A C^0 Finite Element Method for Triharmonic Problems

Joint work with Peimeng Yin[1]

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[1]The University of Texas at El Paso

The Triharmonic Equation

$\Omega \subset \mathbb{R}^2$ – a bounded polygonal domain. $\Gamma := \partial\Omega$.

$H^m(\Omega)$ – the space of functions whose jth ($j \leq m$) derivatives are square integrable.

$S_n \subset H_0^1(\Omega)$ – the linear C^0 finite element space on a quasi-uniform mesh, h = mesh size.

The 6th-order problem with the simply supported boundary condition:

$$-\Delta^3 u = f \quad \text{in} \quad \Omega, \qquad\qquad u = \Delta u = \Delta^2 u = 0 \quad \text{on} \quad \Gamma.$$

High-order models occurs in differential geometry, the thin film equations, and the phase field crystal model, among many others.

The space

$$V := \{\phi \in H^3(\Omega), \phi|_{\partial\Omega} = 0,\ \Delta\phi|_{\partial\Omega} = 0\}.$$

The variational formulation: Find $u \in V$ such that,

$$a(u, \phi) = \int_\Omega \nabla\Delta u \cdot \nabla\Delta\phi dx = \int_\Omega f\phi dx = (f, \phi), \quad \forall\phi \in V.$$

The Triharmonic Equation

A Poincaré inequality for $v \in H_0^1(\Omega) \cap H^2(\Omega)$:

$$||\Delta v||_{L^2(\Omega)} \geq C||v||_{H^2(\Omega)}.$$

Well-posedness [L., & Yin; 2024]

Given $f \in H^{-1}(\Omega)$, the variational formulation admits a unique $u \in V$ and it satisfies

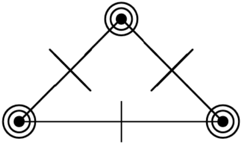
$$||u||_{H^3(\Omega)} \leq C||f||_{H^{-1}(\Omega)}.$$

Uniqueness: for $w \in V \subset H_0^1(\Omega) \cap H^2(\Omega)$,

$$0 = a(w, w) = \int_\Omega \nabla\Delta w \cdot \nabla\Delta w dx = ||\nabla\Delta w||^2_{L^2(\Omega)} \geq C||w||^2_{H^2(\Omega)}.$$

Numerical approximation of the sixth-order problem is difficult.

Conforming FEMs:



Argyris triangle

C^1 | $\mathcal{P}_5(\mathcal{K})$ | 21 degrees of freedom

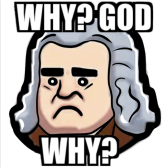
$$\Theta_K = \{D^\alpha p(z_i), |\alpha| \leq 2, 1 \leq i \leq 3, \frac{\partial}{\partial n_i} p(z_{jk}), 1 \leq i \leq 3\}$$

Relevant Works

Numerical approximation of the sixth-order problem is difficult.

Conforming FEMs:

C^2 elements



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Nonconforming FEMs and interior penalty discontinuous Galerkin methods: [Wu & Xu; 2019], [Wu & Xu; 2017], [Gudi & Neilan; 2011].

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Mixed FEMs assuming uniqueness of the solution in H^1: [Droniou, Ilyas, Lamichhane, & Wheeler; 2019].

The equation:

$$-\Delta^3 u = f \quad \text{in} \quad \Omega, \qquad u = \Delta u = \Delta^2 u = 0 \quad \text{on} \quad \Gamma.$$

The direct decomposition with auxiliary functions w, v

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma; \end{cases} \qquad \begin{cases} -\Delta v = w & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma; \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta \bar{u} = v & \text{in } \Omega, \\ \bar{u} = 0 & \text{on } \Gamma. \end{cases}$$

A naive mixed formulation: Find $\bar{u}, v, w \in H_0^1(\Omega)$ such that

$$\begin{aligned} (\nabla w, \nabla \psi) &= (f, \psi), \quad \forall \psi \in H_0^1(\Omega), \\ (\nabla v, \nabla \psi) &= (w, \psi), \quad \forall \psi \in H_0^1(\Omega), \\ (\nabla \bar{u}, \nabla \psi) &= (v, \psi), \quad \forall \psi \in H_0^1(\Omega). \end{aligned}$$

Do Naive Mixed Methods Work?

The equation

$$-\Delta^3 u = f \quad \text{in} \quad \Omega, \qquad u = \Delta u = \Delta^2 u = 0 \quad \text{on} \quad \Gamma.$$

Suppose the largest interior angle ω of Ω is at $(0,0)$, $\omega > \pi/2$ and other angles $< \pi/2$.
(r, θ) – polar coordinates.

Building a solution

$$u(r, \theta) = \tilde{\eta}(r; \tau, R) r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right).$$

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$$u(r,\theta) = \tilde{\eta}(r;\tau,R) r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right).$$

$\tilde{\eta}(r;\tau,R)$ is a cut-off function

$$\tilde{\eta}(r;\tau,R) = \begin{cases} 0, & \text{if } r > R, \\ 1, & \text{if } r < \tau R, \\ \frac{1}{2} + \sum_{i=0}^{6} C_i \left(\frac{2r}{R(1-\tau)} - \frac{1+\tau}{1-\tau}\right)^{2i+1}, & \text{otherwise,} \end{cases}$$

with $R = \frac{32}{5}, \tau = \frac{1}{8}$, and the coefficients C_i are determined by solving the linear system

$$\tilde{\eta}^{(i)}(R;\tau,R) = 0, \quad i = 0, \cdots, 6.$$

The source term f is obtained by calculating $f = -\Delta(\Delta(\Delta u)) \in L^2(\Omega)$.

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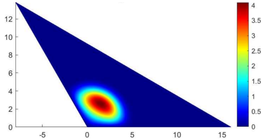

Reference solution u_R (left), the mixed method solution u_{10} (center), $|u_R - u_{10}|$ (right).

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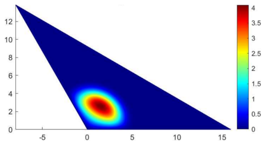

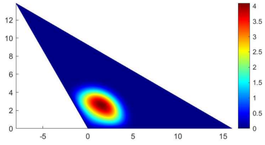

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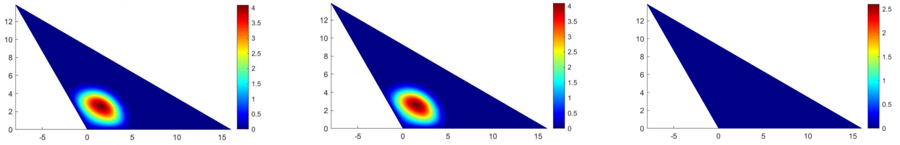

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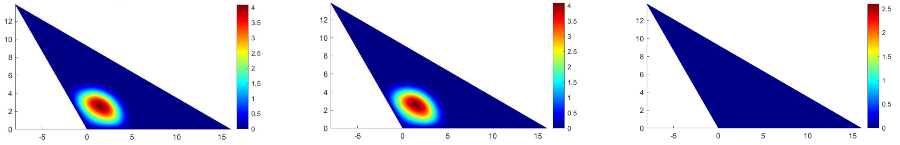
Building a wrong solution

$$u(r,\theta) = \tilde{\eta}(r;\tau,R)r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right) \notin H^3(\Omega).$$



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The naive mixed method converges to the wrong solution even in a convex domain!

The 6th-order problem with the simply supported boundary condition

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Recall $V := \{\phi \in H^3(\Omega), \phi|_{\partial\Omega} = 0,\ \Delta\phi|_{\partial\Omega} = 0\}$.

Theoretical Insights

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The Laplace operator and its image [L. & Yin; 2024]:

- The mapping $-\Delta : V \to H_0^1(\Omega)$ is injective and has a closed range.

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Good news: $\mathcal{H}^\perp$ is finite dimensional! We are able to identify its basis.

Characterization of $\mathcal{H}^\perp$ [L., & Yin; 2024]

A function v belongs to $\mathcal{H}^\perp$ iff $v \in H_0^1(\Omega)$ is the solution of

$$\Delta^2 v = 0 \quad \text{in} \quad \Omega, \qquad v = \Delta v = 0 \quad \text{on} \quad \Gamma.$$

H^1 orthogonality: for any $z \in V$ and $v \in \mathcal{H}^\perp$,

$$(-\nabla\Delta z, \nabla v) = 0.$$

Is it even possible?

Theoretical Insights

The 6th-order problem with the simply supported boundary condition

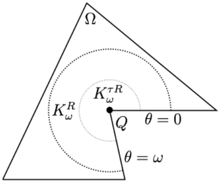
$$-\Delta^3 u = f \quad \text{in} \quad \Omega, \qquad u = \Delta u = \Delta^2 u = 0 \quad \text{on} \quad \Gamma.$$

$K_\omega^R = \{(r,\theta) | 0 \le r \le R, 0 \le \theta \le \omega\}$.

Let $\tau \in (0,1)$ and $\frac{2\omega}{\pi} \notin \mathbb{N}$. Define $N := \lfloor \frac{2\omega}{\pi} \rfloor$.

For $1 \le i \le N$, define

$$s_i^-(r,\theta;\tau,R) = \eta(r;\tau,R) r^{-\frac{i\pi}{\omega}} \sin\left(\frac{i\pi}{\omega}\theta\right) \in H^{-1}(\Omega).$$



Special Functions [L. & Yin; 2024]

(i) For $1 \le i \le N$, we define $\xi_i \in H^{-1}(\Omega)$, such that

$$\xi_i(r,\theta;\tau,R) := s_i^-(r,\theta;\tau,R) + \zeta_i(r,\theta;\tau,R),$$

where $\zeta_i \in H_0^1(\Omega)$ satisfies $-\Delta\zeta_i = \Delta s_i^-$ in Ω, $\zeta_i = 0$ on Γ.
(ii) For $1 \le i \le N$, define $\sigma_i \in H_0^1(\Omega)$, such that

$$-\Delta\sigma_i = \xi_i \text{ in } \Omega, \qquad \sigma_i = 0 \text{ on } \Gamma.$$

$$\Longrightarrow \Delta^2\sigma_i = \Delta\xi_i = 0$$

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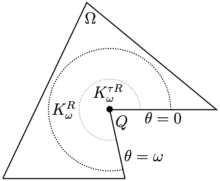
ω	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi)$	$(\pi, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, 2\pi)$
$\frac{\pi}{\omega}$	$(2, \infty)$	$(1, 2)$	$(\frac{2}{3}, 1)$	$(\frac{1}{2}, \frac{2}{3})$
N	0	1	2	3

Table: The value of N for different ω.



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New C^0 Mixed Methods

The dimension of $\mathcal{H}^\perp$ is finite and $\mathcal{H}^\perp = \text{span}\{\sigma_i\}_{1\le i\le N}$.

Any $z \in H_0^1(\Omega)$ can be written as

$$z = z_{\mathcal{H}} + \sum_{i=1}^{N} c_{z,i}\sigma_i,$$

where $z_{\mathcal{H}} \in \mathcal{H}$ and the coefficients $c_{z,i}$ are uniquely determined by

$$\sum_{i=1}^{N} c_{z,i}(\nabla\sigma_i, \nabla\sigma_j) = (\nabla z, \nabla\sigma_j), \quad j = 1, \cdots, N.$$

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Recall the direct decomposition

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma; \end{cases} \qquad \begin{cases} -\Delta v = w & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma; \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta \bar{u} = v & \text{in } \Omega, \\ \bar{u} = 0 & \text{on } \Gamma. \end{cases}$$

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Idea: Modifying the RHS

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma; \end{cases} \qquad \begin{cases} -\Delta v = w & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma; \end{cases} \qquad \begin{cases} -\Delta \tilde{u} = v - \sum_{i=1}^{N} c_{v,i}\sigma_i & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \Gamma, \end{cases}$$

$$\Longrightarrow u = \tilde{u} \in H^3(\Omega).$$

Modified Mixed Methods [L. & Yin; 2024]

Solving the triharmonic problem $\Delta^3 u = f$. Recall s_i^-, $i = 1, \cdots, N$.

- Step 1. $\forall \psi \in S_n$, find the numerical solutions $w_n, v_n, \zeta_{in} \in S_n$

$$(1)\ a(w_n, \psi) = (f, \psi), \qquad (2)\ a(v_n, \psi) = (w_n, \psi), \qquad (3)\ a(\zeta_{in}, \psi) = (\Delta s_i^-, \psi),$$

  and set $\xi_{in} = \zeta_{in} + s_i^-$.
- Step 2. Find the finite element solution $\sigma_{in} \in S_n$, $i = 1, \cdots, N$

$$a(\sigma_{in}, \psi) = (\xi_{in}, \psi).$$

- Step 3. Find the coefficient $c_{v,in} \in \mathbb{R}$ by solving the linear system

$$\sum_{i=1}^{N} c_{v_n,i} (\nabla \sigma_{in}, \nabla \sigma_{jn}) = (\nabla v_n, \nabla \sigma_{jn}), \quad j = 1, \cdots, N.$$

- Step 4. Find the finite element solution $u_n \in S_n$ of the Poisson equation

$$a(u_n, \psi) = \left(v_n - \sum_{i=1}^{N} c_{v_n,i} \sigma_{in}, \psi \right).$$

Error Analysis

Suppose the largest interior angle ω of Ω is at $(0,0)$, $\omega > \pi/2$ and other angles $< \pi/2$.

Recall $N := \lfloor \frac{2\omega}{\pi} \rfloor$. For $1 \le i \le N$, let

$$0 < \alpha < \frac{\pi}{\omega} \quad \text{and} \quad -1 < \beta_i < 1 - \frac{i\pi}{\omega}.$$

Notice $-\Delta\sigma_i = \xi_i \in H^{\beta_i}(\Omega) \Longrightarrow \sigma_i \in H^{\min(2+\beta_i,\alpha)}(\Omega)$. Therefore, for any $\tau \in S_n$,

$$a(u - u_n, \tau) = (v - v_n, \tau) - \sum_{i=1}^{N} (c_{v,i}\sigma_i - c_{v_n,i}\sigma_{in}, \tau)$$

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$\Longrightarrow$

$$\|u - u_n\|_{H^1(\Omega)} \le C\Big(\|u - u_I\|_{H^1(\Omega)} + \|v - v_n\|_{H^{-1}(\Omega)} + \sum_{i=1}^{N} (|c_{in}|\|\sigma_i - \sigma_{in}\|_{H^{-1}(\Omega)} + |c_{v,i} - c_{v_n,i}|\|\sigma_i\|_{H^{-1}(\Omega)})\Big),$$

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where $\|\sigma_i - \sigma_{in}\|_{H^{-1}(\Omega)} \le Ch^{\min(1+\beta_i+\min(\alpha,1),2\alpha)}$ and $|c_{v,i} - c_{v_n,i}| \le Ch^{2\min(1+\beta_i,\alpha)}$.

Error Analysis

Suppose the largest interior angle ω of Ω is at $(0, 0)$, $\omega > \pi/2$ and other angles $< \pi/2$.

Error Analysis [L., & Yin; 2024]

Let $u_n \in S_n$ be the proposed finite element solution of the sixth order problem. Then

$$\|u - u_n\|_{H^1(\Omega)} \leq C_0 h + \sum_{i=1}^{N} C_i h^{\min\{2(1+\beta_i),1\}} \leq Ch^{\gamma}, \quad \beta_i < 1 - \frac{i\pi}{\omega}.$$

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ω	$(0, \frac{\pi}{2}]$	$(\frac{\pi}{2}, \frac{2\pi}{3}]$	$(\frac{2\pi}{3}, \pi]$	$(\pi, \frac{4\pi}{3}]$	$(\frac{4\pi}{3}, \frac{3\pi}{2}]$	$(\frac{3\pi}{2}, 2\pi)$
$\min\{2(1+\beta_1), 1\}$	$--$	$2(1+\beta_1)$	1	1	1	1
$\min\{2(1+\beta_2), 1\}$	$--$	$--$	$--$	$2(1+\beta_2)$	1	1
$\min\{2(1+\beta_3), 1\}$	$--$	$--$	$--$	$--$	$--$	$2(1+\beta_3)$
γ	1	$2(1+\beta_1)$	1	$2(1+\beta_2)$	1	$2(1+\beta_3)$

Table: The values of $\min\{2(1+\beta_i), 1\}$ and γ for different ω.

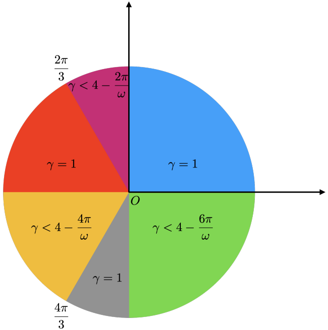
Error Analysis

Suppose the largest interior angle ω of Ω is at $(0,0)$, $\omega > \pi/2$ and other angles $< \pi/2$.

Error Analysis [L., & Yin; 2024]

Let $u_n \in S_n$ be the proposed finite element solution of the sixth order problem. Then

$$\|u - u_n\|_{H^1(\Omega)} \leq C_0 h + \sum_{i=1}^{N} C_i h^{\min\{2(1+\beta_i),1\}} \leq Ch^\gamma, \quad \beta_i < 1 - \frac{i\pi}{\omega}.$$



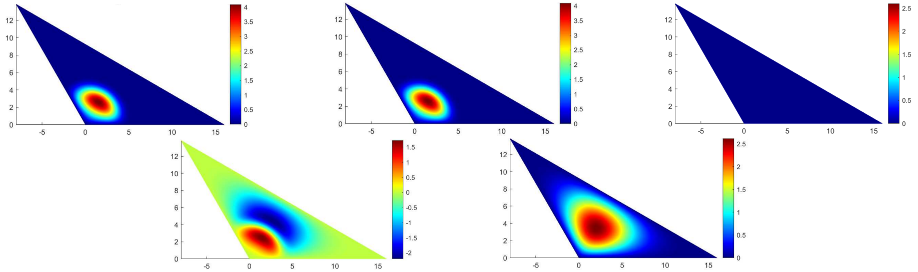
Numerical Illustration I

The equation

$$-\Delta^3 u = f \quad \text{in} \quad \Omega, \qquad u = \Delta u = \Delta^2 u = 0 \quad \text{on} \quad \Gamma.$$

A wrong solution

$$u(r,\theta) = \tilde{\eta}(r;\tau,R)r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right) \notin H^3(\Omega).$$



Top row: reference solution u_R (left), naive mixed method solution u_{10}^N (center), $|u_R - u_{10}^N|$ (right).
Bottom row: proposed method solution u_{10}^A (left), $|u_R - u_{10}^A|$ (right).

Numerical Illustration I

The equation

$$-\Delta^3 u = f \quad \text{in} \quad \Omega, \qquad\qquad u = \Delta u = \Delta^2 u = 0 \quad \text{on} \quad \Gamma.$$

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$$u(r,\theta) = \tilde{\eta}(r;\tau,R) r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right) \notin H^3(\Omega).$$

	$j=7$	$j=8$	$j=9$	$j=10$
$\|\|u-u_j^N\|\|_{H^1(\Omega)}$	2.74964e-01	1.35594e-01	6.77391e-02	3.38605e-02
$\|\|u-u_j^A\|\|_{H^1(\Omega)}$	6.07564	6.02331	6.00958	6.00306

Table: The H^1 error of the numerical solutions on quasi-uniform meshes.

Numerical Illustration II

An H^3 solution

Given $f_0 = -\Delta\left(\tilde{\eta}(r;\tau,R)r^{\frac{N\pi}{\omega}}\sin\left(\frac{N\pi}{\omega}\theta\right)\right) \in H_0^1(\Omega)$, let u be the H^3 solution of $-\Delta^3 u = \Delta^2 f_0$. To ensure $u \in H^3(\Omega)$, we require u satisfies

$$-\Delta u = f_0 - \sum_{i=1}^{N} c_i \sigma_i \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma,$$

where c_i is given by $\sum_{i=1}^{N} c_i(\nabla\sigma_i, \nabla\sigma_j) = (\nabla f_0, \nabla\sigma_j), \quad j = 1, \cdots, N.$

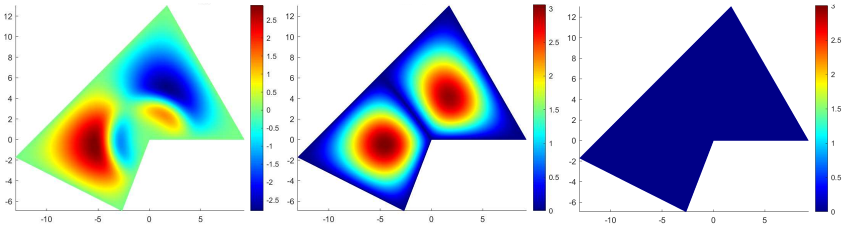

Figure: $\omega \approx 1.383\pi \in (\pi, 3\pi/2)$: the reference solution u_R (left); $|u - u_{10}^N|$ (center); $|u - u_{10}^A|$ (right).

Numerical Illustration II

An H^3 solution

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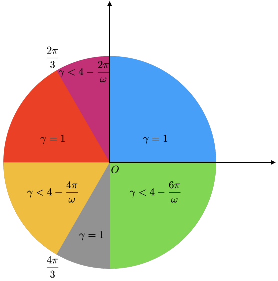
	$j=6$	$j=7$	$j=8$	$j=9$
$\|\|u-u_j^N\|\|_{H^1(\Omega)}$	9.67666	9.64665	9.63404	9.63164
$\|\|u-u_j^A\|\|_{H^1(\Omega)}$	5.27303e-02	2.09405e-02	1.01081e-02	4.20655e-03

Table: The H^1 error of the numerical solutions on quasi-uniform meshes.

Numerical Illustration III

Tests for different values of ω

$$-\Delta^3 u = \sin(\frac{N\pi}{\omega}\theta) \quad \text{in} \quad \Omega, \qquad\qquad u = \Delta u = \Delta^2 u = 0 \quad \text{on} \quad \Gamma.$$



ω	expected rate	$j = 7$	$j = 8$	$j = 9$	$j = 10$
$\approx 0.56409\pi$	0.46	0.75	0.67	0.59	0.54
$\approx 0.63099\pi$	0.83	0.96	0.95	0.94	0.93
$\frac{2\pi}{3}$	1.00	1.03	1.01	1.00	1.00
$\approx 0.70483\pi$	1.00	1.01	1.01	1.01	1.00
$\approx 0.79517\pi$	1.00	1.02	1.01	1.01	1.00

Table: The H^1 error for $\omega \in (\frac{\pi}{2}, \pi)$ on quasi-uniform meshes.

Tests for different values of ω

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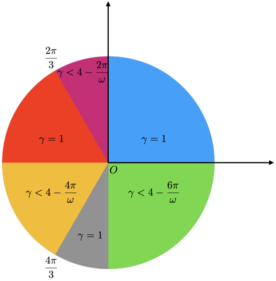


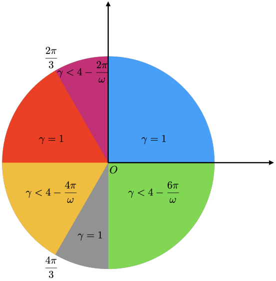
ω	expected rate	$j=6$	$j=7$	$j=8$	$j=9$
$\approx 1.10615\pi$	0.38	0.82	0.72	0.62	0.53
$\approx 1.25612\pi$	0.82	0.96	0.95	0.94	0.93
$\approx 1.30101\pi$	0.93	0.98	0.98	0.98	0.98
$\frac{4\pi}{3}$	1.00	1.00	1.00	1.00	1.00
$\approx 1.38305\pi$	1.00	1.02	1.02	1.01	1.01

Table: The H^1 error for $\omega \in (\pi, \frac{3\pi}{2})$ on quasi-uniform meshes.

Numerical Illustration III

Tests for different values of ω

$$-\Delta^3 u = \sin(\frac{N\pi}{\omega}\theta) \quad \text{in} \quad \Omega, \qquad\qquad u = \Delta u = \Delta^2 u = 0 \quad \text{on} \quad \Gamma.$$



ω	expected rate	$j = 6$	$j = 7$	$j = 8$	$j = 9$
$\approx 1.58946\pi$	0.23	0.87	0.76	0.63	0.50
$\frac{5\pi}{3}$	0.40	0.83	0.75	0.65	0.60
$\frac{7\pi}{4}$	0.57	0.87	0.82	0.77	0.71

Table: The H^1 error for $\omega \in (\frac{3\pi}{2}, 2\pi)$ on quasi-uniform meshes.

Summary

1. $H^3(\Omega)$ well-posedness for the triharmonic problem.
2. The naive mixed formulation does not apply to domains with angles $> \pi/2$.
3. The proposed mixed methods consist of $5 + N$ Poisson equations in general polygonal domains.
4. The correspondence between the largest angle and the convergence rates on quasi-uniform meshes is nonlinear.

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