AS. Simons foundation 27/03/25.

- $Q_X(a,b) = \langle \underbrace{a \cup b}_{\in H^4(X)}, [X] \rangle \in \mathbb{Z}$. (need orientation)

- ~~definite~~ form is hard BUT Donaldson ($Q \simeq \pm I_m$)

  If Q is odd $n\langle 1 \rangle \oplus m\langle -1 \rangle$.

We will focus on:

Thm [Baykur - Stipsicz - Szabó, '24] Suppose that X is closed, connected, oriented, topological, 4-mfd, Q_X is odd & $\pi_1(X) \cong \mathbb{Z}/4m+2$, for some $m \geq 0$. Then (with potential 7 exceptions for fixed m), if X is smoothable then it admits ∞ smooth structures.

Rmk: we don't know for $L_{4m+2} \# n\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ $n = 1, \ldots, 7$.

Who are this L_{4m+2} guys?

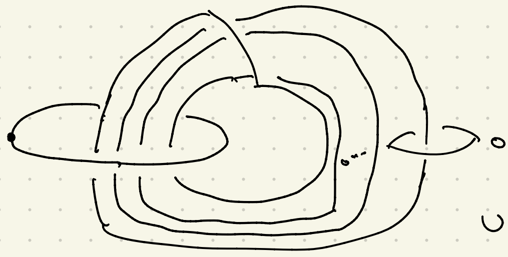
Take $a: S^2 \to S^2$ antipodal map

$a \times a: S^2 \times S^2 \to S^2 \times S^2$ is orientation preserving

$L_2: S^2 \times S^2 / \underbrace{a \times a}_{\substack{\text{orientation} \\ \text{preserving free} \\ \text{involution.}}}$ and $\pi_1(L_2) \cong \mathbb{Z}/2\mathbb{Z}$

More in general $\pi_1 \cong \mathbb{Z}/n\mathbb{Z}$

L_n is



$\cup\, h_3 \cup h_4$

L_2 is 

now i modout by $a \times a$?

now we distinguish? SW

def: The SW function of a closed, oriented, smooth four manifold X is a map $SW_X : H^2(X,\mathbb{Z}) \to \mathbb{Z}$
which counts solutions of particular PDE associated to a cohomology class (via a Spin^c structure) up to simmetry.

rmk: this count can be nontrivial only if

$$d = \frac{1}{4}\left(K^2 - (3\sigma(X) + 2\chi(X)\right) \left(\equiv b_2^+(X) + b_1(X) + 1 \ (\mathrm{mod}(2)\right)$$

is non negative and even. (Some $\mathbb{CP}^\infty$)

Dumb rmk: if $\pi_1(X) = \{e\}$ we need $b_2^+(X)$ odd.

Consider $\pi_1 X = \mathbb{Z}/2\mathbb{Z}$

Suppose that X, Y are 4-mfd with $\pi_1 \simeq \mathbb{Z}/2\mathbb{Z}$. $\tilde{X}, \tilde{Y}$ un.cov.
If X & Y are diffeo then $\tilde{X}, \tilde{Y}$ are diffeo.

Q: Why need $\mathbb{Z}/2\mathbb{Z}$? Something for $\tilde{X}$...

Where to look for exotica?
- $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ & b_2^+ odd
  Rafael.
- $\pi_1(X) = 0$ con $b_2^+ = 4k+1$ con un'involuzione libera che preserva orientazione.

(Il quoziente avrà $b_2^+ = 2k$ pari (perché???) quindi
$d \equiv 1\ (2) \Rightarrow SW = 0$.)

Special Case: A complex surface is a FPP if $b_1=0$ & $b_2^+=b_2=1$ (it means $H^*(\mathbb{CP}^2,\mathbb{Q})\simeq H^*(FPP)$) but is not $\mathbb{CP}^2$ (examples please?)

Exactly 50 (100 up to biholo) Understood through π_1

Mumford
Prasad – Yeung
Cart – Steger

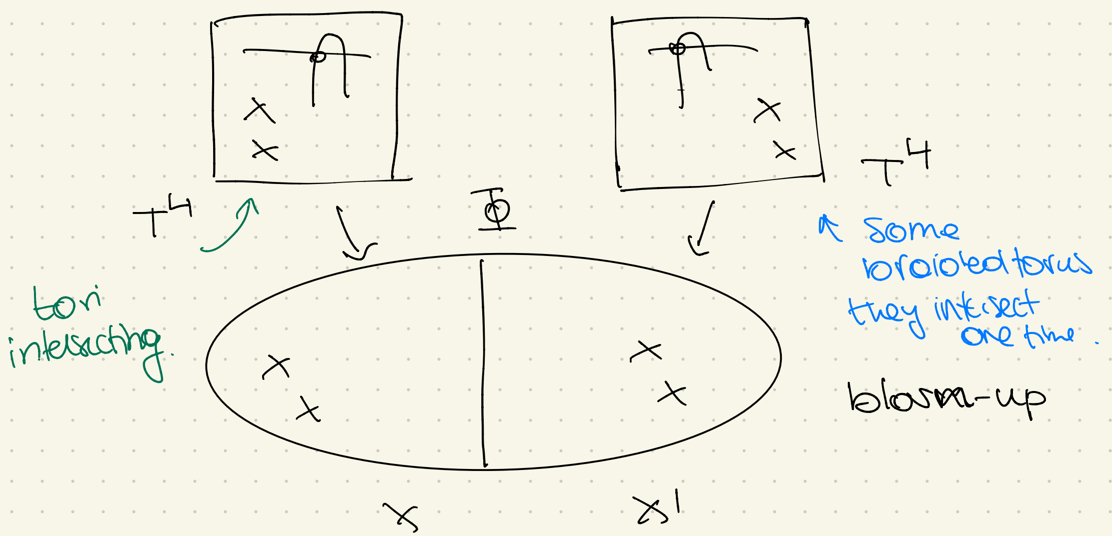
exotic on $L_2\#\mathbb{CP}^2$ or $L_2\#\overline{\mathbb{CP}}^2$
(double cover is homeo to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$)

Why?

Universal Cover of blow-up ?

BK ; FS.

Construction of exotic $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ with free involution



here kind of need to be precise. Check the paper maybe?

Here kind of lost. Calculation $S^1 \times S^1 \times S^1 \times S^1$ might be useful.

thm: There is an infinite family of homeomorphic, pairwise nondiffeomorphic, irreducible FPP with the same f.g. G such that G can be taken to be $R \times \mathbb{Z}/2$ where R is the f.g. of a $\mathbb{Q}HS^3$. ← torsione?

Come buttare dentro gruppi? Circle sum

- (X, τ) s.c. w/ free involution, $\gamma \subset X$ τ-invariant circle.
- Y rat. hom. S^3.
- $S^1 \times Y$ with $a \times id$ and $\gamma' = S^1 \times \{y\} \subseteq S^1 \times Y$

Circle sum, together with free inv $\tau_\gamma = \tau_{\gamma'}$ on

$X \#_{\gamma = \gamma'} S^1 \times Y = X \# S(Y)$ where S is the spun of Y.

there are more examples e.g. Arabadji - Morgan.

Detecting exotica: X^4 smooth.

def: $g_X : H_2(X; \mathbb{Z}) \to \mathbb{Z}$ associates to each hom class the min genus.

Adjunction inequality (Kronheimer-Mrowka): if $\Sigma \subseteq X$ smooth, $[\Sigma]^2 \geq 0$ and $SW_X(K) \neq 0$ then

$$2g(\Sigma)-2 \geq [\Sigma]^2+|k([\Sigma])|.$$

Rmk: • g_X is known for a handful of s.c.

$S^2\times S^2$, $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$

• Does stuff for exotica $\mathbb{CP}^2 \# 2,3\, \overline{\mathbb{CP}^2}$.

• $\mathbb{CP}^2$ $\quad g_{\mathbb{CP}^2}(d) = \frac{1}{2}(d-1)(d-2) \quad (d>0)$.

• In $2\mathbb{CP}^2$ some work from Roy-Marengon-A. Miller-Stipsicz