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Construction of fractional pseudospectral differentiation matrices with applications

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Motiviation

- Fractional Calculus shows potential applications such as anomalous diffusion
- Numerical solutions are challenging caused by singularity and nonlocality
- Fast,higher-order numerical method is needed
- Spectral collocation method is popular

Related works

- Spectral method: Li and Xu (2009,SIAM Numer Anal; 2010, Commun Comput Phys); Chen, Shen and Wang (2016, Math Comput); Mao and Shen (2016, J Comput Phys); Mao, Chen and Shen (2016, Appl Numer Math); Mao and Karniadakis (2018, SIAM Numer Anal)
- Spectral collocation method: Li, Zeng and Liu(2012, FCAA); Doha, Bhrawy and Ezz-Eldien(2012, Appl Math Modelling); Tian, Deng and Wu(2014, Numer Method PDEs); Zayernouri and Karniadakis (2014, SIAM J Sci Comput; 2015, J Comput Phys); Zayernouri, Ainsworth and Karniadakis(2015, Comput Meth Appl Mech Engng); Zeng and Karniadakis (2015,2017, SIAM J Sci Comput); Jiao, Wang and Huang (2016, J Comput Phys)
- Spectral element method(Multi-domian spectral method): Xu and Hesthaven(2014, J Comput Phys); Chen,Xu and Hesthaven(2015, J Comput Phys); Mao and Shen (2018, Adv Comput Math); Zhao, Mao and Karniadakis(2019, Comput Meth Appl Mech Engng)

our work

- ★ The main contribution of this work is to represent some fractional differentiation matrices as a product of several special matrices
- ★ This representation gives not only a direct, fast and stable algorithm of fractional differentiation matrices
- ★ but also more information which can be used for inverse or preconditioning

Definition (left- and right-sided Riemann–Liouville integrals)

For a function $f(x)$ on $(a,b) \subseteq \mathbb{R}$, the μth order left- and right-sided Riemann–Liouville integrals are defined, respectively, as

$$ {}_{RL}\mathrm{D}_{a,x}^{-\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-\eta)^{\mu-1} f(\eta)\mathrm{d}\eta, \quad \mu > 0, \tag{1} $$

and

$$ {}_{RL}\mathrm{D}_{x,b}^{-\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (\eta-x)^{\mu-1} f(\eta)\mathrm{d}\eta, \quad \mu > 0. \tag{2} $$

Definition (left- and right-sided Riemann–Liouville derivatives)

For a function $f(x)$ on $(a, b) \subseteq \mathbb{R}$, the μth order left- and right-sided Riemann–Liouville derivatives are defined as

$$ {}_{RL}\mathrm{D}^{\mu}_{a,x} f(x) = \frac{1}{\Gamma(m-\mu)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m} \int_{a}^{x} (x-\eta)^{m-\mu-1} f(\eta) \mathrm{d}\eta, \tag{3} $$

and

$$ {}_{RL}\mathrm{D}^{\mu}_{x,b} f(x) = \frac{1}{\Gamma(m-\mu)} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m} \int_{x}^{b} (\eta-x)^{m-\mu-1} f(\eta) \mathrm{d}\eta, \tag{4} $$

respectively. Here and in the subsequent sections, m is a positive integer satisfying $m-1 < \mu < m, m \in \mathbb{N}$.

Definitions of Fractional Calculus

Definition (left- and right-sided Caputo derivative)

For a function $f(x)$ defined on $(a, b) \subseteq \mathbb{R}$, the μth order left- and right-sided Caputo derivative is defined, respectively, as

$$ {}_{C}\mathrm{D}^{\mu}_{a,x} f(x) = \frac{1}{\Gamma(m-\mu)} \int_a^x (x-\eta)^{m-\mu-1} f^{(m)}(\eta) \mathrm{d}\eta, \tag{5} $$

and

$$ {}_{C}\mathrm{D}^{\mu}_{x,b} f(x) = \frac{(-1)^m}{\Gamma(m-\mu)} \int_x^b (\eta-x)^{m-\mu-1} f^{(m)}(\eta) \mathrm{d}\eta, \tag{6} $$

Definition (Riesz derivative)

Let $0 < \mu < 2, \mu \neq 1$. The Riesz derivative is defined as

$$ {}_{RZ}\mathrm{D}^{\mu}_{x} f(x) = -\frac{1}{2\cos\left(\frac{\pi\mu}{2}\right)} \left[{}_{RL}\mathrm{D}^{\mu}_{a,x} f(x) + {}_{RL}\mathrm{D}^{\mu}_{x,b} f(x) \right]. \tag{7} $$

2.2 Jacobi Polynomials

Jacobi polynomials $P_n^{\alpha,\beta}(s), s \in I$ with parameters $\alpha, \beta \in \mathbb{R}$ are defined as

$$\begin{aligned} P_n^{\alpha,\beta}(s) = &\frac{(\alpha+1)_n}{n!} + \sum_{j=1}^{n-1} \frac{(n+\alpha+\beta+1)_j(\alpha+j+1)_{n-j}}{j!(n-j)!} \left(\frac{s-1}{2}\right)^j \\ &+ \frac{(n+\alpha+\beta+1)_n}{n!} \left(\frac{s-1}{2}\right)^n, \quad n \geq 1, \end{aligned} \tag{8}$$

and $P_0^{\alpha,\beta}(s) = 1$.

The well-known three-term recurrence relationship of Jacobi polynomials $P_n^{\alpha,\beta}(s)$ with parameters $\alpha, \beta \in \mathbb{R}$ is fulfilled for $-(\alpha+\beta+1) \notin \mathbb{N}^+$:

$$\begin{gathered} P_{n+1}^{\alpha,\beta}(s) = \left(A_n^{\alpha,\beta} s - B_n^{\alpha,\beta}\right) P_n^{\alpha,\beta}(s) - C_n^{\alpha,\beta} P_{n-1}^{\alpha,\beta}(s), n \geqslant 1 \\ P_0^{\alpha,\beta}(s) = 1, \quad P_1^{\alpha,\beta}(s) = \frac{\alpha+\beta+2}{2} s + \frac{\alpha-\beta}{2} \end{gathered} \tag{9}$$

where

$$\begin{cases} A_n^{\alpha,\beta} = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)} \\ B_n^{\alpha,\beta} = \frac{\left(\beta^2-\alpha^2\right)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)} \\ C_n^{\alpha,\beta} = \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)} \end{cases}$$

Jacobi-Jacobi transformation

In [7], Shen, Wang and Xia (2019, Math Comput) proposed an algorithm of Fast structured Jacobi-Jacobi transforms.

Theorem

Let $\alpha_1, \beta_1 > -1$. *If*

$$\sum_{k=0}^{n} a_k P_k^{\alpha_1,\beta_1}(s) = \sum_{k=0}^{n} b_k P_k^{\alpha_2,\beta_2}(s),$$

then there exists a unique transform matrix $\mathbf{T}_n^{J_2\to J_1}$ *with* $J_1 = (\alpha_1, \beta_1)$ *and* $J_2 = (\alpha_2, \beta_2)$ *such that*

$$\mathbf{a} = \mathbf{T}_n^{J_2\to J_1}\mathbf{b},$$

where $\mathbf{a} = [a_0, \cdots, a_n]^T$, $\mathbf{b} = [b_0, \cdots, b_n]^T$ *and the* (i,j)*-th entry, denoted as* $t_{i,j}^{J_1,J_2}$, *of* $\mathbf{T}_n^{J_2\to J_1}$ *can be generated by*

Jacobi-Jacobi transformation(continuous)

Theorem

$$
\begin{aligned}
&t_{i,j}^{J_1,J_2} = 0, \quad j < i,\\
&t_{i,j}^{J_1,J_2} = \widetilde{a}_{i,j}^{J_1,J_2}\ t_{i+1,j-1}^{J_1,J_2} + \widetilde{b}_{i,j}^{J_1,J_2}\ t_{i,j-1}^{J_1,J_2} + \widetilde{c}_{i,j}^{J_1,J_2}\ t_{i-1,j-1}^{J_1,J_2} - C_{j-1}^{J_2}\ t_{i,j-2}^{J_1,J_2}, \quad j \geq i \geq 1,\\
&t_{0,j}^{J_1,J_2} = \widetilde{a}_{0,j}^{J_1,J_2}\ t_{1,j-1}^{J_1,J_2} + \widetilde{b}_{0,j}^{J_1,J_2}\ t_{0,j-1}^{J_1,J_2} - C_{j-1}^{J_2}\ t_{0,j-2}^{J_1,J_2}, \quad j \geq 1,\\
&t_{0,1}^{J_1,J_2} = \widetilde{b}_{0,1}^{J_1,J_2}\ t_{0,0}^{J_1,J_2}, \quad t_{0,0}^{J_1,J_2} = 1
\end{aligned}
$$

with

$$
\begin{aligned}
\widetilde{a}_{i,j}^{J_1,J_2} &= A_{j-1}^{J_2}\frac{1}{A_i^{J_1}}\frac{\gamma_{i+1}^{J_1}}{\gamma_i^{J_1}} = \frac{2(i+\alpha_1+1)(i+\beta_1+1)}{(2i+\alpha_1+\beta_1+2)(2i+\alpha_1+\beta_1+3)}A_{j-1}^{J_2},\\
\widetilde{b}_{i,j}^{J_1,J_2} &= A_{j-1}^{J_2}\frac{B_i^{J_1}}{A_i^{J_1}} - B_{j-1}^{J_2} = \frac{\beta_1^2-\alpha_1^2}{(2i+\alpha_1+\beta_1)(2i+\alpha_1+\beta_1+2)}A_{j-1}^{J_2} - B_{j-1}^{J_2},\\
\widetilde{c}_{i,j}^{J_1,J_2} &= A_{j-1}^{J_2}\frac{C_i^{J_1}}{A_i^{J_1}}\frac{\gamma_{i-1}^{J_1}}{\gamma_i^{J_1}} = \frac{2i(i+\alpha_1+\beta_1)}{(2i+\alpha_1+\beta_1-1)(2i+\alpha_1+\beta_1)}A_{j-1}^{J_2}.
\end{aligned}
$$

Lemma (Spectral relationships)

Let $\mu > 0, \alpha \in \mathbb{R}, \beta > -1$. *Then, the following relations are true*

$$
\begin{aligned}
{}_{RL}\mathrm{D}_{-1,s}^{-\mu}\{(1+s)^{\beta} P_n^{\alpha,\beta}(s)\} &= \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\mu+1)}(1+s)^{\beta+\mu} P_n^{\alpha-\mu,\beta+\mu}(s), \\
{}_{RL}\mathrm{D}_{-1,s}^{\mu}\{(1+s)^{\beta+\mu} P_n^{\alpha-\mu,\beta+\mu}(s)\} &= \frac{\Gamma(n+\beta+\mu+1)}{\Gamma(n+\beta+1)}(1+s)^{\beta} P_n^{\alpha,\beta}(s),
\end{aligned} \tag{10}
$$

and the following relations are true for $\alpha > -1, \beta \in \mathbb{R}$

$$
\begin{aligned}
{}_{RL}\mathrm{D}_{s,1}^{-\mu}\{(1-s)^{\alpha} P_n^{\alpha,\beta}(s)\} &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\mu+1)}(1-s)^{\alpha+\mu} P_n^{\alpha+\mu,\beta-\mu}(s), \\
{}_{RL}\mathrm{D}_{s,1}^{\mu}\{(1-s)^{\alpha+\mu} P_n^{\alpha+\mu,\beta-\mu}(s)\} &= \frac{\Gamma(n+\alpha+\mu+1)}{\Gamma(n+\alpha+1)}(1-s)^{\alpha} P_n^{\alpha,\beta}(s).
\end{aligned} \tag{11}
$$

Pseudospectral Differentiation/Integration Matrix Setup

- ★ Collocation points: Gauss-type quadrature nodes $\{x_j\}_{j=0}^N \subseteq [a, b]$ (e.g., Gauss-Lobatto, Gauss-Radau)
- ★ Interpolate using Lagrange basis functions

$$f(x) \approx f_N(x) = \sum_{j=0}^{N} f(x_j)L_j(x), \tag{12}$$

with $L_j(x) = \prod_{i=0,i\neq j}^{N} \left(\frac{x-x_i}{x_j-x_i}\right), \quad j = 0, 1, \cdots, N.$

- ★ Apply integral/derivative operator on $L_j(x)$ at nodes $\{x_j\}_{j=0}^N$:

$$(\mathbf{F}^\mu)_{ij} = [\mathrm{F}^\mu L_j](x_i), \quad i, j = 0, 1, \cdots, N.$$

Notations

We have 7 fractional Differentiation/Integration matrices as:

- $({}_{RL}\mathbf{D}_l^{-\mu})_{ij} = [{}_{RL}\mathrm{D}_{-1,x}^{-\mu} L_j](x_i)$ –Left Riemann-Liouville integral.
- $({}_{RL}\mathbf{D}_r^{-\mu})_{ij} = [{}_{RL}\mathrm{D}_{x,1}^{-\mu} L_j](x_i)$ –Right Riemann-Liouville integral.
- $({}_{RL}\mathbf{D}_l^{\mu})_{ij} = [{}_{RL}\mathrm{D}_{-1,x}^{\mu} L_j](x_i)$ –Left Riemann-Liouville derivative.
- $({}_{RL}\mathbf{D}_r^{\mu})_{ij} = [{}_{RL}\mathrm{D}_{x,1}^{\mu} L_j](x_i)$ –Right Riemann-Liouville derivative.
- $({}_{C}\mathbf{D}_l^{\mu})_{ij} = [{}_{C}\mathrm{D}_{-1,x}^{\mu} L_j](x_i)$ –Left Caputo derivative.
- $({}_{C}\mathbf{D}_r^{\mu})_{ij} = [{}_{C}\mathrm{D}_{x,1}^{\mu} L_j](x_i)$ –Right Caputo derivative.
- $({}_{RZ}\mathbf{D}^{\mu})_{ij} = [{}_{RZ}\mathrm{D}_{x}^{\mu} L_j](x_i)$ –Riesz derivative.

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Pseudospectral Differentiation/Integration Matrix

Theorem

The first row of ${}_{RL}\mathbf{D}_l^{-\mu}, {}_{C}\mathbf{D}_l^{\mu}$ *and the last row of* ${}_{RL}\mathbf{D}_r^{-\mu}, {}_{C}\mathbf{D}_r^{\mu}$ *are all zeros, e.g., for* $j = 0, \cdots, N$

$$({}_{RL}\mathbf{D}_l^{-\mu})_{0,j} = ({}_{RL}\mathbf{D}_r^{-\mu})_{N,j} = ({}_{C}\mathbf{D}_l^{\mu})_{0,j} = ({}_{C}\mathbf{D}_r^{\mu})_{N,j} = 0,$$

and $({}_{RL}\mathbf{D}_l^{\mu})_{0,0} = ({}_{RL}\mathbf{D}_r^{\mu})_{N,N} = \infty$*. Moreover, if* $0 < \mu < 1$*, we have*

$$({}_{RL}\mathbf{D}_l^{\mu})_{i,j} = ({}_{C}\mathbf{D}_l^{\mu})_{i,j}, \quad j \neq 0, \tag{13}$$

and

$$({}_{RL}\mathbf{D}_r^{\mu})_{i,j} = ({}_{C}\mathbf{D}_r^{\mu})_{i,j}, \quad j \neq N. \tag{14}$$

Pseudospectral Differentiation/Integration Matrix

For $j = 0, 1, \cdots, N$, we expand $L_j(x)$ as

$$L_j(x) = \sum_{k=0}^{N} h_{kj}(x+1)^k = \sum_{k=0}^{N} \overline{h}_{kj}(1-x)^k. \tag{15}$$

We collect the coefficients two matrices $\mathbf{H}_l$ and $\mathbf{H}_r$ as

$$(\mathbf{H}_l)_{ij} = h_{ij}, \quad (\mathbf{H}_r)_{ij} = \overline{h}_{ij}. \tag{16}$$

Let us introduce three $(N+1)$-element vectors as

$$(\mathbf{v}_l^\mu)_i = (x_i+1)^\mu, \quad (\mathbf{v}_r^\mu)_i = (1-x_i)^\mu, \quad (\mathbf{c}^\mu)_i = \frac{\Gamma(i+1)}{\Gamma(i+\mu+1)}, \tag{17}$$

and two $(N+1) \times (N+1)$ matrices

$$(\mathbf{B}_l)_{ij} = (1+x_i)^j, \quad (\mathbf{B}_r)_{ij} = (1-x_i)^j. \tag{18}$$

Theorem

The following representations of the pseudospectral integration matrix are valid,

$$\begin{aligned} {}_{RL}\mathbf{D}_l^{-\mu} &= \mathrm{diag}(\mathbf{v}_l^{\mu})\mathbf{B}_l \mathrm{diag}(\mathbf{c}^{\mu})\mathbf{H}_l, \\ {}_{RL}\mathbf{D}_r^{-\mu} &= \mathrm{diag}(\mathbf{v}_r^{\mu})\mathbf{B}_r \mathrm{diag}(\mathbf{c}^{\mu})\mathbf{H}_r, \end{aligned} \tag{19}$$

where $\mathrm{diag}(\cdot)$ *is a diagonal matrix with the vector in brackets as its diagonal line. Similarly, by replacing* $-\mu$ *with* μ*, the following representations of the pseudospectral differentiation matrix are found to be valid*

$$\begin{aligned} {}_{RL}\mathbf{D}_l^{\mu} &= \mathrm{diag}(\mathbf{v}_l^{-\mu})\mathbf{B}_l \mathrm{diag}(\mathbf{c}^{-\mu})\mathbf{L}_l, \\ {}_{RL}\mathbf{D}_r^{\mu} &= \mathrm{diag}(\mathbf{v}_r^{-\mu})\mathbf{B}_r \mathrm{diag}(\mathbf{c}^{-\mu})\mathbf{L}_r. \end{aligned} \tag{20}$$

Pseudospectral Differentiation/Integration Matrix

In fact, since the Lagrange basis function $L_j(x)$ satisfies $L_j(x_i) = \delta_{ij}$, we also have the inverse relation as

$$\mathbf{B}_l \mathbf{H}_l = \mathbf{I}, \quad \mathbf{B}_r \mathbf{H}_r = \mathbf{I}, \tag{21}$$

where $\mathbf{I}$ is the identity matrix of $(N+1) \times (N+1)$. Noting that $(\mathrm{diag}(\mathbf{v}_l^{\mu}))^{-1} = \mathrm{diag}(\mathbf{v}_l^{-\mu})$, from the above Theorem and the relation (21), the inverse of the pseudospectral integration/differentiation matrix may be formally written as

$$\begin{aligned} ({}_{RL}\mathbf{D}_l^{-\mu})^{-1} &= \mathbf{B}_l (\mathrm{diag}(\mathbf{c}^{\mu}))^{-1} \mathbf{L}_l \mathrm{diag}(\mathbf{v}_l^{-\mu}), \\ ({}_{RL}\mathbf{D}_r^{-\mu})^{-1} &= \mathbf{B}_r (\mathrm{diag}(\mathbf{c}^{\mu}))^{-1} \mathbf{L}_r \mathrm{diag}(\mathbf{v}_r^{-\mu}), \end{aligned} \tag{22}$$

and

$$\begin{aligned} ({}_{RL}\mathbf{D}_l^{\mu})^{-1} &= \mathbf{B}_l (\mathrm{diag}(\mathbf{c}^{-\mu}))^{-1} \mathbf{L}_l \mathrm{diag}(\mathbf{v}_l^{\mu}), \\ ({}_{RL}\mathbf{D}_r^{\mu})^{-1} &= \mathbf{B}_r (\mathrm{diag}(\mathbf{c}^{-\mu}))^{-1} \mathbf{L}_r \mathrm{diag}(\mathbf{v}_r^{\mu}). \end{aligned} \tag{23}$$

Remark

Some remarks are listed as follows:

- It is worth noting that

$$(\mathrm{diag}(\mathbf{c}^{-\mu}))^{-1} \neq (\mathrm{diag}(\mathbf{c}^{\mu})), \quad (\mathrm{diag}(\mathbf{c}^{\mu}))^{-1} \neq (\mathrm{diag}(\mathbf{c}^{-\mu})).$$

- We point out that the two matrices $\mathbf{B}_l$ and $\mathbf{B}_r$ are Vandermonde's type.
- From the above Theorem, the first entry of $\mathbf{v}_l^{\mu}$ and the last entry of $\mathbf{v}_r^{\mu}$ are illogical for $\mu < 0$, which indicates the endpoint singularity of the fractional differential operator. In order to avoid this issue, the nodes $\{x_j\}_{j=0}^{N}$ are altered to the Jacobi–Gauss type.
- As stated above, the matrices $\mathrm{diag}(\mathbf{v}_l^{\mu})$ and $\mathrm{diag}(\mathbf{v}_r^{\mu})$ are singular (or some of their entries are not well-defined). Thus, the inverse of the singular matrix in (22) and (23) should be considered as a generalized inverse or a pseudo-inverse.

Consider expanding $L_j(x)$ as

$$L_j(x) = \sum_{k=0}^{N} l_{kj}^{\alpha,\beta} P_k^{\alpha,\beta}(x) = \sum_{k=0}^{N} l_{kj}^{0,0} P_k^{0,0}(x). \tag{24}$$

where $P_k^{\alpha,\beta}(x)$ is Jacobi orthogonal polynomial of degree k.
Denote

$$(\mathbf{L}^{\alpha,\beta})_{ij} = l_{ij}^{\alpha,\beta}, \quad (\mathbf{L}^{0,0})_{ij} = l_{ij}^{0,0}, \quad (\mathbf{P}^{\alpha,\beta})_{ij} = P_j^{\alpha,\beta}(x_i). \tag{25}$$

From the previous Theorem , we have

$$\mathbf{L}^{0,0} = \mathbf{T}_N^{(\alpha,\beta)\to(0,0)} \mathbf{L}^{\alpha,\beta}. \tag{26}$$

We also have

$$l_{ij}^{\alpha,\beta} = \frac{P_j^{\alpha,\beta}(x_i)\omega_i}{\gamma_j^{\alpha,\beta}}, \quad j = 0, 1, \cdots, N-1, \quad l_{iN}^{\alpha,\beta} = \frac{P_N^{\alpha,\beta}(x_i)\omega_i}{\left(2 + \frac{\alpha+\beta+1}{N}\right) \gamma_N^{\alpha,\beta}},$$

Additionally, it is easy to have that

$$[\mathbf{L}^{\alpha,\beta}]^{-1} = [\mathbf{P}^{\alpha,\beta}]^T. \tag{27}$$

Theorem

The following representations of the pseudospectral integration matrices are valid for $\alpha, \beta > -1$

$$
\begin{aligned}
{}_{RL}\mathbf{D}_l^{-\mu} &= \mathrm{diag}(\mathbf{v}_l^{\mu})\mathbf{P}^{-\mu,\mu}\mathrm{diag}(\mathbf{c}^{\mu})\mathbf{T}_N^{(\alpha,\beta)\to(0,0)}\mathbf{L}^{\alpha,\beta}, \\
{}_{RL}\mathbf{D}_r^{-\mu} &= \mathrm{diag}(\mathbf{v}_r^{\mu})\mathbf{P}^{\mu,-\mu}\mathrm{diag}(\mathbf{c}^{\mu})\mathbf{T}_N^{(\alpha,\beta)\to(0,0)}\mathbf{L}^{\alpha,\beta}.
\end{aligned} \tag{28}
$$

Moreover, the differentiation matrices are valid for $\alpha, \beta > -1$

$$
\begin{aligned}
{}_{RL}\mathbf{D}_l^{\mu} &= \mathrm{diag}(\mathbf{v}_l^{-\mu})\mathbf{P}^{\mu,-\mu}\mathrm{diag}(\mathbf{c}^{-\mu})\mathbf{T}_N^{(\alpha,\beta)\to(0,0)}\mathbf{L}^{\alpha,\beta}, \\
{}_{RL}\mathbf{D}_r^{\mu} &= \mathrm{diag}(\mathbf{v}_r^{-\mu})\mathbf{P}^{-\mu,\mu}\mathrm{diag}(\mathbf{c}^{-\mu})\mathbf{T}_N^{(\alpha,\beta)\to(0,0)}\mathbf{L}^{\alpha,\beta}.
\end{aligned} \tag{29}
$$

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Theorem

Let $\mathbf{P} := \mathbf{P}^{\alpha,\beta}, \mathbf{L} := \mathbf{L}^{\alpha,\beta}$ *and* $\alpha, \beta > -1$. *The representations of the inverses of thepseudospectral integration matrices are valid*

$$\begin{aligned}({}_{RL}\mathbf{D}_l^{-\mu})^{-1} &= \mathbf{P}^T \mathbf{T}_N^{(0,0)\to(\alpha,\beta)} (\mathrm{diag}(\mathbf{c}^{\mu}))^{-1} \mathbf{T}_N^{(-\mu,\mu)\to(\alpha,\beta)} \mathbf{L}^T \mathrm{diag}(\mathbf{v}_l^{-\mu}),\\ ({}_{RL}\mathbf{D}_r^{-\mu})^{-1} &= \mathbf{P}^T \mathbf{T}_N^{(0,0)\to(\alpha,\beta)} (\mathrm{diag}(\mathbf{c}^{\mu}))^{-1} \mathbf{T}_N^{(\mu,-\mu)\to(\alpha,\beta)} \mathbf{L}^T \mathrm{diag}(\mathbf{v}_r^{-\mu}).\end{aligned} \tag{30}$$

Moreover, the pseudospectral differentiation matrices are valid

$$\begin{aligned}({}_{RL}\mathbf{D}_l^{\mu})^{-1} &= \mathbf{P}^T \mathbf{T}_N^{(0,0)\to(\alpha,\beta)} (\mathrm{diag}(\mathbf{c}^{-\mu}))^{-1} \mathbf{T}_N^{(\mu,-\mu)\to(\alpha,\beta)} \mathbf{L}^T \mathrm{diag}(\mathbf{v}_l^{\mu}),\\ ({}_{RL}\mathbf{D}_r^{\mu})^{-1} &= \mathbf{P}^T \mathbf{T}_N^{(0,0)\to(\alpha,\beta)} (\mathrm{diag}(\mathbf{c}^{-\mu}))^{-1} \mathbf{T}_N^{(-\mu,\mu)\to(\alpha,\beta)} \mathbf{L}^T \mathrm{diag}(\mathbf{v}_r^{\mu}).\end{aligned} \tag{31}$$

Theorem

The following representations of the pseudospectral differentiation matrix are valid for $\alpha, \beta > -1$

$$
\begin{aligned}
{}_C\mathbf{D}_l^{\mu} &= \mathrm{diag}(\mathbf{v}_l^{m-\mu})\overline{\mathbf{P}}^{\mu-m,m-\mu}\mathrm{diag}(\overline{\mathbf{c}}^{m-\mu})\mathbf{T}_{N-m}^{(\alpha+m,\beta+m)\to(0,0)}\overline{\mathbf{D}}^{m}\overline{\mathbf{L}}^{\alpha,\beta},\\
{}_C\mathbf{D}_r^{\mu} &= \mathrm{diag}(\mathbf{v}_r^{m-\mu})\overline{\mathbf{P}}^{m-\mu,\mu-m}\mathrm{diag}(\overline{\mathbf{c}}^{m-\mu})\mathbf{T}_{N-m}^{(\alpha+m,\beta+m)\to(0,0)}\overline{\mathbf{D}}^{m}\overline{\mathbf{L}}^{\alpha,\beta},
\end{aligned}
\tag{32}
$$

where $\overline{\mathbf{D}}^{m} = \mathrm{diag}([d_{m,m}^{\alpha,\beta}, d_{m+1,m}^{\alpha,\beta}, \cdots, d_{N,m}^{\alpha,\beta}])$ *with* $d_{k,m}^{\alpha,\beta} = \frac{\Gamma(k+m+\alpha+\beta+1)}{2^m\Gamma(k+\alpha+\beta+1)}$, $(\overline{\mathbf{P}}^{\alpha,\beta})_{i,j} = P_j^{\alpha,\beta}(x_i)$ *and* $(\overline{\mathbf{L}}^{\alpha,\beta})_{j,i} = l_{j+m,i}^{\alpha,\beta}$ *for* $i = 0, \cdots, N; j = 0, \cdots, N-m$.

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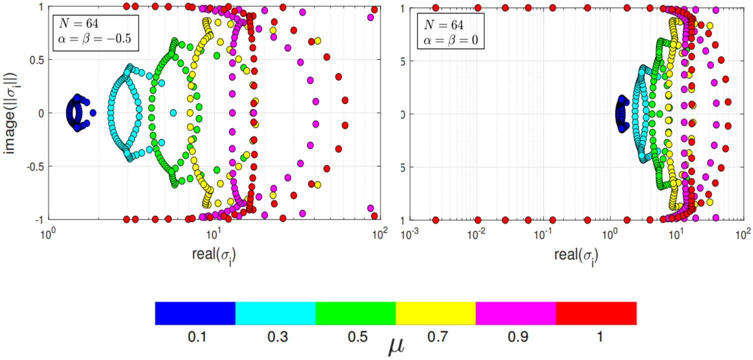
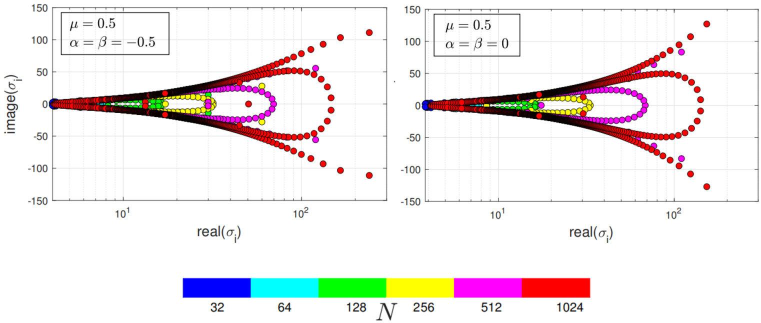
Eigenvalues: Chebyshev vs Legendre$({}_C\mathbf{D}^{\mu}, N = 64)$



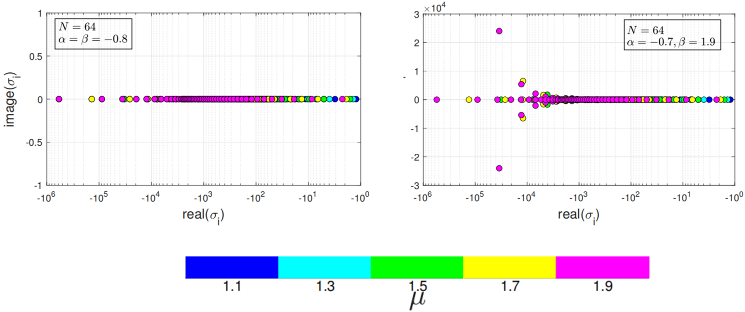
Eigenvalue of Fractional Differentiation Matrix

Eigenvalues: Chebyshev vs Legendre$({}_{C}\mathbf{D}^{0.5})$



Eigenvalue of Fractional Differentiation Matrix

Eigenvalues: Symmetry vs non-symmetric(${}_{RZ}\mathbf{D}^{\mu}$, $N = 64$)



Example (1)

Let $1 < \mu < 2$. Consider the boundary value problem

$$
\begin{aligned}
&{}_C\mathrm{D}^{\mu}_{0,x} u(x) + \lambda u(x) = 0, \\
&pu(0) - ru'(0) = 0, \quad qu(1) + su'(1) = 0.
\end{aligned} \tag{33}
$$

where $p, q, r, s \geq 0$ such that $p^2 + r^2 \neq 0$ and $q^2 + s^2 \neq 0$

This problem is also solved in [3, 6, 5, 4].

Results of fractional eigenvalue problems

Table: The first 9 eigenvalues of example (1), computed with the Chebyshev collocation method ($\alpha = \beta = -0.5$) with $\mu = 1.8$ and $N = 200$.

λ	Our method	[6]	[4]	[5]	[3]
λ_1	9.45685689126	9.4568568891	9.4568568892	9.4997	9.4569
λ_2	28.47687912479	28.4768791170	28.4768791186	28.5116	28.4769
λ_3	62.20037779983	62.2003777331	62.2003777529	62.3239	62.2004
λ_4	97.06323747284	97.0632373708	97.0632377552	97.0896	97.0632
λ_5	155.45013805266	155.4501373840	155.4499080962	155.6972	-
λ_6	196.59593024267	196.5959302453	196.5986985127	196.5152	-
λ_7	301.52706976868	-	-	304.19+3.00i	-
λ_8	306.72685026127	-	-	304.19+3.00i	-
λ_9	461.179+43.050i	-	-	461.24+43.29i	-

Table: The first 6 eigenvalues of example (1), computed with the Chebyshev collocation method ($\alpha = \beta = -0.5$) with $\mu = 1.6$ and $N = 200$.

λ	Our method	[6]	[4]	[3]
λ_1	13.420474051	13.42047399	13.4204739885	13.4205
λ_2	14.645442473	14.64544252	14.6454425351	14.6454
λ_3	47.292859+18.850956i	47.292858+18.850956i	-	-
λ_4	91.705190+43.625498i	91.705189+43.625496i	-	-
λ_5	145.569415+75.805031i	145.569416+75.805025i	-	-
λ_6	207.859129+114.486222i	207.859147+114.486204i	-	-

Table: The first 6 eigenvalues of example (1), computed with the Chebyshev collocation method ($\alpha = \beta = -0.5$) with $N = 200$.

λ	$\mu = 1.9$	$\mu = 1.9$[6]	$\mu = 1.999$	$\mu = 1.999$[5]
λ_1	9.5141431295	9.5141431288	9.8648626632	9.8648
λ_2	33.5956714125	33.5956714089	39.4139459491	39.4139
λ_3	73.0390172335	73.0390172163	88.6441581077	88.6442
λ_4	124.4185311384	124.4185311250	157.5323069955	157.5325
λ_5	191.1460514291	191.1460515330	246.0882332022	246.0886
λ_6	267.9451997398	267.9452006460	354.2916713396	354.2922

Results of fractional eigenvalue problems

Table: The first 6 eigenvalues of example (1), computed with the CCM ($\alpha = \beta = -0.5, N = 200$).

λ	$\mu = 1.1$	$\mu = 1.3$	$\mu = 1.5$
λ_1	1.3531852+7.2497408i	5.4727766+8.2732647i	11.1466676+6.1222836i
λ_2	2.8098353+15.7308531i	13.3926574+21.8112244i	33.3136683+23.1298984i
λ_3	4.3033960+24.6907235i	22.3712536+37.9533918i	61.1959645+46.6510167i
λ_4	5.8291184+33.9686708i	32.1686781+55.9626406i	93.8181576+75.3667762i
λ_5	7.3824934+43.4883769i	42.6472214+75.4712271i	130.5448587+108.4998614i
λ_6	8.9599335+53.2043853i	53.7149929+96.2501648i	170.9467051+145.5427194i

Table: The first 6 eigenvalues of example (1), computed with the CCM ($\alpha = \beta = -0.5, N = 200$).

λ	$\mu = 1.9999$	$\mu = 1.99999$	$\mu = 1.999999$	$\mu = 2$	error
λ_1	9.8691292203	9.8695568729	9.8695996482	9.86960440108	1.073e-11
λ_2	39.4719645919	39.4777722446	39.4783530678	39.47841760434	1.995e-11
λ_3	88.8081881526	88.8246142309	88.8262570696	88.82643960980	7.148e-12
λ_4	157.8754859530	157.9098514892	157.9132885198	157.91367041743	8.527e-14
λ_5	246.6748284659	246.7335809315	246.7394571083	246.74011002723	1.336e-12
λ_6	355.2042026720	355.2956013900	355.3047427196	355.30575843924	2.035e-11

Example (2)

Let $1 < \mu < 2$. Consider the boundary value problem

$$\begin{aligned} {}_{RZ}\mathrm{D}_x^{\mu} u(x) + \lambda u(x) = 0, \\ u(-1) = 0, \quad u(1) = 0. \end{aligned} \tag{34}$$

The same problem has been studied using the Jacobi-Galerkin spectral method [2].

Results of fractional eigenvalue problems

Table: The first 5 eigenvalues of example (2), computed with the Legendre collocation method ($\alpha = \beta = 0, N = 200$).

λ	$\mu = 1.2$		$\mu = 1.4$	
	Our method	[2]	Our method	[2]
λ_1	1.297024021884	1.296995777	1.483262055566	1.4832334320
λ_2	3.486806460504	3.486730536	4.458260013435	4.4581739838
λ_3	5.911808693986	5.911679975	8.150874006594	8.1507167266
λ_4	8.534627231336	8.534441423	12.424593370123	12.424353637
λ_5	11.292675855564	11.29243001	17.162678802344	17.162347657

Table: The first 5 eigenvalues of example (2), computed with the Legendre collocation method ($\alpha = \beta = 0, N = 200$).

λ	$\mu = 1.6$		$\mu = 1.8$	
	Our method	[2]	Our method	[2]
λ_1	1.728321890005	1.72829595710	2.048752746738	2.04873498313
λ_2	5.756434650807	5.75634828003	7.503181981160	7.50311692608
λ_3	11.312063027525	11.3118933010	15.800031154322	15.7998941633
λ_4	18.177615608424	18.1773428791	26.724474991011	26.7242432849
λ_5	26.187596954514	26.1872040516	40.114581604547	40.1142338051

Application of fractional initial value problems

The basic fractional initial value problem:

$$\begin{aligned} {}_C\mathrm{D}_{a,x}^{\mu} u(x) &= f(u,x), \quad m-1<\mu<m, \\ u^{(k)}(a) &= u_k, \quad k=0,\cdots,m-1. \end{aligned} \tag{35}$$

Example (3)

Let $0<\mu<1$. Consider the scalar linear fractional differential equation

$$ {}_C\mathrm{D}_{0,t}^{\mu} u(t) = f(t), \quad t\in(0,T], \quad u(0)=u_0. \tag{36}$$

The term $f(t)$ is chosen so that the exact solution satisfies:

C11. $u(t)=\sum_{k=1}^{5}\frac{t^{k\sigma}}{k}, \quad \sigma>0, \quad t\in(0,2], \quad u_0=0.$

C12. $u(t)=t\sin(t), \quad t\in(0,2\pi], \quad u_0=0.$

C13. $u(t)=E_{\mu,1}(-t^{\mu}), \quad t\in(0,3], \quad u_0=1.$

Results of fractional initial value problems

Table: The error E_∞ and convergence order CO of C11 in example (3), computed with the Chebyshev collocation method $(\alpha = \beta = -0.5, \sigma = 2.5)$

N	$\mu = 0.2$		$\mu = 0.4$		$\mu = 0.6$		$\mu = 0.8$	
	E_∞	CO	E_∞	CO	E_∞	CO	E_∞	CO
12	6.434e-06	-	1.329e-05	-	2.627e-05	-	5.384e-05	-
20	2.356e-07	6.47	5.692e-07	6.17	1.109e-06	6.20	1.968e-06	6.48
28	4.373e-08	5.01	1.055e-07	5.01	2.055e-07	5.01	3.648e-07	5.01
36	1.244e-08	5.00	2.998e-08	5.01	5.840e-08	5.01	1.037e-07	5.01
44	4.559e-09	5.00	1.099e-08	5.00	2.140e-08	5.00	3.800e-08	5.00
52	1.977e-09	5.00	4.764e-09	5.00	9.280e-09	5.00	1.648e-08	5.00
60	9.664e-10	5.00	2.328e-09	5.00	4.536e-09	5.00	8.054e-09	5.00
68	5.168e-10	5.00	1.245e-09	5.00	2.426e-09	5.00	4.307e-09	5.00

Table: The error E_∞ and convergence order CO of C11 in example (3), computed with the Chebyshev collocation method ($\alpha = \beta = -0.5, \mu = 0.5$).

N	$\sigma = 0.5$		$\sigma = 1.2$		$\sigma = 1.8$		$\sigma = 2.2$	
	E_∞	CO	E_∞	CO	E_∞	CO	E_∞	CO
8	4.044e-02	-	1.236e-03	-	1.263e-03	-	7.994e-02	-
16	2.036e-02	0.99	2.410e-04	2.36	1.811e-05	6.12	4.055e-06	14.27
24	1.359e-02	1.00	9.146e-05	2.39	4.193e-06	3.61	6.775e-07	4.41
32	1.020e-02	1.00	4.591e-05	2.40	1.487e-06	3.60	1.908e-07	4.41
40	8.158e-03	1.00	2.689e-05	2.40	6.658e-07	3.60	7.143e-08	4.40
48	6.799e-03	1.00	1.737e-05	2.40	3.453e-07	3.60	3.201e-08	4.40
56	5.828e-03	1.00	1.200e-05	2.40	1.982e-07	3.60	1.624e-08	4.40
64	5.100e-03	1.00	8.708e-06	2.40	1.226e-07	3.60	9.024e-09	4.40

Table: The error E_∞ of C12 in example (3), computed with the Chebyshev collocation method ($\alpha = \beta = -0.5$).

N	$\mu = 0.1$	$\mu = 0.3$	$\mu = 0.5$	$\mu = 0.7$	$\mu = 0.9$	$\mu = 0.99$
6	9.557e-03	2.978e-02	5.208e-02	8.299e-02	1.377e-01	1.931e-01
10	7.758e-06	2.475e-05	4.212e-05	6.440e-05	1.072e-04	1.626e-04
14	1.522e-09	4.709e-09	8.159e-09	1.212e-08	2.043e-08	3.113e-08
18	2.749e-13	5.519e-13	5.405e-13	1.002e-12	1.333e-12	2.055e-12
22	2.398e-13	5.959e-13	3.545e-14	1.024e-12	1.420e-13	1.631e-13

Results of fractional initial value problems

Table: The error E_∞ and convergence order CO of C13 in example (3), computed the Chebyshev collocation method ($\alpha = \beta = -0.5$).

N	$\mu = 0.7$		$\mu = 0.8$		$\mu = 0.9$		$\mu = 0.99$	
	E_∞	CO	E_∞	CO	E_∞	CO	E_∞	CO
4	9.283e-02	-	7.376e-02	-	4.563e-02	-	1.312e-02	-
8	3.180e-02	1.55	1.939e-02	1.93	8.566e-03	2.41	7.498e-04	4.13
16	1.144e-02	1.47	5.948e-03	1.70	2.268e-03	1.92	1.754e-04	2.10
32	4.247e-03	1.43	1.918e-03	1.63	6.379e-04	1.83	4.370e-05	2.01
64	1.597e-03	1.41	6.280e-04	1.61	1.821e-04	1.81	1.103e-05	1.99
128	6.032e-04	1.40	2.066e-04	1.60	5.221e-05	1.80	2.793e-06	1.98
256	2.283e-04	1.40	6.811e-05	1.60	1.499e-05	1.80	7.077e-07	1.98
512	8.647e-05	1.40	2.246e-05	1.60	4.303e-06	1.80	1.794e-07	1.98

Let $1 < \mu < 2$. As a benchmark fractional boundary value problem, we consider the one-dimensional fractional Helmholtz equation as

$$\lambda^2 u(x) - \mathrm{D}_x^{\mu} u(x) = f(x), \quad x \in (a, b), \quad u(a) = u(b) = 0, \tag{37}$$

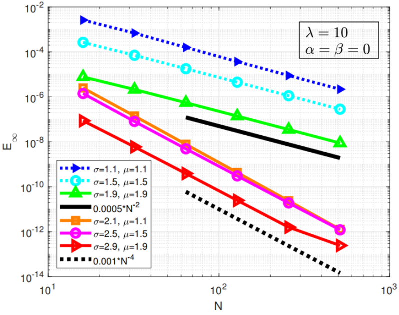
Example (4)

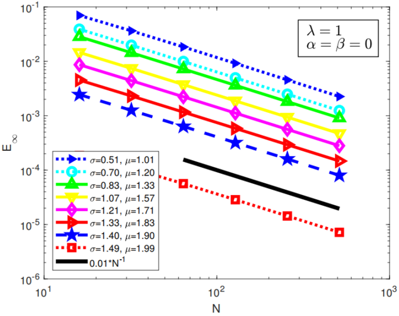
Consider Equation(37) with the Caputo derivative: $\mathrm{D}_x^{\mu} = {}_C\mathrm{D}_{a,x}^{\mu}$. The source term $f(x)$ is chosen so that the exact solution satisfies one of the following cases:

C21. $u(x) = x^{\sigma} - x^{2\sigma}, \quad \sigma > 0, \quad a = 0, b = 1.$

C22. $u(x) = \sin(\pi x), \quad a = -1, b = 1.$

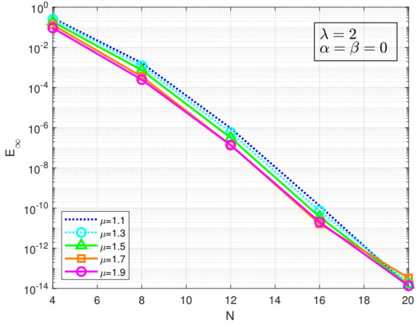
Convergence order: C21 of Example (4)

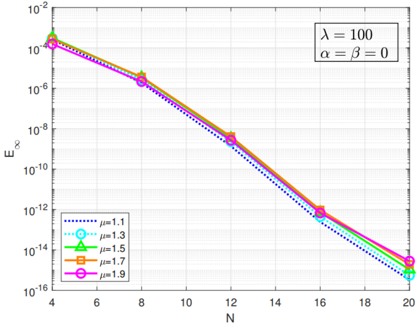




Limited convergence order!!!

Convergence order: C22 of Example (4)





Spectral accuracy!!

Example (5)

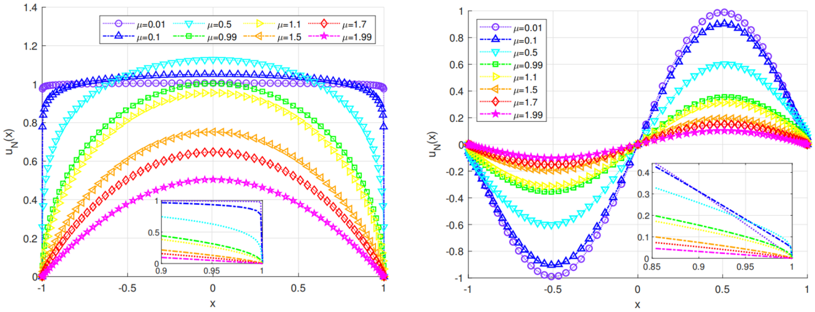
Consider Equation (37) with the Riesz derivative $\mathrm{D}_x^\mu = {}_{RZ}\mathrm{D}_x^\mu$. The numerical test is performed for the fractional Poisson equation of two cases:

C31. $f(x) = 1, \lambda = 0, a = -1, b = 1.$

C32. $f(x) = \sin(\pi x), \lambda = 0, a = -1, b = 1.$

We solve the fractional Poisson problems with the Riesz derivative by employing the Legendre spectral collocation method ($\alpha = \beta = 0$) with $N = 64$. The profiles of the numerical solutions are plotted in the next Figure.

Numerical solution: C31(left) and C32(right) of Example (5)



The fractional Burgers equations (FBEs):

$$\partial_t u + u\partial_x u = \epsilon \mathrm{D}_x^{\mu} u, \quad (x,t) \in (a,b) \times (0,T] \tag{38}$$

with the boundary condition $u(a,t) = u(b,t) = 0$ and the initial profile $u(x,0) = u_0(x)$.
For the time discretization, we employ a semi-implicit time discretization scheme with step size τ, namely the two-step Crank–Nicolson/leapfrog scheme. Then, the full discretization scheme reads:

$$\begin{cases} \left(\mathbf{I} - \epsilon\tau\mathbf{D}^{\mu}\right)\mathbf{u}^{n+1} = \left(\mathbf{I} + \epsilon\tau\mathbf{D}^{\mu}\right)\mathbf{u}^{n-1} - 2\tau(\mathrm{diag}(\mathbf{u}^n)\mathbf{D})\mathbf{u}^n, & n \geq 1, \\ \mathbf{u}^1 = \left(\mathbf{I} + \epsilon\tau\mathbf{D}^{\mu}\right)\mathbf{u}^0 - \tau(\mathrm{diag}(\mathbf{u}^0)\mathbf{D})\mathbf{u}^0 \\ \mathbf{u}^0 = u_0(\mathbf{x}), \end{cases} \tag{39}$$

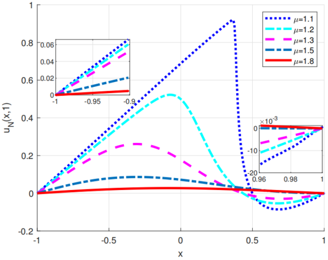
where $\mathbf{D}$ is the first-order differentiation matrix. In the following examples, we always take $\alpha = \beta = 0, N = 360$ and $\tau = 10^{-3}$.

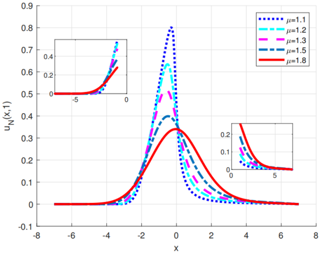
Example (6)

Consider Equation (38) with the Caputo derivative: $\mathrm{D}_x^\mu = {}_C\mathrm{D}_x^\mu$. The numerical test is performed for two cases of initial profiles:

C41. $u_0(x) = \sin(\pi x), a = -1, b = 1.$

C42. $u_0(x) = \exp(-2x^2), a = -7, b = 7.$





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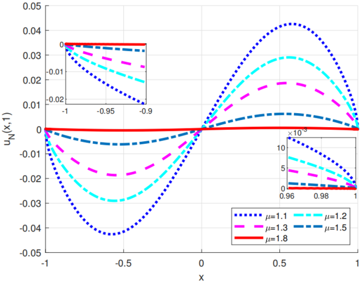
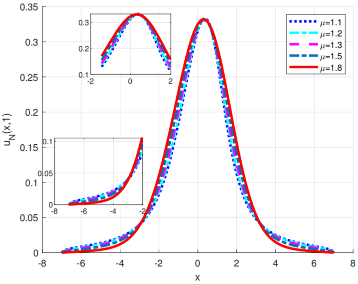
Example (7)

Consider equation (38) with the Riesz derivative: $\mathrm{D}_x^\mu = {}_{RZ}\mathrm{D}_x^\mu$. The numerical test is performed for two cases of initial profiles:

C51. $u_0(x) = \sin(\pi x), a = -1, b = 1.$

C52. $u_0(x) = \exp(-2x^2), a = -7, b = 7.$





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Summary

We present a new algorithm to evaluate the fractional differentiation matrix.

- † Advantages:
  1. High-accuracy, Fast
  2. Applicable to solve various fractional differential equations
  3. Provide more information on fractional differentiation matrix, for, such as, preconditioning
- † Disadvantages:
  1. Can not deal with variable-order case
  2. Need to explore application of the method to fractional differential equations on high dimensional domains.

The main body of this presentation is from the paper [1].

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Thanks for your attention!