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Heat conduction models for the transient hot wire technique†

MOHAMED B. H. SASSI[1], CARLOS A. C. DOS SANTOS[2], ZAQUEU E. DA SILVA[2], JOSE M. GURGEL[2] AND JOSE P. A. JUNIOR[2]

[1]*INSAT, Departement de Physique et Instrumentation, Université 7 Novembre à Carthage, Tunisia*
[2]*Universidade Federal da Paraiba, Departamento de Engenharia Mecânica – Cidade Universitária – João Pessoa – PB – Brazil*

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In the first part of this paper, we present a transient model of radial heat conduction in infinitely large media to model the hot-wire technique. Pulse and a Heaviside heat sources are investigated. The general instantaneous solution is generated using Green's functions and is computed with a hybrid method. Thus, the estimation of the thermal properties is possible using a fitting routine or using the concept of maximum temperature rising at any point of the media. In the second part, a model including three temperature measurements is set to determine the diffusivity with a fitting procedure and an existing inverse scheme is improved to determine the conductivity.

Keywords: Heat conduction, thermal properties, hot wire technique, parameters estimation, inverse scheme, thermal diffusivity, thermal conductivity.

1 INTRODUCTION

Many physical and chemical processes in materials science and engineering are temperature dependent and include heat and mass transfer phenomena. Thus, the determination of the thermal properties of the encountered matter is essential to govern these processes. Typical thermal parameters to be evaluated in this paper are: the thermal conductivity, thermal diffusivity and the specific heat. A vast amount of experimental methods for these parameters estimation exists and each one is more or less appropriate to any type of matter. The first class is using steady-state heat flow. It has often employed the

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geometry of parallel plates or coaxial cylinders with a contained fluid or solid sample in the gap between these two armatures. At a stationary regime, two temperature measurements within the sample are sufficient to determine its conductivity by the use of the Fourier's law and the known heat flux generated by the heat source. This technique has two basic practical difficulties. The first is to take into account the interfacial heat transfer resistance at the contact between the two armatures surfaces and those of the sample. The second concerns the important convective and radiation heat losses especially when the temperature difference between the instrument and the environment is large. In addition, these methods are limited only to the measurement of the thermal conductivity. However, in the second class of methods, simultaneous estimation of the thermal properties is possible using many transient or steady periodic states. The flash method developed firstly by Parker *et al.* in reference [1] consists in irradiating an opaque finite cylindrical sample at the front face and measuring the temperature rise of the back face. Only axial heat transfer is considered in the mathematical model with a restrictive assumption of zero heat loss at the boundaries of the specimen. The mathematical solution of this model can be derived using the thermal impedance method [2]. The maximum sensitivity of this solution to the diffusivity is at the time required to attain half the maximum temperature rising. A general pulse mathematical model taking into account the radial and the axial components of heat flow is proposed by D. A. Watt in reference [3]. The solution, using results and general principles given by Carslaw and Jaeger in reference [4], is established by means of Green's functions with the assumption of constant surface conditions. Obviously, this assumption requires a linear radiation cooling law that is only true for small changes of temperature during the transient. Flash method has the advantage of avoiding the problem of thermal contact resistance between the test sample and the heat source or the detector. Eventually, one can use a laser heat pulse and the temperature rise of the back face can be detected with an infrared remote sensor. This technique is extended to semi-transparent materials using a complex mathematical model taking into account the coupled conduction-radiation heat transfer [5]. In such a case, the solution is generally derived by computational simulation. For further details the reader can refer to [6]. For granular materials such as powders and soil, porous media such as rock, gaseous, (etc), the flash method is not a viable technique. The main reasons are the manner with which the specimen is handled during the experiment and the propagation of radiation through its pores. A concise summary of various techniques used for these kinds of matter can be found in reference [7]. We focus our interest on the transient hot-wire technique; it has a widespread application to a large range of materials. It consists of using a thin cylindrical resistance delivering generally a constant heat flux in an infinitely large medium. The temperature measurement can be done at the surface of the resistance or at any point r of the medium. The mathematical solution of the heat conduction model commonly used is an asymptotic one.

This simple temperature profile is found to be logarithmic and, in terms of Fourier's number, is available only in the time interval respecting $\frac{r^2}{4\alpha t} < 1$. As a consequence, only temperature measurements for large times are explored to estimate the thermal properties and the efficiency of the method is restricted for a small range of diffusivity. The objective of the first part of this work is to develop a mathematical instantaneous solution that can be explored at any range of time. The assumption of an infinitely large medium with respect to the duration of the experiment permits to avoid the consideration of heat losses at the boundary and to obtain a simple mathematical expression of temperature compared to the one of a finite cylindrical geometry proposed in reference [3].

2 THEORETICAL MODELS FOR THE TRANSIENT HOT WIRE TECHNIQUE

2.1 General theory

In this part, the medium is considered infinite and heat transfer is assumed to be radial. To set the model, we make the following assumptions:

- The hot-wire has a negligible heat capacity and a high thermal conductivity,
- The contact between the hot-wire and the considered medium is perfect,
- The thermo-physical proprieties of the medium are constant.

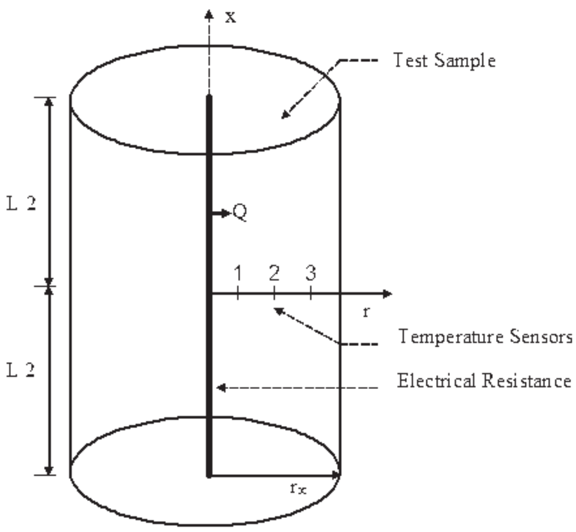

FIGURE 1
Apparatus for measurement of the thermal properties of a granular medium.

At the initial moment the medium is at the temperature T_∞. It is chosen as origin of the temperature $\theta(r,t) = T(r,t) - T_\infty$ and the model is:

$$\frac{1}{\alpha}\frac{\partial\theta}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right) = \frac{\mathrm{g(r,t)}}{\lambda} \text{ in } 0 \le \mathrm{r} \le \infty \tag{1}$$

$$\frac{\partial\theta}{\partial r} = 0 \text{ at } r = 0 \tag{2}$$

$$\theta = 0 \text{ at } r = \infty \tag{3}$$

$$\theta(r, t = 0) = 0 \tag{4}$$

Where α and λ denote respectively the thermal diffusivity and the thermal conductivity.

The term source $g(r,t) = \frac{Q}{2\pi r} f(t)$ will be defined according to the treated case of the dissipated heat flux. The term Q means the heat flux dissipated by meter of the resistance; it has the dimensions [w.s/m].

The Green's function associated with the above model is the solution of the problem [8, 9]:

$$\frac{1}{\alpha}\frac{\partial G}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = \frac{1}{\lambda}\frac{\delta(r-r')\cdot\delta(t-\tau)}{r} \quad \text{in } 0 \le \mathrm{r} \le \infty \tag{5}$$

$$\frac{\partial G}{\partial r} = 0 \quad \text{at } r = 0 \tag{6}$$

$$G = 0 \quad \text{at } r = \infty \tag{7}$$

This Green's function in terms of the modified Bessel function of order zero I_0 is given in reference [8, 9] as:

$$G(r, t/r', \tau) = \frac{1}{2\alpha(t-\tau)} I_0\left(\frac{rr'}{2\alpha(t-\tau)}\right) \exp\left(-\frac{r^2 + r'^2}{4\alpha(t-\tau)}\right) \tag{8}$$

The temperature model has a zero initial temperature field and the homogeneous boundaries conditions are zero too. Thus, the temperature is obtainable from the Green's one as follows:

$$\theta(r,t) = \frac{\alpha}{\lambda}\int_{\tau=0}^{t}\int_{r'=0}^{+\infty} r' \cdot G(r, t/r', \tau) \cdot g(r', \tau) dr' d\tau \tag{9}$$

2.2 Solution for an instantaneous heat pulse

2.2.1 Analytic solution and maximum temperature rising

Let R be the radius of the resistance. The solution will be expressed in terms of the dimensionless coordinates $r^* = r/R$, $a = \frac{r^2+R^2}{2rR} = \frac{1+r^{*2}}{2r^*} \ge 1$ and the Fourier number $\eta = \frac{R^2}{2\alpha t} r^*$. On using an instantaneous heat pulse $f(t) = \delta(t)$

$\delta(r - R)$ in equation 9, the dimensionless temperature $\theta^*(r, t) = \frac{2\pi R^2 \rho c}{Q}\theta(r, t)$ is immediately obtained as follows:

$$\theta^*(r, t) = \frac{1}{r^*}\eta I_0(\eta)\exp(-a\eta) \tag{10}$$

The maximum temperature rising at any point r^* is given by the value of $\eta = \eta_{\max}$ respecting:

$$(1 - a\eta)I_0(\eta) + \eta I_1(\eta) = 0 \tag{11}$$

Where, I_1 is the modified Bessel function of order 1.

We have $0 \leq \frac{I_1(x)}{I_0(x)} \leq 1$ for all positive real x. Thus the above equation leads to the useful inequality:

$$\frac{1}{a} < \eta_{\max} < \frac{1}{a-1} \tag{12}$$

It is interesting to point out that only the relative coordinate $a(r^*)$ is required to determine $\eta_{\max}$ using equation 11. $\eta_{\max}$ is plotted in Figure 2 as a function of the parameter r^*. It is calculated using this iterative scheme:

$$\eta_{k+1} = \frac{1}{a}\left(1 + \frac{\eta_k I_1(\eta_k)}{I_0(\eta_k)}\right) \tag{13}$$

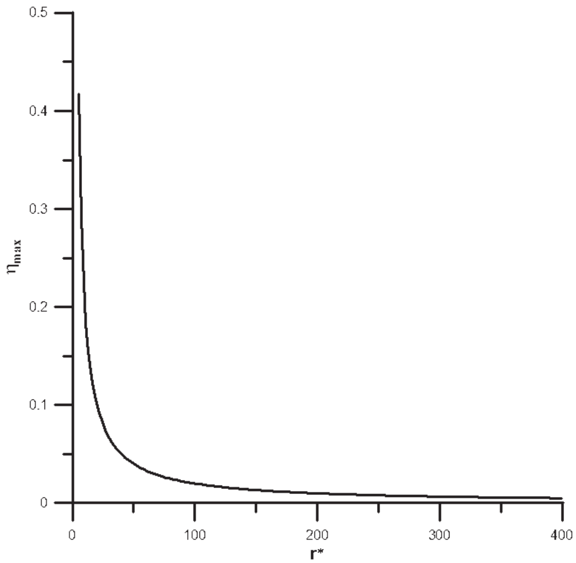

FIGURE 2
$\eta_{\max}$ for pulse heat flux.

The stability of the above scheme is guaranteed for values $a > \frac{1+\sqrt{5}}{2}$, since we have:

$$\frac{\partial \eta_{k+1}}{\partial \eta_k} = \frac{\eta_k}{a}\left(1 - \left(\frac{I_1(\eta_k)}{I_0(\eta_k)}\right)^2\right) < \frac{\eta_k}{a} < \frac{1}{a(a-1)} < 1 \qquad (14)$$

2.2.2 Estimation of the thermal properties

From equation 10, the temperature can be seen as a function of two parameters $\theta(\alpha, \rho c)$. Let $\eta_{\rho c}$ denote the value of η at which the sensibility coefficient $S_{\rho c}(t) = \frac{\partial \theta}{\partial \rho c}$ is maximum. The derivative with respect to time can be expressed $\frac{\partial}{\partial t} = -\frac{\eta}{t}\frac{\partial}{\partial \eta}$. Thus, one can easily demonstrate that $\frac{\partial S_{\rho c}}{\partial t} = 0$ leads to $\frac{\partial \theta}{\partial \eta} = 0$ and so that $\eta_{\rho c} = \eta_{\max}$. Suppose now that one temperature history is measured at a position r. The algorithm of estimation is:

- At first, the parameter a should be calculated using the value of r and R. Then $\eta_{\max}$ is computed using any stable iterative routine to solve equation 11. Finally, the diffusivity is determined using the time $t_{\max}$ required to reach the maximum temperature rise at r as follows:

$$\alpha = \frac{Rr}{2\eta_{\max} t_{\max}} \qquad (15)$$

  It is important to note that is not necessary to know the value of Q in order to determine the thermal diffusivity.

- The measured maximum temperature rise $\theta_{\max}$ and its theoretical value determined using equation 10 at $\eta_{\max}$ permit the determination of the product of the density ρ and heat capacity c from the relationship:

$$\rho c = \frac{Q}{2\pi R^2} \cdot \frac{\eta_{\max} I_0(\eta_{\max}) \exp(-a\eta_{\max})}{r^* \theta_{\max}} \qquad (16)$$

- The conductivity is determined using the follows relationship:

$$\lambda = \alpha \cdot \rho c = \frac{Q}{4\pi} \cdot \frac{I_0(\eta_{\max}) \exp(-a\eta_{\max})}{t_{\max} \theta_{\max}} \qquad (17)$$

It is important to notice that one can also use more than the single maximum temperature measurement to estimate the thermal conductivity. Eventually, integrating the temperature at any given point r over a time interval $[t_1, t_2]$, we obtain a useful equation to determine the conductivity:

$$\int_{t_1}^{t_2} \theta(r, t)dt = \frac{Q}{4\pi\lambda} \int_{\eta_2}^{\eta_1} \frac{1}{x} I_0(x) \exp(-ax)dx \qquad (18)$$

The integral in the left-hand side of this equality should be approximated using measured temperatures at the interval $[t_1, t_2]$. The second integral in the right-hand side is still dependent on the diffusivity via the values of the variables

η_1 and η_2. When using a large interval of time $[t_1 = 0, t]$, in such a case the above relationship becomes as follows:

$$\int_0^t \theta(r,t)dt = \frac{Q}{4\pi\lambda}\int_\eta^{+\infty}\frac{1}{x}I_0(x)\exp(-ax)dx \tag{19}$$

The evaluation of the integral of the right-hand side will be discussed in the subsequent section.

The parameter estimation can be genuinely useful when using a fitting procedure such as least squares as a maximum likelihood estimator [10, 11, 12]. It provides parameters, error estimates on the parameters and a statistical measure of goodness-of-fit. One of the conditions to make the estimation succeed is that the sensibility coefficients of the measured temperature to the parameters to be estimated must be nonlinear dependents. Using the analytic solution, one can easily demonstrate that the coefficients $S_\alpha(t) = \frac{\partial\theta}{\partial\alpha}$ and $S_\lambda(t) = \frac{\partial\theta}{\partial\lambda}$ respect:

$$S_\alpha(t) = -\frac{\lambda}{\alpha}S_\lambda(t) - \frac{Q}{2\pi\lambda rR}\eta\frac{\partial}{\partial\eta}(\eta I_0(\eta)\exp(-a\eta)) \tag{20}$$

These sensibility coefficients S_α and S_λ are linearly dependent when η is zero and at $\eta = \eta_{\max}$. These correspond to infinitely large time and to the time of maximum temperature rise $t_{\max}(r)$. So the inversion is successful when using large number of measurements at the periods of time that correspond to the increasing or decreasing temperature.

2.2.3 Experimental design and optimization

The physical parameters of the apparatus are R, Q, the position of the first sensor r, the geometric large r_∞ with witch the medium can be considered infinitely large and finally the period Γ of the measurements. They should be dimensioned with respect to these considerations:

a) At first, suppose that the resistance radius R is chosen. The hypothesis of the model neglecting its inertial capacity compared to the one of the domain $[0, r]$ implies:

$$r^* \gg \frac{\rho_r c_r}{\rho c} \tag{21}$$

It is important to note that for bad heat conductors such as powders, the relative position error of the sensor can imply crucial errors of the estimated parameters. Thus r^* must be chosen as large as possible to yield a minimum of relative position error.

b) To ensure accuracy, the maximum range of temperature measured by the sensor at the position r must be larger than the measurement noise.

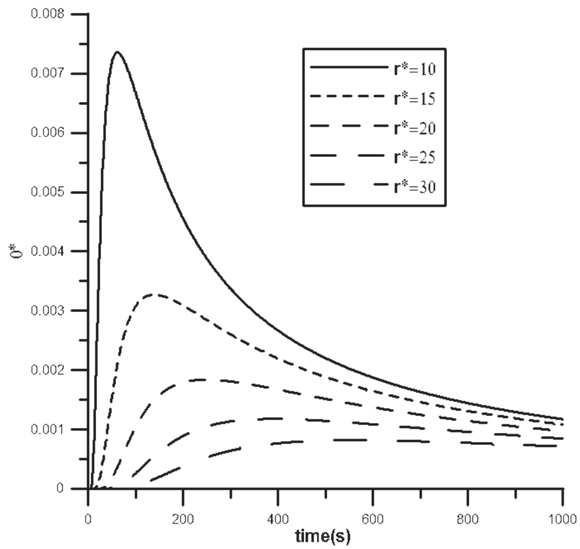

FIGURE 3
Temperature profiles for pulse heat flux.

Thus, fixing a desired maximum of temperature rising $\theta_{\max}(r,t) = \theta_{\max}$, we can determine an appropriate value of Q as follows:

$$Q = \frac{2\pi\rho c R r \theta_{\max}}{\eta_{\max} I_0(\eta_{\max}) \exp(-a\eta_{\max})} \tag{22}$$

Figures 3 and 4 depict the dimensionless temperature histories and the maximum temperature reached for different dimensionless radii.

c) The assumption of a constant temperature at $r = +\infty$ permits to determine r_∞. Eventually, consider that the maximum temperature rise at this point is small compared to the one at the sensor position r:

$$\frac{\theta^*_{\max}(r_\infty)}{\theta^*_{\max}(r)} = \frac{r^*}{r^*_\infty} \frac{\eta_{\infty\max} I_0(\eta_{\infty\max}) \exp(-a(r^*_\infty) \cdot \eta_{\infty\max})}{\eta_{\max} I_0(\eta_{\max}) \exp(-a(r^*) \cdot \eta_{\max})} \ll 1 \tag{23}$$

Considering equation 12 for large value of $r_\infty \gg R$, we have η_∞max $\approx 1/a$. Combining equations 22 and 23 at the limits $a \rightarrow \infty$, a stricter approximation to the above criterion can be obtained:

$$r_\infty \gg \sqrt{\frac{Q}{e\pi\rho c\theta_{\max}}} \approx 0.3422\sqrt{\frac{Q}{\rho c\theta_{\max}}} \tag{24}$$

d) The period of the experiment Γ is determined to satisfy the same assumption of an infinitely medium. It must be taken much less than

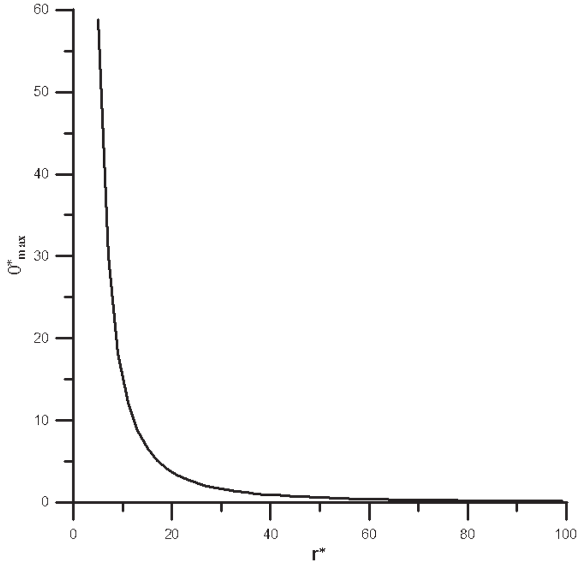

FIGURE 4
Maximum of temperature reached for pulse heat flux.

the time $t_{\infty max}$ at which the temperature at r_∞ will be at its maximum:

$$\Gamma \ll t_{\infty\,\max} = \frac{Rr_\infty}{2\alpha\eta_{\infty\,\max}} \approx \frac{r_\infty^2}{4\alpha} \tag{25}$$

2.3 Solution for a Heaviside heat flux

2.3.1 Analytic solution and maximum temperature rising

Consider now a constant heat flux released by the hot wire during a period of time τ_1. Applying equation 9 for which we set $f(t) = (Y(t) - Y(t - \tau_1))$ $\delta(r - R)$. The dimensionless temperature $\theta^*(r, t) = \frac{4\pi\lambda}{Q}\theta(r, t)$ is found to be:

$$\theta^*(r, t) = \begin{cases} \displaystyle\int_\eta^{+\infty} \frac{1}{x} I_0(x) \exp(-ax)dx & if\ \ t \le \tau_1 \\ \displaystyle\int_{\eta(t)}^{\eta(t-\tau_1)} \frac{1}{x} I_0(x) \exp(-ax)dx & if\ \ t > \tau_1 \end{cases} \tag{26}$$

When the temperature is differentiated with respect to t, one can prove easily that $t_{\max}$ obeys:

$$\eta(t - \tau_1) I_0(\eta(t - \tau_1)) \exp(-a\eta(t - \tau_1)) = \eta(t) I_0(\eta(t)) \exp(-a\eta(t)) \tag{27}$$

Note that the value of $t_{\max}$ is greater than τ_1 and we have the relation:

$$\eta(t_{\max} - \tau_1) = \frac{\eta(t_{\max})}{1 - b\eta(t_{\max})} \tag{28}$$

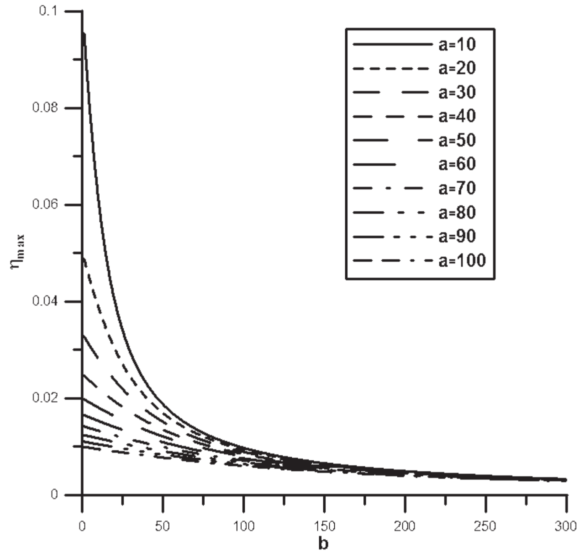

FIGURE 5
$\eta_{\max}$ for Heaviside pulse heat flux.

Where the dimensionless parameter b has a Fourier number mean defined as $b = \frac{2\alpha\tau_1}{rR}$. It is *a priori* still unknown since it depends of the diffusivity. Figure 5 presents $\eta(t_{\max} - \tau_1)$ for different values of the parameters a and b. It is computed using this stable iterative scheme:

$$\eta_{k+1} = \sqrt{\frac{1}{ab}(1 + b\eta_k) \ln\left[(1 + b\eta_k)\frac{I_0(\eta_k)}{I_0\left(\frac{\eta_k}{1+b\eta_k}\right)}\right]} \tag{29}$$

2.3.2 Estimation of the thermal properties

The temperature is a function of the two parameters $\theta(\lambda, \alpha)$. The estimation of the thermal properties using the maximum temperature rise is as follows:

- Assuming that the time of maximum temperature rise is determined. Introducing the parameter $\beta = \frac{\eta(t_{\max}-\tau_1)}{\eta(t_{\max})} = \frac{t_{\max}}{t_{\max}-\tau_1} > 1$, then $\eta_{\max} = \eta(t_{\max})$ can be computed from the equation 27 that in terms of β takes the form:

$$I_0(\eta) \exp(a \cdot (\beta - 1) \cdot \eta) = \beta I_0(\beta\eta) \tag{30}$$

The diffusivity is determined from the relationship:

$$\alpha = \frac{Rr}{2\eta(t_{\max}) \cdot t_{\max}} \tag{31}$$

In this case too, it is not necessary to know the value of Q in order to determine the thermal diffusivity.

- When the first derivative with respect to time of the sensitivity coefficient $S_\lambda(t) = \frac{\partial\theta}{\partial\lambda}$ is set equal to zero, it is easy to prove that this coefficient is maximum at $t = t_{\max}$. Then, the thermal conductivity should be calculated using the measured maximum temperature rise from the relationship:

$$\lambda = \frac{Q}{4\pi} \cdot \frac{\theta^*(t_{\max})}{\theta_{\max}} \tag{32}$$

- The product of the heat capacity and the density is obtained as follows:

$$\rho c = \frac{\lambda}{\alpha} = \frac{Q}{2\pi} \cdot \frac{\eta(t_{\max}) \cdot t_{\max}}{Rr} \cdot \frac{\theta^*(t_{\max})}{\theta_{\max}} \tag{33}$$

It is important to note that since we have $\delta(t) = \frac{\partial Y(t)}{\partial t}$, the mathematical solution of an instantaneous heat pulse can be derived from the Heaviside one by simple derivation with respect to time. As a consequence, when using large value of τ_1, one can calculate from the measured temperature the parameter $\Theta = \frac{\partial\theta}{\partial t}$ and apply the estimation procedure developed for the case of a heat pulse relative to Θ. Eventually, for a Heaviside heat flux, when setting the second derivative equal to zero one can demonstrate the existence of a point of inflection at $t_0 < \tau_1$. Let η_0 denotes the value $\eta(t = t_0)$. The zero second derivative implies that η_0 respects equation 11. As it is seen in the above section, only the parameter a is required to determine η_0 and $\frac{\partial^2\theta}{\partial t^2}$ the thermal diffusivity is $\alpha = \frac{Rr}{2\eta_0 \cdot t_0}$.

The calculation of the sensitivity coefficients leads to the same complication of their linearity at the maximum temperature rise and for large time of measurements. The experiment design and its optimization are the same ones as for a spontaneous heat pulse developed above, except that they should be expressed in terms of the analytical solution given by equation 26.

2.4 A hybrid method to compute the integral $\int_\eta^{+\infty} \frac{1}{x} I_0(x) \exp(-ax) dx$

This integral is encountered in equation 19 that enables to determine the conductivity in the case of a heat pulse. In the case of a heat step and during the time $t \leq \tau_1$, the temperature profile is reduced to:

$$\theta^*(r,t) = \int_\eta^{+\infty} \frac{1}{x} I_0(x) \exp(-ax) dx \tag{34}$$

This integral is improper because its upper limit is $+\infty$. Many techniques can be found in reference [10] to perform its numerical integration. In this work we have developed a concise hybrid solution that consists of expanding the integral in the form:

$$\int_\eta^{+\infty} \frac{1}{x} I_0(x) \exp(-ax) dx = (f_1(\eta) I_0(\eta) + \eta f_2(\eta) I_1(\eta)) \exp(-a\eta) \tag{35}$$

The derivative of both functions of this equality with respect to η leads to:

$$f_1 = -\eta\left(\frac{\partial f_2}{\partial \eta} - af_2\right) \tag{36}$$

$$\eta\left(\frac{\partial f_1}{\partial \eta} - af_1\right) + \eta^2 f_2 + 1 = 0 \tag{37}$$

The combination of these two equations implies:

$$-\eta^2\frac{\partial^2 f_2}{\partial \eta^2} + \eta(2a\eta - 1)\frac{\partial f_2}{\partial \eta} + \eta(a + \eta(1 - a^2))f_2 + 1 = 0 \tag{38}$$

f_2 is determined as an infinite series:

$$f_2(\eta) = \sum_{n=0}^{+\infty}\frac{b_n}{\eta^n} \tag{39}$$

Substituting this series into equation 38, we find:

$$\begin{cases} b_0 = b_1 = 0, b_2 = \dfrac{1}{a^2 - 1} \\ (a^2 - 1)b_{n+2} + a(2n + 1)b_{n+1} + n^2 b_n = 0 \quad \textit{for } n \geq 1 \end{cases} \tag{40}$$

The series $f_2(\eta = 1) = \sum_{n=0}^{+\infty} b_n f_2(\eta = 1) = \sum_{n=0}^{+\infty} b_n$ and $f_2(\eta < 1)$ are tested by computation. They are found to be convergent only for large values of the parameter a, and their rate of convergence are not so powerful. Therefore we have developed a trick by transforming the above series f_1 and f_2 by other equivalent series whose convergences are sufficiently rapid for suitable value of a. Setting $\psi(x) = \frac{I_0(x)}{x}$ and integrating by part the considered integral:

$$\theta^*(r, t) = \int_{\eta}^{+\infty} \psi(x)\exp(-ax)dx = -\frac{1}{a}\left[\sum_{n=0}^{+\infty}\frac{1}{a^n}\psi^{(n)}(x)\exp(-ax)\right]_{\eta}^{+\infty} \tag{41}$$

The successive function ψ derivatives are determined in such form:

$$\psi^{(n)}(x) = \frac{1}{x^{n+1}}[M_n(x)I_0(x) + xN_n(x)I_1(x)] \tag{42}$$

M_n and N_n are polynomial functions having integer coefficients independent of the parameter a. They satisfy the recurrence relations:

$$\begin{cases} M_0(x) = 1, N_0(x) = 0 \\ M_{n+1}(x) = -(n + 1)M_n(x) + x\dfrac{\partial M_n(x)}{\partial x} + x^2 N_n(x) \\ N_{n+1}(x) = -(n + 1)N_n(x) + \dfrac{\partial N_n(x)}{\partial x} + M_n(x) \end{cases} \tag{43}$$

It can be demonstrated by recurrence that the degrees d° of the above polynomials functions respect:

$$\begin{cases} d^\circ M_n(x) \leq n \\ d^\circ N_n(x) \leq n-1 \end{cases} \tag{44}$$

Since we have for all real $x \gg 1$, $I_n(x) \approx \frac{1}{\sqrt{2\pi x}} \exp(x)$ so, in the temperature expression of equation 41, the limit of the series at $+\infty$ is zero so that the temperature profile is:

$$\theta^*(r,t) = \left[I_0(\eta) \sum_{n=0}^{+\infty} \frac{M_n(\eta)}{(a\eta)^{n+1}} + \frac{1}{a} I_1(\eta) \sum_{n=0}^{+\infty} \frac{N_n(\eta)}{(a\eta)^n} \right] \exp(-a\eta) \tag{45}$$

This series is not convergent for all values of $a\eta$, especially for values $\eta < 1$. So, the above formula is recommendable only for values $\eta > 1$. Introducing now the integral:

$$\begin{aligned} \xi(a) &= \int_1^{+\infty} \psi(x) \exp(-ax) dx \\ &= \left[I_0(1) \sum_{n=0}^{+\infty} \frac{M_n(1)}{a^{n+1}} + I_1(1) \sum_{n=0}^{+\infty} \frac{N_n(1)}{a^{n+1}} \right] \exp(-a) \end{aligned} \tag{46}$$

Figure 6 presents the computed value of the logarithm of ξ(a) using the first 100 terms of the summation and the convergence is found to be correct. Finally

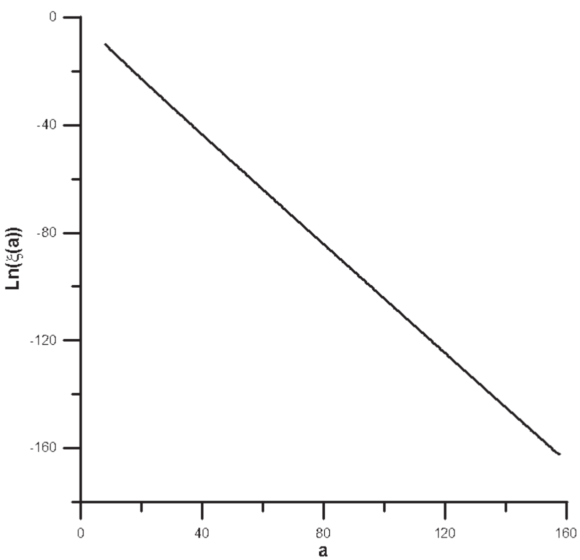

FIGURE 6
Integral $\int_1^{+\infty} \frac{1}{x} I_0(x) \exp(-ax) dx$ as a function of the parameter a.

the temperature can be set in the following form in which the integral is calculated using Gauss-Legendre method [10]:

$$\theta^*(r,t) = \xi(a) + \int_{\eta(t)}^{1} \frac{1}{x} I_0(x) \exp(-ax)dx \tag{47}$$

Temperatures functions of time for different values of the reduced radius using respectively $\tau_1 = 1500$s and $\tau_1 = +\infty$ are given in Figures 7 and 8.

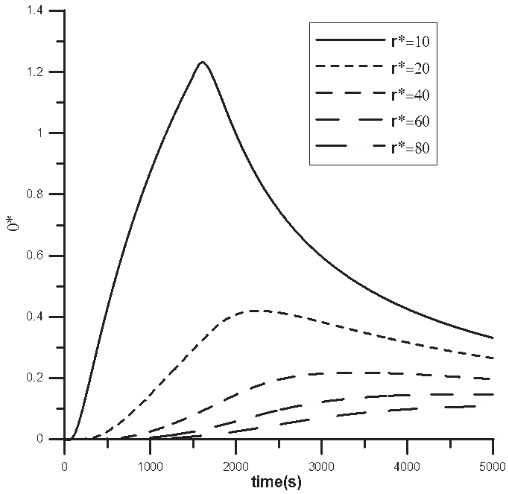

FIGURE 7
Temperature profiles for Heaviside heat flux with $\tau_1 = 1500$s.

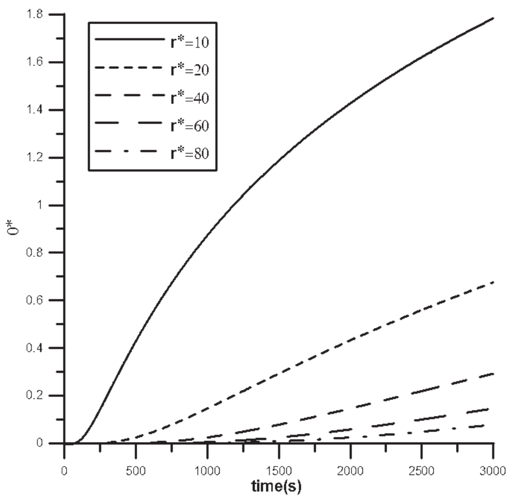

FIGURE 8
Temperature profiles for constant heat flux.

3 ESTIMATION OF THE THEMAL PROPERTIES USING THREE SENSORS

In this second part of the paper, the estimation of the thermal diffusivity and conductivity is not simultaneous, but using two different procedures. Three temperature measurements inside the medium are required to permit the estimation. The advantage of this procedure is that it is not dependent on the dimension of the resistance, the form of the delivered heat flux and the heat losses from the sample.

3.1 The thermal diffusivity estimation

The model consists of using three histories of temperature measurements at different radial positions $r_1 < r_3 < r_2$. The direct model governing the heat conduction transfer in the region between the sensors 1 and 2 is:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{48}$$

The measured temperatures of the sensors 1 and 2 serve as boundary conditions:

$$\begin{cases} T(r_1, t) = F_1(t) \\ T(r_2, t) = F_2(t) \end{cases} \tag{49}$$

The initial condition is:

$$T(r, 0) = F_0(r) \tag{50}$$

The third measured temperature $F_3(t)$ will be used to estimate the diffusivity. The maximum likelihood estimate of the diffusivity is obtained by minimizing the quantity:

$$\chi^2 = \sum_{n=1}^{M}(F_3(t_n) - T(r_3, t_n))^2 \tag{51}$$

We have used the Gauss method of minimization. Eventually to remove instability and reduce oscillations, we add the Levenberg-Marquardt parameter noted μ that is less than unity. The iterative formula to compute the diffusivity is as follows:

$$\alpha_{k+1} = \alpha_k + \sum_{n=1}^{M}\frac{S_n}{\mu^k + \sum_1^M S_n^2}(F_3(t_n) - T(r_3, t_n)) \tag{52}$$

Derivative of the system of equations (48), (49) and (50) with respect to α vanish:

$$\frac{\partial S}{\partial t} - \alpha \Delta S = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{53}$$

$$\begin{cases} S(r_1, t) = 0 \\ S(r_2, t) = 0 \end{cases} \tag{54}$$

$$S(r, 0) = 0 \tag{55}$$

A crucial remark to note is that $\alpha = \infty$ is a stationary point of convergence of the suite α_{k}. Eventually, the system of equations (48), (49) and (50), has a natural constraint on the range of temperature, because T must be between F(r_1,t) and F(r_2,t) independently of the value of the diffusivity α_{k}. That means that $\frac{\partial T}{\partial t}$ is bounded, too. Now, If we set at any step of the iterative procedure $\alpha_{\mathrm{k}} = \infty$, we will have $\frac{1}{\alpha_k}\frac{\partial T}{\partial t} = 0$ and obviously the system of equations (53), (54) and (55), leads to $S_{\mathrm{n}}(\alpha_{\mathrm{k}})$=0. Finally, equation 52 leads to $\alpha_{\mathrm{k+1}} = \alpha_{\mathrm{k}} = \infty$. In conclusion, a good estimate of the initial value of the diffusivity is required to avoid divergence. It is important to note that with such model we can only estimate the thermal diffusivity. To prove this, we take the derivative of the model (Equations 48, 49 and 50) with respect to λ, ρc and α to give the sensitivity equations of the coefficients S_λ, $S_{\rho c}$ and S_α. The result is:

$$\alpha\left[\frac{\partial S_\alpha}{\partial t} - \alpha \Delta S_\alpha\right] = \lambda\left[\frac{\partial S_\lambda}{\partial t} - \alpha \Delta S_\lambda\right] = -\rho c\left[\frac{\partial S_{\rho c}}{\partial t} - \alpha \Delta S_{\rho c}\right] = \frac{\partial T}{\partial t} \tag{56}$$

The boundaries and initial conditions are zero for all the coefficients. So, these coefficients are linearly dependent:

$$\alpha S_\alpha = \lambda S_\lambda = -\rho c S_{\rho c} \tag{57}$$

As a conclusion, we can say that these parameters cannot be simultaneously estimated.

Now, we assume that the temperature measurement errors are additive, having a zero mean, independent, with a normal distribution and a constant standard deviation σ_{T}. The standard deviation of the estimated diffusivity $\hat{\alpha}$ is obtained approximately using the statistical analysis given by Beck and Arnold in reference [11]:

$$\sigma_\alpha^2 = \frac{\sigma^2}{\sum_1^M S_n^2} \tag{58}$$

The confidence interval at the 99% confidence level for the estimated diffusivity is:

$$\text{Probability}(\hat{\alpha} - 2.57 \cdot \sigma_\alpha \le \alpha \le \hat{\alpha} - 2.57 \cdot \sigma_\alpha) = 99\% \tag{59}$$

To illustrate this procedure, we have made measurements using an iron cylindrical tube full of activated carbon. The inner radius of the tube is 50 mm

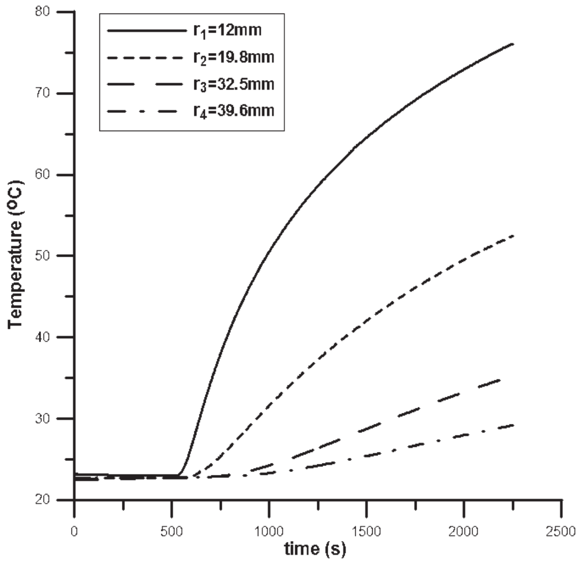

FIGURE 9
Temperature measurements for constant heat flux.

and the resistance at the center of the tube has 0.1 mm of diameter and delivers a constant heat flux $Q = 3\text{W}/250$ mm. The temperature measurements at four points are presented in Figure 9. The use of the temperature history at $r = 12$mm leads to a divergent estimation for different initial estimates of the diffusivity. This is due to the relative position error that is so large. But when using the three radii far from the resistance the estimation is convergent and successful. The variance of the temperature errors is supposed constant $\sigma_T = 0.01°$C and the estimated diffusivity and its covariance are $\hat{\alpha} = 3.498\,10^{-7} m^2 s^{-1}$ and $\hat{\sigma}_\alpha = 3.498\,10^{-10} m^2 s^{-1}$.

The diffusivity is also estimated using the algorithm of heat pulse. Figure 10 presents the derivative with respect to time of the measured temperatures. The time of maximum rising of Θ for the second and the third sensors are obtained from the curves of this figure. Their values are respectively $t_{02} \approx 860$s and $t_{03} \approx 1340$s. Using equation 11, the resulting values of the parameter $\eta_{\max}$ are $\eta_{02} \approx 5.0505\,10^{-3}$ and $\eta_{03} \approx 3.0769\,10^{-3}$. Finally, the diffusivity is calculated using:

$$\alpha = \frac{R}{2} \cdot \frac{1}{t_{03} - t_{02}} \cdot \left(\frac{r_3}{\eta_{03}} - \frac{r_2}{\eta_{02}} \right) \tag{60}$$

The resulting value is $\alpha = 3.4594\,10^{-7} m^2 s^{-1}$. Note that the difference of this value with the one estimated using three sensors is less than 3% .

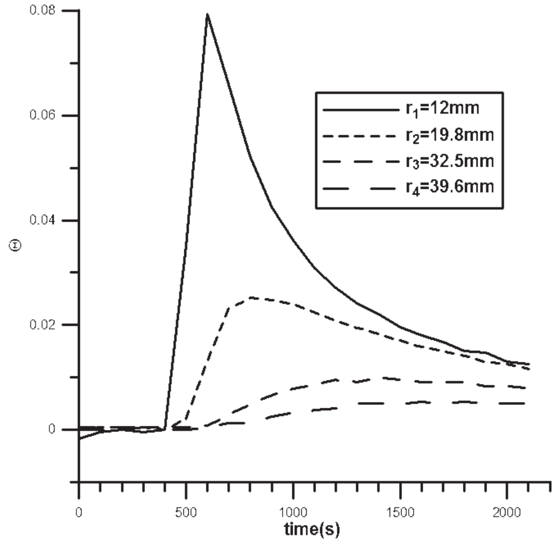

FIGURE 10
Time derivative of measured temperatures.

3.2 The thermal conductivity estimation

The known value of the heat flux dissipated by the resistance at $r = R$ is used as a boundary condition. First we need to determine the temperature profile close to the electric resistance using an inverse scheme. The derivation of this space marching scheme consists at first of the use of a variable transformation $y = lnr$, so the heat conduction equation leads to:

$$\frac{\partial T}{\partial t} = \alpha e^{-2y}\frac{\partial^2 T}{\partial y^2} \tag{61}$$

The thermal conductivity is estimated using the relationship:

$$\lambda = -\frac{Q}{2\pi \left.\frac{\partial T}{\partial y}\right|_{y_0}} \tag{62}$$

To obtain a stable inverse scheme, we introduce a bias for the temporal term used first by M. Sassi and M. Raynaud in reference [12]. It consists to make at each node i of the spatial grid an approximation such as:

$$\left.\frac{\partial T}{\partial t}\right|_i = \frac{1}{2}\left(\left.\frac{\partial T}{\partial t}\right|_{i-1} + \left.\frac{\partial T}{\partial t}\right|_{i+1}\right) \tag{63}$$

In addition, the spatial term is truncated using an explicit scheme, the result is:

$$T_{i-1}^n - \frac{2}{1+a_i}T_i^n + T_{i+1}^n = \frac{a_i}{1+a_i}(T_{i-1}^{n+1} + T_{i+1}^{n+1}) \tag{64}$$

Where n denotes the discreet instant and the set of a_i is given in terms of the time step Δt and the spatial step Δy as follows:

$$a_i = \frac{(\Delta y)^2}{2\alpha \Delta t} e^{2y_i} \tag{65}$$

The space marching molecule is derived using a constant step Δy. As a result, the distribution of the real nodes between r_1 and r_2 has an exponential behavior:

$$r_n = r_1 \cdot e^{(n-1)\Delta y} \tag{66}$$

This scheme is stable, the proofs and calculation details are discussed by Sassi and Raynaud in reference [12]. A concise methodology to compare and evaluate the consistency of inverse schemes is proposed by M. Raynaud and J. V. Beck in reference [13]. Herein, we will evaluate the consistency through the mathematical bias E introduced in this back scheme:

$$E = e^{-2y} \cdot O\big((\Delta y)^2\big) + \frac{1}{\alpha}(O(\Delta y) + O(\Delta t)) \tag{67}$$

It is not a linear function of the grid steps, unless it is a consistent one. In fact, E vanishes when both of the increments are extended to zero. However, the exponential term can be written in terms of the radial coordinates $\frac{1}{r^2}$. It is easily seen that E$\rightarrow \infty$ as r$\rightarrow$ 0. Obviously, this scheme is inaccurate when the inverse calculus is accomplished with reduced radii. In addition, the algorithm of the inverse procedure involves the concept of future temperatures [14, 15]. As a result of the consideration of propagation of bias, the number of nodes of the spatial grid, the time step and the number of future temperature must be optimized with simulated tests. This inverse problem is widely discussed in literature [13, 14], it is found to be efficient under the following condition:

$$\frac{\alpha \Delta t}{(r_1 - R)^2} > 0.02 \tag{68}$$

Simulated experimental temperature measurements are made using Heaviside flux and data given by Table 1.

A noise term $x\upsilon$ is added to the exact temperatures at r_1, r_2 and r_3. To respect the 99% confidence interval, the modulus $|x|$ must be less than 2.57, in this case we have taken $x = 0.2$. The noise υ is randomly generated using the GASDEV-RAN1 subroutine from reference [10]. The standard deviation

Q(w/m)	$\alpha(\mathrm{m}^2 \cdot \mathrm{s}^{-1})$	$\lambda(\mathrm{w} \cdot \mathrm{m}^{-1} \cdot \mathrm{K}^{-1})$	R(mm)	r_1(mm)	r_2(mm)	r_3(mm)	τ_1(s)
10^3	10^{-7}	0.2	5	15	30	40	15000

TABLE 1
Conditions of the experimental simulation.

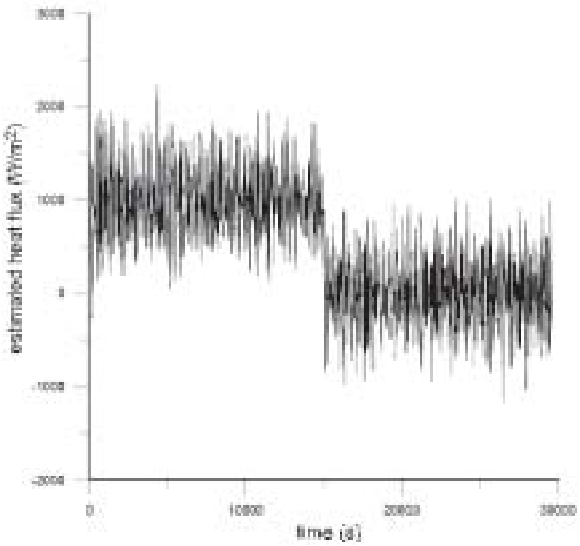

FIGURE 11
Estimated heat flux using noisy data.

of temperature is taken $\sigma_T = 0.01°C$. We have run the inverse procedure with different node number to determine the one that introduces the minimum of bias. 21 nodes are found. Figure 11 presents the estimated heat flux. The estimated conductivity and its standard deviation are respectively $\hat{\lambda} = 0.2487\ \mathrm{w} \cdot \mathrm{m}^{-1} \cdot \mathrm{K}^{-1}$ and $\sigma_\lambda = 0.2060\ \mathrm{w.m}^{-1} \cdot \mathrm{K}^{-1}$. These results show that the estimation of the conductivity is possible using this inverse scheme, unless the procedure is delicate, and its degree of accuracy is not powerful.

4 CONCLUSION

A model of infinitely cylindrical medium is proposed to simulate the transient-hot-wire technique. The instantaneous solution of heat pulse and Heaviside step are derived using Green's functions. They involve the modified Bessel functions of orders 0 and 1 and improper integral formulation. In the first section we have presented the mathematical tools needed to set numerical routines for the thermal parameters estimation. As we can see, the estimation may be done using a fitting procedure or the concept of measured maximum temperature reached in one point of the medium. In the second part, a first model using three sensors is set to estimate only the diffusivity using least-squares sense adding the Levenberg-Marquardt parameter. It has been found accurate. For the second model, the study of the back scheme bias shows that the conductivity estimation is inaccurate when using reduced resistance radius. In addition, the irregular space mesh used in the cylindrical geometry and the time step must be optimized with simulated experiments that make the estimation delicate.

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