$(-\Delta)^s u(\mathbf{x}) = -\frac{1}{2} C(n,s) \int_{\mathbb{R}^n} \frac{u(\mathbf{x}+\mathbf{y}) - 2u(\mathbf{x}) + u(\mathbf{x}-\mathbf{y})}{|\mathbf{y}|^{n+2s}} \mathrm{d}\mathbf{y}. \quad (1.2)$$

Equivalence of Riesz derivative and fractional Laplacian on $\mathbb{R}^n$

Theorem 1 [Cai, Li, 2019]

For any $u(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^n)$, it holds that

$$(-\Delta)^{\frac{\alpha}{2}} u(\mathbf{x}) = {}_{RZ}\mathrm{D}_{\mathbf{x}}^{\alpha} u(\mathbf{x}),\ 0 < \alpha < 2.$$

Sketch of proof for Theorem 1

Denote $\zeta = (\zeta_1, \zeta^{(n-1)}) \in \mathbb{R}^n$ with $\zeta^{(n-1)} \in \mathbb{R}^{n-1}$ $(n = 2, 3, \ldots)$, and define $\eta^{(n-1)} = \zeta^{(n-1)}/|\zeta_1| \in \mathbb{R}^{n-1}$. We have

$$\frac{1}{C(n,s)} = \int_{\mathbb{R}} \frac{1-\cos\zeta_1}{|\zeta_1|^{n+2s}} \mathrm{d}\zeta_1 \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1 + |\zeta^{(n-1)}|^2/|\zeta_1|^2\right)^{\frac{n+2s}{2}}} \mathrm{d}\zeta^{(n-1)}$$

$$= \int_{\mathbb{R}} \frac{1-\cos\zeta_1}{|\zeta_1|^{1+2s}} \mathrm{d}\zeta_1 \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1 + |\eta^{(n-1)}|^2\right)^{\frac{n+2s}{2}}} \mathrm{d}\eta^{(n-1)} = \mathrm{I}_1 \cdot \mathrm{I}_2.$$

Here $\mathrm{I}_1 = \int_{\mathbb{R}} \frac{1-\cos\zeta_1}{|\zeta_1|^{1+2s}} \mathrm{d}\zeta_1 = \frac{\pi^{\frac{1}{2}}\Gamma(1-s)}{2^{2s} s\Gamma\left(\frac{1+2s}{2}\right)}$ and $\mathrm{I}_2 = \int_{\mathbb{R}^{n-1}} \frac{\mathrm{d}\eta^{(n-1)}}{\left(1+|\eta^{(n-1)}|^2\right)^{\frac{n+2s}{2}}}$.
For I_2, we utilize the polar coordinates. Then it holds that

$$\mathrm{I}_2 = \int_{\mathbb{R}^{n-1}} \frac{\mathrm{d}\eta^{(n-1)}}{\left(1 + |\eta^{(n-1)}|^2\right)^{\frac{n+2s}{2}}} = \omega_{n-2} \int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n+2s}{2}}} \mathrm{d}\rho,$$

where $\omega_{n-2} = \frac{(n-1)\pi^{\frac{n-1}{2}}}{\Gamma\left(1+\frac{n-1}{2}\right)}$ represents the $(n-2)$ dimensional measure of the unit sphere S^{n-2}.

Thus we have

$$\begin{aligned}
\mathrm{I}_2 =& \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^{\frac{\pi}{2}} (\sin\theta)^{n-2}(\cos\theta)^{2s}\mathrm{d}\theta \\
=& \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} B\left(\frac{n-1}{2}, \frac{1+2s}{2}\right) = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{1+2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)},
\end{aligned}$$

where $B(x,y)$ denotes the Beta function. Therefore,

$$C(n,s) = \frac{1}{\mathrm{I}_1 \cdot \mathrm{I}_2} = \frac{2^{2s}s\Gamma\left(\frac{1+2s}{2}\right)}{\pi^{\frac{1}{2}}\Gamma(1-s)} \cdot \frac{\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{1+2s}{2}\right)} = \frac{2^{2s}s\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(1-s)}.$$

In view of the above discussion and Remark 2, for $u(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0,1)$, fractional Laplacian can be expressed as

$$(-\Delta)^s u(\mathbf{x}) = -\frac{2^{2s-1}s\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(1-s)} \int_{\mathbb{R}^n} \frac{u(\mathbf{x}+\mathbf{y}) - 2u(\mathbf{x}) + u(\mathbf{x}-\mathbf{y})}{|\mathbf{y}|^{n+2s}}\mathrm{d}\mathbf{y}. \tag{1.3}$$

Comparing the expressions (1.1) and (1.3), we obtain the desired result.

Remark 3

If $u(x) \in \mathcal{S}(\mathbb{R})$, it follows from Theorem 1 that

$$-(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{{}_{RL}\mathrm{D}^{\alpha}_{-\infty,x} u(x) + {}_{RL}\mathrm{D}^{\alpha}_{x,+\infty} u(x)}{2\cos(\frac{\pi\alpha}{2})}, \ \alpha \in (0,1) \cup (1,2),$$

with

$${}_{RL}\mathrm{D}^{\alpha}_{-\infty,x} u(x) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_{-\infty}^{x} \frac{u(t)}{(x-t)^{\alpha-m+1}} \mathrm{d}t, \ \alpha \in (m-1, m),$$

and

$${}_{RL}\mathrm{D}^{\alpha}_{x,+\infty} u(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_{x}^{+\infty} \frac{u(t)}{(t-x)^{\alpha-m+1}} \mathrm{d}t, \ \alpha \in (m-1, m)$$

being the left and right-sided Liouville derivatives.

Sketch of proof for Remark 3

Firstly, we rewrite fractional Laplacian on $\mathbb{R}$ in an explicit form. Setting $n = 1$ in (1.3) yields

$$(-\Delta)^s u(x) = -\frac{2^{2s-1} s\Gamma\left(\frac{1+2s}{2}\right)}{\pi^{\frac{1}{2}}\Gamma(1-s)} \int_{-\infty}^{+\infty} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{1+2s}} \mathrm{d}y. \tag{1.4}$$

The infinite integral in (1.4) makes sense for any $u(x) \in \mathcal{S}(\mathbb{R})$ and arbitrary $s \in (0,1)$. To see this, we recall that Gamma function is a meromorphic function with simple pole only at each negative integer and zero. In addition, a 2-nd order Taylor expansion yields

$$\frac{|u(x+y) - 2u(x) + u(x-y)|}{|y|^{1+2s}} \leqslant \frac{\|u''\|_{L^\infty}}{|y|^{2s-1}},$$

which is integrable near 0 (for any fixed $s \in (0,1)$).
When $0 < \alpha < 1$ and $u(x) \in \mathcal{S}(\mathbb{R})$, it holds that

$$_{RL}\mathrm{D}^{\alpha}_{-\infty,x} u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{+\infty} \frac{u(x-y)}{y^\alpha} \mathrm{d}y,$$

and

$$_{RL}\mathrm{D}^{\alpha}_{x,+\infty} u(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{+\infty} \frac{u(x+y)}{y^\alpha} \mathrm{d}y.$$

Let $s = \frac{\alpha}{2}$ in (1.4). Integration by parts yields

$$
\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} u(x) &= -\frac{2^{\alpha-1}\alpha\Gamma\left(\frac{1+\alpha}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{2-\alpha}{2}\right)} \int_0^{+\infty} \frac{u(x+y)-2u(x)+u(x-y)}{y^{\alpha+1}} \mathrm{d}y \\
&= -\frac{2^{\alpha-1}\Gamma\left(\frac{1+\alpha}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)} \int_0^{+\infty} \frac{u'(x+y)-u'(x-y)}{y^{\alpha}} \mathrm{d}y \\
&= \frac{2^{\alpha-1}\Gamma\left(\frac{1+\alpha}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{+\infty} \frac{u(x-y)-u(x+y)}{y^{\alpha}} \mathrm{d}y,
\end{aligned}
\tag{1.5}
$$

in which $u(x) \in \mathcal{S}(\mathbb{R})$ is utilized. Here the interchange of integration and differentiation is guaranteed by the uniform convergence of the integral $\int_0^{+\infty} \frac{u'(x+y)-u'(x-y)}{y^{\alpha}} \mathrm{d}y$ and convergence of the integral $\int_0^{+\infty} \frac{u(x-y)-u(x+y)}{y^{\alpha}} \mathrm{d}y$ due to $u(x) \in \mathcal{S}(\mathbb{R})$. Since

$$
\frac{1}{2\cos(\frac{\pi\alpha}{2})\Gamma(1-\alpha)} = \frac{\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{1+\alpha}{2})}{2\pi\Gamma(1-\alpha)} = \frac{2^{\alpha-1}\Gamma(\frac{1+\alpha}{2})}{\pi^{\frac{1}{2}}\Gamma(1-\frac{\alpha}{2})},
$$

it holds for $0 < \alpha < 1$ that

$$
-(-\Delta)^{\frac{\alpha}{2}} u(x) = \frac{{}_{RL}\mathrm{D}^{\alpha}_{-\infty,x} u(x) + {}_{RL}\mathrm{D}^{\alpha}_{x,+\infty} u(x)}{-2\cos(\frac{\pi\alpha}{2})}. \tag{1.6}
$$

In the case with $1 < \alpha < 2$, there hold

$$ {}_{RL}\mathrm{D}^{\alpha}_{-\infty,x}u(x) = \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^{+\infty}\frac{u(x-y)}{y^{\alpha-1}}\mathrm{d}y, $$

and

$$ {}_{RL}\mathrm{D}^{\alpha}_{x,+\infty}u(x) = \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^{+\infty}\frac{u(x+y)}{y^{\alpha-1}}\mathrm{d}y. $$

Performing integration by parts on (1.5), we have

$$ \begin{aligned} (-\Delta)^{\frac{\alpha}{2}}u(x) =& \frac{2^{\alpha-1}\Gamma\left(\frac{1+\alpha}{2}\right)}{\pi^{\frac{1}{2}}(1-\alpha)\Gamma\left(1-\frac{\alpha}{2}\right)}\int_0^{+\infty}\frac{u''(x+y)+u''(x-y)}{y^{\alpha-1}}\mathrm{d}y \\ =& \frac{2^{\alpha-1}\Gamma\left(\frac{1+\alpha}{2}\right)}{\pi^{\frac{1}{2}}(1-\alpha)\Gamma\left(1-\frac{\alpha}{2}\right)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^{+\infty}\frac{u(x+y)+u(x-y)}{y^{\alpha-1}}\mathrm{d}y. \end{aligned} $$

Here the interchange of integration and differentiation is ensured by the convergence of the integral $\int_0^{+\infty}\frac{u(x+y)+u(x-y)}{y^{\alpha-1}}\mathrm{d}y$ and the uniform convergence of the integrals $\int_0^{+\infty}\frac{u^{(k)}(x+y)+u^{(k)}(x-y)}{y^{\alpha-1}}\mathrm{d}y$ $(k=1,2)$ due to $u \in \mathcal{S}(\mathbb{R})$. Note that

$$ \frac{1}{2\cos(\frac{\pi\alpha}{2})\Gamma(2-\alpha)} = \frac{2^{\alpha-1}\Gamma(\frac{1+\alpha}{2})}{\pi^{\frac{1}{2}}(1-\alpha)\Gamma(\frac{2-\alpha}{2})}. $$

The relation (1.6) is therefore valid for $1 < \alpha < 2$.

Section 2

Difference between Riesz derivative and fractional Laplacian on the proper subset of $\mathbb{R}$

Riemann-Liouville integral on proper subsets of $\mathbb{R}$

Let $\Omega_{a^-} = (-\infty, a)$ and $\Omega_{a^+} = (a, +\infty)$ be semi-lines; and $\Omega_{a^+b^-} = (a, b)$ be a bounded segment.

Definition 4

For $m - 1 < \alpha < m \in \mathbb{Z}^+$ and a given function $f(x)$ defined on Ω, its α-th left- and right-sided Riemann-Liouville derivatives are defined as follows,

$$
{}_{RL}\mathrm{D}_{a,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(y)}{(x-y)^{1-\alpha}} \mathrm{d}y, \;\; x \in \Omega_{a^+b^-},
$$

$$
{}_{RL}\mathrm{D}_{-\infty,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{f(y)}{(x-y)^{1-\alpha}} \mathrm{d}y, \;\; x \in \Omega_{a^-},
$$

$$
{}_{RL}\mathrm{D}_{a,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(y)}{(x-y)^{1-\alpha}} \mathrm{d}y, \;\; x \in \Omega_{a^+},
$$

$$
{}_{RL}\mathrm{D}_{-\infty,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{f(y)}{(x-y)^{1-\alpha}} \mathrm{d}y, \;\; x \in \mathbb{R}.
$$

Definition 5

For $\alpha > 0$ and a given function $f(x)$ defined on Ω_{a+b-}, Ω_{a-}, Ω_{a+}, and $\mathbb{R}$, its α-th right-sided Riemann-Liouville integral is defined below,

$$ {}_{RL}\mathrm{D}_{x,b}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(y)}{(y-x)^{1-\alpha}} \mathrm{d}y, \;\; x \in \Omega_{a+b-}, $$

$$ {}_{RL}\mathrm{D}_{x,a}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^a \frac{f(y)}{(y-x)^{1-\alpha}} \mathrm{d}y, \;\; x \in \Omega_{a-}, $$

$$ {}_{RL}\mathrm{D}_{x,+\infty}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \frac{f(y)}{(y-x)^{1-\alpha}} \mathrm{d}y, \;\; x \in \Omega_{a+}, $$

$$ {}_{RL}\mathrm{D}_{x,+\infty}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \frac{f(y)}{(y-x)^{1-\alpha}} \mathrm{d}y, \;\; x \in \mathbb{R}. $$

Riemann-Liouville derivative on proper subsets of $\mathbb{R}$

Definition 6

For $m-1<\alpha<m\in\mathbb{Z}^+$ and a given function $f(x)$ defined on Ω, its α-th left- and right-sided Riemann-Liouville derivatives are defined as follows,

$$_{RL}\mathrm{D}^{\alpha}_{a,x}f(x)=\frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_a^x\frac{f(y)}{(x-y)^{\alpha-m+1}}\mathrm{d}y,\ \ x\in\Omega_{a+b-},$$

$$_{RL}\mathrm{D}^{\alpha}_{x,b}f(x)=\frac{(-1)^m}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_x^b\frac{f(y)}{(y-x)^{\alpha-m+1}}\mathrm{d}y,\ \ x\in\Omega_{a+b-};$$

$$_{RL}\mathrm{D}^{\alpha}_{-\infty,x}f(x)=\frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_{-\infty}^x\frac{f(y)}{(x-y)^{\alpha-m+1}}\mathrm{d}y,\ \ x\in\Omega_{a-},$$

$$_{RL}\mathrm{D}^{\alpha}_{x,a}f(x)=\frac{(-1)^m}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_x^a\frac{f(y)}{(y-x)^{\alpha-m+1}}\mathrm{d}y,\ \ x\in\Omega_{a-};$$

Definition 6 (continued)

$$
{}_{RL}\mathrm{D}^{\alpha}_{a,x} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_a^x \frac{f(y)}{(x-y)^{\alpha-m+1}} \mathrm{d}y, \ \ x \in \Omega_{a+},
$$

$$
{}_{RL}\mathrm{D}^{\alpha}_{x,+\infty} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_x^{+\infty} \frac{f(y)}{(y-x)^{\alpha-m+1}} \mathrm{d}y, \ \ x \in \Omega_{a+};
$$

$$
{}_{RL}\mathrm{D}^{\alpha}_{-\infty,x} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_{-\infty}^x \frac{f(y)}{(x-y)^{\alpha-m+1}} \mathrm{d}y, \ \ x \in \mathbb{R},
$$

$$
{}_{RL}\mathrm{D}^{\alpha}_{x,+\infty} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_x^{+\infty} \frac{f(y)}{(y-x)^{\alpha-m+1}} \mathrm{d}y, \ \ x \in \mathbb{R}.
$$

Riesz derivative on proper subsets of $\mathbb{R}$

Definition 7

For $m-1<\alpha<m\in\mathbb{Z}^+$ and function $f(x)$ defined on Ω, its α-th Riesz derivative is defined below,

$$ {}_{RZ}\mathrm{D}^{\alpha}_{x,\Omega_{a+b-}} f(x) = -c_\alpha \left({}_{RL}\mathrm{D}^{\alpha}_{a,x} + {}_{RL}\mathrm{D}^{\alpha}_{x,b} \right) f(x),\ x\in\Omega=\Omega_{a+b-}, $$

$$ {}_{RZ}\mathrm{D}^{\alpha}_{x,\Omega_{a-}} f(x) = -c_\alpha \left({}_{RL}\mathrm{D}^{\alpha}_{-\infty,x} + {}_{RL}\mathrm{D}^{\alpha}_{x,a} \right) f(x),\ x\in\Omega=\Omega_{a-}, $$

$$ {}_{RZ}\mathrm{D}^{\alpha}_{x,\Omega_{a+}} f(x) = -c_\alpha \left({}_{RL}\mathrm{D}^{\alpha}_{a,x} + {}_{RL}\mathrm{D}^{\alpha}_{x,+\infty} \right) f(x),\ x\in\Omega=\Omega_{a+}, $$

$$ {}_{RZ}\mathrm{D}^{\alpha}_{x,\mathbb{R}} f(x) = -c_\alpha \left({}_{RL}\mathrm{D}^{\alpha}_{-\infty,x} + {}_{RL}\mathrm{D}^{\alpha}_{x,+\infty} \right) f(x),\ x\in\Omega=\mathbb{R}, $$

where $c_\alpha = \frac{1}{2\cos\frac{\alpha\pi}{2}}$, $\alpha\neq 1,3,5,\ldots$.

Fractional Laplacian on proper subsets of $\mathbb{R}$

Definition 8

For a given function $f(x)$ defined on Ω, its α-th fractional Laplacians defined below,

$$(-\Delta)_{\Omega}^{\frac{\alpha}{2}} f(x) = C\left(1, \frac{\alpha}{2}\right) \text{P.V.} \int_{\Omega} \frac{f(x) - f(y)}{|x-y|^{1+\alpha}} \mathrm{d}y.$$
$$= C\left(1, \frac{\alpha}{2}\right) \lim_{\varepsilon \to 0^+} \int_{|y-x| \geqslant \varepsilon} \frac{f(x) - f(y)}{|x-y|^{1+\alpha}} \mathrm{d}y, \quad x, y \in \Omega.$$

Here $\Omega = \Omega_{a^+b^-}$, Ω_{a^-} or Ω_{a^+}, $C\left(1, \frac{\alpha}{2}\right) = \left(\int_{-\infty}^{+\infty} \frac{1-\cos t}{|t|^{1+\alpha}} \mathrm{d}t\right)^{-1}$.

Difference between Riesz derivative and fractional Laplacian on proper subsets of $\mathbb{R}$

Theorem 2 [Jiao, Khaliq, Li, Wang, 2021]

Let $C^{\beta}(\Omega)$ be β-Hölder continuous space and $C^{1,\beta}(\Omega) = \{f : f'(x) \in C^{\beta}(\Omega)\}$.

(i) If $f(x) \in C_b^{\lceil\alpha\rceil}(\Omega_{a+b-})$, $\alpha \in (0,1) \cup (1,2)$, then it holds for any fixed $x \in \Omega_{a+b-}$ that

$$-(-\Delta)^{\frac{\alpha}{2}}_{\Omega_{a+b-}} f(x) = {}_{RZ}\mathrm{D}^{\alpha}_{x,\Omega_{a+b-}} f(x) + \frac{c_\alpha f(x)}{\Gamma(1-\alpha)}\left[\frac{1}{(x-a)^{\alpha}} + \frac{1}{(b-x)^{\alpha}}\right].$$

(ii) If $f(x) \in C_b^{\lceil\alpha\rceil}(\Omega_{a-}) \cap L^1(\Omega_{a-})$, $\alpha \in (0,1) \cup (1,2)$, then it holds for any fixed $x \in \Omega_{a-}$ that

$$-(-\Delta)^{\frac{\alpha}{2}}_{\Omega_{a-}} f(x) = {}_{RZ}\mathrm{D}^{\alpha}_{x,\Omega_{a-}} f(x) + \frac{c_\alpha}{\Gamma(1-\alpha)}\frac{f(x)}{(a-x)^{\alpha}}.$$

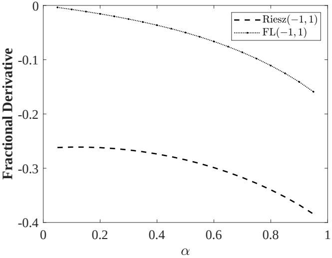
(iii) If $f(x) \in C_b^{\lceil\alpha\rceil}(\Omega_{a+}) \cap L^1(\Omega_{a+})$, $\alpha \in (0,1) \cup (1,2)$, then it holds for any fixed $x \in \Omega_{a+}$ that

$$-(-\Delta)^{\frac{\alpha}{2}}_{\Omega_{a+}} f(x) = {}_{RZ}\mathrm{D}^{\alpha}_{x,\Omega_{a+}} f(x) + \frac{c_\alpha}{\Gamma(1-\alpha)}\frac{f(x)}{(x-a)^{\alpha}}.$$

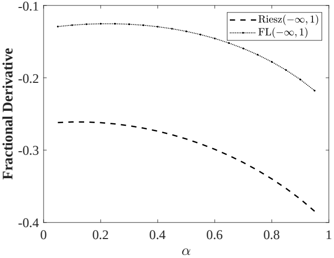
Example 1

$$f(x) = \begin{cases} \exp\left(\dfrac{1}{x^2-1}\right), & x \in (-1,1), \\ 0, & x \in \mathbb{R} \setminus (-1,1). \end{cases} \tag{2.1}$$

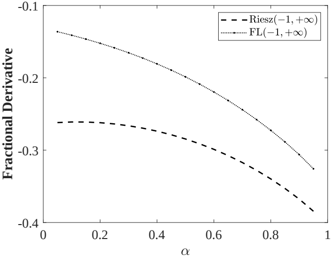
It is easy to verify $f^{(n)}(x) = 0$ for $x \in \mathbb{R} \setminus (-1,1), n = 0, 1, 2 \ldots$ and $f(x) \in \mathcal{S}(\mathbb{R})$.



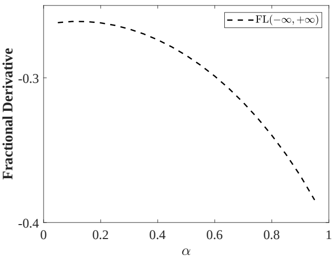
(a) domain $(-1,1)$



(b) domain $(-\infty,1)$

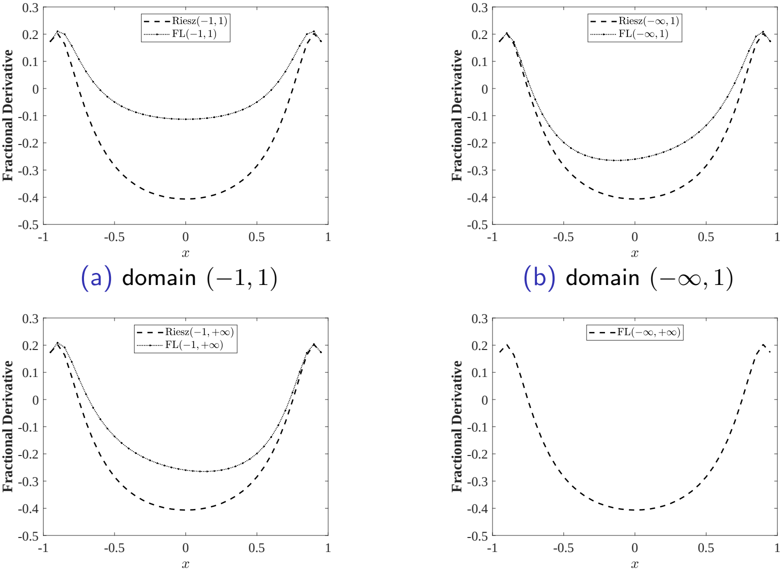


(c) domain $(-1,+\infty)$



(d) domain $(-\infty,+\infty)$

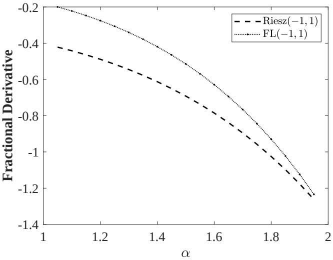
Figure: Fractional Laplacian and Riesz derivative of $f(x)$ at $x=0.5$.



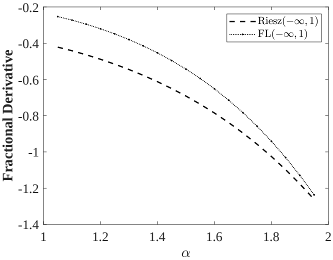
(a) domain $(-1, 1)$ (b) domain $(-\infty, 1)$

(c) domain $(-1, +\infty)$ (d) domain $(-\infty, +\infty)$

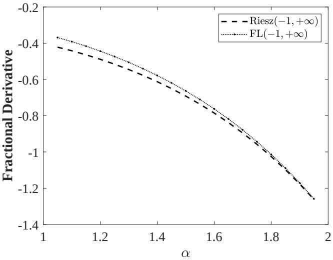
Figure: Fractional Laplacian and Riesz derivative of $f(x)$ for $\alpha = 0.5$.



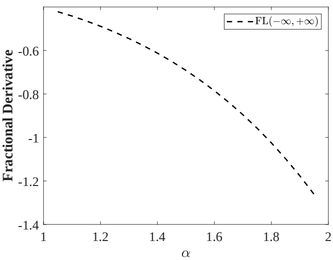
(a) domain $(-1,1)$



(b) domain $(-\infty,1)$

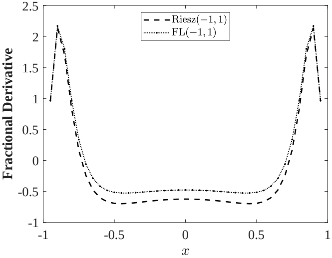


(c) domain $(-1,+\infty)$

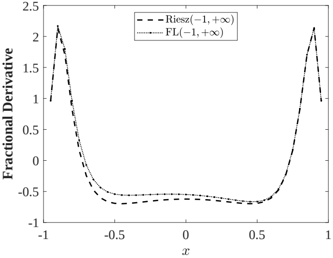


(d) domain $(-\infty,+\infty)$

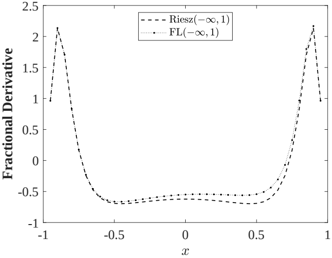
Figure: Fractional Laplacian and Riesz derivative of $f(x)$ at $x = 0.5$.



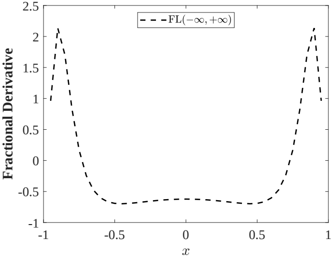
(a) domain $(-1,1)$



(b) domain $(-\infty,1)$



(c) domain $(-1,+\infty)$



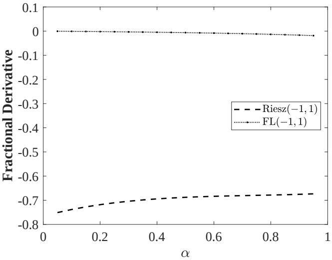
(d) domain $(-\infty,+\infty)$

Figure: Fractional Laplacian and Riesz derivative of $f(x)$ for $\alpha = 1.5$.

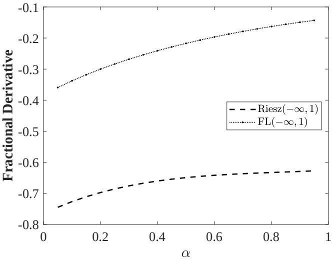
Example 2

$$f(x) = \begin{cases} \exp\left(\frac{1}{x^2-4}\right), & x \in (-2,2), \\ 0, & x \in \mathbb{R} \setminus (-2,2). \end{cases} \tag{2.2}$$

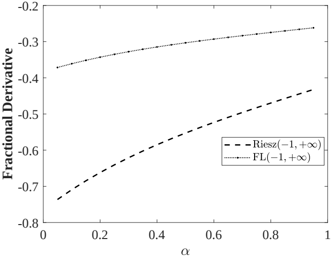
It is also easy to verify $f^{(n)}(x) = 0$ for $x \in \mathbb{R} \setminus (-2,2), n = 0,1,2,\ldots$ and $f(x) \in \mathcal{S}(\mathbb{R})$. Unlike Example 1, the compact support of $f^{(n)}(x)$, $n = 0,1,2,\ldots$ is different, rather than $(-1,1)$, so that differences between fractional Laplacian and Riesz derivative on $(-\infty,1)$ and $(-1,+\infty)$ can be given more obviously.



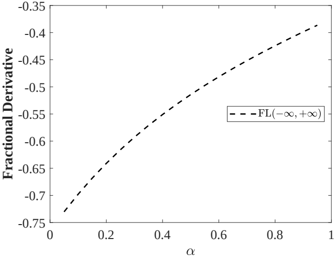
(a) domain $(-1, 1)$



(b) domain $(-\infty, 1)$

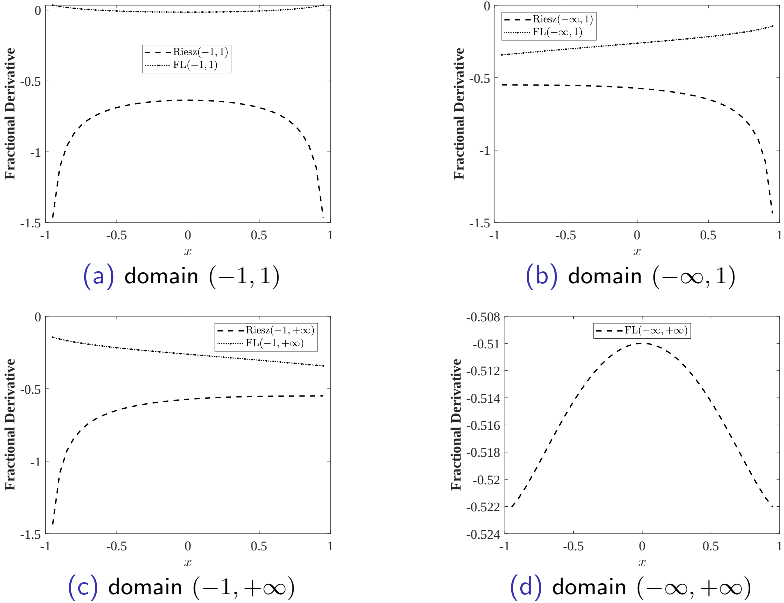


(c) domain $(-1, +\infty)$



(d) domain $(-\infty, +\infty)$

Figure: Fractional Laplacian and Riesz derivative of $f(x)$ at $x = 0.5$.



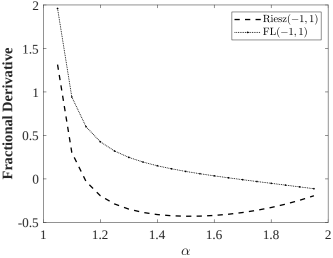
(a) domain $(-1,1)$

(b) domain $(-\infty,1)$

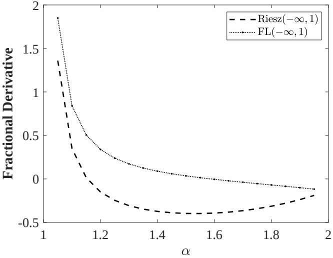
(c) domain $(-1,+\infty)$

(d) domain $(-\infty,+\infty)$

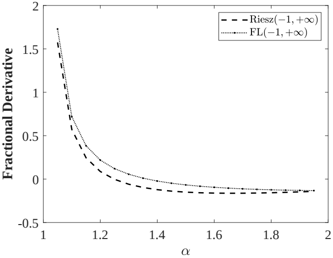
Figure: Fractional Laplacian and Riesz derivative of $f(x)$ for $\alpha = 0.5$.



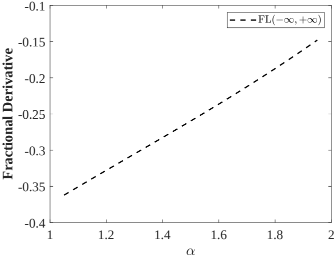
(a) domain $(-1,1)$



(b) domain $(-\infty,1)$

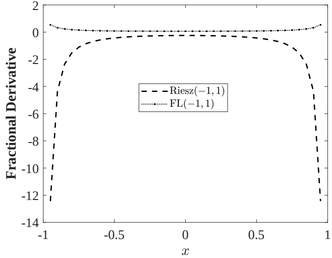


(c) domain $(-1,+\infty)$

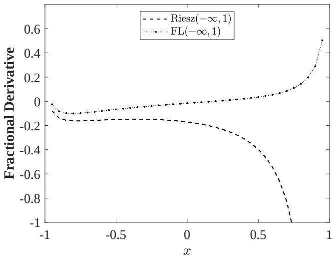


(d) domain $(-\infty,+\infty)$

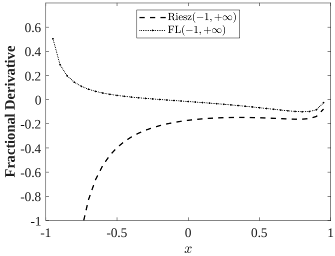
Figure: Fractional Laplacian and Riesz derivative of $f(x)$ at $x=0.5$.



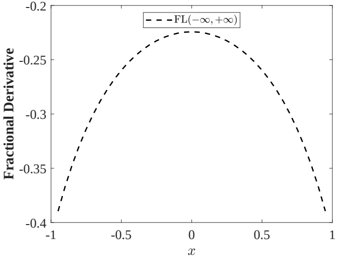
(a) domain $(-1,1)$



(b) domain $(-\infty,1)$



(c) domain $(-1,+\infty)$



(d) domain $(-\infty,+\infty)$

Figure: Fractional Laplacian and Riesz derivative of $f(x)$ for $\alpha = 1.5$.

Section 3

Eigen-spectra of Riesz derivative

Null space of the operator ${}_{RZ}\mathrm{D}_x^{\alpha}$ with $\alpha \in (0,1)$

Theorem 3 [Cai, Li, 2019(b)]

Let $0 < \alpha < 1$ and $x \in \Omega = (0,1)$. Then the null space of the space ${}_{RZ}\mathrm{D}_x^{\alpha}$ is given by

$$\mathcal{N}\{{}_{RZ}\mathrm{D}_x^{\alpha}\} = \mathrm{Span}\left\{x^{\frac{\alpha}{2}}(1-x)^{\frac{\alpha}{2}-1}, x^{\frac{\alpha}{2}-1}(1-x)^{\frac{\alpha}{2}}\right\}.$$

Jacobi polynomials on $[0,1]$

Definition

For $x \in [0,1]$, Jacobi polynomials can be defined as

$$G_n^{(\mu,\nu)}(x) = \sum_{k=0}^{n} g_{n,k}^{(\mu,\nu)} x^k, \ n \geqslant 0,$$

where the coefficients $g_{n,k}^{(\mu,\nu)}$ are given by

$$g_{n,k}^{(\mu,\nu)} = \frac{(-1)^{n+k}\Gamma(n+\nu+1)\Gamma(n+k+\mu+\nu+1)}{\Gamma(k+1)\Gamma(n-k+1)\Gamma(n+\mu+\nu+1)\Gamma(k+\nu+1)}.$$

Let $\Omega = (0,1)$ and $\rho^{(\mu,\nu)}(x) = (1-x)^{\mu}x^{\nu}$ with $\mu, \nu > -1$. Then there holds

$$\int_0^1 \rho^{(\mu,\nu)}(x) G_m^{(\mu,\nu)}(x) G_n^{(\mu,\nu)}(x) \mathrm{d}x = \begin{cases} 0, & m \neq n, \\ \left\| G_n^{(\mu,\nu)}(x) \right\|_{L^2(\Omega,\, \rho^{(\mu,\nu)})}^2, & m = n, \end{cases}$$

where

$$\left\| G_n^{(\mu,\nu)}(x) \right\|_{L^2(\Omega,\, \rho^{(\mu,\nu)})}^2 = \frac{\Gamma(n+\mu+1)\Gamma(n+\nu+1)}{(2n+\mu+\nu+1)\Gamma(n+\mu+\nu+1)\Gamma(n+1)}.$$

Eigen-spectra of Riesz derivative with $\alpha \in (0,1)$

Theorem 4 [Cai, Li, 2019(b)]

Let $0 < \alpha < 1$, $x \in \Omega = (0,1)$, and $\omega(x) = (1-x)^{\frac{\alpha}{2}} x^{\frac{\alpha}{2}}$. For $n = 0, 1, 2, \ldots$, there holds

$$ {}_{RZ}\mathrm{D}_x^{\alpha} \left(\omega(x) G_n^{(\frac{\alpha}{2},\frac{\alpha}{2})}(x) \right) = \lambda_n G_n^{(\frac{\alpha}{2},\frac{\alpha}{2})}(x). $$

Here the constants λ_n, $n \geqslant 0$, are given by

$$ \lambda_n = -\frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)}. $$

Sketch of proof for Theorem 4

Let $\rho^{(\mu,\nu)}(x) = (1-x)^{\mu} x^{\nu}$ and $\omega(x) = \rho^{(\frac{\alpha}{2},\frac{\alpha}{2})}(x) = (1-x)^{\frac{\alpha}{2}} x^{\frac{\alpha}{2}}$. Define

$$\mathrm{I}_x^{1-\alpha} = {}_{RL}\mathrm{D}_{0,x}^{-(1-\alpha)} - {}_{RL}\mathrm{D}_{x,1}^{-(1-\alpha)}, \ 0 < \alpha < 1.$$

Then for arbitrary $\alpha \in (0,1)$, there holds

$$\mathrm{I}_x^{1-\alpha}\left(\omega(x)x^n\right) = \left({}_{RL}\mathrm{D}_{0,x}^{-(1-\alpha)} - {}_{RL}\mathrm{D}_{x,1}^{-(1-\alpha)}\right)\left(\omega(x)x^n\right) = \sum_{k=0}^{n+1} c_{n,k} x^k,$$

and

$$ {}_{RZ}\mathrm{D}_x^{\alpha}\left(\omega(x)x^n\right) = -\frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{I}_x^{1-\alpha}\left(\omega(x)x^n\right) = \sum_{k=0}^{n} d_{n,k} x^k. \quad (3.1)$$

Here

$$c_{n,k} = \frac{(-1)^{n+k+1} 2\cos(\frac{\pi\alpha}{2})\Gamma(1+\frac{\alpha}{2})\Gamma(k+\alpha)}{\Gamma(k-n+\frac{\alpha}{2})\Gamma(n-k+2)\Gamma(k+1)},$$

and

$$d_{n,k} = -\frac{(k+1)c_{n,k+1}}{2\cos\left(\frac{\pi\alpha}{2}\right)} = \frac{(-1)^{n+k+1}\Gamma(1+\frac{\alpha}{2})\Gamma(k+1+\alpha)}{\Gamma(k+1-n+\frac{\alpha}{2})\Gamma(n-k+1)\Gamma(k+1)}.$$

When $n = 0$, there holds

$$G_n^{(\frac{\alpha}{2},\frac{\alpha}{2})}(x) = g_{0,0}^{(\frac{\alpha}{2},\frac{\alpha}{2})} = 1.$$

Then the equality (3.1) with $n = 0$ yields

$${}_{RZ}\mathrm{D}_x^\alpha \left(\omega(x) G_n^{(\frac{\alpha}{2},\frac{\alpha}{2})}(x)\right) = d_{0,0} = -\Gamma(1+\alpha) = \lambda_0.$$

Let $n \geqslant 1$ and choose arbitrary $p(x) \in \mathcal{P}_{n-1}(x)$. In view of relation (3.1), there exists a certain polynomial $\widetilde{p}(x) \in \mathcal{P}_{n-1}(x)$ such that

$${}_{RZ}\mathrm{D}_x^\alpha \left(\omega(x) p(x)\right) = \widetilde{p}(x).$$

In view of

$$\left(G_n^{(\mu,\nu)}(x), p(x)\right)_{L^2\left(\Omega,\, \rho^{(\mu,\nu)}\right)} = 0,\ \forall p(x) \in \mathcal{P}_{n-1}(x),\ n \geqslant 1,$$

and

$$\left({}_{RZ}\mathrm{D}_x^\alpha u, v\right)_{L^2(\Omega)} = \left(u, {}_{RZ}\mathrm{D}_x^\alpha v\right)_{L^2(\Omega)},$$

there holds

$$\begin{aligned}
&\left({}_{RZ}\mathrm{D}_x^{\alpha}\left(\omega(x)G_n^{(\frac{\alpha}{2},\frac{\alpha}{2})}(x)\right), p(x)\right)_{L^2(\Omega,\omega)} \\
=&\left({}_{RZ}\mathrm{D}_x^{\alpha}\left(\omega(x)G_n^{(\frac{\alpha}{2},\frac{\alpha}{2})}(x)\right), \omega(x)p(x)\right)_{L^2(\Omega)} \\
=&\left(\omega(x)G_n^{(\frac{\alpha}{2},\frac{\alpha}{2})}(x), {}_{RZ}\mathrm{D}_x^{\alpha}\left(\omega(x)p(x)\right)\right)_{L^2(\Omega)} \\
=&\left(\omega(x)G_n^{(\frac{\alpha}{2},\frac{\alpha}{2})}(x), \widetilde{p}(x)\right)_{L^2(\Omega)} = 0.
\end{aligned}$$

Note also that ${}_{RZ}\mathrm{D}_x^{\alpha}\left(\omega(x)G_n^{(\frac{\alpha}{2},\frac{\alpha}{2})}(x)\right) \in \mathcal{P}_n(x)$. There exits a constant C such that

$${}_{RZ}\mathrm{D}_x^{\alpha}\left(\omega(x)G_n^{(\frac{\alpha}{2},\frac{\alpha}{2})}(x)\right) = CG_n^{(\frac{\alpha}{2},\frac{\alpha}{2})}(x).$$

Comparing the coefficients of x^k on both sides of the above equation, we have

$$C = d_{n,n} = -\frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)} = \lambda_n.$$

The proof is thus completed.

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Thanks for your attention!