MODELLING COSMIC INFLATION AND OBSERVATIONS

1 Universe Dynamics

Lay down and explain the equations, which govern the dynamics of the Universe:

$$\text{Friedmann Eq.}: \quad H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \tag{1}$$

$$\text{Acceleration Eq.}: \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right), \tag{2}$$

$$\text{Continuity Eq.}: \quad \dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0, \tag{3}$$

$$\text{Eq. of State}: \quad p = w\rho c^2, \tag{4}$$

where

$$H \equiv \frac{\dot{a}}{a} \tag{5}$$

is the Hubble parameter, with a being the scale factor of the Universe, and the dot denotes derivative w.r.t. the cosmic time t: ($\dot{} \equiv \frac{d}{dt}$).

From Eqs. (3) and (4) show that

$$\rho \propto a^{-3(1+w)}. \tag{6}$$

The Universe is approximately spatially flat, i.e. $k \approx 0$. In view of this fact, use Eq. (1) to show that

$$a \propto t^{\frac{2}{3(1+w)}} \;\Rightarrow\; H = \frac{2}{3(1+w)t} \quad \text{and} \quad \rho = \frac{1}{6(1+w)^2\pi G t^2} \quad \text{if} \quad w \neq -1 \tag{7}$$

$$\rho = \text{const.} \;\Rightarrow\; H = \sqrt{(8\pi G/3)\rho} = \text{const.} \quad \text{and} \quad a \propto \exp(Ht) \quad \text{if} \quad w = -1 \tag{8}$$

Finally, discuss the expression for the density of radiation in the hot big bang:

$$\rho_\gamma = \frac{\pi^2 g_*}{30c^2(c\hbar)^3}(k_B T)^4, \tag{9}$$

where T is the temperature of the thermal bath and g_* is the effective relativistic degrees of freedom.

2 Cosmic Inflation

Definition of Inflation: "Brief period of accelerated expansion in Early Universe".

Use Eqs. (2) and (4) to show that:

$$\ddot{a} > 0 \quad \Leftrightarrow \quad w < -\frac{1}{3} \,. \tag{10}$$

Definition of (quasi)-de Sitter inflation: $w \approx -1 \Rightarrow a \propto e^{Ht}$.

Discuss why in quasi-de Sitter inflation ρ is said to be determined by an effective cosmological constant $\Lambda_{\rm eff}$:

$$\rho_{\rm inf} \simeq \frac{\Lambda_{\rm eff} c^2}{8\pi G} \,. \tag{11}$$

Natural units are such that: $c = \hbar = k_B = 1$ and $8\pi G = m_P^{-2}$, where $m_P = 2.43 \times 10^{18}$GeV is the reduced Planck mass. Write the above equations in this system. For example the flat Friedmann equation (1) becomes

$$\rho = 3 m_P^2 H^2. \tag{12}$$

Similalry, the Continuity equation (3) becomes

$$\dot{\rho} + 3H(\rho + p) = 0 \,, \tag{13}$$

while the radiation density in Eq. (9) is given by

$$\rho_\gamma = \frac{\pi^2 g_*}{30} T^4. \tag{14}$$

3 Solving the Horizon and Flatness problems

3.1 The cosmological Horizon

Causal correlations are determined by the notion of the horizon which is formally defined as

$$D_H(t) \equiv a(t) \int_A^B \frac{dt'}{a(t')} \,, \tag{15}$$

where $A(t)$ and $B(t)$ are appropriately defined limits. In the expanding Universe there are two types of horizons:

- When $(A, B) = (0, t)$ Eq. (15) defines the *Particle Horizon*. The physical interpretation of this is that it spans the region of space which has managed to come within causal contact in the life-time of the Universe, i.e. the region which the observer can, in principle, receive signals from.

- When $(A, B) = (t, \infty)$ Eq. (15) defines the *Event Horizon*. The physical interpretation of this is that it spans the region of space which will eventually manage to come within causal contact in the future, i.e. the region which the observer can, in principle, send signals to.

Typically only one of the above is finite during a particular period of the Universe evolution. This is called the *Cosmological Horizon*. Therefore, the cosmological Horizon can be either a particle or an event horizon depending on which of those is finite.

Using Eq. (15) and assuming a flat Universe (in which Eqs. (7) and (8) are valid) show that the cosmological horizon is given by

$$D_H = \frac{2}{|1+3w|} H^{-1} \quad \text{for} \quad -\frac{1}{3} \neq w \geq -1 \,, \tag{16}$$

and corresponds to a particle {event} horizon if $w > -\frac{1}{3}$ $\{-1 \leq w < -\frac{1}{3}\}$. As can be seen by Eq. (7), when $w = -\frac{1}{3}$ we have $a \propto t$ and $H = 1/t$. In this case, the cosmological horizon is both particle and event horizon and it is taken as $D_H(t) = t = H^{-1}$.

3.2 Horizon Problem

Horizon Problem: "The CMB appears to be correlated over scales beyond causal contact (superhorizon scales) at the time of its emission (decoupling)"

Write the comoving scale of the Horizon at present time t_0 as $R_{\rm obs}(t)$. By definition:

$$R_{\rm obs}(t_0) \equiv D_H(t_0) \,, \tag{17}$$

where $D_H(t)$ is the cosmological horizon, for which (c.f. Eq. (16)):

$$D_H(t) \simeq H^{-1}(t) \,. \tag{18}$$

Using that $R_{\rm obs}$ scales with the expansion as any other length-scale: $R_{\rm obs}(t) \propto a(t)$, compare $R_{\rm obs}(t)$ with the horizon $D_H(t)$ at a given time $t < t_0$:

$$\frac{R_{\rm obs}(t)}{D_H(t)} = \frac{D_H(t_0)}{D_H(t)}\frac{a(t)}{a(t_0)} \simeq \frac{H(t)}{H(t_0)}\frac{a(t)}{a(t_0)} \Rightarrow \frac{R_{\rm obs}(t)}{D_H(t)} \simeq \frac{\dot a(t)}{\dot a(t_0)} \,, \tag{19}$$

where we used Eqs. (5), (17) and (18). The above is valid always, i.e. both in the hot big bang and also during inflation. Note that, during quasi-de Sitter inflation $H \approx$ const. and the horizon is given by $D_H \simeq H^{-1} \approx$ const.

In the hot big bang $\ddot a < 0$ and, therefore, $\dot a(t)$ is a decreasing function of time. Consequently, $R_{\rm obs}(t) > D_H(t)$ for all $t < t_0$. Hence, the present horizon is always beyond causal contact. In inflation, however, $\ddot a > 0$ and, therefore, $\dot a(t)$ is an increasing function of time. Hence, provided inflation lasts long enough we may have $R_{\rm obs}(t) < D_H(t)$ at some time during inflation. In this case, the scale of the present horizon used to be in causal contact and the horizon problem can be solved. We need, therefore, $t_{\rm beg} < t < t_{\rm end}$, where 'beg' and 'end' denote the beginning and the end of inflation respectively. The shortest inflationary period that can solve the horizon problem corresponds to $t \to t_{\rm beg}$. Hence, the condition for the solution of the horizon problem is:

$$\frac{\dot{a}(t_{\rm beg})}{\dot{a}(t_0)} < 1 \, . \tag{20}$$

3.3 Flatness Problem

Flatness Problem: "The Universe appears to be spatially flat despite the fact that a flat Universe is unstable. This requires extreme fine-tuning of initial conditions"

The Friedmann equation (1) can be written as:

$$\Omega - 1 = \frac{kc^2}{(aH)^2} \, , \tag{21}$$

where

$$\Omega(t) \equiv \frac{\rho}{\rho_c} \tag{22}$$

with $\rho_c(t)$ being the critical density. A spatially flat Universe ($k = 0$) corresponds to $\Omega = 1$.

Critical Density: "The density that a spatially flat Universe would have for a given expansion rate H": $\rho_c(t) \equiv 3H^2(t)/8\pi G$.

From Eq. (21) we obtain:

$$\frac{|\Omega_0 - 1|}{|\Omega - 1|} = \left[\frac{a(t)H(t)}{a(t_0)H(t_0)}\right]^2 = \left[\frac{\dot{a}(t)}{\dot{a}(t_0)}\right]^2 \, , \tag{23}$$

where $\Omega_0 \equiv \Omega(t_0)$.

In the hot big bang $\ddot{a} < 0$ and, therefore, $\dot{a}(t)$ is a decreasing function of time. Consequently, $|\Omega_0 - 1| > |\Omega - 1|$ for all $t < t_0$. Hence, the Universe increasingly deviates away from spatial flatness in time. However, today observations suggest that $|\Omega_0 - 1| < 0.01$, which implies severe fine tuning for initial times $t \ll t_0$. In inflation, however, $\ddot{a} > 0$ and, therefore, $\dot{a}(t)$ is an increasing function of time. Hence, provided inflation lasts long enough we may have $|\Omega(t) - 1| > |\Omega_0 - 1|$ at some time during inflation. Assuming, on dimensional grounds, an original deviation from flatness close to unity (i.e. $|\Omega(t_{\rm beg}) - 1| \simeq 0.1$) the shortest inflationary period that can solve the horizon problem requires again the constraint in Eq. (20).

4 Inflationary Paradigm

Definition of the Inflationary Paradigm: "The Universe undergoes accelerated expansion when it is dominated by the potential density of a homogeneous scalar field".

Scalar Field: "Spin-zero field; assuming a unique value at each point in space".

Density and pressure of *homogeneous* scalar field ϕ:

$$\begin{array}{l} \rho_\phi = \rho_{\rm kin} + V \\ p_\phi = \rho_{\rm kin} - V \end{array} \quad \text{where} \quad \rho_{\rm kin} \equiv \frac{1}{2}\dot{\phi}^2 \;\; \& \;\; V = V(\phi)\,. \tag{24}$$

Derivating Eq. (12) and combining it with Eq. (13), show that

$$-2\dot{H}m_P^2 = \rho + p = \rho_{\rm kin}\,, \tag{25}$$

where in the last equality we assumed that the scalar field dominates the Universe, such that $\rho = \rho_\phi$ and $p = p_\phi$.

Show that inflation requires:

$$w_\phi < -\frac{1}{3} \quad \Leftrightarrow \quad \rho_{\rm kin} < \frac{1}{2}V\,. \tag{26}$$

Show that (quasi)-de Sitter inflation requires:

$$w_\phi \approx -1 \quad \Leftrightarrow \quad \rho_{\rm kin} \ll V\,. \tag{27}$$

Hence, in quasi-de Sitter inflation $\dot{\phi}$ is very small, which means that ϕ hardly moves and, consequently, $\rho_\phi \simeq V \simeq$ const. $\simeq \Lambda_{\rm eff} m_P^2$.

Field equation (equation of motion) of homogeneous scalar field:

$$\text{Klein} - \text{Gordon Eq.}: \quad \ddot{\phi} + 3H\dot{\phi} + V' = 0\,, \tag{28}$$

where the prime denotes derivative with respect to ϕ: ($' \equiv \frac{\partial}{\partial\phi}$). Obtain the above by inserting Eq. (24) into Eq. (3). Discuss how Eq. (28) resembles the equation of motion of a particle with potential density V in field space (where the coordinate is ϕ) with friction given by $3H$. In the case of quasi-de Sitter inflation the kinetic density of the particle is negligible, so the field "slowly rolls" down its potential.

Show

$$\ddot{\phi} + V' = \frac{1}{\dot{\phi}}\frac{d}{dt}(\rho_{\rm kin} + V)\,. \tag{29}$$

Hence, show that in quasi-de Sitter inflation $\ddot{\phi}$ is negligible in Eq. (28), which becomes

$$\text{Slow-Roll Eq.}: \quad 3H\dot{\phi} \simeq -V'(\phi) \quad \text{(quasi-de Sitter)}. \tag{30}$$

Definition of the *inflaton field*: The scalar field ϕ which is responsible (= determines the dynamics) of inflation. Why is the inflaton homogeneous?

5 Slow-Roll Inflation

Definition of the number of inflation e-foldings (exponential expansions) before the end of inflation:

$$\exp(N) \equiv \frac{a_{\rm end}}{a}\,, \tag{31}$$

where 'end' denotes the end of inflation. For quasi-de Sitter inflation Eq. (8) suggests

$$N = H(t_{\rm end} - t) \quad \Rightarrow \quad dN = -Hdt\,, \tag{32}$$

where $H \approx$ const. Use the above and Eqs. (12) and (30) (considering $\rho \approx V$) to obtain:

$$N = \frac{1}{m_P^2}\int_{\phi_{\rm end}}^{\phi} \frac{V}{V'} d\phi\,. \tag{33}$$

Define the slow-roll parameter ϵ:

$$\epsilon \equiv -\frac{\dot{H}}{H^2}\,. \tag{34}$$

Using Eq. (5) show that

$$\frac{\ddot{a}}{a} = \dot{H} + H^2. \tag{35}$$

Hence, show that inflation is possible only if $\epsilon < 1$.

Discuss the physical interpretation of ϵ (robustness of H): For example, show that Eqs. (32) and (34) suggest:

$$\frac{dH}{H} = \epsilon dN, \tag{36}$$

which means that, if $\epsilon \ll 1$, one needs numerous e-foldings $\Delta N \sim 1/\epsilon \gg 1$ to achieve significant variation in H.

Using Eqs. (12) and (30) show that, in slow-roll inflation:

$$\epsilon \simeq \frac{1}{2} m_P^2 \left(\frac{V'}{V}\right)^2. \tag{37}$$

Then, using Eq. (30) show that

$$\epsilon \ll 1 \Leftrightarrow \rho_{\rm kin} \ll V\,. \tag{38}$$

Hence, slow roll inflation is possible only if $\epsilon \ll 1$. Also, the Friedman equation (12) becomes

$$V \simeq 3H^2 m_P^2\,. \tag{39}$$

Use Eqs. (3), (4), (12) and (34) to show:

$$\epsilon = \frac{3}{2}(1 + w)\,, \tag{40}$$

which implies that $\epsilon \ll 1$ guarantees quasi-de Sitter inflation if $w \geq -1$.

Define the slow-roll parameter η:

$$\eta \equiv m_P^2 \frac{V''}{V} . \tag{41}$$

Use Eq. (12) with $\rho \approx V$ to show that

$$\eta \simeq \frac{1}{3} \frac{V''}{H^2} . \tag{42}$$

Hence,

$$|\eta| \ll 1 \Leftrightarrow |V''| \ll H^2. \tag{43}$$

Using the fact that when quantizing a field the energy and momentum operators (in natural units: $c = \hbar = 1$) are $E \to i\partial/\partial t$ and $\boldsymbol{P} \to -i\boldsymbol{\nabla}$ discuss why V'' is the (mass)^2 of the field.[1]

In order for the field to slow roll the friction has to be dominant to its mass. Hence, slow-roll inflation can occur only if

$$\epsilon, |\eta| < 1 . \tag{44}$$

Inflation is terminated when at least one of the slow roll parameters ϵ and η becomes comparable to 1. Hence, the violation of the condition in Eq. (44) determines the value of the inflaton $\phi_{\rm end}$ when inflation ends.

6 Density Perturbations

6.1 The basic idea

Inflation amplifies the quantum fluctuations of light fields and turns them into classical perturbations, of superhorizon size. They, in turn, may source a perturbation in the density of the Universe, which is responsible for the formation of structures, such as galaxies and galactic clusters.

6.2 Inflaton perturbations

A "light" field is a field whose mass is smaller than H, i.e. its Compton wavelength $\sim$ (mass)$^{-1}$ is larger than the horizon size $\sim H^{-1}$, so that its quantum fluctuations are able to reach and exit the horizon being caught by the superluminal expansion.

[1] To facilitate this consider firstly that, in Eq. (28), friction is negligible (this corresponds to flat space-time but the result remains valid in curved space-time, when the friction term is introduced). Then use that $E^2 = P^2 + m^2$ (where $P^2 \equiv \boldsymbol{P} \cdot \boldsymbol{P}$) and also the fact that ϕ is homogeneous, which means that $\nabla^2 \phi \to 0$.

The inflaton field in slow-roll inflation is appropriately light (because $m_{\rm eff} \sim \sqrt{V''} \ll H$) to obtain a superhorizon spectrum of perturbations.

Each scale, when it exits the horizon obtains a perturbation whose amplitude is given by the Hawking temperature (determined by the uncertainty principle $\Delta E \cdot \Delta t = 1$):

$$\delta\phi = \frac{H}{2\pi}\,. \tag{45}$$

Indeed, a dirty way to undesrand this is as follows. Being light, the scalar field is effectively massless, so its energy density is really its kinetic density, which is

$$\rho_\phi = \frac{1}{2}\left(\frac{\delta\phi}{\delta t}\right)^2 \Rightarrow \Delta E = \rho_\phi \times \mathcal{V} \sim \left(\frac{\delta\phi}{\delta t}\right)^2 H^{-3} \Rightarrow \Delta E\cdot\Delta t \sim \delta\phi^2 H^{-2}, \tag{46}$$

where $\mathcal{V} \sim H^{-3}$ is the horizon volume and we used $\Delta t = \delta t \sim H^{-1}$ (Hubble time). In view of the uncertainty relation, the above suggests $\delta\phi \sim H$, which is in agreement with Eq. (45).

In reality, things are a bit more complicated. Perturbations of the field are, in general, uneven. But, being due to the uncertainty relation, their statistics is Gaussian, reflecting the inherent random nature of quantum fluctuations. Thus, they are characterised by a bell-shaped probability distribution which peaks at the value given by the Hawking temperature. Formally, the above are quantified as follows.

First, we switch to momentum space by Fourier transforming the perturbations as

$$\delta\phi(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \exp(i\boldsymbol{k}\cdot\boldsymbol{x})\delta\phi_{\mathbf{k}}\,, \tag{47}$$

where $\delta\phi_{\mathbf{k}} \equiv \delta\phi(\boldsymbol{k})$. Then, we define the two-point correlator, as

$$\langle\delta\phi_{\mathbf{k}}\delta\phi_{\mathbf{k}'}\rangle \equiv \delta^3(\boldsymbol{k}+\boldsymbol{k}')P_{\delta\phi}(k)\,, \tag{48}$$

where $k \equiv |\boldsymbol{k}|$ and the quantity $P_{\delta\phi}(k)$ is called the power-spectrum. Now, it is more convenient to use the quantity

$$\mathcal{P} \equiv \frac{k^3}{2\pi^2}P\,, \tag{49}$$

such that

$$\langle\delta\phi^2(\boldsymbol{x})\rangle = \frac{1}{(2\pi)^3}\int_0^\infty P_{\delta\phi}d^3k = \int_0^\infty \mathcal{P}_{\delta\phi}\frac{dk}{k}\,. \tag{50}$$

$\mathcal{P}$ is P multiplied by the volume of momentum space k^3 and is confusingly also called power-spectrum even though it is dimensionally different from P. In the following, by spectrum we will mean $\mathcal{P}$. Studying the particle production rigorously, it can be shown that

$$\sqrt{\mathcal{P}_{\delta\phi}} = \frac{H}{2\pi}\,, \tag{51}$$

which you might think as a typical value for the field perturbations per Hubble time.

After becoming superhorizon the perturbations evolve classically following the Klein-Gordon Eq.:

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} + V''\delta\phi = 0 . \tag{52}$$

Obtain the above from Eq. (28) by considering a perturbation of the field: $\phi(x,t) = \bar{\phi}(t) + \delta\phi(x,t)$, where $\bar{\phi}(t)$ is the homogeneous scalar field which also satisfies Eq. (28). Afterwards, set $V'' = m^2$ and, considering $m = \text{const.} \ll H$, solve the above to obtain that the amplitude of a given perturbation does not change after horizon exit using $H \approx$ const. Discuss why this means that the field obtains a "scale-invariant" spectrum of perturbations. Note that the earlier a given fluctuation exits the horizon the larger is the scale of the corresponding perturbation because the more it is enlarged by inflation. Do not confuse the lengthscale with the amplitude of the perturbation in question.

6.3 The curvature perturbation

Perturbations in the value of the inflaton field ϕ means that the field will reach the critical value $\phi_{\rm end}$ at different times at different points in space. As a result inflation will continue a bit more in some places than others.

After inflation, according to Eq. (7) we have $\rho \propto t^{-2}$. This means that the time difference of the end of inflation results in the generation of a density perturbation, with contrast:

$$\frac{\delta\rho}{\rho} \sim \frac{\delta t}{t} \sim \frac{\delta\phi}{\dot{\phi}t} \sim \frac{H^2}{\dot{\phi}} , \tag{53}$$

where we used Eq. (45) and also that, at (and after) the end of inflation $H \sim 1/t$ according to Eq. (7). This is a heuristic proof. The correct estimate for the amplitude of the density perturbations is given by

$$\left.\frac{\delta\rho}{\rho}\right|_{\rm exit} \simeq \left.\frac{H^2}{5\pi\dot{\phi}}\right|_{\rm exit} , \tag{54}$$

where 'exit' denotes the time when a given scale exits the horizon.

Using Eqs, (12) and (30) recast the above as

$$\frac{\delta\rho}{\rho} = \frac{1}{5\sqrt{3}\pi}\frac{V^{3/2}}{m_P^3|V'|} , \tag{55}$$

where the quantities are evaluated at horizon exit.

A related quantity in the literature is the curvature perturbation ζ. As implied by the Friedman equation Eq. (1), the density of the Universe and the curvature of spacetime are connected. In the context of General Relativity one is the source of the other. Hence a density perturbation would correspond to a curvature perturbation. The latter can be defined as:

$$\zeta = -H\frac{\delta\rho}{\dot{\rho}}\,. \tag{56}$$

Use Eqs. (3), (24) and (30) to show that, during inflation:

$$\zeta = \frac{\delta\rho}{3(\rho+p)} = \frac{\delta\rho_\phi}{6\rho_{\rm kin}} = \frac{H\delta\phi}{\dot{\phi}}\,. \tag{57}$$

Then, use Eqs. (45) and (54) to show that, at horizon exit,

$$\zeta = \frac{5}{2}\frac{\delta\rho}{\rho}\,. \tag{58}$$

As before, in reality $\delta\phi$, $\delta\rho$ and ζ are, in general, uneven, satisfying, however, predominantly Gaussian statistics, reflecting the randomness of the original quantum fluctuations. To quantify the above and compare with observations, we need to consider the spectra of the perturbations. Similarly as with the scalar field perturbation, we can define the power-spectrum of the curvature perturbation, through its two-point correlator as

$$\langle \zeta_{\mathbf{k}}\, \zeta_{\mathbf{k}'} \rangle \equiv \delta^3(\boldsymbol{k}+\boldsymbol{k}')\frac{2\pi^2}{k^3}\mathcal{P}_\zeta(k)\,, \tag{59}$$

where the Fourier transform of the curvature perturbation is

$$\zeta(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\,\zeta_{\mathbf{k}}\,, \tag{60}$$

with $\zeta_{\mathbf{k}} \equiv \zeta(\boldsymbol{k})$. Then, in view of Eqs. (55) and (58), we can write

$$\sqrt{\mathcal{P}_\zeta} = \frac{1}{2\sqrt{3}\pi}\frac{V^{3/2}}{m_P^3|V'|}\,, \tag{61}$$

where the quantities are evaluated at horizon exit.

In simple, single-field inflation models the density/curvature perturbation is called *adiabatic* because it is controlled by a single degree of freedom: the inflaton field ϕ. It can be shown that, on superhorizon scales, such an adiabatic curvature perturbation remains constant. This means that the above condition remains valid until the scales which exit the horizon during inflation, reenter the horizon at some time during the hot big bang.

Now, the density perturbations generated by inflation (or equivalently, the curvautre perturbation) a) constitute the sources for structure formation (formation of galaxies and galactic clusters) and b) affect the CMB radiation. Indeed, radiation is redshifted when crossing the gravitational potential wells of growing overdensities (Sachs-Wolfe effect) and blue shifted (heated) when carried into the gravitational wells by in-falling baryonic matter (before recombination). Therefore, at decoupling the density perturbations cause a perturbation in the temperature of the CMB, for which:

$$\frac{1}{5}\,\zeta_{\rm LS} = \frac{1}{2}\,\frac{\delta\rho}{\rho}\bigg|_{\rm H} \approx \frac{\Delta T}{T}\bigg|_{\rm CMB} \sim 10^{-5}, \tag{62}$$

where the subscript 'LS' denotes the surface of Last Scattering, the subscript 'H' denotes horizon reentry and the RHS corresponds to the observations of the Cosmic Background Explorer (COBE) satellite. Since the amplitude of the perturbations (or equivalently, the magnitude of the curvature perturbation) is approximately scale invariant so is the primordial temperature anisotropy in the CMB. The scale of $\delta\rho/\rho$, or equivalently ζ, constrains inflation models as shown in the RHS of Eqs. (55) or (61) respectively. This constraint has to be evaluated N_* e-foldings before the end of inflation, when a particular scale (the so-called pivot scale - see later), which re-enters the horizon after decoupling , has exited the horizon during inflation. In the following '*' denotes quantities evaluated N_* e-foldings before the end of inflation.

At the pivot scale, the latest observations constrain ζ as

$$\sqrt{\mathcal{P}_\zeta} = (4.579 \pm 0.032) \times 10^{-5}, \tag{63}$$

where $\mathcal{P}_\zeta = A_s$ with $\ln(10^{10}A_s) = 3.043 \pm 0.014$ according to the latest Planck satellite observations.

6.4 N_* and the pivot scale

The amplitude of the power spectrum of the curvature perturbations A_s is computed by the Planck team on the pivot comoving scale $k_*/a_0 = 0.05\,{\rm Mpc}^{-1}$, which is somewhat larger than the comoving scale of the horizon at decoupling ($\sim 10^{-2}$ ${\rm Mpc}^{-1}$).[2] This means that, it re-enters the horizon after decoupling, in the matter dominated era.[3]

The value of N_* is estimated by noting that the comoving scale of the horizon at the pivot scale $R_k(t)$ scales as all length-scales $R_k \propto a$. Therefore,

$$R_k(t_k) = \left(\frac{a_k}{a_*}\right) R_k(N_*) \;\Rightarrow\; \exp(N_*) \equiv \frac{a_{\rm end}}{a_*} \simeq \left(\frac{t_k}{t_{\rm dec}}\right)^{1/3} \frac{a_{\rm end}}{a_{\rm dec}}\, H_* t_{\rm dec}\,, \tag{64}$$

where $a_k \equiv a(t_k)$ is the scale factor at the time t_k that the pivot scale re-enters the horizon and we used that $R_k(t_k) \equiv D_H(t_k) \simeq ct_k$ by the definition

[2] k is the wavenumber of a given scale ($k = 2\pi/\ell$, where ℓ is the comoving length-scale).

[3] Note that the '*' in k_* signifies that it corresponds to the scale that left the horizon during inflation N_* e-folds before the end of inflation.

of R_k and also, $R_k(N_*) \equiv D_H(N_*) \simeq cH_*^{-1}$ from the definition of N_*. In the above, the subscript 'dec' denotes decoupling, when the CMB is emitted.

Taking the time of decoupling as $t_{\rm dec} = 5.64 \times 10^{12} h^{-1}$sec (with $h = 0.69$) and the temperature of decoupling as $T_{\rm dec} = 0.26\,$eV (where 1 GeV^{-1} = 6.5822×10^{-25} sec in natural units) show that Eq. (64) becomes

$$N_* = 65.8 - \ln\left(\frac{k_*}{a_0 H_0}\right) + \frac{1}{3}\ln\left(\frac{g_*}{106.75}\right) + \ln\left(\frac{V_{\rm end}^{1/4}}{10^{16}\,{\rm GeV}}\right) + \frac{1}{2}\ln\left(\frac{V_*}{V_{\rm end}}\right), \tag{65}$$

where $V_* \equiv V(\phi_*)$ and $\phi_* \equiv \phi(N_*)$, we assumed that the radiation era begins immediately after the end of inflation (prompt reheating - see below) and we used that

$$\left(\frac{t_k}{t_{\rm dec}}\right)^{1/3} = \left(\frac{t_0}{t_{\rm dec}}\right)^{1/3} \frac{a_0 H_0}{k_*}, \tag{66}$$

where the subscript '0' denotes the present time, we have ignored dark energy and we we considered $t_0 = 13.8\,$Gyrs. Finally, using that the Hubble constant is $H_0 = 67.8\,\frac{\rm km}{\rm sec\,Mpc}$, show that

$$\frac{k_*}{a_0 H_0} = 222\,, \tag{67}$$

where in 1 GeV^{-1} = 1.9733×10^{-19} km in natural units. Thus,

$$N_* = 60.4 + \frac{1}{3}\ln\left(\frac{g_*}{106.75}\right) + \ln\left(\frac{V_{\rm end}^{1/4}}{10^{16}\,{\rm GeV}}\right) + \frac{1}{2}\ln\left(\frac{V_*}{V_{\rm end}}\right). \tag{68}$$

6.5 Spectral indexes

In quasi-de Sitter inflation H is not exactly constant. Hence, the amplitude of the field perturbations [c.f. Eq. (45)] is not exactly the same, but depends slightly on the scale that exits the horizon at the time. The scale dependence is parametrised as

$$\mathcal{P}_\zeta \propto k^{n_s - 1}, \tag{69}$$

where n_s is the spectral index of the curvature perturbation. For $n_s = 1$ the spectrum is exactly scale invariant (because it is independent of k). Hence the deviation from scale invariance is:

$$n_s - 1 = \frac{d\ln\mathcal{P}_\zeta}{d\ln k}\,. \tag{70}$$

Using that $d\ln k = d\ln a = -dN$ and also the expressions in Eqs. (12), (30), (32), (61) as well as Eqs. (34) and (42), recast Eq. (70) as

$$n_s - 1 = 2\eta - 6\epsilon\,. \tag{71}$$

Hence, for slow roll inflation when $\epsilon, \eta \ll 1$ we see that $n_s \approx 1$, i.e. the spectrum is indeed approximately scale invariant. The recent observations of the Planck satellite suggest [1807.06209]

$$n_s = 0.9649 \pm 0.0042\,, \tag{72}$$

at 1-σ and for negligible gravitational waves (see below). The above observed value suggests that $n_s < 1$ (red spectrum) at least at 2-σ, in agreement with the expectations of the inflationary paradigm.

Another by-product of inflation is the generation of gravitational waves (gravitons). The latter can be represented by two scalar degrees of freedom φ_+, $\varphi_\times$ corresponding to the two polarisations. Each one of those behaves as a massless scalar field, very much like the inflaton field ϕ itself (but exactly massless). As a result, gravitational waves are also generated during inflation, due to the same reasons that perturbations of the inflaton $\delta\phi$ are generated. Therefore, the graviton spectrum is also approximately scale invariant, with amplitude [c.f. Eq. (51)]

$$\sqrt{\mathcal{P}_h} = 2\sqrt{16\pi G}\left(\frac{H}{2\pi}\right) \tag{73}$$

where $h_{+,\times} = \frac{1}{2}\sqrt{16\pi G}\,\varphi_{+,\times}$ are the components of the perturbation of the spacetime metric. The scale dependence of the spectrum is parametrised as

$$\mathcal{P}_h \propto k^{n_g} \quad \Leftrightarrow \quad n_g \equiv \frac{d\ln\mathcal{P}_h}{d\ln k}. \tag{74}$$

Using the above and Eqs. (34) and (73) show that the spectral index for the gravity waves generated during inflation, is given by

$$n_g = -2\epsilon\,. \tag{75}$$

Gravitational waves distort the CMB radiation like the adiabatic density perturbations. In fact, their effect is comparable and provides a consistency relation for single field inflation. Indeed, the ratio of the amplitude of the CMB temperature anisotropy due to gravity waves and density perturbations is given by

$$r \equiv \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} = \left[\frac{(\Delta T/T)_{\rm grav}}{(\Delta T/T)_{\delta\rho/\rho}}\right]^2 = 16\epsilon = -8n_g\,. \tag{76}$$

The latest observational requirements impose the upper bound [2208.00188]

$$r < 0.028\,. \tag{77}$$

7 Reheating

Definition of Reheating: "The process, which transforms the inflationary energy density into the matter and radiation of the hot big bang after the end of inflation".

For example, in quasi-de Sitter inflation, reheating transforms the vacuum density corresponding to the effective cosmological constant $\rho_{\rm inf} = \Lambda_{\rm eff} m_P^2$ into radiation: $\rho_{\rm inf} \to \rho_\gamma$.

Reheating in the Inflationary Paradigm: The inflaton field decays into the particles of the thermal bath of the hot big bang.

7.1 Perturbative Reheating

After the end of inflation the inflaton field, typically, oscillates around the bottom of its potential. This is because $\eta > 1 \Rightarrow m_{\rm eff} \sim \sqrt{V''} > H \sim 1/t$, i.e. the friction term in Eq. (28) is negligible. By Taylor expanding the potential around ϕ_0 and considering Eq. (28) show that the oscillation is quasi-harmonic with frequency $\sim m \equiv \sqrt{V''(\phi_0)}$, where ϕ_0 is the field's vacuum expectation value (VEV). This means that, on average, the energy of the inflaton is divided equally between potential and kinetic, i.e. $\overline{\rho_{\rm kin}} \approx \overline{V} \approx \frac{1}{2}\rho_\phi$. Hence, using Eqs. (6) and (24), show that the density of an oscillating scalar field scales as matter: $\rho_\phi \propto a^{-3}$. This can be understood by considering that an oscillating homogeneous scalar field behaves as a collection of massive and motionless (zero momentum) particles (in this case, inflatons). These particles are not stable but decay with decay rate Γ. This decay transfers the inflation density $\rho_\phi \approx V(\phi)$ into the radiation density of the hot big bang when $\Gamma \sim H$.

The equation of motion of the inflaton becomes:

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + V' = 0\,. \tag{78}$$

For $\Gamma \ll H$ the decay of the field is negligible and the hot big bang cannot commence (the Universe remains dominated by the oscillating inflaton). For $\Gamma > H$ the decay term dominates the friction term. In view of this fact and using Eq. (29), show that the above equation may be recast as

$$\dot{\rho}_\phi + 2\Gamma\rho_{\rm kin} = 0\,. \tag{79}$$

Thus, show that the inflaton's energy decays exponentially as

$$\rho_\phi \propto e^{-\Gamma t}, \tag{80}$$

which is extremely efficient. Hence, once $H < \Gamma$ reheating occurs rapidly, by inflaton decay. Thus, we can say that reheating occurs when $H_{\rm reh} \sim \Gamma$, where 'reh' denotes reheating. Then, using the Friedman equation (12) and Eq. (14) (and taking $g_* \sim 1$), show that the reheating temperature is

$$T_{\rm reh} \sim \sqrt{m_P \Gamma}\,, \tag{81}$$

Prompt Reheating: "When $\Gamma \sim H_{\rm end}$ and reheating occurs immediately after inflation". With prompt reheating the hot big bang begins right after the end of inflation. In this case, because $\rho_\phi^{\rm end} = \rho_\gamma^{\rm reh}$, show that we have $V_{\rm end}^{1/4} \sim T_{\rm reh}$.

In general, when $\Gamma < H_{\rm end}$, after the end of inflation but before the onset of the radiation era, the Universe is dominated by the coherently oscillating inflaton field, for which $\rho_\phi \propto a^{-3}$. Show that, in this case, an additional term is added to the RHS of Eq. (68)

$$\Delta N = -\frac{1}{3}\ln\left(\frac{V_{\rm end}^{1/4}}{T_{\rm reh}}\right). \tag{82}$$

Taking $V_{\rm end}^{1/4} \sim 10^{16}\,$GeV and that reheating typically happens before the electroweak phase transition (to allow electroweak baryogenesis) so that $T_{\rm reh} > 100\,$GeV, find: $\Delta N > -10$.

7.2 Preheating

There is another way through which the inflaton field can decay into lighter particles, which could be a part of the HBB thermal bath. This process is called *Preheating* and it corresponds to non-perturbative decay of the inflaton field. Indeed, in preheating the decay occurs in an explosive manner through parametric resonance effects.

To illustrate this consider the following toy-model for the inflaton decay. Suppose that one of the decay products of the inflaton is another scalar particle χ. This means that, during the inflaton oscillations, the equation of motion of the inflaton becomes:

$$\ddot\phi + 3H\dot\phi + m_\phi^2\phi = -g\chi^2\phi\,, \tag{83}$$

where $m_\phi^2 \equiv V''(\phi_0)$ and we ignored Γ since perturbative reheating is typically slower than preheating. The source term in the RHS is due to the coupling between ϕ and χ (quantified by g) which allows the decay $\phi \to \chi$ and corresponds to the inflaton's loss of energy because of the decay. Rearranging the above we can write:

$$\ddot\phi + (m_\phi^2 + g\chi^2)\phi \simeq 0\,, \tag{84}$$

where we ignored the friction term because it is negligible during the inflaton oscillations.

Now, it is important to note that the decay products also satisfy a similar equation:

$$\ddot{\chi} + (m_\chi^2 + g\phi^2)\chi \simeq 0\,, \tag{85}$$

since they are also scalar fields. The above corresponds to a system of coupled oscillators with oscillating frequencies. Hence, they are expected to give rise to resonant effects, which, because of an imbalance in the masses: $m_\phi \gg m_\chi$, will result in explosive production of χ-particles. In turn, these χ-particles will decay further into the HBB thermal bath.

However, it is important to stress here that the above parametric resonance soon becomes inefficient and stops. A fraction of the inflaton energy always survives preheating and continues oscillating until it decays perturbatively.

Since they are light, χ-particles are relativistic and, therefore, their density decays as $\rho_\chi \propto a^{-4}$ after their production. In contrast, after the end of preheating the density of the oscillating inflaton scales as $\rho_\phi \propto a^{-3}$. So, even if only a small fraction of the inflaton's energy escapes preheating, it soon comes to dominate the Universe until its perturbative decay. Thus, preheating can produce a significant fraction of the HBB thermal bath only if perturbative reheating is prompt enough. Illustrate the above with a log-log diagram plotting the density of the inflaton and that of the decay products w.r.t. the scale factor a during the reheating period when perturbative reheating is prompt and when it is not.

8 Starobinsky-type Inflation

Consider the inflaton potential:

$$V(\phi) = V_0 \left(1 - e^{-q\phi/m_P}\right)^2, \tag{86}$$

where $q > 0$ is a real dimensionless constant parametrising a family of models, which we call Starobinsky-type inflation.

You are asked to investigate the above family of inflation models and compare them with observations. By doing so, you will be able to eliminate some of the models of the family and constrain the model parameters, such as V_0. In particular, you are asked to do the following:

- For a given q compute the slow-roll parameters $\epsilon(\phi)$ and $|\eta(\phi)|$. Compare them with each-other and find which one determines the end of inflation, assuming the range $0 < q < \sqrt{2}$. Use your result to compute the value of ϕ_{end} in terms of m_P and q assuming $\phi_{\text{end}} > 0$.
- For a given q obtain $\phi = \phi(N)$. Then, using the result, find $n_s = n_s(N)$ and $r(N)$.
- Employing the approximate expressions obtained of the observables n_s and r, use the observational requirements in Eqs. (72) and (77) to limit the range of the allowed values of q .
- Afterwards, find $\mathcal{P}_\zeta = \mathcal{P}_\zeta(N)$. You will end up with an equation with two unknowns (V_0 and N) plus the parameter q. Evaluate this expression at $N = N_*$, where the value of $\mathcal{P}_\zeta(N_*) = A_s$ is given in Eq. (63). Let us call this equation Eq. (A).
- Now, use Eq. (68) to obtain N_* with respect to V_0 assuming prompt reheating. You will find a second equation with two unknowns (V_0 and N_*) plus the parameter q. Let us call this equation Eq. (B). (Use that, at high energies, the standard model of particle physics suggests: $g_* = 106.75$)
- Eqs. (A) and (B) form a system of two equations with two unknowns (V_0 and N_*), which can in principle be solved for a given value of q. Solve this to obtain N_* and $V_0^{1/4}$ in GeV for the range of q obtained before. Present $N_*(q)$ and $V_0^{1/4}(q)$ at a graph.
- For the allowed range of q, present a graph on the $n_s - r$ plane, with the predictions of $n_s(q)$ and $r(q)$ superimposed on the observations, considering the corresponding value of $N_*(q)$ obtained before. Refine the range of q such that the predictions stay within the 95% c.l. Planck contours.

- As shown, late reheating can reduce N_* by 10 or so. Add on the $n_s - r$ above the predictions of $n_s(q)$ and $r(q)$ with $N_* \to N_* - 10$.

- Afterwards, plot $V_{\rm end}^{1/4} \equiv [V(\phi_{\rm end})]^{1/4}$ in GeV, for the allowed range of q. Furthermore, calculate $w(N_*)$ for a given q. Is Starobiksy-type inflation quasi-de Sitter?

- The original Starobinski model featured $q = \sqrt{2/3}$. For this value, find the values of $V_0^{1/4}$ (in GeV), N_* $n_s(N_*)$ and $r(N_*)$ assuming prompt reheating. Also, solve the Horizon and Flatness problems assuming prompt reheating. In particular, obtain a lower bound on the total number of inflationary e-foldings $N_{\rm tot}$ such that the bound in Eq. (20) is satisfied.

9 Advanced Tasks

9.1 R^2 Inflation

This is the original inflation model suggested by Alexei A. Starobinsky in 1980, before even the term "cosmic inflation" was coined.

Consider the original (Jordan frame) Lagrangian density:

$$\mathcal{L} = \frac{1}{2} m_P^2 R + \beta R^2,$$

where $\beta \equiv \frac{m_P^4}{16V_0}$ and R is the Ricci scalar.

- Show how the model in Eq. (86) with $q = \sqrt{2/3}$ is obtained by a conformal transformation of the (Jordan) metric $g_{\mu\nu}$ to the (Einstein) metric:

$$\tilde{g}_{\mu\nu} = \left(1 + \frac{\varphi m_P^3}{4V_0}\right) g_{\mu\nu}$$

  with the field redefinition

$$\phi = \sqrt{\frac{3}{2}} \ln\left(1 + \frac{\varphi m_P^3}{4V_0}\right) m_P$$

- Based on the observational contraint on $V_0^{1/4}$, estimate β and comment on whether or not its value is reasonable in the context of this modified gravity theory.

- Assuming prompt reheating, use the value of N_* and estimate thus the inflation observables $n_s(N_*)$ and $r(N_*)$. On a $n_s - r$ diagram, plot the predictions of this model for prompt reheating and for late reheating with $N_* \to N_* - 10$.

- Discuss the stability of the model (are there ghosts?).

9.2 Higgs Inflation

In curved spacetime the Lagrangian density of a scalar field assumes a contribution of a direct (non-minimal) coupling ξ of the scalar field to gravity. Thus, the theory is modified gravity, with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \frac{1}{2}(\xi\varphi^2 + m_P^2)R - V(\varphi)$$

Einstein summation is assumed for the space-time index $\mu = 0, 1, 2, 3$, where $\partial_\mu\phi \equiv \partial\phi/\partial x^\mu$ and $\partial^\mu\phi = g^{\mu\nu}\partial_\nu\phi$.

We can consider that the scalar field ϕ is the electroweak Higgs field, which was first observed in CERN in 2012. Then, the potential is

$$V(\varphi) = \frac{1}{4}\lambda(\varphi^2 - v^2)^2, \tag{87}$$

where v is the VEV of the electroweak Higgs field ($v \simeq 246$ GeV) and λ is its self-coupling ($\lambda \simeq 0.13$ in electroweak energy).

- Show how the model in Eq. (86) with $q = \sqrt{2/3}$ is obtained by a conformal transformation of the (Jordan) metric $g_{\mu\nu}$ to the (Einstein) metric:
$$\tilde{g}_{\mu\nu} = \left(1 + \frac{\xi\varphi^2}{m_P^2}\right) g_{\mu\nu}$$
with the field redefinition
$$\phi = \sqrt{\frac{3}{2}}\ln\left(1 + \frac{\xi\varphi^2}{m_P^2}\right) m_P$$

- Based on the observational contraint on $V_0^{1/4}$, estimate ξ and comment on whether or not its value is reasonable in the context of this modified gravity theory.

- The branching ratios of the electrowark Higgs field to the standard model particles is known. Therefore, reheating is fully determined. Briefly discuss reheating in this model and estimate N_* and the resulting values of the inflation observables $n_s(N_*)$ and $r(N_*)$. On a $n_s - r$ diagram, plot the predictions of this model.

- Discuss the issue of the electroweak vacuum stability and how it might be ameliorated in this model.

9.3 α-attractors

In supergravity, a non-trivial Kähler metric may result in a non-canonical kinetic term of the scalar field, which features poles.

- Show that, with a kinetic term of the form

$$\mathcal{L}_{\text{kin}} = \frac{\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi}{\left[1-\frac{1}{6\alpha}\left(\frac{\varphi}{m_P}\right)^2\right]^2}, \tag{88}$$

for a scalar field φ, the associated canonically normalised field ϕ is

$$\varphi = \sqrt{6\alpha}\,m_P \tanh\left(\frac{1}{\sqrt{6\alpha}}\frac{\phi}{m_P}\right). \tag{89}$$

Thus, we see that the pole in the denominator has been transposed to infinity. This is in effect "flattening" the potential, creating thereby the inflationary plateau.

- Using the above, show that we may obtain the α-attractor models by considering the potentials

$$\textbf{T}-\textbf{model}: \qquad V(\varphi) = \frac{1}{2}m^2\varphi^2 \;\Rightarrow\; V(\phi) = 3\alpha m^2 m_P^2 \tanh^2\left(\frac{1}{\sqrt{6\alpha}}\frac{\phi}{m_P}\right) \quad \text{and}$$

$$\textbf{E}-\textbf{model}: V(\varphi) = \frac{\frac{1}{2}m^2\varphi^2}{\left[1+\frac{1}{\sqrt{6\alpha}}\frac{\varphi}{m_P}\right]^2} \;\Rightarrow\; V(\phi) = \frac{3}{4}\alpha m^2 m_P^2\left(1-e^{-\sqrt{\frac{2}{3\alpha}}\phi/m_P}\right)^2,$$

where the last one is the original model with $V_0 = \frac{3}{4}\alpha m^2 m_P^2$ and $q = \sqrt{\frac{2}{3\alpha}}$ so that $\alpha = 1$ is the Starobinsky model.

- In the case of the E-model, assuming prompt reheating, use the value of V_0 as suggested by the observations, to find the value of α. Then obtain the predictions of the model regarding n_s and r.

- In the case of T-model, follow the procedure of the previous section to obtain $n_s(N_*)$ and $r(N_*)$. You must find the same functional forms as with E-model (Starobinsky inflation). This is true for any potential $V(\varphi)$, which does not feature poles at $\varphi = \pm\sqrt{6\alpha}\,m_O$. This is why, this class of models is called α-attractors.

- Assuming prompt reheating, use the value of V_0 as suggested by the observations, to find the value of α. Then obtain the predictions of the model regarding n_s and r.

Additionally, you may investigate the following:

In supergravity (and string theory) one of the fundamental functions is the Kähler potential K, which is a holomorphic function of the scalar fields in the theory. The second derivatives of K determine the geometry in field space. Hence, K_{m^*n} is called the Kähler metric, where and the subscripts denote derivatives with respect the the relevant superfields, e.g. $K_n \equiv \frac{\partial K}{\partial \Phi_n}$. The Kähler metric determines the kinetic part of the Lagrangian density of the scalar fields:

$$\mathcal{L}_{\rm kin} = \sum_{m,n} K_{m^*n} \partial_\mu \Phi_n (\partial^\mu \Phi_m^*), \tag{90}$$

where Einstein summation is assumed for the space-time index $\mu = 0, 1, 2, 3$ and where $\partial_\mu \Phi \equiv \partial \Phi / \partial x^\mu$ and $\partial^\mu \Phi = g^{\mu\nu} \partial_\nu \Phi$.

- Show that the kinetic Lagrangian density in Eq. (88) can be obtained with

$$K = -3\alpha m_P^2 \ln \left(1 - \frac{\Phi \bar{\Phi}}{3m_P^2} \right) \tag{91}$$

where the overbar denotes charge conjugation and we consider a rotation in field space such that $\Phi = \bar{\Phi} = \varphi / \sqrt{2\alpha}$.

- For canonically normalised scalar fields $K = \sum_n \Phi_n \Phi_n^*$, i.e. $K_{m^*n} = \delta_{mn}$, the latter being Krönecker's delta. This is called the minimal Kähler potential.

In supergravity, the F-term scalar potential is given by

$$V_F = e^{K/m_P^2} \sum_{m,n} \left[(W_n + m_P^{-2} W K_n) K^{m^*n} (W_m + m_P^{-2} W K_m)^* - 3m_P^{-2} |W|^2 \right], \tag{92}$$

where W is another fundamental function of the scalar fields of the theory called the superpotential. As above, the subscripts denote derivatives with respect the the relevant superfields, e.g. $W_n \equiv \frac{\partial W}{\partial \Phi_n}$ and K^{m^*n} is the inverse of the Kähler metric K_{m^*n}.

Using the fact that $V_F \propto e^{K/m_P^2}$ and assuming a minimal Kähler potential demonstrate the infamous η-problem of inflation is supergravity.

9.4 Eternal inflation

The scalar field gets a 'kick' from the quantum fluctuations given by $\delta\phi = H/2\pi$ [c.f. Eq. (45)] per Hubble time $\delta t = H^{-1}$.

- By comparing this 'quantum' variation $\delta\phi/\delta t$ with the classical variation of the inflaton $\dot{\phi}$, show that the classical roll becomes subdominant to the quantum fluctuations if:

$$|V'| \lesssim \frac{3}{2\pi} H^3 . \tag{93}$$

- For the Starobinsky-type potential in Eq. (86) show that the critical value $\phi_{\rm x}$, beyond which the inflaton is controlled by the above quantum diffusion, is

$$\exp(-q\phi_{\rm x}/m_P) = \frac{1}{2\sqrt{3}q\pi} \frac{\sqrt{V_0}}{m_P^2} \tag{94}$$

Find also the number $N_{\rm x}$ of remaining e-folds of slow-roll inflation after the inflaton exits the diffusion zone $\phi \lesssim \phi_{\rm x}$.

- For the original Starobinsky inflation, $q = \sqrt{2/3}$ and $V_0 = m_P^4/16\beta$, where β is a large number (of order 10^8). Show that, in this case, $N_{\rm x} = 6\sqrt{2\beta}\,\pi$.
- Based on the above, discuss the initial conditions of Starobinsky-type inflation and ways to overcome any fine-tuning problems. Also, discuss the multiverse hypothesis and how it can be realised in this setup.

9.5 Quintessential inflation

Quintessential inflation is an attempt to unify the theory of inflation with that of quintessence. The latter considers that the Universe is currently engaging in a new phase of late time inflation, driven by the quintessence scalar field. The cosmological constant is zero, as used to be assumed before dark energy was observed.

The basic assumption is that both early and late time inflationary periods are due to the same scalar field, which plays the role both of the inflaton and of the quintessence fields. The scalar potential $V(\phi)$ for the quintessential inflaton field ϕ is of runaway form, because the inflaton should not decay after the end of inflation but it should survive until the present to account for the observed dark energy. Thus, the potential should feature two flat regions: the inflationary plateau and the quintessential tail.

The energy scales of the two flat regions are different by more than a hundred orders of magnitude because

$$\frac{\rho_{\rm inf}}{\rho_{\rm DE}} \sim \left(\frac{10^{15-16}\,{\rm GeV}}{10^{-3}\,{\rm eV}}\right)^4 \sim 10^{108-112}, \tag{95}$$

where $\rho_{\rm inf}$ stands for the density scale of inflation (typically near the energy of grand unification) and $\rho_{\rm DE}$ stands for the energy scale of dark energy at present. Such a huge reduction in scale means that $V(\phi)$ suffers an abrupt massive dip after the end of inflation, behaving almost like a step function.

Consequently, soon after inflation ends, the inflaton energy density is completely dominated by the kinetic part as the potential density becomes negligible. Being oblivious of the potential, the Klein-Gordon Eq. (28) becomes:

$$\ddot{\phi} + 3H\dot{\phi} \simeq 0 \tag{96}$$

As a result, after inflation, the Universe becomes dominated by the kinetic density of the scalar field. This period is called *kination*.

- Show that, during kination, the density scales as $\rho = \rho_{\rm kin} \propto a^{-6}$ with barotropic parameter $w = 1$ (stiff fluid). Show that this implies $a \propto t^{1/3}$ and $H = 1/3t$.

- From Eq. (96) find that

$$\dot{\phi} = \sqrt{2/3}\,(m_P/t) \quad \text{and} \quad \phi = \phi_{\rm end} + \sqrt{\frac{2}{3}}\,m_P \ln\left(\frac{t}{t_{\rm end}}\right) \tag{97}$$

where 'end' denotes the end of inflation and we have assumed abrupt kinetic domination at $t_{\rm end}$.

Kination has to end and the HBB to begin before big bang nucleosynthesis (BBN) takes place. Therefore, the radiation bath of the HBB must be created after inflation. Because for radiation $\rho_\gamma \propto a^{-4}$, radiation eventually takes over, since $\rho_{\rm kin} \propto a^{-6}$. This has to happen before BBN.

There are various possibilities for reheating the Universe in quintessential inflation scenarios, e.g. instant preheating, curvaton reheating and Ricci reheating, which all assume the decay of a spectator field, which does not affect inflationary evolution and is coupled or not to the inflaton. The reheating efficiency is determined by the radiation density parameter at the end of inflation, $\Omega_\gamma^{\rm end} \equiv (\rho_\gamma/\rho)_{\rm end}$, assuming radiation is generated then. In general we have

$$(\Omega_\gamma^{\rm end})_{\rm min} \lesssim \Omega_\gamma^{\rm end} \lesssim 1\,, \tag{98}$$

which can, in principle, approach unity. The lower bound is determined by the fact that kination cannot last too long (see later).

- Show that the HBB begins (i.e. radiation takes over) at time

$$t_{\rm reh} = (\Omega_\gamma^{\rm end})^{-3/2}\,t_{\rm end} \tag{99}$$

where 'reh' stands for "reheating". Then Eq. (97) gives

$$\phi_{\rm reh} = \phi_{\rm end} - \sqrt{\tfrac{3}{2}}\,m_P \ln \Omega_\gamma^{\rm end} \tag{100}$$

- Find that $\rho_{\rm kin}^{\rm reh} = (\Omega_\gamma^{\rm end})^3 \rho_\phi^{\rm end}$, where $\rho_{\rm kin}^{\rm reh} \equiv \rho_{\rm kin}(t_{\rm reh})$. In view of Eq. (14), find that the reheating temperature is

$$T_{\rm reh} = \left[\frac{30}{\pi^2 g_*^{\rm reh}}(\Omega_\gamma^{\rm end})^3 \rho_\phi^{\rm end}\right]^{1/4} \propto (\Omega_\gamma^{\rm end})^{3/4} \tag{101}$$

- After the onset of the HBB, the field continues to roll. Show that the field freezes for $t \gg t_{\rm reh}$ at the value

$$\phi_F = \phi_{\rm end} + 2\sqrt{\tfrac{2}{3}}\left(1 - \tfrac{3}{4}\ln \Omega_\gamma^{\rm end}\right) m_P \tag{102}$$

The furthest that the field can travel in field space corresponds to the minimum value of $\Omega_\gamma^{\rm end}$. Discuss why this cannot be too small (peak in primordial gravitational waves disturbing BBN).

- The remaining potential density $V(\phi_F)$ is to become the dark energy at present. Thus, the rapid roll of the field during and after kination takes it from the inflationary plateau to the quintessential tail. The above suggests that the field runs over super-Planckian distances in field space. Explain why this can be a problem (radiative corrections, $5^{\rm th}$ force, swampland conjecture No 1.).

9.5.1 Quintessential inflation with α-attractors

Consider the following concrete model realizarion.

$$\mathcal{L} = \mathcal{L}_{\rm kin} - V(\varphi) + V_\Lambda \tag{103}$$

where $\mathcal{L}_{\rm kin}$ is given by Eq. (88) and the potential is

$$V(\varphi) = U_0\, e^{-\kappa\varphi/m_P} \tag{104}$$

with κ being the strength of the exponential. In the above, V_Λ is a constant, corresponding to a *negative* cosmological consant, such that the vacuum potential density is zero. Note that a negative cosmological constant is readily admissible in string theory. Considering that in the vacuum $\varphi \to \sqrt{6\alpha}\, m_P$, show that $V_\Lambda = e^{-\kappa\sqrt{6\alpha}} U_0$.

- Write the potential in terms of the canonical scalar field ϕ, related to φ as in Eq. (89). Then show that in the limit of inflation

$$V \simeq e^{\kappa\sqrt{6\alpha}} U_0 \left(1 - 2\kappa\sqrt{6\alpha}\, e^{\sqrt{\frac{2}{3\alpha}}\phi/m_P}\right), \tag{105}$$

  which describes the inflationary plateau, and in the limit of quintessence

$$V \simeq 2\kappa\sqrt{6\alpha}\, e^{-\kappa\sqrt{6\alpha}} U_0\, e^{-\sqrt{\frac{2}{3\alpha}}\phi/m_P} \propto e^{-\lambda\phi/m_P}, \tag{106}$$

  which describes the quintessential tail, with $\lambda \equiv \sqrt{2/3\alpha}$ being the strength of the quintessence exponsential.

- The period of kination adds some e-folds to N_*, which are $-\Delta N$, as given in Eq. (82). Considering $N_* \simeq 65$, obtain the value of n_s and r, the latter as a function of α.

- For quintessence, after thawing, the field follows the dominant exponential attractor with barotropic parameter $w_\phi = -1 + \lambda^2/3$. The observations require $-1 < w_\phi < -0.95$ for quintessence. Thus, obtain a lower bound for α and find a corresponding lower bound on r. Is there allowed parameter space?

- In view of Eqs. (95) and (102) and neglecting $\phi_{\rm end}$, obtain a relation for κ and $\Omega_\gamma^{\rm end}$, using a value of α close to its lower bound (obtained before). Then, consider the values $\Omega_\gamma^{\rm end} \sim 10^{-2}, 10^{-4}, 10^{-6}$ to obtain an estimate for the value of κ. What is the physical implication of the values found?

9.6 Ultra-slow-roll inflation and Primordial Black Holes

Suppose that there is a part of the inflaton potential, traversed during inflation, where the slope is extremely small; effectively zero. This would mean that the Klein-Gordon equation (28) reduces to

$$\ddot{\phi} + 3H\dot{\phi} \simeq 0\,. \tag{107}$$

The above, superficially looks identical to Eq. (96) in kination, but this time the inflaton field is not kinetically dominated as it is during kination. Indeed, even though $V' \to 0$, we still have $V \gg \rho_{\rm kin}$, where $V \simeq V_0 = $ constant. As a result, the Hubble parameter is given by $H \simeq \sqrt{V_0}/\sqrt{3}\, m_P = $ constant (cf. Eq, (39)). The equation of motion (107) is different from Eq. (30), which means that inflation continues but it is no more slow-roll. It is called Ultra-Slow-Roll (USR) inflation.

- Show that $\rho_{\rm kin} \propto a^{-6}$ in USR inflation.
- Find also that during inflation

  $$|\dot{\phi}| = \sqrt{2\epsilon}\, m_P H \;\Rightarrow\; \epsilon = 3\rho_{\rm kin}/\rho\,. \tag{108}$$

  Explain why the above implies $\epsilon \propto a^{-6}$ during USR inflation.
- Show that always

  $$\sqrt{\mathcal{P}_\zeta} = \frac{H}{5\pi\sqrt{2\epsilon}\, m_P}\,. \tag{109}$$

  The above is exact and is valid during USR inflation. Since, ϵ decreases during USR inflation, the curvature perturbation grows.
- Use the fact that $\mathcal{P}_\zeta$ grows during USR inflation to discuss the formation of Primordial Black Holes (PBHs) in the early Universe and how this can be facilitated by a flat patch of the inflaton potential. Discuss the possibility that these PBHs are the dark matter in the Universe, or they can be the seeds for the formation of the supermassive black holes at the centre of galaxies.