Decoupling PDE Computation with Intrinsic or Inertial Type Robin Conditions

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Outline

Introduction

Decoupling

- Domain Decomposition and Parallel Computing
    - Parallel Iterative Methods for Elliptic Problems: h-independent convergence
    - Parallel Marching Schemes for Parabolic Problems: CFL condition/unconditional stability
- Multi-scale and Multi-Physics Computing
    - Fluid-Porous Media Flows: stability and CFL condition
    - Fluid-Structure Interactions (FSI): unconditional instability/added-mass effects

Decoupling vs Coupled Approach

- Monolithic Approach: Fully implicit schemes are usually stable and convergent, but coupled and difficult to solve and implement.
- Decoupling: Appealing practically, but theoretically difficult and challenging!
  - Stability and convergence difficulties often occur if not decoupled appropriately. For instance, "unconditionally unstable" or "artificial added-mass effect" in FSI
  - Dirichlet-Dirichlet doesn't work; Dirichlet-Neumann sometimes works, sometimes doesn't or not effective; Robin-Neumann, Robin-Robin, improving, but puzzling, optimal parameter? WHY??? NOT well understood yet, trying in an ad hoc way ......
  - Don't blame on decoupling! Here is the SOLUTION/ANSWER

Coupled Model (Strong Form)

$$
\begin{aligned}
\rho_1 \frac{\partial u_1}{\partial t} - \nabla \cdot (\beta_1 \nabla u_1) = f_1, &\quad \text{in} \quad \Omega_1, &(1)\\
\rho_2 \frac{\partial u_2}{\partial t} - \nabla \cdot (\beta_2 \nabla u_2) = f_2, &\quad \text{in} \quad \Omega_2, &(2)\\
u_1 = u_2, &\quad \text{on} \quad \Gamma, &(3)\\
\beta_1 \nabla u_1 \cdot \mathbf{n}_1 + \beta_2 \nabla u_2 \cdot \mathbf{n}_2 = q, &\quad \text{on} \quad \Gamma,, &(4)\\
u_1 = u_{1D}, &\quad \text{on} \quad \partial\Omega_1 \backslash \Gamma, &(5)\\
u_2 = u_{2D}, &\quad \text{on} \quad \partial\Omega_2 \backslash \Gamma, &(6)
\end{aligned}
$$

Bridging classical domain decomposition and parallel computing with contemporary multi-physics computing.

- When $\rho_1 = \rho_2$ and $\beta_1 = \beta_2$: underlying single model for non-overlapping domain decomposition so that various domain decomposition time marching schemes or iterative methods may be devised for parabolic or elliptic problems with parallel computing.
- When $\beta_1 \neq \beta_2$, and/or $\rho_1 \neq \rho_2$: to study decoupling techniques for jumping coefficients and multi-scale applications.
- Extension to multi-modeling, multi-physics, multi-scale computation such as FSI.

Weak Formulation

$$
\begin{aligned}
\mathbf{V}_1 &= \left\{ v_1 \in \mathbf{H}^1(\Omega_1) : \ v_1 = 0 \ \text{ on } \ \partial\Omega_1 \backslash \Gamma \right\}, \\
\mathbf{V}_2 &= \left\{ v_2 \in \mathbf{H}^1(\Omega_2) : \ v_2 = 0 \ \text{ on } \ \partial\Omega_2 \backslash \Gamma \right\}, \\
\mathbf{V} &= \{ (v_1, v_2) \in \mathbf{V}_1 \times \mathbf{V}_2 : \ v_1 = v_2 \ \text{ on } \ \Gamma \}, \\
\tilde{\mathbf{V}} &= \{ (v_1, v_2) \in \mathbf{V}_1 \times \mathbf{V}_2 \}, \\
\mathbf{V}^{\Gamma} &= \{ (v_1, v_2) \in \mathbf{V}_1 \times \mathbf{V}_2 : \ v_1 = v_2 = 0 \ \text{ on } \ \Gamma \}, \\
\mathbf{\Lambda}_{\Gamma} &= \{ \xi = v_1|_{\Gamma} : \ v_1 \in \mathbf{V}_1 \} = \{ \xi = v_2|_{\Gamma} : \ v_2 \in \mathbf{V}_2 \}, \\
\mathbf{V}_1^{\Gamma} &= \{ v_1 \in \mathbf{V}_1 : \ v_1 = 0 \ \text{ on } \ \Gamma \}, \\
\mathbf{V}_2^{\Gamma} &= \{ v_2 \in \mathbf{V}_2 : \ v_2 = 0 \ \text{ on } \ \Gamma \}.
\end{aligned} \tag{7}
$$

Coupled Weak Form

Integration by parts directly from (1)-(6) leads to:

Model C: The Weak Form of the Continuous Coupled Model

Find $u^C = (u_1^C, u_2^C) \in \mathbf{V}$, such that

$$\rho_1 \left(\frac{\partial u_1^C}{\partial t}, v_1 \right)_{\Omega_1} + \left(\beta_1 \nabla u_1^C, \nabla v_1 \right)_{\Omega_1} + \rho_2 \left(\frac{\partial u_2^C}{\partial t}, v_2 \right)_{\Omega_2} + \left(\beta_2 \nabla u_2^C, \nabla v_2 \right)_{\Omega_2} = (f_1, v_1)_{\Omega_1} + (f_2, v_2)_{\Omega_2}, \quad \forall v = (v_1, v_2) \in \mathbf{V}. \tag{8}$$

- Dirichlet interface condition is explicitly reinforced in the solution space $\mathbf{V}$, which causes the coupling;
- Neumann interface condition is implicitly guaranteed in the weak form as a natural interface condition.
- There could be various equivalent models as well as interface conditions to be derived for various purposes, just like for root finding and fixed-point. WHICH IS GOOD FOR DECOUPLING???

Semi-Discrete Model and New Interface Conditions

Model S_h

Find $u_h^S = (u_{1,h}^S, u_{2,h}^S) \in \mathbf{V}_h$, such that

$$\rho_1 \left(\frac{\partial u_{1,h}^S}{\partial t}, v_{1,h} \right)_{\Omega_1} + \left(\beta_1 \nabla u_{1,h}^S, \nabla v_{1,h} \right)_{\Omega_1} + \rho_2 \left(\frac{\partial u_{2,h}^S}{\partial t}, v_{2,h} \right)_{\Omega_2} + \left(\beta_2 \nabla u_{2,h}^S, \nabla v_{2,h} \right)_{\Omega_2}$$
$$= \left(f_{1,h}, v_{1,h} \right)_{\Omega_1} + \left(f_{2,h}, v_{2,h} \right)_{\Omega_2}, \quad \forall v_h = (v_{1,h}, v_{2,h}) \in \mathbf{V}_h, \tag{9}$$

where $\mathbf{V}_h$ is a standard finite element subspace of $\mathbf{V}$.

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- **Model** S_h is no longer equivalent to **Model C**, but just an approximation. So, interface conditions are changed;
- There are standard error estimates for the solutions of these two models under certain regularity assumptions.
- No stability issue for semi-discretization yet;
- WHAT ARE THE INTERFACE CONDITIONS FOR u_h^S??? No body cares! It's not essential for coupled schemes.
- It's crucial for decoupling for the sake of stability and convergence!

Mass Matrix Associated with Time Derivative

- Spacial PDE operator and Newmann interface condition are associated with the stiffness matrix by integration by parts;
- How the temporal PDE operator, mass matrix, as well as the interface data transmission are related to each other?
- Lumped-mass, L^2 inner product, and condensed mass:
  - Lumped-mass approximation of inner product $(u_h, v_h)_\Omega$

$$(u_h, v_h)_{\Omega,L} = \sum_{\tau \in \mathcal{T}_h} Q_\tau (u_h v_h) = \sum_{\tau \in \mathcal{T}_h} \left(\frac{1}{3} \text{area}(\tau) \sum_{j=1}^{3} (u_h v_h)(P_{\tau,j}) \right) \tag{10}$$

  - Lifting operators

$$\begin{aligned} \mathcal{L}_{1,h} &: \ \mathbf{\Lambda}_{\Gamma,h} \longrightarrow \mathbf{V}_{1,h}, \\ \mathcal{L}_{2,h} &: \ \mathbf{\Lambda}_{\Gamma,h} \longrightarrow \mathbf{V}_{2,h}, \end{aligned} \tag{11}$$

  - Lumped-mass operators

$$\begin{aligned} (\mathbf{B}_{1,h}\xi_h, \lambda_h)_\Gamma &= (\mathcal{L}_{1,h}\xi_h, \mathcal{L}_{1,h}\lambda_h)_{\Omega_1,L}, \quad \forall \xi_h, \ \lambda_h \in \mathbf{\Lambda}_{\Gamma,h} \\ (\mathbf{B}_{2,h}\xi_h, \lambda_h)_\Gamma &= (\mathcal{L}_{2,h}\xi_h, \mathcal{L}_{2,h}\lambda_h)_{\Omega_2,L}, \quad \forall \xi_h, \ \lambda_h \in \mathbf{\Lambda}_{\Gamma,h}. \end{aligned} \tag{12}$$

Lumped-Mass Semi-Discretization

Model L_h: Lumped-Mass Semi-Discretization

Find $u_h^L = (u_{1,h}^L, u_{2,h}^L) \in \mathbf{V}_h$, such that

$$\rho_1 \left(\frac{\partial u_{1,h}^L}{\partial t}, v_{1,h} \right)_{\Omega_1, L} + \left(\beta_1 \nabla u_{1,h}^L, \nabla v_{1,h} \right)_{\Omega_1} + \rho_2 \left(\frac{\partial u_{2,h}^L}{\partial t}, v_{2,h} \right)_{\Omega_2, L} + \left(\beta_2 \nabla u_{2,h}^L, \nabla v_{2,h} \right)_{\Omega_2}$$
$$= \left(f_{1,h}, v_{1,h} \right)_{\Omega_1} + \left(f_{2,h}, v_{2,h} \right)_{\Omega_2}, \quad \forall v_h = (v_{1,h}, v_{2,h}) \in \mathbf{V}_h. \tag{13}$$

If $u^C = \left(u_1^C, u_2^C \right)$ has H^2 regularity,

$$\begin{aligned} \| u_{i,h}^L - u_i^C \|_{L^2(\Omega_i)} &= O(h^2), \quad i = 1, 2, \\ \| u_{i,h}^L - u_i^C \|_{H^1(\Omega_i)} &= O(h), \quad i = 1, 2. \end{aligned} \tag{14}$$

A Perturbation to Lumped-Mass Semi-Discretization

- There are many important applications where the Dirichlet condition enforced in the solution space $\mathbf{V}_h$ should be relaxed at certain steps for decoupling purposes.
- To understand the interface mechanism, it is fundamentally important to examine the following modified problem by removing the Dirichlet interface condition from the solution space $\mathbf{V}_h$.

Problem P_h: A Perturbation to Model L_h

Consider a function $u_h^{\tilde{L}} = (u_{1,h}^{\tilde{L}}, u_{2,h}^{\tilde{L}}) \in \tilde{\mathbf{V}}_h = \{(v_{1,h}, v_{2,h}) \in \mathbf{V}_{1,h} \times \mathbf{V}_{2,h}\}$ satisfying

$$\rho_1 \left(\frac{\partial u_{1,h}^{\tilde{L}}}{\partial t}, v_{1,h} \right)_{\Omega_1, L} + \left(\beta_1 \nabla u_{1,h}^{\tilde{L}}, \nabla v_{1,h} \right)_{\Omega_1} + \rho_2 \left(\frac{\partial u_{2,h}^{\tilde{L}}}{\partial t}, v_{2,h} \right)_{\Omega_2, L} + \left(\beta_2 \nabla u_{2,h}^{\tilde{L}}, \nabla v_{2,h} \right)_{\Omega_2}$$
$$= \left(f_{1,h}, v_{1,h} \right)_{\Omega_1} + \left(f_{2,h}, v_{2,h} \right)_{\Omega_2}, \quad \forall v_h = (v_{1,h}, v_{2,h}) \in \mathbf{V}_h. \tag{15}$$

Dirichlet Data Transmission Across Interface

Theorem

If $u_h^{\tilde{L}} \in \tilde{\mathbf{V}}_h$ satisfies (15) in **Problem** P_h, then $u_h^{\tilde{L}}$ satisfies the following condition:

$$\rho_1 \left(\mathbf{B}_{1,h} \frac{\partial u_{2,h}^{\tilde{L}}}{\partial t}, \xi_h \right)_\Gamma + \left[\rho_2 \left(\frac{\partial u_{2,h}^{\tilde{L}}}{\partial t}, \mathcal{L}_{2,h}\xi_h \right)_{\Omega_2,L} + \left(\beta_2 \nabla u_{2,h}^{\tilde{L}}, \nabla(\mathcal{L}_{2,h}\xi_h) \right)_{\Omega_2} - \left(f_{2,h}, \mathcal{L}_{2,h}\xi_h \right)_{\Omega_2} \right]$$
$$= \rho_1 \left(\mathbf{B}_{1,h} \frac{\partial u_{1,h}^{\tilde{L}}}{\partial t}, \xi_h \right)_\Gamma - \left[\rho_1 \left(\frac{\partial u_{1,h}^{\tilde{L}}}{\partial t}, \mathcal{L}_{1,h}\xi_h \right)_{\Omega_1,L} + \left(\beta_1 \nabla u_{1,h}^{\tilde{L}}, \nabla(\mathcal{L}_{1,h}\xi_h) \right)_{\Omega_1} - \left(f_{1,h}, \mathcal{L}_{1,h}\xi_h \right)_{\Omega_1} \right]$$
$$+\rho_1 \left(\mathbf{B}_{1,h} \frac{\partial (u_{1,h}^{\tilde{L}} - u_{2,h}^{\tilde{L}})}{\partial t}, \xi_h \right)_\Gamma, \quad \forall \xi_h \in \mathbf{\Lambda}_{\Gamma,h}. \tag{16}$$

Observations

- Mathematically, the Dirichlet interface error e_D at a given time t leads to an interface error of $\frac{\partial e_D}{\partial t}$, which implies how the Dirichlet type of information should be properly related on different time levels when time discretization is applied. It explains why in most of the classical explicit type of decoupled schemes, the explicit use of Dirichlet data from previous time levels, such as the Dirichlet-Dirichlet, Dirichlet-Neumann type of decoupling methods, would usually lead to stability issues.
- Most importantly, when this scalar model problem is extended to other applications such as FSI, the Dirichlet variable will physically correspond to velocity, while the related time derivative interface terms $\rho_1 \mathbf{B}_{1,h} \frac{\partial}{\partial t}$ will correspond to certain interface inertial force quantities due to spacial discretization, which will make the new interface condition to be derived physically meaningful in contrast to the classical Robin interface condition.

A Special Case: When Dirichlet Condition is Enforced

Intrinsic Interface Relationship for Model L_h

If u_h^L is a solution of **Model** L_h, then u_h^L satisfies the following interface condition:

$$\rho_1\left(\mathbf{B}_{1,h}\frac{\partial u_{2,h}^L}{\partial t},\xi_h\right)_\Gamma+\left[\rho_2\left(\frac{\partial u_{2,h}^L}{\partial t},\mathcal{L}_{2,h}\xi_h\right)_{\Omega_2,L}+\left(\beta_2\nabla u_{2,h}^L,\nabla(\mathcal{L}_{2,h}\xi_h)\right)_{\Omega_2}-\left(f_{2,h},\mathcal{L}_{2,h}\xi_h\right)_{\Omega_2}\right]$$
$$=\rho_1\left(\mathbf{B}_{1,h}\frac{\partial u_{1,h}^L}{\partial t},\xi_h\right)_\Gamma-\left[\rho_1\left(\frac{\partial u_{1,h}^L}{\partial t},\mathcal{L}_{1,h}\xi_h\right)_{\Omega_1,L}+\left(\beta_1\nabla u_{1,h}^L,\nabla(\mathcal{L}_{1,h}\xi_h)\right)_{\Omega_1}-\left(f_{1,h},\mathcal{L}_{1,h}\xi_h\right)_{\Omega_1}\right]. \tag{17}$$

- Mathematically, this is equivalent to **Model** L_h since the additional two terms $\rho_1\left(\mathbf{B}_{1,h}\frac{\partial u_{2,h}^L}{\partial t},\xi_h\right)_\Gamma$ and $\rho_1\left(\mathbf{B}_{1,h}\frac{\partial u_{1,h}^L}{\partial t},\xi_h\right)_\Gamma$ are canceled each other due to the Dirichlet interface condition being enforced in the solution space. However, they could make significant differences numerically when further approximation is applied for time discretization as well as decoupling.
- The classical Robin interface condition corresponds to adding two terms $\alpha\left(u_{1,h}^L,\xi_h\right)_\Gamma$ and $\alpha\left(u_{2,h}^L,\xi_h\right)_\Gamma$ to **Model** L_h or **Model** S_h equivalently.

Approximate Intrinsic or Inertial Type Robin Conditions

Under sufficient regularity assumption on the PDE solution:

$$
\begin{aligned}
F = & \\
& \left[\rho_2\left(\frac{\partial u_{2,h}^L}{\partial t}, \mathcal{L}_{2,h}\xi_h\right)_{\Omega_2,L} + \left(\beta_2 \nabla u_{2,h}^L, \nabla(\mathcal{L}_{2,h}\xi_h)\right)_{\Omega_2} - \left(f_{2,h}, \mathcal{L}_{2,h}\xi_h\right)_{\Omega_2}\right] + \\
& \left[\rho_1\left(\frac{\partial u_{1,h}^L}{\partial t}, \mathcal{L}_{1,h}\xi_h\right)_{\Omega_1,L} + \left(\beta_1 \nabla u_{1,h}^L, \nabla(\mathcal{L}_{1,h}\xi_h)\right)_{\Omega_1} - \left(f_{1,h}, \mathcal{L}_{1,h}\xi_h\right)_{\Omega_1}\right] \\
= & \left[\rho_2\left(\frac{\partial u_{2,h}^L}{\partial t}, \mathcal{L}_{2,h}\xi_h\right)_{\Omega_2} + \left(\beta_2 \nabla u_{2,h}^L, \nabla(\mathcal{L}_{2,h}\xi_h)\right)_{\Omega_2} - \left(f_{2,h}, \mathcal{L}_{2,h}\xi_h\right)_{\Omega_2}\right] + \\
& \left[\rho_1\left(\frac{\partial u_{1,h}^L}{\partial t}, \mathcal{L}_{1,h}\xi_h\right)_{\Omega_1} + \left(\beta_1 \nabla u_{1,h}^L, \nabla(\mathcal{L}_{1,h}\xi_h)\right)_{\Omega_1} - \left(f_{1,h}, \mathcal{L}_{1,h}\xi_h\right)_{\Omega_1}\right] + O(h^2) \\
= & \left[\rho_2\left(\frac{\partial u_2^C}{\partial t}, \mathcal{L}_{2,h}\xi_h\right)_{\Omega_2} + \left(\beta_2 \nabla u_2^C, \nabla(\mathcal{L}_{2,h}\xi_h)\right)_{\Omega_2} - \left(f_{2,h}, \mathcal{L}_{2,h}\xi_h\right)_{\Omega_2}\right] + \\
& \left[\rho_1\left(\frac{\partial u_1^C}{\partial t}, \mathcal{L}_{1,h}\xi_h\right)_{\Omega_1} + \left(\beta_1 \nabla u_1^C, \nabla(\mathcal{L}_{1,h}\xi_h)\right)_{\Omega_1} - \left(f_{1,h}, \mathcal{L}_{1,h}\xi_h\right)_{\Omega_1}\right] + O(h) \\
= & \left(\beta_2 \nabla u_2^C \cdot \mathbf{n}_2, \xi_h\right)_\Gamma + \left(\beta_1 \nabla u_1^C \cdot \mathbf{n}_1, \xi_h\right)_\Gamma + O(h), \quad \forall \xi \in \Lambda_h.
\end{aligned} \tag{18}
$$

Approximate Intrinsic or Inertial Type Robin Conditions (Continued)

$$\rho_1 \left(\mathbf{B}_{1,h} \frac{\partial u_{2,h}^L}{\partial t}, \xi_h \right)_\Gamma + \left(\beta_2 \nabla u_{2,h}^L \cdot \mathbf{n}_2, \xi_h \right)_{\Gamma+} = \rho_1 \left(\mathbf{B}_{1,h} \frac{\partial u_{1,h}^L}{\partial t}, \xi_h \right)_\Gamma - \left(\beta_1 \nabla u_{1,h}^L \cdot \mathbf{n}_1, \xi_h \right)_{\Gamma-} + O(h), \quad \forall \xi \in \Lambda_h, \tag{19}$$

which may be formally written as

$$\rho_1 \mathbf{B}_{1,h} \frac{\partial u_{2,h}^L}{\partial t} + \beta_2 \nabla u_{2,h}^L \cdot \mathbf{n}_2 = \rho_1 \mathbf{B}_{1,h} \frac{\partial u_{1,h}^L}{\partial t} - \beta_1 \nabla u_{1,h}^L \cdot \mathbf{n}_1 + O(h), \quad \text{on } \Gamma. \tag{20}$$

Similarly, if we condense the inner product from Ω_2 to the interface and transmit the Dirichlet data $u_{2,h}$ to the other side of the interface, we may have the following similar interface relation

$$\rho_2 \mathbf{B}_{2,h} \frac{\partial u_{2,h}^L}{\partial t} + \beta_2 \nabla u_{2,h}^L \cdot \mathbf{n}_2 = \rho_2 \mathbf{B}_{2,h} \frac{\partial u_{1,h}^L}{\partial t} - \beta_1 \nabla u_{1,h}^L \cdot \mathbf{n}_1 + O(h), \quad \text{on } \Gamma. \tag{21}$$

- Physically, they describe the total exchange of heat or temperature balances on the interface.
- When extended to other applications such as FSI, the related time derivative interface terms $\rho_1 \mathbf{B}_{1,h} \frac{\partial}{\partial t}$ in the new type of Robin condition will correspond to certain interface inertial force quantities according to Newton's second law, because the derivative of velocity gives acceleration, multiplying by the "mass" factor $\rho_1 \mathbf{B}_{1,h}$ leads to a force.

Full Discretization and Decoupling

- Implicit time discretization leads to stable and convergent coupled methods.
- The standard finite element semi-discrete model is often applied to derive implicit coupled scheme by enforcing both Dirichlet and Neumann conditions on the current time level.
- **Problem** P_h illustrates how the Dirichlet data may be exchanged at certain step with the neighboring subdomain by using the computed data from the other side of the interface with the removal of the Dirichlet constraint from the solution space.
- For relaxing the Neumann constraint, take the local PDEs (1)-(2), multiplying by the corresponding test functions, integrating by parts, and then exchanging the Newmann information across the interface by applying the Newmann interface condition explicitly, and finally add them together to get a coupled form defined in the tensor product space of $V_1 \times V_2$.
- **Problem EC: A Quasi-weak Formulation of the Continuous Coupled Model**
  Find $u^{EC} = (u_1^{EC}, u_2^{EC}) \in \mathbf{V}$, such that

$$\rho_1 \left(\frac{\partial u_1^{EC}}{\partial t}, v_1 \right)_{\Omega_1} + \left(\beta_1 \nabla u_1^{EC}, \nabla v_1 \right)_{\Omega_1} + \rho_2 \left(\frac{\partial u_2^{EC}}{\partial t}, v_2 \right)_{\Omega_2} + \left(\beta_2 \nabla u_2^{EC}, \nabla v_2 \right)_{\Omega_2}$$
$$= - \left(\beta_2 \nabla u_2^{EC} \cdot \mathbf{n}_2, v_1 \right)_{\Gamma} - \left(\beta_1 \nabla u_1^{EC} \cdot \mathbf{n}_1, v_2 \right)_{\Gamma} + (f_1, v_1)_{\Omega_1} + (f_2, v_2)_{\Omega_2}, \quad \forall v \in \tilde{\mathbf{V}}. \tag{22}$$

- The two interface terms $\left(\beta_2 \nabla u_2^{EC} \cdot \mathbf{n}_2, v_1 \right)_{\Gamma}$ and $\left(\beta_1 \nabla u_1^{EC} \cdot \mathbf{n}_1, v_2 \right)_{\Gamma}$ are not defined if u^{EC} is only in the H^1 space $\mathbf{V}$, unless the PDE solution has further regularity of at least $H^{3/2}$. However, the corresponding discrete counterparts are well defined.

Problem ES_h: Classical Semi-Discrete Model for Decoupling

Find $u_h^{ES} = (u_{1,h}^{ES}, u_{2,h}^{ES}) \in \mathbf{V}_h$, such that

$$\rho_1 \left(\frac{\partial u_{1,h}^{ES}}{\partial t}, v_{1,h} \right)_{\Omega_1} + \left(\beta_1 \nabla u_{1,h}^{ES}, \nabla v_{1,h} \right)_{\Omega_1} + \rho_2 \left(\frac{\partial u_{2,h}^{ES}}{\partial t}, v_{2,h} \right)_{\Omega_2} + \left(\beta_2 \nabla u_{2,h}^{ES}, \nabla v_{2,h} \right)_{\Omega_2} =$$
$$- \left(\beta_2 \nabla u_{2,h}^{ES} \cdot \mathbf{n}_2, v_{1,h} \right)_{\Gamma^-} - \left(\beta_1 \nabla u_{1,h}^{ES} \cdot \mathbf{n}_1, v_{2,h} \right)_{\Gamma^+} + (f_{1,h}, v_{1,h})_{\Omega_1} + (f_{2,h}, v_{2,h})_{\Omega_2}, \quad \forall v_h \in \tilde{\mathbf{V}}_h. \tag{23}$$

- Dirichlet-Neumann Decoupling
- Robin-Robin or Robin-Neumann based on

$$\beta_1 \nabla u_1 \cdot \mathbf{n}_1 + \beta_2 \nabla u_2 \cdot \mathbf{n}_2 + \alpha_1 (u_1 - u_2) = 0, \tag{24}$$
$$\beta_1 \nabla u_1 \cdot \mathbf{n}_1 + \beta_2 \nabla u_2 \cdot \mathbf{n}_2 + \alpha_2 (u_2 - u_1) = 0. \tag{25}$$

An Equivalent Semi-Discrete Problem Based on the Intrinsic Robin Conditions

Find $u_h^{ESI} = (u_{1,h}^{ESI}, u_{2,h}^{ESI}) \in \mathbf{V}_h$ such that

$$
\begin{aligned}
&\rho_1 \left(\frac{\partial u_{1,h}^{ESI}}{\partial t}, v_{1,h} \right)_{\Omega_1} + \left(\beta_1 \nabla u_{1,h}^{ESI}, \nabla v_{1,h} \right)_{\Omega_1} + \rho_2 \left(\frac{\partial u_{2,h}^{ESI}}{\partial t}, v_{2,h} \right)_{\Omega_2} + \left(\beta_2 \nabla u_{2,h}^{ESI}, \nabla v_{2,h} \right)_{\Omega_2} \\
&\qquad + \rho_2 \left(\mathbf{B}_{2,h} \left(\frac{\partial u_{1,h}^{ESI}}{\partial t} - \frac{\partial u_{2,h}^{ESI}}{\partial t} \right), v_{1,h} \right)_{\Gamma} \\
&= - \left(\beta_2 \nabla u_{2,h}^{ESI} \cdot \mathbf{n}_2, v_{1,h} \right)_{\Gamma-} - \left(\beta_1 \nabla u_{1,h}^{ESI} \cdot \mathbf{n}_1, v_{2,h} \right)_{\Gamma+} + (f_{1,h}, v_{1,h})_{\Omega_1} + (f_{2,h}, v_{2,h})_{\Omega_2}, \quad \forall v_h \in \tilde{\mathbf{V}}_h.
\end{aligned} \tag{26}
$$

Decoupling Based on the Intrinsic Robin Conditions

- Applying the backward Euler scheme gives an implicit and stable, but coupled algorithm.
- The coupling is through the implicit approximation to the interface integral terms as well as the Dirichlet interface condition $u_{1,h}^n = u_{2,h}^n$ enforced in the solution space.
- Our framework described here will allows one to apply various strategies to further decouple the implicit coupled scheme by approximating the interface coupling terms and relaxing the Dirichlet interface condition $u_{1,h}^n = u_{2,h}^n$ enforced in the solution space, while depending on the practical physical properties in more general real applications.
- For instance, as motivated by the thick-wall FSI application [?], we may keep the implicit approximation for the time derivative of the interface Dirichlet data terms $\frac{\partial u_{1,h}^n}{\partial t}$ and $\frac{\partial u_{2,h}^n}{\partial t}$ on one side, say $\frac{\partial u_{1,h}^n}{\partial t} \approx \frac{\delta u_{1,h}^n}{\delta t} = \frac{u_{1,h}^n - u_{1,h}^{n-1}}{\Delta t}$, while approximate the other term by using computed data from previous time levels explicitly on the other side of the interface, say $\frac{\partial u_{2,h}^n}{\partial t} \approx \frac{\delta u_{2,h}^{n-1}}{\delta t} = \frac{u_{2,h}^{n-1} - u_{2,h}^{n-2}}{\Delta t}$.
- Furthermore, to decouple the Dirichlet interface condition, we may remove the enforced condition $u_{1,h}^n = u_{2,h}^n$ in the solution space, while incorporate it into the algorithm by further approximating $u_{1,h}^{n-1}$ in the implicit approximation $\frac{\delta u_{1,h}^n}{\delta t} = \frac{u_{1,h}^n - u_{1,h}^{n-1}}{\Delta t}$ by passing the Dirichlet data from the other side with $u_{2,h}^{n-1}$, which implies to explicitly enforce the Dirichlet interface condition on the previous time level instead, namely, $\frac{\delta u_{1,h}^n}{\delta t} = \frac{u_{1,h}^n - u_{1,h}^{n-1}}{\Delta t} \approx \frac{u_{1,h}^n - u_{2,h}^{n-1}}{\Delta t}$.

Decoupled Intrinsic Robin-Neumann Scheme

For $n = 1, 2, 3 \ldots N$:

- Solve local PDE in subdomain Ω_1 with the intrinsic Robin interface condition:
  Given $u_{2,h}^{n-1}$, $u_{2,h}^{n-2}$ and $\nabla u_{2,h}^{n-1}$, find $u_{1,h}^n \in \mathbf{V}_{1,h}$ such that

$$\left(\rho_1 \frac{u_{1,h}^n - u_{1,h}^{n-1}}{\Delta t}, v_{1,h}\right)_{\Omega_1} + \left(\beta_1 \nabla u_{1,h}^n, \nabla v_{1,h}\right)_{\Omega_1} + \left(\rho_2 \mathbf{B}_{2,h} \frac{\delta u_{1,h}^n}{\delta t}, v_{1,h}\right)_{\Gamma} = \left(f_{1,h}^n, v_{1,h}\right)_{\Omega_1}$$
$$+ \left(\rho_2 \mathbf{B}_{2,h} \frac{\delta u_{2,h}^{n-1}}{\delta t}, v_{1,h}\right)_{\Gamma} - \left(\beta_2 \nabla u_{2,h}^{n-1} \cdot \mathbf{n}_2, v_{1,h}\right)_{\Gamma-}, \quad \forall v_{1,h} \in \mathbf{V}_{1,h},$$

where $\frac{\delta u_{1,h}^n}{\delta t} = \frac{u_{1,h}^n - u_{1,h}^{n-1}}{\Delta t} \approx \frac{u_{1,h}^n - u_{2,h}^{n-1}}{\Delta t}$ and $\frac{\delta u_{2,h}^{n-1}}{\delta t} = \frac{u_{2,h}^{n-1} - u_{2,h}^{n-2}}{\Delta t}$;

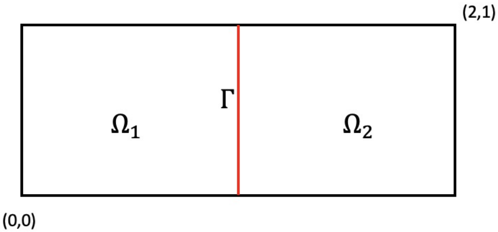
- Solve local PDE in subdomain Ω_2 with the Neumann condition:
  With $\nabla u_{1,h}^n$ computed above, find $u_{2,h}^n \in \mathbf{V}_{2,h}$ such that

$$\left(\rho_2 \frac{u_{2,h}^n - u_{2,h}^{n-1}}{\Delta t}, v_{2,h}\right)_{\Omega_2} + \left(\beta_2 \nabla u_{2,h}^n, \nabla v_{2,h}\right)_{\Omega_2} =$$
$$\left(f_{2,h}^n, v_{2,h}\right)_{\Omega_2} - \left(\beta_1 \nabla u_{1,h}^n \cdot \mathbf{n}_1, v_{2,h}\right)_{\Gamma+}, \quad \forall v_{2,h} \in \mathbf{V}_{2,h}.$$

Numerical Experiments

Set up

- Computational Domains with Interface



- $\beta(x, y) = 2 + x^2 + y^2$
- Source term is defined such that the exact solution of the coupled model is given by

$$u = t\sin(2\pi x)\sin(2\pi y).$$

- The total time is set to be $T = 1s$.

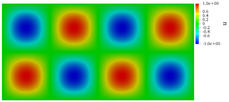
Convergence of Intrinsic Robin-Neumann Scheme

Errors of $\|u_{h,N} - u_{ext}(T)\|_{0,\Omega}$ **with** $\rho_1 = \rho_2 = 1$**,** $\beta_1 = \beta_2 = \beta(x, y)$**,** $\Delta t = h^2$

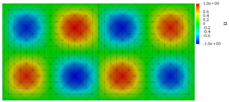
h	Coupled scheme	Decoupled iRN scheme
$\frac{1}{8}$	3.43107e-2	3.75366e-2
$\frac{1}{16}$	9.20988e-3	1.13574e-2
$\frac{1}{32}$	2.34374e-3	2.61407e-3
$\frac{1}{64}$	5.88545e-4	5.77793e-4

- The Decoupled Intrinsic Robin-Neumann Scheme is of the same order of accuracy with the implicit coupled scheme.

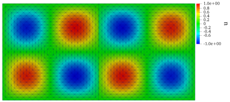
Stability of Intrinsic Robin-Neumann Scheme



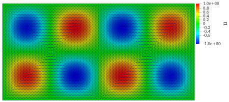
(a) Exact solution



(b) $\Delta t = h = \frac{1}{8}$



(c) $\Delta t = h = \frac{1}{16}$



(d) $\Delta t = h = \frac{1}{32}$

- The Decoupled Intrinsic Robin-Neumann Scheme is still stable with even larger time step size!

Other Decoupling Strategies

- Numerical instability is usually caused by use of Dirichlet data explicitly from the previous time level. It is the introduction of the time derivative of the Dirichlet data that is not only physically justified, but also mathematically allows for the implicit use of Dirichlet data to decouple the Dirichlet condition.
- The ingredients and strategies may vary and be fine tuned, such as whether to apply one inertial or Neumann interface term and how to decouple the Dirichlet and Neumann data by explicit approximation from using computed data, on which side or both sides or on alternating time levels, which would depend on the specific mathematical and physical properties in real applications, such as the ratio of the subdomain sizes, physical parameters like ρ, β, etc.
- The strategy in the Intrinsic Robin-Neumann Algorithm works particularly effectively in the thin-wall FSI application, where it is observed that the implicit interface approximation must be maintained on the bulk fluid side like Ω_1 here, while the interface time derivation may be relaxed by applying an explicit approximation on the thin-wall structure side for decoupling.
- For instance, we may apply the intrinsic Robin conditions (20) and (21) symmetrically on both sides of the interface by adding another term $\rho_1 \left(\mathbf{B}_{1,h}(\frac{\partial u_{2,h}}{\partial t} - \frac{\partial u_{1,h}}{\partial t}), v_{2,h}\right)_\Gamma$ to **Problem** ESI_h.

Decoupled Intrinsic Robin-Robin Scheme

For $n = 1, 2, 3 \dots N$:

- Solve local PDE in subdomain Ω_1 with the intrinsic Robin interface condition:
  Given $u_{2,h}^{n-1}$, $u_{2,h}^{n-2}$ and $\nabla u_{2,h}^{n-1}$, find $u_{1,h}^{n} \in \mathbf{V}_{1,h}$ such that

$$\left(\rho_1 \frac{u_{1,h}^{n} - u_{1,h}^{n-1}}{\Delta t}, v_{1,h}\right)_{\Omega_1} + \left(\beta_1 \nabla u_{1,h}^{n}, \nabla v_{1,h}\right)_{\Omega_1} + \left(\rho_2 \mathbf{B}_{2,h} \frac{\delta u_{1,h}^{n}}{\delta t}, v_{1,h}\right)_{\Gamma} =$$

$$\left(f_{1,h}^{n}, v_{1,h}\right)_{\Omega_1} + \left(\rho_2 \mathbf{B}_{2,h} \frac{\delta u_{2,h}^{n-1}}{\delta t}, v_{1,h}\right)_{\Gamma} - \left(\beta_2 \nabla u_{2,h}^{n-1} \cdot \mathbf{n}_2, v_{1,h}\right)_{\Gamma-}, \quad \forall v_{1,h} \in \mathbf{V}_{1,h},$$

  where $\frac{\delta u_{1,h}^{n}}{\delta t} = \frac{u_{1,h}^{n} - u_{1,h}^{n-1}}{\Delta t} \approx \frac{u_{1,h}^{n} - u_{2,h}^{n-1}}{\Delta t}$ and $\frac{\delta u_{2,h}^{n-1}}{\delta t} = \frac{u_{2,h}^{n-1} - u_{2,h}^{n-2}}{\Delta t}$;

- Solve local PDE in subdomain Ω_2 with the intrinsic or inertial Robin condition:
  With $u_{1,h}^{n}$, $u_{1,h}^{n-1}$ and $\nabla u_{1,h}^{n-1}$ computed above, find $u_{2,h}^{n} \in \mathbf{V}_{2,h}$ such that

$$\left(\rho_2 \frac{u_{2,h}^{n} - u_{2,h}^{n-1}}{\Delta t}, v_{2,h}\right)_{\Omega_2} + \left(\beta_2 \nabla u_{2,h}^{n}, \nabla v_{2,h}\right)_{\Omega_2} + \left(\rho_1 \mathbf{B}_{1,h} \frac{\delta u_{2,h}^{n}}{\delta t}, v_{2,h}\right)_{\Gamma} =$$

$$\left(f_{2,h}^{n}, v_{2,h}\right)_{\Omega_2} + \left(\rho_1 \mathbf{B}_{1,h} \frac{\delta u_{1,h}^{n}}{\delta t}, v_{2,h}\right)_{\Gamma} - \left(\beta_1 \nabla u_{1,h}^{n} \cdot \mathbf{n}_1, v_{2,h}\right)_{\Gamma+}, \quad \forall v_{2,h} \in \mathbf{V}_{2,h},$$

  where $\frac{\delta u_{2,h}^{n}}{\delta t} = \frac{u_{2,h}^{n} - u_{2,h}^{n-1}}{\Delta t} \approx \frac{u_{2,h}^{n} - u_{1,h}^{n-1}}{\Delta t}$ and $\frac{\delta u_{1,h}^{n}}{\delta t} = \frac{u_{1,h}^{n} - u_{1,h}^{n-1}}{\Delta t}$.

Robustness of Decoupling

Errors of $\|u_{h,N} - u_{ext}(T)\|_{0,\Omega}$ with $\rho_1 = \rho_2 = 1$, $\beta_1 = \beta_2 = \beta(x, y)$, $\Delta t = h^2$

h	Coupled scheme	Decoupled DN scheme
$\frac{1}{8}$	3.43107e-2	3.53891e-2
$\frac{1}{16}$	9.20988e-3	9.46749e-3
$\frac{1}{32}$	2.34374e-3	2.40475e-3

h	Decoupled iRR scheme	Decoupled RR scheme	
		$\alpha_1 = 10, \alpha_2 = 5$	$\alpha_1 = 1, \alpha_2 = 1$
$\frac{1}{8}$	3.53436e-2	3.24908e-2	3.80028e-2
$\frac{1}{16}$	9.55398e-3	8.81404e-3	1.39637e-2
$\frac{1}{32}$	2.25541e-3	2.26911e-3	5.08893e-3

- Sensitivity of the classical Robin-Robin approach on the relaxation parameters
- parameter-free performance of the intrinsic Robin approach

Robustness of Decoupling

Errors of $\|u_{h,N} - u_{ext}(T)\|_{0,\Omega}$ with $\rho_1 = 1, \rho_2 = 10$, $\beta_1 = \beta_2 = \beta(x, y)$, $\Delta t = h^2$

h	Coupled scheme	Decoupled DN scheme
$\frac{1}{8}$	3.39101e-2	3.21780e-2
$\frac{1}{16}$	9.09826e-3	9.38034e-3
$\frac{1}{32}$	2.31506e-3	2.38207e-3

h	Decoupled iRR scheme	Decoupled RR scheme	
		$\alpha_1 = 10, \alpha_2 = 5$	$\alpha_1 = 1, \alpha_2 = 1$
$\frac{1}{8}$	3.23888e-2	3.21780e-2	3.76394e-2
$\frac{1}{16}$	8.82536e-3	8.72945e-3	1.38221e-2
$\frac{1}{32}$	2.28889e-3	2.24812e-3	5.04232e-3

- Varying physical parameters
- Robustness of the intrinsic Robin approach with multi-scaling

Robustness of Decoupling

Errors of $\|u_{h,N} - u_{ext}(T)\|_{0,\Omega}$ with $\rho_1 = 10,\ \rho_2 = 1$, $\beta_1 = \beta_2 = \beta(x,y)$, $\Delta t = h^2$

h	Coupled scheme	Decoupled DN scheme
$\frac{1}{8}$	3.36012e-2	∞
$\frac{1}{16}$	9.01136e-3	∞
$\frac{1}{32}$	2.29268e-3	∞

h	Decoupled iRR scheme	Decoupled RR scheme	
		$\alpha_1 = 10, \alpha_2 = 5$	$\alpha_1 = 1, \alpha_2 = 1$
$\frac{1}{8}$	3.24630e-2	3.18627e-2	3.72954e-2
$\frac{1}{16}$	8.80706e-3	8.64307e-3	1.37006e-2
$\frac{1}{32}$	2.27580e-3	2.26911e-3	5.00114e-3

- Failure of the Dirichlet-Neumann approach with multi-scaling
- Intrinsic Robin approach is always robust

Concluding Remarks

- We have derived an intrinsic or inertial type Robin condition for multi-modeling problems, which is justified both mathematically and physically in contrast to the classical Robin condition.
- Based on this new interface condition, a decoupling approach is presented for devising effective decoupled numerical methods for applications from classical parallel computing to multi-physics applications.
- Numerical experiments show the effectiveness and robustness of our decoupling approach, its advantages over the existing decoupling approaches, as well as the promising potential for its application to complicated real problems in science and technology.
- Theoretical analysis for stability and convergence is under investigation.