Continuity and Differentiability of Eigenvalues of Laplacian with respect to Domain Perturbations

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Numerical Methods for Spectral Problems: Theory and Applications

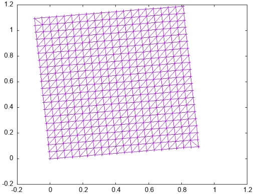
New Century Hotel, Gui'an Guizhou

August xx, 2024

Motivation

Let $\Omega := (0,1) \times (0,1)$, and

$$\mathcal{T}_t(x) = x + \left(t + \frac{1}{2}t^2\right)\begin{pmatrix} x_1 + x_2 \\ -x_1 - x_2 \end{pmatrix}, \quad -0.1 \leq t \leq 0.1.$$

Then, $\Omega_t := \mathcal{T}_t(\Omega)$ looks like 

We consider the eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega_t, \qquad u = 0 \quad \text{on} \quad \partial\Omega_t.$$

It has eigenvalues

$$0 < \lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_k(t) \leq \cdots.$$

Motivation 2

We would like to know if $\lambda_k(t)$ is continuous and differentiable.

Hadamard considered the problem under the conditions that $\partial\Omega$ is sufficiently smooth, and the perturbation is written as $t\rho(s)\boldsymbol{\nu}$, where $s \in \partial\Omega$, and $\boldsymbol{\nu}$ is the outer unit normal vector.
He found that $\lambda_1'(0)$ exists and is written as

$$\lambda_1'(0) = -\left\langle \frac{\partial u_1(0)}{\partial \boldsymbol{\nu}}, \rho \frac{\partial u_1(0)}{\partial \boldsymbol{\nu}} \right\rangle_{\partial\Omega}.$$

This is called the **Hadamard's variational formula** for the first eigenvalue. (P.R. Garabedian, *Partial Differential Equations*, 2nd edition, Chelsea. New York, 1986.)

We would like to consider this problem for any $\lambda_k(t)$ under the condition that $\partial\Omega$ is Lipschitz and the perturbation is in general form.

The Domain perturbations

$\Omega \subset \mathbb{R}^n$: a bounded domain with the Lipschitz (or $C^{0,1}$) boundary $\partial\Omega$ ($n \geq 2$).
$\mathcal{T}_t : \Omega \to \mathbb{R}^n$: bi-Lipschitz (or $C^{0,1}$) diffeomorphism such that

- $\mathcal{T}_0(x) = x$: identity map.
- $\mathcal{T}_t$ is twice continuously differentiable w.r.t. t, and has the Taylor expansion
$$\mathcal{T}_t(x) = x + t\mathrm{S}(x) + \frac{1}{2}t^2\mathrm{R}(x) + o(t^2),$$
where S and R are Lipschitz vector fields.

Set $\Omega_t := \mathcal{T}_t(\Omega)$. Let $\widetilde{\Omega} \subset \mathbb{R}^n$ be a sufficiently large domain such that $\Omega_t \subset \widetilde{\Omega}$ for t, $|t| < \varepsilon$.
Let γ^0, $\gamma^1 \subset \partial\Omega$ such that $\overline{\gamma^0} \cup \overline{\gamma^1} = \partial\Omega$, $\gamma^0 \cap \gamma^1 = \emptyset$. We set
$$\gamma_t^i := \mathcal{T}_t(\gamma^i), \quad i = 0, 1.$$

The eigenvalue problem of Laplacian on Ω_t

We cosider the eigenvalue problem of Laplacian:

$$
\begin{gathered}
-\Delta u_t = \lambda(t) u_t \ \text{ in } \ \Omega_t, \\
u_t = 0 \ \text{ on } \ \gamma_t^0, \qquad \frac{\partial u_t}{\partial \boldsymbol{\nu}} = 0 \ \text{ on } \ \gamma_t^1.
\end{gathered}
$$

There exist the eigenvalues

$$
0 < \lambda_1(t) \le \lambda_2(t) \le \cdots,
$$

and the corresponding eigenfunctions $u_k(t)$ with $\|u_k(t)\|_{L^2(\Omega_t)} = 1$.

Main Questions

(1) Are $\lambda_k(t)$ and $u_k(t)$ continuous and differentiable with respect to t?
(2) If so, how do we compute $\lambda_k'(t)$ and $\lambda_k''(t)$?

Weak form

The weak form of

$$-\Delta u_t = \lambda(t) u_t \ \text{ in } \ \Omega_t, \quad u_t = 0 \ \text{ on } \ \gamma_t^0, \quad \frac{\partial u_t}{\partial \boldsymbol{\nu}} = 0 \ \text{ on } \ \gamma_t^1$$

is

$$\int_{\Omega_t} \nabla u \cdot \nabla v \mathrm{d}x = \lambda \int_{\Omega_t} uv \mathrm{d}x, \quad \forall v \in V_t \quad \text{with } \|u\|_{L^2(\Omega_t)} = 1,$$
$$V_t := \{v \in H^1(\Omega_t) \mid v|_{\gamma_t^0} = 0\}.$$

To make the problem easier, we pullback the weak form to Ω as

$$u \in V, \ A_t(u,v) = \lambda B_t(u,v), \quad \forall v \in V, \quad B_t(u,u) = 1,$$
$$A_t(u,v) := \int_\Omega Q_t[\nabla u, \nabla v] \det D\mathcal{T}_t \,\mathrm{d}x, \quad B_t(u,v) := \int_\Omega uv \det D\mathcal{T}_t \,\mathrm{d}x,$$
$$V := \{v \in H^1(\Omega) \mid v|_{\gamma^0} = 0\}, \quad Q_t := (D\mathcal{T}_t)^{-1}(D\mathcal{T}_t)^{-\top}.$$

General framework

In the following, we consider the eigenvalue problem in the general framework.

Let $X := L^2(\Omega)$ and $I \subset \mathbb{R}$ be an open interval.
Let $A_t : V \times V \to \mathbb{R}$ and $B_t : X \times X \to \mathbb{R}$ be symmetric bilinear forms depends on $t \in I$ with

$$|A_t(u,v)| \le C\|u\|_V\|v\|_V, \quad A_t(v,v) \ge \delta\|v\|_V^2, \quad \forall u, v \in V,$$
$$|B_t(u,v)| \le C|u|_X|v|_X, \quad B_t(v,v) \ge \delta|v|_X^2, \quad \forall u, v \in X.$$

We consider the following eigenvalue problem:

$$u \in V,\ A_t(u,v) = \lambda B_t(u,v), \quad \forall v \in V, \quad B_t(u,u) = 1.$$

The known results

We reformulate the problem in the general framework, many known results are available.

- S.-N. Chow and J.K. Hale, *Methods of Bifurcation Theory*, Springer-Verlag, New York, 1982.
- T. Kato, *Perturbation Theory for Linear Operators*, second edition, Springer-Verlag, Berlin, 1976.
- F. Rellich, *Perturbation Theory of Eigenvalue Problems*, Lecture Notes, New York Univ. 1953.

However, the proofs of the known results are quite complicated and difficult to follow. Our aim is to give a simpler proof for the continuity and differentiability of eigenvalues.

Continuity of the eigenvalues

Theorem

Suppose that the symmetric bilinear forms A_t and B_t satisfy

$$\lim_{h\to 0} \sup_{\|u\|_V \le 1, \|v\|_V \le 1} |(A_{t+h} - A_t)(u,v)| = 0,$$

$$\lim_{h\to 0} \sup_{|u|_X \le 1, |v|_X \le 1} |(B_{t+h} - B_t)(u,v)| = 0$$

for any $t \in I$. Then, the eigenvalues $\lambda_k(t)$ is continuous with respect to t:

$$\lim_{h\to 0} \lambda_k(t+h) = \lambda_k(t), \quad \forall t \in I, \quad k = 1, 2, \cdots .$$

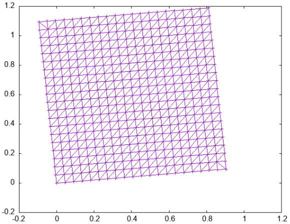
Let $u_k(t)$ be the eigenfunction of $\lambda_k(t)$. In general, $u_k(t)$ is NOT continuous with respect to t even if $\lambda_k(t)$ is simple. We need to consider on "the continuity of the space $\mathrm{span}\{u_k(t)\}$".

Differentiability of the eigenvalues, Numerical Example 1

We expect that $\lambda_k(t)$ is differentiable with respect to t.
However, this is NOT the case in general. Let $\Omega := (0,1) \times (0,1)$, and

$$\mathcal{T}_t(x) = x + \left(t + \frac{1}{2}t^2\right) \begin{pmatrix} x_1 + x_2 \\ -x_1 - x_2 \end{pmatrix}, \quad -0.1 \le t \le 0.1.$$

Then, $\Omega_t := \mathcal{T}_t(\Omega)$ looks like



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Differentiability of the eigenvalues, Numerical Example 1

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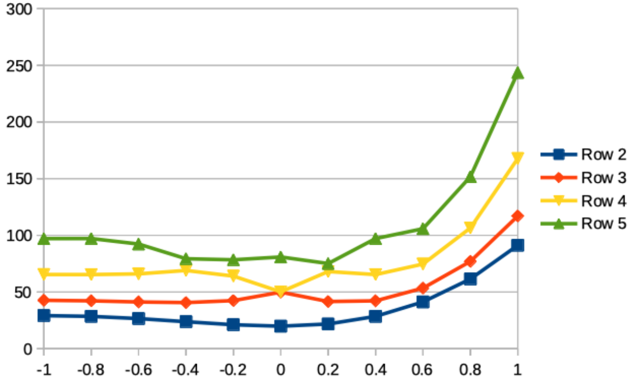
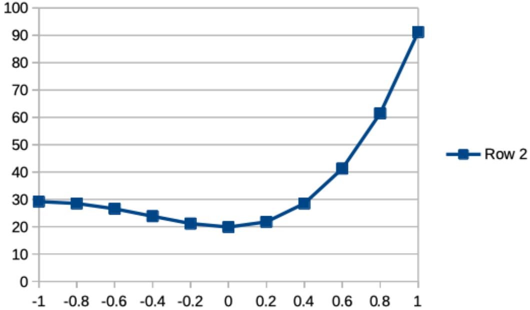


The profile of the first eigenvalue is following:



Differentiability of the eigenvalues, Numerical Example 1

Let $(\lambda_1(t), u_1(t))$ be the first eigen pair.

The approximated value of the Hadamard variation

$$\lambda_1'(0) = -\left\langle \frac{\partial u_1(0)}{\partial \boldsymbol{\nu}}, (\mathsf{S}\cdot\boldsymbol{\nu})\frac{\partial u_1(0)}{\partial \boldsymbol{\nu}} \right\rangle_{\partial\Omega}.$$

is `-1.18E-16`. (Note that the original form of the Hadamard variation cannot be applied to this case.)

Differentiability of the eigenvalues, 1

Define

$$\dot{A}_t(u,v) := \frac{\mathrm{d}}{\mathrm{d}t} A_t(u,v), \quad \dot{B}_t(u,v) := \frac{\mathrm{d}}{\mathrm{d}t} B_t(u,v)$$

We suppose that the bilinear forms A_t, B_t satisfy the assumption of Theorem 1 (continuity) and the followings (differentiability):

$$|\dot{A}_t(u,v)| \le C\|u\|_V\|v\|_V,\ u,v \in V,$$
$$|\dot{B}_t(u,v)| \le C|u|_X|v|_X,\ u,v \in X,$$
$$\lim_{h\to 0}\frac{1}{h}\sup_{\|u\|_V\le 1,\|v\|_V\le 1}\left|(A_{t+h}-A_t-h\dot{A}_t)(u,v)\right| = 0,$$
$$\lim_{h\to 0}\frac{1}{h}\sup_{|u|_X\le 1,|v|_X\le 1}\left|(B_{t+h}-B_t-h\dot{B}_t)(u,v)\right| = 0.$$

Differentiability of the eigenvalues, 2

Theorem

Suppose that A_t, B_t satisfy the above conditions. Let $\lambda_k(t)$ be of multiplicity $m \geq 1$. Then, there exists

$$\dot{\lambda}_j^{\pm}(t) := \lim_{h \to \pm 0} \frac{\lambda_j(t+h) - \lambda_j(t)}{h},$$

and we have

$$\dot{\lambda}_j^{+} = \mu_{j-k+1}^{\lambda}, \quad \dot{\lambda}_j^{-} = \mu_{k+m-j}^{\lambda}, \quad k \leq j \leq k+m-1,$$

where $\mu_1^{\lambda} \leq \mu_2^{\lambda} \leq \cdots \leq \mu_m^{\lambda}$ are the eigenvalues of

$$u \in Y_\lambda, \qquad (\dot{A}_t - \lambda \dot{B}_t)(u,v) = \mu B_t(u,v), \quad \forall v \in Y_\lambda,$$

and Y_λ is the m-dimensional subspace spanned by the eigenfunctions of $\lambda := \lambda_k(t)$.

Differentiability of the eigenvalues, 3

Corollary

(1) *Suppose that* $\lambda_k(t)$ *is a simple eigenvalue. Then,* $\lambda_k(t)$ *is differentaible at* t*: there exists*

$$\lambda_j'(t) := \lim_{h\to 0} \frac{\lambda_j(t+h) - \lambda_j(t)}{h}.$$

(2) *Suppose that* $\lambda_k(t)$ *is of multiplicity* 2*. Define* $\tilde{\lambda}_{k+p}$*, for* $p = 0, 1$*, by*

$$\tilde{\lambda}_{k+p}(s) := \begin{cases} \lambda_{k+p}(s) & s \geq t \\ \lambda_{k+1-p}(s) & s \leq t \end{cases}, \qquad p = 0, 1.$$

Then, $\tilde{\lambda}_i(s)$ *is differentiable at* t*, and we have, with* $\lambda := \lambda_k(t)$*,*

$$\tilde{\lambda}_{k+p}'(t) := \lim_{h\to 0} \frac{\tilde{\lambda}_{k+p}(t+h) - \lambda}{h} = \dot{\lambda}_{k+p}^{+}(t),\ p = 0, 1.$$

Differentiability of the eigenvalues, 4

Corollary

Suppose that $\lambda_k(t)$ is of multiplicity 3.
(1) *Then, $\lambda_{k+1}(t)$ is differentiable at t.*

(2) *Define $\tilde{\lambda}_{k+p}$, for $p = 0, 2$, by*

$$\tilde{\lambda}_{k+p}(s) := \begin{cases} \lambda_{k+p}(s) & s \geq t \\ \lambda_{k+2-p}(s) & s \leq t \end{cases}, \qquad p = 0, 2.$$

Then, $\tilde{\lambda}_i(s)$ is differentiable at t, and we have, with $\lambda := \lambda_k(t)$,

$$\tilde{\lambda}'_{k+p}(t) := \lim_{h \to 0} \frac{\tilde{\lambda}_{k+p}(t+h) - \lambda}{h} = \dot{\lambda}^{+}_{k+p}(t),\ p = 0, 2.$$

Continuity of the derivatives, 1

Theorem

Suppose that A_t, B_t satisfy the assumptions of Theorem 2. Supoose also that A_t, B_t satisfy

$$\lim_{h\to 0} \sup_{\|u\|_V \le 1, \|v\|_V \le 1} |(\dot{A}_{t+h} - \dot{A}_t)(u,v)| = 0,$$
$$\lim_{h\to 0} \sup_{|u|_X \le 1, |v|_X \le 1} |(\dot{B}_{t+h} - \dot{B}_t)(u,v)| = 0.$$

Then, the one-sided derivatives $\dot{\lambda}_i^{\pm}(s)$ are one-sided continuous. That is,

$$\lim_{h\to +0} \dot{\lambda}_k^{+}(t+h) = \dot{\lambda}_k^{+}(t), \quad \lim_{h\to -0} \dot{\lambda}_k^{-}(t+h) = \dot{\lambda}_k^{-}(t)$$

hold.

Continuity of the derivatives, 2

Theorem

Let $I \subset \mathbb{R}$ be an open interval.
Suppose that A_t, B_t satisfy the assumptions of Theorem 3 for any $t \in I$.
Then, defining $\tilde{\lambda}_k(t)$ by appropriate rearrangements (change of the order) of $\lambda_k(t)$ at most countably many times, $\tilde{\lambda}_k(t)$ are of $C^1(I)$.

Higher derivatives 1

Define

$$\ddot{A}_t(u,v) := \frac{\mathrm{d}^2}{\mathrm{d}t^2}A_t(u,v), \quad \ddot{B}_t(u,v) := \frac{\mathrm{d}^2}{\mathrm{d}t^2}B_t(u,v).$$

We suppose that the bilinear forms A_t, B_t satisfy the assumption of Theorem 3 and the followings:

$$|\ddot{A}_t(u,v)| \le C\|u\|_V\|v\|_V,\ u,v\in V,$$
$$|\ddot{B}_t(u,v)| \le C|u|_X|v|_X,\ u,v\in X,$$
$$\lim_{h\to 0}\frac{1}{h^2}\sup_{\|u\|_V\le 1,\|v\|_V\le 1}\left|\left(A_{t+h}-A_t-h\dot{A}_t-\frac{h^2}{2}\ddot{A}_t\right)(u,v)\right| = 0,$$
$$\lim_{h\to 0}\frac{1}{h^2}\sup_{|u|_X\le 1,|v|_X\le 1}\left|\left(B_{t+h}-B_t-h\dot{B}_t-\frac{h^2}{2}\ddot{B}_t\right)(u,v)\right| = 0.$$

Higher derivatives 2

Theorem

Let $I \subset \mathbb{R}$ be an open interval.
Suppose that A_t, B_t satisfy the assumptions of Theorem 3 for any $t \in I$. Suppose also that the above mentioned conditions hold. Then, defining $\tilde{\lambda}_k(t)$ by appropriate rearrangements (change of the order) of $\lambda_k(t)$ at most countably many times, $\tilde{\lambda}_k(t)$ are of $C^2(I)$.

Summary of the first part

- If a perturbation is continuous, the perturbed eigenvalues are continuous wrt the perturbation.
- If a perturbation is continuously differentiable, the perturbed eigenvalues with appropriate rearrangements (change of the order) are continuously differentiable wrt the perturbation.
- If a perturbation is of C^2, the perturbed eigenvalues with appropriate rearrangements (change of the order) are of C^2 wrt the perturbation.
- We have given a simple proof of these facts.
- Therefore, with respect to general domain perturbations of C^2 the perturbed eigenvalues with appropriate rearrangements (change of the order) are of C^2 as well,

T. Suzuki, T. Tsuchiya, Hadamard variation of eigenvalues with respect to general domain perturbations, *to appear in* Journal of Mathematical Society of Japan. arXiv:2309.00273

The convexity of the first eigenvalue.

In the paper

Garabedian, Schiffer, Convexity of domain functionals, J. Anal. Math., 2 (1952-53) 281–368,

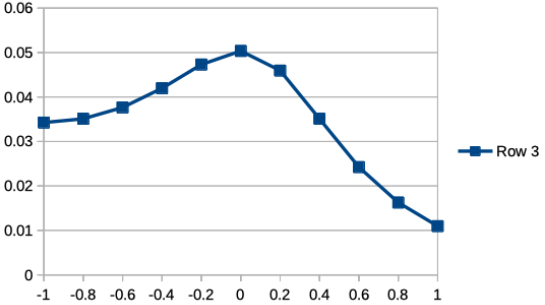
Garabedian and Schiffer claimed that in some cases we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{\lambda_1(t)}\right) \geq 0,$$

that is, the reciprocal of the first eigenvalue is convex with respect to t. We would like to know the sufficient conditions for the convexity.

The reciprocal of the first eigenvalue, Numerical Example 1

The profile of the reciprocal of the first eigenvalue is following.
Clearly, the reciprocal of the first eigenvalue is not convex with respect to t.



First Theorem

Theorem

Let $\lambda_t(t)$ *be the first eigenvalue and* ϕ_t *is the eigen function of* $\lambda_1(t)$ *with* $\int_\Omega \phi_t^2 \mathrm{d}\mathbf{x}$, *and*

$$\dot{A} := \dot{A}_t(\phi_1, \phi_1), \quad \ddot{A} := \ddot{A}_t(\phi_1, \phi_1), \quad \dot{B} := \ddot{B}_t(\phi_1, \phi_1), \quad \ddot{B} := \ddot{B}_t(\phi_1, \phi_1).$$

Then,

$$\dot{\lambda} = \dot{A} - \lambda \dot{B}, \qquad \ddot{\lambda} \leq \ddot{A} - \lambda \ddot{B} - 2\dot{\lambda}\dot{B}.$$

Sufficient condition for the convexity

Theorem

Let $\lambda_t(t)$ be the first eigenvalue and ϕ_t is the eigen function of $\lambda_1(t)$ with $\int_\Omega \phi_t^2 \mathrm{d}\mathbf{x}$, and

$$\dot{A} := \dot{A}_t(\phi_1, \phi_1), \quad \ddot{A} := \ddot{A}_t(\phi_1, \phi_1), \quad \dot{B} := \ddot{B}_t(\phi_1, \phi_1), \quad \ddot{B} := \ddot{B}_t(\phi_1, \phi_1).$$

Then,

$$\dot{\lambda} = \dot{A} - \lambda \dot{B}, \qquad \ddot{\lambda} \le \ddot{A} - \lambda \ddot{B} - 2\dot{\lambda}\dot{B}.$$

Sufficient condition for the convexity

Because

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{\lambda_1(t)}\right) = -\frac{\ddot{\lambda}\lambda^2 - 2\lambda\dot{\lambda}}{\lambda^2},$$

we have the following theorem.

Theorem

Suppose that

$$2\dot{A}^2 + \lambda^2\ddot{B} \geq \lambda(\ddot{A} + 2\dot{A}\dot{B}).$$

Then, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{\lambda_1(t)}\right) \geq 0.$$

For the case of conformal transformations, $n = 2$.

Let $n = 2$, and $\mathcal{T}_t : \Omega \to \Omega_t$ is conformal. Then, we have the following theorem.

Theorem

If

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \det D\mathcal{T}_t \geq 0 \quad \text{in } \Omega,$$

then, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\frac{1}{\lambda_1(t)} \right) \geq 0.$$

For the case of conformal transformations, $n = 2$.

Lemma

Let $\mathbb{R}^2 \cong \mathbb{C}$, *and* $\mathcal{T}_t(z) := z + a_2 t z^2 + a_3 t^2 z^3 + \cdots$.
If $2|a_2|^2 \geq 3|a_3|$, *then* $\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} \det D\mathcal{T}_t \geq 0$ *in* Ω.

Lemma

Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ *be the unit disk, and*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots , \qquad a_1 \in \mathbb{R}$$

be univalent in D. *If we define the transformation by*
$\mathcal{T}_t(z) = (1 - t)z + t f(z)$, *then* $\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} \det D\mathcal{T}_t \geq 0$.

For the case of conformal transformations, $n = 2$.

Theorem

Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk, and

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \qquad a_1 \in \mathbb{R}$$

be univalent in D. If we define the transformation by $\mathcal{T}_t(z) = (1-t)z + tf(z)$, then we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\det D\mathcal{T}_t \geq 0,$$

$$\lambda_1(D) \geq \lambda_1(\mathcal{T}_t(D))\left(2\int_D \phi_1^2 \mathrm{Re} f'(z)\mathrm{d}z - 1\right).$$

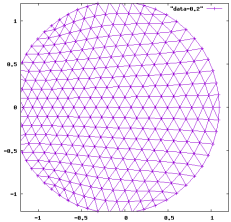
Suzuki, Tsuchiya, Hadamard's variational formula for simple eigenvalues, *in preparation*.

Convexity of the first eigennvalue, Numerical Example 2

Let $D \subset \mathbb{C}$ be the unit disk, and

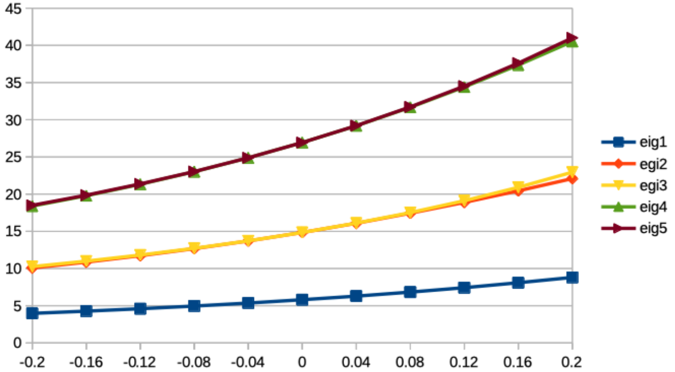
$$\mathcal{T}_t(z) = (1-t)z + t\cos z.$$

Then, $\Omega_t := \mathcal{T}_t(D)$ looks like



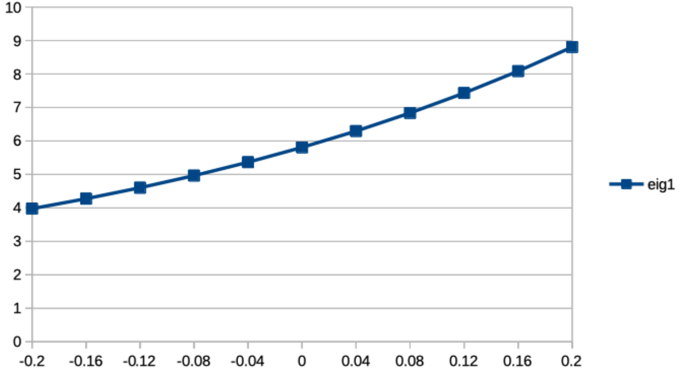
Convexity of the first eigennvalue, Numerical Example 2

The profile of the eigenvalues is following:



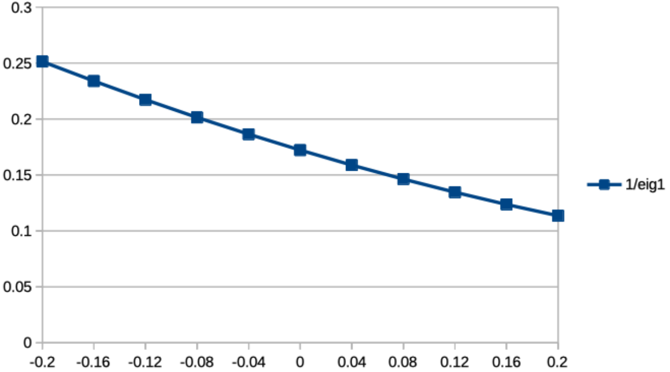
Convexity of the first eigenvalue, Numerical Example 2

The profile of the first eigenvalues is following:



Convexity of the first eigenvalue, Numerical Example 2

The profile of the reciprocal of the first eigenvalues is following:

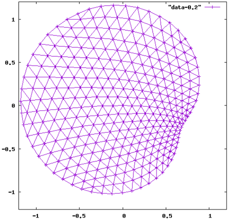


Convexity of the first eigenvalue, Numerical Example 3

Let $D \subset \mathbb{C}$ be the unit disk, and

$$\mathcal{T}_t(z) = z + a_2 t z^2 + a_3 t^2 z^3 + a_4 t^3 z^4 + a_5 t^4 z^5$$

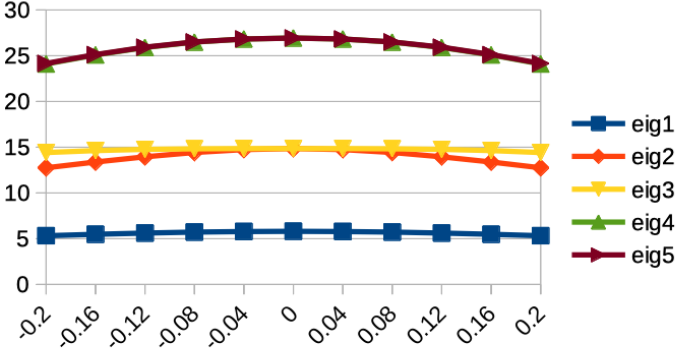
with $a_2 = 1 + 0.5i$, $a_3 = -a_2$, $a_4 = 1 + i$, and $a_5 = -a_4$. Then, $\Omega_t := \mathcal{T}_t(D)$ looks like



Note that this transformation does not satisfy the sufficient condition.

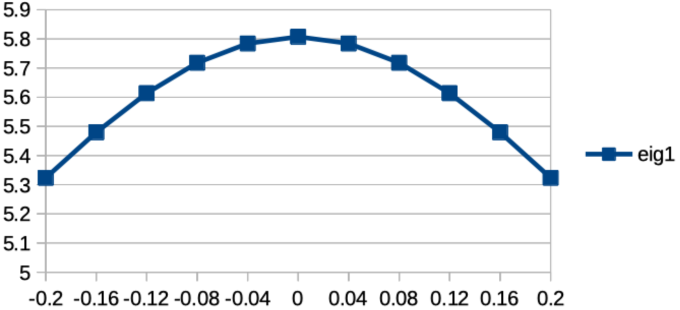
Differentiability of the eigenvalues, Numerical Example 3

The profile of the eigenvalues is following:



Differentiability of the eigenvalues, Numerical Example 3

The profile of the first eigenvalue is following:



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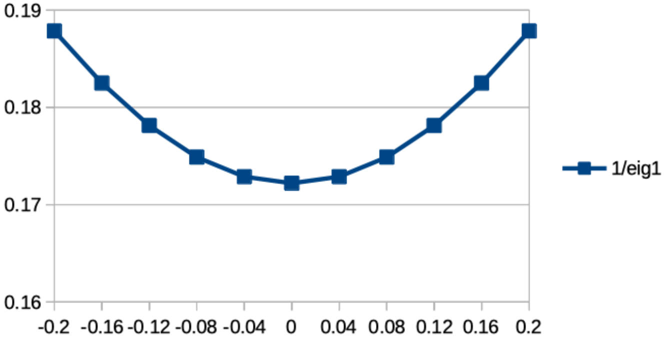

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