A variational Crank–Nicolson ensemble Monte Carlo algorithm for a heat equation under uncertainty

Changlun Ye
Guizhou Normal University

Joint work with Hai Bi and Xianbing Luo

Numerical methods for spectral problems: theory and applications

Guiyang, China, August 5-9,2024

Main Contents

1. Problem
2. FVE/CN scheme and FVEMC/CN algorithm
3. Stability and error analysis
4. Some numerical examples

Problem

- Find a random function ϕ conforming to models:

$$\begin{cases} \phi_t - \nabla \cdot [\alpha(t,\mathbf{x},\omega)\nabla\phi] = f(t,\mathbf{x},\omega), \text{ in } [0,T] \times D \times \Omega, \\ \alpha\nabla\phi(t,\mathbf{x},\omega)\cdot\mathbf{n} = 0, \text{ on } [0,T] \times \partial D_0 \times \Omega, \\ \alpha\nabla\phi(t,\mathbf{x},\omega)\cdot\mathbf{n} = \beta(t,\mathbf{x},\omega)(g(t,\mathbf{x},\omega) - \phi(t,\mathbf{x},\omega)), \\ \text{on } [0,T] \times \partial D_1 \times \Omega, \\ \phi(0,\mathbf{x}) = \phi^0(\mathbf{x},\omega)), \text{ in } D \times \Omega, \end{cases} \tag{1}$$

where Robin coefficients β, diffusion coefficients α, boundary condition g , source term f and initial condition ϕ^0 are random fields with continuous and bounded covariance functions.

Introduction:Ensemble method

- For problems involving random parameters, there are several methods currently available, including the stochastic Galerkin method, stochastic collocation, reduced basis methods, MC, QMC.
- In the numerical simulation by MC sampling, a large number of linear equations $\mathbf{A}_j\mathbf{z}_j = \mathbf{b}_j, j \in \{1, 2, \ldots, J\}$ need to be solved. The calculation cost is very high since the number J is relatively large.

[1] Jiang, N., Layton, W.. ... Int. J. Uncertain. Quan. **4**, 2014

[2] Y. Luo and Z. Wang. ... SIAM Journal on Numerical Analysis, 56 , 859-876(2018)

Introduction:Ensemble method

- In order to improve computing efficiency, The ensemble method was first proposed in reference [1]. The ensemble method change solving $\mathbf{A}_j\mathbf{z}_j = \mathbf{b}_j, j \in \{1, 2, \ldots, J\}$ to solving $\tilde{\mathbf{A}}\mathbf{z}_j = \tilde{\mathbf{b}}_j, j \in \{1, 2, \ldots, J\}$. In [2], the authors studied the parabolic problem with random coefficients by using an ensemble method with first-order accuracy in time and obtained an error estimate.
- The authors of [3]combined the ensemble with HDG method to obtain an optimal error estimate in space.
- For the heat conduction problem with a random Robin coefficient, refer to [4].

[3] Li, M., Luo, X. ... Numer.Methods Partial Differ. Eq. **39**, 2840-2864(2023).

[4] Yao, T., Ye, C., Luo, X. et al. ... Numer. Algorithms (2023).

FVE/CN scheme

We use an uniform time partition on $[0, T]$ with $t_n = n\Delta t$, where Δt represent the step size. In the numerical simulation of equations (1), it is well known the following non-ensemble one-step time-stepping method based on Crank-Nicolson (CN), denoted as NE/CN:

$$\left(\frac{\phi_j^n - \phi_j^{n-1}}{\Delta t}, v\right) + \left(\alpha_j{}^{n-\frac{1}{2}} \nabla \widetilde{\phi}_j^{n-\frac{1}{2}}, \nabla v\right) + \left(\beta_j{}^{n-\frac{1}{2}} \widetilde{\phi}_j^{n-\frac{1}{2}}, v\right)_{\partial D_1}$$
$$= \left(f_j^{n-1/2}, v\right) + \left(\beta_j^{n-\frac{1}{2}} g_j^{n-1/2}, v\right)_{\partial D_1}, j = 1, \ldots, J, \tag{2}$$

where

$$\widetilde{\phi}_j^{n-\frac{1}{2}} = \frac{\phi_j^n + \phi_j^{n-1}}{2}.$$

A great number of linear equations $\mathbf{A}_j \mathbf{z}_j = \mathbf{b}_j, j \in \{1, 2, \ldots, J\}$ need to be solved for the system (2).

FVE/CN scheme

- Denote $\bar{\alpha}(t,\mathbf{x}) := \frac{1}{J}\sum_{j=1}^{J}\alpha_j(t,\mathbf{x})$ and $\bar{\beta}(t,\mathbf{x}) := \frac{1}{J}\sum_{j=1}^{J}\beta_j(t,\mathbf{x})$, be the ensemble average for the Robin coefficients and the diffusion coefficient, respectively.
- Let $\mathcal{T}_h$ be a quasi-uniform triangulation of D. Furthermore, we have $D = \bigcup_{K\in\mathcal{T}_h} K$ where each element K is assigned a diameter denoted as h_K. Additionally, we denote $h = \max_{K\in\mathcal{T}_h} h_K$. The collection of polynomials with degree l is represented by $\mathbf{P}_l$, then the definite elements space V_h for the domain D is given as follows:

$$V_h = \left\{ v \in C(\bar{D}) : v|_K \in \mathbf{P}_l, \forall K \in \mathcal{T}_h \right\}. \tag{3}$$

FVE/CN scheme

We describe the ensemble-based time-stepping second-order scheme as follows: for $j = 1, \dots, J$, introduce a new fully variational CN ensemble scheme (FVE/CN): seek $\phi_{j,h}^n \in V_h$ such that

$$\left(\frac{\phi_{j,h}^n - \phi_{j,h}^{n-1}}{\Delta t}, v_h\right) + \left(\bar{\alpha}^{n-\frac{1}{2}} \nabla \widetilde{\phi}_{j,h}^{n-\frac{1}{2}}, \nabla v_h\right) + \left(\left(\alpha_j^{n-\frac{1}{2}} - \bar{\alpha}^{n-\frac{1}{2}}\right) \nabla (2\widetilde{\phi}_{j,h}^{n-\frac{3}{2}} - \widetilde{\phi}_{j,h}^{n-\frac{5}{2}}),\right.$$
$$\left.\nabla v_h\right) + \left(\bar{\beta}^{n-\frac{1}{2}} \widetilde{\phi}_{j,h}^{n-\frac{1}{2}}, v_h\right)_{\partial D_1} + \left(\left(\beta_{j,h}^{n-\frac{1}{2}} - \bar{\beta}^{n-\frac{1}{2}}\right)\left(2\widetilde{\phi}_{j,h}^{n-\frac{3}{2}} - \widetilde{\phi}_{j,h}^{n-\frac{5}{2}}\right), v_h\right)_{\partial D_1}$$
$$= \left(f_j^{n-1/2}, v_h\right) + \left(\beta_j^{n-\frac{1}{2}} g_j^{n-1/2}, v_h\right)_{\partial D_1}, j = 1, \dots, J, \ \forall\ v_h \in V_h, \tag{4}$$

where

$$\widetilde{\phi}_{j,h}^{n-\frac{3}{2}} = \frac{\phi_{j,h}^{n-1} + \phi_{j,h}^{n-2}}{2}, \quad \widetilde{\phi}_{j,h}^{n-\frac{5}{2}} = \frac{\phi_{j,h}^{n-2} + \phi_{j,h}^{n-3}}{2}.$$

Eq. (4) is a fully discrete variational Crank-Nicolson ensemble (FVE/CN) scheme for the heat equation with random Robin and diffusion

FVE/CN scheme

- The FVE/CN scheme is a three-step method aimed at solving equation for $\phi_{j,h}^{n}$, given $\phi_{j,h}^{n-3}$, $\phi_{j,h}^{n-2}$, and $\phi_{j,h}^{n-1}$. The initial conditions provide $\phi_{j,h}^{0}$. To obtain $\phi_{j,h}^{1}$ and $\phi_{j,h}^{2}$, a one-step stable ensemble algorithm [2] or any other non-ensemble one-step stable time-stepping method, such as NE/CN, can be used.
- In the FVE/CN scheme, a special linear extrapolation $2\widetilde{\phi}_j^{n-\frac{3}{2}} - \widetilde{\phi}_j^{n-\frac{5}{2}} = 2\frac{\phi_j^{n-1}+\phi_j^{n-2}}{2} - \frac{\phi_j^{n-2}+\phi_j^{n-3}}{2}$ is used in the fluctuation term to approximate $\phi_j(t_{n-\frac{1}{2}})$, denoted by $\phi_j^{n-\frac{1}{2}}$. Through this approach, we can prove that the FVE/CN scheme is long-time energetically stable under two-parameter conditions.

FVE/CN scheme

After arrangement, we obtain

$$
\begin{aligned}
&\left(\frac{\phi_j^n}{\Delta t}, v\right) + \frac{1}{2}\left(\bar{\alpha}^{n-\frac{1}{2}}\nabla\phi_j^n, \nabla v\right) + \frac{1}{2}\left(\bar{\beta}^{n-\frac{1}{2}}\phi_j^n, v\right)_{\partial D_1} \\
&= \left(\frac{\phi_j^{n-1}}{\Delta t}, v\right) - \frac{1}{2}\left(\bar{\alpha}^{n-\frac{1}{2}}\nabla\phi_j^{n-1}, \nabla v\right) \\
&- \left(\left(\alpha_j^{n-\frac{1}{2}} - \bar{\alpha}^{n-\frac{1}{2}}\right)\nabla(2\widetilde{\phi}_j^{n-\frac{3}{2}} - \widetilde{\phi}_j^{n-\frac{5}{2}}), \nabla v\right) \\
&- \frac{1}{2}\left(\bar{\beta}^{n-\frac{1}{2}}\phi_j^{n-1}, v\right)_{\partial D_1} - \left(\left(\beta_j^{n-\frac{1}{2}} - \bar{\beta}^{n-\frac{1}{2}}\right)\left(2\widetilde{\phi}_j^{n-\frac{3}{2}} - \widetilde{\phi}_j^{n-\frac{5}{2}}\right), v\right)_{\partial D_1} \\
&+ \left(f_j^{n-1/2}, v\right) + \left(\beta_j^{n-\frac{1}{2}} g_j^{n-1/2}, v\right)_{\partial D_1}, j = 1, \dots, J.
\end{aligned} \tag{5}
$$

It is observed that the coefficient matrix of the resulting linear system is independent of j, which is a key characteristic of the ensemble scheme.

FVEMC/CN algorithm

Utilizing the FVE/CN scheme for the stochastic partial differential equation (SPDE) (1), first we employ the MC method for random sampling and obtain independent identically distributed (i.i.d.) samples, then solve the deterministic equation using the FVE/CN scheme.

Step1. Generate a collection of samples for the Robin and diffusion coefficients, the source term, the initial conditions and boundary conditions $\beta_j = \beta(\cdot,\cdot,\omega_j)$, $\alpha_j = \alpha(\cdot,\cdot,\omega_j)$, $f_j = f(\cdot,\cdot,\omega_j)$, $\phi_j^0 = \phi^0(\cdot,\omega_j)$ and $g_j = g(\cdot,\cdot,\omega_j)$, respectively, with the j-th sample. Hence, the solutions $\phi(\cdot,\cdot,\omega_j)$ are i.i.d..

Step2. Compute $\bar{\beta}^n = \frac{1}{J}\sum_{j=1}^{J}\beta(t_n,\mathbf{x},\omega_j)$, and $\bar{\alpha}^n = \frac{1}{J}\sum_{j=1}^{J}\alpha(t_n,\mathbf{x},\omega_j)$. For the j-th sample, solve (2) to obtain $\phi_{j,h}^1, \phi_{j,h}^2$, then find $\phi_{j,h}^n$ that satisfies the FVE/CN scheme (4) when $n = 3,\ldots,N$.

Step3. For $n = 3,\cdots,N$, compute $\frac{1}{J}\sum_{j=1}^{J}\phi_{j,h}^n$ to approximate the expectation $\mathbb{E}[\phi^n]$.

Using the FVEMC/CN algorithm, one just needs to solve a single linear system with multiple right-hand side vectors in a group at each time step.

Assumptions

To derive the stability and error estimate of the FVEMC/CN algorithm, we refer to [2] and assume that the following two hypotheses are valid:
(**H1**) There are four positive constants denoted as λ_{max}, λ_{min}, μ_{min}, μ_{max} such that for any $t \in [0, T]$, the probability is given by

$$\mathbb{P}\left\{\omega \in \Omega;\ \min_{x\in\bar{D}} \alpha(t, \mathbf{x}, \omega) > \lambda_{min}\right\} = 1, \tag{6}$$

$$\mathbb{P}\left\{\omega \in \Omega;\ \max_{x\in\bar{D}} \alpha(t, \mathbf{x}, \omega) < \lambda_{max}\right\} = 1, \tag{7}$$

and

$$\mathbb{P}\left\{\omega \in \Omega;\ \min_{x\in\bar{D}} \beta(t, \mathbf{x}, \omega) > \mu_{min}\right\} = 1, \tag{8}$$

$$\mathbb{P}\left\{\omega \in \Omega;\ \max_{x\in\bar{D}} \beta(t, \mathbf{x}, \omega) < \mu_{max}\right\} = 1. \tag{9}$$

Assumptions

(**H2**) There are four positive constants denoted as λ_+, μ_+ such that for any $t \in [0, T]$, the probability is given by

$$\mathbb{P}\left\{\omega_j \in \Omega;\ \ \left|\alpha\left(t, \mathbf{x}, \omega_j\right) - \bar{\alpha}\right|_\infty \leq \lambda_+\right\} = 1, \tag{10}$$

and

$$\mathbb{P}\left\{\omega_j \in \Omega;\ \ \left|\beta\left(t, \mathbf{x}, \omega_j\right) - \bar{\beta}\right|_\infty \leq \mu_+\right\} = 1. \tag{11}$$

At present, the hypothesis (**H1**) ensures the uniform coercivity, a.s, while the hypothesis (**H2**) represents the uniform bound of $|\alpha\left(t, \mathbf{x}, \omega_j\right) - \bar{\alpha}(t, \mathbf{x})|$, a.s.,. Similar properties ,also, hold for $\beta\left(t, \mathbf{x}, \omega_j\right)$.

Stability

Theorem

Assuming $f_j \in \widehat{L}^2\left(0, T; H^{-1}(D)\right)$, $g_j \in \widehat{L}^2\left(0, T; L^2(\partial D_1)\right)$, $\phi_j^0 \in \widehat{L}^2\left(H^1(D)\right)$ *and hypothesises* (**H1**) *and* (**H2**) *are satisfied, then the FVE/CN scheme* (4) *is long time energetically stable if*

$$\lambda_{min} - 3\lambda_+ > 0 \qquad \text{and} \qquad \mu_{min} - 3\mu_+ > 0. \tag{12}$$

Furthermore, the numerical solution to (4) *satisfies*

$$\begin{aligned}
&\mathbb{E}\left[\|\phi_{j,h}^N\|^2\right] + \frac{\lambda_{min} - 3\lambda_+}{4}\Delta t \sum_{n=1}^{N} \mathbb{E}\left[\|\nabla(\phi_{j,h}^{n-1} + \phi_{j,h}^n)\|^2\right] \\
&+ \frac{\mu_{min} - 3\mu_+}{8}\Delta t \sum_{n=1}^{N} \mathbb{E}\left[\|\phi_{j,h}^{n-1} + \phi_{j,h}^n\|_{\partial D_1}^2\right] \le C\Bigg(\Delta t \sum_{n=3}^{N} \mathbb{E}\left[\|f_j^{n-\frac{1}{2}}\|_{-1}^2\right] \\
&+ \Delta t \sum_{n=3}^{N} \mathbb{E}\left[\|g_j^{n-\frac{1}{2}}\|_{\partial D_1}^2\right] + \mathbb{E}\left[\|\phi_{j,h}^2\|^2\right] + \Delta t\mathbb{E}\left[\|\nabla\phi_{j,h}^2\|^2\right] + \Delta t\mathbb{E}\left[\|\nabla\phi_{j,h}^1\|^2\right] \\
&+ \Delta t\mathbb{E}\left[\|\nabla\phi_{j,h}^0\|^2\right] + \Delta t\mathbb{E}\left[\|\phi_{j,h}^0\|_{\partial D_1}^2\right] + \Delta t\mathbb{E}\left[\|\phi_{j,h}^1\|_{\partial D_1}^2\right] + \Delta t\mathbb{E}\left[\|\phi_{j,h}^2\|_{\partial D_1}^2\right]\Bigg).
\end{aligned} \tag{13}$$

Convergence Analysis

•Let $\Phi_h^n \equiv \frac{1}{J}\sum_{j=1}^{J}\phi_{j,h}^n$ be obtained by the FVEMC/CN algorithm which approximates the exact solution's expect. Now we estimate the error $\mathbb{E}\left[\phi_j(t_n)\right] - \Phi_h^n$ in some averaged norms. We omit the subscript j in $\mathbb{E}\left[\phi_j(t_n)\right] - \Phi_h^n$ and divide it into two terms:

$$\begin{aligned}\mathbb{E}\left[\phi(t_n)\right] - \Phi_h^n &= \left(\mathbb{E}\left[\phi(t_n)\right] - \mathbb{E}\left[\phi_h^n\right]\right) + \left(\mathbb{E}\left[\phi_h^n\right] - \Phi_h^n\right) \\ &= \mathcal{E}_h^n + \mathcal{E}_J^n,\end{aligned}$$

where the first term $\mathcal{E}_h^n = \mathbb{E}\left[\phi(t_n) - \phi_h^n\right]$ corresponds to the finite element discretization error, and the second, $\mathcal{E}_J^n = \mathbb{E}\left[\phi_h^n\right] - \Phi_h^n$, represents the statistical error. In the following, we derive the bounds of $\mathcal{E}_h^n$ and $\mathcal{E}_J^n$ to establish an error estimate for the FVEMC/CN algorithm

Convergence Analysis: $\mathcal{E}_J^n$

Theorem

Assume that (**H1**) , (**H2**), *and the stability condition* (12) *holds, and suppose that* $f_j \in \widehat{L}^2\left(0, T; H^{-1}(D)\right)$, $g_j \in \widehat{L}^2\left(0, T; L^2(\partial D_1)\right)$, $\phi_j^0 \in \widehat{L}^2\left(H^{l+1}(D)\right)$. *Then there exists a constant* $C > 0$ *satisfying*

$$\begin{aligned}
&\mathbb{E}\left[\|\mathcal{E}_J^N\|^2\right] + \frac{\lambda_{min} - 3\lambda_+}{8}\Delta t \sum_{n=1}^{N} \mathbb{E}\left[\|\nabla(\mathcal{E}_J^{n-1} + \mathcal{E}_J^n)\|^2\right] \\
&\quad + \frac{\mu_{min} - 3\mu_+}{8}\Delta t \sum_{n=1}^{N} \mathbb{E}\left[\|\mathcal{E}_J^{n-1} + \mathcal{E}_J^n\|_{\partial D_1}^2\right] \le \frac{C}{J}\left(\Delta t \sum_{n=3}^{N} \mathbb{E}\left[\|f_j^{n-\frac{1}{2}}\|_{-1}^2\right]\right. \\
&\quad + \Delta t \sum_{n=3}^{N} \mathbb{E}\left[\|g_j^{n-\frac{1}{2}}\|_{\partial D_1}^2\right] + \mathbb{E}\left[\|\phi_{j,h}^2\|^2\right] + \Delta t \mathbb{E}\left[\|\nabla\phi_{j,h}^2\|^2\right] + \Delta t \mathbb{E}\left[\|\nabla\phi_{j,h}^1\|^2\right] \\
&\quad \left. + \Delta t \mathbb{E}\left[\|\nabla\phi_{j,h}^0\|^2\right] + \Delta t \mathbb{E}\left[\|\phi_{j,h}^0\|_{\partial D_1}^2\right] + \Delta t \mathbb{E}\left[\|\phi_{j,h}^1\|_{\partial D_1}^2\right] + \Delta t \mathbb{E}\left[\|\phi_{j,h}^2\|_{\partial D_1}^2\right]\right). \qquad (14)
\end{aligned}$$

4.Convergence Analysis: $\mathcal{E}_h^n$

Theorem

Let ϕ_j^n and $\phi_{j,h}^n$ represent the solutions of systems (1) *and* (4) *at t_n, respectively. Suppose $f_j \in \widehat{L}^2\left(0,T;H^{-1}(D)\right)$, $g_j \in \widehat{L}^2\left(0,T;L^2(\partial D_1)\right)$, $\phi^0 \in \widehat{L}^2\left(H^{l+1}(D)\right)$ and hypothesis* (**H1**) *and* (**H2**) *are satisfied. Assume that the initial errors $\|\phi_j(t_0)-\phi_{j,h}^0\|_{1,D}$, $\|\phi_j(t_1)-\phi_{j,h}^1\|_{1,D}$, and $\|\phi_j(t_2)-\phi_{j,h}^2)\|_{1,D}$ are all at least $\mathcal{O}(h^l)$. Then, there exists a positive constant C such that*

$$\mathbb{E}\left[\|\mathcal{E}_h^N\|^2\right]+\frac{\lambda_{min}-3\lambda_+}{8}\Delta t\sum_{n=1}^{N}\mathbb{E}\left[\|\nabla(\mathcal{E}_h^{n-1}+\mathcal{E}_h^n)\|^2\right]$$

$$+\frac{\mu_{min}-3\mu_+}{8}\Delta t\sum_{n=1}^{N}\mathbb{E}\left[\|\mathcal{E}_h^{n-1}+\mathcal{E}_h^n\|_{\partial D_1}^2\right]\le C\left(\Delta t^4+h^{2l}\right), \quad (15)$$

if the stability conditions $\lambda_{min}-3\lambda_+>0$ and $\mu_{min}-3\mu_+>0$ hold.

Convergence Analysis: The total error

Combining the MC sampling error and FE error, one can acquire the total error of FVEMC/CN algorithm.

Theorem

Suppose $f \in \widehat{L}^2\left(0, T; H^{-1}(D)\right)$, *the boundary function* $g \in \widehat{L}^2\left(0, T; L^2(\partial D_1)\right)$, *and* $\phi^0 \in \widehat{L}^2\left(H^1(D) \cap H^{l+1}(D)\right)$. *Assume that the hypotheses* (**H1**), (**H2**) *and the stability condition* (12) *are valid. Then there is a constant* $C > 0$ *satisfying*

$$\begin{aligned}
&\mathbb{E}\left[\left\|\mathbb{E}\left[\phi(t_N)\right] - \Phi_h^N\right\|^2\right] \\
&+ \frac{\lambda_{min} - 3\lambda_+}{8}\Delta t \sum_{n=1}^{N} \|\nabla\left(\mathbb{E}\left[\phi(t_n)\right] - \Phi_h^n + \mathbb{E}\left[\phi(t_{n-1})\right] - \Phi_h^{n-1}\right)\|^2 \\
&+ \frac{\mu_{min} - 3\mu_+}{8}\Delta \sum_{n=1}^{N} \|\mathbb{E}\left[\phi(t_n)\right] - \Phi_h^n + \mathbb{E}\left[\phi(t_{n-1})\right] - \Phi_h^{n-1}\|_{\partial D_1} \\
&\leq C\left(\frac{1}{J} + \Delta t^4 + h^{2l}\right). \qquad (16)
\end{aligned}$$

Numerical examples

The FVE/CN scheme is used to simulate the group in this experiment with $J = 30$. Define

$$\mathbf{E}_{\mathcal{N}} := \max_{1\le n\le N} \sqrt{\frac{1}{J}\sum_{j=1}^{J} \|\phi_j(t_n) - \phi_{j,\mathcal{N}}^n\|^2}, \quad \mathbf{E}_{\mathcal{E}} := \max_{1\le n\le N} \sqrt{\frac{1}{J}\sum_{j=1}^{J} \|\phi_j(t_n) - \phi_{j,\mathcal{E}}^n\|^2},$$

$$\mathbf{E}_{\mathcal{N},\mathcal{E}} := \sqrt{\frac{1}{J}\sum_{j=1}^{J}\Delta t\sum_{n=1}^{N}\left(\|\nabla(\phi_{j,\mathcal{E}}^n - \phi_{j,\mathcal{N}}^n)\|^2 + \|\phi_{j,\mathcal{E}}^n - \phi_{j,\mathcal{N}}^n\|^2\right)},$$

where $\phi_{j,\mathcal{E}}^n$ and $\phi_{j,\mathcal{N}}^n$ denote the ensemble solution and the non-ensemble solution, respectively.

convergence order

We adopt linear finite elements ($l = 1$), an isometric time partition and uniform space partition, and we set the time step $\Delta t = h$ and choose the space step h from $\frac{1}{2^3}$ to $\frac{1}{2^7}$.

Table 1. Numerical errors and convergence rates ($\Delta t = h, J = 30$)

h	0.125	0.0625	0.03125	0.015625	0.0078125
$\mathbf{E}_{\mathcal{E}}$	1.1179E+00	3.2010E-01	8.3001E-02	2.0946E-02	5.2489E-03
Rate		1.804	1.947	1.986	1.997
$\mathbf{E}_{\mathcal{N}}$	1.0944E+00	3.2000E-01	8.3024E-02	2.0954E-02	5.2512E-03
Rate		1.774	1.946	1.986	1.997
$\mathbf{E}_{\mathcal{N},\mathcal{E}}$	1.4931E+00	2.5310E-01	5.2494E-02	1.2288E-02	2.9859E-03
Rate		2.561	2.269	2.095	2.041

Example 2.

Example 2. In this example, we set the diffusion coefficient and the Robin coefficient as follows:

$$\begin{aligned}\beta(\mathbf{x},\omega_1) =& 2+(1+\omega_1)\sin(x_1x_2)\sin(t),\\ \alpha(\mathbf{x},\omega_2,\cdots,\omega_5) =& 4+\exp\Big[(\omega_2\cos(\pi x_2)+\omega_3\sin(\pi x_2))\exp(-\frac{3}{4})\\ &+(\omega_4\cos(\pi x_1)+\omega_5\sin(\pi x_1))\exp(-\frac{3}{4})\Big]\,t,\end{aligned}$$

and the initial condition, Robin boundary condition and the source term are so chosen such that the exact solution is

$$\phi(t,\mathbf{x},\omega)=(1+\sum_{i=1}^{5}\omega_i)\cos(2\pi x_1)\cos(2\pi x_2)\exp(t+1).$$

Here, the random variables $\omega_1,\ldots,\omega_5$ are independent; ω_1 is uniformly distributed on the interval $[0,1]$, while $\omega_2,\ldots,\omega_5$ are uniformly distributed in the range $[-\frac{1}{2},\frac{1}{2}]$.

convergence order

Table 2. Numerical errors and convergence rates ($\Delta t = h, J = 30$)

h	0.125	0.0625	0.03125	0.015625	0.0078125
$\mathbf{E}_{\mathcal{E}}$	9.5953E-01	2.8938E-01	7.5508E-02	1.9075E-02	4.7814E-03
Rate		1.7294	1.9383	1.9849	1.9962
$\mathbf{E}_{\mathcal{N}}$	9.5960E-01	2.8939E-01	7.5510E-02	1.9076E-02	4.7815E-03
Rate		1.7294	1.9383	1.9849	1.9962
$\mathbf{E}_{\mathcal{N},\mathcal{E}}$	7.6223E-03	1.4761E-03	1.8942E-04	3.7595E-05	8.9745E-06
Rate		2.3684	2.9621	2.3330	2.0666

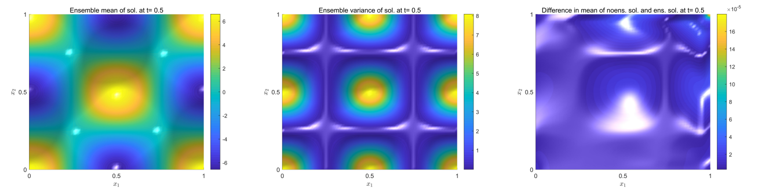

Figure 1. FVEMC/CN algorithm simulations. Left: mean. Middle: variance of the solution at t = 0.5. Right: difference between the mean of FVEMC/CN and FNEMC/CN algorithm simulation.

Cost

Since the discrete system's scale is not excessively large, we use the command **decomposition** in MATLAB for its sparse Cholesky decomposition.

Table 3. All CPU time (s) comparisons ($\Delta t = h = \frac{1}{2^6}$)

J	10	20	40	80	160	320	640
FNEMC/CN	45.73	97.13	194.87	473.74	718.50	1450.00	3076.63
FVEMC/CN	27.49	58.89	116.58	281.56	431.72	871.36	1838.44

Table 4. Comparisons of the CPU time (s) required for solving only the algebraic systems. ($\Delta t = h = \frac{1}{2^6}$)

J	10	20	40	80	160	320	640
FNEMC/CN	20.54	40.61	82.38	208.34	299.88	603.61	1295.34
FVEMC/CN	1.21	2.37	4.79	11.90	17.67	35.49	75.36

Convergence rate of samples J

We use the FVEMC/CN algorithm to calculate the mean of solution using $J_0 = 10000$ samples as our benchmark, with $h = \Delta t = \frac{1}{2^3}$. Let $\Phi_{J,h}^{n,m}$ represent the FVEMC/CN solution at time t_n in the m-th independent replica which is defined by $\Phi_{J,h}^{n,m} \equiv \frac{1}{J}\sum_{j=1}^{J} \phi_{j,h}^{n,m}$ where $\phi_{j,h}^{n,m}$ is the output of the FVE/CN scheme in the m-th experiment. We further define

$$\mathcal{E}_{L^2}^{*} := \max_{1\le n\le N} \sqrt{\frac{1}{M}\sum_{m=1}^{M} \|\Phi_{J_0,h}^{n} - \Phi_{J,h}^{n,m}\|^2}.$$

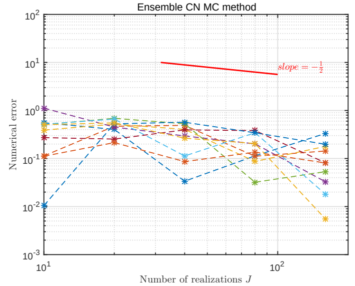

Figure 2. Convergence rate of MC sample .

References

[1] Jiang, N., Layton, W.. *An algorithm for fast calculation of flow ensembles.* Int. J. Uncertain. Quan. **4**, 273-301(2014).

[2] Luo, Y., Wang, Z. *An ensemble algorithm for numerical solutions to deterministic and random parabolic PDEs.* SIAM J. Numer. Anal. **56**, 859-876(2018).

[3] Luo, Y., Wang, Z. *A multilevel Monte Carlo ensemble scheme for solving random parabolic PDEs.* SIAM J. Sci. Comput. **41**, A622-A642(2019).

[4] J. A. Fiordilino. A second order ensemble timestepping algorithm for natural convection. SIAM J. Numer. Anal., 56 (2018): 816-837.

[5] Li, M., Luo, X. *An ensemble Monte Carlo HDG method for parabolic PDEs with random coefficients.* Int. J. Comput. Math., 100, 405-421 (2022).

[6] Li, M., Luo, X. *A multilevel Monte Carlo ensemble and hybridizable discontinuous Galerkin method for a stochastic parabolic problem.* Numer. Methods Partial Differ. Eq. **39**, 2840-2864(2023).

[7] Yao, T., Ye, C., Luo, X. et al. *An ensemble scheme for the numerical solution of a random transient heat equation with uncertain inputs.* Numer.

References

- [8] I. Babuska, F. Nobile, and R. Tempone. *A stochastic collocation method for elliptic partial differential equations with random input data.* SIAM Review, 52(2)(2010): 317-355.
- [9] M. B. Giles. *Multilevel Monte Carlo methods.* Acta Numerica, 24 (2015): 259-328.
- [10] B. Jin and J. Zou. *Numerical identification of a Robin coefficient in parabolic problems.* Math. Comp., 81(279)(2012): 1369-1398.
- [11] M. Gunzburger, C. G. Webster, and G. Zhang. *Stochastic finite element methods for partial differential equations with random input data.* Acta Numer., 23(2014):521-650.
- [12] T. Zhou, T. Tang.*Galerkin Methods for Stochastic Hyperbolic Problems Using Bi-Orthogonal Polynomials.* J. Sci.Comput.,51(2012): 274-292.
- [13] G. Fishman. *Monte Carlo: Concepts, Algorithms, and Applications.* Springer, New York. (1996).

Thanks for your attention.