The fully mixed methods for the Signorini Problem

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- **Signorini problem:** simulate the interaction between a linearly elastic body and a rigid foundation without friction.
- **Application:** electropainting, the elastic contact problem, the artificial outflow boundary condition of viscous flow.

Signorini problem (P)

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \end{aligned}$$

with Signorini condition

$$u \geq 0,\ \nabla u \cdot \boldsymbol{n} \geq 0,\ u(\nabla u \cdot \boldsymbol{n}) = 0 \ \text{ on } \Gamma_S.$$

The penalty approach for (P)

$$\begin{aligned} -\Delta u_\varepsilon &= f && \text{in } \Omega, \\ u_\varepsilon &= 0 && \text{on } \Gamma_D, \\ \nabla u_\varepsilon \cdot \boldsymbol{n} &= \frac{1}{\varepsilon}[u_\varepsilon]_- && \text{on } \Gamma_S. \end{aligned}$$

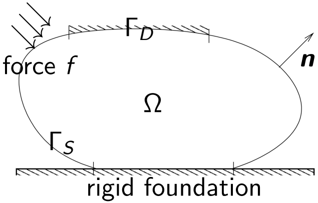


Figure 1: region Ω, Γ_D and Γ_S

Notations

- Ω: smooth and bbd' domain,
- $\partial\Omega := \overline{\Gamma_D \cup \Gamma_S}$, $\Gamma_D \cap \Gamma_S = \emptyset$,
- $f \in L^2(\Omega)$: given external force,
- u: $\Omega \to \mathbb{R}$,
- $\boldsymbol{n}$: unit outer normal vector,
- ε: the penalty parameter $(0 < \varepsilon \ll 1)$,
- $[s]_- := \max(0, -s)$.

Variational inequality and Lagrange multiplier form

(VI) Find $u \in \tilde{X}$ such that

$$(\nabla u, \nabla(v-u)) \geq (f, v-u) \qquad \forall v \in \tilde{X}.$$

Set **Lagrange multiplier** $\lambda := \nabla u \cdot \boldsymbol{n} \in H^{-\frac{1}{2}}(\Gamma_S)$. Note that $\lambda \geq 0$.

(VL) Find $(u, \lambda) \in X \times \tilde{\Lambda}_{-1/2}$, such that

$$(\nabla u, \nabla v) - \langle \lambda, v \rangle_{\Gamma_S} = (f, v) \qquad \forall v \in X,$$

$$\langle \mu - \lambda, u \rangle_{\Gamma_S} \geq 0 \qquad \forall \mu \in \tilde{\Lambda}_{-1/2}.$$

$X := H^1_{0,\Gamma_D}(\Omega)$, $\tilde{X} := \{v \in X : v|_{\Gamma_S} \geq 0\}$, $\tilde{\Lambda}_{-1/2} := \{\mu \in H^{-\frac{1}{2}}(\Gamma_S) : \mu \geq 0\}$.

Iteration algorithms for solving (VI) or (VL):

- Uzawa algorithm [Glowinski 84],
- the over-relaxation method [Glowinski 84],
- the penalty method [Haslinger 04],
- the Active/Inactive set method (Primal-dual Active set) [Ito 08].

The variational form of penalty method

$\textbf{(P)}_\varepsilon$ Find $u_\varepsilon \in X$ such that

$$(\nabla u_\varepsilon, \nabla v) - \frac{1}{\varepsilon} \int_{\Gamma_S} [u_\varepsilon]_- v \, ds = (f, v) \quad (\forall v \in X).$$

Introduce $\boldsymbol{\sigma} := \nabla u$ as an additional unknown.

Fully mixed problem and variational inequality

(MP)

$$\begin{aligned} \boldsymbol{\sigma} - \nabla u &= 0 && \text{in } \Omega, && \text{(2a)} \\ \nabla \cdot \boldsymbol{\sigma} + f &= 0 && \text{in } \Omega, && \text{(2b)} \\ u &= 0 && \text{on } \Gamma_D, && \text{(2c)} \\ u \geq 0, \quad \boldsymbol{\sigma} \cdot \boldsymbol{n} \geq 0, \quad u \boldsymbol{\sigma} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_S. && \text{(2d)} \end{aligned}$$

(MVI) Find $(\boldsymbol{\sigma}, u) \in \tilde{V} \times Q$ s.t.

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau} - \boldsymbol{\sigma}) + (u, \nabla \cdot (\boldsymbol{\tau} - \boldsymbol{\sigma})) &\geq 0 && \forall \boldsymbol{\tau} \in \tilde{V}, && \text{(3a)} \\ (\nabla \cdot \boldsymbol{\sigma}, v) + (f, v) &= 0 && \forall v \in Q. && \text{(3b)} \end{aligned}$$

$\tilde{V} := \{ \boldsymbol{\tau} \in H(\mathrm{div}; \Omega) : \boldsymbol{\tau} \cdot \boldsymbol{n} \geq 0 \text{ on } \Gamma_S \}$, $Q := L^2(\Omega)$.

Compare (MVI) with (VI):

- Unknown $\boldsymbol{\sigma}$ has a clear physical meaning (stress), and is computed directly.
- **(MVI)** is weaker than **(VI)**, the flexible choice of finite element subspace for u (like P0-element).

Perious research and Motivation

Perious research

- the theories of abstract mixed variational inequalities [Brezzi 78]
- mixed methods for the elastoplasticity problem [Reddy 92, Han 95]
- unilateral elastic contact problem [Slimane 04] with the penalty-type stabilization term $\frac{1}{\varepsilon}(\nabla \cdot \boldsymbol{\sigma}_\varepsilon + f)$
- Uzawa-type iteration based on the minimization problem [Wang 09]
- a modified Uzawa algorithm, using the BEM to compute the discrete $H^{\frac{1}{2}}$-inner product [Gatica 11]
- mixed (hemi-)VI for Stokes/NS with subgradient type bc [Han 21]

Motivation (No discrete $H^{\frac{1}{2}}$-inner product involved)

- propose two equivalent mixed multiplier formulations for **(MVI)** and revisit the well-posedness
- investigate two penalty approaches and establish the well-posedness
- finite element approximation (RT0/P0 for $\boldsymbol{\sigma}/u$),
- the Active/Inactive set algorithms

Outline

Proposition 1 (MVI) $\Rightarrow$ (MP)

Let $(\boldsymbol{\sigma}, u) \in \tilde{V} \times Q$ be the solution of **(MVI)**. If $u \in H^1(\Omega)$ and $\boldsymbol{\sigma} \cdot \boldsymbol{n} \in L^2(\Gamma_S)$, then $(\boldsymbol{\sigma}, u)$ satisfies (2) where the Signorini condition (2d) holds a.e. on Γ_S.

Theorem 1 (The well-posedness of (MVI))

Given $f \in Q$, **(MVI)** admits a unique solution $(\boldsymbol{\sigma}, u) \in \tilde{V} \times Q$ with the estimate

$$\|\boldsymbol{\sigma}\|_V + \|u\|_Q \leq C\|f\|_Q.$$

Remark: The unique existence of **(MVI)** has been proved by [Gatica 11]. We present an alternative proof revealing that $\boldsymbol{\sigma}$ can be seen as a projection onto a closed convex subspace of V.

Two equivalent mixed multiplier formulations

(MLM-I) Find $(\boldsymbol{\sigma}, u, p) \in \tilde{V} \times Q \times \Lambda_{1/2}$ such that

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \boldsymbol{\tau} \cdot \boldsymbol{n}, p \rangle_{\Gamma_S} = 0 \qquad \forall \boldsymbol{\tau} \in V, \tag{4a}$$

$$(\nabla \cdot \boldsymbol{\sigma}, v) + (f, v) = 0 \qquad \forall v \in Q, \tag{4b}$$

$$\langle \boldsymbol{\tau} \cdot \boldsymbol{n} - \boldsymbol{\sigma} \cdot \boldsymbol{n}, p \rangle_{\Gamma_S} \geq 0 \qquad \forall \boldsymbol{\tau} \in \tilde{V}; \tag{4c}$$

(MLM-II) Find $(\boldsymbol{\sigma}, u, p) \in V \times Q \times \tilde{\Lambda}_{1/2}$ such that

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \boldsymbol{\tau} \cdot \boldsymbol{n}, p \rangle_{\Gamma_S} = 0 \qquad \forall \boldsymbol{\tau} \in V, \tag{5a}$$

$$(\nabla \cdot \boldsymbol{\sigma}, v) + (f, v) = 0 \qquad \forall v \in Q, \tag{5b}$$

$$\langle \boldsymbol{\sigma} \cdot \boldsymbol{n}, q - p \rangle_{\Gamma_S} \geq 0 \qquad \forall q \in \tilde{\Lambda}_{1/2}. \tag{5c}$$

$\Lambda_{1/2} := H_{00}^{\frac{1}{2}}(\Gamma_S)$, $\tilde{\Lambda}_{1/2} := \{p \in \Lambda_{1/2} : p \geq 0 \text{ a.e.}\}$, $\Lambda_{-1/2} := H^{-\frac{1}{2}}(\Gamma_S) = \Lambda_{1/2}^*$.

Assume that $\boldsymbol{\sigma} \cdot \boldsymbol{n} \in L^2(\Gamma_S)$,

$$\begin{cases} (4c) \rightsquigarrow \quad \boldsymbol{\sigma} \cdot \boldsymbol{n} = \max(0, \boldsymbol{\sigma} \cdot \boldsymbol{n} - \rho p) \text{ to ensure } \boldsymbol{\sigma} \cdot \boldsymbol{n} \geq 0 \text{ on } \Gamma_S, \\ (5c) \rightsquigarrow \quad p = \max(0, p - \rho \boldsymbol{\sigma} \cdot \boldsymbol{n}) \text{ to ensure } u \geq 0 \text{ on } \Gamma_S. \end{cases}$$

(MVI) $\Leftrightarrow$ **(MLM-I)** $\Leftrightarrow$ **(MLM-II)**

Outline

Finite element approximate

Mesh and the finite element spaces(RT0/P0 + P0(e))

- Ω is a polygon ($d = 2$) or polyhedron ($d = 3$),
- $\mathcal{T}_h$: a quasi-uniform regular triangulation to Ω, $h = \max_{K \in \mathcal{T}_h} \operatorname{diam}(K)$,
- $\mathcal{E}_h$: the set of elements faces, $\mathcal{E}_{h,\Gamma_S}$ the mesh on Γ_S inherited from $\mathcal{T}_h$.

$$V_h := \{\boldsymbol{\tau}_h \in V : \boldsymbol{\tau}_h|_T \in RT_0(T)\}, \quad \tilde{V}_h := \{\boldsymbol{\tau}_h \in V_h : \boldsymbol{\tau}_h \cdot \boldsymbol{n} \geq 0 \text{ on } \Gamma_S\},$$
$$\Lambda_h := \{p_h \in \Lambda : p_h|_e \in P_0(e)\} = \{\boldsymbol{\tau}_h \cdot \boldsymbol{n}|_{\Gamma_S} : \boldsymbol{\tau}_h \in V_h\},$$
$$\tilde{\Lambda}_h := \{p_h \in \Lambda : p_h \geq 0\}, \quad Q_h := \{v_h \in Q : v_h|_T \in P_0(T)\}.$$

$P_0(T)$ and $P_0(e)$: the set of piecewise constant functions on T and e,

$$RT_0(T) := \{(a,b)^\top + c(x,y)^\top : a,b,c \in \mathbb{R},\ (x,y)^\top \in T\} \quad (d=2),$$
$$RT_0(T) := \{(a,b,c)^\top + d(x,y,z)^\top : a,b,c,d \in \mathbb{R},\ (x,y,z)^\top \in T\} \quad (d=3).$$

(MVI)$_h$ Find $(\boldsymbol{\sigma}_h, u_h) \in \tilde{V}_h \times Q_h$ such that

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h) + (u_h, \nabla \cdot (\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h)) \geq 0 \qquad \forall \boldsymbol{\tau}_h \in \tilde{V}_h, \tag{6a}$$

$$(\nabla \cdot \boldsymbol{\sigma}_h, v_h) + (f, v_h) = 0 \qquad \forall v_h \in Q_h. \tag{6b}$$

Theorem 2 (The well-posedness of (MVI)$_h$)

(MVI)$_h$ admits a unique solution $(\boldsymbol{\sigma}_h, u_h) \in \tilde{V}_h \times Q_h$ with the estimate

$$\|\boldsymbol{\sigma}_h\|_V + \|u_h\|_Q \leq C\|f\|_Q. \tag{7}$$

Key: $\boldsymbol{\sigma}_h$ can be seen as a projection onto a closed convex subspace of V_h.

Theorem 3 (The error estimates)

Let $(\boldsymbol{\sigma}, u)$ and $(\boldsymbol{\sigma}_h, u_h)$ be the unique solutions of **(MVI)** and **(MVI)**$_h$, respectively. Let p and p_h satisfy (4a) and (9a), respectively. Assume that $\boldsymbol{\sigma} \in H^1(\Omega)^d$, $\nabla \cdot \boldsymbol{\sigma} \in H^1(\Omega)$, $\boldsymbol{\sigma} \cdot \boldsymbol{n}|_{\Gamma_S} \in H^1(\Gamma_S)$, $u \in H^1(\Omega)$ and $p \in H^1(\Gamma_S)$. Then, we have

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_V + \|u - u_h\|_Q + \|p - p_h\|_\Lambda \leq Ch. \tag{8}$$

(MLM-I)$_h$ Find $(\boldsymbol{\sigma}_h, u_h, p_h) \in \tilde{V}_h \times Q_h \times \Lambda_h$ such that

$$
\begin{aligned}
(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (u_h, \nabla \cdot \boldsymbol{\tau}_h) - (p_h, \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\Gamma_S} &= 0 \qquad \forall \boldsymbol{\tau}_h \in V_h, && (9a)\\
(\nabla \cdot \boldsymbol{\sigma}_h, v_h) + (f, v_h) &= 0 \qquad \forall v_h \in Q_h, && (9b)\\
(\boldsymbol{\tau}_h \cdot \boldsymbol{n} - \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, p_h)_{\Gamma_S} &\geq 0 \qquad \forall \boldsymbol{\tau}_h \in \tilde{V}_h. && (9c)
\end{aligned}
$$

(MLM-II)$_h$ Find $(\boldsymbol{\sigma}_h, u_h, p_h) \in V_h \times Q_h \times \tilde{\Lambda}_h$ such that

$$
\begin{aligned}
(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (u_h, \nabla \cdot \boldsymbol{\tau}_h) - (p_h, \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\Gamma_S} &= 0 \qquad \forall \boldsymbol{\tau}_h \in V_h, && (10a)\\
(\nabla \cdot \boldsymbol{\sigma}_h, v_h) + (f, v_h) &= 0 \qquad \forall v_h \in Q_h, && (10b)\\
(q_h - p_h, \boldsymbol{\sigma}_h \cdot \boldsymbol{n})_{\Gamma_S} &\geq 0 \qquad \forall q_h \in \tilde{\Lambda}_h. && (10c)
\end{aligned}
$$

Note that $\boldsymbol{\sigma}_h \cdot \boldsymbol{n} \in L^2(\Gamma_S)$.

$$
\begin{cases}
(9c) \rightsquigarrow & \boldsymbol{\sigma}_h \cdot \boldsymbol{n} = P_{\tilde{\Lambda}_h}(\boldsymbol{\sigma}_h \cdot \boldsymbol{n} - \rho p_h) = \max(0, \boldsymbol{\sigma}_h \cdot \boldsymbol{n} - \rho p_h) \\
 & \text{iteration } \boldsymbol{\sigma}_h^{(n)} \cdot \boldsymbol{n} = P_{\tilde{\Lambda}_h}(\boldsymbol{\sigma}_h^{(n-1)} \cdot \boldsymbol{n} - \rho p_h^{(n-1)}). \\
(10c) \rightsquigarrow & p_h = P_{\tilde{\Lambda}_h}(p_h - \rho \boldsymbol{\sigma}_h \cdot \boldsymbol{n}) = \max(0, p_h - \rho \boldsymbol{\sigma}_h \cdot \boldsymbol{n}) \\
 & \text{iteration } p_h^{(n)} = P_{\tilde{\Lambda}_h}(p_h^{(n-1)} - \rho \boldsymbol{\sigma}_h^{(n-1)} \cdot \boldsymbol{n}).
\end{cases}
$$

Remark (Two ways to define the projection: $H^{\frac{1}{2}}$ V.S. L^2)

1. (cf. [Gatica 11]) **Riesz representation** $R : (H^{\frac{1}{2}}_{00}(\Gamma_S))^* \to H^{\frac{1}{2}}_{00}(\Gamma_S)$.
$\langle \boldsymbol{\tau}\cdot\boldsymbol{n} - \boldsymbol{\sigma}\cdot\boldsymbol{n}, p\rangle_{\Gamma_S} \geq 0 \rightsquigarrow (R\boldsymbol{\tau}\cdot\boldsymbol{n} - R\boldsymbol{\sigma}\cdot\boldsymbol{n}, p)_{H^{\frac{1}{2}}(\Gamma_S)} \geq 0$
$$\rightsquigarrow (R\boldsymbol{\tau}\cdot\boldsymbol{n} - R\boldsymbol{\sigma}\cdot\boldsymbol{n}, -\rho p + R\boldsymbol{\sigma}\cdot\boldsymbol{n} - R\boldsymbol{\sigma}\cdot\boldsymbol{n})_{H^{\frac{1}{2}}(\Gamma_S)} \leq 0 \quad (\forall \rho > 0)$$
$$\rightsquigarrow R\boldsymbol{\sigma}\cdot\boldsymbol{n} = P_{\tilde{\Lambda}_{1/2}}(-\rho p + R\boldsymbol{\sigma}\cdot\boldsymbol{n}) \quad \text{(Projection in } H^{\frac{1}{2}}(\Gamma_S)\text{)},$$
$$\langle \boldsymbol{\sigma}\cdot\boldsymbol{n}, p - q\rangle_{\Gamma_S} \geq 0 \rightsquigarrow (R\boldsymbol{\sigma}\cdot\boldsymbol{n}, p - q)_{H^{\frac{1}{2}}(\Gamma_S)} \geq 0$$
$$\rightsquigarrow (-\rho R\boldsymbol{\sigma}\cdot\boldsymbol{n} + p - p, p - q)_{H^{\frac{1}{2}}(\Gamma_S)} \leq 0 \quad (\forall \rho > 0)$$
$$\rightsquigarrow p = P_{\tilde{\Lambda}_{1/2}}(-\rho R\boldsymbol{\sigma}\cdot\boldsymbol{n} + p).$$
Discrete $H^{\frac{1}{2}}(\Gamma_S)$ inner product. Discretization of Riesz projection, and the discrete version of the $H^{\frac{1}{2}}(\Gamma_S)$ inner product, the discrete version of projection of $P_{\tilde{\Lambda}_{1/2}}$!

2. (Our proposal): No discrete Riesz projection and discrete $H^{\frac{1}{2}}(\Gamma_S)$ inner product. Just L^2 inner product in the discrete problem. But we can still prove the **well-posedness, error analysis (similar to [Gatica 11]) and efficient iteration scheme using projection**.

Algorithm 1 The Active/Inactive set method based on **(MLM-I)**$_h$

(Step 1) Choose a $\rho > 0$. Find $(\boldsymbol{\sigma}_h^{(0)}, u_h^{(0)}) \in V_h \times Q_h$ such that

$$\begin{aligned} \boldsymbol{\sigma}_h^{(0)} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_S, \\ (\boldsymbol{\sigma}_h^{(0)}, \boldsymbol{\tau}_h) + (u_h^{(0)}, \nabla \cdot \boldsymbol{\tau}_h) &= 0 && \forall \boldsymbol{\tau}_h \in V_h, \\ (\nabla \cdot \boldsymbol{\sigma}_h^{(0)}, v_h) + (f_h, v_h) &= 0 && \forall v_h \in Q_h. \end{aligned}$$

Set $p_h^{(0)} = 0$ and $n = 1$.

(Step 2) Decompose Γ_S into two subsets:

$$\Gamma_S^{1,(n)} := \{e \in \Gamma_S : \boldsymbol{\sigma}_h^{(n-1)} \cdot \boldsymbol{n} - \rho p_h^{(n-1)} < 0 \text{ on } e\} \qquad \text{(The active set)},$$

$$\Gamma_S^{2,(n)} := \Gamma_S \backslash \Gamma_S^{1,(n)} \qquad \text{(The inactive set)}.$$

(Step 3) Find $(\boldsymbol{\sigma}_h^{(n)}, u_h^{(n)}) \in V_h \times Q_h$ such that

$$\boldsymbol{\sigma}_h^{(n)} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_S^{1,(n)}, \tag{11a}$$

$$(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_h) + (u_h^{(n)}, \nabla \cdot \boldsymbol{\tau}_h) = 0 \qquad \forall \boldsymbol{\tau}_h \in V_{h,\Gamma_S^{1,(n)}}, \tag{11b}$$

$$(\nabla \cdot \boldsymbol{\sigma}_h^{(n)}, v_h) + (f_h, v_h) = 0 \qquad \forall v_h \in Q_h. \tag{11c}$$

(Step 4) Find $p_h^{(n)} \in \Lambda_h$ such that

$$(p_h^{(n)}, \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\Gamma_S} = (\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_h) + (u_h^{(n)}, \nabla \cdot \boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in V_h. \tag{12}$$

(Step 5) $n \to n + 1$. Iterate (Step2)-(Step4) until convergence.

Algorithm 2 The Active/Inactive set method based on $\textbf{(MLM-II)}_h$

(Step 1) Same to Algorithm 1.

(Step 2) Decompose Γ_S into two subsets:

$$\Gamma_S^{1,(n)} := \{e \in \Gamma_S : p_h^{(n-1)} - \rho \boldsymbol{\sigma}_h^{(n-1)} \cdot \boldsymbol{n} < 0 \text{ on } e\} \qquad \text{(The active set)},$$

$$\Gamma_S^{2,(n)} := \Gamma_S \backslash \Gamma_S^{1,(n)} \qquad \text{(The inactive set)}.$$

(Step 3) Find $(\boldsymbol{\sigma}_h^{(n)}, u_h^{(n)}) \in V_h \times Q_h$ such that

$$\boldsymbol{\sigma}_h^{(n)} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_S^{2,(n)}, \tag{13a}$$

$$(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_h) + (u_h^{(n)}, \nabla \cdot \boldsymbol{\tau}_h) = 0 \qquad \forall \boldsymbol{\tau}_h \in V_{h,\Gamma_S^{2,(n)}}, \tag{13b}$$

$$(\nabla \cdot \boldsymbol{\sigma}_h^{(n)}, v_h) + (f_h, v_h) = 0 \qquad \forall v_h \in Q_h. \tag{13c}$$

(Step 4) Find $p_h^{(n)} \in \Lambda_h$ such that

$$(p_h^{(n)}, \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\Gamma_S} = (\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_h) + (u_h^{(n)}, \nabla \cdot \boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in V_h. \tag{14}$$

(Step 5) $n \to n+1$. Iterate (Step2)-(Step4) until convergence.

Remark: In Alg. 1, the projection $\boldsymbol{\sigma}_h^{(n)} \cdot \boldsymbol{n} = \max(0, \boldsymbol{\sigma}_h^{(n-1)} \cdot \boldsymbol{n} - \rho p_h^{(n-1)})$ is carried out only on $\Gamma_S^{1,(n)}$. From (11b) and (12), $p_h^{(n)} = 0$ on $\Gamma_S^{2,(n)}$.
In Alg. 2, according to (13b) and (14), we have $p_h^{(n)} = 0$ on $\Gamma_S^{1,(n)}$, same as the projection $p_h^{(n)} = \max(0, p_h^{(n-1)} - \rho \boldsymbol{\sigma}_h^{(n-1)} \cdot \boldsymbol{n})$ on $\Gamma_S^{1,(n)}$.

Convergence results

Proposition 2

Assume that after n_0 iteration steps, the active(also inactive) sets becomes invariant, i.e., $\Gamma_S^{i,(n_0+1)} = \Gamma_S^{i,(n_0)} (i = 1, 2)$. Then Alg. 1 and Alg. 2 are convergent,

$$(\boldsymbol{\sigma}_h^{(n+1)}, u_h^{(n+1)}, p_h^{(n+1)}) = (\boldsymbol{\sigma}_h^{(n)}, u_h^{(n)}, p_h^{(n)}) \quad (\forall n \geq n_0).$$

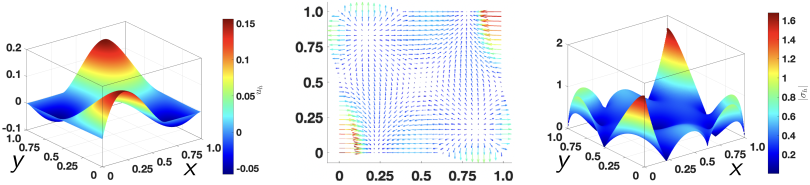
Moreover, we have $\boldsymbol{\sigma}_h^{(n)} \cdot \boldsymbol{n} \geq 0$, $p_h^{(n)} \geq 0$, $\boldsymbol{\sigma}_h^{(n)} \cdot \boldsymbol{n} p_h^{(n)} = 0$ on Γ_S.

Remark: In Proposition 2, we assume the active (also inactive) sets are invariant after a couple of iterations, which is the same assumption referred to [Kashiwabara 13, Zhou 23]. In fact, this assumption cannot be prescribed. Nevertheless, in our numerical experiments, the algorithm performed well, and the iteration stopped after only a few steps.

Example 1 (for Alg. 1 and 2)

Setting: Square region $\Omega = (0,1)^2 \quad f = 50(\frac{1}{2} - x)(\frac{1}{2} - y)$,
$\Gamma_S = \Gamma_{S1} \cup \Gamma_{S2} = (0,1) \times \{1\} \cup (0,1) \times \{0\}, \quad \Gamma_D = \partial\Omega \setminus \Gamma_S$.

The simulation results for Alg. 1 and 2 are almost identical. So we only display the result of Alg. 1.



(a) u_h (b) $\boldsymbol{\sigma}_h$ (c) $|\boldsymbol{\sigma}_h|$

Figure 2: Example 1: The numerical solution $(\boldsymbol{\sigma}_h, u_h)$.

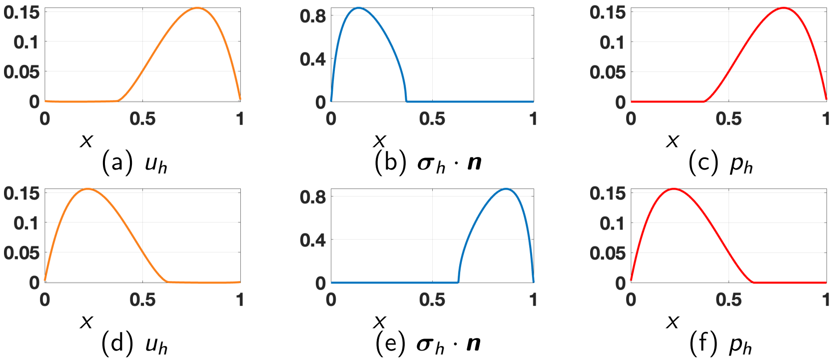


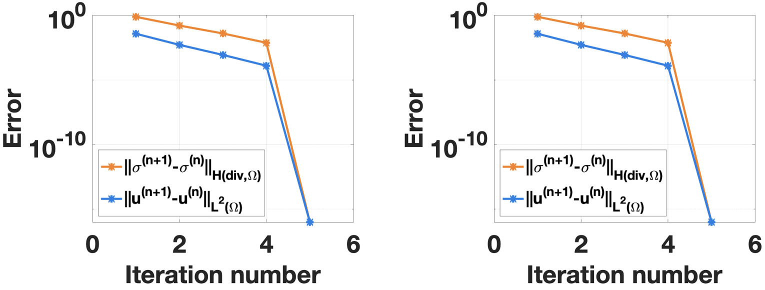
Figure 3: Example 1: $(u_h, \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, p_h)$ on Γ_{S1} (top) and Γ_{S2} (bottom) of **(MLM-I)**$_h$ by using Alg. 1.

Table 1: Example 1: The errors $\|\boldsymbol{\sigma}_{ref} - \boldsymbol{\sigma}_h\|_V$ and $\|u_{ref} - u_h\|_Q$ for Alg. 1

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$\|\boldsymbol{\sigma}_{ref} - \boldsymbol{\sigma}_h\|_V$	1.65	0.795	0.383	0.178	0.0764
Order	-	1.05	1.06	1.10	1.22
$\|u_{ref} - u_h\|_Q$	0.0465	0.0247	0.0121	0.00568	0.00244
Order	-	0.91	1.03	1.10	1.22

Table 2: Example 1: The errors $\|\boldsymbol{\sigma}_{ref} - \boldsymbol{\sigma}_h\|_V$ and $\|u_{ref} - u_h\|_Q$ for Alg. 2

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$\|\boldsymbol{\sigma}_{ref} - \boldsymbol{\sigma}_h\|_V$	1.65	0.795	0.383	0.178	0.0764
Order	-	1.05	1.06	1.10	1.22
$\|u_{ref} - u_h\|_Q$	0.0465	0.0247	0.0121	0.00568	0.00244
Order	-	0.91	1.03	1.10	1.22

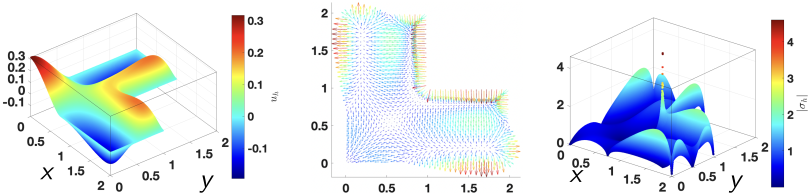


(a) Alg. 1 (b) Alg. 2

Figure 4: Example 1: The iteration errors $\|\boldsymbol{\sigma}_h^{(n+1)} - \boldsymbol{\sigma}_h^{(n)}\|_V$ and $\|u_h^{(n+1)} - u_h^{(n)}\|_Q$ by Alg. 1 and 2 with $h = 2^{-6}$.

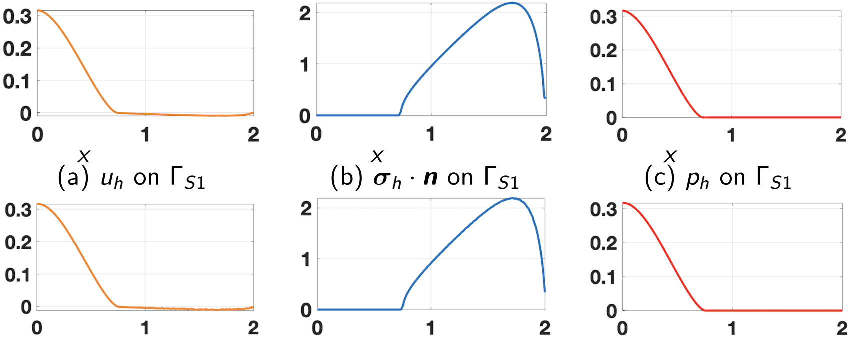
Example 2 (for Alg. 1 and 2)

Setting: L-shaped region $\Omega = (0,2)^2 \setminus [1,2]^2$, $\quad f = 25(\frac{1}{2} - x)(\frac{1}{2} - y)$.
$\Gamma_S = \Gamma_{S_1} \cup \Gamma_{S_2} = (0,2) \times \{0\} \cup \{0\} \times (0,2)$, $\quad \Gamma_D = \partial\Omega \setminus \Gamma_S$.



(a) u_h (b) $\boldsymbol{\sigma}_h$ (c) $|\boldsymbol{\sigma}_h|$

Figure 5: Example 2: The numerical solution $(\boldsymbol{\sigma}_h, u_h)$.



(a) u_h on Γ_{S1} (b) $\boldsymbol{\sigma}_h \cdot \boldsymbol{n}$ on Γ_{S1} (c) p_h on Γ_{S1}

(d) u_h on Γ_{S2} (e) $\boldsymbol{\sigma}_h \cdot \boldsymbol{n}$ on Γ_{S2} (f) p_h on Γ_{S2}

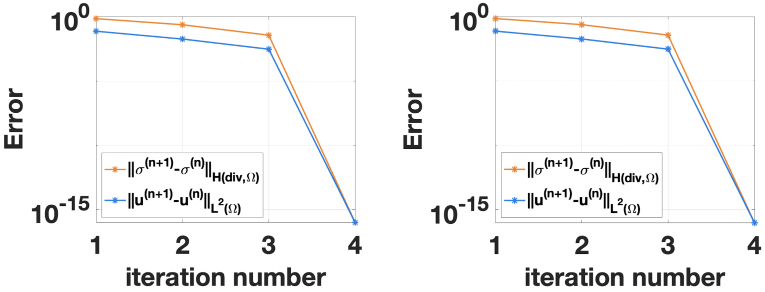
Figure 6: Example 2: $(u_h, \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, p_h)$ on Γ_{S1} (top) and Γ_{S2} (bottom) by using Alg. 1.

Table 3: Example 2: The errors $\|\boldsymbol{\sigma}_{ref} - \boldsymbol{\sigma}_h\|_V$ and $\|u_{ref} - u_h\|_Q$ for Alg. 1

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$\|\boldsymbol{\sigma}_{ref} - \boldsymbol{\sigma}_h\|_V$	2.17	1.02	0.479	0.252	0.139
Order	-	1.14	1.30	0.86	0.86
$\|u_{ref} - u_h\|_Q$	0.114	0.0539	0.0229	0.0103	0.00511
Order	-	1.13	1.48	1.06	1.02

Table 4: Example 2: The errors $\|\boldsymbol{\sigma}_{ref} - \boldsymbol{\sigma}_h\|_V$ and $\|u_{ref} - u_h\|_Q$ for Alg. 2

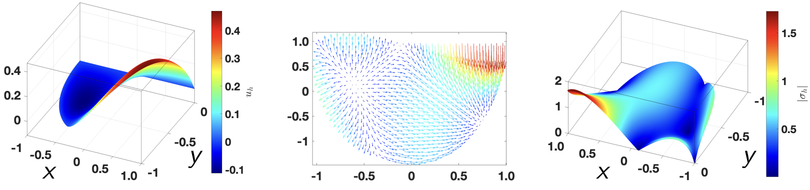
h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$\|\boldsymbol{\sigma}_{ref} - \boldsymbol{\sigma}_h\|_V$	2.17	1.02	0.479	0.252	0.139
Order	-	1.14	1.30	0.86	0.86
$\|u_{ref} - u_h\|_Q$	0.114	0.0539	0.0229	0.0103	0.00511
Order	-	1.13	1.48	1.06	1.02



(a) Alg. 1 (b) Alg. 2

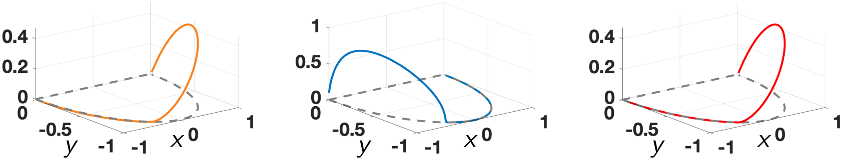
Figure 7: Example 2: The iteration errors $\|\boldsymbol{\sigma}_h^{(n+1)} - \boldsymbol{\sigma}_h^{(n)}\|_V$ and $\|u_h^{(n+1)} - u_h^{(n)}\|_Q$ by Alg. 1 and 2 with $h = 2^{-3}$.

Setting: Half-disk region $\Omega = \{(x, y) : x^2 + y^2 < 1, y < 0\}$, $\quad f = 5x$.
$\Gamma_S = \{(x, -\sqrt{1 - x^2}) : x \in (-1, 1)\}$ and $\Gamma_D = \partial\Omega \setminus \Gamma_S$.



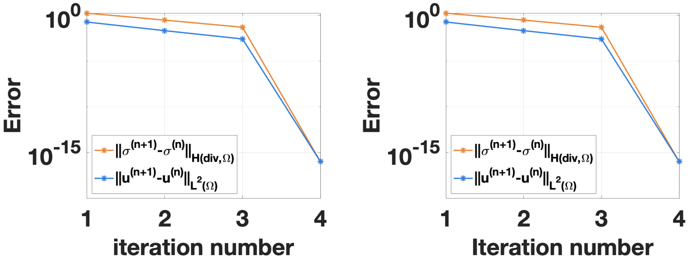
(a) u_h (b) $\boldsymbol{\sigma}_h$ (c) $|\boldsymbol{\sigma}_h|$

Figure 8: Example 3: The numerical solution $(\boldsymbol{\sigma}_h, u_h)$.



(a) u_h (b) $\boldsymbol{\sigma}_h \cdot \boldsymbol{n}$ (c) p_h

Figure 9: Example 3: $(u_h, \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, p_h)$ on Γ_S by using Alg. 1.



(a) Alg. 1 (b) Alg. 2

Figure 10: Example 3: The iteration errors $\|\boldsymbol{\sigma}_h^{(n+1)} - \boldsymbol{\sigma}_h^{(n)}\|_V$ and $\|u_h^{(n+1)} - u_h^{(n)}\|_Q$ by Alg. 1 and 2 with $h = 2^{-3}$.

Table 5: Example 3: The errors $\|\boldsymbol{\sigma}_{ref} - \boldsymbol{\sigma}_h\|_V$ and $\|u_{ref} - u_h\|_Q$ for Alg. 1

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$\|\boldsymbol{\sigma}_{ref} - \boldsymbol{\sigma}_h\|_V$	0.311	0.161	0.0720	0.0331	0.0160
Order	-	0.97	1.27	1.03	1.17
$\|u_{ref} - u_h\|_Q$	0.0528	0.0254	0.0111	0.00509	0.00252
Order	-	1.07	1.32	1.03	1.12

Table 6: Example 3: The errors $\|\boldsymbol{\sigma}_{ref} - \boldsymbol{\sigma}_h\|_V$ and $\|u_{ref} - u_h\|_Q$ for Alg. 2

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
$\|\boldsymbol{\sigma}_{ref} - \boldsymbol{\sigma}_h\|_V$	0.311	0.161	0.0720	0.0331	0.0160
Order	-	0.97	1.27	1.03	1.17
$\|u_{ref} - u_h\|_Q$	0.0528	0.0254	0.0111	0.00509	0.00252
Order	-	1.07	1.32	1.03	1.12

(MP$_\varepsilon$-I) Find $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon, p_\varepsilon) \in V \times Q \times \Lambda_{1/2}$ such that:

$$(\boldsymbol{\sigma}_\varepsilon, \boldsymbol{\tau}) + (u_\varepsilon, \nabla\cdot\boldsymbol{\tau}) - \langle \boldsymbol{\tau}\cdot\boldsymbol{n}, p_\varepsilon\rangle_{\Gamma_S} = 0 \qquad (\forall \boldsymbol{\tau} \in V), \tag{15a}$$

$$(\nabla\cdot\boldsymbol{\sigma}_\varepsilon, v) + (f, v) = 0 \qquad (\forall v \in Q), \tag{15b}$$

$$\langle \boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}, q\rangle_{\Gamma_S} - \frac{1}{\varepsilon}([p_\varepsilon]_-, q)_{\Gamma_S} = 0 \qquad (\forall q \in \Lambda_{1/2}); \tag{15c}$$

(MP$_\varepsilon$-II) Find $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon) \in \hat{V} \times Q$ such that:

$$(\boldsymbol{\sigma}_\varepsilon, \boldsymbol{\tau}) + (u_\varepsilon, \nabla\cdot\boldsymbol{\tau}) - \frac{1}{\varepsilon}([\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_-, \boldsymbol{\tau}\cdot\boldsymbol{n})_{\Gamma_S} = 0 \qquad (\forall \boldsymbol{\tau} \in \hat{V}), \tag{16a}$$

$$(\nabla\cdot\boldsymbol{\sigma}_\varepsilon, v) + (f, v) = 0 \qquad (\forall v \in Q). \tag{16b}$$

$\hat{V} := \{\boldsymbol{\tau} \in V : \boldsymbol{\tau}\cdot\boldsymbol{n}|_{\Gamma_S} \in \Lambda = L^2(\Gamma_S)\} \subset V$.

The strong forms of (MP$_\varepsilon$-I) and (MP$_\varepsilon$-II)

(MP$_\varepsilon$-I$'$):

$$-\nabla\cdot\boldsymbol{\sigma}_\varepsilon = f, \quad \boldsymbol{\sigma}_\varepsilon - \nabla u_\varepsilon = 0 \quad \text{in } \Omega, \tag{17a}$$

$$u_\varepsilon = 0 \quad \text{on } \Gamma_D, \tag{17b}$$

$$\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n} = \frac{1}{\varepsilon}[u_\varepsilon]_- \quad \text{on } \Gamma_S; \tag{17c}$$

(MP$_\varepsilon$-II$'$):

$$-\nabla\cdot\boldsymbol{\sigma}_\varepsilon = f, \quad \boldsymbol{\sigma}_\varepsilon - \nabla u_\varepsilon = 0 \quad \text{in } \Omega, \tag{18a}$$

$$u_\varepsilon = 0 \quad \text{on } \Gamma_D, \tag{18b}$$

$$u_\varepsilon = \frac{1}{\varepsilon}[\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_- \quad \text{on } \Gamma_S. \tag{18c}$$

Well-posedness of **(MP$_\varepsilon$-I)** and **(MP$_\varepsilon$-II)**

Theorem 4 (The well-posedness of (MP$_\varepsilon$-I))

There exists a unique solution $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon, p_\varepsilon) \in V \times Q \times \Lambda_{1/2}$ of **(MP$_\varepsilon$-I)** satisfying

$$\|\boldsymbol{\sigma}_\varepsilon\|_V^2 + \|u_\varepsilon\|_Q^2 + \frac{1}{\varepsilon}\|[p_\varepsilon]_-\|_{L^2(\Gamma_S)}^2 + \|p_\varepsilon\|_{\Lambda_{1/2}}^2 \leq C\|f\|_Q^2. \qquad (19)$$

Theorem 5 (The well-posedness of (MP$_\varepsilon$-II))

There exists a unique solution $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon) \in \hat{V} \times Q$ of **(MP$_\varepsilon$-II)** with the estimate

$$\|\boldsymbol{\sigma}_\varepsilon\|_V^2 + \|u_\varepsilon\|_Q^2 + \frac{1}{\varepsilon}\|[\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}]_-\|_\Lambda^2 \leq C\|f\|_Q^2. \qquad (20)$$

Remark

$$\textbf{(MP}_\varepsilon\textbf{-I)} \quad \not\Leftrightarrow \quad \textbf{(MP}_\varepsilon\textbf{-II)}$$

To investigate the penalty error, we set

$$\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} := \boldsymbol{\sigma} - \boldsymbol{\sigma}_\varepsilon, \quad e_{p,\varepsilon} := p - p_\varepsilon, \quad e_{u,\varepsilon} := u - u_\varepsilon.$$

Theorem 6 (The penalty error of (MP$_\varepsilon$-I))

Let $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon, p_\varepsilon)$ and $(\boldsymbol{\sigma}, u, p)$ be the unique solution of **(MP$_\varepsilon$-I)** and **(MLM-I)** respectively. Assume that $\boldsymbol{\sigma} \cdot \boldsymbol{n} \in \Lambda$. Then we have

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_V + \|e_{u,\varepsilon}\|_Q + \|e_{p,\varepsilon}\|_{\Lambda_{1/2}} + \sqrt{\varepsilon}\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} \cdot \boldsymbol{n}\|_\Lambda \leq C\sqrt{\varepsilon}\|\boldsymbol{\sigma} \cdot \boldsymbol{n}\|_\Lambda. \quad (21)$$

Additionally, if $\boldsymbol{\sigma} \cdot \boldsymbol{n} \in \Lambda_{1/2}$, then we have

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_V + \|e_{u,\varepsilon}\|_Q + \|e_{p,\varepsilon}\|_{\Lambda_{1/2}} + \sqrt{\varepsilon}\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} \cdot \boldsymbol{n}\|_\Lambda \leq C\varepsilon\|\boldsymbol{\sigma} \cdot \boldsymbol{n}\|_{\Lambda_{1/2}}. \quad (22)$$

Theorem 7 (The penalty error of (MP$_\varepsilon$-II))

Assume that $\boldsymbol{\sigma} \cdot \boldsymbol{n} \in \Lambda$. Let $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon)$ and $(\boldsymbol{\sigma}, u)$ be the unique solution of **(MP$_\varepsilon$-II)** and **(MVI)**. We have the error estimate

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_V + \|e_{u,\varepsilon}\|_Q + \sqrt{\varepsilon}\|e_{p,\varepsilon}\|_\Lambda \leq C\sqrt{\varepsilon}\|p\|_\Lambda. \quad (23)$$

In addition, if $[\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}]_- \in \Lambda_{1/2}$ (or equivalently, $p_\varepsilon \in \Lambda_{1/2}$), then we have

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_V + \|e_{u,\varepsilon}\|_Q + \sqrt{\varepsilon}\|e_{p,\varepsilon}\|_\Lambda + \|e_{p,\varepsilon}\|_{\Lambda_{1/2}} \leq C\varepsilon\|p\|_{\Lambda_{-1/2}}. \quad (24)$$

Mesh and the finite element spaces(RT0/P0 + P0(e))

- Ω is a polygon ($d = 2$) or polyhedron ($d = 3$),
- $\mathcal{T}_h$: a quasi-uniform regular triangulation to Ω, $h = \max_{K \in \mathcal{T}_h} \mathrm{diam}(K)$,
- $\mathcal{E}_h$: the set of elements faces, $\mathcal{E}_{h,\Gamma_S}$ the mesh on Γ_S inherited from $\mathcal{T}_h$.

$$
\begin{aligned}
V_h &:= \{\boldsymbol{\tau}_h \in V : \boldsymbol{\tau}_h|_E \in RT_0(T)\ (\forall T \in \mathcal{T}_h)\},\\
Q_h &:= \{v_h \in Q : v_h|_T \in P_0(T)\ (\forall T \in \mathcal{T}_h)\},\\
\Lambda_h &:= \{p_h \in \Lambda : p_h|_e \in P_0(e)\,(\forall e \in \mathcal{T}_{h,\Gamma_S})\} = \{\boldsymbol{\tau}_h \cdot \boldsymbol{n}|_{\Gamma_S} : \boldsymbol{\tau}_h \in V_h\},
\end{aligned}
$$

$P_0(T)$ and $P_0(e)$: the set of piecewise constant functions on T and e,

$RT_0(T) := \{(a, b)^\top + c(x, y)^\top : a, b, c \in \mathbb{R},\ (x, y)^\top \in T\} \quad (d = 2)$,

$RT_0(T) := \{(a, b, c)^\top + d(x, y, z)^\top : a, b, c, d \in \mathbb{R},\ (x, y, z)^\top \in T\} \quad (d = 3)$.

Discretization of **(MP$_\varepsilon$-I)** and **(MP$_\varepsilon$-II)**

(MP$_\varepsilon$-I)$_h$ Find $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}, p_{\varepsilon h}) \in V_h \times Q_h \times \Lambda_h$ such that

$$(\boldsymbol{\sigma}_{\varepsilon h}, \boldsymbol{\tau}_h) + (u_{\varepsilon h}, \nabla \cdot \boldsymbol{\tau}_h) - (\boldsymbol{\tau}_h \cdot \boldsymbol{n}, p_{\varepsilon h})_{\Gamma_S} = 0 \qquad (\forall \boldsymbol{\tau}_h \in V_h), \tag{25a}$$

$$(\nabla \cdot \boldsymbol{\sigma}_{\varepsilon h}, v_h) + (f_h, v_h) = 0 \qquad (\forall v_h \in Q_h), \tag{25b}$$

$$\left(\boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n} - \frac{1}{\varepsilon}[p_{\varepsilon h}]_-, q_h\right)_{\Gamma_S} = 0 \qquad (\forall q_h \in \Lambda_h). \tag{25c}$$

(MP$_\varepsilon$-II)$_h$ Find $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}) \in V_h \times Q_h$ such that

$$(\boldsymbol{\sigma}_{\varepsilon h}, \boldsymbol{\tau}_h) + (u_{\varepsilon h}, \nabla \cdot \boldsymbol{\tau}_h) - \frac{1}{\varepsilon}([\boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n}]_-, \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\Gamma_S} = 0 \;\; (\forall \boldsymbol{\tau}_h \in V_h), \tag{26a}$$

$$(\nabla \cdot \boldsymbol{\sigma}_{\varepsilon h}, v_h) + (f_h, v_h) = 0 \;\; (\forall v_h \in Q_h). \tag{26b}$$

Theorem 8 (The well-posedness of (MP$_\varepsilon$-I)$_h$)

There exists a unique solution $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}, p_{\varepsilon h}) \in V_h \times Q_h \times \Lambda_h$ to **(MP$_\varepsilon$-I)$_h$** with the estimate

$$\|\boldsymbol{\sigma}_{\varepsilon h}\|_V + \|u_{\varepsilon h}\|_Q + \frac{1}{\sqrt{\varepsilon}}\|[p_{\varepsilon h}]_-\|_\Lambda + \|p_{\varepsilon h}\|_\Lambda \le C\|f_h\|_Q. \tag{27}$$

Theorem 9 (The well-posedness of (MP$_\varepsilon$-II)$_h$)

There exists a unique solution $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}) \in V_h \times Q_h$ to **(MP$_\varepsilon$-II)$_h$** with the boundedness

$$\|\boldsymbol{\sigma}_{\varepsilon h}\|_V + \|u_{\varepsilon h}\|_Q + \frac{1}{\sqrt{\varepsilon}}\|[\boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n}]_-\|_\Lambda \le C\|f_h\|_Q. \tag{28}$$

The error estimates

Set the notations:

$$\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h} := \boldsymbol{\sigma} - \boldsymbol{\sigma}_{\varepsilon h} = (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}) + (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_{\varepsilon h}) =: \boldsymbol{\rho}_h + \boldsymbol{\theta}_h,$$

$$e_{u,\varepsilon h} := u - u_{\varepsilon h}, \quad e_{p,\varepsilon h} := p - p_{\varepsilon h}.$$

Theorem 10 ((MP$_\varepsilon$-I)$_h$ and (MLM-I))

Let $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}, p_{\varepsilon h})$ and $(\boldsymbol{\sigma}, u, p)$ be the solutions of **(MP$_\varepsilon$-I)$_h$** and **(MLM-I)** respectively. Assume that $\boldsymbol{\sigma} \in H^1(\Omega)^d$, $\nabla \cdot \boldsymbol{\sigma} \in H^1(\Omega)$, $u \in H^1(\Omega)$, $p \in H^1(\Gamma_S)$ and $\boldsymbol{\sigma} \cdot \boldsymbol{n} \in H^1(\Gamma_S)$. Then we have:

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}\|_V + \|e_{u,\varepsilon h}\|_Q + \sqrt{\varepsilon}\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h} \cdot \boldsymbol{n}\|_\Lambda \le C(h + \sqrt{\varepsilon}), \tag{29a}$$

$$\|e_{p,\varepsilon h}\|_\Lambda \le C(\sqrt{\varepsilon} + \sqrt{h}). \tag{29b}$$

Theorem 11 ((MP$_\varepsilon$-I)$_h$ and (MP$_\varepsilon$-I))

Let $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}, p_{\varepsilon h})$ and $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon, p_\varepsilon)$ be the solutions of **(MP$_\varepsilon$-I)$_h$** and **(MP$_\varepsilon$-I)**, respectively. Assume that $\boldsymbol{\sigma}_\varepsilon \in H^1(\Omega)^d$, $\nabla \cdot \boldsymbol{\sigma}_\varepsilon \in H^1(\Omega)$, $u_\varepsilon \in H^1(\Omega), p_\varepsilon \in H^1(\Gamma_S)$, $\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n} \in H^1(\Gamma_S)$. We have the error estimates:

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},h}\|_V + \|e_{u,h}\|_Q + \sqrt{\varepsilon}\|\boldsymbol{e}_{\boldsymbol{\sigma},h} \cdot \boldsymbol{n}\|_\Lambda \le C_\varepsilon h, \tag{30a}$$

$$\|e_{p,h}\|_\Lambda \le C_\varepsilon \sqrt{h}. \tag{30b}$$

Theorem 12 ($(\textbf{MP}_\varepsilon\textbf{-II})_h$ and (MLM-II))

Let $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h})$ and $(\boldsymbol{\sigma}, u, p)$ be the solutions of $(\textbf{MP}_\varepsilon\textbf{-II})_h$ and **(MLM-II)** respectively. Assume that $\boldsymbol{\sigma} \in H^1(\Omega)^d,\ \nabla \cdot \boldsymbol{\sigma} \in H^1(\Omega),\ u \in H^1(\Omega),\ p \in H^1(\Gamma_S)$ and $\boldsymbol{\sigma} \cdot \boldsymbol{n} \in H^1(\Gamma_S)$. Then we have the error estimate

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}\|_V + \|e_{u,\varepsilon h}\|_Q \le C(h + \sqrt{\varepsilon}), \tag{31a}$$

$$\|e_{p,\varepsilon h}\|_\Lambda \le C(\sqrt{\varepsilon} + \sqrt{h}). \tag{31b}$$

Theorem 13 ($(\textbf{MP}_\varepsilon\textbf{-II})_h$ and $(\textbf{MP}_\varepsilon\textbf{-II})$)

Let $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h})$ and $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon)$ be the solutions of $(\textbf{MP}_\varepsilon\textbf{-II})_h$ and $(\textbf{MP}_\varepsilon\textbf{-II})$, respectively. Under the same assumption of Theorem 11, we have the error estimates:

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},h}\|_V + \|e_{u,h}\|_Q \le C_\varepsilon h, \tag{32a}$$

$$\|e_{p,h}\|_\Lambda \le C_\varepsilon \sqrt{h}. \tag{32b}$$

Remark

- Convergence rates are similar to the non-mixed method
- The idea of **prime-dual method** can be used to design the iteration algorithms.

Algorithm 3 An iteration algorithm for $\mathbf{(MP_\varepsilon\text{-}I)}_h$

(Step 1) Find $(\boldsymbol{\sigma}_{\varepsilon h}^{(0)}, u_{\varepsilon h}^{(0)}) \in V_h \times Q_h$ such that

$$\boldsymbol{\sigma}_{\varepsilon h}^{(0)} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_S, \tag{33a}$$

$$(\boldsymbol{\sigma}_{\varepsilon h}^{(0)}, \boldsymbol{\tau}_h) + (u_{\varepsilon h}^{(0)}, \nabla \cdot \boldsymbol{\tau}_h) = 0 \qquad (\forall \boldsymbol{\tau}_h \in V_{h,\Gamma_S}), \tag{33b}$$

$$(\nabla \cdot \boldsymbol{\sigma}_{\varepsilon h}^{(0)}, v_h) + (f_h, v_h) = 0 \qquad (\forall v_h \in Q_h). \tag{33c}$$

Find $p_{\varepsilon h}^{(0)} \in \Lambda_h$ such that

$$(p_{\varepsilon h}^{(0)}, \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\Gamma_S} = (\boldsymbol{\sigma}_{\varepsilon h}^{(0)}, \boldsymbol{\tau}_h) + (u_{\varepsilon h}^{(0)}, \nabla \cdot \boldsymbol{\tau}_h) \quad (\forall \boldsymbol{\tau}_h \in V_h). \tag{34}$$

Set $n = 1$.

(Step 2) Divide Γ_S into two parts:

$\Gamma_S^{1,(n)} := \{e \in \Gamma_S : p_{\varepsilon h}^{(n-1)} > 0\}$ (The active set), $\quad \Gamma_S^{2,(n)} := \Gamma_S \backslash \Gamma_S^{1,(n)}$ (The inactive set).

Find $(\boldsymbol{\sigma}_{\varepsilon h}^{(n)}, u_{\varepsilon h}^{(n)}) \in V_h \times Q_h$ such that

$$\boldsymbol{\sigma}_{\varepsilon h}^{(n)} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_S^{1,(n)}, \tag{35a}$$

$$(\boldsymbol{\sigma}_{\varepsilon h}^{(n)}, \boldsymbol{\tau}_h) + (u_{\varepsilon h}^{(n)}, \nabla \cdot \boldsymbol{\tau}_h) + \varepsilon(\boldsymbol{\tau}_h \cdot \boldsymbol{n}, \boldsymbol{\sigma}_{\varepsilon h}^{(n)} \cdot \boldsymbol{n})_{\Gamma_S^{2,(n)}} = 0 \qquad (\forall \boldsymbol{\tau}_h \in V_{h,\Gamma_S^{1,(n)}}), \tag{35b}$$

$$(\nabla \cdot \boldsymbol{\sigma}_{\varepsilon h}^{(n)}, v_h) + (f_h, v_h) = 0 \qquad (\forall v_h \in Q_h). \tag{35c}$$

(Step 3) Modify the value of $\boldsymbol{\sigma}_{\varepsilon h}^{(n)}$ such that

$$\boldsymbol{\sigma}_{\varepsilon h}^{(n)} \cdot \boldsymbol{n} \geq 0 \quad \text{on } \Gamma_S^{2,(n)}. \tag{36}$$

Find $p_{\varepsilon h}^{(n)} \in \Lambda_h$ such that

$$(p_{\varepsilon h}^{(n)}, \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\Gamma_S} = (\boldsymbol{\sigma}_{\varepsilon h}^{(n)}, \boldsymbol{\tau}_h) + (u_{\varepsilon h}^{(n)}, \nabla \cdot \boldsymbol{\tau}_h) \quad (\forall \boldsymbol{\tau}_h \in V_h). \tag{37}$$

(Step 4) If $\Gamma_S^{1,(n)} = \Gamma_S^{1,(n-1)}$ $(n > 1)$, then the algorithm is finished. Otherwise, $n \to n+1$ and then iterate (Step2) and (Step3).

Algorithm 4 An iteration algorithm for $\mathbf{(MP_\varepsilon\text{-II})}_h$

(Step 1) Find $(\boldsymbol{\sigma}_{\varepsilon h}^{(0)}, u_{\varepsilon h}^{(0)}) \in V_h \times Q_h$ such that

$$(\boldsymbol{\sigma}_{\varepsilon h}^{(0)}, \boldsymbol{\tau}_h) + (u_{\varepsilon h}^{(0)}, \nabla \cdot \boldsymbol{\tau}_h) = 0 \qquad (\forall \boldsymbol{\tau}_h \in V_h), \tag{38a}$$

$$(\nabla \cdot \boldsymbol{\sigma}_{\varepsilon h}^{(0)}, v_h) + (f_h, v_h) = 0 \qquad (\forall v_h \in Q_h). \tag{38b}$$

Set $n = 1$.

(Step 2) Divide Γ_S into two parts:

$$\Gamma_S^{1,(n)} := \{e \in \Gamma_S : \boldsymbol{\sigma}_{\varepsilon h}^{(n-1)} \cdot \boldsymbol{n} > 0\} \text{ (The active set)},$$

$$\Gamma_S^{2,(n)} := \Gamma_S \backslash \Gamma_S^{1,(n)} \text{ (The inactive set)}.$$

Find $(\boldsymbol{\sigma}_{\varepsilon h}^{(n)}, u_{\varepsilon h}^{(n)}) \in V_h \times Q_h$ such that

$$(\boldsymbol{\sigma}_{\varepsilon h}^{(n)}, \boldsymbol{\tau}_h) + (u_{\varepsilon h}^{(n)}, \nabla \cdot \boldsymbol{\tau}_h) + \frac{1}{\varepsilon}(\boldsymbol{\sigma}_{\varepsilon h}^{(n)} \cdot \boldsymbol{n}, \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\Gamma_S^{2,(n)}} = 0 \quad (\forall \boldsymbol{\tau}_h \in V_h),$$

$$(\nabla \cdot \boldsymbol{\sigma}_{\varepsilon h}^{(n)}, v_h) + (f_h, v_h) = 0 \quad (\forall v_h \in Q_h).$$

(Step 3) If $\Gamma_S^{1,(n)} = \Gamma_S^{1,(n-1)}$ $(n > 1)$, then the algorithm is finished. Otherwise, $n \to n + 1$ and then go to (Step2).

Proposition 3

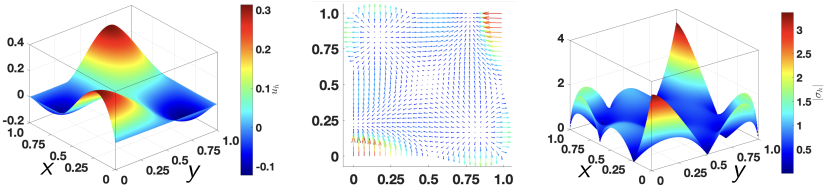
Assume that after n_0 iteration steps, the active(also inactive) sets becomes invariant, i.e., $\Gamma_S^{i,(n_0+1)} = \Gamma_S^{i,(n_0)} (i = 1, 2)$. Then Alg. 3 and Alg. 4 are convergent,

$$(\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)}, u_{\varepsilon h}^{(n+1)}) = (\boldsymbol{\sigma}_{\varepsilon h}^{(n)}, u_{\varepsilon h}^{(n)}) = (\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}).$$

Moreover, we have $\boldsymbol{\sigma}_{\varepsilon h}^{(n)} \cdot \boldsymbol{n} \geq 0$, $u_{\varepsilon h}^{(n)} \geq 0$, $\boldsymbol{\sigma}_{\varepsilon h}^{(n)} \cdot \boldsymbol{n} u_{\varepsilon h}^{(n)} = 0$ on Γ_S.

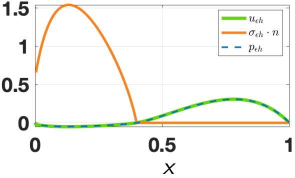
Outline

Setting: Square region $\Omega = (0,1)^2 \quad f = 100(\frac{1}{2} - x)(\frac{1}{2} - y)$,
$\Gamma_S = \Gamma_{S1} \cup \Gamma_{S2}$, $\Gamma_{S1} = (0,1) \times \{1\}$, $\Gamma_{S2} = \{0\} \times (0,1)$, $\quad \Gamma_D = \partial\Omega \setminus \Gamma_S$.

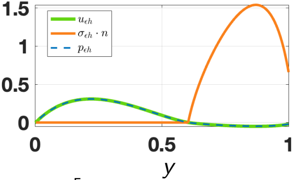


(a) u_h (b) $\boldsymbol{\sigma}_h$ (c) $|\boldsymbol{\sigma}_h|$

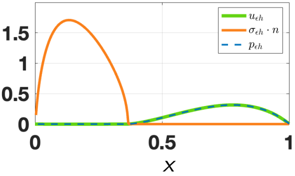
Figure 11: Example 4: The numerical solution $(\boldsymbol{\sigma}_h, u_h)$.



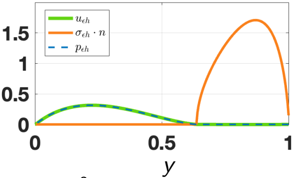
(a) $\varepsilon = 2^{-5}$: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n}, p_{\varepsilon h})$ on Γ_{S1}



(b) $\varepsilon = 2^{-5}$: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n}, p_{\varepsilon h})$ on Γ_{S2}

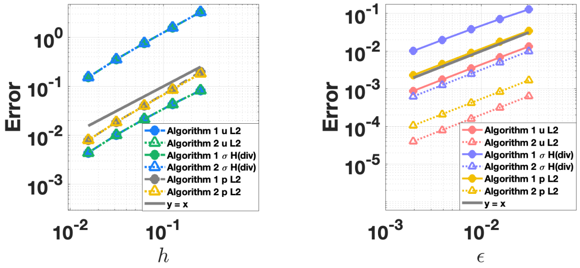


(c) $\varepsilon = 10^{-8}$: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n}, p_{\varepsilon h})$ on Γ_{S1}



(d) $\varepsilon = 10^{-8}$: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n}, p_{\varepsilon h})$ on Γ_{S2}

Figure 12: Example 4: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n}, p_{\varepsilon h})$ on Γ_S with $(h, \varepsilon) = (2^{-8}, 2^{-5})$ (see (a)–(b)) and $(h, \varepsilon) = (2^{-8}, 10^{-8})$ (see (c)–(d)) of $\textbf{(MP}_\varepsilon\textbf{-I)}_h$ by using Algorithm 3.



(a) Fixed $\varepsilon = 10^{-8}$ (b) Fixed $h = 2^{-8}$

Figure 13: Example 4: The errors $\|\boldsymbol{\sigma}_{\varepsilon ref} - \boldsymbol{\sigma}_{\varepsilon h}\|_V$, $\|u_{\varepsilon ref} - u_{\varepsilon h}\|_Q$ and $\|p_{\varepsilon ref} - p_{\varepsilon h}\|_\Lambda$ computed by Algorithms 3 and 4.

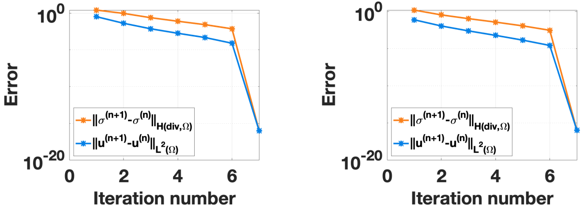
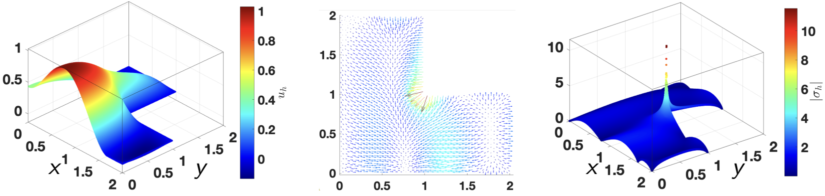


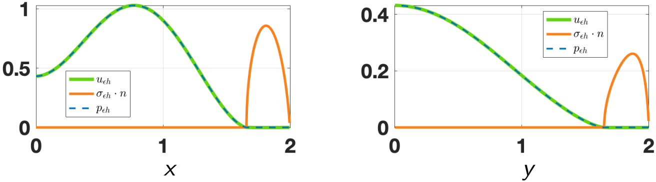
Figure 14: Example 4: The iteration errors $\|\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\|_V$ and $\|u_{\varepsilon h}^{(n+1)} - u_{\varepsilon h}^{(n)}\|_Q$ of Algorithms 3 (left) and 4 (right)with $(h, \varepsilon) = (2^{-8}, 10^{-8})$.

Setting: L-shaped region $\Omega = (0,2)^2 \setminus [1,2]^2$, $\quad f = 6\sin(\pi(x-\frac{1}{3}))\cos(\frac{y}{3})$.
$\Gamma_S = \Gamma_{S1} \cup \Gamma_{S2}$, $\Gamma_{S1} = (0,2)\times\{0\}$, $\Gamma_{S2} = \{0\}\times(0,2)$.



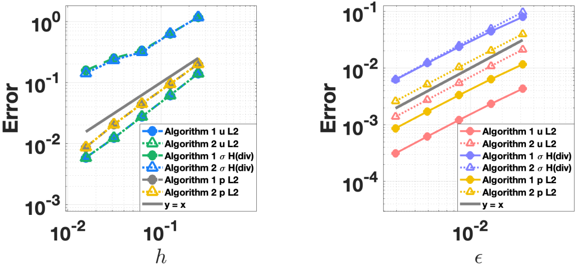
(a) u_h (b) $\boldsymbol{\sigma}_h$ (c) $|\boldsymbol{\sigma}_h|$

Figure 15: Example 5: The numerical solution $(\boldsymbol{\sigma}_h, u_h)$.



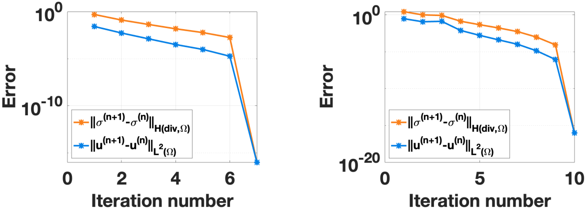
(a) $(u_h, \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, p_h)$ on Γ_{S1} (b) $(u_h, \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, p_h)$ on Γ_{S2}

Figure 16: Example 5: $(u_h, \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, p_h)$ on Γ_S.



(a) Fixed $\varepsilon = 10^{-8}$ (b) Fixed $h = 2^{-8}$

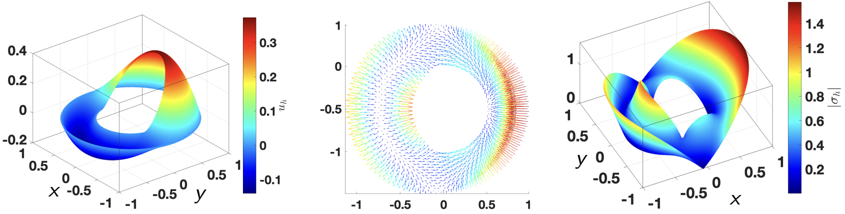
Figure 17: Example 5: The errors $\|\boldsymbol{\sigma}_{\varepsilon ref} - \boldsymbol{\sigma}_{\varepsilon h}\|_V$, $\|u_{\varepsilon ref} - u_{\varepsilon h}\|_Q$ and $\|p_{\varepsilon ref} - p_{\varepsilon h}\|_\Lambda$ computed by Algorithms 3 and 4.



(a) Algorithm 3 (b) Algorithm 4

Figure 18: Example 5: The iteration errors $\|\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\|_V$ and $\|u_{\varepsilon h}^{(n+1)} - u_{\varepsilon h}^{(n)}\|_Q$ of Algorithms 3 and 4 with $(h, \varepsilon) = (2^{-8}, 10^{-8})$.

Setting: $\Omega = \{(x,y) : \frac{1}{2} < \sqrt{x^2+y^2} < 1\}$, $\quad f = 5\sin(\frac{1}{2}\pi x)$.
$\Gamma_S = \{(x,y) : \sqrt{x^2+y^2} = \frac{1}{2}\}$ and $\Gamma_D = \partial\Omega \setminus \Gamma_S$.



(a) u_h (b) $\boldsymbol{\sigma}_h$ (c) $|\boldsymbol{\sigma}_h|$

Figure 19: Example 6: The numerical solution $(\boldsymbol{\sigma}_h, u_h)$.

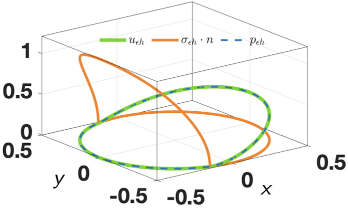
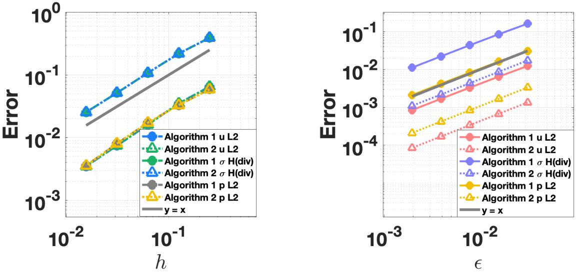
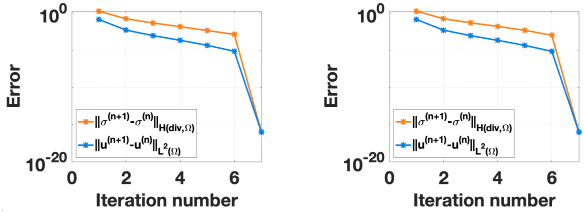


Figure 20: Example 6: $(u_h, \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, p_h)$ on Γ_S.



(a) Fixed $\varepsilon = 10^{-8}$ (b) Fixed $h = 2^{-8}$

Figure 21: Example 6: The errors $\|\boldsymbol{\sigma}_{\varepsilon ref} - \boldsymbol{\sigma}_{\varepsilon h}\|_V$, $\|u_{\varepsilon ref} - u_{\varepsilon h}\|_Q$ and $\|p_{\varepsilon ref} - p_{\varepsilon h}\|_\Lambda$ computed by Algorithms 3 and 4.



(a) Algorithm 3 (b) Algorithm 4

Figure 22: Example 6: The iteration errors $\|\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\|_V$ and $\|u_{\varepsilon h}^{(n+1)} - u_{\varepsilon h}^{(n)}\|_Q$ of Algorithms 3 and 4 with $(h, \varepsilon) = (2^{-8}, 10^{-8})$.

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THANKS!