Adaptive FEM for Elliptic Eigenvalue Optimization in Phase Field

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NMSP2024@Guizhou August 7, 2024

Outline

1. Introduction
2. Adaptive Algorithm
3. Numerical Example
4. Theoretical Result
5. Concluding Remark

Current Section

1. Introduction
2. Adaptive Algorithm
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Mathematical Formulation

$$\inf_{\Omega\in\mathcal{A}} \{\mathcal{J}(\Omega) := \Psi(\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_l})\} \tag{1.1}$$

subject to eigenvalues λ_{i_j} $(1 \le j \le l)$ by

$$\int_D \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \,\mathrm{d}x + \alpha \int_D \chi_\Omega uv \,\mathrm{d}x = \lambda \int_D uv \,\mathrm{d}x \quad \forall v \in H_0^1(D), \tag{1.2}$$

for $(u, \lambda) \in H_0^1(D) \times \mathbb{R}$ with $\|u\|_{L^2(D)} = 1$.

$\Psi : (\mathbb{R}_{>0})^l \to \mathbb{R}$: C^1 function, $\mathcal{A} := \{$open set $\Omega \subset D \mid |\Omega| = V\}$,
$D \subset \mathbb{R}^2$: open bounded polygonal domain,
$V \in (0, |D|)$: given volume, $l \in \mathbb{N}$,
χ_Ω: characteristic function of Ω, $\alpha > 0$: given constant.

Typical example:

$$\inf_{\Omega\in\mathcal{A}} \lambda_1, \quad \inf_{\Omega\in\mathcal{A}} -\lambda_1 \ (\sup_{\Omega\in\mathcal{A}} \lambda_1), \quad \inf_{\Omega\in\mathcal{A}} (\lambda_1 - \lambda_2) \ (\sup_{\Omega\in\mathcal{A}} (\lambda_2 - \lambda_1)).$$

Application: design of an inhomogeneous/composite drum

Mathematical Formulation

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Application: design of an inhomogeneous/composite drum

Common Approach

- Boundary Variation Technique J. Sokolowski & J–P. Zolesio (Springer, 1992), A. Henrot (Birkhäuser Verlag, 2006);
- Variational Method D. Bucur & G. Buttazzo (Birkhäuser, 2005), A. Henrot (Birkhäuser Verlag, 2006);
- Homogenization Method G. Allaire (Springer, 2002);
- Level Set Method S.J. Osher & F. Santosa (JCP, 2001), G. Allaire & F. Jouve (CMAME, 2005);
- Phase Field Method B. Bourdin & A. Chambolle (ESAIM:COCV, 2003), Garcke et al. (ESAIM:COCV, 2023).

Mathematical Formulation in Phase–Field Setting

$$\inf_{\phi\in\mathcal{U}} \left\{\mathcal{J}^\varepsilon(\phi) := \Psi(\lambda^\varepsilon_{i_1}, \lambda^\varepsilon_{i_2}, \cdots, \lambda^\varepsilon_{i_l}) + \gamma\mathcal{F}_\varepsilon(\phi)\right\} \tag{1.3}$$

subject to $\lambda^\varepsilon_{i_1}, \lambda^\varepsilon_{i_2}, \cdots, \lambda^\varepsilon_{i_l}$ different eigenvalues given by

$$\int_D \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} v \,\mathrm{d}x + \alpha \int_D \phi w v \,\mathrm{d}x = \lambda^\varepsilon(\phi) \int_D w v \,\mathrm{d}x \tag{1.4}$$

$$\forall v \in H^1_0(D) \text{ with } \|w\|_{L^2(D)} = 1.$$

ε and γ: positive and fixed constants,
$\mathcal{U} := \{\phi \in H^1(D) \mid \phi \in [0,1] \text{ a.e. in } D, \int_D \phi \,\mathrm{d}x = V\}$,
$\mathcal{F}_\varepsilon$ is a Ginzburg–Landau energy functional over $\mathcal{U}$

$$\mathcal{F}_\varepsilon(\phi) := \frac{\varepsilon}{2}\|\boldsymbol{\nabla}\phi\|^2_{L^2(D)} + \frac{1}{\varepsilon}\int_D f(\phi)\,\mathrm{d}x \quad \varepsilon > 0$$

$f(\phi) = \frac{1}{2}\phi(1-\phi)$ or $\frac{1}{4}\phi^2(1-\phi)^2$.

Existence of one minimizer: Garcke et al. (ESAIM:COCV, 2023).

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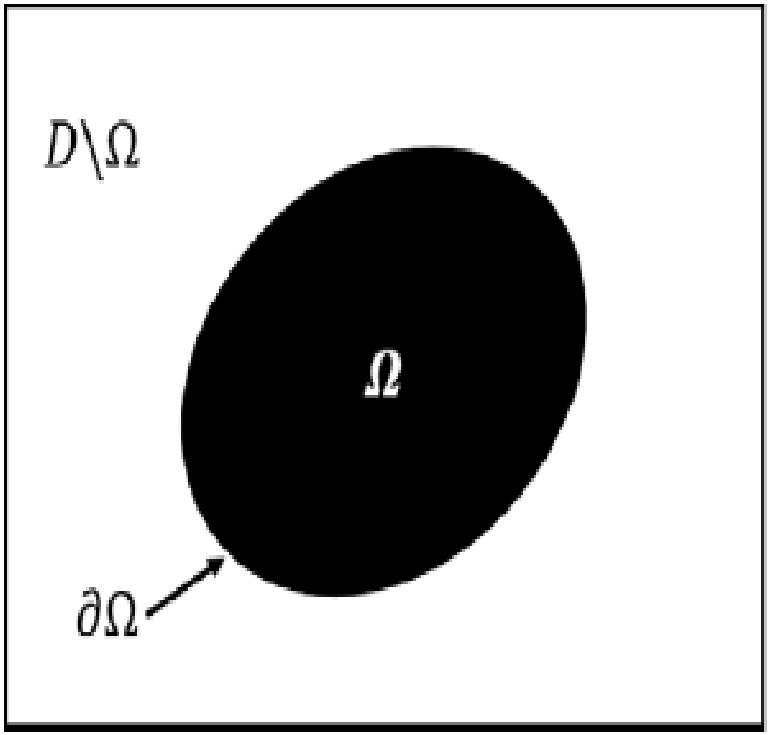
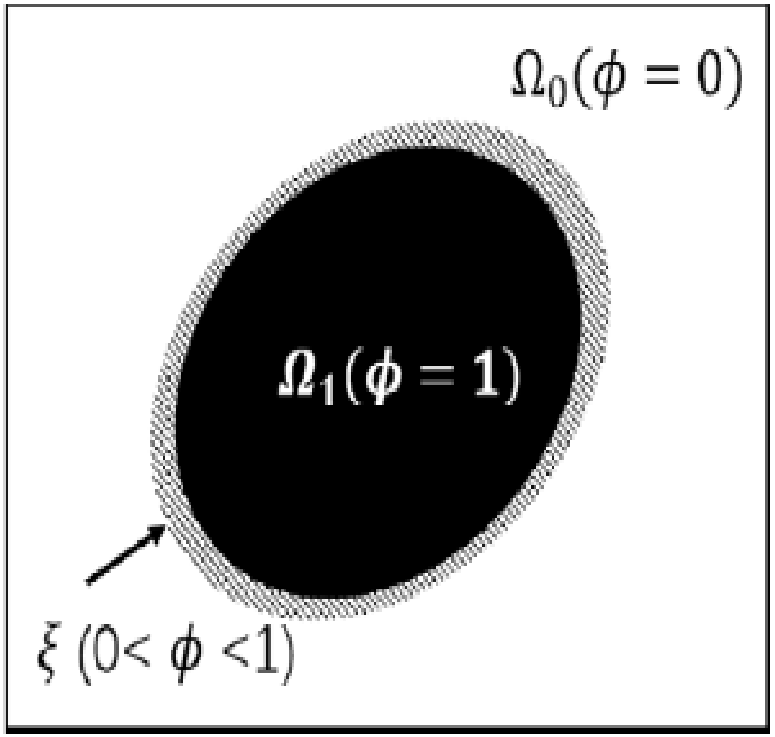


(a) original domain



(b) domain represented by phase field

- The thickness of ξ is of order ε;
- The basis of this approximation: Γ–convergence.

Finite Element Method

$\mathcal{T}$: a shape-regular conforming triangulation of $\bar{D}$ into a set of closed triangles,
$h_T := |T|^{1/2}$ for each element $T \in \mathcal{T}$,
$V_{\mathcal{T}}$ is a H^1-conforming linear element space over $\mathcal{T}$,
$\mathcal{U}_{\mathcal{T}} := V_{\mathcal{T}} \cap \mathcal{U}$ is the discrete admissible set,
$V_{\mathcal{T}}^0 : V_{\mathcal{T}} \cap H_0^1(D)$.
The discrete minimization problem:

$$\min_{\phi_{\mathcal{T}} \in \mathcal{U}_{\mathcal{T}}} \left\{ \mathcal{J}_{\mathcal{T}}^{\varepsilon}(\phi_{\mathcal{T}}) := \Psi(\lambda_{\mathcal{T},i_1}^{\varepsilon}, \lambda_{\mathcal{T},i_2}^{\varepsilon}, \cdots, \lambda_{\mathcal{T},i_l}^{\varepsilon}) + \gamma \mathcal{F}_{\varepsilon}(\phi_{\mathcal{T}}) \right\} \tag{1.5}$$

subject to $\lambda_{\mathcal{T},i_1}^{\varepsilon}, \lambda_{\mathcal{T},i_2}^{\varepsilon}, \cdots, \lambda_{\mathcal{T},i_l}^{\varepsilon}$ different eigenvalues given by

$$\int_D \boldsymbol{\nabla} w_{\mathcal{T}} \cdot \boldsymbol{\nabla} v_{\mathcal{T}} \, \mathrm{d}x + \alpha \int_D \phi_{\mathcal{T}} w_{\mathcal{T}} v_{\mathcal{T}} \, \mathrm{d}x = \lambda_{\mathcal{T}}^{\varepsilon}(\phi_{\mathcal{T}}) \int_D w_{\mathcal{T}} v_{\mathcal{T}} \, \mathrm{d}x, \tag{1.6}$$

$\forall v_{\mathcal{T}} \in V_{\mathcal{T}}^0$ with $\|w_{\mathcal{T}}\|_{L^2(D)} = 1$.

Finite Element Method

Optimality system: the minimizer $\phi_{\mathcal{T}}^*$ and the associated eigenpairs $\left(\lambda_{k,i_j}^{\varepsilon,*}, w_{k,i_j}^*\right)$ $(1 \le j \le l)$ satisfy, on the assumption that each $\lambda_{k,i_j}^{\varepsilon,*}$ is simple, for any $(\phi_{\mathcal{T}}, v_{\mathcal{T}}) \in \mathcal{U}_{\mathcal{T}} \times V_{\mathcal{T}}^0$

$$\begin{cases} \gamma \mathcal{F}_\varepsilon'(\phi_{\mathcal{T}}^*)(\phi_{\mathcal{T}} - \phi_{\mathcal{T}}^*) \\ +\alpha \displaystyle\sum_{j=1}^{l} \frac{\partial \Psi(\lambda_{\mathcal{T},i_1}^{\varepsilon,*}, \lambda_{k,i_2}^{\varepsilon,*}, \cdots, \lambda_{k,i_l}^{\varepsilon,*})}{\partial \lambda_{\mathcal{T},i_j}^{\varepsilon}} \int_D (w_{\mathcal{T},i_j}^*)^2 (\phi_{\mathcal{T}} - \phi_{\mathcal{T}}^*)\,\mathrm{d}x \ge 0, \\ \displaystyle\int_D \boldsymbol{\nabla} w_{\mathcal{T},i_j}^* \cdot \boldsymbol{\nabla} v_{\mathcal{T}}\,\mathrm{d}x + \alpha \int_D \phi_{\mathcal{T}}^* w_{\mathcal{T},i_j}^* v_{\mathcal{T}}\,\mathrm{d}x = \lambda_{\mathcal{T},i_j}^{\varepsilon,*} \int_D w_{\mathcal{T},i_j}^* v_{\mathcal{T}}\,\mathrm{d}x. \end{cases}$$

$$\mathcal{F}_\varepsilon'(\phi_{\mathcal{T}}^*)\phi_{\mathcal{T}} = \varepsilon \int_D \boldsymbol{\nabla} \phi_{\mathcal{T}}^* \cdot \boldsymbol{\nabla} \phi_{\mathcal{T}}\,\mathrm{d}x + \frac{1}{\varepsilon} \int_D f'(\phi_{\mathcal{T}}^*)\phi_{\mathcal{T}}\,\mathrm{d}x.$$

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Adaptive Cycle

Adaptive finite element methods (AFEM) consist of successive loops of the form:

$$\text{SOLVE/OPTIMIZE} \to \text{ESTIMATE} \to \text{MARK} \to \text{REFINE}.$$

SOLVE/OPTIMIZE: numerical solution of the discrete (minimization) problem given by FEM;

ESTIMATE: a posteriori error estimation;

MARK: marking strategies based on the error estimator;

REFINE: refinement process of the current mesh.

Motivation of AFEM: the minimizing phase–field function is mainly equal to 0 or 1 in D and transits from 0 to 1 in a thin layer of order ε.

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A Posteriori Error Estimate

$\phi_{\mathcal{T}}^*$: minimizer of (1.5)–(1.6),
$\{(\lambda_{\mathcal{T},i_j}^{\varepsilon,*}, w_{\mathcal{T},i_j}^*)\}_{j=1}^l$: l associated eigenpairs.
Some computable quantities

$$R_0(\phi_{\mathcal{T}}^*)|_T := \alpha \sum_{j=1}^{l} \frac{\partial \Psi(\lambda_{\mathcal{T},i_1}^{\varepsilon,*}, \cdots, \lambda_{\mathcal{T},i_l}^{\varepsilon,*})}{\partial \lambda_{\mathcal{T},i_j}^{\varepsilon}} (w_{\mathcal{T},i_j}^*)^2 - \gamma\varepsilon\Delta\phi_{\mathcal{T}}^* + \frac{\gamma}{\varepsilon} f'(\phi_{\mathcal{T}}^*),$$

$$R_j(\phi_{\mathcal{T}}^*, \lambda_{\mathcal{T},i_j}^{\varepsilon,*}, w_{\mathcal{T},i_j}^*)|_T := -\Delta w_{\mathcal{T},i_j}^* + \alpha\phi_{\mathcal{T}}^* w_{\mathcal{T},i_j}^* - \lambda_{\mathcal{T},i_j}^{\varepsilon,*} w_{\mathcal{T},i_j}^* \quad 1 \le j \le l,$$

$$J_0(\phi_{\mathcal{T}}^*)|_F := \begin{cases} \gamma\varepsilon[\boldsymbol{\nabla}\phi_{\mathcal{T}}^* \cdot \boldsymbol{n}_F] & F \in \mathcal{F}_{\mathcal{T}}(D), \\ \gamma\varepsilon\boldsymbol{\nabla}\phi_{\mathcal{T}}^* \cdot \boldsymbol{n} & F \in \mathcal{F}_{\mathcal{T}} \setminus \mathcal{F}_{\mathcal{T}}(D), \end{cases}$$

$$J_j(w_{\mathcal{T},i_j}^*)|_F := [\boldsymbol{\nabla} w_{\mathcal{T},i_j}^* \cdot \boldsymbol{n}_F] \quad F \in \mathcal{F}_{\mathcal{T}}(D) \quad 1 \le j \le l,$$

where $[\cdot]$ denotes jumps across interior faces F.

A Posteriori Error Estimate

Local error indicators

$$\eta^2_{\mathcal{T},0}(\phi^*_{\mathcal{T}};T) := h_T^2\|R_0(\phi^*_{\mathcal{T}})\|^2_{L^2(T)} + h_T \sum_{F\subset\partial T} \|J_0(\phi^*_{\mathcal{T}})\|^2_{L^2(F)},$$

$$\eta^2_{\mathcal{T},j}(\phi^*_{\mathcal{T}}, \lambda^{\varepsilon,*}_{\mathcal{T},i_j}, w^*_{\mathcal{T},i_j};T) := h_T^2\|R_j(\phi^*_{\mathcal{T}}, \lambda^{\varepsilon,*}_{\mathcal{T},i_j}, w^*_{\mathcal{T},i_j})\|^2_{L^2(T)} + h_T \sum_{F\subset\partial T} \|J_j(w^*_{\mathcal{T},i_j})\|^2_{L^2(F)} \quad 1\le j\le l.$$

Global estimators

$$\eta^2_{\mathcal{T},0}(\phi^*_{\mathcal{T}};\mathcal{M}_{\mathcal{T}}) := \sum_{T\in\mathcal{M}_{\mathcal{T}}} \eta^2_{\mathcal{T},0}(\phi^*_{\mathcal{T}};T),$$

$$\eta^2_{\mathcal{T},j}(\phi^*_{\mathcal{T}}, \lambda^{\varepsilon,*}_{\mathcal{T},i_j}, w^*_{\mathcal{T},i_j};\mathcal{M}_{\mathcal{T}}) := \sum_{T\in\mathcal{M}_{\mathcal{T}}} \eta^2_{\mathcal{T},j}(\phi^*_{\mathcal{T}}, \lambda^{\varepsilon,*}_{\mathcal{T},i_j}, w^*_{\mathcal{T},i_j};T) \quad 1\le j\le l.$$

Algorithm 2.1 AFEM for Elliptice Eigenvalue Optimization

1: Given $\theta_j \in (0,1]$ $(0 \le j \le l)$, fix $\varepsilon > 0$ and $\gamma > 0$, specify an initial mesh $\mathcal{T}_0$ and set K of maximum refinements.
2: **for** $k = 0 : K-1$ **do**
3: (OPTIMIZE) Solve (1.5)–(1.6) over $\mathcal{T}_k$ for the minimizer $\phi_k^* \in \mathcal{U}_k$ and l associated eigenpairs $(\lambda_{k,i_j}^{\varepsilon,*}, w_{k,i_j}^*) \in \mathbb{R} \times V_k^0$.
4: (ESTIMATE) Compute $l+1$ error estimators $\eta_{k,0}(\phi_k^*)$ and $\eta_{k,j}(\phi_k^*, \lambda_{k,i_j}^{\varepsilon,*}, w_{k,i_j}^*)$ $(1 \le j \le l)$.
5: (MARK) Mark $l+1$ subsets $\mathcal{M}_k^j \subseteq \mathcal{T}_k$ $(0 \le j \le l)$ such that $\eta_{k,j}(\mathcal{M}_k^j) \ge \theta_j \eta_{k,j}$.Then $\mathcal{M}_k := \cup_{j=0}^l \mathcal{M}_k^j$.
6: (REFINE) Refine each element T in $\mathcal{M}_k$ by bisection to get $\mathcal{T}_{k+1}$.
7: Check the stopping criterion.
8: **end for**
9: Output ϕ_k^* and $(\lambda_{k,i_j}^{\varepsilon,*}, w_{k,i_j}^*)$ $(1 \le j \le l)$.

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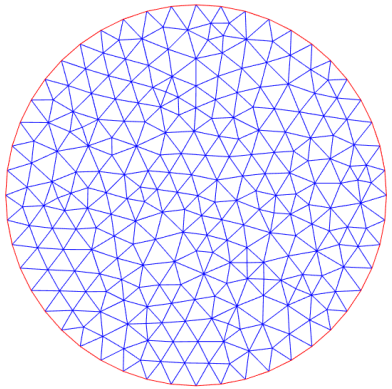
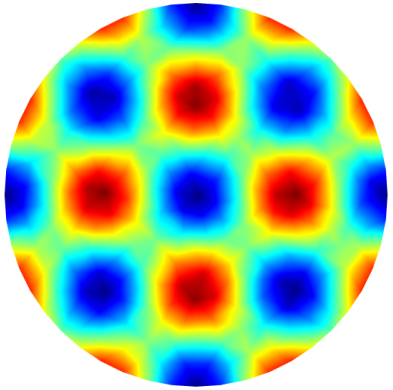
Computing environment: Matlab 2020a in a laptop with i5-7500 CPU & 8GB memory.
$f(\phi) = \frac{1}{4}\phi^2(\phi-1)^2$ in all experiments.

Example 3.1 (Circle)

Minimize λ_1 in $D = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$ with $\alpha = 1$ and volume fraction $|V|/|\Omega| = 0.5$.
Optimal shape: $\{(x_1, x_2) \mid R^2 < x_1^2 + x_2 < 1\}$ with $R = 1/\sqrt{2}$.
$K = 6$, $\gamma = 10^{-3}$, $\varepsilon = 10^{-2}$, $\theta_0 = 0.7$, $\theta_1 = 0.2$, $\phi_0 = \frac{1}{2}\cos(1 - \cos(2\pi x_1)\cos(2\pi x_2))$.

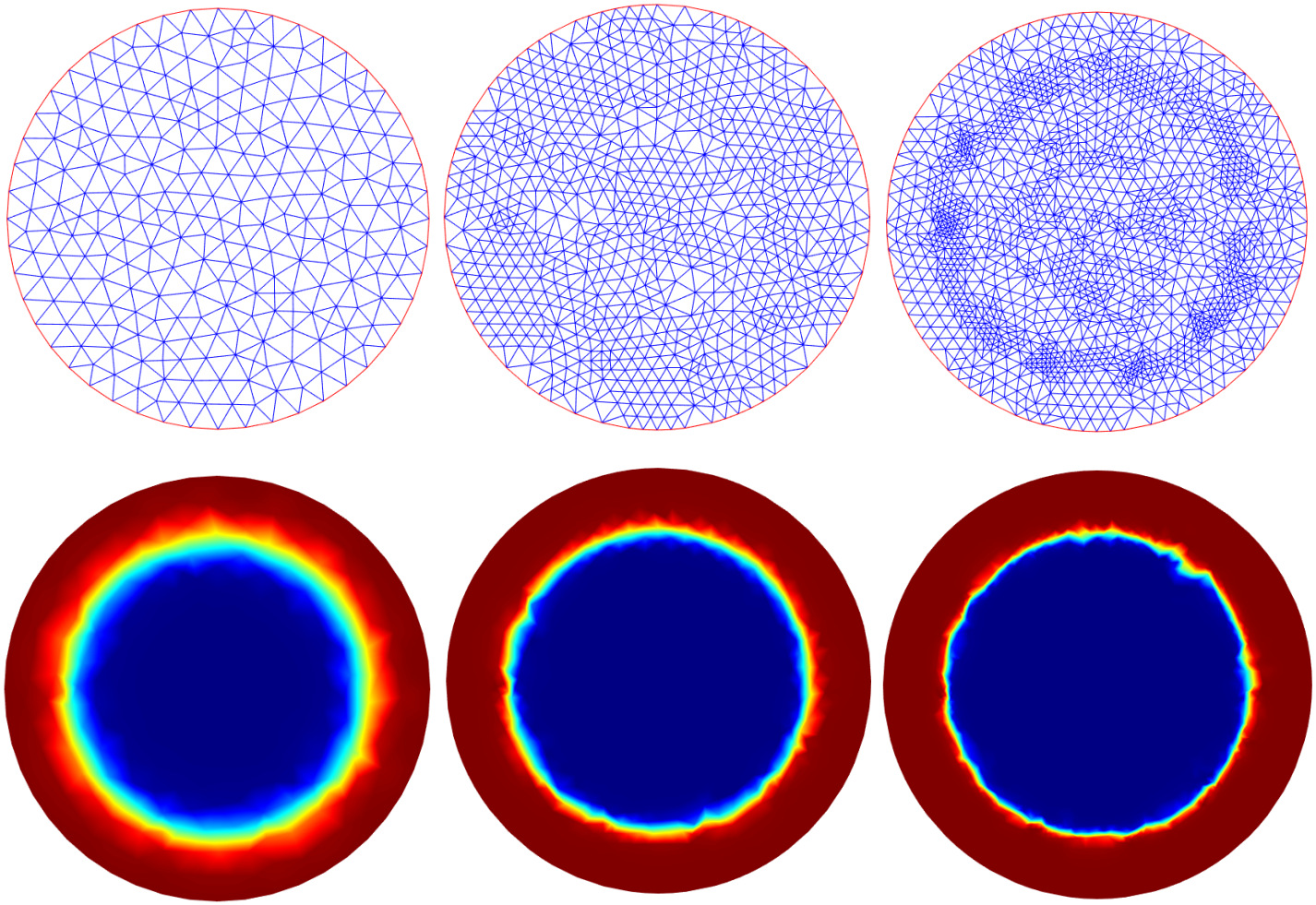


$\mathcal{T}_0$ with 269 vertices



Initial guess of ϕ_0^* over $\mathcal{T}_0$

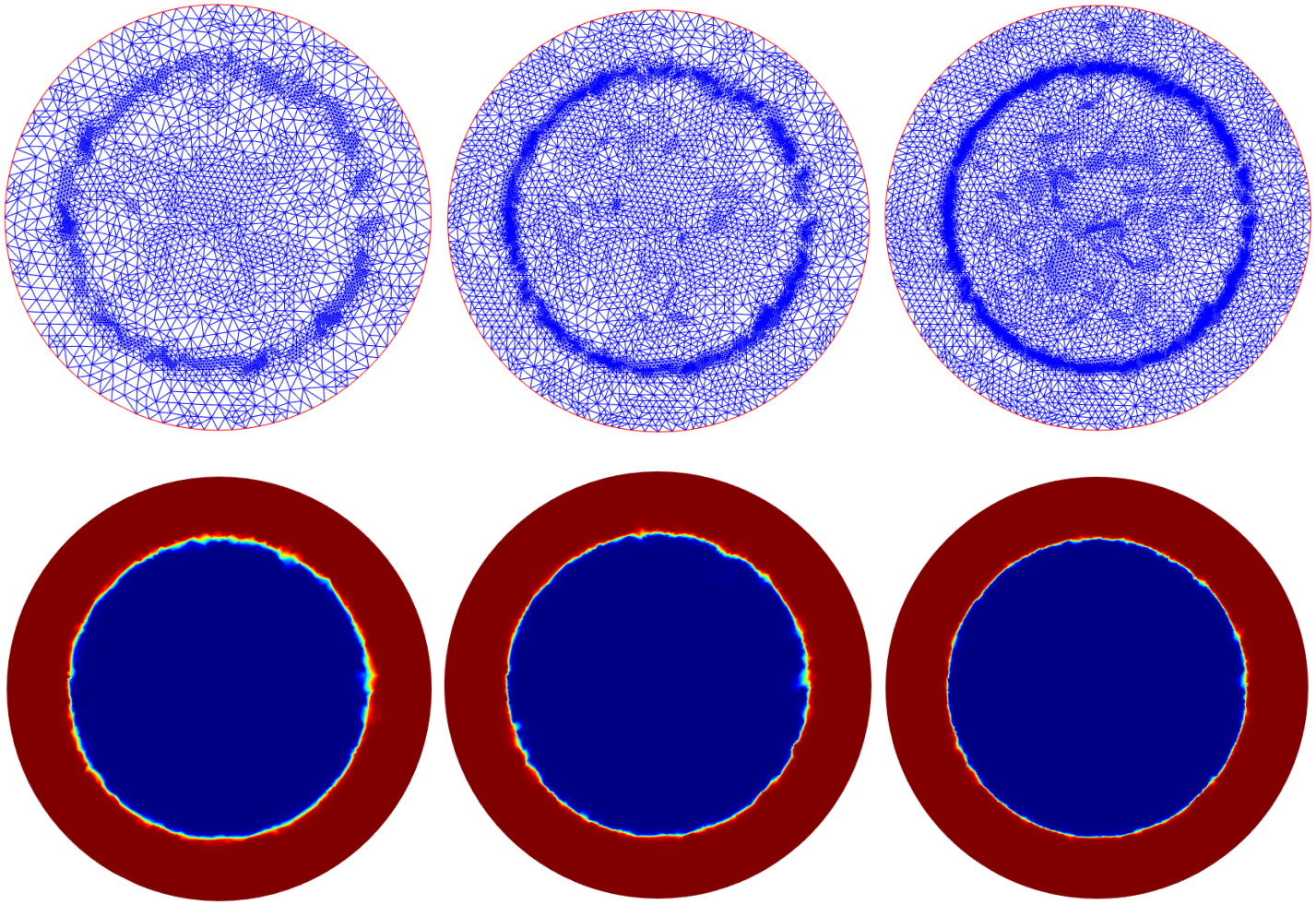
Minimization of λ_1 in Example 3.1



$k = 0(269)$ $k = 1(817)$ $k = 2(1655)$

BLUE–0, RED–1

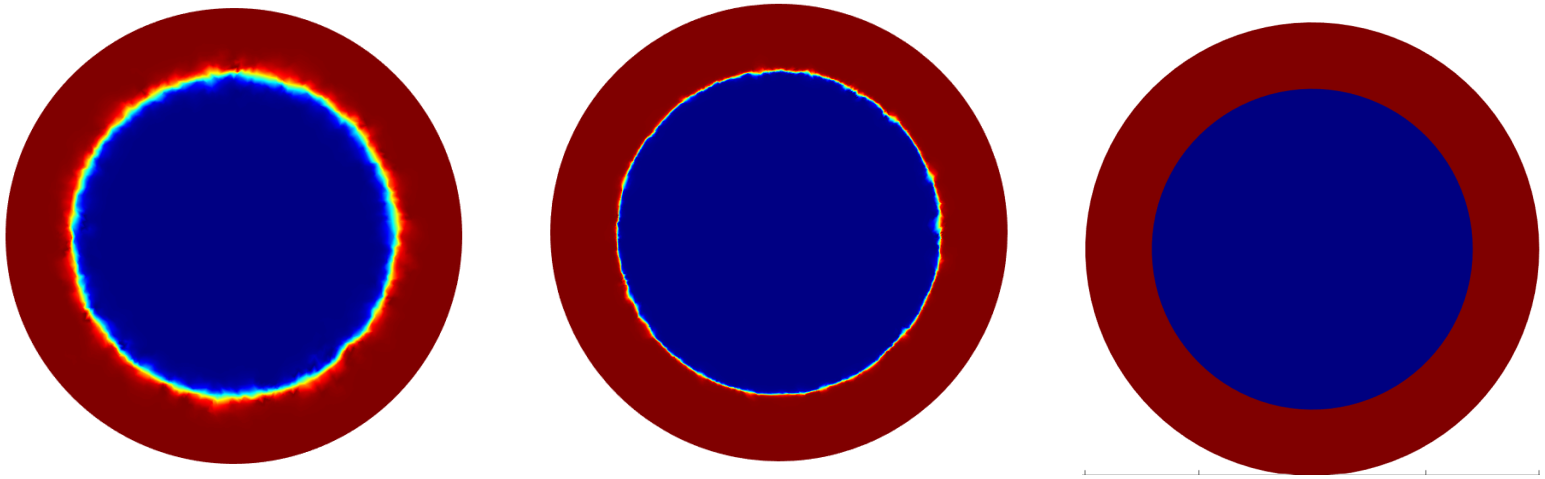
Minimization of λ_1 in Example 3.1



$k = 3(3095)$ $k = 4(5659)$ $k = 5(10102)$

BLUE–0, RED–1

Example 3.1: Comparison with Uniform Refinement

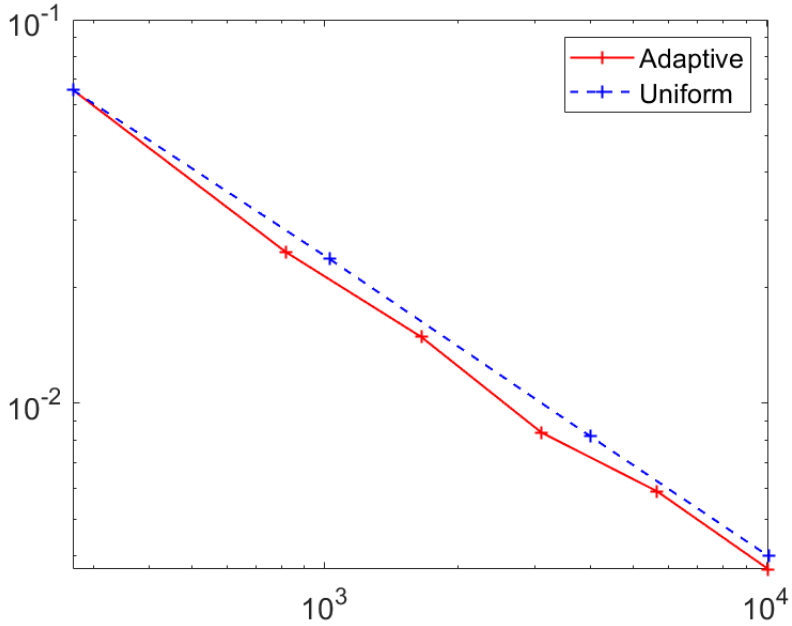


Uniform Adaptive Optimal

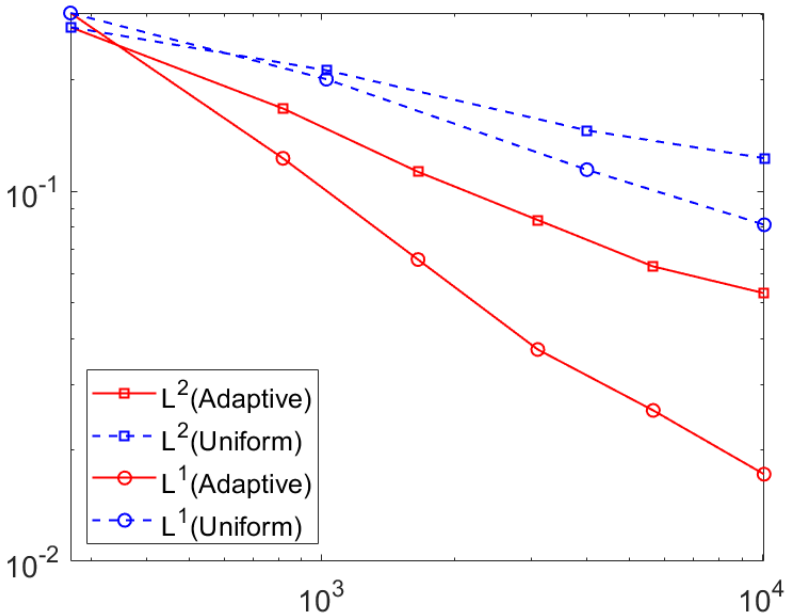
	vertices	objective	time (sec.)
Uniform	10111	5.8732	2796.11
Adaptive	10102	5.8729	2041.64

Time decrease: 26.98%

Convergence History



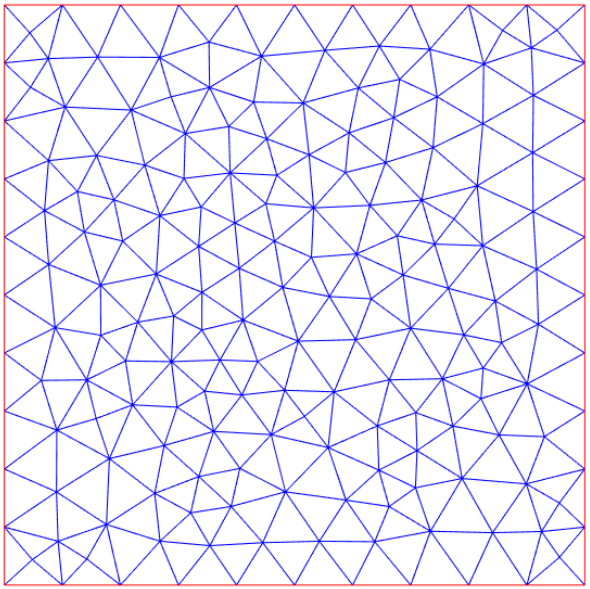
Errors in minimizing λ_1



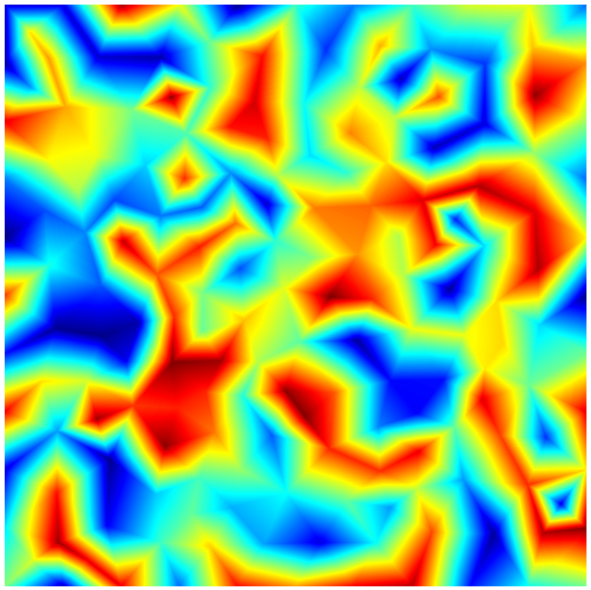
$L^1(D)$ and $L^2(D)$ errors

Example 3.2 (Square)

Maximize λ_1 in $(0,1)^2$ with $\alpha = 10.21$ and volume fraction $|\Omega|/|D| = 0.5$.
$K = 6$, $\gamma = 5 \times 10^{-3}$, $\varepsilon = 5 \times 10^{-2}$, $\theta_0 = 0.8$, $\theta_1 = 0.3$.

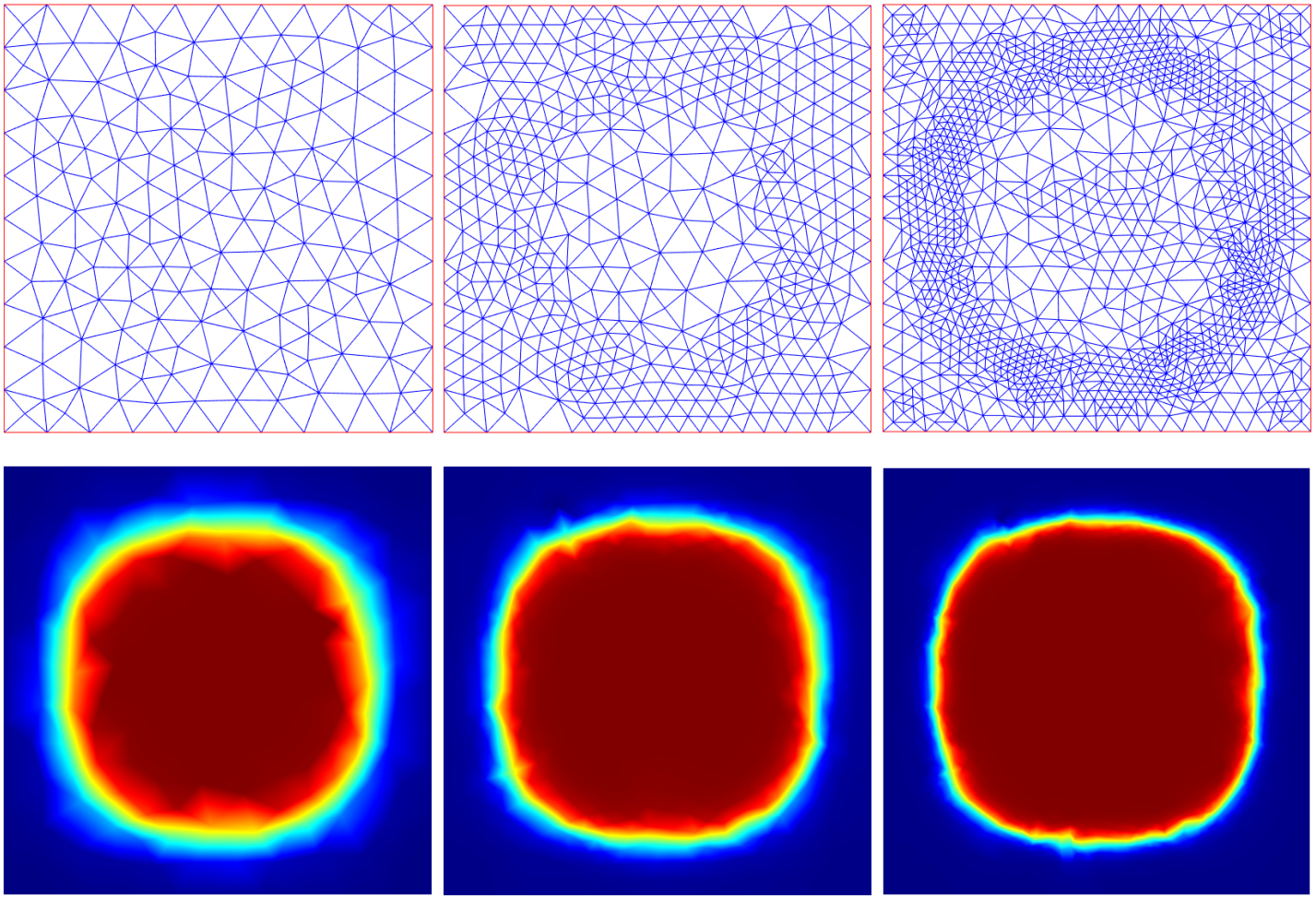


$\mathcal{T}_0$ with 185 vertices



Initial guess of ϕ_0^* over $\mathcal{T}_0$

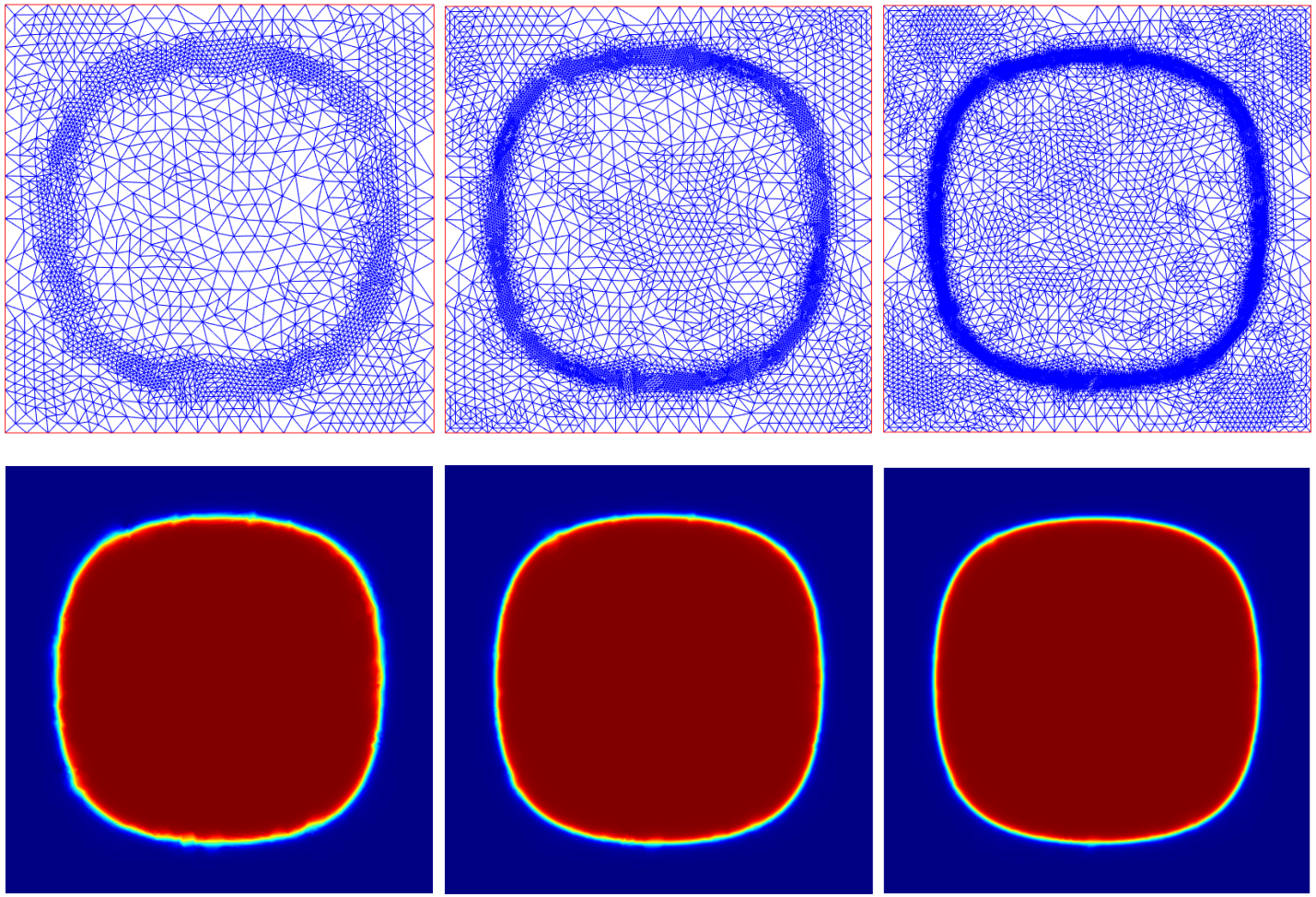
Maximization of λ_1 in Example 3.2



$k = 0(185)$ $k = 1(494)$ $k = 2(1220)$

BLUE–0, RED–1

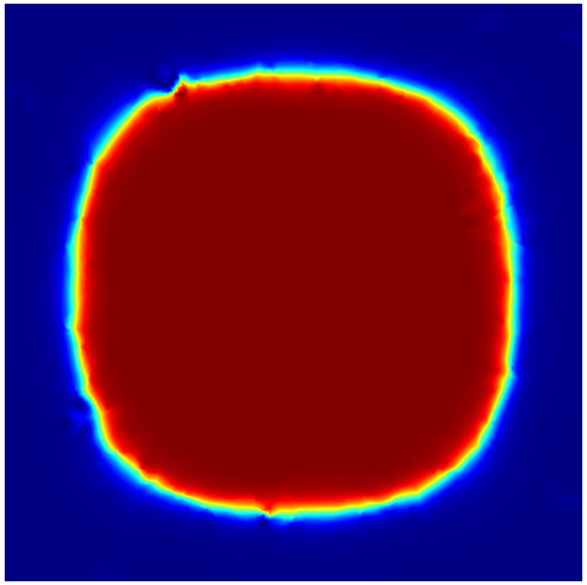
Maximization of λ_1 in Example 3.2



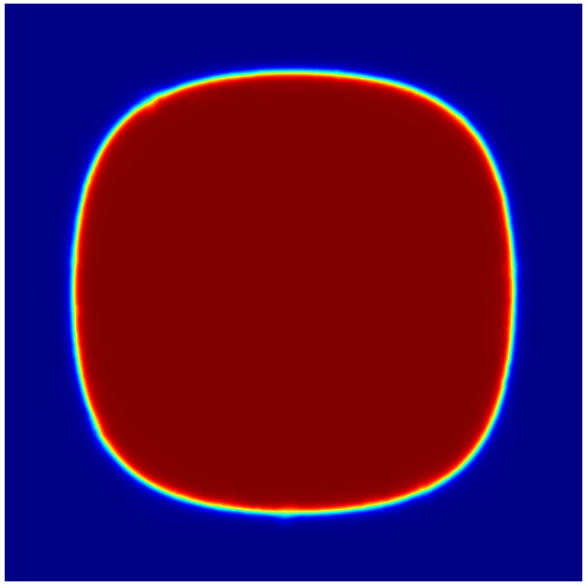
$k = 3(2702)$ $k = 4(5608)$ $k = 5(13331)$

BLUE–0, RED–1

Example 3.2: Comparison with Uniform Refinement



Uniform



Adaptive

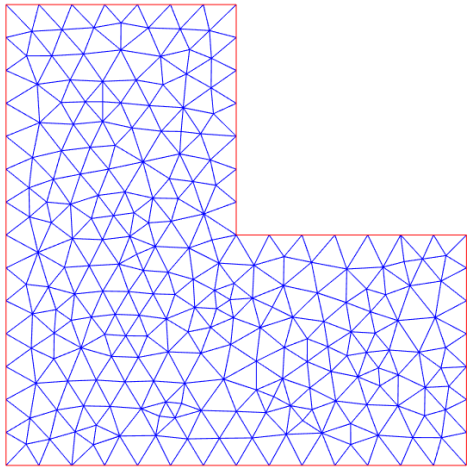
	vertices	objective	time (sec.)
Uniform	13347	29.1420	2230.13
Adaptive	13331	29.1916	1944.73

Time decrease: 12.80%

Example 3.3: L-Shaped Domain

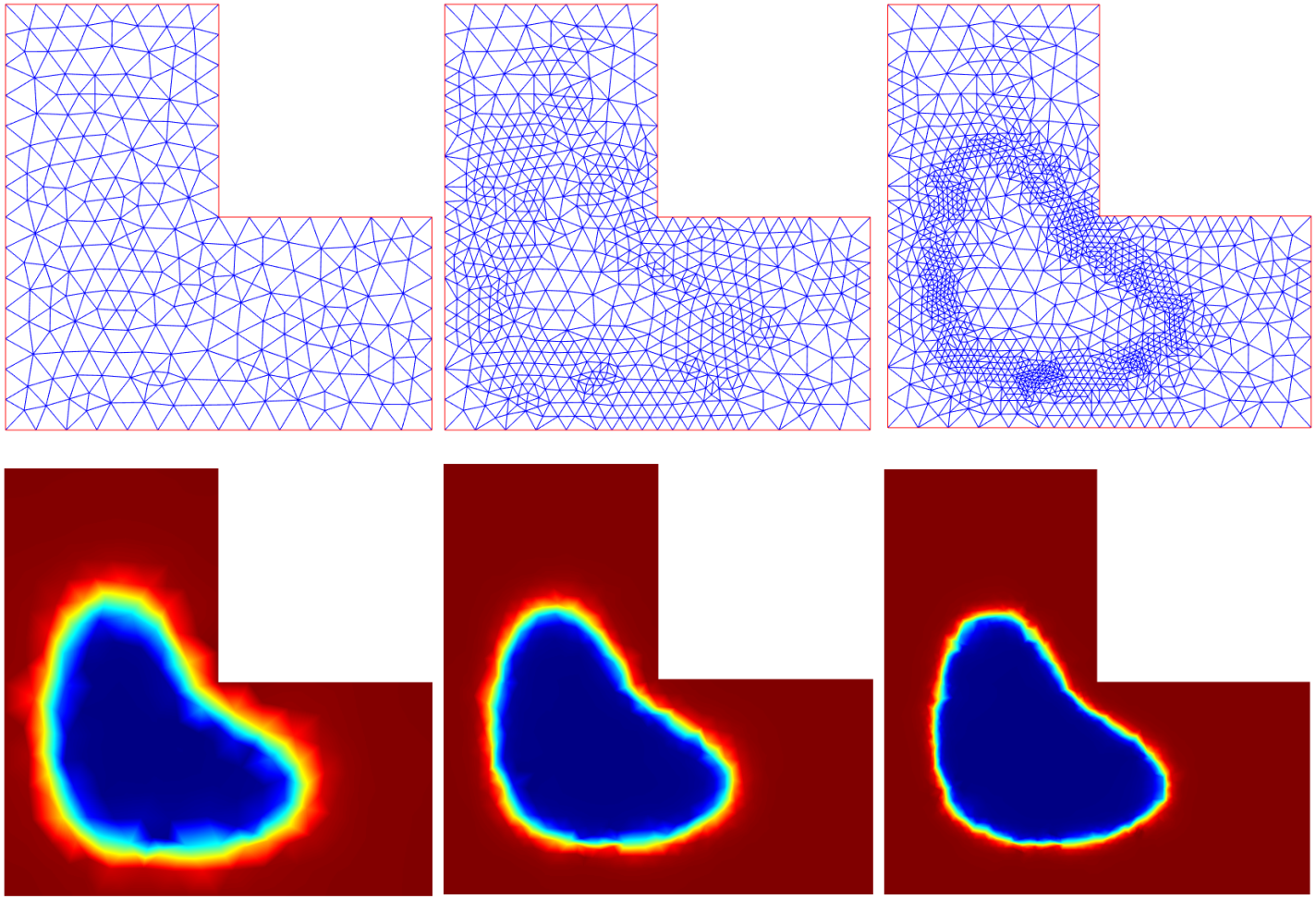
Example 3.3

Minimize and maximize λ_1 in a L-shaped domain $(0,2)^2 \setminus [1,2]^2$ respectively with $\alpha = 1$.
$K = 6$, $\gamma = 8 \times 10^{-3}$, $\varepsilon = 10^{-3}$, $\theta_0 = 0.8$, $\theta_1 = 0.3$.
Volume fraction $|\Omega|/|D| = 0.75$ (min) & 0.2 (max);
Initial guess of ϕ_0^* over $\mathcal{T}_0$: 0.75 (min) & 0.2 (max).



$\mathcal{T}_0$ with 259 vertices

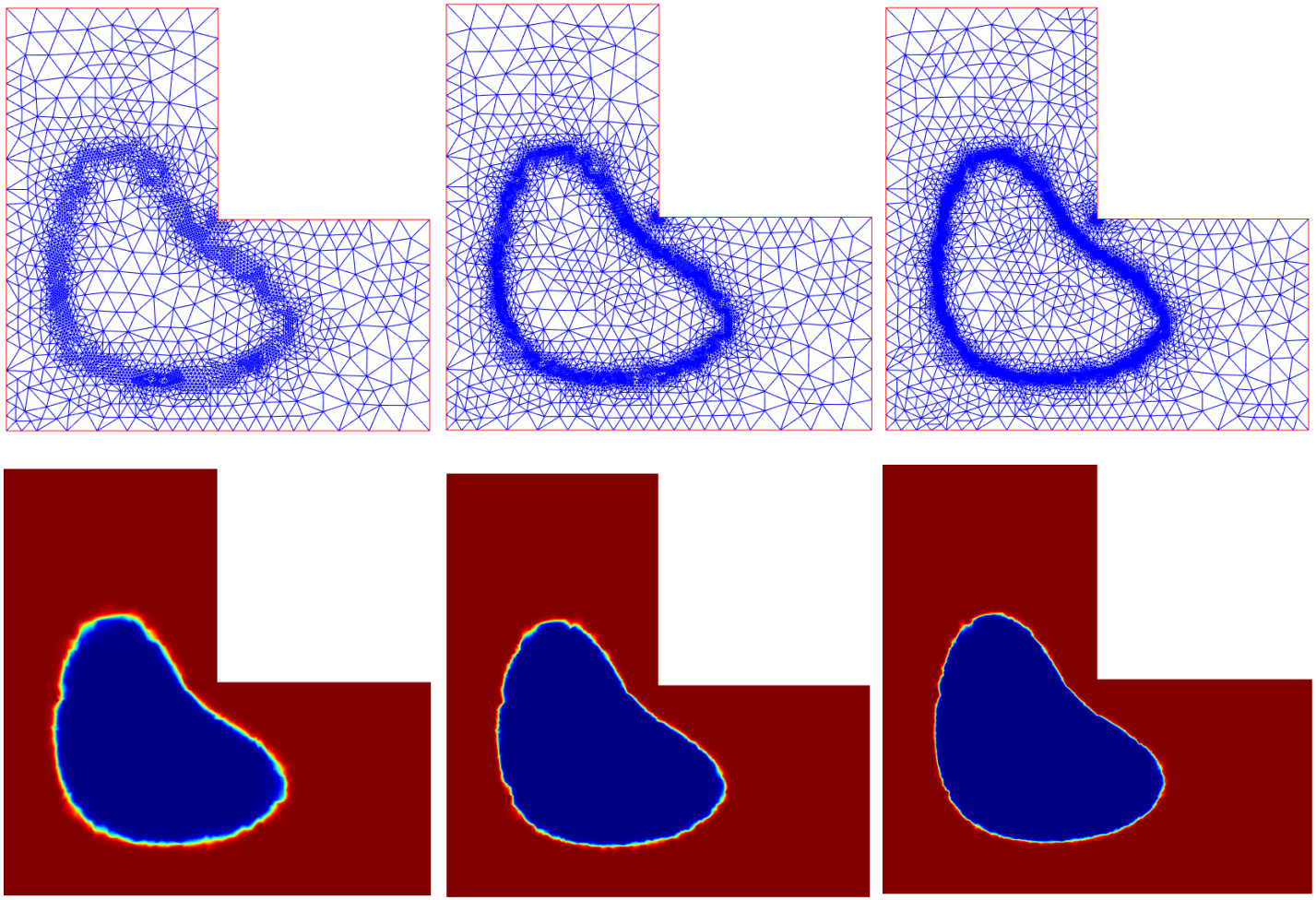
Minimization of λ_1 in Example 3.3



$k = 0(259)$ $k = 1(663)$ $k = 2(1226)$

BLUE–0, RED–1

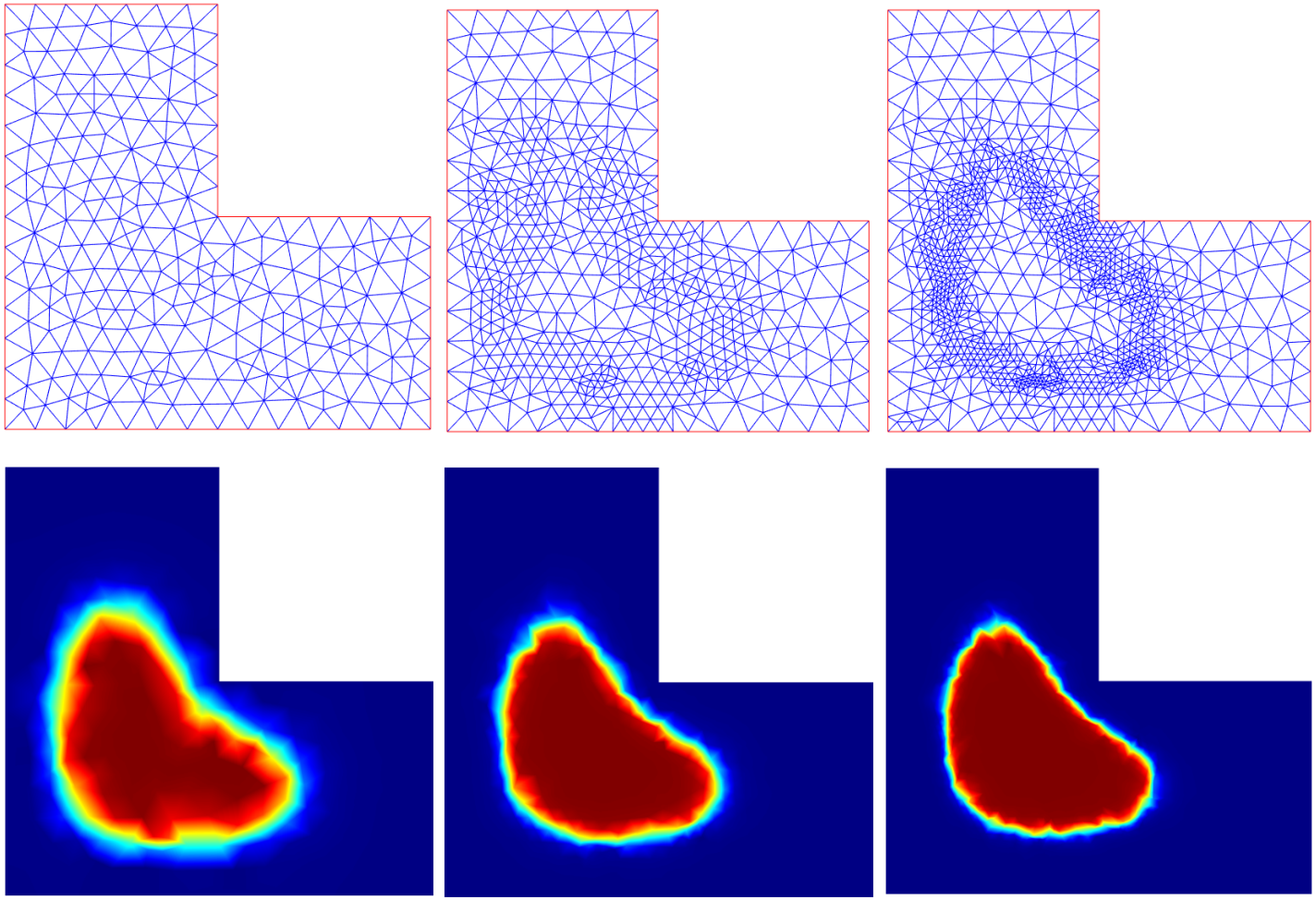
Minimization of λ_1 in Example 3.3



$k = 3(2462)$ $k = 4(4779)$ $k = 5(9083)$

BLUE–0, RED–1.

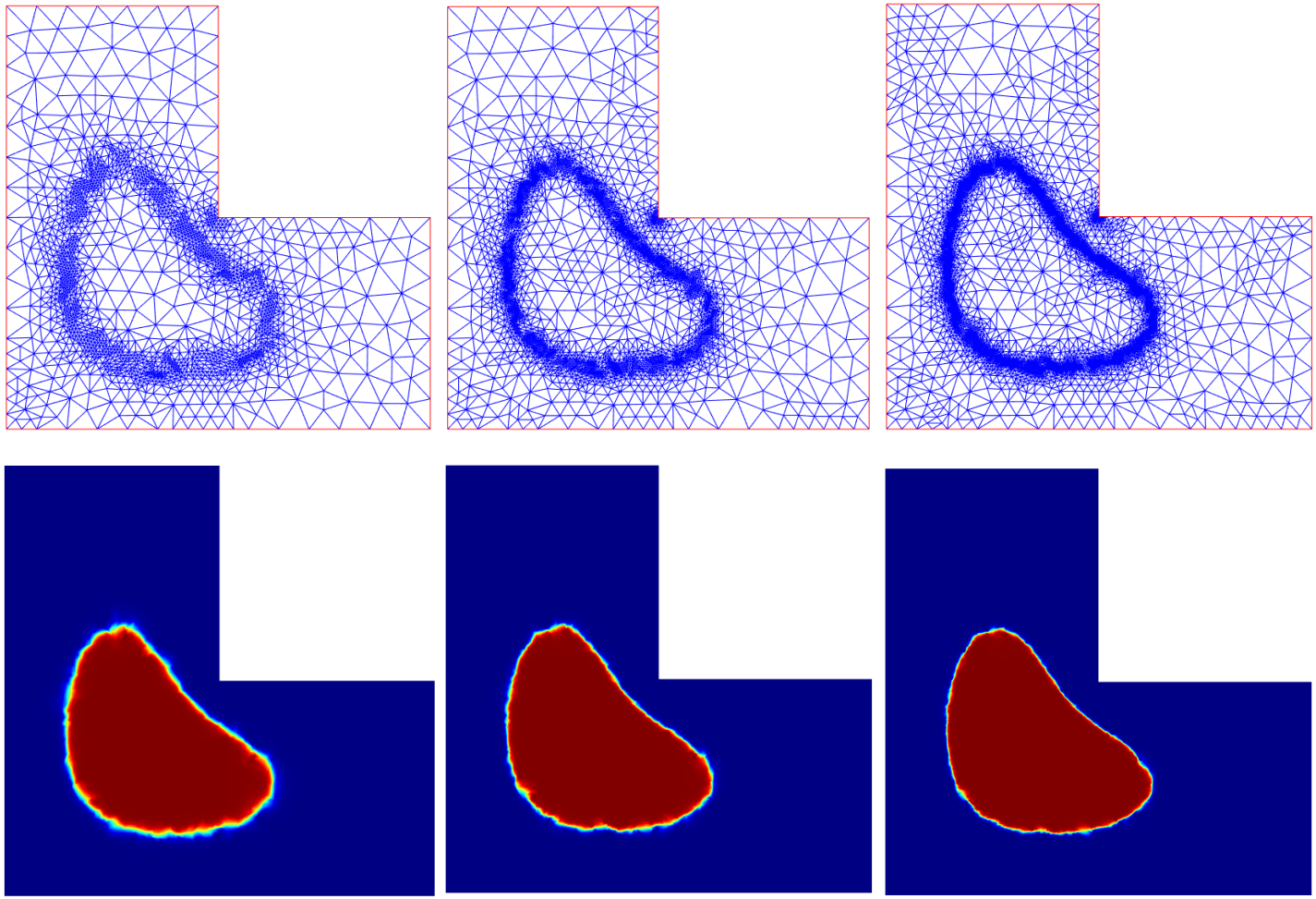
Maximization of λ_1 in Example 3.3



$k = 0(259)$ $k = 1(559)$ $k = 2(1053)$

BLUE–0, RED–1.

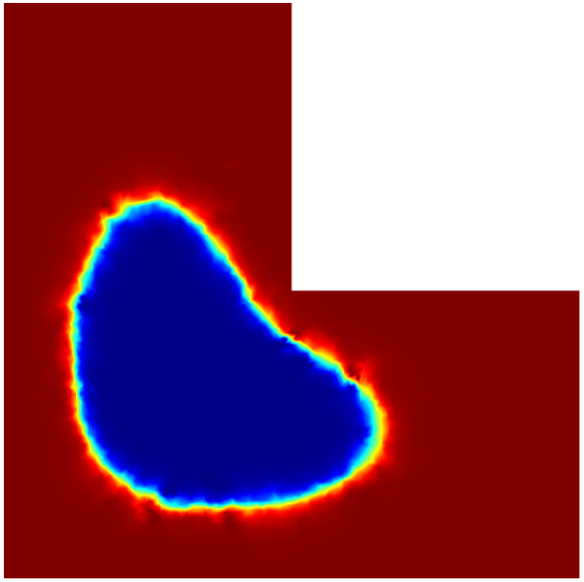
Maximization of λ_1 in Example 3.3



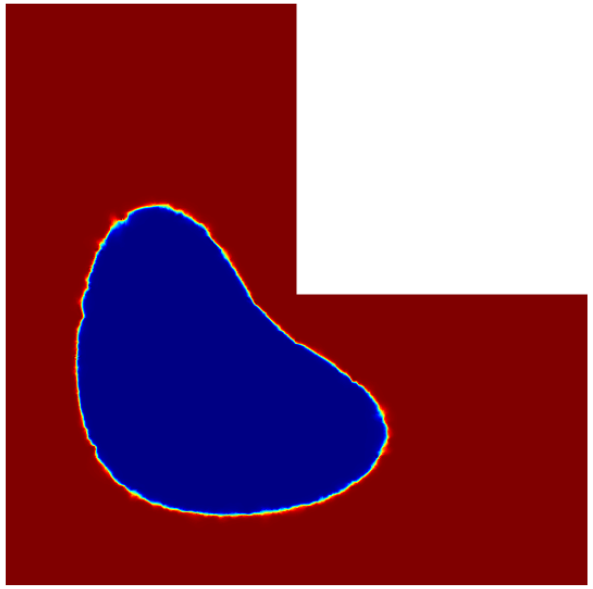
$k = 3(2022)$ $k = 4(3886)$ $k = 5(7543)$

BLUE–0, RED–1.

Example 3.3: Comparison with Uniform Refinement



Uniform $(\min \lambda_1)$

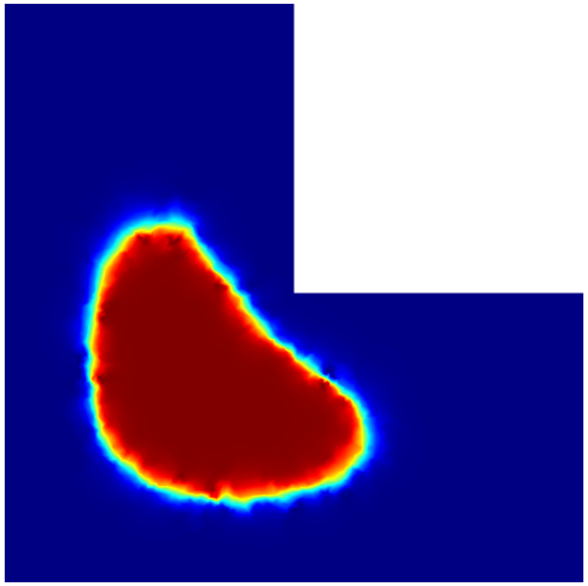


Adaptive $(\min \lambda_1)$

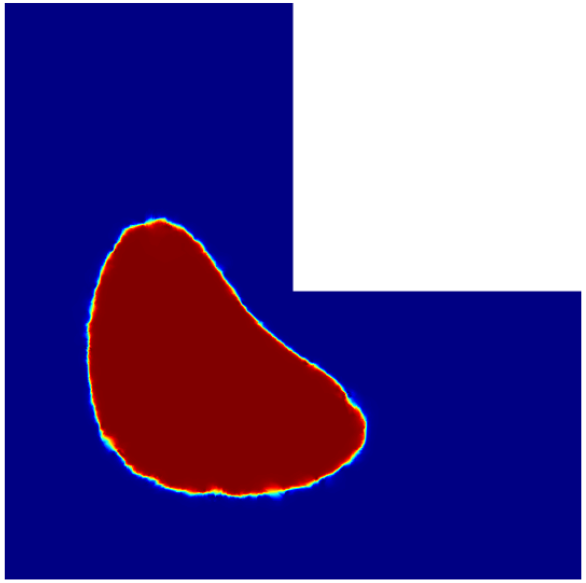
	vertices	objective	time (sec.)
Uniform	9096	9.9310	2051.03
Adaptive	9083	9.9310	1174.42

Time decrease: 42.74%

Example 3.3: Comparison with Uniform Refinement



Uniform ($\max \lambda_1$)



Adaptive ($\max \lambda_1$)

	vertices	objective	time (sec.)
Uniform	7563	10.2839	1022.06
Adaptive	7543	10.3026	754.92

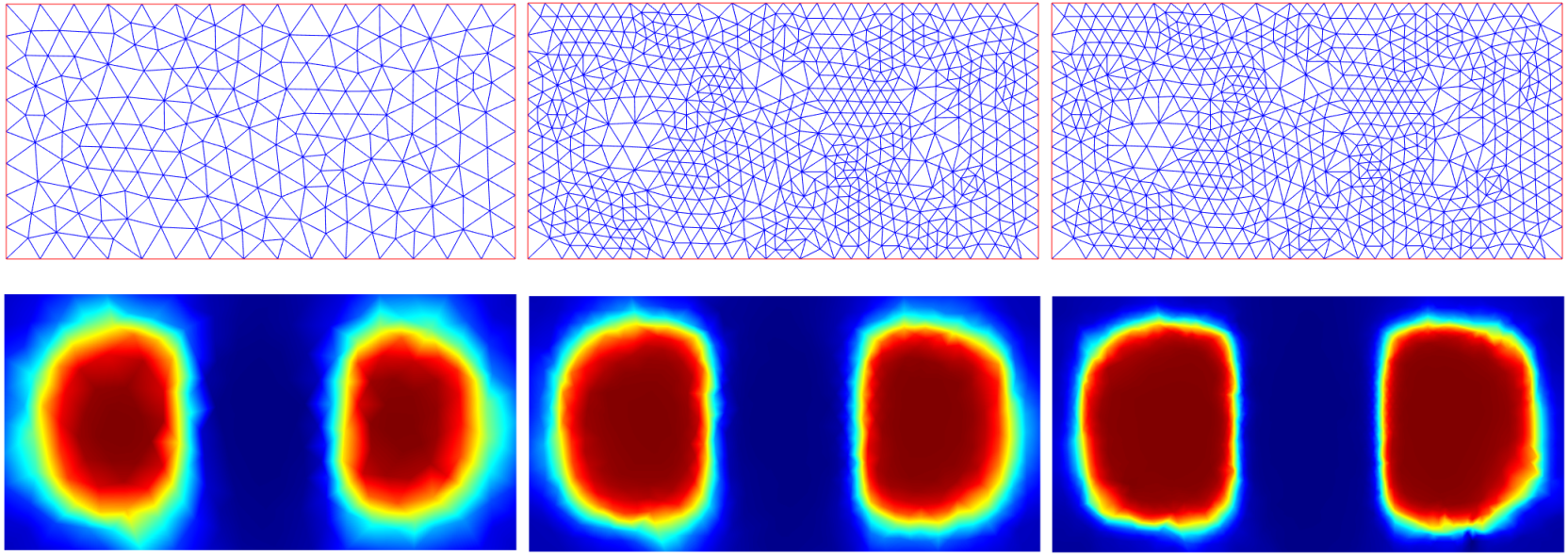
Time decrease: 26.14%

Example 3.4: Band-Gap in Rectangle

Example 3.4

Maximize $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_2$ in a rectangle $(-1, 2) \times (-0.5, 1)$ with $\alpha = 9$.
$\max(\lambda_2 - \lambda_1)$: $\varepsilon = 10^{-2}$, $\gamma = 5 \times 10^{-3}$, $K = 6$, $\theta_0 = 0.8$, $\theta_1 = \theta_2 = 0.2$, $|\Omega|/|D| = 0.45$.
$\max(\lambda_3 - \lambda_2)$: $\varepsilon = 5 \times 10^{-3}$, $\gamma = 10^{-3}$, $K = 4$, $\theta_0 = 0.8$, $\theta_1 = \theta_2 = 0.1$, $|\Omega|/|D| = 0.35$.
Initial guess of ϕ_0^* over $\mathcal{T}_0$: 0.55 & 0.35.

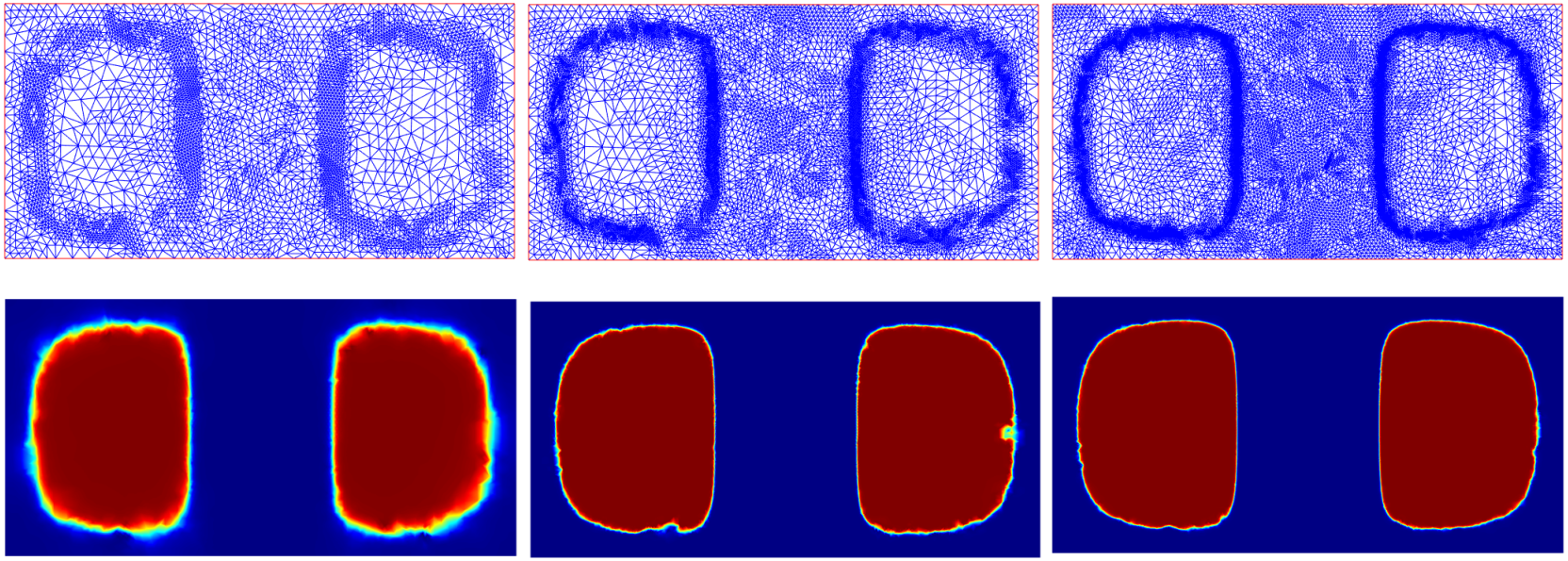
Maximization of $\lambda_2 - \lambda_1$ in Example 3.4



$k = 0(219)$ $k = 1(711)$ $k = 2(1962)$

BLUE–0, RED–1

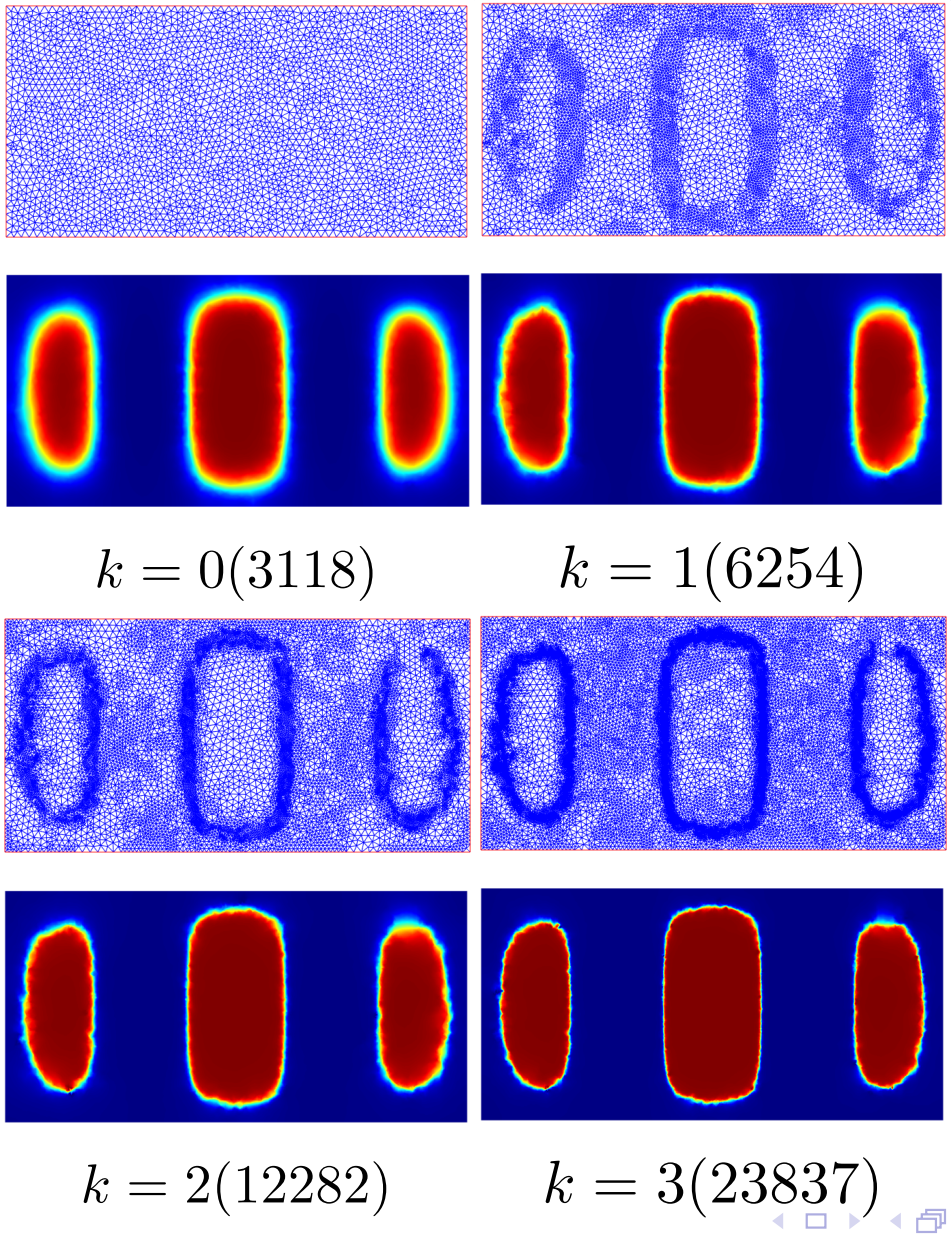
Maximization of $\lambda_2 - \lambda_1$ in Example 3.4



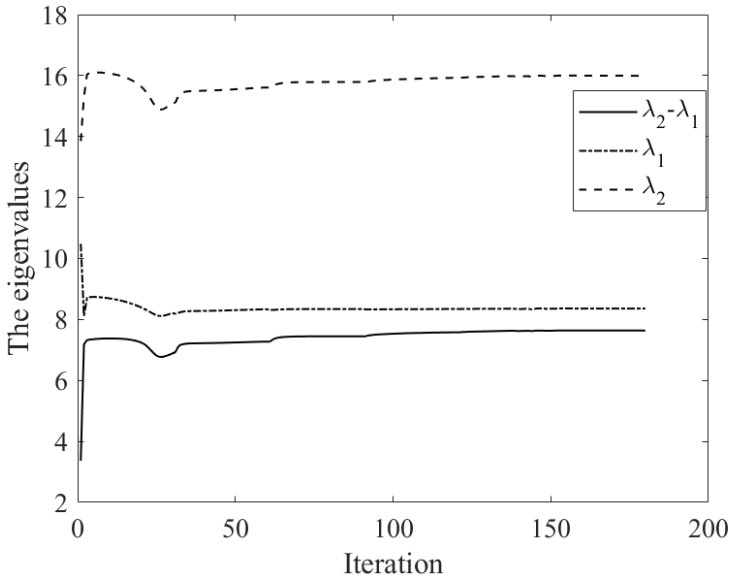
$k = 3(4528)$ $k = 4(9140)$ $k = 5(16441)$

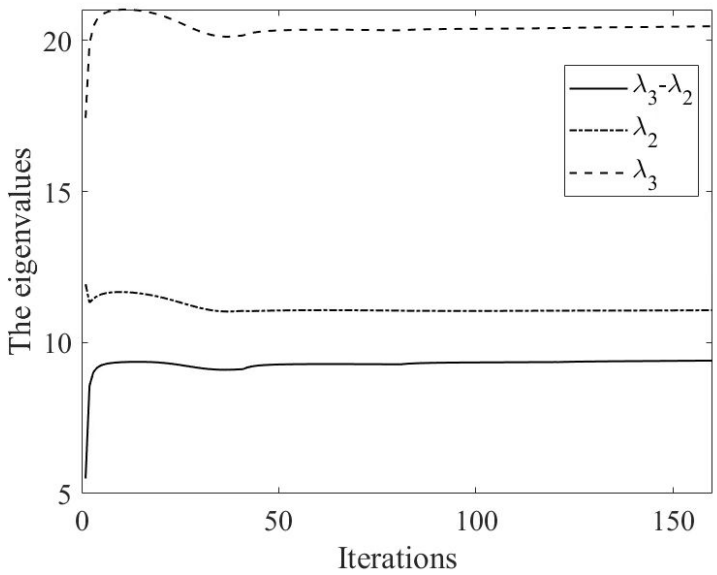
BLUE–0, RED–1

Maximization of $\lambda_3 - \lambda_2$ in Example 3.4



Example 3.4: Convergence History



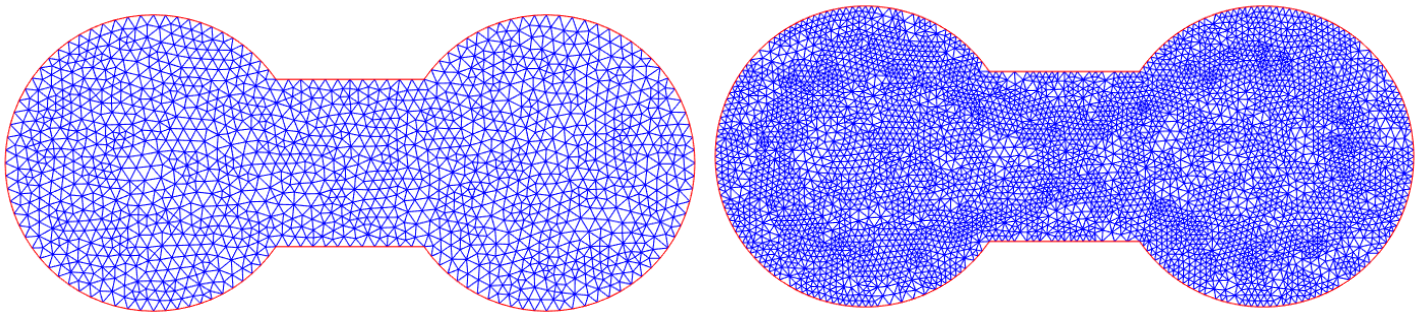


Convergence histories of $\lambda_2 - \lambda_1$ (left) and $\lambda_3 - \lambda_2$ (right) as functions of iteration numbers.

Example 3.5: Dumb-Bell Domain

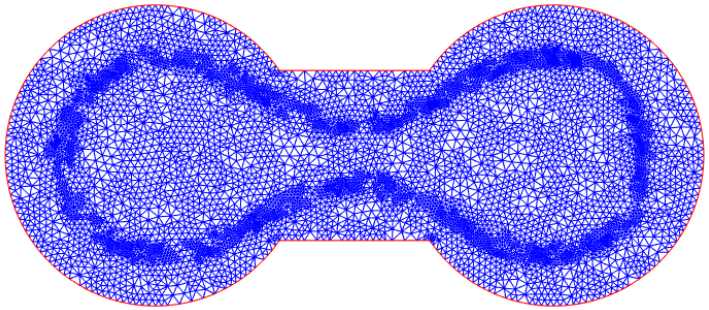
Example 3.5

Minimize λ_1 in two dumb-bell domains $D_{\mathcal{H}} = B(-1,0) \cup ((-1,1) \times (-\mathcal{H}, \mathcal{H})) \cup B(1,0)$ ($\mathcal{H} = 0.6$ or 0.3) respectively with $\alpha = 1$.
$K = 4$, $\varepsilon = 10^{-2}$.
$\mathcal{H} = 0.6$: $\gamma = 10^{-3}$, $|\Omega|/|D| = 0.5$, $\theta_0 = 0.7$, $\theta_1 = 0.1$.
Initial guess of ϕ_0^* over $\mathcal{T}_0$: 0.8.

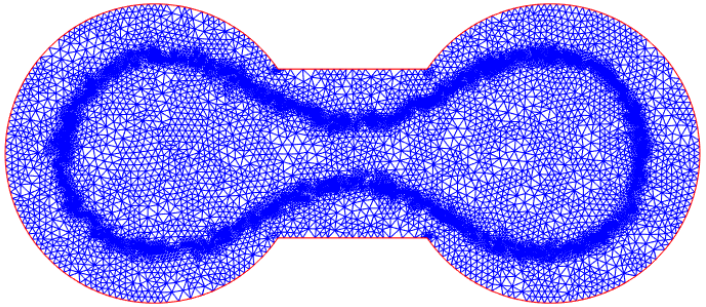


$2\mathcal{H} = 1.2: \ k = 0 \ (1546)$ $\quad$ $2\mathcal{H} = 1.2: \ k = 1 \ (4531)$

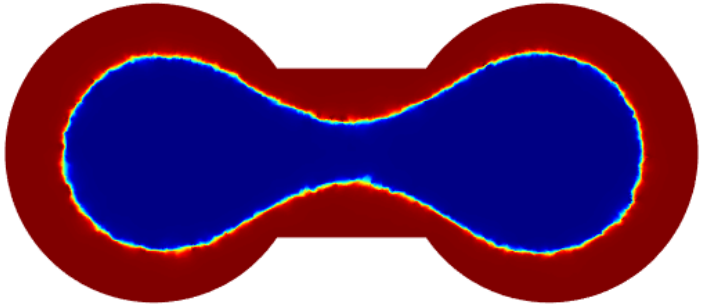
Example 3.5: Symmetry Preserving



$2\mathcal{H} = 1.2: \ k = 2 \ (7062)$



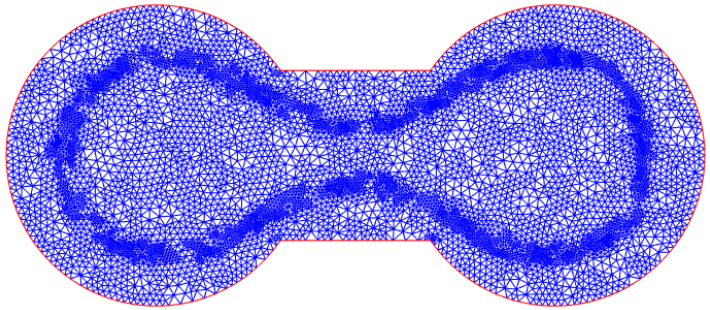
$2\mathcal{H} = 1.2: \ k = 3 \ (11184)$



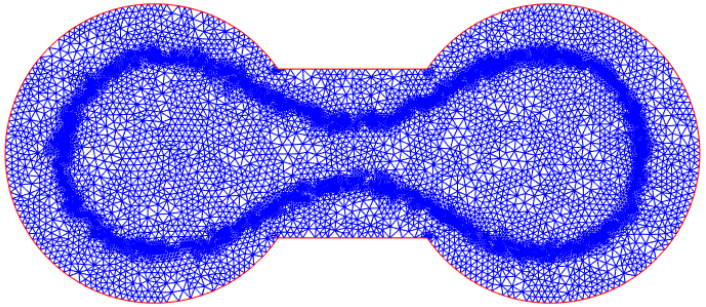
optimized design over $\mathcal{T}_3$ with BLUE–0 and RED–1.

$\mathcal{H} = 0.3$: $\gamma = 5 \times 10^{-3}$, $|\Omega|/|D| = 0.85$, $\theta_0 = 0.8$, $\theta_1 = 0.3$.
Initial guess of ϕ_0^* over $\mathcal{T}_0$: 0.5.

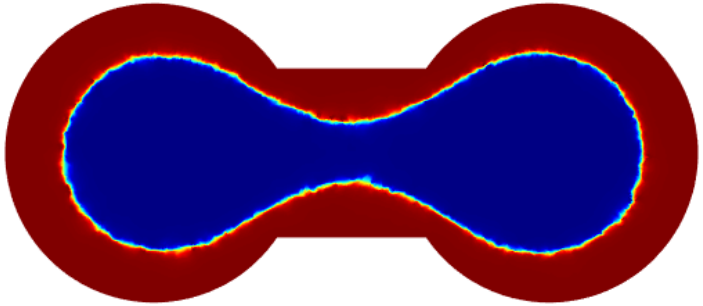
Example 3.5: Symmetry Preserving



$2\mathcal{H} = 1.2: \ k = 2 \ (7062)$



$2\mathcal{H} = 1.2: \ k = 3 \ (11184)$

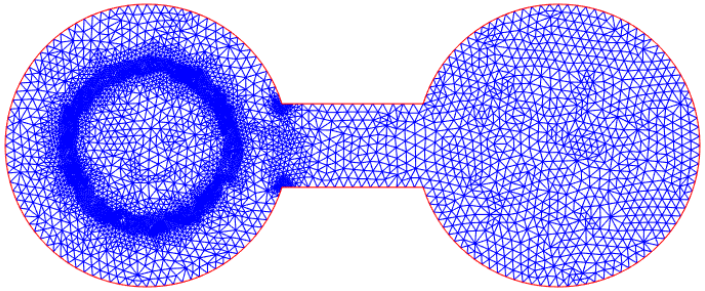


optimized design over $\mathcal{T}_3$ with BLUE–0 and RED–1.

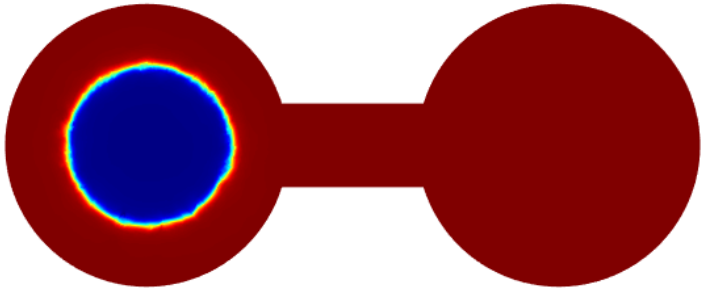
$\mathcal{H} = 0.3$: $\gamma = 5 \times 10^{-3}$, $|\Omega|/|D| = 0.85$, $\theta_0 = 0.8$, $\theta_1 = 0.3$.
Initial guess of ϕ_0^* over $\mathcal{T}_0$: 0.5.

Example 3.5: Symmetry Breaking

BLUE–0, RED–1.



$2\mathcal{H} = 0.6: \ k = 3 \ (5062)$



optimized design

- S. Chanillo, D. Grieser, M. Imai, K. Kurata & I. Ohnishi, *Symmetry breaking and other phenomena in the optimization of eigenvalues for composite membranes*, Comm. Math. Phys., 214:315-337, 2000.

Current Section

1. Introduction
2. Adaptive Algorithm
3. Numerical Example
4. Theoretical Result
5. Concluding Remark

Theorem 4.1

Assuming all eigenvalues $\left\{\lambda_{k,i_j}^{\varepsilon,*}\right\}_{k\geq 0}$ $(1 \leq j \leq l)$ *are simple, the* $l+1$ *sequences of* $\{\phi_k^*\}_{k\geq 0}$ *and* l *associated eigenpairs generated by Algorithm 2.1 have convergent subsequences in the sense* $\|\phi_{k_n}^* - \bar{\phi}\|_{H^1(D)}$, $\lambda_{k_n,i_j}^{\varepsilon,*} - \bar{\lambda}_{i_j}^{\varepsilon}$, $\|w_{k_n,i_j}^* - \bar{w}_{i_j}\|_{H^1(D)} \to 0$ *as* $n \to \infty$, *where* $\bar{\phi} \in H_1(D)$ *and* $(\bar{\lambda}_{i_j}^{\varepsilon}, \bar{w}_{i_j})_{j=1}^{l} \subset \mathbb{R} \times H_0^1(D)$ *solve*

$$\begin{cases} \gamma \mathcal{F}_{\varepsilon}'(\bar{\phi})(\phi - \bar{\phi}) \\ + \alpha \displaystyle\sum_{j=1}^{l} \frac{\partial \Psi}{\partial \lambda_{i_j}^{\varepsilon}}(\bar{\lambda}_{i_1}^{\varepsilon}, \bar{\lambda}_{i_2}^{\varepsilon}, \cdots, \bar{\lambda}_{i_l}^{\varepsilon}) \int_D (\bar{w}_{i_j})^2 (\phi - \bar{\phi})\, \mathrm{d}x \geq 0, \\ \displaystyle\int_D \boldsymbol{\nabla} \bar{w}_{i_j} \cdot \boldsymbol{\nabla} v\, \mathrm{d}x + \alpha \int_D \bar{\phi} \bar{w}_{i_j} v\, \mathrm{d}x = \bar{\lambda}_{i_j}^{\varepsilon} \int_D \bar{w}_{i_j} v\, \mathrm{d}x \end{cases} \tag{4.1}$$

$\forall (\phi, v) \in \mathcal{U} \times H_0^1(D)$. $\mathcal{F}_{\varepsilon}'(\bar{\phi})\phi = \varepsilon \int_D \boldsymbol{\nabla}\bar{\phi} \cdot \boldsymbol{\nabla}\phi\, \mathrm{d}x + \frac{1}{\varepsilon}\int_D f'(\bar{\phi})\phi\, \mathrm{d}x$.

Idea of Convergence Proof: Step 1

1. There exist subsequences $\{\phi^*_{k_n}\}_{n\geq 0}$ and $\left\{\left(\lambda^{\varepsilon,*}_{k_n,i_j}, w^*_{k_n,i_j}\right)_{j=1}^{l}\right\}_{n\geq 0}$ $\|\phi^*_{k_n} - \phi^*_\infty\|_{H^1(D)}$, $\lambda^{\varepsilon,*}_{k_n,i_j} - \lambda^{\varepsilon,*}_{\infty,i_j}$, $\|w^*_{k_n,i_j} - w^*_{\infty,i_j}\|_{H^1(D)} \to 0$. ϕ^*_∞ and $\left(\lambda^{\varepsilon,*}_{\infty,i_j}, w^*_{\infty,i_j}\right)$: minimizer and l eigenpairs of a limiting problem

For $\{\mathcal{U}_k\}_{k\geq 0}$ and $\{V^0_k\}_{k\geq 0}$ generated by AFEM,

$$\mathcal{U}_\infty := \overline{\bigcup_{k\geq 0} \mathcal{U}_k}^{H^1(D)}, \quad V^0_\infty := \overline{\bigcup_{k\geq 0} V^0_k}^{H^1_0(D)}.$$

$$\inf_{\phi_\infty \in \mathcal{U}_\infty} \left\{\mathcal{J}^\varepsilon_\infty(\phi_\infty) := \Psi(\lambda^\varepsilon_{\infty,i_1}, \lambda^\varepsilon_{\infty,i_2}, \cdots, \lambda^\varepsilon_{\infty,i_l}) + \gamma \mathcal{F}_\varepsilon(\phi_\infty)\right\}$$

$$\text{subject to } \{\lambda^\varepsilon_{\infty,i_j}\}_{j=1}^{l} \text{ given by} \tag{4.2}$$

$$\int_D \nabla w_\infty \cdot \nabla v_\infty \,\mathrm{d}x + \alpha \int_D \phi_\infty w_\infty v_\infty \,\mathrm{d}x = \lambda^\varepsilon_\infty(\phi_\infty) \int_D w_\infty v_\infty \,\mathrm{d}x$$

$$\forall v_\infty \in V^0_\infty \text{ for } (w_\infty, \lambda^\varepsilon_\infty) \in V^0_\infty \times \mathbb{R} \text{ with } \|w_\infty\|_{L^2(D)} = 1. \tag{4.3}$$

Idea of Convergence Proof: Step 1

1. There exist subsequences $\{\phi^*_{k_n}\}_{n\geq 0}$ and $\left\{\left(\lambda^{\varepsilon,*}_{k_n,i_j}, w^*_{k_n,i_j}\right)_{j=1}^{l}\right\}_{n\geq 0}$ $\|\phi^*_{k_n} - \phi^*_\infty\|_{H^1(D)}$, $\lambda^{\varepsilon,*}_{k_n,i_j} - \lambda^{\varepsilon,*}_{\infty,i_j}$, $\|w^*_{k_n,i_j} - w^*_{\infty,i_j}\|_{H^1(D)} \to 0$. ϕ^*_∞ and $\left(\lambda^{\varepsilon,*}_{\infty,i_j}, w^*_{\infty,i_j}\right)$: minimizer and l eigenpairs of a limiting problem

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$$\forall v_\infty \in V^0_\infty \text{ for } (w_\infty, \lambda^\varepsilon_\infty) \in V^0_\infty \times \mathbb{R} \text{ with } \|w_\infty\|_{L^2(D)} = 1. \tag{4.3}$$

Idea of Convergence Proof: Step 2 & Step 3

1. $\|\phi^*_{k_n} - \phi^*_\infty\|_{H^1(D)}$, $\lambda^{\varepsilon,*}_{k_n,i_j} - \lambda^{\varepsilon,*}_{\infty,i_j}$, $\|w^*_{k_n,i_j} - w^*_{\infty,i_j}\|_{H^1(D)} \to 0$ as $n \to \infty$ $(1 \le j \le l)$.
2. $\eta_{k_n,j} \to 0$ as $n \to \infty$ $(0 \le j \le l)$.
   $\{\eta_{k_n,j}\}_{n\ge 0}$ $(0 \le j \le l)$: $l+1$ sequences of estimators associated with the convergent subsequences in Step 1.
   Argument: the convergence in Step 1, properties of estimators and the marking strategy in Algorithm 2.1.
3. The limits ϕ^*_∞ and $\left(\lambda^{\varepsilon,*}_{\infty,i_j}, w^*_{\infty,i_j}\right)$ $(1 \le j \le l)$ solve (4.1).
   Argument: residuals are dominated by estimators, e.g.

$$\left|\int_D \nabla w^*_{k,i_j} \cdot \nabla v \,\mathrm{d}x + \alpha \int_D \phi^*_k w^*_{k,i_j} v \,\mathrm{d}x - \lambda^{\varepsilon,*}_{k,i_j} \int_D \phi^*_k w^*_{k,i_j} v \,\mathrm{d}x\right|$$
$$\le c\eta_{k,j}\|v\|_{H^1(D)} \quad \forall v \in H^1_0(D), 1 \le j \le l.$$

   and convergence in Steps 1-2. This observation motivates the estimators used in ESTIMATE of Algorithm 2.1.

Idea of Convergence Proof: Step 2 & Step 3

1. $\|\phi^*_{k_n} - \phi^*_\infty\|_{H^1(D)}$, $\lambda^{\varepsilon,*}_{k_n,i_j} - \lambda^{\varepsilon,*}_{\infty,i_j}$, $\|w^*_{k_n,i_j} - w^*_{\infty,i_j}\|_{H^1(D)} \to 0$ as $n \to \infty$ $(1 \le j \le l)$.
2. $\eta_{k_n,j} \to 0$ as $n \to \infty$ $(0 \le j \le l)$.
   $\{\eta_{k_n,j}\}_{n\ge 0}$ $(0 \le j \le l)$: $l+1$ sequences of estimators associated with the convergent subsequences in Step 1.
   Argument: the convergence in Step 1, properties of estimators and the marking strategy in Algorithm 2.1.
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   Argument: residuals are dominated by estimators, e.g.

$$\left|\int_D \boldsymbol{\nabla} w^*_{k,i_j} \cdot \boldsymbol{\nabla} v \,\mathrm{d}x + \alpha \int_D \phi^*_k w^*_{k,i_j} v \,\mathrm{d}x - \lambda^{\varepsilon,*}_{k,i_j} \int_D \phi^*_k w^*_{k,i_j} v \,\mathrm{d}x\right|$$
$$\le c\eta_{k,j}\|v\|_{H^1(D)} \quad \forall v \in H^1_0(D), 1 \le j \le l.$$

   and convergence in Steps 1-2. This observation motivates the estimators used in ESTIMATE of Algorithm 2.1.

Idea of Convergence Proof: Step 2 & Step 3

1. $\|\phi^*_{k_n} - \phi^*_\infty\|_{H^1(D)}$, $\lambda^{\varepsilon,*}_{k_n,i_j} - \lambda^{\varepsilon,*}_{\infty,i_j}$, $\|w^*_{k_n,i_j} - w^*_{\infty,i_j}\|_{H^1(D)} \to 0$ as $n \to \infty$ $(1 \le j \le l)$.
2. $\eta_{k_n,j} \to 0$ as $n \to \infty$ $(0 \le j \le l)$.
   $\{\eta_{k_n,j}\}_{n\ge 0}$ $(0 \le j \le l)$: $l+1$ sequences of estimators associated with the convergent subsequences in Step 1.
   Argument: the convergence in Step 1, properties of estimators and the marking strategy in Algorithm 2.1.
3. The limits ϕ^*_∞ and $\left(\lambda^{\varepsilon,*}_{\infty,i_j}, w^*_{\infty,i_j}\right)$ $(1 \le j \le l)$ solve (4.1).
   Argument: residuals are dominated by estimators, e.g.

$$\left|\int_D \boldsymbol{\nabla} w^*_{k,i_j} \cdot \boldsymbol{\nabla} v \,\mathrm{d}x + \alpha \int_D \phi^*_k w^*_{k,i_j} v \,\mathrm{d}x - \lambda^{\varepsilon,*}_{k,i_j} \int_D \phi^*_k w^*_{k,i_j} v \,\mathrm{d}x\right|$$
$$\le c\eta_{k,j}\|v\|_{H^1(D)} \quad \forall v \in H^1_0(D), 1 \le j \le l.$$

   and convergence in Steps 1-2. This observation motivates the estimators used in ESTIMATE of Algorithm 2.1.

Current Section

1. Introduction
2. Adaptive Algorithm
3. Numerical Example
4. Theoretical Result
5. Concluding Remark

References

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Related Work

The minimum compliance problem in structural topology optimization

$$\inf_{\rho\in\mathcal{A}}\left\{\mathcal{J}(\rho)=\int_\Omega \mathcal{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u})\,\mathrm{d}x\right\},$$

where $\mathcal{A}:=\{\rho\in L^\infty(\Omega):\rho\in[\underline{\rho},1]$ a.e. in $\Omega,\int_\Omega\rho\,\mathrm{d}x=C\}$, $\boldsymbol{u}\in\boldsymbol{H}^1_D(\Omega):=\{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega)\mid\boldsymbol{v}|_{\Gamma_D}=\boldsymbol{0}\}$ solves

$$\int_\Omega \mathcal{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{v})\,\mathrm{d}x=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x+\int_{\Gamma_N}\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}x\quad\forall\,\boldsymbol{v}\in\boldsymbol{H}^1_D(\Omega)$$

with $\varepsilon(\boldsymbol{u})=\frac{1}{2}(\boldsymbol{\nabla u}+\boldsymbol{\nabla u}^{\mathrm{T}})$, $\mathcal{C}\varepsilon(\boldsymbol{u})=\rho^p(2\mu\varepsilon(\boldsymbol{u})+\lambda\mathrm{tr}(\varepsilon(\boldsymbol{u}))\mathcal{I})$, $p>1$.

Jin, Li, Xu & Zhu, JCP article 112932, 2024.

AFEM for the above problem regularized by the phase field model.

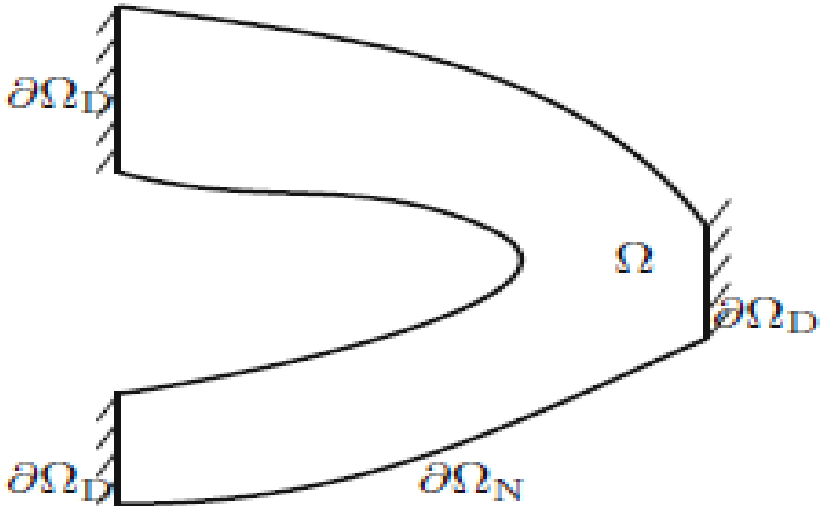
Future work: structural eigenfrequency optimization, e.g. maximization of eigenvalues associated with the elasticity problem

$$\max_{|\Omega|=C} \lambda_1 \text{ or } \min_{|\Omega|=C} -\sum_{k=1}^{n} w_k \lambda_k \ (w_k > 0, \sum_{k=1}^{n} w_k = 1) \quad \text{subject to}$$

$$-\boldsymbol{\nabla} \cdot (\mathcal{A}\varepsilon(\boldsymbol{u})) = \lambda \rho \boldsymbol{u} \quad \text{in } \Omega,$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega_D, \ \mathcal{A}\varepsilon \cdot \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \partial\Omega_N.$$

$\mathcal{A}\varepsilon(\boldsymbol{u}) = (2\mu\varepsilon(\boldsymbol{u}) + \lambda \mathrm{tr}(\varepsilon(\boldsymbol{u}))\mathcal{I})$.
ρ: the density function, C : the prescribed volume.



This work is available at arxiv.org/abs/2310.03970.

Thanks for your attention !