MA8151 Engineering Mathematics - I

1 Differential calculus

1.1 Representation of a function.

Definition. A function f is a rule that assigns to each element x in a set $\mathbb{D}$ exactly one element, called $f(x)$, in a set E.

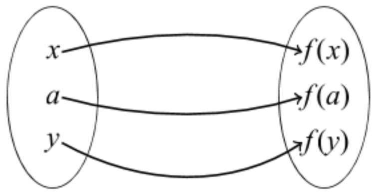
The set $\mathbb{D}$ is called the domain of the function. The number $f(x)$ is the value of f at x. The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the domain of a function f is called an independent variable. A symbol that represents a number in the range of f is called a dependent variable.

Example. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r, there is associated one value of A and we say that A is a function of r. Here, r is the independent variable and A is the dependent variable.

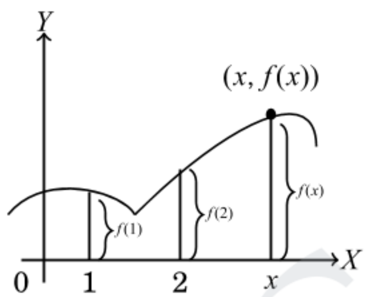
We can think of a function as a machine. If x is in the domain of the function f, then when x enters the machine, it is accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus, we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

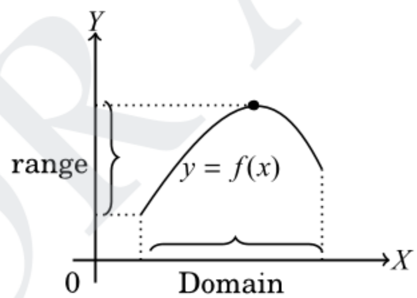
$$\text{input } x \to \boxed{f} \to f(x) \text{ output}$$

Another way to represent a function is by an Arrow diagram. Each arrow connects an element of $\mathbb{D}$ to an element of $\mathbb{E}$. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a and so on.



The most common method for visualizing a function is its graph. If f is a function with domain $\mathbb{D}$, then its graph is the set of ordered pairs $\{(x, f(x))/x \in \mathbb{D}\}$. In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$, and x is in the domain of f.



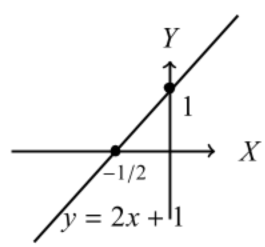


The graph of a function f gives us a useful picture of the behavior of a function. Since the y–coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x. The graph of f also allows us to picture the domain of f on the x–axis and its range on the y–axis.

Example 1.1. Sketch the graph of $f(x) = 2x + 1$ and find the domain and range.

Solution.

x	0	$\frac{-1}{2}$
y	1	0

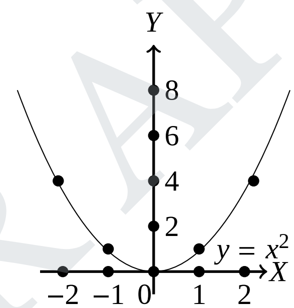


The equation of the graph is $y = 2x + 1$. This represents a line with slope 2 and y–intercept 1. With these informations, we can sketch the portion of the graph of f. The expression $2x + 1$ is defined for all real numbers, and hence the domain of f is the set of all real numbers, which we denote by R. The graph shows that the range is also $\mathbb{R}$.

Example 1.2. Sketch the graph of $g(x) = x^2$ and find the domain and range.

Solution. The equation of the graph is $y = x^2$.

x	-2	-1	0	1	2
y	4	1	0	1	4

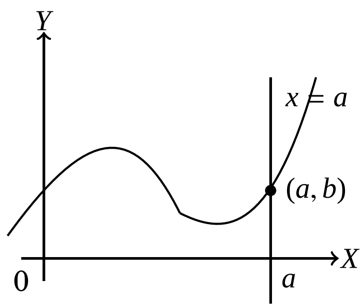


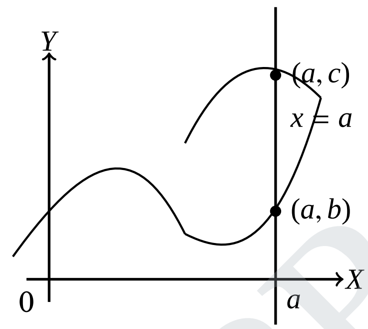
From the tabular column we can plot the points $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$ and $(2, 4)$ and join them to produce the graph, which represents a parabola. The domain of g is $\mathbb{R}$. The range of g consists of all values of $g(x)$, that is, all numbers of the form x^2. But $x^2 \geq 0$ for all numbers x and any positive number y is a square. Hence, the range of g is $\{y : y \geq 0\} = [0, \infty]$.

The graph of a function is a curve in the xy–plane. But whether all curves in the xy–plane represent the graph of some function. This can be answered by the vertical line test.

The vertical line test. A curve in the xy–plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

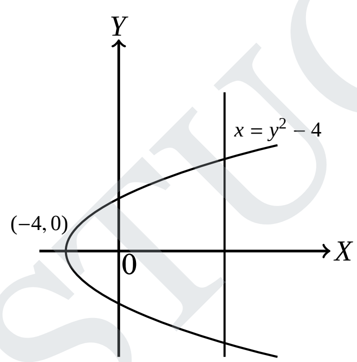
Consider the following graphs.





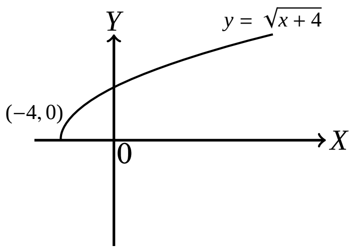
In the first graph, each vertical line $x = a$ intersects the curve only once, at (a, b), then exactly one function value is defined by $f(a) = b$. But in the second graph, the line $x = a$ intersects the curve twice at (a, b) and (a, c), that is, the function assigns two different values to a. Hence, the first curve represents the graph of a function while the second graph is not.

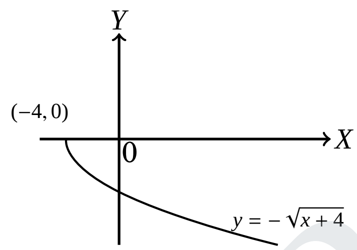
As an illustration consider the parabola $x = y^2 - 4$. The curve that represents this equation is give below.



This curve does not represent the graph of a function of x because, we can see that every vertical line intersects the parabola twice. This parabola is the union of the graphs of two functions of x. Rewriting the equation as $y^2 = x + 4$ we get $y = \pm\sqrt{x + 2}$.

Hence, the upper and lower halves of the parabola are the graphs of the functions $f(x) = \sqrt{x+4}$ and $g(x) = -\sqrt{x+4}$. The separate graphs are given below.





If we reverse the roles of x and y, then we get the equation $x = h(y) = y^2 - 4$ definitely define x as a function of y with y as the independent variables and x as the dependent variable and in this case, the parabola represent the graph of the function h.

Piecewise difined functions.

Example 1.3. Consider the function f defined by $f(x) = \begin{cases} 1 - x & , \text{ if } x \leq -1 \\ x^2 & , \text{ if } x > -1 \end{cases}$.

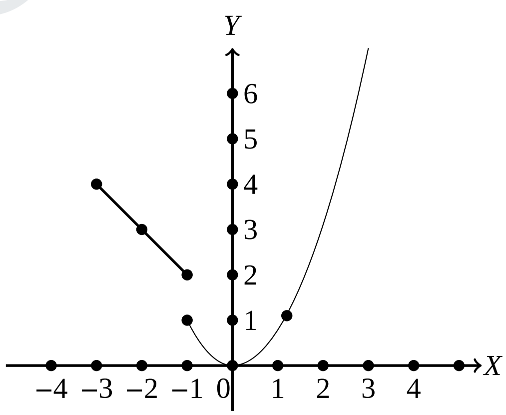
For all $x \leq -1$.

x	-3	-2	-1
$y = 1 - x$	4	3	2

For all $x > -1$.

x	-1	0	1	2	3
$y = x^2$	1	0	1	4	9

The combined graph of the function f is as follows



Notice that the point $(-1, 1)$ in $y = x^2$ is not included and there is a discontinuity between the graph for all $x \leq 1$ (i.e.$y = 1 - x$) and the graph for all $x > 1$ (i.e. $y = x^2$).

Example 1.4. Consider the absolute value function $f(x) = |x|$, which is defined as follows $f(x) = \begin{cases} x & , \text{ if } x \geq 0 \\ -x & , \text{ if } x < 0 \end{cases}$. The graph of $f(x)$ consists of two branches namely $y = x$ for all $x \geq 0$ and $y = -x$ for all $x < 0$.

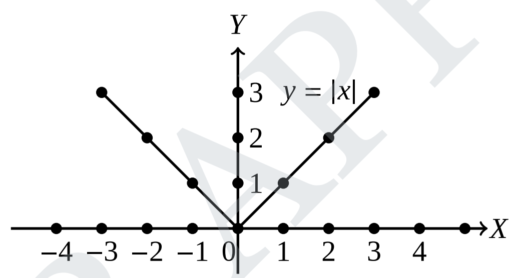
$y = x \ (x \geq 0)$

x	0	1	2	3
y	0	1	2	3

$y = -x \ (x < 0)$

x	-3	-2	-1	0
y	3	2	1	0

The combined graph of f is given below.



Notice that the point $(0, 0)$ is not included in the graph of $y = -x \ (x < 0)$.

Example 1.5. Consider the function $f(x) = \begin{cases} x & , \text{ if } 0 \leq x \leq 1 \\ 2 - x & , \text{ if } 1 < x \leq 2 \\ 0 & , \text{ if } x > 2 \end{cases}$.

$y = x \ (0 \leq x \leq 1)$

x	0	1
y	0	1

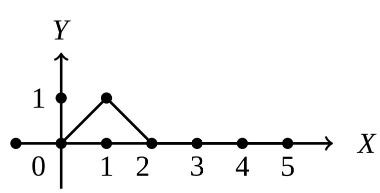
$y = 2 - x \ (1 \leq x \leq 2)$

x	1	2
y	1	0

$y = 0 \ (x > 2)$

x	2	3	4	5
y	0	0	0	0

The graph of f is given below.



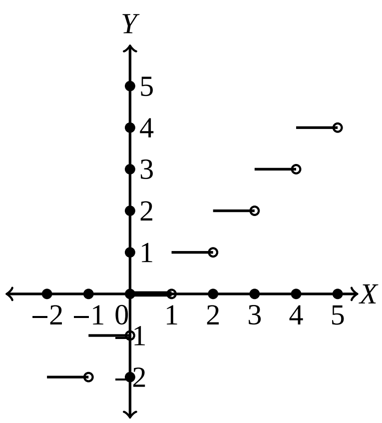
Example 1.6. Consider the greatest integer function defined by $f(x) = [x]$.

By definition $[x]$= Largest integer that is less than or equal to x

$= \max\{m \in \mathbb{Z}; m \leq x\}$.

Example. $[2.7] = 2$, $[-1.3] = -2$, $[4] = 4$

The graph of the function is as follows



The open dots are not included at the corresponding points.

The Examples 1.3 to 1.6 indicate that there is jump from one value to the next. Hence, the above functions are called piecewise functions. The function discussed in Example 1.6 is called step function.

symmetry

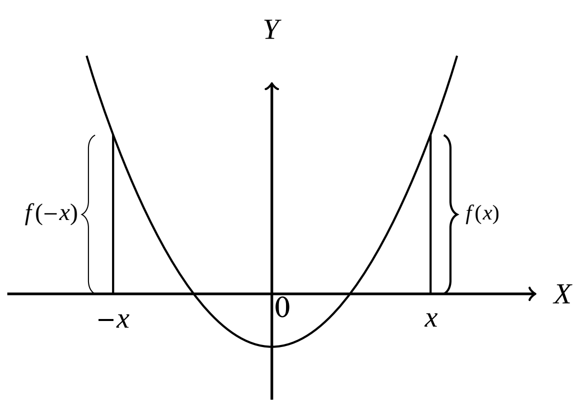
Even function. If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an even function.

Example. $f(x) = x^2$ is even,

since $f(-x) = (-x)^2 = x^2 = f(x)$.

Geometrically, tha graph of an even function is symmetric about the y–axis.

Consider the curve of $y = x^2 - 4$.

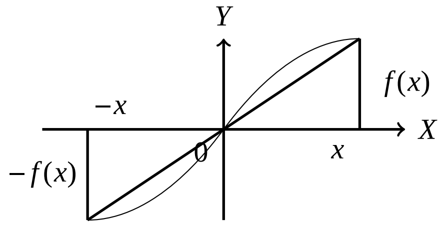


Odd function. If f statifies $f(-x) = -f(x)$ for every number x in the domain, then f is called an odd function.

Example. Consider $f(x) = x^3$

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

$\therefore$ $f(x) = x^3$ is an odd function.



The graph of the odd function is symmetric about the origin.

Example 1.7. Determine the nature of the following functions.

(i) $f(x) = x^5 - x$. (ii) $g(x) = 1 + x^4$. (iii) $h(x) = x + x^2$.

Solution.

(i) $f(x) = x^5 - x$.

$$f(-x) = (-x)^5 - (-x) = -x^5 + x = -(x^5 - x) = -f(x).$$

$\therefore$ $f(x)$ is odd.

(ii) $g(x) = 1 + x^4$.

$$g(-x) = 1 + (-x)^4 = 1 + x^4 = g(x).$$

$\therefore$ $g(x)$ is even.

(iii) $h(x) = x + x^2$.

$$h(-x) = (-x) + (-x)^2 = -x + x^2.$$

$h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$

$\therefore$ $h(x)$ is neither even nor odd.

Increasing and decreasing functions. A function f is called increasing on an interval I, if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

It is called decreasing on I, if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

1.2 Limit of a function

Definition. Intuitive definition of a limit. Suppose $f(x)$ is defined. When x is near the number a, then we write $\lim_{x\to a} f(x) = L$. This means that the limit of $f(x)$, as x approaches a equals L.

$\lim_{x\to a} f(x) = L$ can also be represented as $f(x) \to L$ as $x \to a$.

One sided limits. The left hand limit of $f(x)$ as x approaches a is written as $\lim_{x\to a-} f(x) = L$ which means that the limit of $f(x)$ as x approaches a from the left is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a with x less than a.

Similarly, if we require that x greater than a, we get the right hand limit of $f(x)$ as x approaches a is equal to L and we write $\lim_{x\to a+} f(x) = L$.

Result. $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a-} f(x) = L$ and $\lim_{x\to a+} f(x) = L$.

Infinite limits.

(i) Let f ba a function defined on both sides of a, except possibly at a itself. Then $\lim_{x\to a} f(x) = \infty$ means that the value of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

(ii) Let f be a function defined on both sides of a, expect possibly at a itself. Then $\lim_{x\to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

Example. $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

$$\lim_{x\to 0} \left(-\frac{1}{x^2}\right) = -\infty.$$

Asymptote. The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if atleast one of the following statements is true:

$\lim_{x\to a} f(x) = \infty,\ \lim_{x\to a-} f(x) = \infty,\ \lim_{x\to a+} f(x) = \infty.$

$\lim_{x\to a} f(x) = -\infty,\ \lim_{x\to a-} f(x) = -\infty,\ \lim_{x\to a+} f(x) = -\infty.$

Example. (i) For the function $f(x) = \frac{2x}{x-3}$, $x = 3$ is a vertical asymptote.

(ii) For the curve $f(x) = \tan x$, $x = \dfrac{\pi}{2}$ is a vertical asymptote.

Laws of limits. Suppose that c is a constant and the limits $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist, then the following laws are true.

1. $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$.
2. $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$.
3. $\lim\limits_{x \to a} [c f(x)] = c \lim\limits_{x \to a} f(x)$.
4. $\lim\limits_{x \to a} [f(x) g(x)] = \lim\limits_{x \to a} f(x) \times \lim\limits_{x \to a} g(x)$.
5. $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$. if $\lim\limits_{x \to a} g(x) \neq 0$.
6. $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$, where n is a positive integer.
7. $\lim\limits_{x \to a} c = c$.
8. $\lim\limits_{x \to a} x = a$.
9. $\lim\limits_{x \to a} x^n = a^n$, where n is a positive integer
10. $\lim\limits_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is a positive integer and $a > 0$.
11. $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$.

Direct substitution property. If f is a polynomial or a rational function and a is in the domain of f, then $\lim\limits_{x \to a} f(x) = f(a)$.

Theorem. If $f(x) \leq g(x)$, when x is near a and the limits of f and g both exist as x approaches a, then $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$.

The Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$, when x is near a and $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L$, then $\lim\limits_{x \to a} g(x) = L$.

The above theorem is sometimes called **Sandwich Theorem** or the **Pinching Theorem**.

Worked Examples

Example 1.8. Find $\lim_{x\to5}(2x^2-3x+4)$.

Solution.
$$\lim_{x\to5}(2x^2-3x+4) = \lim_{x\to5}2x^2 - \lim_{x\to5}3x + \lim_{x\to5}4$$
$$= 2\lim_{x\to5}x^2 - 3\lim_{x\to5}x + 4$$
$$= 2\times5^2 - 3\times5 + 4 = 50 - 15 + 4 = 39.$$

Example 1.9. Find $\lim_{x\to-2}\dfrac{x^3+2x^2-1}{5-3x}$.

Solution.
$$\lim_{x\to-2}\frac{x^3+2x^2-1}{5-3x} = \frac{\lim_{x\to-2} x^3+2x^2-1}{\lim_{x\to-2} 5-3x}$$
$$= \frac{\lim_{x\to-2} x^3 + \lim_{x\to-2} 2x^2 - \lim_{x\to-2} 1}{\lim_{x\to-2} 5 - \lim_{x\to-2} 3x}$$
$$= \frac{\lim_{x\to-2} x^3 + 2\lim_{x\to-2} x^2 - 1}{\lim_{x\to-2} 5 - 3\lim_{x\to-2} x}$$
$$= \frac{-8+2(-2)^2-1}{5-3(-2)}$$
$$= \frac{-8+8-1}{5+6} = -\frac{1}{11}.$$

Example 1.10. Find $\lim_{x\to1}\dfrac{x^2-1}{x-1}$.

Solution.
$$\lim_{x\to1}\frac{x^2-1}{x-1} = \lim_{x\to1}\frac{(x+1)(x-1)}{x-1} = \lim_{x\to1}(x+1) = 1+1 = 2.$$

Example 1.11. Find $\lim_{x\to1} f(x)$ where $f(x) = \begin{cases} x+1 & ,\text{if} x\neq1 \\ \pi & ,\text{if} x=1 \end{cases}$.

Solution. f is defined at $x=1$ and $g(1)=\pi$.

But the value of a limit as x approahces 1 does not depend on the value of the function at 1.

Since $g(x) = x+1$ for $x\neq1$, we have

$$\lim_{x\to1} g(x) = \lim_{x\to1}(x+1) = 2.$$

Example 1.12. Evaluate $\lim_{x\to 0} \frac{(3+x)^2-9}{x}$.

Solution.
$$\lim_{x\to 0} \frac{(3+x)^2-9}{x} = \lim_{x\to 0} \frac{(3+x-3)(3+x+3)}{x} = \lim_{x\to 0} \frac{x(6+x)}{x} = \lim_{x\to 0}(6+x) = 6.$$

Example 1.13. Find $\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}$.

Solution. On rationalizing the numerator we get

$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2} = \lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2} \times \frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}$$
$$= \lim_{x\to 0} \frac{x^2+9-9}{x^2\sqrt{x^2+9}+3}$$
$$= \lim_{x\to 0} \frac{x^2}{x^2\sqrt{x^2+9}+3}$$
$$= \lim_{x\to 0} \frac{1}{\sqrt{x^2+9}+3} = \frac{1}{3+3} = \frac{1}{6}.$$

Example 1.14. Show that $\lim_{x\to 0} |x| = 0$.

Solution. We know that $|x| = \begin{cases} x & ,\text{if} x \geq 0 \\ -x & ,\text{if} x < 0 \end{cases}$.

Since $|x| = -x$ for $x < 0$, we have $\lim_{x\to 0-} |x| = \lim_{x\to 0-} -x = 0$.

Also $|x| = x$ for $x > 0$, we have $\lim_{x\to 0+} |x| = \lim_{x\to 0+} x = 0$.

$\therefore \lim_{x\to 0-} |x| = \lim_{x\to 0+} |x| = 0$.

Hence, $\lim_{x\to 0} |x| = 0$.

Example 1.15. Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Solution. We know that $|x| = \begin{cases} x & ,\text{if} x \geq 0 \\ -x & ,\text{if} x < 0 \end{cases}$.

Now, $\lim\limits_{x\to 0+} \frac{|x|}{x} = \lim\limits_{x\to 0+} \frac{x}{x} = 1.$
$\lim\limits_{x\to 0-} \frac{|x|}{x} = \lim\limits_{x\to 0-} \frac{-x}{x} = -1.$
Since $\lim\limits_{x\to 0+} \frac{|x|}{x} \neq \lim\limits_{x\to 0-} \frac{|x|}{x}$, $\lim\limits_{x\to 0} \frac{|x|}{x}$ does not exist.

Example 1.16. If $f(x) = \begin{cases} \sqrt{x-4} & ,\text{if} x > 4 \\ 8-2x & ,\text{if} x < 4 \end{cases}$, determine whether $\lim\limits_{x\to 4} f(x)$ exists.

Solution. We have $f(x) = \sqrt{x-4}$ for $x > 4$.

$\therefore \lim\limits_{x\to 4+} f(x) = \lim\limits_{x\to 4+} \sqrt{x-4} = 0.$
Also, $f(x) = 8-2x$ for $x < 4$.
$\therefore \lim\limits_{x\to 4-} f(x) = \lim\limits_{x\to 4-} (8-2x) = 0.$
Since $\lim\limits_{x\to 4+} f(x) = \lim\limits_{x\to 4-} f(x) = 0$, we have $\lim\limits_{x\to 4} f(x)$ exists and $\lim\limits_{x\to 4} f(x) = 0$.

Example 1.17. Show that for the greatest integer function $[x]$, $\lim\limits_{x\to 3}[x]$ does not exist.

Solution. By definition $[x] = 3$ for $3 \leq x < 4$.

$\therefore \lim\limits_{x\to 3+} [x] = \lim\limits_{x\to 3+} 3 = 3.$
Also $[x] = 2$ for $2 \leq x < 3$.
$\therefore \lim\limits_{x\to 3-} [x] = \lim\limits_{x\to 3-} 2 = 2.$
Since $\lim\limits_{x\to 3+} [x] \neq \lim\limits_{x\to 3-} [x]$, $\lim\limits_{x\to 3}[x]$ does not exist.

1.3 Continuity

Definition. A function f is continuous at a number a, if $\lim\limits_{x\to a} f(x) = f(a)$.

f is discontinuous at a if it is not continuous at a.

Note. For proving a function f to be continuous, we have to prove the following.

1. $f(a)$ must be defined.

2. $\lim\limits_{x\to a} f(x)$ exists.

3. $\lim_{x \to a} f(x) = f(a)$.

Example. Consider $f(x) = \dfrac{x^2 - x - 2}{x - 2}$.

Here, $f(x)$ is not defined at $x = 2$.

$\therefore$ f is not continuous at $x = 2$.

If we define $f(x)$ as $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & , \text{if} x \neq 2 \\ 1 & , \text{if} x = 2 \end{cases}$.

Then

(i) $f(2)$ is defined and $f(2) = 1$.

(ii) $\lim_{x \to 2} f(x) = \lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \dfrac{(x-2)(x+1)}{x-2} = \lim_{x \to 2}(x+1) = 3$

(iii) Since $f(2) = 1 \neq \lim_{x \to 2} f(x)$, $f(x)$ is not continuous at $x = 2$.

Definition. A function f is said to be continuous from the right at a number a if $\lim_{x \to a+} f(x) = f(a)$, and f is said to be continuous from the left at a if $\lim_{x \to a-} f(x) = f(a)$.

Example. Consider the greatest integer function $f(x) = [x]$. We notice that

$$\lim_{x \to n+} f(x) = \lim_{x \to n+} [x] = n = f(n)$$
$$\lim_{x \to n-} f(x) = \lim_{x \to n-} [x] = n - 1 \neq f(n).$$

$\therefore$ $f(x) = [x]$ is continuous from the right but not continuous from the left.

Definition. A function f is continuous on an interval if it is continuous at every number in the interval.

Theorem. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a.

1. $f + g$ 2. $f - g$ 3. cf 4. fg 5. $\dfrac{f}{g}$ $g(a) \neq 0$.

Theorem.

(a) Any polynomial is continuous everywhere. i.e., it iscontinuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined, i.e., it is continuous on its domain.

Theorem. The following types of functions are continuous at every number in their domains.

1. Polynomials.
2. Rational functions.
3. Root functions.
4. Trigonometric functions.
5. Inverse trigonometric functions.
6. Exponential functions.
7. Logarithmic functions.

Theorem. If f is continuous at b, and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$.

i.e., $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$.

Theorem. If g is continuous at a and f is continuous at $g(a)$, then the composite function fog given by $(fog)(x) = f(g(x))$ is continuous at a.

The intermediate value theorem. Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$.

Result. The intermediate value theorem is useful in the location of the roots of the given equation.

Example 1.18. Find the domain where the fnction f is continuous. Also find the numbers at which the function f is discontinous where

$$f(x) = \begin{cases} 1 + x^2 & , \text{ if } x \leq 0 \\ 2 - x & , \text{ if } 0 < x \leq 2 \\ (x-2)^2 x & , \text{ if } x > 2 \end{cases}.$$ [A.U. Dec. 2015]

Solution. The function f changes its value at $x = 0$, and $x = 2$.

$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (1 + x^2) = 1.$

$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (2 - x) = 2.$

Since, $\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x)$, f is not continuous at $x = 0$.

$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} (2 - x) = 0.$

$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} (x - 2)^2 = 0.$

Since, $\lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x) = 0 = f(2)$, $f(x)$ is continuous at $x = 2$.

$\therefore$ The only number at which the function is discontinuous is at $x = 0$.

The domain of continuity of f is $\{(-\infty, 0) \cup (0, \infty)$.

The graph of f is given below

For all $x \leq 0$.

$y = f(x) = 1 + x^2$.

x	-3	-2	-1	0
y	10	5	2	1

For all $0 < x \leq 2$.

$y = f(x) = 2 - x$.

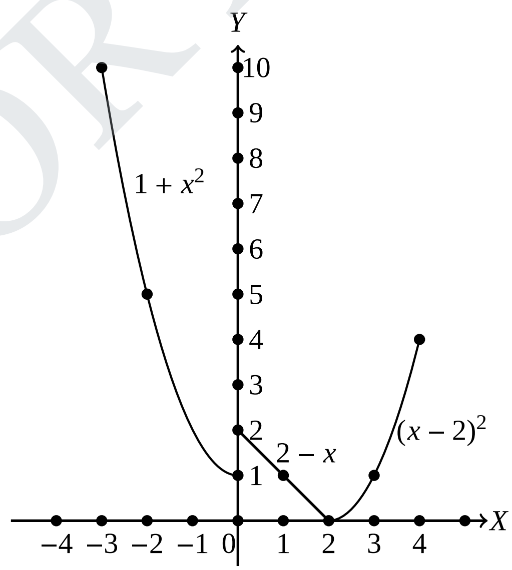
x	0	1	2
y	2	1	0

For all $x > 2$.

$y = f(x) = (x - 2)^2$.

x	2	3	4
y	0	1	4

The combined graph of the function f is as follows



Notice that the point $(-1, 1)$ in $y = x^2$ is not included and there is a discontinuity between the graph for all $x \leq 1$ (i.e.$y = 1 - x$) and the graph for all $x > 1$ (i.e. $y = x^2$).

Example 1.19. Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2$ between

1 and 2.

Solution. Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.

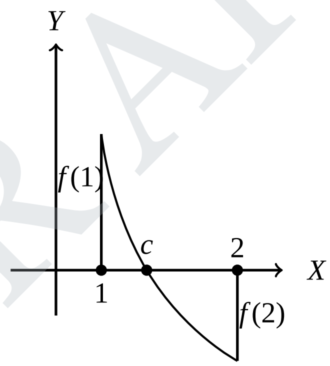
$$f(1) = 4(1)^3 - 6(1)^2 + 3(1) - 2 = 4 - 6 + 3 - 2 = -1 < 0.$$

$$f(2) = 4(2)^3 - 6(2)^2 + 3(2) - 2$$

$$= 4 \times 8 - 6 \times 4 + 3 \times 2 - 2 = 32 - 24 + 6 - 2 = 12 > 0.$$

The function changes its sign between 1 and 2.

Since $f(x)$ is a polynomial, which is continuous in its domain the graph of $y = f(x)$ must cross the x-axis atleast at one point say $x = c$ between $x = 1$ and $x = 2$ such that $f(c) = 0$, which is the root of the equation $f(x) = 0$. This proves that the given equation $f(x) = 0$ has a root between 1 and 2.



Example 1.20. Prove that the equation $x^3 - 15x + 1 = 0$ has atmost one real root in the interval $[-2, 2]$. [A.U.Nov.2016]

Solution. Let $f(x) = x^3 - 15x + 1$

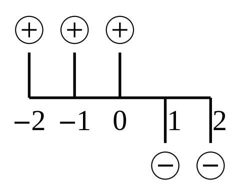
$$f(-2) = -8 + 30 + 1 = 23 = +ve.$$

$$f(-1) = -1 + 15 + 1 = 15 = +ve.$$

$$f(0) = 1 = +ve.$$

$$f(1) = 1 - 15 + 1 = -13 = -ve.$$

$$f(2) = 8 - 30 + 1 = -21 = -ve.$$



Hence, $f(0) > 0 > f(1)$.

f changes sign between 0 and 1.

$\therefore$ By intermediate value theorem, there is a number c between 1 and 2 such that $f(c) = 0$.

$\therefore$ The given equation has atleast one root c in the interval $(0, 1)$.

Since $f(x)$ changes sign in the interval $[-2, 2]$ only once between 0 and 1, $y = f(x)$ crosses the x axis only once.

i.e., $f(x)$ attains 0 only once in $[-2, 2]$

Hence, there is atmost one real root in the interval $[-2, 2]$.

1.4 Derivatives and differentiation rules

Definition. The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope $m = \lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$.

Definition. The derivative of a function f at a number a, denoted by $f'(a)$ is defined as $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$, if the limit exists.

Note. Let $x = a+h$. As $h \to 0$, $x \to a$. The equivalent definition for the derivative is $f'(a) = \lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$.

Worked Examples

Example 1.21. Find an equation to the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution. Given $a = 1$, $f(a) = 1$, $f(x) = x^2$.

$$\text{Slope} \quad m = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x\to 1}(x+1) = 1 + 1 = 2.$$

Equation of the tangent line through $(1, 1)$ is

$$
\begin{aligned}
y - y_1 &= m(x - x_1). \\
y - 1 &= 2(x - 1) \\
y &= 2x - 2 + 1 \\
y &= 2x - 1.
\end{aligned}
$$

Example 1.22. Find an equation to the tangent line to the hyperbola $y = \frac{3}{x}$ at the point $(3, 1)$.

Solution. Given $a = 3,\ f(a) = f(3) = 1$.

$$
\begin{aligned}
\text{We have}\quad m &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \\
&= \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h} \\
&= \lim_{h \to 0} \frac{3 - 3 - h}{h(3+h)} \\
&= \lim_{h \to 0} \frac{-h}{h(3+h)} = \lim_{h \to 0} \frac{-1}{(3+h)} = \frac{-1}{3}.
\end{aligned}
$$

$\therefore$ The equation of the tangent line is

$$
\begin{aligned}
y - y_1 &= m(x - x_1). \\
y - 1 &= \frac{-1}{3}(x - 3) \\
3y - 3 &= -x + 3 \\
x + 3y - 3 - 3 &= 0 \\
x + 3y - 6 &= 0.
\end{aligned}
$$

Example 1.23. Find the derivative of the function $f(x) = x^2 - 8x + 9$ at a number

a. Find also the equation the tangent line at the point $(3, -6)$.

Solution.
$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= \lim_{h\to 0} \frac{(a+h)^2 - 8(a+h) + 9 - (a^2 - 8a + 9)}{h} \\ &= \lim_{h\to 0} \frac{a^2 + h^2 + 2ah - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 2ah - 8h}{h} \\ &= \lim_{h\to 0} (h + 2a - 8) = 2a - 8. \end{aligned}$$

We have $m = f'(a) = f'(3) = 2 \times 3 - 8 = -2$.

$\therefore$ The equation of the tangent line is

$$\begin{aligned} y - y_1 &= m(x - x_1). \\ y + 6 &= -2(x-3) \\ &= -2x + 6 \\ 2x + y &= 0. \end{aligned}$$

Derivative of a function. Let $f(x)$ be a given function. The derivative of $f(x)$ at any variable point x is defined by $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Example 1.24. If $f(x) = x^3 - x$, find a formula for $f'(x)$.

Solution. Given $f(x) = x^3 - x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - (x+h) - (x^3 - x)}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 1 = 3x^2 - 1. \end{aligned}$$

Example 1.25. If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

Solution. Given $f(x) = \sqrt{x}$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

From this we notice that $f'(x)$ exists if $x > 0$.

$\therefore$ The domain of f' is $(0, \infty)$.

Example 1.26. Where is the function $f(x) = |x|$ differentiable. [A.U. Nov. 2015]

Solution. We have that $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

Consider $x > 0$. Then $|x| = x$. We choose h small enough such that $x + h > 0$.

$\therefore |x+h| = x+h$.

$\therefore$ When $x > 0$,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{|x+h|-|x|}{h} \\ &= \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1. \end{aligned}$$

$\therefore f$ is differentiable for $x > 0$.

Consider $x = 0$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{|0+h| - |x|}{h}$$
$$= \lim_{h \to 0} \frac{|h|}{h} \quad \text{[if it exists]}$$

Let us find the left and right limits.

$\lim\limits_{h \to 0+} \frac{|h|}{h} = \lim\limits_{h \to 0+} \frac{h}{h} = \lim\limits_{h \to 0+} 1 = 1.$

$\lim\limits_{h \to 0-} \frac{|h|}{h} = \lim\limits_{h \to 0-} \frac{-h}{h} = \lim\limits_{h \to 0} -1 = -1.$

Since the two limits are different, $f'(0)$ does not exist.

$\therefore$ f is not differentiable at $x = 0$.

Consider $x < 0$.

In this case $|x| = -x$.

Choose h small so that $x + h < 0$.

$\therefore |x+h| = -(x+h)$

$\therefore$ For $x < 0$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$
$$= \lim_{h \to 0} \frac{-x - h + x}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} -1 = -1.$$

$\therefore$ f is differentiable for $x < 0$.

Combining all the three cases we obtain that f is differentiable at all x except 0.

$$\therefore f'(x) = \begin{cases} 1 & \text{if} x > 0 \\ -1 & \text{if} x < 0 \end{cases}.$$

Theorem. If f is differentiable at a, then f is continuous at a.

Proof. Given that f is differentiable at a.

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exits.} \tag{1}$$

Now, $f(x) - f(a) = \dfrac{f(x) - f(a)}{x - a}(x - a)$

Taking limits as $x \to a$ both sides we get

$$\begin{aligned}\lim_{x\to a} f(x) - f(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a)\\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \times \lim_{x\to a}(x - a)\\ &= f'(a) \times 0 \qquad \text{[by (1)]}\end{aligned}$$

$$\lim_{x\to a} f(x) - f(a) = 0 \tag{2}$$

$$\begin{aligned}\text{Also} \quad \lim_{x\to a} f(x) &= \lim_{x\to a} [f(a) + (f(x) - f(a))]\\ &= \lim_{x\to a} + \lim_{x\to a}(f(x) - f(a))\\ &= f(a) + 0 \qquad \text{[by (1)]}\\ \lim_{x\to a} f(x) &= f(a)\end{aligned}$$

$\therefore$ $f(x)$ is continuous at a.

Result. The converse of the above theorem is false.

i.e., If a function is continuous at a need not be differentiable at a.

For example consider the function $f(x) = |x|$.

$\lim_{x\to a} f(x) = \lim_{x\to a} |x| = 0 = f(0)$.

But by example 1.26, $f(x) = |x|$ is not differentiable at 0.

Higher derivatives. If a function f is differentiable, then its derivative f' is also a function. Hence, f' may have a derivative of its own denoted by $(f')' = f''$. f'' is called the second derivative of f. The second derivative of $y = f(x)$ is denoted by $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$. In a similar way we can have the third derivative of f denoted by $(f'')' = f'''$. If $y = f(x)$ then we have $y''' = f'''(x) = \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{d^3y}{dx^3}$. If this process is continued we can get in general, the n^{th} derivative of f denoted by $f^{(n)}$ is obtained by differentiating f n times. Generally if $y = f(x)$ then we write $y^{(n)} = f^{(n)}(x) = \dfrac{d^ny}{dx^n}$.

Rules on differentiation

1. **The sum rule.** If f and g are both differentiable, then

$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$.

Proof. Let $F(x) = f(x) + g(x)$. Then,

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - [f(x) + g(x)]}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) - f(x)]}{h} + \lim_{h\to 0} [g(x+h) - g(x)]h \\
&= f'(x) + g'(x).
\end{aligned}$$

2. **Constant multiple rule.** If c is a constant and f is a differentiable function, then $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$.

   Proof. Let $F(x) = cf(x)$. Then,

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} \\
&= \lim_{h\to 0} \frac{c[f(x+h) - f(x)]}{h} \\
&= c \lim_{h\to 0} \frac{[f(x+h) - f(x)]}{h} \\
&= cf'(x) = c\frac{d}{dx}[f(x)].
\end{aligned}$$

3. **The difference rule.** If f and g are both differentiable, then $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$.

   Proof. We have $f(x) - g(x) = f(x) + (-)g(x)$.

$$\begin{aligned}
\frac{d}{dx}[f(x) - g(x)] &= \frac{d}{dx}[f(x) + (-1)g(x)] \\
&= \frac{d}{dx}[f(x)] + \frac{d}{dx}[(-1)g(x)] \quad \text{[by sum rule]} \\
&= \frac{d}{dx}[f(x)] + (-1)\frac{d}{dx}[g(x)] \quad \text{[by constant multiple rule]}
\end{aligned}$$

$$= \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)].$$

4. **The product rule.** If f and g are both differentiable, then $\frac{d}{dx}[f(x) \times g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$.

Proof.
$$\begin{aligned}
(f(x)g(x))' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)\{g(x+h) - g(x)\} + g(x)\{f(x+h) - f(x)\}}{h} \\
&= \lim_{h\to 0} \left\{ f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h} \right\} \\
&= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}
\end{aligned}$$
$$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x).$$

5. **Quotient rule.** If f and g are both differentiable, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$.

Proof.
$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \lim_{h\to 0} \left\{ \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \right\} \\
&= \lim_{h\to 0} \left\{ \frac{f(x+h)g(x) - g(x+h)f(x)}{hg(x+h)g(x)} \right\} \\
&= \lim_{h\to 0} \left\{ \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - g(x+h)f(x)}{hg(x+h)g(x)} \right\} \\
&= \lim_{h\to 0} \left\{ \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{hg(x+h)g(x)} \right\} \\
&= \lim_{h\to 0} \left\{ \frac{g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)} \right\} \\
&= \frac{\lim_{h\to 0} g(x)\frac{f(x+h)-f(x)}{h} - \lim_{h\to 0} f(x)\frac{g(x+h)-g(x)}{h}}{\lim_{h\to 0} g(x+h)g(x)}
\end{aligned}$$

$$= \frac{\lim_{h\to 0} g(x) \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} - \lim_{h\to 0} f(x) \lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h}}{g(x)g(x)}$$
$$= \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}.$$

Derivatives of simple functions

1. If c is a constant, prove that $\frac{d}{dx}(c) = 0$.

   Proof. Let $f(x) = c$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0. \end{aligned}$$

2. If n is a positive integer, then prove that $\frac{d}{dx}(x^n) = nx^{n-1}$.

   Proof. Let $f(x) = x^n$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + {}^nC_3 x^{n-3}h^3 + \cdots + h^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{6}x^{n-3}h^3 + \cdots + h^n}{h} \\ &= \lim_{h\to 0} nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h^1 + \frac{n(n-1)(n-2)}{6}x^{n-3}h^2 + \cdots + h^{n-1} \\ &= \lim_{h\to 0} nx^{n-1} + \lim_{h\to 0} \frac{n(n-1)}{2}x^{n-2}h^1 + \lim_{h\to 0} \frac{n(n-1)(n-2)}{6}x^{n-3}h^2 + \cdots + \lim_{h\to 0} h^{n-1} \\ &= nx^{n-1} + 0 + 0 + \cdots + 0 = nx^{n-1}. \end{aligned}$$

General rule. If n ia any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

3. **Derivative of e^x.** Prove that $\frac{d}{dx}(e^x) = e^x$.

   Definition of the number e. e is a number such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.

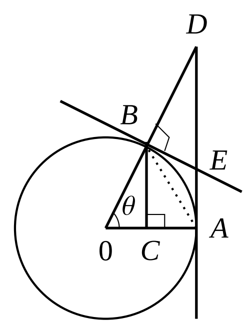
   Proof. Let $f(x) = e^x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h\to 0} e^x \left(\frac{e^h - 1}{h}\right) \\ &= \lim_{h\to 0} e^x \times \lim_{h\to 0} \left(\frac{e^h - 1}{h}\right) = e^x \times 1 = e^x. \end{aligned}$$

Two important limits

1. Prove that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

   Proof. Consider a circle with center O and radius 1 unit. Let $arc(AB)$ subtends an angle θ at the centre and assume that $0 < \theta < \frac{\pi}{2}$. Draw BC perpendicular to OA. Let the tangents at A and B meet at E. Produce AE and OB to intersect at D.



   Now we have $arcAB = \theta$.

   From ΔOAB, $\sin\theta = \frac{BC}{OB} = \frac{BC}{1}$.

   $\therefore |BC| = \sin\theta$.

   From the figure we have

   $|BC| < |AB| < arcAB$.

$\therefore \sin\theta < \theta.$

$$\Rightarrow \frac{\sin\theta}{\theta} < 1. \tag{1}$$

Again form the figure,

$arcAB < |AE| + |EB|$

$\therefore \theta = arcAB < |AE| + |EB| < |AB| + |ED| = |AD| = |OA|\tan\theta = \tan\theta.$

$$\therefore \theta < \tan\theta.$$

$$\text{i.e.,} \quad \theta < \frac{\sin\theta}{\cos\theta}$$

$$\theta\cos\theta < \sin\theta$$

$$\cos\theta < \frac{\sin\theta}{\theta} < 1 \quad [by(1)]$$

Taking limit as $\theta \to 0$ we get

$\lim_{\theta\to 0}\cos\theta < \lim_{\theta\to 0}\frac{\sin\theta}{\theta} < \lim_{\theta\to 0} 1$

$1 < \lim_{\theta\to 0}\frac{\sin\theta}{\theta} < 1$

By squeeze theorem $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$

2. Prove that $\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0.$

Proof.
$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} = \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}\times\frac{\cos\theta+1}{\cos\theta+1}$$
$$= \lim_{\theta\to 0}\frac{\cos^2\theta-1}{\theta(\cos\theta+1)} = \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)}$$
$$= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\times\lim_{\theta\to 0}\frac{\sin\theta}{(\cos\theta+1)} = -1\times\frac{0}{1+1} = 0.$$

$$\therefore \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} = 0.$$

Derivatives of trigonometric functions

1. Prove that $\frac{d}{dx}(\sin x) = \cos x.$

Proof. Let $f(x) = \sin(x)$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \left(\frac{\sin(x)(\cos(h)-1)}{h} + \frac{\cos(x)\sin(h)}{h}\right) \\
&= \lim_{h\to 0} \sin(x)\left(\frac{(\cos(h)-1)}{h} + \lim_{h\to 0} \cos(x)\frac{\sin(h)}{h}\right) \\
&= \sin(x)\lim_{h\to 0}\left(\frac{(\cos(h)-1)}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}\right) \\
&= \sin(x)\times 0 + \cos(x)\times 1 = 0 + \cos(x) = \cos(x).
\end{aligned}$$

2. Prove that $\frac{d}{dx}(\cos x) = -\sin x$.

Proof. Let $f(x) = \cos(x)$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)(\cos(h)-1)-\sin(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \left(\frac{\cos(x)(\cos(h)-1)}{h} - \frac{\sin(x)\sin(h)}{h}\right) \\
&= \lim_{h\to 0} \cos(x)\left(\frac{(\cos(h)-1)}{h} - \lim_{h\to 0} \sin(x)\frac{\sin(h)}{h}\right) \\
&= \cos(x)\times 0 - \sin(x)\times 1 = -\sin(x).
\end{aligned}$$

$$f'(x) = -\sin x.$$

3. Prove that $\frac{d}{dx}(\tan x) = \sec^2 x$.

Proof. Let $f(x) = \tan(x)$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$
$$= \frac{\cos x \times \cos x - \sin x \times (-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

4. Prove that $\frac{d}{dx}(cosec x) = -cosec x \cot x$.

Proof. $$\frac{d}{dx}(cosec x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(1) - 1 \times \frac{d}{dx}(\sin x)}{\sin^2 x}$$
$$= \frac{\sin x \times 0 - 1 \times (\cos x)}{\sin^2 x}$$
$$= \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x} \times \frac{\cos x}{\sin x} = -cosec x \cot x.$$

5. Prove that $\frac{d}{dx}(\sec x) = \sec x \tan x$.

Proof. $$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(1) - 1 \times \frac{d}{dx}(\cos x)}{\cos^2 x}$$
$$= \frac{\cos x \times 0 - 1 \times (-\sin x)}{\cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \sec x \tan x.$$

6. Prove that $\frac{d}{dx}(\cot x) = -cosec^2 x$.

Proof. $$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(\cos x) - \cos x \times \frac{d}{dx}(\sin x)}{\sin^2 x}$$
$$= \frac{\sin x \times (-\sin x) - \cos x \times (\cos x)}{\sin^2 x}$$
$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -cosec^2 x.$$

The chain rule. If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = fog$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product $F'(x) = f'(g(x)) \times g'(x)$.

Proof. Let $y = F(x) = f(g(x))$

Let $u = g(x) \Rightarrow y = f(u)$

Let Δx be a small change in x, then Δu is the corresponding change in u.

$\Delta u = g(x + \Delta x) - g(x)$.

Then the corresponding change in y is given by

$\Delta y = f(u + \Delta u) - f(u)$

$$\begin{aligned}
\text{Now,}\quad \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \times \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \times \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \qquad [\Delta u \to 0 \text{ as } \Delta x \to 0] \\
&= \frac{dy}{du} \times \frac{du}{dx} \\
&= \frac{d}{du}(f(u)) \times \frac{d}{dx}(g(x)) = f'(u) \times g'(x)
\end{aligned}$$

$$\frac{d}{dx}[F(x)] = f'(g(x)) \times g'(x)$$

$$F'(x) = f'(g(x)) \times g'(x)$$

Worked Examples

Example 1.27. Evaluate $\lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2}$. [A.U.Jan. 2015]

Solution. $\lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{x \to \frac{\pi}{2}} \frac{2\cos^2 x}{(\pi - 2x)^2}$

Let $\frac{\pi}{2} - x = y$

As $x \to \frac{\pi}{2}, y \to 0$

$$\therefore \lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{x \to \frac{\pi}{2}} \frac{2\cos^2 x}{4\left(\frac{\pi}{2} - x\right)^2}$$

$$= \frac{1}{2} \lim_{y \to 0} \frac{\cos^2\left(\frac{\pi}{2} - y\right)}{y^2}$$
$$= \frac{1}{2} \lim_{y \to 0} \frac{\sin^2 y}{y^2}$$
$$= \frac{1}{2} \lim_{y \to 0} \left(\frac{\sin y}{y}\right)^2$$
$$= \frac{1}{2} \left(\lim_{y \to 0} \frac{\sin y}{y}\right)^2 = \frac{1}{2} \times 1^2 = \frac{1}{2}.$$

Example 1.28. If $y = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$, find $\frac{dy}{dx}$.

Solution. Given, $y = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5\right)$$
$$= \frac{d}{dx}(x^8) + \frac{d}{dx}(12x^5) - \frac{d}{dx}(4x^4) + \frac{d}{dx}(10x^3) - \frac{d}{dx}(6x) + \frac{d}{dx}(5)$$
$$= 8x^{8-1} + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + 0$$
$$= 8x^7 + 12 \times 5(x^{5-1}) - 4 \times 4(x^{4-1}) + 10 \times 3(x^{3-1}) - 6 \times 1$$
$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6.$$

Example 1.29. If $y = e^x - x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution. Given $y = e^x - x$.

$$\frac{dy}{dx} = e^x - 1$$
$$\frac{d^2y}{dx^2} = e^x.$$

Example 1.30. If $y = x^2 \sin x$, find $\frac{dy}{dx}$.

Solution. Given $y = x^2 \sin x$.

Applying the product rule we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^2 \sin x\right)\\ &= x^2\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x^2)\\ &= x^2 \times (\cos x) + \sin x \times (2x)\\ &= x^2 \cos x + 2x \sin x.\end{aligned}$$

Example 1.31. If $f(x) = \dfrac{\sec x}{1+\tan x}$, find $\dfrac{dy}{dx}$. For what values of x, does the graph of f, has a horizontal tangent.

Solution. Given $f(x) = \dfrac{\sec x}{1+\tan x}$.
By the quotient rule, we have

$$\begin{aligned}f'(x) &= \frac{(1+\tan x)\frac{d}{dx}(\sec x) - \sec x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}\\ &= \frac{(1+\tan x)(\sec x \tan x) - \sec x\left(0+\sec^2 x\right)}{(1+\tan x)^2}\\ &= \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1+\tan x)^2}\\ &= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}\\ &= \frac{\sec x(\tan x - (\sec^2 x - \tan^2 x))}{(1+\tan x)^2}\\ &= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}\\ f'(x) &= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}.\end{aligned}$$

The points where the graph of f has a horizontal tangent are given by $f'(x) = 0$

$$\begin{aligned}\text{i.e.,}\frac{\sec x(\tan x - 1)}{(1+\tan x)^2} &= 0\\ \Rightarrow \sec x(\tan x - 1) &= 0\\ \text{But} \sec x &\neq 0\end{aligned}$$

$$\therefore \tan x - 1 = 0$$

$$\text{i.e., } \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}, \quad n \text{ is an integer.}$$

Example 1.32. Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Solution. Horizontal tangent occur at points where $\dfrac{dy}{dx} = 0$.

$$\text{i.e., } 4x^3 - 6 \times 2x = 0$$

$$4x^3 - 12x = 0$$

$$4x(x^2 - 3) = 0$$

$$\text{i.e., } x = 0 \text{ or } x^2 - 3 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}.$$

When $x = 0$, $y = 4$.

The point is $(0, 4)$.

When $x = \sqrt{3}$, $y = \left(\sqrt{3}\right)^4 - 6\left(\sqrt{3}\right)^2 + 4 = 9 - 18 + 4 = -5$.

The point is $(\sqrt{3}, -5)$.

When $x = -\sqrt{3}$, $y = \left(-\sqrt{3}\right)^4 - 6\left(-\sqrt{3}\right)^2 + 4 = 9 - 18 + 4 = -5$.

The point is $(-\sqrt{3}, -5)$.

The given curve has horizontal tangents at the points $(0, 4)$, $(\sqrt{3}, -5)$ and $(-\sqrt{3}, -5)$.

Example 1.33. At what point on the cure $y = e^x$, is the tangent line parallel to the line $y = 2x$.

Solution. Given $y = e^x \Rightarrow y' = e^x$.

slope of the tangent at $x = e^x$.

Since the tangent line is parallel to the line $y = 2x$,

Slope of the tangent = 2.

i.e., $e^x = 2 \quad \Rightarrow \quad x = \log 2$.

When $x = \log 2$, $y = e^{\log 2} = 2$.

$\therefore$ The required point is $(\log 2, 2)$.

Example 1.34. If $f(x) = xe^x$ find $f^{(n)}(x)$.

Solution. Given $f(x) = xe^x$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(xe^x) \\
&= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) = xe^x + e^x.1 = e^x(x+1). \\
f''(x) &= \frac{d}{dx}((x+1)e^x) \\
&= (x+1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+1) \\
&= (x+1)e^x + e^x.1 = e^x(x+1+1) = e^x(x+2). \\
f'''(x) &= \frac{d}{dx}((x+2)e^x) \\
&= (x+2)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+2) \\
&= (x+2)e^x + e^x.1 = e^x(x+2+1) = e^x(x+3).
\end{aligned}$$

Applying this process successively n times we get

$$f^{(n)}(x) = e^x(x+n).$$

Example 1.35. If $y = \dfrac{x^2+x-2}{x^3+6}$, find $\dfrac{dy}{dx}$.

Solution. Given $y = \dfrac{x^2+x-2}{x^3+6}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(x^3+6)\frac{d}{dx}(x^2+x-2) - (x^2+x-2)\frac{d}{dx}(x^3+6)}{(x^3+6)^2} \\
&= \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2} \\
&= \frac{2x^4+x^3+12x+6-3x^4-3x^3+6x^2}{(x^3+6)^2} \\
&= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}.
\end{aligned}$$

Example 1.36. Find the equation of the tangent line to the curve $y = \dfrac{e^x}{1+x^2}$ at the point $(1, \frac{e}{2})$.

Solution. Given $y = \dfrac{e^x}{1+x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{1+x^2}\right)$$
$$= \frac{(1+x^2)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$
$$= \frac{(1+x^2)e^x - e^x 2x}{(1+x^2)^2}$$
$$= \frac{e^x(1+x^2-2x)}{(1+x^2)^2} = \frac{e^x(1-x)^2}{(1+x^2)^2}$$
$$\left(\frac{dy}{dx}\right)_{\left(1,\frac{e}{2}\right)} = \frac{e(1-1)}{(1+1)^2} = 0.$$

$\therefore$ Slope of the tangent $m = 0$.

Equation of the tangent line is

$$y - y_1 = m(x - x_1)$$
$$y - \frac{e}{2} = 0(x-1)$$
$$y - \frac{e}{2} = 0.$$

Example 1.37. Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

Solution. Let $y = \sqrt{x^2+1}$, let $u = x^2+1$ and $y = \sqrt{u}$.

$$\text{Now} F'(x) = \frac{dy}{du} \times \frac{du}{dx}.$$
$$= \frac{1}{2}u^{\frac{-1}{2}} \times 2x = \frac{1}{\sqrt{u}}x = \frac{x}{\sqrt{x^2+1}}.$$

Example 1.38. Find $\dfrac{dy}{dx}$ if (i) $y = \sin(x^2)$ (ii) $y = \sin^2 x$.

Solution. (i) Given $y = \sin(x^2)$.

Let $u = x^2$. Then $y = \sin u$.

$$\text{Now} \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$
$$= \cos u \times 2x = \cos(x^2)2x = 2x\cos(x^2).$$

(ii) Given $y = \sin^2 x$

Let $u = \sin x$. Then $y = u^2$.

$$\text{Now}\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$
$$= 2u\cos x = 2\sin x\cos x = \sin 2x.$$

Example 1.39. Differentiate the following functions

(i) $y = (x^3 - 1)^{100}$.

(ii) $y = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

(iii) $y = (2x + 1)^5(x^3 - x + 1)^4$.

(iv) $y = e^{\sin x}$.

(v) $y = \sin(\cos(\tan x))$.

(vi) $y = e^{\sec 3x}$.

(vii) $y = \left(\dfrac{x - 2}{2x + 1}\right)^9$.

Solution. (i) Given $y = (x^3 - 1)^{100}$

Let $u = x^3 - 1$. Then $y = u^{100}$.

$$\text{Now}\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$
$$= 100u^{99} \times 3x^2 = 300x^2(x^3 - 1)^{99}.$$

(ii) Given $y = \frac{1}{\sqrt[3]{x^2+x+1}}$

Let $u = x^2 + x + 1$. Then $y = u^{\frac{-1}{3}}$.

$$\text{Now}\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$
$$= \frac{-1}{3}u^{\frac{-1}{3}-1} \times (2x + 1)$$
$$= \frac{-1}{3}u^{\frac{-4}{3}} \times (2x + 1) = \frac{-1}{3u^{\frac{4}{3}}} \times (2x + 1) = \frac{-(2x + 1)}{3(x^2 + x + 1)^{\frac{4}{3}}}.$$

(iii) Given $y = (2x + 1)^5(x^3 - x + 1)^4$

Let $u = x^2 + x + 1$. Then $y = u^{\frac{-1}{3}}$.

$$\begin{aligned}
\text{Now} \frac{dy}{dx} &= \frac{d}{dx}\left((2x+1)^5(x^3-x+1)^4\right) \\
&= (2x+1)^5\frac{d}{dx}\left((x^3-x+1)^4\right) + (x^3-x+1)^4\frac{d}{dx}\left((2x+1)^5\right) \\
&= (2x+1)^5\left(4(x^3-x+1)^3(3x^2-1)\right) + (x^3-x+1)^4\left(5(2x+1)^4\times 2\right) \\
&= 4(2x+1)^5(x^3-x+1)^3(3x^2-1) + 10(x^3-x+1)^4(2x+1)^4 \\
&= 2(2x+1)^4(x^3-x+1)^3\{2(2x+1)(3x^2-1) + 5(x^3-x+1)\} \\
&= 2(2x+1)^4(x^3-x+1)^3\{12x^3-4x+6x^2-2+5x^3-5x+5)\} \\
&= 2(2x+1)^4(x^3-x+1)^3\{17x^3+6x^2-9x+3)\}.
\end{aligned}$$

(iv) Given $y = e^{\sin x}$

$$\frac{dy}{dx} = e^{\sin x}\cos x.$$

(v) Given $y = \sin(\cos(\tan x))$.

$$\begin{aligned}
\frac{dy}{dx} &= \cos(\cos(\tan x))(-\sin(\tan x))\sec^2 x \\
&= -\cos(\cos(\tan x))\sin(\tan x)\sec^2 x.
\end{aligned}$$

(vi) Given $y = e^{\sec 3x}$.

$$\frac{dy}{dx} = e^{\sec 3x}\times\sec 3x\times\tan 3x\times 3 = 3\sec 3x\tan 3x e^{\sec 3x}.$$

(vii) Given $y = \left(\frac{x-2}{2x+1}\right)^9$.

$$\begin{aligned}
\frac{dy}{dx} &= 9\left(\frac{x-2}{2x+1}\right)^8\left(\frac{(2x+1)\times 1-(x-2)\times 2}{(2x+1)^2}\right) \\
&= 9\left(\frac{x-2}{2x+1}\right)^8\left(\frac{2x+1-2x+4}{(2x+1)^2}\right) = 45\frac{(x-2)^8}{(2x+1)^{10}}.
\end{aligned}$$

1.5 Implicit differentiation, logarithmic differentiation and Inverse trigonometric function.

Whenever an equation containing two variables x and y, it is not always possible to express y interns of x and it is not easy to find $\frac{dy}{dx}$ directly. In such case, we differentiate both sides with respect to x and then solve the resultant equation for $\frac{dy}{dx}$.

Worked Examples

Example 1.40. If $x^2+y^2 = 25$, find $\frac{dy}{dx}$. Find also the equation of the tangent to the circle at the point $(3,4)$.

Solution. Given $x^2 + y^2 = 25$.

Differentiating both sides w.r.to x we get

$$2x + 2y\frac{dy}{dx} = 0$$
$$x + y\frac{dy}{dx} = 0$$
$$y\frac{dy}{dx} = -x \quad \Rightarrow \frac{dy}{dx} = \frac{-x}{y}.$$

At $(3,4)$ $\frac{dy}{dx} = \frac{-3}{4}$.

Slope of the tangent at $(3,4) = m = \frac{-3}{4}$.

Equation of the tangent is

$$y - 4 = \frac{-3}{4}(x-3)$$
$$4y - 16 = -3x + 9$$
$$3x + 4y = 16 + 9$$
$$3x + 4y = 25.$$

Example 1.41. If $x^3+y^3 = 6xy$, find $\frac{dy}{dx}$. Find the tangent to the Folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3,3)$. At what point in the first quadrant is the tangent line horizontal.

Solution. Given $x^3 + y^3 = 6xy$.

Differentiating both sides w.r.to x we get

$$3x^2 + 3y^2\frac{dy}{dx} = 6\left[x\frac{dy}{dx} + y \times 1\right]$$

$$x^2 + y^2\frac{dy}{dx} = 2\left[x\frac{dy}{dx} + y\right] = 2x\frac{dy}{dx} + 2y$$

$$y^2\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - x^2$$

$$\frac{dy}{dx}(y^2 - 2x) = 2y - x^2$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}.$$

$$\text{At}(3,3) \quad \frac{dy}{dx} = \frac{2 \times 3 - 3^2}{3^2 - 2 \times 3} = -1.$$

Slope of the tangent at $(3, 3) = m = -1$.

The equation of the tangent is $y - 3 = -1(x - 3) = -x + 3 \Rightarrow x + y = 6$.

The points at which the tangent line is horizontal are given by $\frac{dy}{dx} = 0$.

i.e., $\frac{2y - x^2}{y^2 - 2x} = 0$

i.e., $2y - x^2 = 0$

i.e., $y = \frac{x^2}{2}$.

Substituting $y = \frac{x^2}{2}$ in the equation of the curve we obtain

$$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x \times \frac{x^2}{2}$$

$$x^3 + \left(\frac{x^6}{8}\right) = 3x^3$$

$$\frac{x^6}{8} - 2x^3 = 0$$

$$x^3\left(\frac{x^3}{8} - 2\right) = 0$$

$$x^3\left(x^3 - 16\right) = 0$$

$$x^3 = 0 \text{ or } x^3 - 16 = 0$$
$$x^3 = 0 \text{ or } x^3 = 16$$

But $x \neq 0$

$\therefore x^3 = 16 = 2^4 \Rightarrow x = 2^{\frac{4}{3}}$.

when $x = 2^{\frac{4}{3}}, y = \frac{1}{2}\left(2^{\frac{4}{3}}\right)^2 = \frac{1}{2}2^{\frac{8}{3}} = 2^{\frac{8}{3}-1} = 2^{\frac{5}{3}}$.

$\therefore$ In the first quadrant, the tangent line is horizontal at $(2^{\frac{4}{3}}, 2^{\frac{5}{3}})$.

Example 1.42. Find $\dfrac{dy}{dx}$ if $\sin(x+y) = y^2\cos x$.

Solution. Given $\sin(x+y) = y^2\cos x$.

Differentiating both sides w.r.to x we get

$$\cos(x+y)\left(1+\frac{dy}{dx}\right) = y^2(-\sin x) + \cos x \times 2y\frac{dy}{dx}$$
$$\cos(x+y) + \cos(x+y)\frac{dy}{dx} = -y^2\sin x + 2y\cos x\frac{dy}{dx}$$
$$\cos(x+y)\frac{dy}{dx} - 2y\cos x\frac{dy}{dx} = -y^2\sin x - \cos(x+y)$$
$$\frac{dy}{dx}\left(\cos(x+y) - 2y\cos x\right) = -\left(y^2\sin x + \cos(x+y)\right)$$
$$\frac{dy}{dx} = \frac{-\left(y^2\sin x + \cos(x+y)\right)}{(\cos(x+y) - 2y\cos x)}$$
$$= \frac{y^2\sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}.$$

Example 1.43. If $\sin y = x\sin(a+y)$ find $\dfrac{dy}{dx}$.

Solution. Given $\sin y = x\sin(a+y)$

$\therefore x = \dfrac{\sin y}{\sin(a+y)}$

Differentiating with respect to y we get

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)} = \frac{\sin(a+y-y)}{\sin^2(a+y)} = \frac{\sin(a)}{\sin^2(a+y)}$$
$$\therefore \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin(a)}.$$

Example 1.44. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, prove that $\dfrac{dy}{dx} = -\dfrac{1}{(1+x)^2}$.

Solution. Given $x\sqrt{1+y} + y\sqrt{1+x} = 0 \Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

squaring both sides we get

$$x^2(1+y) = y^2(1+x)$$
$$x^2 + x^2y = y^2 + y^2x$$
$$x^2 + x^2y - y^2 - y^2x = 0$$
$$x^2 - y^2 + x^2y - y^2x = 0$$
$$(x-y)(x+y) + xy(x-y) = 0$$
$$(x-y)(x+y+xy) = 0$$
$$\therefore x - y = 0 \text{ or } x+y+xy = 0$$
$$\text{i.e.,} x = y \text{ or } x+y+xy = 0$$

But $x \neq y$.

$\therefore x + y + xy = 0$ (1)

Differentiating both sides w.r.to x we get

$$1 + \frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right) = 0$$
$$1 + \frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$
$$\frac{dy}{dx}(1+x) = -y - 1$$
$$\frac{dy}{dx} = \frac{-(y+1)}{(1+x)}.$$

But form (1) we have $y(1+x) = -x \Rightarrow y = \dfrac{-x}{1+x}$.

$$\therefore \frac{dy}{dx} = -\frac{1 - \frac{x}{1+x}}{1+x} = -\frac{1+x-x}{(1+x)^2} = -\frac{1}{(1+x)^2}.$$

Example 1.45. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots \infty}}}$, find $\dfrac{dy}{dx}$.

Solution. Given $y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots \infty}}}$.

$$y = \sqrt{x+y}$$

$$y^2 = x + y.$$

Differentiating w.r.to x we get

$$2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y-1) = 1$$

$$\frac{dy}{dx} = \frac{1}{(2y-1)}.$$

Example 1.46. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots \infty}}}$, find $\frac{dy}{dx}$.

Solution. Given $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots \infty}}}$.

$$y = \sqrt{\sin x + y}.$$

Squaring on both sides we get

$$y^2 = \sin x + y.$$

Differentiating w.r.to x we get

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx}(2y-1) = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{(2y-1)}.$$

Example 1.47. Find y'' if $x^4 + y^4 = 16$.

Solution. Given $x^4 + y^4 = 16$

Differentiating w.r.to x we get

$$4x^3 + 4y^3\frac{dy}{dx} = 0$$

$$x^3 + y^3\frac{dy}{dx} = 0$$

$$y^3\frac{dy}{dx} = -x^3$$

$$\frac{dy}{dx} = -\frac{x^3}{y^3}.$$

Again differentiating w.r.to x

$$\frac{d^2y}{dx^2} = -\frac{y^3 \times 3x^2 - x^3 \times 3y^2\frac{dy}{dx}}{y^6}$$
$$= \frac{-3x^2y^2\left(y - x\frac{dy}{dx}\right)}{y^6} = \frac{-3x^2y^2\left(y - x\frac{dy}{dx}\right)}{y^6}$$
$$= \frac{-3x^2\left(y - x\left(\frac{-x^3}{y^3}\right)\right)}{y^4} = \frac{-3x^2\left(y^4 + x^4\right)}{y^7} = \frac{-3x^2 \times 16}{y^7} = \frac{-48x^2}{y^7}.$$

Derivative of Inverse Trigonometric functions

1. Find $\frac{dy}{dx}$ if $y = \sin^{-1} x$.

   Solution. Given $y = \sin^{-1} x$.

   $\sin y = x$.

   Differentiating w.r.to x

   $$\cos y\frac{dy}{dx} = 1$$
   $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{\cos^2 y}} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

2. If $y = \cos^{-1} x$, find $\frac{dy}{dx}$.

   Solution. Given $y = \cos^{-1} x$.

   $\cos y = x$.

   Differentiating w.r.to x

   $$-\sin y\frac{dy}{dx} = 1$$
   $$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{\sin^2 y}} = \frac{-1}{\sqrt{1 - \cos^2 y}} = \frac{-1}{\sqrt{1 - x^2}}.$$

3. If $y = \tan^{-1} x$, find $\frac{dy}{dx}$.

   Solution. Given $y = \tan^{-1} x$.

   $\tan y = x$.

   Differentiating w.r.to x

$$\sec^2 y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

4. If $y = \cot^{-1} x$, find $\frac{dy}{dx}$.

Solution. Given $y = \cot^{-1} x$.

$\cot y = x.$

Differentiating w.r.to x

$$-cosec^2 y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{-1}{cosec^2 y} = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}.$$

5. If $y = \sec^{-1} x$, find $\frac{dy}{dx}$.

Solution. Given $y = \sec^{-1} x$.

$\sec y = x.$

Differentiating w.r.to x

$$\sec y \tan y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{\tan^2 y}} = \frac{1}{x\sqrt{\sec^2 y - 1}} = \frac{1}{x\sqrt{x^2-1}}.$$

6. If $y = cosec^{-1} x$, find $\frac{dy}{dx}$.

Solution. Given $y = cosec^{-1} x$.

$cosec y = x.$

Differentiating w.r.to x

$$-cosec y \cot y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{-1}{cosec y \cot y} = \frac{-1}{x\sqrt{\cot^2 y}} = \frac{-1}{x\sqrt{cosec^2 y - 1}} = \frac{-1}{x\sqrt{x^2-1}}.$$

Worked Examples

Example 1.48. Find the derivatives of (i) $\frac{1}{\sin^{-1} x}$ (ii) $x \tan^{-1}\left(\sqrt{x}\right)$.

Solution. (i) Let $y = \frac{1}{\sin^{-1} x} = \left(\sin^{-1} x\right)^{-1}$.

$$\frac{dy}{dx} = (-1)\left(\sin^{-1} x\right)^{-2} \frac{1}{\sqrt{1-x^2}} = \frac{1}{\left(\sin^{-1} x\right)^2 \sqrt{1-x^2}}.$$

(ii) Let $y = x\tan^{-1}\left(\sqrt{x}\right)$.

$$\frac{dy}{dx} = (x)\frac{1}{1+(\sqrt{x})^2} \times \frac{1}{2\sqrt{x}} + \tan^{-1}\left(\sqrt{x}\right).1 = \frac{\sqrt{x}}{2(1+x)} + \tan^{-1}\left(\sqrt{x}\right).$$

Example 1.49. Differentiate w.r.to x (i) $\sin^{-1}\left(\sqrt{x}\right)$ (ii) $(1+x^2)\tan^{-1} x$ (iii) $\sec^{-1}(x^4)$ (iv) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$.

Solution. (i) Let $y = \sin^{-1}(\sqrt{x})$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x(1-x)}}.$$

(ii) Let $(1+x^2)\tan^{-1} x$.

$$\frac{dy}{dx} = (1+x^2) \times \frac{1}{1+x^2} + \tan^{-1} x \times 2x = 1 + 2x\tan^{-1} x.$$

(iii) Let $\sec^{-1}(x^4)$.

$$\frac{dy}{dx} = \frac{1}{x^4\sqrt{(x^4)^2-1}} \times 4x^3 = \frac{4}{x\sqrt{x^8-1}}.$$

(iv) Let $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$.

$$= \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right).$$

$$= \tan^{-1}\left(\frac{\sin 2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{1+\cos 2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right).$$

$$= \tan^{-1}\left(\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right).$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right).$$

$$y = \left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \frac{-1}{2}.$$

Example 1.50. If $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ find $\frac{dy}{dx}$.

Solution. Given $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

Let $x = \tan\theta$

$\therefore y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta$

$y = 2\tan^{-1} x$.

$\frac{dy}{dx} = \frac{2}{1+x^2}$.

Example 1.51. If $\sin^{-1}\left(3x - 4x^3\right)$ find $\frac{dy}{dx}$.

Solution. Let $y = \sin^{-1}\left(3x - 4x^3\right)$.

Let $x = \sin\theta$.

$y = \sin^{-1}(3\sin\theta - 4\sin^3\theta) = \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1} x$.

$\frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$.

Example 1.52. If $y = \tan^{-1}\left(\frac{a-x}{1+ax}\right)$ find $\frac{dy}{dx}$.

Solution. $y = \tan^{-1}\left(\frac{a-x}{1+ax}\right)$

Let $a = \tan\alpha$, $x = \tan\theta$.

$\therefore y = \tan^{-1}\left(\frac{\tan\alpha - \tan\theta}{1+\tan\alpha\tan\theta}\right) = \tan^{-1}\left(\tan(\alpha-\theta)\right) = \alpha - \theta = \alpha - \tan^{-1}(x)$.

$\frac{dy}{dx} = \frac{-1}{1+x^2}$.

Logarithmic Differentiation

Result. If $y = \log_e x$, find $\frac{dy}{dx}$.

Solution. Given $y = \log_e x$

$\therefore x = e^y$.

Differentiating w.r.to y

$\frac{dx}{dy} = e^y = x$

$\frac{dy}{dx} = \frac{1}{x}$.

Note. (i) If $y = b^x$ find $\frac{dy}{dx}$.

Solution. Given $y = b^x = e^{x\log b}$

Differentiating w.r.to x

$$\frac{dy}{dx} = e^{x\log b}\log b = b^x \log b.$$

(ii) If $y = \log_b x$ find $\frac{dy}{dx}$.

Solution. Given $y = \log_b x$

$\therefore b^y = x.$

Differentiating w.r.to x

$$b^y \log b \frac{dy}{dx} = 1.$$

$$\frac{dy}{dx} = \frac{1}{b^y \log b} = \frac{1}{x\log b}.$$

Worked Examples

Example 1.53. Differntiate (i) $\log(x^3+1)$ (ii) $\log(\sin x)$ (iii) $\log(2+\sin x)$ (iv) $\log\left(\frac{x+1}{\sqrt{x+2}}\right)$ (v) $\log|x|$ (vi) $\sqrt{\log(x)}$.

Solution. (i) Let $y = \log(x^3+1)$.

$$\frac{dy}{dx} = \frac{1}{x^3+1}3x^2 = \frac{3x^2}{x^3+1}.$$

(ii) Let $y = \log(\sin x)$.

$$\frac{dy}{dx} = \frac{1}{\sin x}\cos x = \cot x.$$

(iii) Let $y = \log_{10}(2+\sin x)$.

$$\frac{dy}{dx} = \frac{1}{(2+\sin x)\log 10}\cos x = \frac{\cos x}{(2+\sin x)\log 10}.$$

(iv) Let $y = \log\left(\frac{x+1}{\sqrt{x+2}}\right)$.

$$= \log(x+1) - \log\left(\sqrt{x+2}\right).$$

$$= \log(x+1) - \log(x+2)^{1/2}.$$

$$= \log(x+1) - \frac{1}{2}\log(x+2).$$

$$\frac{dy}{dx} = \frac{1}{(x+1)} - \frac{1}{2(x+2)}.$$

(v) Let $y = \log|x| = \begin{cases} \log x & ,\text{if } \ x > 0 \\ \log(-x) & ,\text{if } \ x < 0 \end{cases}$.

$$\frac{dy}{dx} = \begin{cases} \frac{1}{x} & ,\text{if } \ x > 0 \\ \frac{1}{-x}(-1) & ,\text{if } \ x < 0 \end{cases} = \begin{cases} \frac{1}{x} & ,\text{if } \ x > 0 \\ \frac{1}{x} & ,\text{if } \ x < 0 \end{cases}.$$

$$\therefore \frac{dy}{dx} = \frac{1}{x} \text{for all} x \neq 0.$$

(vi) Let $y = \sqrt{\log x} = (\log x)^{1/2}$.

$$\frac{dy}{dx} = \frac{1}{2}(\log x)^{-1/2}\frac{1}{x} = \frac{1}{2x\sqrt{\log x}}.$$

Example 1.54. If $y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$, find $\dfrac{dy}{dx}$.

Solution. Given $y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

Taking log on both sides we get

$$\log y = \log\left(x^{3/4}\right) + \log\sqrt{x^2+1} - \log(3x+2)^5$$
$$= \frac{3}{4}\log x + \frac{1}{2}\log(x^2+1) - 5\log(3x+2).$$

Differentiating w.r.to x we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\times\frac{1}{x} + \frac{1}{2}\times\frac{1}{x^2+1}2x - 5\frac{1}{3x+2}\times 3 = \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right] = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right].$$

Example 1.55. If $y = x^{\sqrt{x}}$ find $\dfrac{dy}{dx}$.

Solution. Given $y = x^{\sqrt{x}}$.

$$\log y = \sqrt{x}\log x.$$

Differentiating w.r.t x

$$\frac{1}{y}\frac{dy}{dx} = \sqrt{x}\frac{1}{x} + \log x\frac{1}{2\sqrt{x}}$$
$$= \frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{x}}\log x$$

$$= \frac{1}{\sqrt{x}}\left(1 + \frac{1}{2}\log x\right)$$
$$= \frac{1}{2\sqrt{x}}(2 + \log x)$$

$$\frac{dy}{dx} = y\frac{1}{2\sqrt{x}(2 + \log x)} = \frac{x^{\sqrt{x}}}{2\sqrt{x}(2 + \log x)}.$$

Example 1.56. Find $\frac{d}{dx}((\sin x)^{\cos x})$. [A.U. Jan. 2015]

Solution. Let $y = (\sin x)^{\cos x}$.

Taking logarithms on both sides we get

$\log y = \log(\sin x)^{\cos x} = \cos x \log(\sin x)$.

Differentiating w.r.to x.

$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{1}{\sin x}\cos x + \log(\sin x)(-\sin x)$$
$$= \frac{\cos^2 x}{\sin x} - \sin x \log(\sin x)$$
$$\frac{dy}{dx} = y\left[\frac{\cos^2 x}{\sin x} - \sin x\log(\sin x)\right] = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \log(\sin x)\right].$$

Example 1.57. If $y = (\sin x)^x$, find $\frac{dy}{dx}$.

Solution. Let $y = (\sin x)^x$.

Taking log both sides we get

$\log y = \log(\sin x)^x = x\log\sin x$.

Differentiating w.r.t. x.

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{\sin x}\cos x + \log\sin x.1$$
$$= x\cot x + \log\sin x.$$
$$\frac{dy}{dx} = y\left[x\cot x + \log\sin x\right] = (\sin x)^x\left[x\cot x + \log\sin x\right].$$

Example 1.58. If $y = x^{x^{x^{x^{\cdots\infty}}}}$, find $\frac{dy}{dx}$.

Solution. Let $y = x^{x^{x^{x^{\cdots\infty}}}} \Rightarrow y = x^y$.

Taking log both sides we get

$\log y = \log x^y = y \log x.$

Differentiating w.r.t. x.

$\frac{1}{y}\frac{dy}{dx} = y\frac{1}{x} + \log x \frac{dy}{dx}.$

$\frac{1}{y}\frac{dy}{dx} - \log x \frac{dy}{dx} = \frac{y}{x}.$

$\frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x}.$

$\frac{dy}{dx}\left(\frac{1 - y\log x}{y}\right) = \frac{y}{x}.$

$\frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}.$

Hyperbolic functions

We define $\sinh x = \frac{e^x - e^{-x}}{2}$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$cosechx = \frac{1}{\sinh x}$$

$$sechx = \frac{1}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

Hyperbolic identities

1. $\sinh(-x) = -\sinh x.$

2. $\cosh(-x) = \cosh x.$

3. $\cosh^2 x - \sinh^2 x = 1.$

4. $1 - \tanh^2 x = sech^2 x.$

5. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y.$

6. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

Derivatives of hyperbolic functions.

1. Prove that $\frac{d}{dx}(\sinh x) = \cosh x$.

   Proof. $\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}(e^x - e^{-x}(-1)) = \left(\frac{e^x + e^{-x}}{2}\right) = \cosh x.$

   In the same way we can establish the following.

2. $\frac{d}{dx}(\cosh x) = \sinh x$.

3. $\frac{d}{dx}(\tanh x) = sech^2 x$.

4. $\frac{d}{dx}(cosech x) = -cosech x \coth x$.

5. $\frac{d}{dx}(sech x) = sech x \tanh x$.

6. $\frac{d}{dx}(\coth x) = -cosech^2 x$.

Inverse hyperbolic functions.

1. Prove that $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$.

   Proof. Let $y = \sinh^{-1} x$.

   $\therefore x = \sinh y = \frac{e^y - e^{-y}}{2}$.

   i.e., $e^y - e^{-y} = 2x$.

   $e^y - \frac{1}{e^y} - 2x = 0$.

   $e^{2y} - 2xe^y - 1 = 0$.

   This is a quadratic in e^y. Solving for e^y we get

   $e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{x \pm \sqrt{x^2+1}}{1} = (x \pm \sqrt{x^2+1})$.

   We have $e^y > 0$ always.

   But $x - \sqrt{x^2-1} < 0$.

   $\therefore e^y = x - \sqrt{x^2+1}$ is not possible.

   $\therefore e^y = x + \sqrt{x^2+1}$ $\qquad y = \log(x + \sqrt{x^2+1})$.

   In a similar way we can prove the following.

2. $\cosh^{-1} x) = \log\left(x + \sqrt{x^2 - 1}\right)$.

3. $\tanh^{-1} x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$.

Differentiation of Inverse hyperbolic functions.

1. Prove that $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$.

Proof. $y = \sinh^{-1} x$.

$\therefore \sinh y = x$.

Differentiating w.r.t. x we get

$\cosh y \frac{dy}{dx} = 1$.

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{\cosh^2 y}} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

In a similar way we can prove the following.

2. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$.

3. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$.

4. $\frac{d}{dx}(cosech^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$.

5. $\frac{d}{dx}(sech^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$.

6. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$.

Exercise 1 (E)

1. Find the derivatives of the following functions.

(i) $x^2 - 4xy + y^2 = 4$

(ii) $x^4 + x^2y^2 + y^3 = 5$

(iii) $\frac{x^2}{x+y} = y^2 + 1$

(iv) $y\cos x = x^2 + y^2$

(v) $\sqrt{x+y} = x^4 + y^4$

(vi) $e^{\frac{x}{y}} = x - y$

(vii) $\tan^{-1}(x^2y) = x + xy^2$

(viii) $\sin(xy) = \cos(x + y)$

(ix) $\cos(xy) = 1 + \sin y$

(x) $x\sin y + y\sin x = 1$

2. If $(\sin x)^{\cos y} = (\sin y)^{\cos x}$, find $\dfrac{dy}{dx}$.

3. If $y = (\sin x)^{\sin x^{\sin x^{\cdots\infty}}}$, find $\dfrac{dy}{dx}$.

4. If $x^{1+y} + y^{1+x} = a$, where a is a constant, find $\dfrac{dy}{dx}$.

5. Find the derivatives of the following using logarithmic differentiation.

(i) $(x^2 + 2)^2(x^4 + 4)^4$

(ii) $\sqrt{\frac{(x-1)}{(x^4+1)}}$

(iii) x^x

(iv) $x^{\sin x}$

(v) $(\cos x)^x$

(vi) $(\tan x)^{\frac{1}{x}}$

6. Find the equation of the tangent line to the curve $y\sin 2x = x\cos 2y$ at $(\frac{\pi}{2}, \frac{\pi}{4})$.

7. Find the equation of the tangent line to the hyperbola $x^2 - xy - y^2 = 1$ at the point $(2, 1)$.

8. Find the equation to the tangent line to the Cardioid $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at $\left(0, \frac{1}{2}\right)$.

9. Find the derivative of the following functions.

(i) $\tanh\sqrt{x}$.

(ii) $\sinh(\log(x))$.

(iii) $e^{\cosh 3x}$.

(iv) $x\coth\sqrt{1 + x^2}$.

(v) $\cosh^{-1}(\sqrt{x})$.

(vi) $x\sinh^{-1}\left(\frac{x}{3}\right) - \sqrt{9 + x^2}$.

(vii) $\coth^{-1}(\sec x)$.

(viii) $\tanh^{-1}(\sin x)$.

10. At what point of the curve $y = \cosh x$ does the tangent have slope 1.

Definition. Let c be a number in the domain $\mathbb{D}$ of a function f. Then $f(c)$ is the absolute maximum value of f on $\mathbb{D}$ if $f(c) \geq f(x)$ for all x in $\mathbb{D}$. $f(c)$ is the absolute minimum value of f on D if $f(c) \leq x$ for all x in $\mathbb{D}$.

Maximum and minimum values of f are called extreme values of f.

Definition. The number $f(c)$ is a local maximum value of f if $f(c) \geq f(x)$ when x is near c. $f(c)$ is the local minimum value of f if $f(c) \leq f(x)$ when x is near c.

Increasing and decreasing test

(i) If $f'(x) > 0$, on an interval, then f is increasing on that interval.

(ii) If $f'(x) < 0$, on an interval, then f is decreasing on that interval.

Fermat's theorem. If f has a local maximum or minimum at c, and $f'(c)$ exits, then $f'(c) = 0$.

Critical number. A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Result. If f has a local maximum or minimum at c, then c is a Critical number of f.

First derivative test. Suppose that c is a critical number of a continuous function f.

(i) If f' changes from positive to negative at c, then f has a local maximum at c.

(ii) If f' changes from negative to positive at c, then f has a local minimum at c.

(iii) If f' is positive to the left and right of c or negative to the left and right of c, then f has no local maximum or minimum at c.

Second derivative test. Suppose f'' is continuous near c.

(i) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(ii) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Concavity. If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.

Concavity test

(i) If $f''(x) > 0$, for all x in I, then the graph of f is concave upward on I.

(ii) If $f''(x) < 0$, for all x in I, then the graph of f is concave downward on I.

Point of inflexion. A point P on a curve $y = f(x)$ is called a point of inflexion if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Note. Points of inflection occurs at points where $f''(x) = 0$.

Worked Examples

Example 1.59. Find the intervals on which the function $f(x) = \sin x + \cos x$ is increasing or decreasing. Also find the curve is concave upwards or concave downwards. [A.U.Jan.2015]

Solution. $f(x) = \sin x + \cos x.$

$f'(x) = \cos x - \sin x.$

$f''(x) = -\sin x - \cos x.$

At critical points $f'(x) = 0$

i.e., $\cos x - \sin x = 0.$

i.e., $\sin x = \cos x.$

i.e., $\tan x = 1.$

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}.$

Let us evaluate the intervals at which f decrease or increases.

Interval	$f'(x)$	f
$0 < x < \frac{\pi}{4}$	+	increases in $(0, \frac{\pi}{4})$
$\frac{\pi}{4} < x < \frac{5\pi}{4}$	-	decreases in $(\frac{\pi}{4}, \frac{5\pi}{4})$
$\frac{5\pi}{4} < x < 2\pi$	+	increases in $(\frac{5\pi}{4}, 2\pi)$

Since f' changes from positive to negative at $x = \frac{\pi}{4}$,

f has a maximum at $x = \frac{\pi}{4}$.

Maximum value of $f = f(\frac{\pi}{4}) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$.

Also f' changes from negative to positive at $x = \frac{5\pi}{4}$.

Minimum value of $f = f(\frac{5\pi}{4}) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$.

Let us evaluate the intervals at which the curve is concave upwards or concave downwards.

Making $f'' = 0$ we get

$-\sin x - \cos x = 0$.

i.e., $\sin x + \cos x = 0$.

$\cos x = -\sin x = \cos\left(\frac{\pi}{2} + x\right)$ (or) $\cos x = \cos\left(\frac{3\pi}{2} - x\right)$

Taking $\cos x = \cos\left(\frac{\pi}{2} + x\right)$ implies $x = \frac{\pi}{2} + x$

i.e., $\frac{\pi}{2} = 0$ which is absurd.

Taking $\cos x = \cos\left(\frac{3\pi}{2} - x\right)$ implies $x = \frac{3\pi}{2} - x$

i.e., $2x = \frac{3\pi}{2} \quad \Rightarrow x = \frac{3\pi}{4}$

The interval to be considered are $\left(0, \frac{3\pi}{4}\right), \left(\frac{3\pi}{4}, 2\pi\right)$

Interval	$f'(x)$	f
$0 < x < \frac{3\pi}{4}$	-	concave downwards
$\frac{3\pi}{4} < x < 2\pi$	+	concave upwards

Since f'' changes sign from negative to positive at $x = \frac{3\pi}{4}$, $x = \frac{3\pi}{4}$ is a point of inflexion.

Example 1.60. Find the intervals on which the function $f(x) = x^4 - 2x^2 + 3$ is increasing or decreasing. Also find the local maximum and minimum values of f by using first derivative test. [A.U.Nov.2016]

Solution. $f(x) = x^4 - 2x^2 + 3.$

$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1).$

Critical numbers occur at points where $f'(x) = 0$

i.e., $4x(x-1)(x+1) = 0.$

$x = -1, x = 0, x = 1.$

The critical numbers are $x = -1, x = 0, x = 1.$

Let us evaluate the intervals at which f decreases or increases.

Interval	$4x$	$x-1$	$x+1$	$f'(x)$	f
$x < -1$	-	-	-	-	decreases in $(-\infty, -1)$
$-1 < x < 0$	-	-	+	+	increases in $(-1, 0)$
$0 < x < 1$	+	-	+	-	decreases in $(0, 1)$
$x > 1$	+	+	+	+	increases in $(1, \infty)$

Since f' changes from negative to positive at $x = -1$,

f has a local minimum at $x = -1$.

Minimum value of $f = f(-1) = 1 - 2 + 3 = 1.$

Also f' changes from positive to negative at $x = 0$.

$\therefore f$ has a maximum at $x = 0$.

Maximum value of $f = f(0) = 3.$

Again f' changes from negative to positive at $x = 1$.

$\therefore f$ has a local minimum at $x = 1$.

Minimum value of $f = 1 - 2 + 3 = 2.$

Example 1.61. Find the intervals on which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and decreasing. Also find the local maximum and minimum values of f by using first derivative test. [A.U.Nov.2016]

Solution. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5.$

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$.

Critical numbers occur at points where $f'(x) = 0$

i.e., $12x(x-2)(x+1) = 0$.

$x = -1,\ x = 0,\ x = 2$.

The critical numbers are $x = -1,\ x = 0,\ x = 2$.

Let us evaluate the intervals at which f decreases or increases.

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x < -1$	-	-	-	-	decreases in $(-\infty, -1)$
$-1 < x < 0$	-	-	+	+	increases in $(-1, 0)$
$0 < x < 2$	+	-	+	-	decreases in $(0, 2)$
$x > 2$	+	+	+	+	increases in $(2, \infty)$

Since f' changes from negative to positive at $x = -1$,

f has a local minimum at $x = -1$.

Minimum value of $f = f(-1) = 3 + 4 - 12 + 5 = 20$.

Also f' changes from positive to negative at $x = 0$.

$\therefore f$ has a local maximum at $x = 0$.

Maximum value of $f = f(0) = 5$.

Again f' changes from negative to positive at $x = 2$.

$\therefore f$ has a minimum at $x = 2$.

Minimum value of $f = 3 \times 16 - 4 \times 8 - 12 \times 4 + 5 = 48 - 32 - 48 + 5 = -27$.

Example 1.62. Find the local maximum and minimum values of the function $f(x) = x + 2\sin x,\ 0 \le x \le 2\pi$.

Solution. $f(x) = x + 2\sin x$.

$f'(x) = 1 + 2\cos x$.

$f''(x) = -2\sin x$.

Local maximum and minimum occur at points where $f'(x) = 0$.

i.e., $1 + 2\cos x = 0$.

$2\cos x = -1$.

$\cos x = \frac{-1}{2}$.

$\cos x$ is negative in the II and III quadrants.

$x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$.

First derivative test

The intervals to be considered are $0 < x < \frac{2\pi}{3}, \frac{2\pi}{3} < x < \frac{4\pi}{3}, \frac{4\pi}{3} < x < 2\pi$.

Let us evaluate the intervals at which f decreases and increases.

Interval	$f'(x) = 1 + 2\cos x$	f
$0 < x < \frac{2\pi}{3}$	+	increasing
$\frac{2\pi}{3} < x < \frac{4\pi}{3}$	-	decreasing
$\frac{4\pi}{3} < x < 2\pi$	+	increasing

Since f' changes from positive to negative at $\frac{2\pi}{3}$,

f has a local maximum at $x = \frac{2\pi}{3}$.

$\therefore$ Maximum value of $f = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} = \frac{2\pi}{3} + 2\frac{\sqrt{3}}{2} = \frac{2\pi}{3} + \sqrt{3}$.

Since f' changes from negative to positive at $x = \frac{4\pi}{3}$,

f has a minimum at $x = \frac{4\pi}{3}$.

$\therefore$ Minimum value of $f = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3} = \frac{4\pi}{3} + 2\left(\frac{-\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3}$.

Second derivative test

When $x = \frac{2\pi}{3}$, $f''(x) = -2\sin\frac{2\pi}{3} = -2\frac{\sqrt{3}}{2} = -\sqrt{3} < 0$.

$\therefore f$ has a maximum at $x = \frac{2\pi}{3}$.

Maximum value of $f = \frac{2\pi}{3} + \sqrt{3}$.

When $x = \frac{4\pi}{3}$, $f''(x) = -2\sin\frac{4\pi}{3} = -2\frac{-\sqrt{3}}{2} = \sqrt{3} > 0$.

$\therefore f$ has a minimum at $x = \frac{4\pi}{3}$.

$\therefore$ Minimum value of $f = \frac{4\pi}{3} - \sqrt{3}$.

Example 1.63. Find the local maxima and minima and the points of inflexion for the function $f(x) = x^4 - 4x^3$.

Solution. $f(x) = x^4 - 4x^3$.

$f'(x) = 4x^3 - 12x^2$.

$f''(x) = 12x^2 - 24x.$

Local maximum and minimum occur at points where $f'(x) = 0$

i.e., $4x^3 - 12x^2 = 0.$

$4x^2(x-3) = 0.$

$x = 0,\ x = 3.$

When $x = 0,\ f''(x) = 0.$

$\therefore\ f(x)$ does not have either mxaimum or minimum.

When $x = 3,\ f''(x) = 12 \times 9 - 24 \times 3 = 108 - 72 = 36 > 0$

$\therefore\ f$ has a local minimum at $x = 3.$

Minimum value of $f = 81 - 4 \times 27 = 81 - 108 = -27.$

At the points of inflexion, $f''(x) = 0$

$12x^2 - 24x = 0 \Rightarrow 12x(x-2) = 0 \Rightarrow x = 0, x = 2.$

When $x = 0,\ f(0) = 0.$

$\therefore$ $(0,0)$ is a point of inflexion.

When $x = 2,\ f(2) = 2^4 - 4 \times 2^3 = 16 - 32 = -16.$

Another point of inflexion is $(2,-16)$.

Example 1.64. Find the maxima and minima of the function $f(x) = x^{2/3}(6-x)^{1/3}$.

Solution. $f(x) = x^{2/3}(6-x)^{1/3}$.

$$\begin{aligned}
f'(x) &= x^{2/3}\frac{1}{3}(6-x)^{-2/3}(-1) + (6-x)^{1/3}\frac{2}{3}x^{-1/3}. \\
&= \frac{1}{3}\left[\frac{-x^{2/3}}{(6-x)^{2/3}} + \frac{2(6-x)^{1/3}}{x^{1/3}}\right]. \\
&= \frac{1}{3}\left[\frac{-x+2(6-x)}{x^{1/3}(6-x)^{2/3}}\right] \\
&= \frac{1}{3}\left[\frac{-x+12-2x}{x^{1/3}(6-x)^{2/3}}\right] \\
&= \frac{1}{3}\left[\frac{12-3x}{x^{1/3}(6-x)^{2/3}}\right] = \left[\frac{4-x}{x^{1/3}(6-x)^{2/3}}\right].
\end{aligned}$$

Now, $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ and $x = 6$.

$\therefore$ The critical numbers are 0, 4, 6.

Let us evaluate the intervals at which f increases or decreases.

Interval	$4-x$	$x^{1/3}$	$(6-x)^{2/3}$	$f'(x)$	f
$-\infty < x < 0$	+	-	+	-	decreases on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increases on $(0, 4)$
$4 < x < 6$	-	+	+	-	decreases on $(4, 6)$
$x > 6$	-	+	+	-	decreases on $(6, \infty)$

Since f' changes from negative to positive at $x = 0$, f has a local minimum at $x = 0$.

$\therefore$ Minimum value of $f = f(0) = 0$.

Also f' changes from positive to negative at $x = 4$.

$\therefore$ f has a local maximum at $x = 0$.

$\therefore$ Maximum value of $f = f(4) = 4^{2/3}(6-4)^{1/3} = (2^2)^{2/3}(2^{1/3}) = 2^{4/3}2^{1/3} = 2^{5/3}$.

Again f' does not changes sign at $x = 6$.

$\therefore$ f has no maximum or minimum at $x = 6$.

Example 1.65. Using first derivative test, examine for maximum and minimum of the function $f(x) = x^3 - 3x + 3$, $x \in \mathbb{R}$.

Solution. $f(x) = x^3 - 3x + 3$.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$.

Critical numbers occur at points where $f'(x) = 0$.

i.e., $3(x-1)(x+1) = 0$.

i.e., $x = -1$, $x = 1$.

The critical numbers are $x = -1, 1$.

Let us evaluate the intervals at which f decreases or increases.

Interval	$x-1$	$x+1$	$f'(x)$	f
$x < -1$	-	-	+	increases in $(-\infty, -1)$
$-1 < x < 1$	-	+	-	decreases in $(-1, 1)$
$x > 1$	+	+	-	increases in $(1, \infty)$

Since f' changes from positive to negative at $x = -1$.

f has a local maximum at $x = -1$.

$\therefore$ Maximum value of $f = f(-1) = (-1)^3 - 3(-1) + 3 = -1 + 3 + 3 = 5.$

Also f' changes from negative to positive at $x = 1$.

$\therefore f$ has a local minimum at $x = 1$.

$\therefore$ Minimum value of $f = f(1) = 1^3 - 1 \times 3 + 3 = 1 - 3 + 3 = 1.$

Example 1.66. Using first derivative test examine the maximum and minimum of $f(x) = \sin^2 x$, $0 < x < \pi$.

Solution. $f(x) = \sin^2 x$.

$f'(x) = 2\sin x \cos x = \sin 2x$.

Critical numbers occur at points where $f'(x) = 0$.

i.e., $\sin 2x = 0$.

$\therefore 2x = 0$, or $2x = \pi$.

$\therefore x = 0$, or $x = \frac{\pi}{2}$.

The critical numbers are $x = 0$, $x = \frac{\pi}{2}$.

Let us evaluate the intervals at which f decreases or increases.

Interval	$\sin 2x$	$f'(x)$	f
$0 < x < \frac{\pi}{2}$	+	+	increases in $(0, \frac{\pi}{2})$
$x > \frac{\pi}{2}$	-	-	decreases in $(\frac{\pi}{2}, \pi)$

Since f' changes from positive to negative at $x = \frac{\pi}{2}$.

$\therefore f$ has a maximum at $x = \frac{\pi}{2}$.

$\therefore$ Maximum value of $f = f(\frac{\pi}{2}) = \sin^2\left(\frac{\pi}{2}\right) = 1.$

Example 1.67. Find the maximum and minimum of the function $f(x) = 2x^3 - 3x^2 - 36x + 10$.

Solution. $f(x) = 2x^3 - 3x^2 - 36x + 10$.

$f'(x) = 6x^2 - 6x - 36$.

$f''(x) = 12x - 6$.

Critical pints occur at $f'(x) = 0$.

i.e., $6x^2 - 6x - 36 = 0$.

$\therefore x^2 - x - 6 = 0$.

$\therefore (x-3)(x+2) = 0.$

$\therefore x = -2,\ x = 3.$

The critical points are at $x = -2,\ x = 3$.

At $x = -2,\ f''(x) = 12 \times (-2) - 6 = -24 - 6 = -30 = -ve$

$\therefore f$ is maximum at $x = -2$.

Maximum value of

$$f = 2(-2)^3 - 3(-2)^2 - 36(-2) + 10 = -16 - 12 + 72 + 10 = 54.$$

At $x = 3,\ f''(x) = 12 \times 3 - 6 = 36 - 6 = 30 = +ve.$

$\therefore f$ has minimum at $x = 3$.

$\therefore$ Minimum value of

$$f = f(3) = 2 \times 3^3 - 3 \times 3^2 - 36 \times 3 + 10 = 54 - 27 - 108 + 10 = -71.$$

Example 1.68. Find the maxima and minima of the function $f(x) = \sin x(1 + \cos x)$, $0 \leq x \leq 2\pi$.

Solution. Given $f(x) = \sin x(1 + \cos x)$

$$f'(x) = \sin x(0 - \sin x) + (1 + \cos x)(\cos x) = -\sin^2 x + \cos x + \cos^2 x = \cos x + \cos 2x.$$

$$f''(x) = -\sin x - 2\sin 2x.$$

At critical points $f'(x) = 0$.

$$\cos x + \cos 2x = 0$$

$$\cos x + 2\cos^2 x - 1 = 0$$

$$\text{i.e.,}\quad 2\cos^2 x + \cos x - 1 = 0$$

$$\cos x = \frac{-1 \pm \sqrt{1 - 4 \times 2 \times (-1)}}{2 \times 2} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4}$$

$$\cos x = \frac{-1+3}{4} = \frac{1}{2} \quad \text{or} \quad \cos x = \frac{-1-3}{4} = -1.$$

$\cos x = \frac{1}{2}$ gives $x = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.

$\cos x = -1$ gives $x = \pi$.

When $x = \frac{\pi}{3},\ f''(x) = -\sin\frac{\pi}{3} - 2\sin\frac{2\pi}{3} = -\frac{\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2} < 0.$

$\therefore f$ has a maximum at $x = \frac{\pi}{3}$.

Maximum value of $f = f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)\left(1+\cos\left(\frac{\pi}{3}\right)\right) = \dfrac{\sqrt{3}}{2}(1+\frac{1}{2}) = \frac{3\sqrt{3}}{4}$.

When $x = \dfrac{5\pi}{3}$, $f''(x) = -\sin\frac{5\pi}{3} - 2\sin\frac{10\pi}{3} = -\left(\frac{-\sqrt{3}}{2}\right) - 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} + 2\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} > 0$.

$\therefore$ f has a minimum at $x = \dfrac{5\pi}{3}$.

$\therefore$ Minimum value of $f = f\left(\frac{5\pi}{3}\right) = \sin\left(\frac{5\pi}{3}\right)\left(1+\cos\left(5\frac{\pi}{3}\right)\right) = \dfrac{-\sqrt{3}}{2}(1+\frac{1}{2}) = \frac{-\sqrt{3}}{2}\times\frac{3}{2} = \frac{-3\sqrt{3}}{4}$.

When $x = \pi$, $f''(x) = -\sin\pi - 2\sin 2\pi = 0$.

$\therefore$ f has neiher maximum nor minimum at $x = \pi$.

$\therefore$ Maximum value $= \frac{3\sqrt{3}}{4}$.

$\therefore$ Minimum value $= \frac{-3\sqrt{3}}{4}$.

2 Function of Several Variables

2.1 Limits and Continuity

Limit. Let $f(x, y)$ be a function of two independent variables x and y. $f(x, y)$ is said to have the limit ℓ as (x, y) tends to (x_0, y_0), if corresponding to any positive number ϵ, however small, there is a positive number η such that

$$|f(x, y) - \ell| < \epsilon \text{ where } 0 < d < \eta,$$

where d is the distance between (x, y) and (x_0, y_0) given by $d^2 = (x - x_0)^2 + (y - y_0)^2$.

Note. If the limit exists, then it can be written as $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = \ell$.

Continuity. $f(x, y)$ is said to be continuous at (x_0, y_0) if, to any positive number ϵ, however small, there corresponds a positive number η such that

$$|f(x, y) - f(x_0, y_0)| < \epsilon \text{ where } 0 < d < \eta,$$

where d is the distance between (x, y) and (x_0, y_0) given above.

Result 1. The above definition is equivalent to $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(x_0, y_0)$.

Result 2. Usually the limit is the same irrespective of the path along which the point (x, y) approaches (x_0, y_0).

$$\text{i.e., } \lim_{x \to x_0} \left\{ \lim_{y \to y_0} f(x, y) \right\} = \lim_{y \to y_0} \left\{ \lim_{x \to x_0} f(x, y) \right\}.$$

But this is not true always.

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But this is not true always.

Consider the following example.

Let $f(x,y) = \dfrac{x-2y}{x+2y}$.

As $(x,y) \to (0,0)$ along the line $y = mx$, we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{x-2y}{x+2y}\\ &= \lim_{x\to 0} \frac{x-2mx}{x+2mx}\\ &= \lim_{x\to 0} \frac{x(1-2m)}{x(1+2m)}\\ &= \frac{1-2m}{1+2m},\end{aligned}$$

which is different for lines with different slopes.

Also $\lim\limits_{x\to 0}\left\{\lim\limits_{y\to 0} \dfrac{x-2y}{x+2y}\right\} = \lim\limits_{x\to 0}\left(\dfrac{x}{x}\right) = 1$

and $\lim\limits_{y\to 0}\left\{\lim\limits_{x\to 0} \dfrac{x-2y}{x+2y}\right\} = \lim\limits_{y\to 0}\left(\dfrac{-2y}{2y}\right) = -1.$

Hence, as (x,y) approaches $(0,0)$ along different paths, $f(x,y)$ approaches different limits. Hence, the two repeated limits are not equal and hence $f(x,y)$ is discontinuous at the origin.

Note. If a function is continuous at every point of the domain $a \le x \le a'$, $b \le y \le b'$, it is said to be continuous in that domain.

Worked Examples

Example 2.1. Find $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{2x^2y}{x^2+y^2+1}$.

Solution.

$$\lim_{\substack{x\to 1\\ y\to 2}} \frac{2x^2y}{x^2+y^2+1} = \frac{2\times 1\times 2}{1+4+1} = \frac{4}{6} = \frac{2}{3}.$$

Example 2.2. Find $\lim\limits_{\substack{x\to \infty\\ y\to 2}} \dfrac{xy+1}{x^2+2y^2}$.

Solution. $\lim\limits_{\substack{x\to \infty\\ y\to 2}} \dfrac{xy+1}{x^2+2y^2} = \lim\limits_{\substack{x\to \infty\\ y\to 2}} \dfrac{x(y+\frac{1}{x})}{x^2\left(1+2(\frac{y}{x})^2\right)} = \lim\limits_{\substack{x\to \infty\\ y\to 2}} \dfrac{y+\frac{1}{x}}{x\left(1+2(\frac{y}{x})^2\right)} = \dfrac{2}{\infty} = 0.$

Example 2.3. If $f(x,y) = \dfrac{x-y}{2x+y}$, show that $\lim\limits_{x\to 0}\left\{\lim\limits_{y\to 0} f(x,y)\right\} \neq \lim\limits_{y\to 0}\left\{\lim\limits_{x\to 0} f(x,y)\right\}$.

Solution.

$$\begin{aligned}\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} &= \lim_{x\to 0}\left\{\lim_{y\to 0} \frac{x-y}{2x+y}\right\}\\ &= \lim_{x\to 0}\left(\frac{x}{2x}\right) = \frac{1}{2}.\\ \lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\} &= \lim_{y\to 0}\left\{\lim_{x\to 0} \frac{x-y}{2x+y}\right\}\\ &= \lim_{y\to 0}\left(\frac{-y}{y}\right) = -1.\end{aligned}$$

$\therefore$ The two limits are not equal.

Example 2.4. If $f(x,y) = \dfrac{x^3y^2 + x^2y^3 - 3}{2-xy}$, show that $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim\limits_{\substack{y\to 0\\ x\to 0}} f(x,y)$.

Solution. $$\begin{aligned}\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) &= \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)\\ &= \lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^3y^2 + x^2y^3 - 3}{2-xy}\right)\\ &= \lim_{x\to 0}\left(\frac{-3}{2}\right) = \frac{-3}{2}.\\ \lim_{\substack{y\to 0\\ x\to 0}} f(x,y) &= \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)\\ &= \lim_{y\to 0}\left(\lim_{x\to 0} \frac{x^3y^2 + x^2y^3 - 3}{2-xy}\right)\\ &= \lim_{y\to 0}\left(\frac{-3}{2}\right) = \frac{-3}{2}.\end{aligned}$$

$$\therefore \lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{y\to 0\\ x\to 0}} f(x,y) = \frac{-3}{2}.$$

2.2 Partial Derivatives

We know that, if y is a continuous function of the independent variable x, and Δy the increment in y corresponding to an increment Δx in x, then $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$ if exists, is called the differential coefficient or the derivative of y with respect to x.

Now let us consider a function of several independent variables.

Let u be a function of two independent variables x and y and let Δu be the increment in u corresponding to an increment in x, y remaining constant. If u is a continuous function of x, the limit $\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$ if exists is called the partial derivative of u with respect to x, and is represented by $\frac{\partial u}{\partial x}$ or u_x.

Hence, $\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$.

In the same way, if u is a continuous function of y, the increment Δu in u, corresponding to an increment Δy in y, x remaining constant, then the limit $\lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}$ if exists is called the partial derivative of u with respect to y and is represented by $\frac{\partial u}{\partial y}$ or u_y.

It can be easily seen that the change in the value of u corresponding to a change in the value of x will not in general be the same as the change in the value of u corresponding to an equal change in the value of y. For example, the area A of a rectangular box is a function of the length x and the breadth y, being given by the relation $A = xy$. When x alone changes,

$$\Delta A = (x + \Delta x)y - xy = y\Delta x.$$

When y alone changes, $\Delta A = x(y + \Delta y) - xy = x\Delta y$.

Though $\Delta x = \Delta y$, the values of ΔA will not be equal unless $x = y$. i.e., unless the box is in the form of a square. Hence, we can see that, in general $\frac{\partial A}{\partial x} \neq \frac{\partial A}{\partial y}$.

Likewise, if u is a continuous function of several independent variables $x, y, z, \ldots$, we can define the partial derivatives of u with respect to each of these independent variables. As an example, let us consider a function with three independent variables x, y and z.

Let $u = f(x, y, z)$. The first partial derivative of u denoted by $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ treating x, y and z respectively alone as variables can be obtained. We can also find the higher order derivatives $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial x \partial z}, \ldots$.

They are evaluated as follows.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \text{ and } \frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right).$$

$$\text{Also, } \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$\text{and } \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right).$$

Generally, $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

The third and higher orders of the partial derivatives can be obtained similarly.

Worked Examples

Example 2.5. If $V = \pi r^2 h$ where r and h are independent variables, find $\dfrac{\partial V}{\partial r}$ and $\dfrac{\partial V}{\partial h}$.

Solution. Given: $V = \pi r^2 h$.

Differentiating partially w.r.t. r,

$$\frac{\partial V}{\partial r} = \pi h \times 2r = 2\pi r h.$$

Differentiating partially w.r.t. h,

$$\frac{\partial V}{\partial h} = \pi r^2 \times 1 = \pi r^2.$$

Example 2.6. If $u = \log(e^x + e^y)$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 1$.

Solution. Given: $u = \log(e^x + e^y)$.

$$\frac{\partial u}{\partial x} = \frac{e^x}{e^x + e^y}.$$

$$\frac{\partial u}{\partial y} = \frac{e^y}{e^x + e^y}.$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{e^x + e^y}{e^x + e^y} = 1.$$

Example 2.7. If $z = \tan^{-1}\left(\dfrac{x}{y}\right)$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$.

Solution. Given: $z = \tan^{-1}\left(\frac{x}{y}\right)$.

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{x^2}{y^2}}\frac{1}{y} = \frac{y^2}{x^2+y^2}\frac{1}{y} = \frac{y}{x^2+y^2}.$$

$$x\frac{\partial z}{\partial x} = \frac{xy}{x^2+y^2}.$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) = -\frac{xy^2}{y^2(x^2+y^2)} = -\frac{x}{x^2+y^2}$$

$$y\frac{\partial z}{\partial y} = -\frac{xy}{x^2+y^2}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{xy}{x^2+y^2} - \frac{xy}{x^2+y^2} = 0.$$

Example 2.8. If $u = (x-y)^4 + (y-z)^4 + (z-x)^4$, find the value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

Solution. $u = (x-y)^4 + (y-z)^4 + (z-x)^4$.

$$\frac{\partial u}{\partial x} = 4(x-y)^3 + 4(z-x)^3(-1) = 4(x-y)^3 - 4(z-x)^3.$$

$$\frac{\partial u}{\partial y} = 4(y-z)^3 - 4(x-y)^3.$$

$$\frac{\partial u}{\partial z} = 4(z-x)^3 - 4(y-z)^3.$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Example 2.9. If $u = \log(x^3+y^3+z^3-3xyz)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$.

Solution. $u = \log(x^3+y^3+z^3-3xyz)$.

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}.$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}.$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3yx}{x^3+y^3+z^3-3xyz}.$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3x^2+3y^2+3z^2-3xy-3yz-3zx}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}=\frac{3}{x+y+z}.$$

Example 2.10. If $u=(x-y)(y-z)(z-x)$, show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.

Solution. Given, $u=(x-y)(y-z)(z-x)$.

$$\begin{aligned}\frac{\partial u}{\partial x}&=(y-z)\{(x-y)(-1)+(z-x)\cdot 1\}\\&=(y-z)(-x+y+z-x)\\&=(y-z)(y+z-2x)=y^2-z^2-2x(y-z).\\\frac{\partial u}{\partial y}&=(z-x)\{(x-y)\cdot 1+(y-z)(-1)\}\\&=(z-x)\{x-y-y+z\}\\&=(z-x)(z+x-2y)\\&=z^2-x^2-2y(z-x).\\\frac{\partial u}{\partial z}&=(x-y)\{(y-z)\cdot 1+(z-x)\cdot(-1)\}\\&=(x-y)(y-z-z+x)\\&=(x-y)(x+y-2z)\\&=x^2-y^2-2z(x-y).\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}&=y^2-z^2-2x(y-z)+z^2-x^2-2y(z-x)+x^2-y^2-2z(x-y)\\&=-2\{xy-xz+yz-xy+zx-yz\}=-2\times 0=0.\end{aligned}$$

Example 2.11. If $u=\log\sqrt{x^2+y^2}$, show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

Solution. $u=\log\sqrt{x^2+y^2}=\frac{1}{2}\log(x^2+y^2)$.

Differentiating partially w.r.t. x we get

$$\frac{\partial u}{\partial x}=\frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x=\frac{x}{x^2+y^2}.$$

Again differentiating partially w.r.t. x we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Differentiating u partially w.r.t. y we get

$$\frac{\partial u}{\partial y} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2y = \frac{y}{x^2+y^2}.$$

Again differentiating partially w.r.t. y we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0.$$

Example 2.12. If $u = \log(x^2+y^2+z^2)$, prove that $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 2.$

Solution. Given, $u = \log(x^2+y^2+z^2)$

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{2x}{x^2+y^2+z^2}.\\
\frac{\partial^2 u}{\partial x^2} &= \frac{(x^2+y^2+z^2)\cdot 2 - 2x\cdot 2x}{(x^2+y^2+z^2)^2}\\
&= \frac{2x^2+2y^2+2z^2-4x^2}{(x^2+y^2+z^2)^2}\\
&= \frac{2y^2+2z^2-2x^2}{(x^2+y^2+z^2)^2}\\
&= \frac{2(y^2+z^2-x^2)}{(x^2+y^2+z^2)^2}.
\end{aligned}$$

Similarly, we have

$$\frac{\partial^2 u}{\partial y^2} = \frac{2(x^2+z^2-y^2)}{(x^2+y^2+z^2)^2}$$

$$\text{and } \frac{\partial^2 u}{\partial z^2} = \frac{2(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}.$$

$$\text{Now, } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{2[y^2+z^2-x^2+x^2+z^2-y^2+x^2+y^2-z^2]}{(x^2+y^2+z^2)^2}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$$

$$= \frac{2}{x^2+y^2+z^2}.$$

$$\therefore\ (x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) = 2$$

Example 2.13. If $u = e^{xy}$, show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{1}{u}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$. [Jan 2013]

Solution. $u = e^{xy}$

$$\frac{\partial u}{\partial x} = e^{xy}\cdot y = uy.$$

$$\frac{\partial^2 u}{\partial x^2} = y\cdot\frac{\partial u}{\partial x} = yuy = uy^2.$$

$$\frac{\partial u}{\partial y} = e^{xy}\cdot x = ux.$$

$$\frac{\partial^2 u}{\partial y^2} = x\cdot\frac{\partial u}{\partial y} = x\cdot ux = ux^2.$$

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2 = u^2y^2+u^2x^2$$

$$= u[uy^2+ux^2]$$

$$= u\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

$$\frac{1}{u}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

Example 2.14. If $u = yf\left(\frac{x}{y}\right)+g\left(\frac{y}{x}\right)$ find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$. [Dec 2012]

Solution. $u = yf\left(\frac{x}{y}\right)+g\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = y f'\left(\frac{x}{y}\right) \cdot \frac{1}{y} + g'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = x f'\left(\frac{x}{y}\right) - \frac{y}{x} g'\left(\frac{y}{x}\right). \tag{1}$$

$$\frac{\partial u}{\partial y} = y f'\left(\frac{x}{y}\right) \cdot \left(\frac{-x}{y^2}\right) + f\left(\frac{x}{y}\right) \cdot 1 + g'\left(\frac{y}{x}\right) \cdot \frac{1}{x}.$$

$$y\frac{\partial u}{\partial y} = -x f'\left(\frac{x}{y}\right) + y f\left(\frac{x}{y}\right) + \frac{y}{x} g'\left(\frac{y}{x}\right) \tag{2}$$

$$(1) + (2) \Rightarrow \; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = y f\left(\frac{x}{y}\right).$$

Example 2.15. If $r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$.

Solution. Given: $r^2 = x^2 + y^2 + z^2$.

Differentiating w.r.t. x partially we get,

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r - x\frac{\partial r}{\partial x}}{r^2} = \frac{r - x\frac{x}{r}}{r^2} = \frac{r^2 - x^2}{r^3}.$$

$$\text{Similarly, } \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3}$$

$$\text{and } \frac{\partial^2 r}{\partial z^2} = \frac{r^2 - z^2}{r^3}.$$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{3r^2 - (x^2 + y^2 + z^2)}{r^3}$$

$$= \frac{3r^2 - r^2}{r^3} = \frac{2r^2}{r^3} = \frac{2}{r}.$$

Example 2.16. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

Solution. Differentiating u partially w.r.t. y, we get

$$\begin{aligned}
\frac{\partial u}{\partial y} &= x^2 \frac{1}{1+\frac{y^2}{x^2}} \frac{1}{x} - \left[y^2 \frac{1}{1+\frac{x^2}{y^2}} \left(\frac{-x}{y^2}\right) + \tan^{-1}\left(\frac{y}{x}\right) 2y\right] \\
&= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\left(\frac{y}{x}\right) \\
&= \frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) = x - 2y\tan^{-1}\left(\frac{x}{y}\right)
\end{aligned}$$

Differentiating this partially w.r.t. x, we get

$$\begin{aligned}
\frac{\partial^2 u}{\partial x \partial y} &= 1 - 2y \frac{1}{1+\frac{x^2}{y^2}} \frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2} \\
&= \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}.
\end{aligned}$$

Example 2.17. If $u = x^y$, show that $u_{xxy} = u_{xyx}$. [Jan 2012, Jun 2008]

Solution. Given: $u = e^{y \log x}$

$$\begin{aligned}
u_y &= e^{y\log x} \log x = x^y \log x \\
u_{xy} &= yx^{y-1} \log x + x^y \frac{1}{x} \\
&= yx^{y-1} \log x + x^{y-1} = x^{y-1}(y \log x + 1) \\
u_{xxy} &= x^{y-1} \frac{y}{x} + (y\log x + 1)(y-1)x^{y-2} \\
&= x^{y-2} y + x^{y-2}(y \log x + 1)(y-1) \\
&= x^{y-2}(y - y\log x - 1 + y^2 \log x + y) \\
&= x^{y-2}(2y - 1 + y \log x(y-1)) \\
&= x^{y-2}(y(y-1)\log x + 2y - 1) \qquad (1) \\
u &= x^y \\
u_x &= yx^{y-1} = ye^{(y-1)\log x} \\
u_{yx} &= ye^{(y-1)\log x} \log x + e^{(y-1)\log x}.1
\end{aligned}$$

$$= yx^{y-1}\log x + x^{y-1} = x^{y-1}(y\log x + 1)$$

$$u_{xyx} = x^{y-1}\left(\frac{y}{x}\right) + (y\log x + 1)(y-1)x^{y-2}$$

$$= x^{y-2}y + x^{y-2}(y-1)(y\log x + 1)$$

$$= x^{y-2}(y + y^2\log x + y - y\log x - 1)$$

$$= x^{y-2}(2y - 1 + y\log x(y-1)) \tag{2}$$

From (1) and (2) we have $u_{xxy} = u_{xyx}$.

Example 2.18. If $u = (x-y)f\left(\frac{y}{x}\right)$, find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.
[May 2001]

Solution. $u = (x-y)f\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = (x-y)f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right).$$

$$\frac{\partial^2 u}{\partial x^2} = -(x-y)\left(\frac{y}{x^2}\right)\cdot f''\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) - (x-y)f'\left(\frac{y}{x}\right)\cdot y(-2)x^{-3}$$
$$-\frac{y}{x^2}\cdot f'\left(\frac{y}{x}\right)\cdot 1 + f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right).$$

$$= (x-y)\frac{y^2}{x^4}f''\left(\frac{y}{x}\right) + 2\frac{y}{x^3}(x-y)f'\left(\frac{y}{x}\right) - \frac{y}{x^2}f'\left(\frac{y}{x}\right) - \frac{y}{x^2}f'\left(\frac{y}{x}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} = \frac{y^2}{x^2}(x-y)f''\left(\frac{y}{x}\right) + \frac{2y}{x}(x-y)f'\left(\frac{y}{x}\right) - 2yf'\left(\frac{y}{x}\right). \tag{1}$$

$$\frac{\partial u}{\partial y} = (x-y)f'\left(\frac{y}{x}\right)\cdot\frac{1}{x} + f\left(\frac{y}{x}\right)(-1)$$

$$= \left(1-\frac{y}{x}\right)f'\left(\frac{y}{x}\right) - f\left(\frac{y}{x}\right).$$

$$\frac{\partial^2 u}{\partial y^2} = \left(1-\frac{y}{x}\right)f''\left(\frac{y}{x}\right)\cdot\frac{1}{x} + f'\left(\frac{y}{x}\right)\left(\frac{-1}{x}\right) - f'\left(\frac{y}{x}\right)\cdot\frac{1}{x}.$$

$$= \frac{(x-y)}{x^2}f''\left(\frac{y}{x}\right) - \frac{2}{x}f'\left(\frac{y}{x}\right)$$

$$y^2\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{x^2}(x-y)f''\left(\frac{y}{x}\right) - \frac{2y^2}{x}f'\left(\frac{y}{x}\right). \tag{2}$$

Differentiating $\frac{\partial u}{\partial y}$ w.r.t. x partially we get

$$\frac{\partial^2 u}{\partial x \partial y} = \left(1 - \frac{y}{x}\right) f''\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f'\left(\frac{y}{x}\right)(-y)\left(\frac{-1}{x^2}\right) - f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$= -\frac{(x-y)y}{x^3} f''\left(\frac{y}{x}\right) + \frac{y}{x^2} f'\left(\frac{y}{x}\right) + \frac{y}{x^2} f'\left(\frac{y}{x}\right)$$

$$= -\frac{(x-y)\cdot y}{x^3} f''\left(\frac{y}{x}\right) + \frac{2y}{x^2} f'\left(\frac{y}{x}\right).$$

$$2xy\frac{\partial^2 u}{\partial x \partial y} = \frac{-2y^2}{x^2}(x-y) f''\left(\frac{y}{x}\right) + \frac{4y^2}{x} f'\left(\frac{y}{x}\right). \tag{3}$$

$$(1)+(2)+(3) \Rightarrow \; x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

Homogeneous function. A function $f(x, y)$ is said to be a homogeneous function in x and y of degree n if $f(tx, ty) = t^n f(x, y)$ for any positive t.

Example

(i) $f(x, y) = x^2 + y^2 + 2xy$ is a homogeneous function of degree 3 in x and y.

(ii) If $\tan^{-1}\left(\frac{x}{y}\right) = u$ then $\tan u$ is a homogeneous function of degree 0.

Note. If $u = f(x, y)$ is a homogeneous function of degree n in x and y, then u can be written as $u = x^n F\left(\frac{y}{x}\right)$.

Example. Let $u = \frac{x^3 + y^3}{x + y}$.

$$u(tx, ty) = \frac{t^3x^3 + t^3y^3}{tx + ty} = \frac{t^3(x^3 + y^3)}{t(x + y)} = t^2\frac{x^3 + y^3}{x + y} = t^2 u.$$

u is a homogeneous function of degree 2.

$$\text{Now } u = \frac{x^3\left(1 + \frac{y^3}{x^3}\right)}{x\left(1 + \frac{y}{x}\right)}$$

$$= x^2\left(\frac{1 + \left(\frac{y}{x}\right)^3}{1 + \left(\frac{y}{x}\right)}\right)$$

$$= x^2 F\left(\frac{y}{x}\right) \text{ where } F\left(\frac{y}{x}\right) = \frac{1 + \left(\frac{y}{x}\right)^3}{1 + \left(\frac{y}{x}\right)}.$$

$\therefore$ u is a homogeneous function of degree 2 in x and y.

2.3 Euler's Theorem

Statement. If $f(x, y)$ is a homogeneous function of degree n in x and y having continuous partial derivatives, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

Proof.

Given that $f(x, y)$ is a homogeneous function of degree n in x and y. Hence $f(x, y)$ can be written as

$$\begin{aligned}
f(x, y) &= x^n F\left(\frac{y}{x}\right). \\
\frac{\partial f}{\partial x} &= nx^{n-1}F\left(\frac{y}{x}\right) + x^n F'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \\
&= nx^{n-1}F\left(\frac{y}{x}\right) - x^{n-2}yF'\left(\frac{y}{x}\right). \\
\frac{\partial f}{\partial y} &= x^n F'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = x^{n-1}F'\left(\frac{y}{x}\right). \\
x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= nx^n F\left(\frac{y}{x}\right) - x^{n-1}yF'\left(\frac{y}{x}\right) + x^{n-1}yF'\left(\frac{y}{x}\right) \\
&= nx^n F\left(\frac{y}{x}\right) \\
x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= nf(x, y).
\end{aligned}$$

Extension. If $f(x_1, x_2, \ldots, x_n)$ is a homogeneous function of degree n in $x_1, x_2, \ldots, x_n$, then $x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots + x_n\frac{\partial f}{\partial x_n} = nf$.

Euler's theorem on higher partial derivatives

Statement. If $f(x, y)$ is a homogeneous function of degree n in x and y then, $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$.

Proof. Given, $f(x, y)$ is a homogeneous function of degree n in x and y.

$\therefore$ By Euler's theorem we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf. \tag{1}$$

Differentiating this partially w.r.t. x we obtain

$$x\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} + y\frac{\partial^2 f}{\partial x \partial y} = n\frac{\partial f}{\partial x}.$$

$$\implies x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} = n\frac{\partial f}{\partial x} - \frac{\partial f}{\partial x} = (n-1)\frac{\partial f}{\partial x}.$$

Multiplying both sides by x we get

$$x^2\frac{\partial^2 f}{\partial x^2} + xy\frac{\partial^2 f}{\partial x \partial y} = (n-1)x\frac{\partial f}{\partial x}. \tag{2}$$

Differentiating (1) partially w.r.t. y we get

$$x\frac{\partial^2 f}{\partial y \partial x} + y\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y} = n\frac{\partial f}{\partial y}$$

$$x\frac{\partial^2 f}{\partial y \partial x} + y\frac{\partial^2 f}{\partial y^2} = n\frac{\partial f}{\partial y} - \frac{\partial f}{\partial y} = (n-1)\frac{\partial f}{\partial y}.$$

Since the partial derivatives of f are continuous we have $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

$$\therefore x\frac{\partial^2 f}{\partial x \partial y} + y\frac{\partial^2 f}{\partial y^2} = (n-1)\frac{\partial f}{\partial y}.$$

Multiplying both sides by y we get

$$xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = (n-1)y\frac{\partial f}{\partial y}. \tag{2}$$

$$\begin{aligned}(1)+(2) \implies x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} &= (n-1)\Big(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\Big) \\ &= (n-1)nf \qquad \text{[By Euler's theorem]} \\ &= n(n-1)f.\end{aligned}$$

Worked Examples

Example 2.19. If $u = (x-y)(y-z)(z-x)$ prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$ and $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3u$.

Solution. $u = (x-y)(y-z)(z-x)$.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= (y-z)((x-y)(-1) + z - x) \\
&= (y-z)(-x+y+z-x) \\
&= (y-z)(y+z-2x).
\end{aligned}$$

Similarly, $\dfrac{\partial u}{\partial y} = (z-x)(z+x-2y)$ and $\dfrac{\partial u}{\partial z} = (x-y)(x+y-2z)$.

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= y^2 - z^2 - 2x(y-z) + z^2 - x^2 - 2y(z-x) + x^2 - y^2 - 2z(x-y). \\
&= -2xy + 2xz - 2yz + 2xy - 2xz + 2yz = 0
\end{aligned}$$

$$\begin{aligned}
\text{Also, } u(tx, ty, tz) &= (tx-ty)(ty-tz)(tz-tx) \\
&= t^3(x-y)(y-z)(z-x) = t^3 u.
\end{aligned}$$

$\therefore$ u is a homogeneous function of degree 3 in x, y, z.

$\therefore$ By Euler's theorem, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3u$.

Example 2.20. If $u = \log\left(\dfrac{x^3+y^3}{x+y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2$.

Solution. Given: $u = \log\left(\dfrac{x^3+y^3}{x+y}\right) \Rightarrow e^u = \dfrac{x^3+y^3}{x+y}$

$$\begin{aligned}
e^{u(tx,ty)} &= \frac{t^3x^3 + t^3y^3}{tx+ty} = \frac{t^3(x^3+y^3)}{t(x+y)} \\
&= t^2\frac{x^3+y^3}{x+y} = t^2 e^u.
\end{aligned}$$

$\therefore$ e^u is a homogeneous function of degree 2.

Here $n = 2$.

$\therefore$ By Euler's theorem,

$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 2e^u.$$

$$\implies xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 2e^u$$

$$e^u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2e^u.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2.$$

Example 2.21. If $u = \frac{y}{x} + \frac{z}{x}$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Solution. Given: $u = \frac{y}{x} + \frac{z}{x}$.

$$u(tx, ty, tz) = \frac{ty}{tx} + \frac{tz}{tx} = t^0\left(\frac{y}{x} + \frac{z}{x}\right) = t^0 u.$$

$\implies u$ is a homogeneous function in x, y, z of degree 0.

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 0.u = 0.$$

Example 2.22. If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$. [Jun 2012]

Solution. $u(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$.

$$\begin{aligned} u(tx, ty, tz) &= \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx} \\ &= \frac{t}{t}\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right) \\ &= t^0 u(x, y, z). \end{aligned}$$

u is a homogeneous function of degree 0 in x, y, z.

$\therefore$ By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.u = 0$.

Example 2.23. If $\sin u = \frac{x^2y^2}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$.

Solution. $\sin u(x, y) = \frac{x^2y^2}{x+y}$.

$$\sin u(tx, ty) = \frac{t^2x^2t^2y^2}{tx+ty} = \frac{t^4(x^2y^2)}{t(x+y)} = t^3 \sin u.$$

$\sin u$ is a homogeneous function of degree 3 in x and y.

By Euler's theorem,

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 3\sin u.$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 3\sin u$$

$$\cos u\Big(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\Big) = 3\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3\sin u}{\cos u} = \frac{3\sin u}{\cos u} = 3\tan u.$$

Example 2.24. If $u = f\Big(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\Big)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$

Solution.

$$u(x,y,z) = f\Big(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\Big)$$

$$u(tx,ty,tz) = f\Big(\frac{tx}{ty}, \frac{ty}{tz}, \frac{tz}{tx}\Big)$$

$$= f\Big(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\Big)$$

$$= u(x,y,z) = t^0 u(x,y,z)$$

u is a homogeneous function of degree 0 in x, y, z.

$\therefore$ By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.u = 0.$

Example 2.25. If $u = \tan^{-1}\Big(\frac{x^3+y^3}{x-y}\Big)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$

Solution. $\tan u = \frac{x^3+y^3}{x-y}.$

$\tan u$ is a homogeneous function of degree 2 in x and y.

By Euler's theorem

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$\sec^2 u x\frac{\partial u}{\partial x} + \sec^2 u y\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = \frac{2\sin u}{\cos u}\cos^2 u$$

$$= \sin 2u.$$

Example 2.26. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u.$ [Jun 2009]

Solution. $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) \Rightarrow \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}.$

$\sin u$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

By Euler's theorem,

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u.$$
$$\cos u x\frac{\partial u}{\partial x} + \cos u y\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u.$$

Example 2.27. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{2}\cot u.$ [Dec 2011]

Solution. $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right).$

$$\cos u = \left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right).$$

$\cos u$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

$\therefore$ By Euler's theorem

$$x\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cdot\cos u.$$
$$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u.$$
$$-\sin u\left(x\frac{\partial u}{\partial x} + y\frac{\partial}{\partial y}\right) = \frac{1}{2}\cos u.$$
$$\therefore\quad x\frac{\partial u}{\partial x} + y\frac{\partial}{\partial y} = \frac{-1}{2}\frac{\cos u}{\sin u} = \frac{-1}{2}\cot u.$$

Example 2.28. Verify Euler's theorem for the function $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right).$

Solution. Given: $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right).$

$$u(tx, ty) = \sin^{-1}\left(\frac{tx}{ty}\right) + \tan^{-1}\left(\frac{ty}{tx}\right)$$

$$= t^0\left(\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$= t^0 u(x, y).$$

$\Longrightarrow$ u is a homogeneous function of degree 0 in x and y.

$\therefore$ By Euler's theorem, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

Verification.

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$$

$$= \frac{y}{\sqrt{y^2 - x^2}}\frac{1}{y} - \frac{x^2 y}{x^2(x^2 + y^2)}$$

$$= \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}.$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} \tag{1}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}}\left(\frac{-x}{y^2}\right) + \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$= \frac{-xy}{y^2\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

$$= -\frac{x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} \tag{2}$$

$$(1) + (2) \Longrightarrow = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Hence, Euler's theorem is verified.

Example 2.29. Verify Euler's theorem for the function $u = (x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^n + y^n)$.

Solution. It is easy to verify that u is a homogeneous function of degree $\left(n + \frac{1}{2}\right)$ in x and y.

Hence, by Euler's theorem we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right)u. \tag{1}$$

Verification

$$\frac{\partial u}{\partial x} = (x^{\frac{1}{2}} + y^{\frac{1}{2}})nx^{n-1} + (x^n + y^n)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$x\frac{\partial u}{\partial x} = n(x^{\frac{1}{2}} + y^{\frac{1}{2}})x^n + \frac{1}{2}(x^n + y^n)x^{\frac{1}{2}}.$$

Since the function is symmetric in x and y we have

$$y\frac{\partial u}{\partial y} = n(x^{\frac{1}{2}} + y^{\frac{1}{2}})y^n + \frac{1}{2}(x^n + y^n)y^{\frac{1}{2}}.$$
$$\begin{aligned} x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= n(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^n + y^n) + \frac{1}{2}(x^n + y^n)(x^{\frac{1}{2}} + y^{\frac{1}{2}}) \\ &= (n + \frac{1}{2})(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^n + y^n) \\ &= (n + \frac{1}{2})u, \end{aligned}$$

which verifies Euler's theorem.

Example 2.30. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Solution. u is a homogeneous function of degree 2 in x and y.

By Euler's theorem, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2(2-1)u = 2u \quad [n = 2]$.

Example 2.31. If $z = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, show that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$.

Solution. Let $u = xf\left(\frac{y}{x}\right)$ and $v = g\left(\frac{y}{x}\right)$.

$\therefore z = u + v$.

u is a homogeneous function of degree 1 in x and y.

$\therefore$ By Euler's theorem,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 1(1-1)u = 0. \tag{1}$$

v is a homogeneous function of degree 0 in x and y.

By Euler's theorem,

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 0(0-1)v = 0. \tag{2}$$

$$(1)+(2) \Rightarrow x^2\frac{\partial^2(u+v)}{\partial x^2} + 2xy\frac{\partial^2(u+v)}{\partial x\partial y} + y^2\frac{\partial^2(u+v)}{\partial y^2} = 0$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0.$$

Example 2.32. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\sin u\cos 3u.$$

Solution. $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$.

$$\tan u = \left(\frac{x^3+y^3}{x-y}\right)$$

$\tan u$ is an homogeneous function of degree 2 in x and y.

By Euler's theorem,

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u.$$

$$x\cdot\sec^2 u\frac{\partial u}{\partial x} + y\cdot\sec^2 u\frac{\partial u}{\partial y} = 2\tan u.$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = 2\cdot\frac{\sin u}{\cos u}\cdot\cos^2 u$$

$$= 2\sin u\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u. \tag{1}$$

Differentiating (1) partially w.r.t. x we get

$$x\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = 2\cos 2u\frac{\partial u}{\partial x}.$$

Multiplying by x we get

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} = 2\cos 2u\cdot x\frac{\partial u}{\partial x}. \tag{2}$$

Differentiating (1) partially w.r.t. y we get

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2\cos 2u\frac{\partial u}{\partial y}.$$

Multiplying w.r.t. y we get

$$xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 2\cos 2u \cdot y\frac{\partial u}{\partial y}. \qquad (3)$$

$(2) + (3) \Rightarrow$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cos 2u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\begin{aligned} x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= (2\cos 2u - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \\ &= (2\cos 2u - 1)\sin 2u \\ &= \sin 2u\{2(2\cos^2 u - 1) - 1\} \\ &= \sin 2u\{4\cos^2 u - 3\} \\ &= 2\sin u\cos u(4\cos^2 u - 3) \\ &= 2\sin u(4\cos^3 u - 3\cos u) \\ &= 2\sin u \cdot \cos 3u. \end{aligned}$$

2.4 Total Derivative - Implicit functions

We know that if $y = f(u)$ and $u = \phi(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. We shall now extend this result to function of several variables.

Let $z = f(x, y)$ possess continuous partial derivatives and let x, y be differentiable functions of an independent variable t. Let $\Delta x, \Delta y$ and Δz be increments in x, y and z respectively, corresponding to an increment Δt in t. Then the total increment in z is given by

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y). \end{aligned}$$

By mean value theorem, $f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) = f_x(x+\theta_1 x, y+\Delta y)\Delta x$ and $f(x, y+\Delta y) - f(x,y) = f_y(x, y+\theta_2\Delta y)\Delta y, 0 < \theta_1, \theta_2 < 1$.

$$\Delta z = f_x(x+\theta_1 x, y+\Delta y)\Delta x + f_y(x, y+\theta_2\Delta y)\Delta y \qquad (1)$$

where θ_1 and θ_2 are positive proper fractions.

$$\therefore \frac{\Delta z}{\Delta t} = f_x(x+\theta_1 x, y+\Delta y)\frac{\Delta x}{\Delta t} + f_y(x, y+\theta_2\Delta y)\frac{\Delta y}{\Delta t}.$$

Taking limits as $\Delta t \to 0$, we get the total derivative

$$\frac{dz}{dt} = f_x(x,y)\frac{dx}{dt} + f_y(x,y)\frac{dy}{dt}.$$

$$\text{i.e., } \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \qquad (2)$$

In the same way if $u = f(x_1, x_2, \ldots, x_n)$ possesses continuous partial derivatives with respect to each of the independent variables and if $x_1, x_2, \ldots, x_n$ are differentiable functions of an independent variable t, then the total derivative of u with respect to t is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt}. \qquad (3)$$

Note. (2) expresses the rate of change of z w.r.t. t along the curve $x = x(t), y = y(t)$.

Composite functions. Let $z = f(u,v)$ and u and v are themselves functions of the independent variables x, y so that $u = \phi(x,y)$ and $v = \psi(x,y)$.

To find $\frac{\partial z}{\partial x}$, we consider y as a constant so that u and v may be supposed to be functions of x only. Hence, by the above result we have
$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$. Similarly $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$.
Now, in (2), if we replace t by x then y is a function of x we get

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

Similarly $\frac{dz}{dy} = \frac{\partial z}{\partial x}\frac{dx}{dy} + \frac{\partial z}{\partial y}$.

Implicit Functions. Let $f(x, y) = c$, where c is a constant, define y as an implicit function of x.

If $u = f(x, y)$, then we have

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}.$$

But $\frac{du}{dx} = 0$, since $u = f(x, y) = c$

$$\text{i.e., } \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}.$$

Similarly $\frac{dx}{dy} = -\frac{f_y}{f_x}$.

Total differential. The total differential dz of z is defined as the principal part of the increment Δz which is given by

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Worked Examples

Example 2.33. If $z = x^n y^m$ where $x = \cos at, y = \sin bt$, find $\frac{dz}{dt}$.

Solution. We have $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$.

Now $\frac{\partial z}{\partial x} = nx^{n-1}y^m, \frac{\partial z}{\partial y} = mx^n y^{m-1}$

$\frac{dx}{dt} = -a\sin at, \frac{dy}{dt} = b\cos bt.$

$$\begin{aligned}\therefore \frac{dz}{dt} &= -anx^{n-1}y^m \sin at + bmx^n y^{m-1}\cos bt \\ &= -x^n y^m\left(\frac{an}{x}\sin at - \frac{bm}{y}\cos bt\right) \\ &= -\cos^n at \sin^m bt(an\tan at - bm\cot bt).\end{aligned}$$

Example 2.34. If $u = x^2y^3, x = \log t, y = e^t$, find $\frac{du}{dt}$. [Jan 2000]

Solution. We have $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$.

$$\frac{\partial u}{\partial x} = y^3 2x = 2xy^3$$
$$\frac{\partial u}{\partial y} = x^2 3y^2 = 3x^2y^2$$
$$\frac{dx}{dt} = \frac{1}{t} \qquad \frac{dy}{dt} = e^t$$

$$\begin{aligned}\frac{du}{dt} &= 2xy^3\frac{1}{t} + 3x^2y^2e^t \\ &= \frac{2\log te^{3t}}{t} + 3(\log t)^2e^{2t}e^t \\ &= e^{3t}\log t\Big(\frac{2}{t} + 3\log t\Big) \\ &= \frac{e^{3t}\log t}{t}(2 + 3t\log t).\end{aligned}$$

Example 2.35. Find $\frac{du}{dt}$ when $u = x^2y, x = t^2, y = e^t$.

Solution.

$$\frac{\partial u}{\partial x} = 2xy, \qquad \frac{\partial u}{\partial y} = x^2$$
$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = e^t$$

$$\text{we have } \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = 2xy.2t + x^2e^t$$
$$= 4t^2e^tt + t^4e^t = 4t^3e^t + t^4e^t = t^3e^t(4 + t).$$

Example 2.36. If $u = xy + yz + zx$ where $x = \frac{1}{t}, y = e^t$ and $z = e^{-t}$ find $\frac{du}{dt}$. [Jun 2013]

Solution.

$$\frac{\partial u}{\partial x} = y + z, \qquad \frac{\partial u}{\partial y} = x + z, \qquad \frac{\partial u}{\partial z} = x + y$$
$$\frac{dx}{dt} = \frac{-1}{t^2}, \qquad \frac{dy}{dt} = e^t, \qquad \frac{dz}{dt} = -e^{-t}$$

$$\text{we have } \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$
$$= (y+z)\Big(\frac{-1}{t^2}\Big) + (x+z)e^t + (x+y)(-e^{-t})$$
$$= -\frac{e^t+e^{-t}}{t^2} + (e^{-t} + \frac{1}{t})e^t - \Big(\frac{1}{t} + e^t\Big)e^{-t}$$
$$= \frac{-2}{t^2}\cos ht + 1 + \frac{e^t}{t} - \frac{e^{-t}}{t} - 1$$
$$= \frac{-2}{t^2}\cos ht + \frac{2\sin ht}{t}$$
$$= \frac{2}{t^2}(2t\sin ht - \cos ht).$$

Example 2.37. Find $\dfrac{du}{dt}$ when $u = \sin\left(\frac{x}{y}\right), x = e^t, y = t^2$.

Solution.

$$\frac{\partial u}{\partial x} = \cos\Big(\frac{x}{y}\Big)\frac{1}{y}, \qquad \frac{\partial u}{\partial y} = -\frac{x}{y^2}\cos\Big(\frac{x}{y}\Big)$$

$$\frac{dx}{dt} = e^t \qquad \frac{dy}{dt} = 2t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \frac{1}{y}\cos\Big(\frac{x}{y}\Big)e^t - \frac{x}{y^2}\cos\Big(\frac{x}{y}\Big)2t$$
$$= \cos\Big(\frac{x}{y}\Big)\left(\frac{e^t}{t^2} - \frac{2te^t}{t^4}\right) = \cos\Big(\frac{x}{y}\Big)e^t\left(\frac{1}{t^2} - \frac{2t}{t^4}\right) = \frac{e^t(t-2)}{t^3}\cos\Big(\frac{e^t}{t^2}\Big).$$

Example 2.38. If $u = \sin^{-1}(x-y)$ where $x = 3t, y = 4t^3$, show that $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1-t^2}}$.

Solution.

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}}, \qquad \frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-(x-y)^2}}$$

$$\frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = 12t^2$$

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \\
&= \frac{1}{\sqrt{1-(x-y)^2}}3 - \frac{1}{\sqrt{1-(x-y)^2}}12t^2 \\
&= \frac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}} = \frac{3-12t^2}{\sqrt{1-(9t^2+16t^6-24t^4)}} \\
&= \frac{3-12t^2}{\sqrt{1-9t^2-16t^6+24t^4}} = \frac{3-12t^2}{\sqrt{-(16t^6-24t^4+9t^2-1)}} \\
&= \frac{3(1-4t^2)}{\sqrt{-(t^2-1)(16t^4-8t^2+1)}} = \frac{3(1-4t^2)}{\sqrt{-(t^2-1)(1-4t^2)^2}} \\
&= \frac{3(1-4t^2)}{(1-4t^2)\sqrt{(1-t^2)}} = \frac{3}{\sqrt{1-t^2}}.\end{aligned}$$

Example 2.39. Find $\frac{du}{dx}$ if $u = \cos(x^2+y^2)$ and $a^2x^2 + b^2y^2 = c^2$. [Jan 2001]

Solution. We have $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = -\sin(x^2+y^2)2x - \sin(x^2+y^2)2y\frac{dy}{dx}$$

Given that $a^2x^2 + b^2y^2 = c^2$.

Diff. w.r.t. x we get

$$\begin{aligned}2a^2x + 2b^2y\frac{dy}{dx} = 0 \Rightarrow b^2y\frac{dy}{dx} = -a^2x \Rightarrow \frac{dy}{dx} = -\frac{a^2x}{b^2y} \\
\therefore \frac{du}{dx} = -2\sin(x^2+y^2)\left(x + y\frac{dy}{dx}\right) = -2\sin(x^2+y^2)\left(x + y\left(\frac{-a^2x}{b^2y}\right)\right) \\
= 2\sin(x^2+y^2)\left(\frac{a^2-b^2}{b^2y}xy\right) = \frac{2(a^2-b^2)x}{b^2}\sin(x^2+y^2).\end{aligned}$$

Example 2.40. If $z = \sqrt{x^2+y^2}$ and $x^3+y^3+3axy = 5a^2$, find the value of $\frac{dz}{dx}$ when $x = y = a$.

Solution. Given $z = \sqrt{x^2+y^2}$.

We have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\text{Now } \frac{\partial z}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}.$$

Also $x^3 + y^3 + 3axy = 5a^2$

Diff.w.r.t. x we get

$$3x^2 + 3y^2\frac{dy}{dx} + 3a\left[x\frac{dy}{dx} + y\right] = 0$$

$$x^2 + y^2\frac{dy}{dx} + ax\frac{dy}{dx} + ay = 0$$

$$\frac{dy}{dx}(y^2 + ax) = -ay - x^2$$

$$\frac{dy}{dx} = \frac{-(x^2+ay)}{y^2+ax}.$$

$$\therefore \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\left(\frac{-(x^2+ay)}{y^2+ax}\right)$$

$$\left(\frac{dz}{dx}\right)_{x=y=a} = \frac{a}{a\sqrt{2}} + \frac{a}{a\sqrt{2}}\left(\frac{-2a^2}{2a^2}\right) = \frac{a}{a\sqrt{2}} - \frac{a}{a\sqrt{2}} = 0.$$

Example 2.41. If z is a function of x and y and $x = e^u + e^{-v}, y = e^{-u} - e^v$, show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$. [Jun 2001]

Solution. z is a composite function of u and v.

We have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}(-e^{-u}) = \frac{\partial z}{\partial x}e^u - \frac{\partial z}{\partial y}(e^{-u}).$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}) = -\frac{\partial z}{\partial x}e^{-v} - \frac{\partial z}{\partial y}(e^{v}).$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^v) = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Example 2.42. If $z = f(x, y)$ where $x = e^u \cos v, y = e^u \sin v$, show that $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$. [Jan 2000]

Solution. z is a composite function of u and v.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^u \cos v + \frac{\partial z}{\partial y}e^u \sin v.$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^u \sin v) + \frac{\partial z}{\partial y}e^u \cos v.$$

$$y\frac{\partial z}{\partial u} = y\frac{\partial z}{\partial x}e^u \cos v + y\frac{\partial z}{\partial y}e^u \sin v = e^u \sin v e^u \cos v\frac{\partial z}{\partial x} + e^u \sin v e^u \sin v\frac{\partial z}{\partial y}$$

$$= e^{2u} \sin v \cos v\frac{\partial z}{\partial x} + e^{2u} \sin^2 v\frac{\partial z}{\partial y}.$$

$$x\frac{\partial z}{\partial v} = -e^u \sin v e^u \cos v\frac{\partial z}{\partial x} + e^u \cos v e^u \cos v\frac{\partial z}{\partial y}$$

$$= -e^{2u} \sin v \cos v\frac{\partial z}{\partial x} + e^{2u} \cos^2 v\frac{\partial z}{\partial y}$$

Adding we get $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$.

Example 2.43. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$ prove that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

Solution. Given: $u = f\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$

Let $r = \frac{y-x}{xy}, s = \frac{z-x}{zx}, r = \frac{1}{x} - \frac{1}{y}, s = \frac{1}{x} - \frac{1}{z}$.

$\therefore u = f(r, s)$.

$\therefore$ u is a function of r and s and r and s are functions of x, y and z.

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial s}\left(\frac{-1}{x^2}\right)$$

$$\therefore x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}. \quad (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{-1}{y^2}\right) + \frac{\partial u}{\partial s}(0)$$

$$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}. \quad (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(\frac{1}{z^2}\right)$$

$$z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}. \tag{3}$$

$$(1) + (2) + (3) \Longrightarrow x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0.$$

Example 2.44. If $u^2 + 2v^2 = 1 - x^2 + y^2$ and $u^2 + v^2 = x^2 + y^2 - 2$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$.

Solution. Consider u and v as functions of x and y.

$$u^2 + 2v^2 = 1 - x^2 + y^2.$$

Differentiating partially w.r.t. x

$$\begin{aligned} 2u\frac{\partial u}{\partial x} + 4v\frac{\partial v}{\partial x} &= -2x \\ u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} &= -x. \qquad (1) \\ u^2 + v^2 &= x^2 + y^2 - 2 \end{aligned}$$

Differentiating partially w.r.t. x

$$\begin{aligned} 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} &= 2x \\ u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} &= x. \qquad (2) \\ (1) - (2) \Rightarrow v\frac{\partial v}{\partial x} &= -2x \\ \Rightarrow \frac{\partial v}{\partial x} &= \frac{-2x}{v} \\ (1) \Rightarrow u\frac{\partial u}{\partial x} + 2v\frac{(-2x)}{v} &= -x. \\ u\frac{\partial u}{\partial x} &= 3x \\ \Rightarrow \frac{\partial u}{\partial x} &= \frac{3x}{u}. \end{aligned}$$

Example 2.45. If z is a function of x and y and $x = u\cos\alpha - v\sin\alpha, y = u\sin\alpha + v\cos\alpha$, show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 u}{\partial v^2}$. [Jun 2000]

Solution.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha.$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial u^2} &= \frac{\partial}{\partial x}\Big(\frac{\partial z}{\partial u}\Big)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\Big(\frac{\partial z}{\partial u}\Big)\frac{\partial y}{\partial u}\\
&= \frac{\partial}{\partial x}\Big(\frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha\Big)\cos\alpha + \frac{\partial}{\partial y}\Big(\frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha\Big)\sin\alpha\\
&= \frac{\partial^2 z}{\partial x^2}\cos^2\alpha + \frac{\partial^2 z}{\partial x\partial y}\sin\alpha\cos\alpha + \frac{\partial^2 z}{\partial y\partial x}\cos\alpha\sin\alpha + \frac{\partial^2 z}{\partial y^2}\sin^2\alpha. \qquad (1)
\end{aligned}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-\sin\alpha) + \frac{\partial z}{\partial y}\cos\alpha.$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial v^2} &= \frac{\partial}{\partial x}\Big(\frac{\partial z}{\partial v}\Big)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\Big(\frac{\partial z}{\partial v}\Big)\frac{\partial y}{\partial v}\\
&= \frac{\partial}{\partial x}\Big(-\frac{\partial z}{\partial x}\sin\alpha + \frac{\partial z}{\partial y}\cos\alpha\Big)(-\sin\alpha) + \frac{\partial}{\partial y}\Big(-\frac{\partial z}{\partial x}\sin\alpha + \frac{\partial z}{\partial y}\cos\alpha\Big)\cos\alpha.\\
&= \frac{\partial^2 z}{\partial x^2}\sin^2\alpha - \frac{\partial^2 z}{\partial x\partial y}\sin\alpha\cos\alpha - \frac{\partial^2 z}{\partial y\partial x}\cos\alpha\sin\alpha + \frac{\partial^2 z}{\partial y^2}\cos^2\alpha. \qquad (2)
\end{aligned}$$

$$\begin{aligned}
(1)+(2) \Rightarrow \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} &= \frac{\partial^2 z}{\partial x^2}(\sin^2\alpha + \cos^2\alpha) + \frac{\partial^2 z}{\partial y^2}(\sin^2\alpha + \cos^2\alpha)\\
&= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}.
\end{aligned}$$

Example 2.46. If F is a function of x and y and if $x = e^u \sin v, y = e^u \cos v$, prove that $\dfrac{\partial^2 F}{\partial x^2} + \dfrac{\partial^2 F}{\partial y^2} = e^{-2u}\left[\dfrac{\partial^2 F}{\partial u^2} + \dfrac{\partial^2 F}{\partial v^2}\right]$ [Jan 2013]

Solution. $\dfrac{\partial F}{\partial u} = \dfrac{\partial F}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial F}{\partial y}\cdot\dfrac{\partial y}{\partial u}$

$$= \frac{\partial F}{\partial x}\cdot e^u\sin v + \frac{\partial F}{\partial y}\cdot e^u\cos v.$$

$$\begin{aligned}
\frac{\partial^2 F}{\partial u^2} &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u}\right)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u}\right)\frac{\partial y}{\partial u}\\
&= \frac{\partial}{\partial x}\left[\frac{\partial F}{\partial x}\cdot e^u\sin v + \frac{\partial F}{\partial y}\cdot e^u\cos v\right]\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left[\frac{\partial F}{\partial x}\cdot e^u\sin v + \frac{\partial F}{\partial y}\cdot e^u\cos v\right]\frac{\partial y}{\partial u}\\
&= \left[\frac{\partial^2 F}{\partial x^2}\cdot e^u\sin v + \frac{\partial^2 F}{\partial x\partial y}\cdot e^u\cos v\right]\cdot e^u\sin v + \left[\frac{\partial^2 F}{\partial y^2}\cdot e^u\cos v + \frac{\partial^2 F}{\partial y\partial x}\cdot e^u\sin v\right]\cdot e^u\cos v\\
&= \frac{\partial^2 F}{\partial x^2}\cdot e^{2u}\sin^2 v + \frac{\partial^2 F}{\partial x\partial y}\cdot e^{2u}\sin v\cos v + \frac{\partial^2 F}{\partial y\partial x}\cdot e^{2u}\sin v\cos v + \frac{\partial^2 F}{\partial y^2}\cdot e^{2u}\cos^2 v \qquad (1)
\end{aligned}$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial v}.$$
$$= \frac{\partial F}{\partial x}\cdot e^u\cos v - \frac{\partial F}{\partial y}\cdot e^u \sin v.$$
$$\frac{\partial^2 F}{\partial v^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v}\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v}\right)\frac{\partial y}{\partial v}$$
$$= \frac{\partial}{\partial x}\left[\frac{\partial F}{\partial x}\cdot e^u\cos v - \frac{\partial F}{\partial y}\cdot e^u\sin v\right]e^u\cos v + \frac{\partial}{\partial y}\left[\frac{\partial F}{\partial x}\cdot e^u\cos v - \frac{\partial F}{\partial y}\cdot e^u\sin v\right](-e^u\sin v)$$
$$= \frac{\partial^2 F}{\partial x^2}\cdot e^{2u}\cos^2 v - \frac{\partial^2 F}{\partial x\partial y}\cdot e^{2u}\sin v\cos v - \frac{\partial^2 F}{\partial y\partial x}\cdot e^{2u}\sin v\cos v + \frac{\partial^2 F}{\partial y^2}\cdot e^{2u}\sin^2 v. \quad (2)$$
$$(1)+(2) \Rightarrow \frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} = \frac{\partial^2 F}{\partial x^2}e^{2u} + \frac{\partial^2 F}{\partial y^2}\cdot e^{2u}$$
$$= e^{2u}\left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right]$$
$$e^{-2u}\left[\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2}\right] = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}.$$

Example 2.47. Transform the equation $z_{xx} + 2z_{xy} + z_{yy} = 0$ by changing the independent variables using $u = x - y$ and $v = x + y$. [Jun 2012]

Solution. Consider z as a function of u and v and u and v are functions of x and y.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$= \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}\cdot 1$$
$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}.$$
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial v}{\partial x}$$
$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\cdot 1 + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\cdot 1$$
$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v\partial u} + \frac{\partial^2 z}{\partial v^2}$$

i.e.,$z_{xx} = z_{uu} + z_{uv} + z_{vu} + z_{vv}$ (1)

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$
\begin{aligned}
&= \frac{\partial z}{\partial u}(-1) + \frac{\partial z}{\partial v}\cdot 1 \\
&= -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial y}\right)\frac{\partial v}{\partial y} \\
&= \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)(-1) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\cdot 1 \\
&= \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u\partial v} - \frac{\partial^2 z}{\partial v\partial u} + \frac{\partial^2 z}{\partial v^2}.
\end{aligned}
$$

i.e., $z_{yy} = z_{uu} - z_{uv} - z_{vu} + z_{vv}$ (2)

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \\
&= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial y}\right)\frac{\partial v}{\partial x} \\
&= \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\cdot 1 + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\cdot 1 \\
&= -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\partial v} - \frac{\partial^2 z}{\partial v\partial u} + \frac{\partial^2 z}{\partial v^2}.
\end{aligned}
$$

i.e., $z_{xy} = -z_{uu} + z_{uv} - z_{vu} + z_{vv}$ (3)

$(1) + (2) + 2\times(3) \Rightarrow$

$$
\begin{aligned}
z_{xx} + 2z_{xy} + z_{yy} &= 2z_{uv} - 2z_{vu} + 4z_{vv} \\
0 &= 2z_{uv} - 2z_{vu} + 4z_{vv}.
\end{aligned}
$$

$$\therefore z_{uv} - z_{vu} + 2z_{vv} = 0$$

which is the required equation.

Example 2.48. If $z = f(x,y)$ where $x = u^2 - v^2$ and $y = 2uv$, prove that $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$. [Jan 2012, Jan 2010]

Solution. z is a function of x and y and x and y are functions of u and v.

$\therefore$ z is a composite function of u and v.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\cdot 2u + \frac{\partial z}{\partial y}\cdot 2v.$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial y}{\partial u}.$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\cdot 2u + \frac{\partial z}{\partial y}\cdot 2v\right]2u + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\cdot 2u + \frac{\partial z}{\partial y}\cdot 2v\right]2v$$

$$= \frac{\partial^2 z}{\partial x^2}\cdot 4u^2 + \frac{\partial^2 z}{\partial x\partial y}4uv + \frac{\partial^2 z}{\partial y\partial x}4uv + \frac{\partial^2 z}{\partial y^2}\cdot 4v^2.$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}\cdot(-2v) + \frac{\partial z}{\partial y}\cdot 2u$$

$$= -2v\frac{\partial z}{\partial x} + 2u\frac{\partial z}{\partial y}.$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)\frac{\partial y}{\partial v}.$$

$$= \frac{\partial}{\partial x}\left[-2v\frac{\partial z}{\partial x} + 2u\frac{\partial z}{\partial y}\right](-2v) + \frac{\partial}{\partial y}\left[-2v\cdot\frac{\partial z}{\partial x} + 2u\cdot\frac{\partial z}{\partial y}\right]2u$$

$$= 4v^2\frac{\partial^2 z}{\partial x^2} - 4uv\frac{\partial^2 z}{\partial x\partial y} - 4uv\frac{\partial^2 z}{\partial y\partial x} + 4v^2\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2}(4u^2 + 4v^2) + \frac{\partial^2 z}{\partial y^2}(4u^2 + 4v^2)$$

$$= 4(u^2 + v^2)\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right].$$

Example 2.49. If $g(x,y) = \chi(u,v)$ where $u = x^2 - y^2, v = 2xy$, prove that $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2 \chi}{\partial u^2} + \frac{\partial^2 \chi}{\partial v^2}\right)$.

Solution.

Similar to the previous problem.

Example 2.50. If $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$, find $\dfrac{du}{dt}$.
[Dec 2011]

Solution.

$$\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}\\
&= 2x\cdot 2e^{2t} + 2y\left[-3e^{2t}\sin 3t + 2e^{2t}\cos 3t\right]\\
&\qquad + 2z\left[3e^{2t}\cos 3t + 2e^{2t}\sin 3t\right]\\
&= 2e^{2t}\cdot 2e^{2t} + 2e^{2t}\cos 3t\left[-3e^{2t}\sin 3t + 2e^{2t}\cos 3t\right]\\
&\qquad + 2e^{2t}\sin 3t\left[3e^{2t}\cos 3t + 2e^{2t}\sin 3t\right]\\
&= 4e^{4t} - 6e^{4t}\cos 3t\sin 3t + 4e^{4t}\cos^2 3t + 6e^{4t}\sin 3t\cos 3t + 4e^{4t}\sin^2 3t\\
&= 4e^{4t} + 4e^{4t}(\cos^2 3t + \sin^2 3t) = 4e^{4t} + 4e^{4t} = 8e^{4t}.
\end{aligned}$$

Example 2.51. Find $\dfrac{dy}{dx}$ if x and y are connected by the relation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution. Let $f(x,y) = x^{\frac{2}{3}} + y^{\frac{2}{3}} - a^{\frac{2}{3}}$

$$f_x = \frac{\partial f}{\partial x} = \frac{2}{3}x^{-\frac{1}{3}}.$$

$$f_y = \frac{\partial f}{\partial y} = \frac{2}{3}y^{-\frac{1}{3}}.$$

$$\text{Now } \frac{dy}{dx} = -\frac{f_x}{f_y} = -\left(\frac{x}{y}\right)^{-\frac{1}{3}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}.$$

Example 2.52. Find $\dfrac{dy}{dx}$ if $f(x,y) = \log(x^2 + y^2) + \tan^{-1}\left(\dfrac{y}{x}\right)$.

Solution.

$$f_x = \frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} + \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{2x}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{2x-y}{x^2+y^2}.$$

$$f_y = \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{2y+x}{x^2+y^2}.$$

$$\text{Now } \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x-y}{2y+x} = \frac{y-2x}{x+2y}.$$

2.5 Jacobian and its Properties

Definition. If u and v are continuous functions of two independent variables x and y having first order partial derivatives, then the Jacobian determinant or the Jacobian of u and v is defined by

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ or } J\left(\frac{u,v}{x,y}\right) \text{ or } J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

If u, v, w are continuous functions of three independent variables x, y, z having first order partial derivatives, then the Jacobian of u, v, w w.r.t. x, y, z is defined as

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

Properties of Jacobians

Property I. If u and v are functions of r and s where r and s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)}\frac{\partial(r,s)}{\partial(x,y)}.$$

Proof. Since u and v are functions of x and y, we have

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = u_r r_x + u_s s_x.$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = u_r r_y + u_s s_y.$$

$$v_x = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} = v_r r_x + v_s s_x.$$

$$v_y = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} = v_r r_y + v_s s_y.$$

$$\text{Now } \frac{\partial(u,v)}{\partial(r,s)}\frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix}\begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$$
$$= \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix}\begin{vmatrix} r_x & s_x \\ r_y & s_y \end{vmatrix}$$
$$= \begin{vmatrix} u_r r_x + u_s s_x & u_r r_y + u_s s_y \\ v_r r_x + v_s s_x & v_r r_y + v_s s_y \end{vmatrix}$$
$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$
$$= \frac{\partial(u,v)}{\partial(x,y)}.$$

Property II. If J_1 is the Jacobian of u, v with respect to x, y and J_2 is the Jacobian of x, y w.r.t. u, v then $J_1 J_2 = 1$. i.e., $\dfrac{\partial(u,v)}{\partial(x,y)} \bullet \dfrac{\partial(x,y)}{\partial(u,v)} = 1$.

Proof. u is a function of x and y.

Differentiating partially with respect to u and v we get

$$1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} = u_x x_u + u_y y_u.$$

$$0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} = u_x x_v + u_y y_v.$$

v is a function of x and y.

Differentiating with respect to u and v partially we get

$$0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} = v_x x_u + v_y y_v.$$

$$1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} = v_x x_v + v_y y_v.$$

$$\text{Now } \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}$$

$$= \begin{vmatrix} u_x x_u + u_y y_v & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$J_1 J_2 = 1.$$

Property III. If the functions u, v, w of three independent variables x, y, z are not independent, then the Jacobian of u, v, w with respect to x, y, z vanishes.

Solution. Given: u, v and w are not independent variables.

$\Longrightarrow$ There exists a relation $F(u, v, w) = 0$.

Differentiating this w.r.t x, y and z we get

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial x} = 0$$

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial y} = 0$$

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial z} = 0.$$

Eliminating $\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}$ and $\frac{\partial F}{\partial w}$ from the above three equations we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = 0.$$

$$\Longrightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$$

Functional relationship. Let $u_1, u_2, \ldots, u_n$ be functions of $x_1, x_2, x_3, \ldots, x_n$. Then, a necessary and sufficient condition that $u_1, u_2, \ldots, u_n$ are not independent is that the Jacobian $\dfrac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = 0$.

Worked Examples

Example 2.53. If $x = r\cos\theta, y = r\sin\theta$, find the Jacobian of x and y w.r.t r and θ.

[Jan 2001]

Solution.

$$x = r\cos\theta \implies \frac{\partial x}{\partial r} = \cos\theta, \frac{\partial x}{\partial\theta} = -r\sin\theta.$$

$$y = r\sin\theta \implies \frac{\partial u}{\partial r} = \sin\theta, \frac{\partial u}{\partial\theta} = r\cos\theta.$$

$$\text{Now, } \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Example 2.54. The transformation equations in spherical polar coordinates is $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$. Compute the Jacobian of x, y, z w.r.t. r, θ, φ.

[May 2011, Jan 2009]

Solution.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\sin\varphi\cos\theta & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta(\sin\theta\cos\varphi(0 + \sin\theta\cos\varphi) - \cos\theta\cos\varphi(0 - \cos\theta\cos\varphi)$$
$$- \sin\varphi(-\sin^2\theta\sin\varphi - \cos^2\theta\sin\varphi))$$

$$= r^2\sin\theta(\sin^2\theta\cos^2\varphi + \cos^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta\sin^2\varphi)$$

$$= r^2\sin\theta(\cos^2\varphi + \sin^2\varphi) = r^2\sin\theta.$$

Example 2.55. In cylindrical polar coordinates, $x = \rho\cos\phi, y = \rho\sin\phi, z = z$. Show that $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho$.
Solution.

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= 1(\rho\cos^2\phi + \rho\sin^2\phi) = \rho.$$

Example 2.56. If $u = 2xy, v = x^2 - y^2, x = r\cos\theta, y = r\sin\theta$, evaluate $\dfrac{\partial(u,v)}{\partial(r,\theta)}$ without actual substitution. [Jan 2009]

Solution. Given that u and v are functions of x and y. x and y are functions of r and θ.

$\therefore$ By property (1) of the Jacobians we have

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(r,\theta)}.$$
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix}$$
$$= -4y^2 - 4x^2 = -4(x^2+y^2) = -4r^2.$$
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= r\cos^2\theta + r\sin^2\theta = r$$
$$\frac{\partial(u,v)}{\partial(r,\theta)} = -4(r^2)r = -4r^3.$$

Example 2.57. If $x = uv, y = \dfrac{u+v}{u-v}$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$.

Solution. We know that $\dfrac{\partial(u,v)}{\partial(x,y)}\dfrac{\partial(x,y)}{\partial(u,v)} = 1 \Rightarrow \dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\dfrac{\partial(x,y)}{\partial(u,v)}}$.

$$x = uv.$$

$$\frac{\partial x}{\partial u} = v, \frac{\partial x}{\partial v} = u.$$

$$y = \frac{u+v}{u-v}.$$

$$\frac{\partial y}{\partial u} = \frac{u-v-(u+v)}{(u-v)^2} = \frac{u-v-u-v}{(u-v)^2} = \frac{-2v}{(u-v)^2}.$$

$$\frac{\partial y}{\partial v} = \frac{(u-v).1-(u+v)(-1)}{(u-v)^2} = \frac{2u}{(u-v)^2}.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2u}{(u-v)^2} \end{vmatrix}$$

$$= \frac{2uv}{(u-v)^2} + \frac{2uv}{(u-v)^2} = \frac{4uv}{(u-v)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{(u-v)^2}{4uv}.$$

Example 2.58. If $x = u(1-v), y = uv$ then compute J_1 and J_2 and prove that $J_1 J_2 = 1$.

Solution. We have

$$J_1 = \frac{\partial(x,y)}{\partial(u,v)}, J_2 = \frac{\partial(u,v)}{\partial(x,y)}$$

$$x = u - uv, y = uv.$$

$$J_1 = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u.$$

We shall express u and v interms of x and y.

$$x = u - uv = u - y \Rightarrow x + y = u.$$

$$y = uv \Rightarrow v = \frac{y}{u} = \frac{y}{x+y}.$$

$$\text{we get } J_2 = \frac{1}{u}.$$

$$\text{Now } J_1 J_2 = u\frac{1}{u} = 1.$$

Example 2.59. If $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_3x_1}{x_2}, y_3 = \frac{x_1x_2}{x_3}$, prove that $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = 4$. [Jan 2014]

Solution. $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$

$$= \begin{vmatrix} \frac{-x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{-x_2x_3}{x_1^2}\left(\frac{x_1^2x_2x_3}{x_2^2x_3^2} - \frac{x_1^2}{x_2x_3}\right) - \frac{x_3}{x_1}\left(-\frac{x_1x_2x_3}{x_2x_3^2} - \frac{x_1x_2}{x_2x_3}\right) + \frac{x_2}{x_1}\left(\frac{x_1x_3}{x_2x_3} + \frac{x_1x_2x_3}{x_2^2x_3}\right)$$

$$= -1 + 1 + 1 + 1 + 1 + 1 = 4.$$

Example 2.60. If $u = x + y + z, uv = y + z, uvw = z$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$. [Jan 2012, Jan 2010]

Solution. Given $u = x + y + z \Rightarrow u = x + uv \Rightarrow x = u - uv = u(1 - v)$.

$uv = y + z \Rightarrow uv = y + uvw \Rightarrow y = uv - uvw = uv(1 - w), uvw = z$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 - v & -u & 0 \\ v(1 - w) & u(1 - w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= u(uv)\begin{vmatrix} 1 - v & -1 & 0 \\ v(1 - w) & 1 - w & -1 \\ vw & w & 1 \end{vmatrix}$$

$$= u^2v\{(1 - v)(1 - w + w) + 1(v(1 - w) + vw)\}$$

$$= u^2v\{(1 - v) + (v - vw + vw)\}$$

$$= u^2v\{1 - v + v\} = u^2v.$$

Example 2.61. If $u = xyz, v = x^2 + y^2 + z^2, w = x + y + z$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

Solution. We have $\frac{\partial(x,y,z)}{\partial(u,v,w)}\frac{\partial(u,v,w)}{\partial(x,y,z)} = 1 \Rightarrow \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}}$.

$$\begin{aligned}
\frac{\partial(u,v,w)}{\partial(x,y,z)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} \\
&= yz(2y-2z) - xz(2x-2z) + xy(2x-2y) \\
&= 2(yz(y-z) - xz(x-z) + xy(x-y)) \\
&= 2(y^2z - yz^2 - x^2z + xz^2 + xy(x-y)) \\
&= 2(z^2(x-y) + xy(x-y) - z(x^2-y^2)) \\
&= 2(z^2(x-y) + xy(x-y) - z(x-y)(x+y)) \\
&= 2(x-y)(z^2 + xy - xz - yz) \\
&= 2(x-y)(z(z-x) - y(z-x)) \\
&= 2(x-y)(z-x)(z-y) \\
&= -2(x-y)(y-z)(z-x).
\end{aligned}$$

Now, $\Rightarrow \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}} = \frac{1}{-2(x-y)(y-z)(z-x)}$.

Example 2.62. If $u = x + 2y + z$, $v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$, show that they are not independent. Find the relation between u, v and w.

Solution. Given: $u = x + 2y + z$.

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 2, \frac{\partial u}{\partial z} = 1.$$

$$v = x - 2y + 3z.$$

$$\frac{\partial v}{\partial x} = 1, \frac{\partial v}{\partial y} = -2, \frac{\partial v}{\partial z} = 3.$$

$$w = 2xy - xz + 4yz - 2z^2.$$

$$\frac{\partial w}{\partial x} = 2y - z, \frac{\partial w}{\partial y} = 2x + 4z, \frac{\partial w}{\partial z} = -x + 4y - 4z.$$

$$\begin{aligned}
\frac{\partial(u,v,w)}{\partial(x,y,z)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y - z & 2x + 4z & -x + 4y - 4z \end{vmatrix} \\
&= 1(-2(-x + 4y - 4z) - 3(2x + 4z)) - 2(-x + 4y - 4z - 3(2y - z)) \\
&\quad + 1(2x + 4z + 2(2y - z)) \\
&= 2x - 8y + 8z - 6x - 12z + 2x - 8y + 8z + 12y - 6z + 2x + 4z + 4y - 2z = 0.
\end{aligned}$$

Hence, u, v, w are not independent.

Now $u + v = 2x + 4z, u - v = 4y - 2z$.

$$\begin{aligned}
(u + v)(u - v) &= 2(x + 2z).2(2y - z) \\
u^2 - v^2 &= 4(2xy - xz + 4yz - 2z^2) \\
u^2 - v^2 &= 4w.
\end{aligned}$$

2.6 Taylor's expansion for functions of two variables

We know that for a function $f(x)$ of one single variable x, the Taylor's expansion is

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

Now let $f(x, y)$ be a function of two independent variables x, y defined in a region R of the $xy-$plane and let (a, b) be a point in R. Suppose $f(x, y)$ has all its partial derivatives in a neighbourhood of (a, b) then

$$\begin{aligned}
f(a+h, b+k) = {} & f(a,b) + \Big(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\Big) f(a,b) \\
& + \frac{1}{2!}\Big(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\Big)^2 f(a,b) + \frac{1}{3!}\Big(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\Big)^3 f(a,b) + \cdots + \\
= {} & f(a,b) + \big(h f_x(a,b) + k f_y(a,b)\big) \\
& + \frac{1}{2!}\big(h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\big) \\
& + \frac{1}{3!}\big(h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) \\
& + k^3 f_{yyy}(a,b)\big) + \cdots .
\end{aligned}$$

Put $x = a + h, y = b + k$ then $h = x - a, k = y - b$.

$\therefore$ The Taylor's series can be written as

$$\begin{aligned}
f(x,y) = {} & f(a,b) + \big((x-a) f_x(a,b) + (y-b) f_y(a,b)\big) \\
& + \frac{1}{2!}\big((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\big) \\
& + \frac{1}{3!}\big((x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) \\
& + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\big) + \cdots .
\end{aligned}$$

This is known as the Taylor's expansion of $f(x, y)$ in the neighbourhood of (a, b) or about the point (a, b).

Put $a = 0, b = 0.$ we get

$$\begin{aligned}
f(x,y) = {} & f(0,0) + \big(x f_x(0,0) + y f_y(0,0)\big) \\
& + \frac{1}{2!}\big(x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\big) \\
& + \frac{1}{3!}\big(x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\big) + \cdots .
\end{aligned}$$

This is called Maclaurin's series for $f(x, y)$ in powers of x and y.

Worked Examples

Example 2.63. Expand $e^x \sin y$ in powers of x and y as far as the terms of third degree. [Jun 2013]

Solution.

$$\begin{aligned}
f(x,y) &= e^x \sin y & f(0,0) &= 0\\
f_x(x,y) &= e^x \sin y & f_x(0,0) &= 0\\
f_y(x,y) &= e^x \cos y & f_y(0,0) &= 1\\
f_{xx}(x,y) &= e^x \sin y & f_{xx}(0,0) &= 0\\
f_{xy}(x,y) &= e^x \cos y & f_{xy}(0,0) &= 1\\
f_{yy}(x,y) &= -e^x \sin y & f_{yy}(0,0) &= 0\\
f_{xxx}(x,y) &= e^x \sin y & f_{xxx}(0,0) &= 0\\
f_{xxy}(x,y) &= e^x \cos y & f_{xxy}(0,0) &= 1\\
f_{xyy}(x,y) &= -e^x \sin y & f_{xyy}(0,0) &= 0\\
f_{yyy}(x,y) &= -e^x \cos y & f_{yyy}(0,0) &= -1.
\end{aligned}$$

$$\begin{aligned}
\text{Now } f(x,y) = {} & f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left(x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right)\\
& + \frac{1}{3!}\left(x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right) + \cdots\\
= {} & 0 + x.0 + y.1 + \frac{1}{2}\left(x^2.0 + 2xy.1 + y^2.0\right)\\
& \frac{1}{6}\left(x^3.0 + 3x^2y.1 + 3xy^2.0 + y^3(-1)\right) + \cdots\\
= {} & y + xy + \frac{x^2y}{2} - \frac{y^3}{6} + \cdots.
\end{aligned}$$

Example 2.64. Expand $e^x \log_e(1+y)$ in powers of x and y upto terms of third degree. [Jan 2014, Dec 2011, Jan 2003]

Solution.

$$f_{yyy} = \frac{2e^x}{(1+y)^3}$$

$$f(x, y) = e^x \log_e(1+y) \qquad f(0,0) = 0$$

$$f_x(x, y) = e^x \log_e(1+y) \qquad f_x(0,0) = 0$$

$$f_y(x, y) = \frac{e^x}{1+y} \qquad f_y(0,0) = 1$$

$$f_{xx} = e^x \log(1+y) \qquad f_{xx}(0,0) = 0$$

$$f_{xy} = \frac{e^x}{1+y} \qquad f_{xy}(0,0) = 1$$

$$f_{yy} = \frac{-e^x}{(1+y)^2} \qquad f_{yy}(0,0) = -1$$

$$f_{xxx} = e^x \log(1+y) \qquad f_{xxx}(0,0) = 0$$

$$f_{xxy} = \frac{e^x}{1+y} \qquad f_{xxy}(0,0) = 1$$

$$f_{xyy} = \frac{-e^x}{(1+y)^2} \qquad f_{xyy}(0,0) = -1$$

$$f_{yyy}(0,0) = 2$$

By Maclaurin's series we have,

$$\begin{aligned} f(x, y) = {} & f(0,0) + x f_x(0,0) + y f_y(0,0) \\ & + \frac{1}{2!}\left(x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right) \\ & + \frac{1}{3!}\left(x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0)\right. \\ & \left. + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right) + \cdots . \end{aligned}$$

$$\begin{aligned} e^x \log_e(1+y) &= 0 + x.0 + y.1 + \frac{1}{2}(x^2.0 + 2xy.1 + y^2(-1)) + \\ & \quad \frac{1}{6}[x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2(-1) + y^3(2)] + \ldots . \\ &= y + xy - \frac{y^2}{2} + \frac{x^2 y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} - \cdots \end{aligned}$$

Example 2.65. Expand $x^2 y + 3y - 2$ in powers of $x - 1$ and $y + 2$ upto third degree terms. [Jun 2012]

Solution. $f(x,y) = x^2y + 3y - 2$. $\quad f(1,-2) = 1 \times (-2) + 3(-2) - 2$

$$= -2 - 6 - 2 = -10.$$

$$\begin{aligned}
f_x(x,y) &= 2xy. & f_x(1,-2) &= -4.\\
f_y(x,y) &= x^2 + 3. & f_y(1,-2) &= 4.\\
f_{xx}(x,y) &= 2y. & f_{xx}(1,-2) &= -4.\\
f_{xy}(x,y) &= 2x. & f_{xy}(1,-2) &= 2.\\
f_{yy}(x,y) &= 0. & f_{yy}(1,-2) &= 0.\\
f_{xxx}(x,y) &= 0. & f_{xxx}(1,-2) &= 0.\\
f_{xxy}(x,y) &= 2. & f_{xxy}(1,-2) &= 2.\\
f_{xyy}(x,y) &= 0. & f_{xyy}(1,-2) &= 0.\\
f_{yyy}(x,y) &= 0. & f_{yyy}(1,-2) &= 0.
\end{aligned}$$

By Taylor's theorem we have,

$$\begin{aligned}
f(x,y) = \; & f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)\\
& + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]\\
& + \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b)\right.\\
& \left. + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b) + \cdots\right]\\
x^2y + 3y - 2 = \; & f(1,-2) + (x-1)f_x(1,-2) + (y+2)f_y(1,-2)\\
& + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,-2) + 2(x-1)(y+2)f_{xy}(1,-2) + (y+2)^2 f_{yy}(1,-2)\right]\\
& + \frac{1}{3!}\left[(x-1)^3 f_{xxx}(1,-2) + 3(x-1)^2(y+2)f_{xxy}(1,-2)\right.\\
& \left. + 3(x-1)(y+2)^2 f_{xyy}(1,-2) + (y+2)^3 f_{yyy}(1,-2)\right] + \cdots
\end{aligned}$$

$$= -10 - 4(x-1) + 4(y+2) + \frac{1}{2}\left[-4(x-1)^2 + 2(x-1)(y+2)\right]$$
$$+ \frac{1}{6}\left[6(x-1)^2(y+2)\right] + \cdots$$
$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + (x-1)(y+2) + (x-1)^2(y+2) + \cdots$$

Example 2.66. Find the Taylor's series expansion of $x^2y^2 + 2x^2y + 3y^2$ in powers of $(x+2)$ and $y-1$ upto third degree terms. [Jan 2012, Jun 2010, Jan 2010]

Solution.

$$f(x,y) = x^2y^2 + 2x^2y + 3y^2. \qquad f(-2,1) = 4 + 8 + 3 = 15.$$
$$f_x(x,y) = 2xy^2 + 4xy. \qquad f_x(-2,1) = -4 - 8 = -12.$$
$$f_y(x,y) = 2x^2y + 2x^2 + 6y. \qquad f_y(-2,1) = 8 + 8 + 6 = 22.$$
$$f_{xx}(x,y) = 2y^2 + 4y. \qquad f_{xx}(-2,1) = 2 + 4 = 6.$$
$$f_{xy}(x,y) = 4xy + 4x. \qquad f_{xy}(-2,1) = -8 - 8 = -16.$$
$$f_{yy}(x,y) = 2x^2 + 6. \qquad f_{yy}(-2,1) = 8 + 6 = 14.$$
$$f_{xxx}(x,y) = 0. \qquad f_{xxx}(-2,1) = 0.$$
$$f_{xxy}(x,y) = 4y + 4. \qquad f_{xxy}(-2,1) = 4 + 4 = 8.$$
$$f_{xyy}(x,y) = 4x. \qquad f_{xyy}(-2,1) = -8.$$
$$f_{yyy}(x,y) = 0. \qquad f_{yyy}(-2,1) = 0.$$

By Taylor's theorem we have,

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b)\right.$$
$$\left. + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots$$

$$
\begin{aligned}
x^2y^2 + 2x^2y + 3y^2 = 15 - 12(x+2) + 22(y-1) \\
+ \frac{1}{2}\left[6(x+2)^2 - 2\times 16(x+2)(y-1) + 14(y-1)^2\right] \\
+ \frac{1}{6}\left[24(x+2)^2(y-1) - 24(x+2)(y-1)^2\right] + \cdots \\
= 15 - 12(x+2) + 22(y-1) + 3(x+2)^2 - 16(x+2)(y-1) + 7(y-1)^2 \\
+ 4(x+2)^2(y-1) - 4(x+2)(y-1)^2 + \cdots
\end{aligned}
$$

Example 2.67. Use Taylor's formula to expand the function defined by $f(x,y) = x^3 + y^3 + xy^2$ in powers of $(x-1)$ and $(y-2)$. [May 2011]

Solution.

$$
\begin{aligned}
f(x,y) &= x^3 + y^3 + xy^2. & f(1,2) &= 1 + 8 + 4 = 13. \\
f_x(x,y) &= 3x^2 + y^2. & f_x(1,2) &= 3 + 4 = 7. \\
f_y(x,y) &= 3y^2 + 2xy. & f_y(1,2) &= 12 + 4 = 16. \\
f_{xx}(x,y) &= 6x. & f_{xx}(1,2) &= 6. \\
f_{xy}(x,y) &= 2y. & f_{xy}(1,2) &= 4. \\
f_{yy}(x,y) &= 6y + 2x. & f_{yy}(1,2) &= 12 + 2 = 14. \\
f_{xxx}(x,y) &= 6. & f_{xxx}(1,2) &= 6. \\
f_{xxy}(x,y) &= 0. & f_{xxy}(1,2) &= 0. \\
f_{xyy}(x,y) &= 2. & f_{xyy}(1,2) &= 2. \\
f_{yyy}(x,y) &= 6. & f_{yyy}(1,2) &= 6.
\end{aligned}
$$

By Taylor's theorem we have,

$$
\begin{aligned}
f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) \\
+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]
\end{aligned}
$$

$$+\frac{1}{3!}\Big[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b)$$
$$+3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\Big] + \cdots$$

$$x^3+y^3+xy^2 = 13 + 7(x-1) + 16(y-2) + \frac{1}{2}\Big[6(x-1)^2 + 8(x-1)(y-2) + 14(y-2)^2\Big]$$
$$+\frac{1}{6}\Big[6(x-1)^3 + 6(x-1)(y-2)^2 + 6(y-2)^3\Big] + \cdots$$
$$= 13 + 7(x-1) + 16(y-2) + 3(x-1)^2 + 4(x-1)(y-2) + 7(y-2)^2$$
$$+(x-1)^3 + (x-1)(y-2)^2 + (y-2)^3 + \cdots$$

Example 2.68. Expand $e^{-x}\log y$ as a Taylor's series in powers of x and $y-1$ upto third degree terms. [Jun 2011]

Solution.

$$f(x,y) = e^{-x}\log y. \qquad f(0,1) = 0.$$
$$f_x(x,y) = -e^{-x}\log y. \qquad f_x(0,1) = 0.$$
$$f_y(x,y) = \frac{e^{-x}}{y}. \qquad f_y(0,1) = 1.$$
$$f_{xx}(x,y) = e^{-x}\log y. \qquad f_{xx}(0,1) = 0.$$
$$f_{xy}(x,y) = -\frac{e^{-x}}{y}. \qquad f_{xy}(0,1) = -1.$$
$$f_{yy}(x,y) = -\frac{e^{-x}}{y^2}. \qquad f_{yy}(0,1) = -1.$$
$$f_{xxx}(x,y) = -e^{-x}\log y. \qquad f_{xxx}(0,1) = 0.$$
$$f_{xxy}(x,y) = \frac{e^{-x}}{y}. \qquad f_{xxy}(0,1) = 1.$$
$$f_{xyy}(x,y) = \frac{e^{-x}}{y^2}. \qquad f_{xyy}(0,1) = 1.$$
$$f_{yyy}(x,y) = \frac{2e^{-x}}{y^3}. \qquad f_{yyy}(0,1) = 2.$$

By Taylor's theorem we have,

$$\begin{aligned} f(x,y) = {} & f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) \\ & + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] \\ & + \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b)\right. \\ & \left. +3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots \end{aligned}$$

$$e^{-x}\log y = y - 1 + \frac{1}{2!}[-2x(y-1) - (y-1)^2] + \frac{1}{3!}[3x^2(y-1) + 3x(y-1)^2 + 2(y-1)^3] + \cdots$$

Example 2.69. Expand $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ about $(1,1)$ upto the second degree terms. Hence compute $f(1.1, 0.9)$ approximately. [Jan 2005]

Solution. Given, $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$$(a,b) = (1,1) \qquad a = 1, b = 1$$

$$f(1,1) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$f_x = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = -\frac{x^2 y}{(x^2+y^2)x^2} = \frac{-y}{x^2+y^2} \qquad f_x(1,1) = \frac{-1}{2}$$

$$f_y = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x^2}{(x^2+y^2)x} = \frac{x}{x^2+y^2} \qquad f_y(1,1) = \frac{1}{2}$$

$$f_{xx} = -y(-1)(x^2+y^2)^{-2}2x = \frac{2xy}{(x^2+y^2)^2} \qquad f_{xx}(1,1) = \frac{2}{4} = \frac{1}{2}$$

$$f_{xy} = \frac{x^2+y^2-x2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad f_{xy}(1,1) = 0$$

$$f_{yy} = x(-1)(x^2+y^2)^{-2}2y = \frac{-2xy}{(x^2+y^2)^2} \qquad f_{yy}(1,1) = \frac{-1}{2}$$

By Taylor's theorem we have

$$\begin{aligned} f(x,y) = {} & f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) \\ & + \frac{1}{2!}\Big((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big) + \cdots \end{aligned}$$

$$\tan^{-1}\left(\frac{y}{x}\right) = f(1,1) + (x-1)f_x(1,1) + (y-1)f_y(1,1)$$
$$+\frac{1}{2!}\Big((x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\Big)$$
$$= \frac{\pi}{4} + (x-1)\Big(\frac{-1}{2}\Big) + (y-1)\frac{1}{2} + \frac{1}{2}\Big((x-1)^2\frac{1}{2} + 2(x-1)(y-1)0$$
$$+ (y-1)^2\frac{(-1)}{2}\Big) + \cdots$$
$$= \frac{\pi}{4} - \frac{1}{2}(x-1-y+1) + \frac{1}{2}\Big(\frac{1}{2}(x-1)^2 - \frac{1}{2}(y-1)^2\Big) + \cdots$$
$$= \frac{\pi}{4} - \frac{1}{2}(x-y) + \frac{1}{4}(x^2-y^2-2x+2y) + \cdots$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \cdots.$$

$$f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(0.1) + \frac{1}{2}(-0.1) + \frac{1}{4}(0.1)^2 - \frac{1}{4}(-0.1)^2 \text{ approximately}$$
$$= \frac{\pi}{4} - 0.1 = 0.685 \text{ approximately.}$$

Example 2.70. Find the Taylor's series expansion of $e^x \sin y$ at the point $\left(-1, \frac{\pi}{4}\right)$ upto third degree terms. [Jan 2009]

Solution.

$$f(x,y) = e^x \sin y \qquad f\Big(-1, \frac{\pi}{4}\Big) = \frac{1}{e\sqrt{2}}$$
$$f_x(x,y) = e^x \sin y \qquad f_x\Big(-1, \frac{\pi}{4}\Big) = \frac{1}{e\sqrt{2}}$$
$$f_y(x,y) = e^x \cos y \qquad f_y\Big(-1, \frac{\pi}{4}\Big) = \frac{1}{e\sqrt{2}}$$
$$f_{xx} = e^x \sin y \qquad f_{xx}\Big(-1, \frac{\pi}{4}\Big) = \frac{1}{e\sqrt{2}}$$
$$f_{xy} = e^x \cos y \qquad f_{xy}\Big(-1, \frac{\pi}{4}\Big) = \frac{1}{e\sqrt{2}}$$
$$f_{yx} = e^x \cos y \qquad f_{yx}\Big(-1, \frac{\pi}{4}\Big) = \frac{1}{e\sqrt{2}}$$
$$f_{yy} = -e^x \sin y \qquad f_{yy}\Big(-1, \frac{\pi}{4}\Big) = \frac{-1}{e\sqrt{2}}$$
$$f_{xxx} = e^x \sin y \qquad f_{xxx}\Big(-1, \frac{\pi}{4}\Big) = \frac{1}{e\sqrt{2}}$$

$$f_{xxy} = e^x \cos y \qquad f_{xxy}(-1, \frac{\pi}{4}) = \frac{1}{e\sqrt{2}}$$

$$f_{xyy} = -e^x \sin y \qquad f_{xyy}(-1, \frac{\pi}{4}) = \frac{-1}{e\sqrt{2}}$$

$$f_{yyx} = -e^x \sin y \qquad f_{yyx}(-1, \frac{\pi}{4}) = \frac{-1}{e\sqrt{2}}$$

$$f_{yyy} = -e^x \sin y \qquad f_{yyy}(-1, \frac{\pi}{4}) = \frac{-1}{e\sqrt{2}}$$

By Taylor's theorem we have

$$\begin{aligned}
f(x,y) = {} & f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) \\
& + \frac{1}{2!}\Big((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big) \\
& + \frac{1}{3!}\Big((x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) \\
& + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\Big) + \cdots . \\
e^x \sin y = {} & f\Big(-1, \frac{\pi}{4}\Big) + (x+1) f_x(-1, \frac{\pi}{4}) + (y - \frac{\pi}{4}) f_y(-1, \frac{\pi}{4}) \\
& + \frac{1}{2!}\Big((x+1)^2 f_{xx}(-1, \frac{\pi}{4}) + 2(x+1)(y-\frac{\pi}{4}) f_{xy}(-1, \frac{\pi}{4}) \\
& + (y-\frac{\pi}{4})^2 f_{yy}(-1, \frac{\pi}{4})\Big) \\
& + \frac{1}{6}\Big((x+1)^3 f_{xxx}(-1, \frac{\pi}{4}) + 3(x+1)^2(y-\frac{\pi}{4}) f_{xxy}(-1, \frac{\pi}{4}) \\
& + 3(x+1)(y-\frac{\pi}{4})^2 f_{xyy}(-1, \frac{\pi}{4}) + (y-\frac{\pi}{4})^3 f_{yyy}(-1, \frac{\pi}{4})\Big) + \cdots . \\
= {} & \frac{1}{e\sqrt{2}} + \frac{1}{e\sqrt{2}}(x+1) + \frac{1}{e\sqrt{2}}(y - \frac{\pi}{4}) + \frac{1}{2\sqrt{2}e}(x+1)^2 \\
& + \frac{1}{\sqrt{2}e}(x+1)(y-\frac{\pi}{4}) - \frac{1}{2\sqrt{2}e}(y-\frac{\pi}{4})^2 + \frac{1}{6\sqrt{2}e}(x+1)^3 \\
& + \frac{\sqrt{2}}{e}(x+1)^2(y-\frac{\pi}{4}) - \frac{\sqrt{2}}{e}(x+1)(y-\frac{\pi}{4})^2 - \frac{1}{6\sqrt{2}e}(y-\frac{\pi}{4})^3 + \cdots .
\end{aligned}$$

Example 2.71. Expand $e^x \cos y$ near the point $\left(1, \frac{\pi}{4}\right)$ by Taylor's series as far as quadratic terms. [Jan 1996]

Solution.

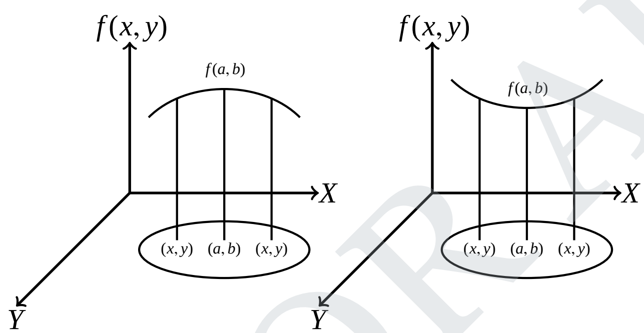
$$\begin{aligned}
f(x,y) &= e^x \cos y & f(1, \tfrac{\pi}{4}) &= \frac{e}{\sqrt{2}} \\
f_x(x,y) &= e^x \cos y & f_x(1, \tfrac{\pi}{4}) &= e \cos \frac{\pi}{4} = \frac{e}{\sqrt{2}} \\
f_y(x,y) &= -e^x \sin y & f_y(1, \tfrac{\pi}{4}) &= -\frac{e}{\sqrt{2}} \\
f_{xx} &= e^x \cos y & f_{xx}(1, \tfrac{\pi}{4}) &= \frac{e}{\sqrt{2}} \\
f_{xy} &= -e^x \sin y & f_{xy}(1, \tfrac{\pi}{4}) &= \frac{-e}{\sqrt{2}} \\
f_{yy} &= -e^x \cos y & f_{yy}(1, \tfrac{\pi}{4}) &= \frac{-e}{\sqrt{2}}
\end{aligned}$$

By Taylor's theorem we have

$$\begin{aligned}
f(x,y) = {} & f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) \\
& + \frac{1}{2!}\Big((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big) \\
e^x \cos y = {} & f(1, \tfrac{\pi}{4}) + (x-1) f_x(1, \tfrac{\pi}{4}) + (y - \tfrac{\pi}{4}) f_y(1, \tfrac{\pi}{4}) \\
& + \frac{1}{2!}\Big((x-1)^2 f_{xx}(1, \tfrac{\pi}{4}) + 2(x-1)(y - \tfrac{\pi}{4}) f_{xy}(1, \tfrac{\pi}{4}) \\
& + (y - \tfrac{\pi}{4})^2 f_{yy}(1, \tfrac{\pi}{4})\Big) + \cdots \\
= {} & \frac{e}{\sqrt{2}} + (x-1)\frac{e}{\sqrt{2}} + (y - \frac{\pi}{4})\frac{(-e)}{\sqrt{2}} \\
& + \frac{1}{2}\Big((x-1)^2 \frac{e}{\sqrt{2}} + 2(x-1)(y - \frac{\pi}{4})\frac{(-e)}{\sqrt{2}} + (y - \frac{\pi}{4})^2 \frac{(-e)}{\sqrt{2}}\Big) + \cdots \\
= {} & \frac{e}{\sqrt{2}}\left(1 + (x-1) - (y - \frac{\pi}{4}) + \frac{1}{2}(x-1)^2 - (y - \frac{\pi}{4})(x-1) - \frac{1}{2}(y - \frac{\pi}{4})^2\right) + \cdots .
\end{aligned}$$

2.7 Maxima and Minima for functions of two variables

Definition. Let $f(x, y)$ be a continuous function defined in a closed and bounded domain D of the xy plane and let (a, b) be an interior point of D.
(i) $f(a, b)$ is said to be a local maximum value of $f(x, y)$ at the point (a, b) if there exists a neighborhood N of (a, b) such that $f(x, y) < f(a, b)$ for all points (x, y) in N.
(ii) $f(a, b)$ is said to be a local minimum if $f(x, y) > f(a, b)$ for all points (x, y) in N other than (a, b).



Local maximum or local minimum values are called extreme values.

Stationary point of $f(x, y)$

A point (a, b) satisfying $f_x = 0$ and $f_y = 0$ is called a stationary point of $f(x, y)$.

Necessary conditions for Maximum or minimum

If $f(a, b)$ is an extreme value of $f(x, y)$ at (a, b), then (a, b) is a stationary point of $f(x, y)$ if f_x and f_y exist at (a, b) and $f_x(a, b) = 0$, $f_y(a, b) = 0$.

Sufficient conditions for extreme values of $f(x, y)$

Let (a, b) be a stationary point of the differentiable function $f(x, y)$.

i.e., $f_x(a, b) = 0$, $f_y = (a, b) = 0$.

Let us define $f_{xx}(a, b) = r$, $f_{xy}(a, b) = s$, $f_{yy}(a, b) = t$.

(i) If $rt - s^2 > 0$ and $r < 0$, then $f(a, b)$ is a maximum value.

(ii) If $rt - s^2 > 0$ and $r > 0$ then $f(a, b)$ is a minimum value.

(iii) If $rt - s^2 < 0$, then $f(a, b)$ is not an extreme value but (a, b) is a saddle point of

$f(x, y)$.

(iv) If $rt - s^2 = 0$, then no conclusion is possible and further investigation is required.

Working rule to find maxima and minima of $f(x, y)$

step (1). Find $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ and solve for $f_x = 0$ and $f_y = 0$ as simultaneous equations in x and y.

Let $(a, b), (a_1, b_1), \ldots$ be the solutions which are stationary points of $f(x, y)$.

step (2). Find $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$.

step (3). Evaluate r, s, t at each stationary point.

At the point (a, b) if

(i) $rt - s^2 > 0$ and $r < 0$ then $f(a, b)$ is a maximum value of $f(x, y)$.

(ii) $rt - s^2 > 0$ and $r > 0$ then $f(a, b)$ is a minimum value of $f(x, y)$.

(iii) $rt - s^2 < 0$ then (a, b) is called a saddle point.

(iv) $rt - s^2 = 0$, no conclusion can be made, further investigation is required.

Critical Point

A point (a, b) is a critical point of $f(x, y)$ if $f_x = 0$ and $f_y = 0$ at (a, b) or f_x and f_y do not exist at (a, b).

Maxima or Minima occur at a critical point.

Worked Examples

Example 2.72. Examine $f(x, y) = x^3 + y^3 - 12x - 3y + 20$ for its extreme values.

[Jun 2013, Jan 2012, May 2011, Jun 2010]

Solution. Given $f(x, y) = x^3 + y^3 - 12x - 3y + 20$.

$$f_x = 3x^2 - 12 \qquad f_y = 3y^2 - 3$$

$$r = f_{xx} = 6x \qquad s = f_{xy} = 0 \qquad t = f_{yy} = 6y$$

For stationary points, solve $f_x = 0$ and $f_y = 0$.

$f_x = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.

$f_y = 0 \Rightarrow 3y^2 - 3 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$.

Stationary points are $(2,1), (2,-1), (-2,1)$ and $(-2,-1)$.

$rt - s^2 = 6x6y - 0 = 36xy$.

At $(2,1)$, $rt - s^2 = 36 \times 2 \times 1 = 72 > 0$.

At $(2,1)$, $r = 6(2) = 12 > 0$.

$\therefore (2,1)$ is a minimum point.

Minimum value is $f(2,1) = 8 + 1 - 24 - 3 + 20 = 29 - 27 = 2$.

At $(2,-1)$, $rt - s^2 = 36 \times 2 \times -1 = -72 < 0$.

$\therefore (2,-1)$ is a saddle point.

At $(-2,1)$, $rt - s^2 = 36 \times (-2) \times 1 = -72 < 0$.

$\therefore (-2,1)$ is a saddle point.

At $(-2,-1)$, $rt - s^2 = 36 \times -2 \times -1 = 72 > 0$.

$r = 6(-2) = -12 < 0$.

$\therefore (-2,-1)$ is a maximum point.

Maximum value $f(-2,-1) = -8 - 1 + 24 + 3 + 20 = 47 - 9 = 38$.

Example 2.73. Examine $f(x,y) = x^3 + y^3 - 3axy$ for maximum and minimum values. [Jan 1999]

Solution.

Given $f(x,y) = x^3 + y^3 - 3axy$.

$$f_x = 3x^2 - 3ay$$

$$f_y = 3y^2 - 3ax$$

$$r = f_{xx} = 6x$$

$$s = f_{xy} = -3a \quad t = f_{yy} = 6y.$$

For stationary points, solve $f_x = 0$ and $f_y = 0$.

$f_x = 0 \Rightarrow 3x^2 - 3ay = 0 \Rightarrow ay = x^2 \Rightarrow y = \dfrac{x^2}{a}$.

$f_y = 0 \Rightarrow 3y^2 - 3ax = 0 \Rightarrow \dfrac{x^4}{a^2} - ax = 0$.

$x\left(\dfrac{x^3}{a^2} - a\right) = 0 \Rightarrow x(x^3 - a^3) = 0 \Rightarrow x = 0$ or $x = a$.

$x = 0 \Rightarrow y = 0$

$x = a \Rightarrow y = \frac{a^2}{a} = a.$

Stationary points are $(0,0)$ and (a,a).

At $(0,0)$, $rt - s^2 = 6x6y - 9a^2 = 36xy - 9a^2 = -9a^2 < 0.$

$r = 6x = 0.$

$\therefore$ No maximum or minimum at $(0,0)$.

$\therefore$ $(0,0)$ is a saddle point.

At (a,a), $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$ if $a \neq 0$.

$r = 6x = 6a.$

If $a < 0, r < 0.$

$\therefore$ (a,a) is a maximum point if $a < 0$.

If $a > 0, r > 0.$

$\therefore$ (a,a) is a minimum point if $a > 0$.

Maximum value $= a^3 + a^3 - 3a^3 = -a^3$ if $a < 0$

Minimum value $= -a^3$ if $a > 0$.

Example 2.74. Discuss the maxima and minima of $f(x,y) = x^3y^2(1-x-y)$.

[Jan 2014]

Solution. Given: $f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$.

$$f_x = y^2[x^3(-1) + (1-x-y)3x^2] = x^2y^2[-x+3-3x-3y]$$
$$= x^2y^2[-4x-3y+3].$$
$$f_y = 2x^3y - 2x^4y - 3x^3y^2.$$
$$r = f_{xx} = y^2[-12x^2 - 6xy + 6x].$$
$$s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2.$$
$$t = f_{yy} = 2x^3 - 2x^4 - 6x^3y.$$

For stationary points, solve $f_x = 0$ and $f_y = 0$.

$f_x = 0 \Rightarrow x^2y^2[-4x-3y+3] = 0.$

$\Rightarrow x = 0, y = 0, 4x + 3y = 3.$

$f_y = 0 \Rightarrow x^3 y(2 - 2x - 3y) = 0.$

$$x = 0, y = 0, 2x + 3y = 2.$$

$$\text{Solving} \quad 4x + 3y = 3 \tag{1}$$

$$2x + 3y = 2 \tag{2}$$

we get $2x = 1 \Rightarrow x = \frac{1}{2}$.

When $x = \frac{1}{2}$, $(1) \Rightarrow 2 + 3y = 3 \Rightarrow 3y = 1 \Rightarrow y = \frac{1}{3}$.

$\therefore$ The stationary points are $(0,0), \left(\frac{1}{2}, \frac{1}{3}\right)$.

At $(0,0)$, $rt - s^2 = 0.0 - 0 = 0$.

We can not say maximum or minimum. Further investigation is required.

At $\left(\frac{1}{2}, \frac{1}{3}\right)$,

$$\begin{aligned}
r &= \frac{1}{9}\left(-12 \times \frac{1}{4} - 6 \times \frac{1}{2} \times \frac{1}{3} + 6 \times \frac{1}{2}\right) \\
&= \frac{1}{9}\left(-3 - 1 + 3\right) = \frac{1}{9}(-1) - \frac{1}{9}. \\
t &= 2\frac{1}{8} - 2\frac{1}{16} - 6\frac{1}{8}\frac{1}{3} = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}. \\
s^2 &= \left(6\frac{1}{4}\frac{1}{3} - 8\frac{1}{8}\frac{1}{3} - 9\frac{1}{4}\frac{1}{9}\right)^2 = \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)^2 \\
&= \left(\frac{1}{4} - \frac{1}{3}\right)^2 = \left(-\frac{1}{12}\right)^2 = \frac{1}{144}. \\
rt - s^2 &= \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \frac{1}{144} \\
&= \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} > 0 \text{ and } r < 0.
\end{aligned}$$

$\therefore \left(\frac{1}{2}, \frac{1}{3}\right)$ is a maximum point.

Maximum value is $f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{8}\frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{72}\left(\frac{6-3-2}{6}\right) = \frac{1}{72 \times 6} = \frac{1}{432}$.

Example 2.75. Find the extreme values of the function $f(x,y) = x^4 + y^2 + x^2 y$.

Solution. Given: $f(x,y) = x^4 + y^2 + x^2 y$.

$$f_x = 4x^3 + 2xy. \qquad r = f_{xx} = 12x^2 + 2y.$$

$$f_y = 2y + x^2. \qquad s = f_{xy} = 2x. \qquad t = f_{yy} = 2.$$

For stationary points, solve $f_x = 0$ and $f_y = 0$.

$f_x = 0 \Rightarrow 4x^3 + 2xy = 0 \Rightarrow 2x(y + 2x^2) = 0 \Rightarrow x = 0, y + 2x^2 = 0 \Rightarrow y = -2x^2$.

$f_y = 0 \Rightarrow 2y + x^2 = 0 \Rightarrow -4x^2 + x^2 = 0 \Rightarrow -3x^2 = 0 \Rightarrow x = 0$.

When $x = 0, y = 0$.

$\therefore$ The only stationary point is $(0, 0)$.

$$rt - s^2 = (12x^2 + 2y)2 - 4x^2.$$

At $(0, 0)$, $rt - s^2 = 0$.

We can not say maximum or minimum.

We shall investigate the nature of the function in a neighbourhood of $(0, 0)$.

We have $f(0, 0) = 0$. In a neighbourhood of $(0, 0)$ on the $x-$axis, take the point $(h, 0)$, $f(h, 0) = h^4 > 0$.

On the $y-$axis take the point $(0, k)$, $f(0, k) = k^2 > 0$.

On $y = mx$, for any m, take the point (h, mh).

$$f(h, mh) = h^4 + m^2h^2 + mh^3 = h^2[h^2 + m^2 + mh].$$

For the quadratic in m, $m^2 + mh + h^2$

discriminant $= B^2 - 4AC = h^2 - 4.1.h^2 = -3h^2 < 0$ if $m \neq 0$.

$\therefore$ $f(h, mh) > 0$ for all $m \neq 0$.

$\therefore$ In a neigbourhood of $(0, 0)$ for all points (x, y), $f(x, y) > 0$.

$\therefore$ $f(0, 0)$ is minimum and the minimum value $= 0$.

Example 2.76. Find the maximum and minimum values of $x^2 - xy + y^2 - 2x + y$.

[Jun 2012, Jun 2010]

Solution. $f(x, y) = x^2 - xy + y^2 - 2x + y$.

$f_x = 2x - y - 2.$

$f_y = -x + 2y + 1.$

$r = f_{xx} = 2.$

$s = f_{xy} = -1.$

$t = f_{yy} = 2.$

For stationary points, solve $f_x = 0, f_y = 0$

$$2x - y - 2 = 0. \tag{1}$$

$$-x + 2y + 1 = 0. \tag{2}$$

$$(1) \Rightarrow \; y = 2x - 2.$$

$$(2) \Rightarrow \; -x + 2(2x - 2) + 1 = 0$$

$$-x + 4x - 4 + 1 = 0$$

$$3x - 3 = 0$$

$$3x = 3$$

$$x = 1.$$

$$\therefore \; y = 2 - 2 = 0.$$

The stationary point is $(1, 0)$.

$$rt - s^2 = 4 + 1 = 5 > 0 \quad \text{and} \quad r = 2 > 0.$$

$\therefore$ $(1, 0)$ is a minimum point.

Minimum value of $f = 1 - 2 = -1$.

Example 2.77. Find the extreme values of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$. [Jan 2012]

Solution. $f(x, y) = x^3 + y^3 - 3x - 12y + 20.$

$$f_x = 3x^2 - 3.$$

$$f_y = 3y^2 - 12.$$

$$r = f_{xx} = 6x.$$

$$s = f_{xy} = 0.$$

$$t = f_{yy} = 6y.$$

For stationary points, solve $f_x = 0, f_y = 0$.

$$3x^2 - 3 = 0. \qquad 3y^2 - 12 = 0.$$

$$3x^2 = 3. \qquad 3y^2 = 12.$$

$$x^2 = 1. \qquad y^2 = 4.$$

$$x = \pm 1. \qquad y = \pm 2.$$

The stationary points are $(1, 2), (-1, 2), (1, -2), (-1, -2)$.

At $(1, 2)$, $rt - s^2 = 36xy = 36 \times 1 \times 2 = 72 > 0$.

$$r = 6 > 0.$$

$\therefore$ $(1, 2)$ is a minimum point.

Minimum value of $f = 1 + 8 - 3 - 24 + 20 = 2$.

At $(-1, 2)$, $rt - s^2 = 36xy = 36 \times (-1) \times 2 = -72 < 0$.

$\therefore$ $(-1, 2)$ is a saddle point.

At $(1, -2)$, $rt - s^2 = 36xy = 36 \times 1 \times (-2) = -72 < 0$.

$\therefore$ $(1, -2)$ is a saddle point.

At $(-1, -2)$, $rt - s^2 = 36xy = 72 > 0$.

$$r = 6 \times (-1) = -6 < 0.$$

$\therefore$ $(-1, -1)$ is a maximum point.

Maximum value of $f = -1 - 8 + 3 + 24 + 20 = 38$.

Maxima $= 38$.

Minima $= 2$.

Example 2.78. Test for maxima and minima of the function $f(x,y) = x^3y^2(6-x-y)$.

[Jan 2013]

Solution. $f(x,y) = x^3y^2(6-x-y)$

$= 6x^3y^2 - x^4y^2 - x^3y^3.$

$f_x = 18x^2y^2 - 4x^3y^2 - 3x^2y^3.$

$f_y = 12x^3y - 2x^4y - 3x^3y^2.$

$r = f_{xx} = 36xy^2 - 12x^2y^2 - 6xy^3.$

$s = f_{xy} = 36x^2y - 8x^3y - 9x^2y^2.$

$t = f_{yy} = 12x^3 - 2x^4 - 6x^3y.$

For stationary points, $f_x = 0$, $f_y = 0$.

$$f_x = 0 \Rightarrow 18x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$x^2y^2(18-4x-3y) = 0$$

$$\text{i.e., } 4x + 3y = 18. \quad (1)$$

$$f_y = 0 \Rightarrow 12x^3y - 2x^4y - 3x^3y^2 = 0$$

$$x^3y(12-2x-3y) = 0$$

$$2x + 3y = 12. \quad (2)$$

$(1) - (2) \Rightarrow 2x = 6$

$x = 3.$

$(1) \Rightarrow 12 + 3y = 18$

$3y = 6$

$y = 2.$

The stationary point is $(3,2)$

At $(3,2)$,

$$\begin{aligned}
r &= 36 \times 3 \times 4 - 12 \times 9 \times 4 - 6 \times 3 \times 8 \\
&= 432 - 432 - 144 \\
&= -144 < 0. \\
t &= 12 \times 9 - 2 \times 81 - 6 \times 27 \times 2 \\
&= 108 - 162 - 324 \\
&= -378. \\
s &= 34 \times 9 \times 2 - 8 \times 27 \times 2 - 9 \times 9 \times 4 \\
&= 612 - 432 - 324 \\
&= -144. \\
rt - s^2 &= (-144)(-378) - (-144)^2 \\
&= 54432 - 20736 \\
&= 33696 > 0
\end{aligned}$$

Since $rt - s^2 > 0$ and $r < 0$, $(3, 2)$ is a maximum point.

$\therefore$ Maximum value of $f = 27 \times 4(6 - 3 - 2) = 108$.

Example 2.79. Examine for minimum and maximum values $\sin x + \sin y + \sin(x + y)$.

Solution. We have $f(x, y) = \sin x + \sin y + \sin(x + y)$.

$$f_x = \cos x + \cos(x + y) \qquad f_y = \cos y + \cos(x + y).$$

$$\begin{aligned}
r = f_{xx} &= -\sin x - \sin(x + y) \\
s = f_{xy} &= -\sin(x + y) \\
t = f_{yy} &= -\sin y - \sin(x + y).
\end{aligned}$$

For stationary points, solve $f_x = 0$ and $f_y = 0$.

$$\text{i.e., } \cos x + \cos(x + y) = 0. \tag{1}$$

$$\cos y + \cos(x + y) = 0. \tag{2}$$

$$(1)-(2) \Longrightarrow \cos x - \cos y = 0.$$
$$\text{i.e., } \cos x = \cos y$$
$$\Longrightarrow x = y$$
$$\text{Now } (1) \Longrightarrow \cos x + \cos 2x = 0$$
$$\text{i.e., } \cos 2x = -\cos x.$$
$$\cos 2x = \cos(\pi - x)$$
$$\Longrightarrow 2x = \pi - x.$$
$$\text{i.e., } 3x = \pi$$
$$x = \frac{\pi}{3}.$$

When $x = \frac{\pi}{3}, y = \frac{\pi}{3}$.
$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a stationary point.
At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$r = -\sin\frac{\pi}{3} - \sin\frac{2\pi}{3} = \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3} < 0.$$
$$s = -\sin\frac{2\pi}{3} = -\frac{\sqrt{3}}{2}, t = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}.$$
$$rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0.$$

Since $rt - s^2 > 0$ and $r < 0$, $f(x,y)$ has a maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$\therefore$ Maximum value $= f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$.

Example 2.80. Find the maximum and minimum values of $\sin x \sin y \sin(x+y), 0 < x, y < \pi$. [Jan 1997]

Solution. Given $f(x,y) = \sin x \sin y \sin(x+y)$.

$$f_x = \sin y[\sin x \cos(x+y) + \sin(x+y)\cos x] = \sin y \sin(2x+y).$$

$$r = f_{xx} = 2\sin y \cos(2x+y).$$

$$f_y = \sin x[\sin y\cos(x+y) + \sin(x+y)\cos y] = \sin x\sin(x+2y).$$

$$s = f_{xy} = \sin x\cos(x+2y) + \sin(x+2y)\cos x = \sin(2x+2y).$$

$$t = f_{yy} = 2\sin x\cos(x+2y).$$

For stationary points, solve $f_x = 0$ and $f_y = 0$.

$f_x = 0 \Rightarrow \sin y\sin(2x+y) = 0.$

$f_y = 0 \Rightarrow \sin x\sin(x+2y) = 0.$

Since, $x, y \neq 0 \& \neq \pi \Rightarrow \sin x \neq 0, \sin y \neq 0.$

$\therefore \sin(2x+y) = 0$ and $\sin(x+2y) = 0$.

Since, $0 < x < \pi, 0 < 2x < 2\pi$

$0 < y < \pi \Rightarrow 0 < 2x+y < 3\pi.$

Similarly $0 < x+2y < 3\pi$. Since, $\sin(2x+y) = 0 \Rightarrow 2x+y = \pi$ or 2π.

Similarly $x+2y = \pi$ or 2π.

$$\text{If} \quad 2x+y = \pi \tag{1}$$

$$\text{and} \quad x+2y = \pi \tag{2}$$

then $x - y = 0 \Rightarrow x = y$.

$\therefore (1) \Rightarrow 3x = \pi \Rightarrow x = \dfrac{\pi}{3}$.

$\therefore y = \dfrac{\pi}{3}$.

$\therefore$ one stationary point is $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$.

If $2x+y = \pi$ and $x+2y = 2\pi$ then, $x - y = -\pi \Rightarrow x = y - \pi$.

$\therefore 2(y-\pi) + y = \pi \Rightarrow 3y - 2\pi = \pi \Rightarrow 3y = 3\pi \Rightarrow y = \pi,$

which is not admissible since $y \neq \pi$.

Similarly, $2x+y = 2\pi$ and $x+2y = \pi$ is also not possible.

Now take $2x+y = 2\pi$ and $x+2y = 2\pi$

$\Rightarrow x - y = 0 \Rightarrow x = y$

$\Rightarrow 3x = 2\pi \Rightarrow x = \dfrac{2\pi}{3}$

$\therefore y = \dfrac{2\pi}{3}$.

$\therefore$ Another stationary point is $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$.

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$r = 2\sin\frac{\pi}{3}\cos\pi = 2\frac{\sqrt{3}(-1)}{2} < 0.$$

$$s = \sin\frac{4\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

$$t = 2\sin\frac{\pi}{3}\cos\pi = \frac{2\sqrt{3}(-1)}{2} = -\sqrt{3}.$$

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{-\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0.$$

$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a maximum point.

Maximum value $= f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\frac{\pi}{3}.\sin\frac{\pi}{3}\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\sin\left(\pi - \frac{\pi}{3}\right)$
$= \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}.$

At $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$

$r = 2\sin\frac{2\pi}{3}\cos\frac{6\pi}{3} = 2\frac{\sqrt{3}}{2}.1 = \sqrt{3} > 0.$

$t = 2\sin\frac{2\pi}{3}\cos\frac{6\pi}{3} = \sqrt{3}.$

$s = \sin\frac{8\pi}{3} = \sin\left(3\pi - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$

$rt - s^2 = \sqrt{3}\sqrt{3} - \left(\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0.$

$\therefore \left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ is a minimum point.

Minimum value $= f\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) = \sin\frac{2\pi}{3}\sin\frac{2\pi}{3}\sin\frac{4\pi}{3} = \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}\right) = \frac{-3\sqrt{3}}{8}.$

Example 2.81. In a plane triangle, find the maximum value of $\cos A\cos B\cos C$.

[Jan 2000]

Solution. The angles of the $\Delta^{le}ABC$ satisfy $0 < A, B, C < \pi$ and $A + B + C = \pi$,

$\Longrightarrow C = \pi - (A + B)$. Replacing C, we get

$f(A, B) = \cos A\cos B\cos(\pi - (A + B)) = -\cos A\cos B\cos(A + B), \quad 0 < A, B < \pi.$

$f_A = -\cos B[\cos A(-\sin(A + B)) + \cos(A + B)(-\sin A)] = \cos B\sin(2A + B).$

$f_B = -\cos A[-\sin(A+2B)] = \cos A \sin(A+2B)$.

$r = f_{AA} = 2\cos B \cos(2A+B)$.

$s = f_{AB} = \cos A \cos(A+2B) + \sin(A+2B)(-\sin A) = \cos(2A+2B)$.

$t = f_{BB} = 2\cos A \cos(A+2B)$.

For stationary points, solve $f_A = 0, f_B = 0$.

$$\text{i.e., } \cos B \sin(2A+B) = 0 \text{ and } \cos A \sin(A+2B) = 0.$$

$$\cos B = 0 \text{ or } \sin(2A+B) = 0 \text{ and } \cos A = 0 \text{ or } \sin(A+2B) = 0$$

$$\Longrightarrow B = \frac{\pi}{2} \text{ or } 2A+B = \pi \text{ or } 2\pi$$

and

$$A = \frac{\pi}{2} \text{ or } A+2B = \pi \text{ or } 2\pi.$$

Different possibilities

case (i) Let $B = \frac{\pi}{2}$ and $A = \frac{\pi}{2}$.

$\Rightarrow A + B = \pi$

$\Longrightarrow C = 0$ not possible.

case (ii) If $B = \frac{\pi}{2}$ and $A + 2B = \pi$.

$\Rightarrow A + \pi = \pi \Rightarrow A = 0$ not possible.

case (iii) If $B = \frac{\pi}{2}$ and $A + 2B = 2\pi$.

$\Rightarrow A + \pi = 2\pi \Rightarrow A = \pi$ not possible.

case (iv) $A = \frac{\pi}{2}, 2A + B = \pi$.

$\Rightarrow \pi + B = \pi \Rightarrow B = 0$ not possible.

case (v) $A = \frac{\pi}{2}, 2A + B = 2\pi \Rightarrow B = \pi$ not possible.

case (vi) If $2A + B = \pi, A + 2B = \pi$.

Subtracting $A - B = 0 \Rightarrow A = B$.

$\therefore 3A = \pi \Rightarrow A = \frac{\pi}{3}$.

$\Longrightarrow B = \frac{\pi}{3}, C = \frac{\pi}{3}$.

case (vii) If $2A + B = \pi$ and $A + 2B = 2\pi \Rightarrow A - B = -\pi$ not possible.

Finally $2A + B = 2\pi$ and $A + 2B = 2\pi \Rightarrow A - B = 0 \Rightarrow A = B$.

$3A = 2\pi \Rightarrow A = \frac{2\pi}{3}, B = \frac{2\pi}{3}$.

$A + B = \frac{2\pi}{3} + \frac{2\pi}{3} = 4\frac{\pi}{3} > \pi$ not possible.

$\therefore$ The only stationary point is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$,

$r = 2\cos\frac{\pi}{3}\cos\pi = 2\frac{1}{2}(-1) < 0$.

$t = 2\cos\frac{\pi}{3}\cos\pi = -1$.

$s = \cos\left(\frac{4\pi}{3}\right) = \cos\left(\pi + \frac{\pi}{3}\right) = -\frac{1}{2}$.

$rt - s^2 = (-1)(-1) - \left(\frac{-1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0$.

$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a maximum point.

$\therefore$ Maximum value of f is $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{\pi}{3} = \frac{1}{8}$.

Example 2.82. A flat circular plate is heated so that the temperature at any point (x, y) is $U(x, y) = x^2 + 2y^2 - x$. Find the coldest point on the plate. [Jan 2005]

Solution. Given $U = x^2 + 2y^2 - x$.

$$U_x = 2x - 1 \qquad U_y = 4y.$$

$$r = U_{xx} = 2. \qquad s = U_{xy} = 0. \qquad t = U_{yy} = 4.$$

$rt - s^2 = 8 > 0$ and $r = 2 > 0$.

$\therefore$ All points are minimum points.

At minimum, $U_x = 0$ and $U_y = 0$.

$U_x = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

$U_y = 0 \Rightarrow 4y = 0 \Rightarrow y = 0$.

$\therefore$ The minimum point is $\left(\frac{1}{2}, 0\right)$.

$\therefore$ The coldest point on the plate is $\left(\frac{1}{2}, 0\right)$.

2.8 Constrained Maxima and Minima - Lagrange's Method

Lagrange's Method

Let $f(x, y, z)$ be the function for which the extreme values are to be found subject to the condition

$$\phi(x, y, z) = 0. \tag{1}$$

Construct the auxiliary function $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$, where λ is an undetermined parameter independent of x, y, z which is called the Lagrange's multiplier. Any relative extremum of $f(x, y, z)$ subject to (1) must occur at a stationary point of $F(x, y, z)$.

The stationary points of F are given by $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial \lambda} = 0.$

$\Rightarrow f_x + \lambda\phi_x = 0,\ f_y + \lambda\phi_y = 0,\ f_z + \lambda\phi_z = 0, \phi(x, y, z) = 0.$

$\frac{f_x}{\phi_x} = \frac{f_y}{\phi_y} = \frac{f_z}{\phi_z} = -\lambda$ and $\phi(x, y, z) = 0.$

Solving these equations we can find the values of x, y, z which are stationary points of F and the values of f at these points give the maximum and minimum values of $f(x, y, z)$.

Worked Examples

Example 2.83. Find the maximum value of $x^m y^n z^p$ subject to $x + y + z = a$.

[Jan 2009]

Solution. Let $f = x^m y^m z^p$.

$$\phi = x + y + z - a = 0. \tag{1}$$

We have to maximise f subject to (1).

Let $F = f + \lambda\phi$ where λ is the Lagrange's multiplier.

$$F = x^m y^n z^p + \lambda(x + y + z - a).$$

$$F_x = mx^{m-1}y^n z^p + \lambda, F_y = nx^m y^{n-1} z^p + \lambda, F_z = px^m y^n z^{p-1} + \lambda.$$

To find the stationary points, solve $F_x = 0, F_y = 0, F_z = 0, \phi = 0.$

$$F_x = 0 \Rightarrow mx^{m-1}y^n z^p = -\lambda.$$

$$F_y = 0 \Rightarrow nx^m y^{n-1} z^p = -\lambda.$$

$$F_x = 0 \Rightarrow px^m y^n z^{p-1} = -\lambda.$$

From the above three equations we get

$$mx^{m-1} y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}.$$

Dividing by $x^m y^n z^p$ we get, $\dfrac{m}{x} = \dfrac{n}{y} = \dfrac{p}{z} = \dfrac{m+n+p}{x+y+z} = \dfrac{m+n+p}{a}$.

$$x = \frac{ma}{m+n+p}, y = \frac{na}{m+n+p}, z = \frac{pa}{m+n+p}.$$

The stationary point is $\left(\dfrac{ma}{m+n+p}, \dfrac{na}{m+n+p}, \dfrac{pa}{m+n+p}\right)$.

$\therefore$ Max. value of $f = \dfrac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}$.

Example 2.84. Find the minimum value of x^2yz^3 subject to $2x + y + 3z = a$.

[Jan 2007]

Solution. Given $f = x^2yz^3$.

$$\phi = 2x + y + 3z - a = 0. \tag{1}$$

Let $F = f + \lambda\phi$ where λ is the Lagrange's multiplier.

$$F = x^2yz^3 + \lambda(2x + y + 3z - a).$$

$$F_x = 2xyz^3 + 2\lambda, F_y = x^2z^3 + \lambda, F_z = 3x^2yz^2 + 3\lambda.$$

To find the stationary points, solve $F_x = 0, F_y = 0, F_z = 0, \phi = 0$.

$$F_x = 0 \Rightarrow 2xyz^3 + 2\lambda = 0 \Rightarrow xyz^3 = -\lambda.$$

$$F_y = 0 \Rightarrow x^2z^3 = -\lambda.$$

$$F_z = 0 \Rightarrow 3x^2yz^2 + 3\lambda = 0 \Rightarrow x^2yz^2 = -\lambda.$$

Therefore

$$xyz^3 = x^2z^3 = x^2yz^2$$

$$xyz^3 = x^2z^3 \Rightarrow y = x.$$

$$x^2z^3 = x^2yz^2 \Rightarrow y = z.$$

$$x = y = z.$$

$$(1) \Rightarrow 2x + x + 3x = a \Rightarrow 6x = a \Rightarrow x = \frac{a}{6} = y = z.$$

$\therefore$ The stationary point is $\left(\frac{a}{6}, \frac{a}{6}, \frac{a}{6}\right)$.

Minimum value $= \left(\frac{a}{6}\right)^2 \frac{a}{6} \left(\frac{a}{6}\right)^3 = \frac{a^6}{6^6} = \left(\frac{a}{6}\right)^6$.

Example 2.85. If $u = x^2 + y^2 + z^2$ where $ax + by + cz - p = 0$, find the stationary value of u. [Jan 2006]

Solution. Given $f = x^2 + y^2 + z^2$.

$$\phi = ax + by + cz - p = 0. \tag{1}$$

Let $F = f + \lambda\phi$ where λ is the Lagrange's multiplier.

$$F = x^2 + y^2 + z^2 + \lambda(ax + by + cz - p).$$

$$F_x = 2x + a\lambda, F_y = 2y + b\lambda, F_z = 2z + c\lambda.$$

To find the stationary points, solve $F_x = 0, F_y = 0, F_z = 0, \phi = 0$.

$$2x + \lambda a = 0 \Rightarrow x = \frac{-a\lambda}{2} \Rightarrow ax = \frac{-a^2\lambda}{2}.$$

Similarly

$$by = \frac{-b^2\lambda}{2}, cz = \frac{-c^2\lambda}{2}.$$

$$(1) \Rightarrow \frac{-a^2\lambda}{2} - \frac{b^2\lambda}{2} - \frac{c^2\lambda}{2} = p$$

$$\frac{a^2 + b^2 + c^2}{2}\lambda = -p \Rightarrow \lambda = \frac{-2p}{a^2 + b^2 + c^2}.$$

$$\therefore x = \frac{ap}{a^2 + b^2 + c^2}, y = \frac{bp}{a^2 + b^2 + c^2}, z = \frac{cp}{a^2 + b^2 + c^2}.$$

Stationary value of u is

$$u = \frac{a^2p^2}{(a^2+b^2+c^2)^2} + \frac{b^2p^2}{(a^2+b^2+c^2)^2} + \frac{c^2p^2}{(a^2+b^2+c^2)^2}$$

$$= \frac{p^2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2} = \frac{p^2}{a^2+b^2+c^2}.$$

Example 2.86. The temperature T at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

[Jan 2005]

Solution. We have to maximize

$$T = 400xyz^2 \tag{1}$$

$$\text{subject to} \qquad \phi = x^2 + y^2 + z^2 - 1 = 0. \tag{2}$$

Consider $F = T + \lambda\phi$, where λ is the Lagrange multiplier.
$F = 400xyz^2 + \lambda(x^2 + y^2 + z^2 - 1)$.

$$F_x = 400yz^2 + 2\lambda x, F_y = 400xz^2 + 2\lambda y, F_z = 400xyz + 2\lambda z.$$

To find the stationary points, solve $F_x = 0, F_y = 0, F_z = 0, \phi = 0$.
$F_x = 0 \Rightarrow 400yz^2 + 2\lambda x = 0 \Rightarrow -\lambda = \dfrac{200yz^2}{x}$.
$F_y = 0 \Rightarrow 400xz^2 + 2\lambda y \Rightarrow -\lambda = \dfrac{200xz^2}{y}$.
$F_z = 0 \Rightarrow 800xyz + 2\lambda z = 0 \Rightarrow -\lambda = 400xy$.
$\therefore \dfrac{200yz^2}{x} = \dfrac{200xz^2}{y} = 400xy$.
Taking $\dfrac{200yz^2}{x} = \dfrac{200xz^2}{y}$ we get $x^2 = y^2 \Rightarrow y = \pm x$.
Taking $\dfrac{200yz^2}{x} = 400xy$ we get $\dfrac{z^2}{x} = 2x \Rightarrow z^2 = 2x^2 \Rightarrow z = \pm\sqrt{2}x$.
Taking $\dfrac{200xz^2}{y} = 400xy$ we get $\dfrac{z^2}{y} = 2y \Rightarrow z^2 = 2y^2 \Rightarrow z = \pm\sqrt{2}y$.
Substituting in $x^2 + y^2 + z^2 = 1$ we get
$x^2 + x^2 + 2x^2 = 1 \Rightarrow 4x^2 = 1 \Rightarrow x^2 = \dfrac{1}{4} \Rightarrow x = \pm\dfrac{1}{2}$

$\therefore y = \pm\frac{1}{2}, z = \pm\sqrt{2}\frac{1}{2} = \pm\frac{1}{\sqrt{2}}$.

The stationary points are given by $x = \pm\frac{1}{2}, y = \pm\frac{1}{2}, z = \pm\frac{1}{\sqrt{2}}$.

To have maximum value, we must have xy positive.

$\therefore$ The points are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-1}{2}, \frac{-1}{2}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{\sqrt{2}}\right)$.

$\therefore$ Maximum $T = 400\frac{1}{2}\frac{1}{2}\frac{1}{2} = 50^0 C$.

Example 2.87. Find the shortest and longest distance from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$ using Lagrange's method of constrained maxima and minima. [Jun 2011, Jan 2002]

Solution. Let $P(x, y, z)$ be any point on the sphere.

Let A be the point $(1, 2, -1)$.

$$AP = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}.$$

$$\text{Let } f(x, y, z) = (x-1)^2 + (y-2)^2 + (z+1)^2. \qquad (1)$$

AP is minimum or maximum if f is minimum or maximum.

$\therefore$ The problem is now reduced to minimise or maximise f subject to

$\phi(x, y, z) = x^2 + y^2 + z^2 - 24 = 0$.

Consider the auxiliary function

$F = f + \lambda\phi = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2 + y^2 + z^2 - 24)$ where λ is the Lagrange's multiplier.

$F_x = 2(x-1) + 2\lambda x, F_y = 2(y-2) + 2\lambda y, F_z = 2(z+1) + 2\lambda z$.

To find the stationary points solve

$F_x = 0, F_y = 0, F_z = 0, \phi = 0$.

$F_x = 0 \Rightarrow 2(x-1) + 2\lambda x = 0 \Rightarrow x - 1 = -\lambda x \Rightarrow -\lambda = \frac{x-1}{x} = 1 - \frac{1}{x}$.

$F_y = 0 \Rightarrow 2(y-2) + 2\lambda y = 0 \Rightarrow y - 2 = -\lambda y \Rightarrow -\lambda = \frac{y-2}{y} = 1 - \frac{2}{y}$.

$F_z = 0 \Rightarrow 2(z+1) + 2\lambda z = 0 \Rightarrow z + 1 = -\lambda z \Rightarrow -\lambda = \frac{z+1}{z} = 1 + \frac{1}{z}$.

$\therefore 1 - \frac{1}{x} = 1 - \frac{2}{y} = 1 + \frac{1}{z}$.

Taking $1 - \frac{1}{x} = 1 - \frac{2}{y}$ we get $\frac{1}{x} = \frac{2}{y} \Rightarrow y = 2x$.

Taking $1 - \frac{1}{x} = 1 + \frac{1}{z}$ we get $z = -x$.

Taking $1 - \frac{2}{y} = 1 + \frac{1}{z}$ we get $z = \frac{-y}{2} = -x$.

We have $x^2 + y^2 + z^2 = 24 \Rightarrow x^2 + 4x^2 + x^2 = 24 \Rightarrow 6x^2 = 24 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.

When $x = 2, y = 4, z = -2$, the first point is $(2, 4, -2)$. Let this point be P_1.

When $x = -2, y = -4, z = 2$, the second point is $(-2, -4, 2)$. Let this point be P_2.

$P_1A = \sqrt{1 + 4 + 1} = \sqrt{6}, P_2A = \sqrt{9 + 36 + 9} = \sqrt{54} = 3\sqrt{6}$

$\therefore$ Shortest distance $= \sqrt{6}$

Longest distance $= 3\sqrt{6}$.

Example 2.88. A rectangular box open at the top is to have a volume of $32cc$. Find the dimensions of the box which requires least amount of material for its construction. [Jun 2012, Dec 2011, Jun 2010, Jan 2005]

Solution. Let the dimensions of the box be Length $= x$, Breadth $= y$, height $= z$.

Given: Volume $= 32cc$.

$$\implies xyz = 32, x, y, z > 0. \tag{1}$$

We have to minimize the amount of material used for the construction of the box.

Let S be the surface area of the box whose top is open

$$\therefore S = xy + 2xz + 2yz \tag{2}$$

By Lagrange's method

Let $F = s + \lambda\phi = xy + 2yz + 2xz + \lambda(xyz - 32)$ where λ is the Lagrange's multiplier.

$F_x = y + 2z + \lambda yz, F_y = x + 2z + \lambda xz, F_z = 2y + 2x + \lambda xy$.

To find the stationary points, solve

$$F_x = 0, F_y = 0, F_z = 0, \phi = 0$$

$$F_x = 0 \Rightarrow y + 2z + \lambda yz = 0$$

$$\implies y + 2z = -\lambda yz$$

$$\Rightarrow \frac{y}{yz} + \frac{2z}{yz} = -\lambda$$

$$\Rightarrow \frac{1}{z} + \frac{2}{y} = -\lambda \qquad (3)$$

$F_y = 0 \Rightarrow x + 2z + \lambda xz = 0$

$$\Longrightarrow x + 2z = -\lambda xz$$

$$\text{i.e., } xy + 2yz = -\lambda xyz.$$

$$\Rightarrow \frac{xy}{xyz} + \frac{2yz}{xyz} = -\lambda$$

$$\Rightarrow \frac{1}{z} + \frac{2}{x} = -\lambda \qquad (4)$$

$F_z = 0 \Rightarrow 2y + 2x + \lambda xy = 0$

$$\Longrightarrow 2x + 2y = -\lambda xy$$

$$2xz + 2yz = -\lambda xyz.$$

$$\Rightarrow \frac{2xz}{xyz} + \frac{2yz}{xyz} = -\lambda$$

$$\Rightarrow \frac{2}{y} + \frac{2}{x} = -\lambda \qquad (5)$$

$(3) - (4) \Rightarrow$

$$\frac{2}{y} - \frac{2}{x} = 0$$

$$\frac{1}{y} = \frac{1}{x}$$

$$\Rightarrow x = y \qquad (6)$$

$(3) - (5) \Rightarrow$

$$\frac{1}{z} - \frac{2}{x} = 0$$

$$\frac{1}{z} = \frac{2}{x}$$

$$\Rightarrow x = 2z \qquad (7)$$

From (6)and(7) we obtain $x = y = 2z$.

(1)$\Rightarrow 2z.2z.z = 32 \Rightarrow 4z^3 = 32 \Rightarrow z^3 = 8 \Rightarrow z = 2.$

$\therefore x = 4, y = 4.$

The stationary point is $(4,4,2)$.

The dimensions are $4cm, 4cm, 2cm$.

Example 2.89. Find the dimensions of the rectangular box open at the top of maximum capacity whose surface area is 432 sq.m. [Jun 2013]

Solution. Let the dimensions of the box be x, y, z.

Given, surface area$= 432$.

$$xy + 2xz + 2yz = 432 \tag{1}$$

Let V be the volume of the box.

We have to maximize V.

$$V = xyz \tag{2}$$

By lagrange's method

$F = V + \lambda\phi$, where λ is the lagrange multiplier.

$$F = xyz + \lambda(xy + 2xz + 2yz - 432).$$

$$F_x = yz + \lambda(y + 2z).$$

$$F_y = xz + \lambda(x + 2z).$$

$$F_z = xy + \lambda(2x + 2y).$$

$$F_\lambda = xy + 2xz + 2yz - 432.$$

For stationary points, $F_x = 0, F_y = 0, F_z = 0, F_\lambda = 0$.

$$F_x = 0 \Rightarrow \; yz + \lambda(y + 2z) = 0$$

$$\Rightarrow \; xyz + \lambda(xy + 2xz) = 0. \tag{1}$$

$$F_y = 0 \Rightarrow \; xz + \lambda(x + 2z) = 0$$

$$\Rightarrow \; xyz + \lambda(xy + 2yz) = 0. \tag{2}$$

$$F_z = 0 \Rightarrow \; xy + \lambda(2x + 2y) = 0$$

$$xyz + \lambda(2xz + 2yz) = 0. \tag{3}$$

$(1) + (2) + (3) \Rightarrow$

$$3xyz + \lambda(2xy + 4xz + 4yz) = 0$$

$$3xy + 2\lambda(xy + 2xz + 2yz) = 0$$

$$3xyz + 2\lambda \times 432 = 0$$

$$3xyz = -864\lambda$$

$$\lambda = -\frac{3xyz}{864} = -\frac{xyz}{288}$$

Substituting the value of λ in $F_x = 0$ we get

$$yz - \frac{xyz}{288}(y + 2z) = 0$$

$$1 - \frac{x}{288}(y + 2z) = 0$$

$$1 = \frac{x}{288}(y + 2z)$$

$$xy + 2xz = 288 \tag{4}$$

$$F_y = 0 \Rightarrow \ xz - \frac{xyz}{288}(x + 2z) = 0$$

$$1 - \frac{y}{288}(x + 2z) = 0$$

$$1 - \frac{y}{288}(x + 2z) = 0, 1 = \frac{y}{288}(x + 2z)$$

$$xy + 2yz = 288 \tag{5}$$

$$F_z = 0 \Rightarrow \ xy - \frac{xyz}{288}(2x + 2y) = 0$$

$$1 - \frac{z}{288}(2x + 2y) = 0, 1 = \frac{z}{288}(2x + 2y)$$

$$2xz + 2yz = 288 \tag{6}$$

$$(4) - (5) \Rightarrow \; 2z(x-y) = 0, z \neq 0$$

$$\therefore \; x - y = 0$$

$$\Rightarrow \; x = y.$$

$$(5) - (6) \Rightarrow \; xy - 2xz = 0$$

$$x(y - 2z) = 0, x \neq 0$$

$$\therefore y - 2z = 0$$

$$y = 2z.$$

$$\therefore x = y = 2z$$

$$(4) \Rightarrow \; 2z \times 2z + 2 \cdot 2z \cdot z = 288$$

$$4z^2 + 4z^2 = 288$$

$$z^2 = 36.$$

$$z = \pm 6.$$

$$z = -6 \text{ is not possible}$$

$$\therefore \; z = 6.$$

$$\therefore \; y = 2z = 12$$

$$x = 2z = 12$$

The dimensions of the box to have maximum capacity are

$$\text{Length} = 12m$$

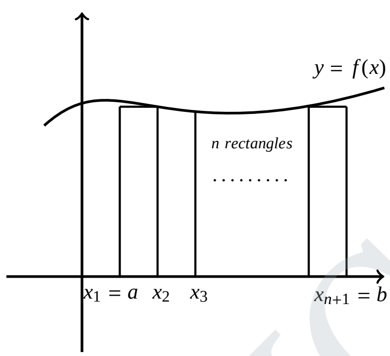
$$\text{Breadth} = 12m$$

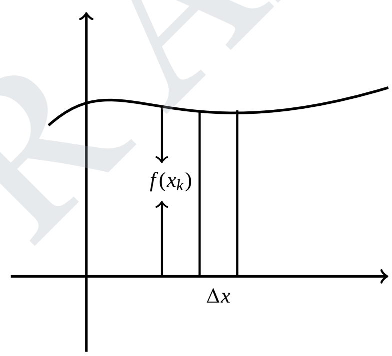
$$\text{Height} = 6m.$$

3 Integral Calculus

3.1 The definite integral

Integration as the limit of a sum





Consider the graph of the positive function $y = f(x)$. Let us find the area under the curve $y = f(x)$,between the $x-$axis and the ordinates $x = a$ and $x = b$. Divide this area into n rectangles of equal width $\Delta x = \frac{b-a}{n}$. Let the $x-$ co ordinates at the left hand side of the rectangles be $x_1 = a, x_2, x_3, \ldots x_{n+1} = b$. Consider a typical rectangle, the k^{th} one with height $f(x_k)$. The area of this rectangle is $f(x_k)\Delta x$. The sum of all the areas of the n rectangles is

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

In summation convention form, this can be written as $\sum_{k=1}^{n} f(x_k)\Delta x$. This gives an estimate of the area under the curve but it is not exact. To improve the estimate we must take a large number of very thin rectangles. This can be achieved by allowing $n \to \infty$ and making $\Delta x \to 0$.

$$\therefore \text{ Area under the curve } = \lim_{n=\infty} \sum_{k=1}^{n} f(x_k)\Delta x.$$

The lower and upper limits on the sum correspond to the first and the last rectangle where $x = a$ and $x = b$ respectively. Hence the above limit can be written as $\lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\Delta x$. Since the number of rectangles increase without bound, we drop the subsript k from x_k and write $f(x)$ which is the value of f at a typical value of x. If this can actually be found, it is called the definite integral of $f(x)$ from $x = a$ and $x = b$ and it is written as $\int_a^b f(x)dx$.

$$\therefore \int_a^b f(x)dx = \lim_{n=\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\Delta x.$$

Note. If we approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip; (i.e., x_k is taken at the right end points of the k^{th} rectangle and $f(x_k)$ as the height) then also the above result will be achieved.

Definition of a definite integral. If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0(= a), x_1, x_2, \cdots x_n(= b)$ be the end points of these subintervals and if we choose $x_1^*, x_2^*, \cdots x_n^*$ as any sample ponits in these subintervals, so that x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is defined as $\int_a^b f(x)dx = \lim_{n=\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$, provided that the limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a, b]$.

Result (1). $\int_a^b f(x)dx = \lim_{n=\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ can also be written as, for $\epsilon > 0$, there is an integer N such that $\left| \int_a^b f(x)dx - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \epsilon$ for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

Result (2). The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called the Reimann sum.

Theorem 1. If f is continuos on $[a,b]$, (or) if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$.[i.e., $\int_a^b f(x)dx$ exists].

Theorem 2. If f is integrable on $[,b]$, then $\int_a^b f(x)dx = \lim_{n=\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Worked Examples

Example 3.1. Using the area property evaluate $\int_0^1 x^2 dx$. Show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$.

Solution. Consider the function $y = f(x) = x^2$. Divide the area under the curve $y = x^2$ between $x = 0$ and $x = 1$ into 4 rectangles of equal width $\Delta x = \frac{1-0}{4} = \frac{1}{4}$.

The width of each rectangle is $\frac{1}{4}$.

The subintervals are $\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right]$.

The height of the rectangles is approximated to the value of the ordinates at $x = 0, \frac{1}{4}, \frac{1}{2} \& \frac{3}{4}$.

$$\begin{aligned}
\therefore \text{ Area of the required portion } &= f(0)\Delta x + f\left(\frac{1}{4}\right)\Delta x + f\left(\frac{1}{2}\right)\Delta x + f\left(\frac{3}{4}\right)\Delta x. \\
&= 0^2.\frac{1}{4} + \left(\frac{1}{4}\right)^2.\frac{1}{4} + \left(\frac{1}{2}\right)^2.\frac{1}{4} + \left(\frac{3}{4}\right)^2.\frac{1}{4} \\
&= \frac{1}{4}\left[0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right] \\
&= \frac{1}{4}\left[\frac{1+4+9}{16}\right] \\
&= \frac{1}{4} \times \frac{14}{16} = \frac{7}{32}.
\end{aligned}$$

If the height of the rectangles is approximated to the value of the ordinates at the right edge of each rectangle then,

$$\begin{aligned}
\text{Area} &= f\left(\frac{1}{4}\right)\Delta x + f\left(\frac{1}{2}\right)\Delta x + f\left(\frac{3}{4}\right)\Delta x + f(1)\Delta x.\\
&= \left(\frac{1}{4}\right)^2.\frac{1}{4} + \left(\frac{1}{2}\right)^2.\frac{1}{4} + \left(\frac{3}{4}\right)^2.\frac{1}{4} + 1^2\cdot\frac{1}{4}\\
&= \frac{1}{4}\left[\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right] = \frac{1}{4}\left[\frac{1+4+9+16}{16}\right] = \frac{1}{4}\times\frac{30}{16} = \frac{15}{32}.
\end{aligned}$$

Now, let us divide the area into n rectangles each of their with width $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. The subintervals are $\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \cdots, \left[\frac{n-1}{n}, \frac{n}{n} = 1\right]$.

The heights of the rectangle are the values of the function $f(x) = x^2$ at the points $\frac{1}{n}, \frac{2}{n}, \frac{3}{n} \cdots \frac{n}{n}$.

$$\begin{aligned}
\therefore\ \text{Area} &= \left(\frac{1}{n}\right)^2.\frac{1}{n} + \left(\frac{2}{n}\right)^2.\frac{1}{n} + \left(\frac{3}{n}\right)^2.\frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^2.\frac{1}{n}\\
&= \frac{1}{n}\left[\frac{1^2}{n^2} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \cdots + \frac{n^2}{n^2}\right]\\
&= \frac{1}{n^3}\left[1^2 + 2^2 + 3^2 + \cdots + n^2\right]\\
&= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{(n+1)(2n+1)}{6n^2}\\
\lim_{n=\infty}(\text{Area}) &= \lim_{n=\infty}\frac{(n+1)(2n+1)}{6n^2}\\
&= \lim_{n=\infty}\frac{n^2(1+\frac{1}{n})(2+\frac{1}{n})}{6n^2} = \frac{2}{6} = \frac{1}{3}.
\end{aligned}$$

Properties of definite integrals.

The following properties can be easily proved using Riemann sums.

(i) $\int_b^a f(x)dx = -\int_a^b f(x)dx.$

(ii) $\int_a^a f(x)dx = 0.$

(iii) $\int_a^b c\,dx = c(b-a)$, where c is any constant.

(iv) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$

(v) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is any constant.

(vi) $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$

(vii) If $a<c<b$ then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

Comparison properties of the integral.

(i) If $f(x)\ge 0$ for $a\le x\le b$, then $\int_a^b f(x)dx \ge 0.$

(ii) If $f(x)\ge g(x)$ for $a\le x\le b$, then $\int_a^b f(x)dx \ge \int_a^b g(x)dx.$

(iii) If $m\le f(x)\le M$ for $a\le x\le b$, then $m(b-a)\le \int_a^b f(x)dx \le M(b-a).$

Proof. Given $m\le f(x)\le M$

Integrating between a and b we get

$$\int_a^b m dx \le \int_a^b f(x)dx \le \int_a^b M dx$$

$$m\int_a^b dx \le \int_a^b f(x)dx \le M\int_a^b dx$$

$$m(b-a)\le \int_a^b f(x)dx \le M(b-a).$$

The first fundamental theorem of calculus

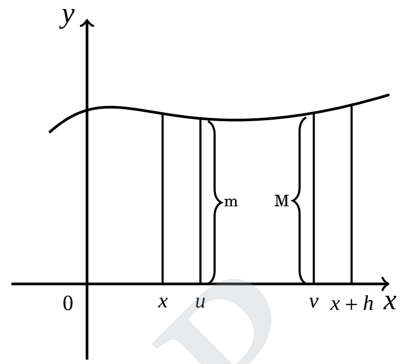
Statement. If f is continuous on $[a,b]$, then the function g defined by

$$g(x)=\int_a^x f(t)dt, a\le x\le b$$

is continuous on $[a,b]$ and is differentiable on (a,b) and $g'(x)=f(x)$.

Proof.Choose x and $x+b\in(a,b)$.

$$\text{Now, } g(x+h) - g(x) = \int_a^{x+h} f(t)dt - \int_a^{x} f(t)dt$$
$$= \int_a^{x} f(t)dt + \int_x^{x+h} f(t)dt - \int_a^{x} f(t)dt$$
$$= \int_x^{x+h} f(t)dt.$$



Hence, for $h \neq 0$, we have $\dfrac{g(x+h) - g(x)}{h} = \dfrac{1}{h}\int_x^{x+h} f(t)dt.$ (1)

Let us assume that $h > 0$. Since f is continuous on $[x, x+h]$, by the extreme value theorem, there exists numbers u and v in $[x, x+h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x+h]$. By the comparison property of the definite integral we have

$$mh \leq \int_x^{x+h} f(t)dt \leq Mh.$$
$$f(u)h \leq \int_x^{x+h} f(t)dt \leq f(v)h.$$

Since $h > 0$, dividing throughout by h we get

$$f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)dt \leq f(v). \qquad (2)$$

(2) can be proved in a similar way for $h < 0$.

Now, allowing $h \to 0$ and since u and v lie in $[x, x+h]$, we have $u \to x$ and $v \to x$. Since f is continuous at x,we have

$$\therefore \lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x)$$
$$\text{and } \lim_{h\to 0} f(v) = \lim_{v\to x} f(v) = f(x).$$

By (2) we have

$$\lim_{h\to 0} f(u) \leq \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt \leq \lim_{h\to 0} f(v)$$

$$f(x) \le \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \le f(x) \quad \text{[by (1)]}$$

$$\text{i.e.,} f(x) \le g'(x) \le f(x).$$

By Squeeze Theorem we get

$$g'(x) = f(x). \tag{3}$$

Since every differentiable function is continuous,

we obtain g is also continuous on (a, b).

If $x = a$ or $x = b$, from (3) we get the one sided limits at a and b.

This shows that g is continuous on $[a, b]$.

Example 3.2. Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$.

Solution. Since $f(t) = \sqrt{1+t^2}$ is continuous on $[0, x]$, by part 1 of the Fundamental theorem of calculus we get $g'(x) = \sqrt{1+x^2}$

Example 3.3. Find $\frac{d}{dx}\left[\int_1^{x^4} \sec t\,dt\right]$.

Solution. Let $u = x^4$

$$\therefore \frac{d}{dx}\left[\int_1^{x^4} \sec t\,dt\right] = \frac{d}{dx}\left[\int_1^{u} \sec t\,dt\right]$$

$$= \frac{d}{du}\left[\int_1^{u} \sec t\,dt\right] \cdot \frac{du}{dx} \quad \text{[By Chain rule]}$$

$$= \sec u \cdot 4x^3 \quad \text{[by FTC 1]}$$

$$= \sec x^4 \cdot 4x^3.$$

Second Fundamental theorem of calculus

Statement.If f is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f. [i.e, a function such that $F' = f$].

Proof. Let $g(x) = \int_a^x f(t)dt$.

By the First Fundamental theorem of calculus we have $g'(x) = f(x)$. i.e,

g is an antiderivative of f.

If F is any other antiderivative of f on $[a,b]$, we know that F and g differ by a constant.

$\therefore F(x) = g(x) + c$ for $a < x < b$.

Since both F and g are continuous on $[a,b]$ the above relation holds when $x = a$ and $x = b$.

i.e $\lim_{x \to a+} F(x) = F(a)$ & $\lim_{x \to b-} F(x) = F(b)$.

Also $\lim_{x \to a+} g(x) = g(a)$ & $\lim_{x \to b-} g(x) = g(b)$.

Hence we obtain $F(x) = g(x) + c$ for all x in $[a,b]$ (1)

When $x = a$, we have $g(a) = \int_a^a f(t)dt = 0$.

Using (1) for $x = b$ and $x = a$ we get

$$\begin{aligned} F(b) - F(a) &= [g(b) + c] - [g(a) + c] \\ &= g(b) - g(a) \\ &= g(b) = \int_a^b f(t)dt. \end{aligned}$$

Result. The First and Second Fundamental Theorem of Calculus are combined together is called the Fundamental Theorem of Calculus.

Statement of Fundamental Theorem of Calculus.

Suppose f is continuous on $[a,b]$

(i) If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.

(ii) $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f. (i.e., $F' = f$.)

Note (1). The Fundamental theorem of calculus says that differentiation and integration are inverse processors.

Note (2). By part(ii) of the Fundamental theorem of calculus we have

$\int_a^b f(x)dx = F(b) - F(a)$ where $F' = f$.

i.e $\int_a^b F'(x)dx = F(b) - F(a)$.

The above result is sometimes called as the Net Change Theorem.

Net Change Theorem

Statement. The integral of a rate of change is the net change.

$$\text{i.e} \int_a^b F'(x)dx = F(b) - F(a).$$

Worked Examples

Example 3.4. Evaluate $\int_0^3 (x^2 - 2x)dx$ by using Reimann sum by taking right end points as the sample points. Hence verify it by using fundamental theorem of calculus.

Solution. Let $f(x) = x^2 - 2x$.

Divide the interval $[0, 3]$ into n sub intevals with equal width $\Delta x = \frac{3-0}{n} = \frac{3}{n}$.

The intervals are $\left[0, \frac{3}{n}\right], \left[\frac{3}{n}, \frac{6}{n}\right], \left[\frac{6}{n}, \frac{9}{n}\right], \cdots \left[\frac{3n-3}{n}, \frac{3n}{n}\right]$.

The sample points x_i^* are the right end points of the sub intervals.

$$\therefore x_i^* = \frac{3}{n}, \frac{6}{n}, \frac{9}{n}, \cdots \frac{3n}{n}.$$

The corresponding $f(x_i^*) = x_i^{*2} - 2x_i^*$ are calculated and tabulated as follows.

x_i^*	$\frac{3}{n}$	$\frac{6}{n}$	$\frac{9}{n}$	$\cdots$	$\frac{3n}{n}$
$f(x_i^*)$	$\frac{9}{n^2} - \frac{6}{n}$	$\frac{36}{n^2} - \frac{12}{n}$	$\frac{81}{n^2} - \frac{18}{n}$	$\cdots$	$\frac{9n^2}{n^2} - \frac{6n}{n}$

By Reimann's sum we have,

$$\int_0^3 (x^2 - 2x)dx = \lim_{n=\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \lim_{n=\infty} \left[\left(\frac{9}{n^2} - \frac{6}{n}\right)\frac{3}{n} + \left(\frac{36}{n^2} - \frac{12}{n}\right)\frac{3}{n} + \left(\frac{81}{n^2} - \frac{18}{n}\right)\frac{3}{n} + \cdots + \left(\frac{9n^2}{n^2} - \frac{6n}{n}\right)\frac{3}{n}\right]$$

$$= \lim_{n=\infty} \left[\left(\frac{3^3}{n^3} \cdot 1 - \frac{3^2}{n^2} \cdot 2\right) + \left(\frac{3^3}{n^3} \cdot 2^2 - \frac{3^2}{n^2} \cdot 4\right) + \left(\frac{3^3}{n^3} \cdot 3^2 - \frac{3^2}{n^2} \cdot 6\right) + \cdots + \left(\frac{3^3}{n^3} \cdot n^2 - \frac{3^2}{n^2} \cdot 2n\right)\right]$$

$$= \lim_{n=\infty} \frac{3^3}{n^3}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right) - \lim_{n=\infty} \frac{3^2}{n^2}(2 + 4 + 6 + \cdots + 2n)$$

$$= \lim_{n=\infty} \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] - \lim_{n=\infty} \frac{9}{n^2} \cdot 2\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{27}{6} \lim_{n=\infty}\left[\frac{\cancel{n^3}(1+\frac{1}{n})(2+\frac{1}{n})}{\cancel{n^3}}\right] - 9 \lim_{n=\infty}\left[\frac{\cancel{n^2}(1+\frac{1}{n})}{\cancel{n^2}}\right] = \frac{27}{6} \times 2 - 9 = 0.$$

By the Fundamental Theorem of Calculus we have

$$\int_0^3 (x^2 - 2x)dx = \left(\frac{x^3}{3} - 2 \cdot \frac{x^2}{2}\right)_0^3 = \frac{27}{3} - 9 = 9 - 9 = 0.$$

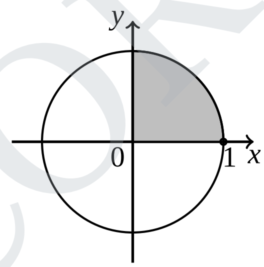
Example 3.5. Evaluate the integral $\int_0^1 \sqrt{1 - x^2}dx$ interms of areas.

Solution. $\int_0^1 \sqrt{1 - x^2}dx$ represents the area under the curve $y = \sqrt{1 - x^2}$ between the x–axis and the lines $x = 0$ and $x = 1$.

$$\text{Now}, y = \sqrt{1 - x^2}$$

$$y^2 = 1 - x^2$$

$$x^2 + y^2 = 1.$$



This is a circle with centre at the origin and radius = 1 unit. Since we need the area between $x = 0$ and $x = 1$, the required area is the area of the portion of the circle that lies in the first quadrant $= \frac{1}{4} \cdot \pi \times 1^2 = \frac{\pi}{4}$.

$$\therefore \int_0^1 \sqrt{1 - x^2}dx = \frac{\pi}{4}.$$

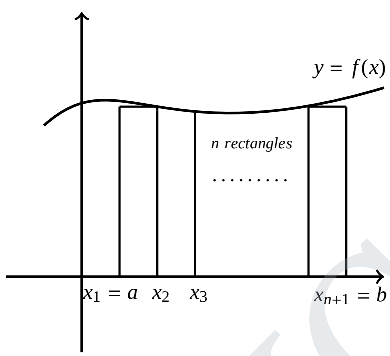
Example 3.6. Evaluate the integral $\int_0^3 (x - 1)dx$ interms of areas.

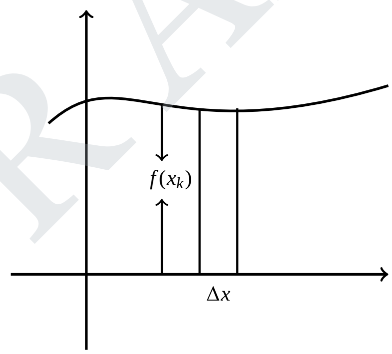
Solution. $\int_0^3 (x - 1)dx$ represents the area under the curve $y = (x - 1)$ between the x–axis and the lines $x = 0$ and $x = 3$.

3 Integral Calculus

3.1 The definite integral

Integration as the limit of a sum





Consider the graph of the positive function $y = f(x)$. Let us find the area under the curve $y = f(x)$,between the $x-$axis and the ordinates $x = a$ and $x = b$. Divide this area into n rectangles of equal width $\Delta x = \dfrac{b-a}{n}$. Let the $x-$ co ordinates at the left hand side of the rectangles be $x_1 = a, x_2, x_3, \ldots x_{n+1} = b$. Consider a typical rectangle, the k^{th} one with height $f(x_k)$. The area of this rectangle is $f(x_k)\Delta x$. The sum of all the areas of the n rectangles is

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

In summation convention form, this can be written as $\sum\limits_{k=1}^{n} f(x_k)\Delta x$. This gives an estimate of the area under the curve but it is not exact. To improve the estimate we must take a large number of very thin rectangles. This can be achieved by allowing $n \to \infty$ and making $\Delta x \to 0$.

$\therefore$ Area under the curve $= \lim\limits_{n=\infty} \sum\limits_{k=1}^{n} f(x_k)\Delta x$.

The lower and upper limits on the sum correspond to the first and the last rectangle where $x = a$ and $x = b$ respectively. Hence the above limit can be written as $\lim\limits_{\Delta x \to 0} \sum\limits_{x=a}^{b} f(x)\Delta x$. Since the number of rectangles increase without bound, we drop the subsript k from x_k and write $f(x)$ which is the value of f at a typical value of x. If this can actually be found, it is called the definite integral of $f(x)$ from $x = a$ and $x = b$ and it is written as $\int\limits_a^b f(x)dx$.

$$\therefore \int\limits_a^b f(x)dx = \lim_{n=\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\Delta x.$$

Note. If we approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip; (i.e., x_k is taken at the right end points of the k^{th} rectangle and $f(x_k)$ as the height) then also the above result will be achieved.

Definition of a definite integral. If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \dfrac{b-a}{n}$. We let $x_0(= a), x_1, x_2, \cdots x_n(= b)$ be the end points of these subintervals and if we choose $x_1^*, x_2^*, \cdots x_n^*$ as any sample ponits in these subintervals, so that x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is defined as $\int\limits_a^b f(x)dx = \lim\limits_{n=\infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$, provided that the limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a, b]$.

Result (1). $\int\limits_a^b f(x)dx = \lim\limits_{n=\infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$ can also be written as, for $\epsilon > 0$, there is an integer N such that $\left| \int\limits_a^b f(x)dx - \sum\limits_{i=1}^{n} f(x_i^*)\Delta x \right| < \epsilon$ for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

Result (2). The sum $\sum\limits_{i=1}^{n} f(x_i^*)\Delta x$ is called the Reimann sum.

Theorem 1. If f is continuos on $[a,b]$, (or) if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$.[i.e., $\int\limits_a^b f(x)dx$ exists].

Theorem 2. If f is integrable on $[,b]$, then $\int\limits_a^b f(x)dx = \lim\limits_{n=\infty} \sum\limits_{i=1}^{n} f(x_i)\Delta x$ where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Worked Examples

Example 3.1. Using the area property evaluate $\int\limits_0^1 x^2 dx$. Show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$.

Solution. Consider the function $y = f(x) = x^2$. Divide the area under the curve $y = x^2$ between $x = 0$ and $x = 1$ into 4 rectangles of equal width $\Delta x = \frac{1-0}{4} = \frac{1}{4}$.

The width of each rectangle is $\frac{1}{4}$.

The subintervals are $\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right]$.

The height of the rectangles is approximated to the value of the ordinates at $x = 0, \frac{1}{4}, \frac{1}{2} \& \frac{3}{4}$.

$$\begin{aligned}
\therefore \text{ Area of the required portion } &= f(0)\Delta x + f\left(\frac{1}{4}\right)\Delta x + f\left(\frac{1}{2}\right)\Delta x + f\left(\frac{3}{4}\right)\Delta x. \\
&= 0^2 . \frac{1}{4} + \left(\frac{1}{4}\right)^2 . \frac{1}{4} + \left(\frac{1}{2}\right)^2 . \frac{1}{4} + \left(\frac{3}{4}\right)^2 . \frac{1}{4} \\
&= \frac{1}{4}\left[0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right] \\
&= \frac{1}{4}\left[\frac{1+4+9}{16}\right] \\
&= \frac{1}{4} \times \frac{14}{16} = \frac{7}{32}.
\end{aligned}$$

If the height of the rectangles is approximated to the value of the ordinates at the right edge of each rectangle then,

$$\begin{aligned}
\text{Area} &= f\left(\frac{1}{4}\right)\Delta x + f\left(\frac{1}{2}\right)\Delta x + f\left(\frac{3}{4}\right)\Delta x + f(1)\Delta x.\\
&= \left(\frac{1}{4}\right)^2 .\frac{1}{4} + \left(\frac{1}{2}\right)^2 .\frac{1}{4} + \left(\frac{3}{4}\right)^2 .\frac{1}{4} + 1^2 \cdot \frac{1}{4}\\
&= \frac{1}{4}\left[\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right] = \frac{1}{4}\left[\frac{1+4+9+16}{16}\right] = \frac{1}{4} \times \frac{30}{16} = \frac{15}{32}.
\end{aligned}$$

Now, let us divide the area into n rectangles each of their with width $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. The subintervals are $\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \cdots, \left[\frac{n-1}{n}, \frac{n}{n} = 1\right]$.

The heights of the rectangle are the values of the function $f(x) = x^2$ at the points $\frac{1}{n}, \frac{2}{n}, \frac{3}{n} \cdots \frac{n}{n}$.

$$\begin{aligned}
\therefore\ \text{Area} &= \left(\frac{1}{n}\right)^2 .\frac{1}{n} + \left(\frac{2}{n}\right)^2 .\frac{1}{n} + \left(\frac{3}{n}\right)^2 .\frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^2 .\frac{1}{n}\\
&= \frac{1}{n}\left[\frac{1^2}{n^2} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \cdots + \frac{n^2}{n^2}\right]\\
&= \frac{1}{n^3}\left[1^2 + 2^2 + 3^2 + \cdots + n^2\right]\\
&= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{(n+1)(2n+1)}{6n^2}\\
\lim_{n=\infty}(\text{Area}) &= \lim_{n=\infty}\frac{(n+1)(2n+1)}{6n^2}\\
&= \lim_{n=\infty}\frac{n^2(1+\frac{1}{n})(2+\frac{1}{n})}{6n^2} = \frac{2}{6} = \frac{1}{3}.
\end{aligned}$$

Properties of definite integrals.

The following properties can be easily proved using Riemann sums.

(i) $\int_b^a f(x)dx = -\int_a^b f(x)dx.$

(ii) $\int_a^a f(x)dx = 0.$

(iii) $\int_a^b cdx = c(b-a)$, where c is any constant.

(iv) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$

(v) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is any constant.

(vi) $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$

(vii) If $a < c < b$ then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

Comparison properties of the integral.

(i) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0.$

(ii) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx.$

(iii) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$

Proof. Given $m \leq f(x) \leq M$

Integrating between a and b we get

$$\int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx$$

$$m\int_a^b dx \leq \int_a^b f(x)dx \leq M\int_a^b dx$$

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

The first fundamental theorem of calculus

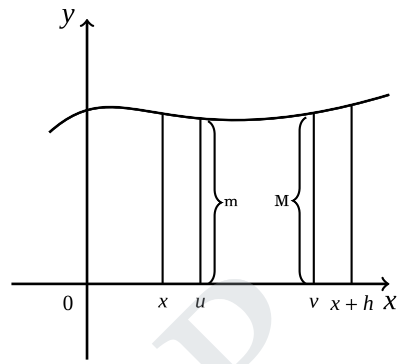
Statement. If f is continuous on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt, a \leq x \leq b$$

is continuous on $[a,b]$ and is differentiable on (a,b) and $g'(x) = f(x)$.

Proof.Choose x and $x+b \in (a,b)$.

$$\text{Now, } g(x+h)-g(x) = \int_a^{x+h} f(t)dt - \int_a^{x} f(t)dt$$
$$= \int_a^{x} f(t)dt + \int_x^{x+h} f(t)dt - \int_a^{x} f(t)dt$$
$$= \int_x^{x+h} f(t)dt.$$



Hence, for $h \neq 0$, we have $$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt. \quad (1)$$

Let us assume that $h > 0$. Since f is continuous on $[x, x+h]$, by the extreme value theorem, there exists numbers u and v in $[x, x+h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x+h]$. By the comparison property of the definite integral we have

$$mh \leq \int_x^{x+h} f(t)dt \leq Mh.$$

$$f(u)h \leq \int_x^{x+h} f(t)dt \leq f(v)h.$$

Since $h > 0$, dividing throughout by h we get

$$f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)dt \leq f(v). \quad (2)$$

(2) can be proved in a similar way for $h < 0$.

Now, allowing $h \to 0$ and since u and v lie in $[x, x+h]$, we have $u \to x$ and $v \to x$. Since f is continuous at x,we have

$$\therefore \lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x)$$

$$\text{and } \lim_{h\to 0} f(v) = \lim_{v\to x} f(v) = f(x).$$

By (2) we have

$$\lim_{h\to 0} f(u) \leq \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt \leq \lim_{h\to 0} f(v)$$

$$f(x) \le \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \le f(x) \quad \text{[by (1)]}$$

$$\text{i.e.,} f(x) \le g'(x) \le f(x).$$

By Squeeze Theorem we get

$$g'(x) = f(x). \tag{3}$$

Since every differentiable function is continuous,

we obtain g is also continuous on (a, b).

If $x = a$ or $x = b$, from (3) we get the one sided limits at a and b.

This shows that g is continuous on $[a, b]$.

Example 3.2. Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$.

Solution. Since $f(t) = \sqrt{1+t^2}$ is continuous on $[0, x]$, by part 1 of the Fundamental theorem of calculus we get $g'(x) = \sqrt{1+x^2}$

Example 3.3. Find $\frac{d}{dx}\left[\int_1^{x^4} \sec t dt\right]$.

Solution. Let $u = x^4$

$$\begin{aligned}
\therefore \frac{d}{dx}\left[\int_1^{x^4} \sec t dt\right] &= \frac{d}{dx}\left[\int_1^{u} \sec t dt\right] \\
&= \frac{d}{du}\left[\int_1^{u} \sec t dt\right] \cdot \frac{du}{dx} \quad \text{[By Chain rule]} \\
&= \sec u \cdot 4x^3 \quad \text{[by FTC 1]} \\
&= \sec x^4 \cdot 4x^3.
\end{aligned}$$

Second Fundamental theorem of calculus

Statement.If f is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f. [i.e, a function such that $F' = f$].

Proof. Let $g(x) = \int_a^x f(t)dt$.

By the First Fundamental theorem of calculus we have $g'(x) = f(x)$. i.e,

g is an antiderivative of f.

If F is any other antiderivative of f on $[a,b]$, we know that F and g differ by a constant.

$\therefore F(x) = g(x) + c$ for $a < x < b$.

Since both F and g are continuous on $[a,b]$ the above relation holds when $x = a$ and $x = b$.

i.e $\lim\limits_{x\to a+} F(x) = F(a)$ & $\lim\limits_{x\to b-} F(x) = F(b)$.

Also $\lim\limits_{x\to a+} g(x) = g(a)$ & $\lim\limits_{x\to b-} g(x) = g(b)$.

Hence we obtain $F(x) = g(x) + c$ for all x in $[a,b]$ (1)

When $x = a$, we have $g(a) = \int\limits_a^a f(t)dt = 0$.

Using (1) for $x = b$ and $x = a$ we get

$$\begin{aligned} F(b) - F(a) &= [g(b) + c] - [g(a) + c] \\ &= g(b) - g(a) \\ &= g(b) = \int\limits_a^b f(t)dt. \end{aligned}$$

Result. The First and Second Fundamental Theorem of Calculus are combined together is called the Fundamental Theorem of Calculus.

Statement of Fundamental Theorem of Calculus.

Suppose f is continuous on $[a,b]$

(i) If $g(x) = \int\limits_a^x f(t)dt$, then $g'(x) = f(x)$.

(ii) $\int\limits_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f. (i.e., $F' = f$.)

Note (1). The Fundamental theorem of calculus says that differentiation and integration are inverse processors.

Note (2). By part(ii) of the Fundamental theorem of calculus we have

$$\int\limits_a^b f(x)dx = F(b) - F(a) \text{ where } F' = f.$$

$$\text{i.e } \int\limits_a^b F'(x)dx = F(b) - F(a).$$

The above result is sometimes called as the Net Change Theorem.

Net Change Theorem

Statement. The integral of a rate of change is the net change.

$$\text{i.e } \int_a^b F'(x)dx = F(b) - F(a).$$

Worked Examples

Example 3.4. Evaluate $\int_0^3 (x^2 - 2x)dx$ by using Reimann sum by taking right end points as the sample points. Hence verify it by using fundamental theorem of calculus.

Solution. Let $f(x) = x^2 - 2x$.

Divide the interval $[0, 3]$ into n sub intevals with equal width $\Delta x = \frac{3-0}{n} = \frac{3}{n}$.

The intervals are $\left[0, \frac{3}{n}\right], \left[\frac{3}{n}, \frac{6}{n}\right], \left[\frac{6}{n}, \frac{9}{n}\right], \cdots \left[\frac{3n-3}{n}, \frac{3n}{n}\right]$.

The sample points x_i^* are the right end points of the sub intervals.

$$\therefore x_i^* = \frac{3}{n}, \frac{6}{n}, \frac{9}{n}, \cdots \frac{3n}{n}.$$

The corresponding $f(x_i^*) = x_i^{*2} - 2x_i^*$ are calculated and tabulated as follows.

x_i^*	$\frac{3}{n}$	$\frac{6}{n}$	$\frac{9}{n}$	$\cdots$	$\frac{3n}{n}$
$f(x_i^*)$	$\frac{9}{n^2} - \frac{6}{n}$	$\frac{36}{n^2} - \frac{12}{n}$	$\frac{81}{n^2} - \frac{18}{n}$	$\cdots$	$\frac{9n^2}{n^2} - \frac{6n}{n}$

By Reimann's sum we have,

$$\int_0^3 (x^2 - 2x)dx = \lim_{n=\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \lim_{n=\infty} \left[\left(\frac{9}{n^2} - \frac{6}{n}\right)\frac{3}{n} + \left(\frac{36}{n^2} - \frac{12}{n}\right)\frac{3}{n} + \left(\frac{81}{n^2} - \frac{18}{n}\right)\frac{3}{n} + \cdots + \left(\frac{9n^2}{n^2} - \frac{6n}{n}\right)\frac{3}{n}\right]$$

$$= \lim_{n=\infty} \left[\left(\frac{3^3}{n^3} \cdot 1 - \frac{3^2}{n^2} \cdot 2\right) + \left(\frac{3^3}{n^3} \cdot 2^2 - \frac{3^2}{n^2} \cdot 4\right) + \left(\frac{3^3}{n^3} \cdot 3^2 - \frac{3^2}{n^2} \cdot 6\right) + \cdots + \left(\frac{3^3}{n^3} \cdot n^2 - \frac{3^2}{n^2} \cdot 2n\right)\right]$$

$$= \lim_{n=\infty} \frac{3^3}{n^3}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right) - \lim_{n=\infty} \frac{3^2}{n^2}(2 + 4 + 6 + \cdots + 2n)$$
$$= \lim_{n=\infty} \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] - \lim_{n=\infty} \frac{9}{n^2} \cdot 2\left[\frac{n(n+1)}{2}\right]$$
$$= \frac{27}{6} \lim_{n=\infty}\left[\frac{\cancel{n^3}(1+\frac{1}{n})(2+\frac{1}{n})}{\cancel{n^3}}\right] - 9 \lim_{n=\infty}\left[\frac{\cancel{n^2}(1+\frac{1}{n})}{\cancel{n^2}}\right] = \frac{27}{6} \times 2 - 9 = 0.$$

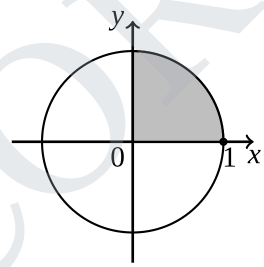
By the Fundamental Theorem of Calculus we have

$$\int_0^3 (x^2 - 2x)dx = \left(\frac{x^3}{3} - 2 \cdot \frac{x^2}{2}\right)_0^3 = \frac{27}{3} - 9 = 9 - 9 = 0.$$

Example 3.5. Evaluate the integral $\int_0^1 \sqrt{1-x^2}dx$ interms of areas.

Solution. $\int_0^1 \sqrt{1-x^2}dx$ represents the area under the curve $y = \sqrt{1-x^2}$ between the x–axis and the lines $x = 0$ and $x = 1$.

$$\text{Now}, y = \sqrt{1-x^2}$$
$$y^2 = 1 - x^2$$
$$x^2 + y^2 = 1.$$

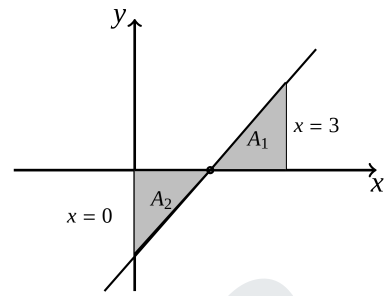


This is a circle with centre at the origin and radius = 1 unit. Since we need the area between $x = 0$ and $x = 1$, the required area is the area of the portion of the circle that lies in the first quadrant $= \frac{1}{4} \cdot \pi \times 1^2 = \frac{\pi}{4}$.

$$\therefore \int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}.$$

Example 3.6. Evaluate the integral $\int_0^3 (x-1)dx$ interms of areas.

Solution. $\int_0^3 (x-1)dx$ represents the area under the curve $y = (x-1)$ between the x–axis and the lines $x = 0$ and $x = 3$.



The required area $= A_1 - A_2$.

A_1 is a triangle with base = 2units

and height = 2units.

[Put $x = 3$ in $y = x - 1$.]

$\therefore$ Area of $A_1 = \frac{1}{2} \times 2 \times 2 = 2$.

A_2 is a triangle with base = 1 unit and height = 1unit.

$\therefore$ Area of $A_2 = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$.

$\int_0^3 (x-1)dx = 2 - \frac{1}{2} = \frac{3}{2}$.

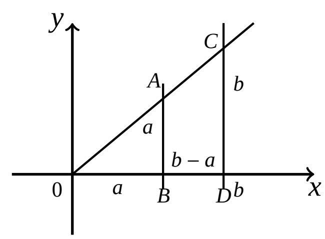
Example 3.7. Prove that $\int_a^b xdx = \frac{b^2 - a^2}{2}$ by interpretting interms of areas.

[A.U Nov 2016]

Solution. $\int_a^b xdx$ represents the area under the curve $y = x$ between the x–axis and the lines $x = a$ and $x = b$.

Required area = Area of ΔOCD– Area of ΔOAB

$$= \frac{1}{2} \times OD \times CD - \frac{1}{2} \times OB \times AB$$
$$= \frac{1}{2} b \cdot b - \frac{1}{2} a \cdot a$$
$$= \frac{b^2}{2} - \frac{a^2}{2} = \frac{b^2 - a^2}{2}$$



$$\therefore \int_a^b xdx = \frac{b^2 - a^2}{2}.$$

Example 3.8. Express $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\Delta x$ as an integral on the interval $[0, \pi]$.

Solution. According to Riemann sum,the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\Delta x = \int_0^{\pi} (x^3 + x \sin x)dx.$$

Example 3.9. Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be the right end points and $a = 0, b = 3, n = 6$. Also Evaluate $\int_0^3 (x^3 - 6x)dx$.

Solution. Divide the interval $[0, 3]$ into 6 subintervals with equal width $\frac{3-0}{6} = \frac{1}{2}$.

Hence, the intervals are $\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right], \left[2, \frac{5}{2}\right], \left[\frac{5}{2}, 3\right]$

The right end points are $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$.

The value of $f(x)$ at these points are as follows.

$x(k)$	x^3	$-6x$	$f(x_k)$	$f(x_k)\Delta x$
$\frac{1}{2}$	$\frac{1}{8}$	-3	-$\frac{23}{8}$	-$\frac{23}{16}$
1	1	-6	-5	-$\frac{5}{2}$
$\frac{3}{2}$	$\frac{27}{8}$	-9	-$\frac{45}{8}$	-$\frac{45}{16}$
2	8	-12	-4	-2
$\frac{5}{2}$	$\frac{125}{8}$	-15	$\frac{5}{8}$	$\frac{5}{16}$
3	27	-18	9	$\frac{9}{2}$

$$\begin{aligned}
\text{Reimann Sum} &= \sum_{k=1}^{n} f(x_k)\Delta x_k \\
&= -\frac{23}{16} - \frac{5}{2} - \frac{45}{16} - 2 + \frac{5}{16} + \frac{9}{2} \\
&= \frac{-23 - 40 - 45 - 32 + 5 + 72}{16} \\
&= \frac{-140 + 77}{16} = \frac{-63}{16}.
\end{aligned}$$

Let us evaluate $\int_0^3 (x^3 - 6x)dx$.

Divide the interval $[0, 3]$ into n subintervals with equal width $\Delta x = \frac{3-0}{n} = \frac{3}{n}$.

Hence the intervals are $\left[0, \frac{3}{n}\right], \left[\frac{3}{n}, \frac{6}{n}\right], \left[\frac{6}{n}, \frac{9}{n}\right], \cdots \left[\frac{3n-3}{n}, \frac{3n}{n}\right]$

The sample points x_i^* are the right end points of the sub intervals

$\therefore x_i^* = \frac{3}{n}, \frac{6}{n}, \frac{9}{n}, \cdots \frac{3n}{n}$.

The corresponding $f(x_i^*) = x_i^3 - 6x_i$ are calculated and tabulated as follows.

x_i^*	$\frac{3}{n}$	$\frac{6}{n}$	$\frac{9}{n}$	$\cdots$	$\frac{3n}{n}$
$f(x_i^*)$	$\frac{3^3}{n^3} - 6 \cdot \frac{3}{n}$	$\frac{6^3}{n^3} - 6 \cdot \frac{6}{n}$	$\frac{9^3}{n^3} - 6 \cdot \frac{9}{n}$	$\cdots$	$\frac{3^3 n^3}{n^3} - 6 \cdot \frac{3n}{n}$

By Reimann sum

$$\int_0^3 (x^3 - 6x)dx = \lim_{n=\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \lim_{n=\infty} \left[\left(\frac{3^3}{n^3} - 6 \cdot \frac{3}{n} \right) \frac{3}{n} + \left(\frac{6^3}{n^3} - 6 \cdot \frac{6}{n} \right) \frac{3}{n} + \cdots + \left(\frac{3^3 n^3}{n^3} - 6 \cdot \frac{3n}{n} \right) \frac{3}{n} \right]$$

$$= \lim_{n=\infty} \left[\frac{3^4}{n^4} \left(1^3 + 2^3 + 3^3 + \cdots + n^3 \right) - 6 \cdot \frac{3^2}{n^2} (1 + 2 + 3 + \cdots + n) \right]$$

$$= \lim_{n=\infty} \left[\frac{3^4}{n^4} \left(\frac{n(n+1)}{2} \right)^2 - 6 \cdot \frac{3^2}{n^2} \left(\frac{n(n+1)}{2} \right) \right]$$

$$= \lim_{n=\infty} \left[\frac{3^4}{n^4} \frac{n^2(n+1)^2}{4} - 6 \cdot \frac{3^2}{n^2} \frac{n(n+1)}{2} \right]$$

$$= \lim_{n=\infty} \left[\frac{3^4}{n^4} \cdot \frac{n^4 \left(1 + \frac{1}{n} \right)^2}{4} - 6 \cdot \frac{3^2}{n^2} \frac{n^2 \left(1 + \frac{1}{n} \right)}{2} \right]$$

$$= \lim_{n=\infty} \left[\frac{81}{4} \cdot \left(1 + \frac{1}{n} \right)^2 - 27 \cdot \left(1 + \frac{1}{n} \right) \right]$$

$$= \frac{81}{4} - 27 = \frac{81 - 108}{4} = -\frac{27}{4}.$$

Indefinite Integrals.

The Fundamental theorem of calculus gives the relation between antiderivatives and integrals. The notation $\int f(x)dx$ is traditionally used for an antiderivative of f and is called an indefinite integral. Thus $\int_a^b f(x)dx = F(x)$ means $F'(x) = f(x)$.

$$\frac{d}{dx}[F(x)] = f(x) \text{ then } \int f(x)dx = F(x)$$

In other words if $\frac{d}{dx}[F(x)] = f(x)$, then $\int f(x)dx = F(x)$.

Also we have $\frac{d}{dx}[F(x) + c] = f(x)$ then

$\int f(x)dx = F(x) + c$,where c is called the constant of integration.

With this notation, we can derive the basic results in integration, using the basic results in differentiation.

Basic results.

1. We have $\dfrac{d}{dx}(x^n) = nx^{n-1}$.
$$\text{(or)}\ \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n.$$
$$\therefore \int x^n dx = \frac{x^{n+1}}{n+1} + c.$$

2. $\dfrac{d}{dx}(\log x) = \dfrac{1}{x} \Rightarrow \int \dfrac{1}{x} dx = \log x + c.$

3. $\dfrac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + c.$

4. $\int b^x dx = \dfrac{b^x}{\log b} + c.$

5. $\int \sin x dx = -\cos x + c.$

6. $\int \cos x dx = \sin x + c.$

7. $\int \sec^2 x dx = \tan x + c.$

8. $\int cosec^2 x dx = -\cot x + c.$

9. $\int \sec x \tan x dx = \sec x + c.$

10. $\int cosecx \cot x dx = -cosecx + c.$

11. $\int \dfrac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c.$

12. $\int \dfrac{1}{1+x^2} dx = \tan^{-1} x + c.$

13. $\int \dfrac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c.$

14. $\int \sinh x dx = \cos hx + c.$

15. $\int \cosh x dx = \sin hx + c.$

16. $\int \frac{1}{\sqrt{x^2 - 1}} dx = \cos h^{-1} x + c$ or $\log(x + \sqrt{x^2 - 1}) + c.$

17. $\int \frac{1}{\sqrt{x^2 + 1}} dx = \sin h^{-1} x + c$ or $\log(x + \sqrt{x^2 + 1}) + c.$

Basic properties of indefinite integrals.

If $f(x)$ and $g(x)$ are functions of x and k is a constant, then the following properties are true.

(i) $\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx + c.$

(ii) $\int [f(x) - g(x)] dx = \int f(x) dx - \int g(x) dx + c.$

(iii) $\int k f(x) dx = k \int f(x) dx + c.$

Worked Examples

Example 3.10. Evaluate $\int (10x^4 - 2\sec^2 x) dx.$

Solution. $\int (10x^4 - 2\sec^2 x) dx = 10 \int x^4 dx - 2 \int \sec^2 x dx + c$

$$= 10 \cdot \frac{x^5}{5} - 2 \tan x + c = 2x^5 - 2\tan x + c.$$

Example 3.11. Evaluate $\int_1^2 \left(x^2 - 3\sqrt{x} + \frac{1}{x^2}\right) dx.$

Solution. $\int_1^2 \left(x^2 - 3\sqrt{x} + \frac{1}{x^2}\right) dx = \int_1^2 x^2 dx - 3 \int_1^2 x^{\frac{1}{2}} dx + \int_1^2 x^{-2} dx$

$$= \left(\frac{x^3}{3}\right)_1^2 - 3\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_1^2 + \left(\frac{x^{-1}}{-1}\right)_1^2$$

$$= \frac{1}{3}[2^3 - 1^3] - 3 \times \frac{2}{3}[2^{\frac{3}{2}} - 1] - \left(\frac{1}{x}\right)_1^2$$

$$= \frac{1}{3}[8 - 1] - 2[2\sqrt{2} - 1] - [\frac{1}{2} - 1]$$

$$= \frac{7}{3} - 4\sqrt{2} + 2 + \frac{1}{2}$$

$$= \frac{21 + 12 + 3}{6} - 4\sqrt{2} = \frac{36}{6} - 4\sqrt{2} = 6 - 4\sqrt{2}.$$

Example 3.12. Evaluate $\int_0^3 (x^3 - 6x)dx$.

Solution. $$\int_0^3 (x^3 - 6x)dx = \int_0^3 x^3 dx - 6\int_0^3 x dx = \left(\frac{x^4}{4}\right)_0^3 - 6\left(\frac{x^2}{2}\right)_0^3$$

$$= \frac{1}{4}[3^4 - 0] - 3[3^2 - 0]$$

$$= \frac{1}{4} \times 81 - 27 = \frac{81}{4} - 27 = \frac{81 - 108}{4} = -\frac{27}{4}.$$

Example 3.13. Evaluate $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$.

Solution. $$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx = 2\int_0^2 x^3 dx - 6\int_0^2 x dx + 3\int_0^2 \frac{1}{x^2+1} dx$$

$$= 2 \cdot \left(\frac{x^4}{4}\right)_0^2 - 6\left(\frac{x^2}{2}\right)_0^2 + 3\left(\tan^{-1} x\right)_0^2$$

$$= \frac{1}{2}[2^4 - 0] - 3[2^2 - 0] + 3\left(\tan^{-1} 2 - \tan^{-1} 0\right)$$

$$= 8 - 12 + 3\tan^{-1} 2 = 3\tan^{-1} 2 - 4.$$

Example 3.14. Evaluate $\int_0^{\pi/6} \cos^2\left(\frac{x}{2}\right) dx$.

Solution. $$\int_0^{\pi/6} \cos^2\left(\frac{x}{2}\right) dx = \frac{1}{2}\int_0^{\pi/6} (1 + \cos x) dx$$

$$= \frac{1}{2}\left[\int_0^{\pi/6} dx + \int_0^{\pi/6} \cos x dx\right]$$

$$= \frac{1}{2}\left[(x)_0^{\pi/6} dx + \left(\sin x\right)_0^{\pi/6}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - 0 + \sin\frac{\pi}{6} - \sin 0\right] = \frac{1}{2}\left[\frac{\pi}{6} + \frac{1}{2}\right] = \frac{\pi}{12} + \frac{1}{4}.$$

Example 3.15. Evaluate $\int_1^9 \frac{2x^2 + x^2\sqrt{x} - 1}{x^2} dx$.

Solution.
$$\int_1^9 \frac{2x^2 + x^2\sqrt{x} - 1}{x^2} dx = \int_1^9 \left(\frac{2x^2}{x^2} + \frac{x^2\sqrt{x}}{x^2} - \frac{1}{x^2}\right) dx$$
$$= \int_1^9 \left(2 + x^{\frac{1}{2}} - x^{-2}\right) dx$$
$$= 2\int_1^9 dx + \int_1^9 x^{\frac{1}{2}} dx - \int_1^9 x^{-2} dx$$
$$= 2(x)_1^9 + \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_1^9 - \left(\frac{x^{-1}}{-1}\right)_1^9$$
$$= 2[9-1] + \frac{2}{3}\left(9^{\frac{3}{2}} - 1\right) + \left(\frac{1}{9} - 1\right)$$
$$= 16 + \frac{2}{3} \times 26 - \frac{8}{9}$$
$$= 16 + \frac{52}{3} - \frac{8}{9} = \frac{144 + 156 - 8}{9} = \frac{292}{9}.$$

Example 3.16. Evaluate $\int \frac{1}{\sin^2 x \cos^2 x} dx$

Solution.
$$\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$
$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$
$$= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx$$
$$= \tan x - \cot x + c.$$

3.2 The Substitution Rule

Integrals of functions of the form $\int f(ax+b)dx$.

This type of integrals can be evaluated by making a substitution $ax + b = t$.

For example consider $\int (2x+3)^2 dx$.

$$\text{Let } t = 2x + 3$$
$$dt = 2dx$$
$$\therefore dx = \frac{dt}{2}$$

$$\text{Now, } \int (2x+3)^2 dx = \int t^2 \cdot \frac{dt}{2} = \frac{1}{2}\int t^2 dt = \frac{1}{2}\cdot\frac{t^3}{3} + c = \frac{1}{6}\cdot(2x+3)^3 + c.$$

Standard results.

1. Evaluate $\int (ax+b)^n dx$.

Solution. Let $ax + b = t$.

$$adx = dt \Rightarrow dx = \frac{dt}{a}.$$

$$\int (ax+b)^n dx = \int t^n \frac{dt}{a} = \frac{1}{a}\cdot\frac{t^{n+1}}{n+1} + c = \frac{(ax+b)^{n+1}}{a(n+1)} + c.$$

In a similar way, the following results can be easily derived.

2. $\int \frac{1}{ax+b} dx = \frac{\log(ax+b)}{a} + c.$

3. $\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + c.$

4. $\int \sin(ax+b) dx = -\frac{\cos(ax+b)}{a} + c.$

5. $\int \cos(ax+b) dx = \frac{\sin(ax+b)}{a} + c.$

6. $\int \sec^2(ax+b) dx = \frac{\tan(ax+b)}{a} + c.$

7. $\int \operatorname{cosec}^2(ax+b) dx = -\frac{\operatorname{cosec}(ax+b)}{a} + c.$

8. $\int \sec(ax+b)\tan(ax+b) dx = \frac{\sec(ax+b)}{a} + c.$

9. $\int \operatorname{cosec}(ax+b)\cot(ax+b) dx = -\frac{\cos(ax+b)}{a} + c.$

Worked Examples

Example 3.17. Evaluate $\int \sin^2 3x dx$.

Solution.
$$\int \sin^2 3x dx = \int \frac{1}{2}(1 - \cos 6x)dx$$
$$= \frac{1}{2}\left[\int dx - \int \cos 6x dx\right]$$
$$= \frac{1}{2}\left[x - \frac{\sin 6x}{6}\right] + c$$
$$= \frac{x}{2} - \frac{\sin 6x}{12} + c.$$

Example 3.18. Evaluate $\int \cos mx \cos nx dx$.

Solution. case(i) when $m \neq n$.
$$\int \cos mx \cos nx dx = \int \frac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right) dx$$
$$= \frac{1}{2}\left(\int \cos(m+n)x dx + \int \cos(m-n)x\right) dx$$
$$= \frac{1}{2}\left(\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right) + c.$$

case(ii) If $m = n$
$$\int \cos mx \cos nx dx = \int \cos mx \cos mx dx$$
$$= \int \cos^2 mx dx$$
$$= \int \frac{1}{2}(\cos 2mx + 1)dx$$
$$= \frac{1}{2}\left[\int \cos 2mx dx + \int dx\right]$$
$$= \frac{1}{2}\left[\frac{\sin 2mx}{2m} + x\right] + c$$
$$= \frac{\sin 2mx}{4m} + \frac{x}{2} + c.$$

Example 3.19. Evaluate $\int \cos^3 2x dx$.

Solution. $\displaystyle\int \cos^3 2x dx = \int \left(\frac{1}{4}\cos 6x + \frac{3}{4}\cos 2x\right) dx$

$$= \frac{1}{4}\int \cos 6x dx + \frac{3}{4}\int \cos 2x dx$$

$$= \frac{1}{4}\cdot\frac{\sin 6x}{6} + \frac{3}{4}\cdot\frac{\sin 2x}{2} + c$$

$$= \frac{\sin 6x}{24} + 3\frac{\sin 2x}{8} + c.$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

$$\cos^3 2x = \frac{1}{4}\cos 6x + \frac{3}{4}\cos 2x.$$

Example 3.20. Evaluate $\displaystyle\int \sin^4 x dx$.

Solution. $\displaystyle\int \sin^4 x dx = \int \left(\sin^2 x\right)^2 dx$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx.$$

$$= \frac{1}{4}\int (1 + \cos^2 2x - 2\cos 2x) dx$$

$$= \frac{1}{4}\int \left(1 + \frac{1+\cos 4x}{2} - 2\cos 2x\right) dx$$

$$= \frac{1}{4}\int (1 + \frac{1}{2} + \frac{1}{2}\cos 4x - 2\cos 2x) dx$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\cos^2 2x = \frac{1+\cos 4x}{2}.$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right) dx$$

$$= \frac{1}{4}\left[\frac{3}{2}\int dx + \frac{1}{2}\int \cos 4x dx - 2\int \cos 2x dx\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} + \frac{1}{2}\frac{\sin 4x}{4} - 2\frac{\sin 2x}{2}\right] + c = \frac{3}{8}x + \frac{\sin 4x}{32} - \frac{\sin 2x}{4} + c.$$

Example 3.21. Evaluate $\displaystyle\int \sin^2 4x dx$.

Solution. $\displaystyle\int \sin^2 4x dx = \int \left(\frac{1-\cos 8x}{2}\right) dx.$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 8x\right) dx$$

$$= \frac{1}{2}\left[\int dx - \int \cos 8x dx\right]$$

$$= \frac{1}{2}\left[x - \frac{\sin 8x}{8}\right] + c = \frac{x}{2} - \frac{\sin 8x}{16} + c.$$

$$2\sin^2 x = 1 - \cos 2x$$

$$2\sin^2 4x = 1 - \cos 8x$$

$$\sin^2 4x = \frac{1-\cos 8x}{2}$$

Example 3.22. Evaluate $\int \frac{x^2}{(ax+b)^3}dx$.

Solution. Let $ax + b = t \Rightarrow ax = t - b \Rightarrow x = \frac{t-b}{a}$

$$adx = dt \quad \Rightarrow dx = \frac{dt}{a}.$$

$$\begin{aligned}
\int \frac{x^2}{(ax+b)^3}dx &= \int \frac{\left(t-\frac{b}{a}\right)^2}{t^3}\frac{dt}{a} \\
&= \frac{1}{a}\int \frac{t^2 + \frac{b^2}{a^2} - \frac{2b}{a}t}{t^3}dt \\
&= \frac{1}{a}\int \left(\frac{1}{t} + \frac{b^2}{a^2}\cdot\frac{1}{t^3} - \frac{2b}{a}\frac{1}{t^2}\right)dt \\
&= \frac{1}{a}\left[\int \frac{1}{t}dt + \frac{b^2}{a^2}\int t^{-3}dt - \frac{2b}{a}\int t^{-2}dt\right] \\
&= \frac{1}{a}\left[\log t + \frac{b^2}{a^2}\cdot\frac{t^{-2}}{(-2)} - \frac{2b}{a}\times\frac{t^{-1}}{(-1)}\right] + c \\
&= \frac{1}{a}\left[\log t - \frac{b^2}{2a^2}\cdot\frac{1}{t^2} + \frac{2b}{a}\frac{1}{t}\right] + c \\
&= \frac{1}{a}\left[\log(ax+b) - \frac{b^2}{a^2}\cdot\frac{1}{(ax+b)^2} + \frac{2b}{a}\frac{1}{(ax+b)}\right] + c \\
&= \frac{1}{a}\log(ax+b) - \frac{b^2}{2a^3}\cdot\frac{1}{(ax+b)^2} + \frac{2b}{a^2}\frac{1}{(ax+b)} + c
\end{aligned}$$

3.3 Evaluation of integrals of the form $\int f(g(x))g'(x)dx$.

Method. Let $u = g(x) \quad du = g'(x)dx$.

Now, $\int f(g(x))g'(x)dx = \int f(u)du$.

This can be evaluated by earlier methods.

Worked Examples

Example 3.23. Evaluate $\int 2x\sqrt{1+x^2}dx$.

Solution. Let $1 + x^2 = u$.

$$2xdx = du.$$

$$\therefore \int 2x\sqrt{1+x^2}dx = \int \sqrt{u}du = \int u^{\frac{1}{2}}du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c.$$

Example 3.24. Evaluate $\int x^2\sqrt{1-4x^3}dx$.

Solution. Let $u = 1-4x^3$.

$$du = -4 \times 3x^2 dx.$$

$$x^2 dx = -\frac{du}{12}$$

$$\therefore \int x^2\sqrt{1-4x^3}dx = \int \sqrt{u}\left(\frac{-du}{12}\right)$$

$$= -\frac{1}{12}\int u^{\frac{1}{2}}du$$

$$= -\frac{1}{12}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= -\frac{1}{12}\frac{2}{3}(1-4x^3)^{\frac{3}{2}} + c = -\frac{1}{18}(1-4x^2)^{\frac{3}{2}} + c.$$

Example 3.25. Evaluate $\int \frac{1}{(1+e^x)(1+e^{-x})}dx$.

Solution. $\int \frac{1}{(1+e^x)(1+e^{-x})}dx = \int \frac{1}{(1+e^x)(1+\frac{1}{e^x})}dx$

$$= \int \frac{e^x}{(1+e^x)(1+e^x)}dx$$

$$= \int \frac{e^x}{(1+e^x)^2}dx \qquad 1+e^x = t$$

$$= \int \frac{dt}{t^2} \qquad e^x dx = dt$$

$$= \int t^{-2}dt = \frac{t^{-1}}{-1} + c = -\frac{1}{t} + c = -\frac{1}{1+e^x} + c.$$

Example 3.26. Evaluate $\int x^3\cos(x^4+2)dx$.

Solution. Let $x^4 + 2 = t$.

$$4x^3 dx = dt$$

$$x^3 dx = \frac{dt}{4}$$

$$\therefore \int x^3 \cos(x^4 + 2) dx = \int \cos t \frac{dt}{4}$$

$$= \frac{1}{4} \int \cos t dt = \frac{1}{4} \sin t + c = \frac{\sin(x^4 + 2)}{4} + c.$$

Example 3.27. Evaluate $\int \frac{x}{\sqrt{1 - 4x^2}} dx$.

Solution. Let $1 - 4x^2 = t$.

$$-8x dx = dt$$

$$x dx = -\frac{dt}{8}$$

$$\therefore \int \frac{x}{\sqrt{1 - 4x^2}} dx = \int \frac{1}{\sqrt{t}} \left(-\frac{dt}{8}\right)$$

$$= -\frac{1}{8} \int t^{-\frac{1}{2}} dt$$

$$= -\frac{1}{8} \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = -\frac{1}{8} \times 2\sqrt{t} + c = -\frac{1}{4}\sqrt{1 - 4x^2} + c.$$

Example 3.28. Evaluate $\int \frac{x^3}{\sqrt{4 + x^2}} dx$. [A.U. Dec. 2015]

Solution. Let $4 + x^2 = t$.

$$2xdx = dt$$

$$xdx = \frac{dt}{2}$$

$$\therefore \int \frac{x^3}{\sqrt{4+x^2}}dx = \int \frac{x^2}{\sqrt{4+x^2}}xdx$$

$$= \int \frac{t-4}{\sqrt{t}}\frac{dt}{2}$$

$$= \frac{1}{2}\int \left(t^{\frac{1}{2}} - 4t^{-\frac{1}{2}}\right)dt$$

$$= \frac{1}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right] + c$$

$$= \frac{1}{2}\left[\frac{2}{3}\left(4+x^2\right)^{\frac{3}{2}} - 8\sqrt{4+x^2}\right] + c.$$

Example 3.29. Evaluate $\int \sqrt{1+x^2}\ \ x^5 dx$.

Solution. Let $1 + x^2 = t$.

$$2xdx = dt$$

$$xdx = \frac{dt}{2}$$

$$\therefore \int \sqrt{1+x^2}\ \ x^5dx = \int \sqrt{1+x^2}\ \ x^4 \cdot xdx$$

$$= \int \sqrt{1+x^2}(x^2)^2 \cdot xdx$$

$$= \int \sqrt{t}(t-1)^2 \cdot \frac{dt}{2}$$

$$= \frac{1}{2}\int \sqrt{t}(t^2+1-2t)dt$$

$$= \frac{1}{2}\int \left(t^{\frac{5}{2}} + t^{\frac{1}{2}} - 2t^{\frac{3}{2}}\right)dt$$

$$= \frac{1}{2}\left(\int t^{\frac{5}{2}}dt + \int t^{\frac{1}{2}}dt - 2\int t^{\frac{3}{2}}dt\right)$$

$$= \frac{1}{2}\left[\frac{t^{\frac{7}{2}}}{\frac{7}{2}} + \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 2\frac{t^{\frac{5}{2}}}{\frac{5}{2}}\right] + c.$$

$$= \frac{1}{2}\left[\frac{2}{7}(1+x^2)^{\frac{7}{2}} + \frac{2}{3}(1+x^2)^{\frac{3}{2}} - \frac{4}{5}(1+x^2)^{\frac{5}{2}}\right] + c$$
$$= \frac{1}{7}(1+x^2)^{\frac{7}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + c.$$

Example 3.30. Evaluate $\int \frac{e^x}{e^{\frac{x}{2}} - 1} dx.$

Solution. Let $e^{\frac{x}{2}} = t.$

$$e^{\frac{x}{2}} \cdot \frac{dx}{2} = dt$$
$$\therefore \int \frac{e^x}{e^{\frac{x}{2}} - 1} dx = \int \frac{e^{\frac{x}{2}} e^{\frac{x}{2}}}{e^{\frac{x}{2}} - 1} dx$$
$$= \int \frac{e^{\frac{x}{2}}}{e^{\frac{x}{2}} - 1} \cdot e^{\frac{x}{2}} dx$$
$$= \int \frac{1+t}{t} 2dt$$
$$= 2\int \left(\frac{1}{t} + 1\right) dt$$
$$= 2\left[\int \frac{1}{t} dt + \int dt\right] + c$$
$$= 2[\log t + t] + c$$
$$= 2[\log\left(e^{\frac{x}{2}} - 1\right) + e^{\frac{x}{2}} - 1\big)] + c.$$

Example 3.31. Evaluate $\int \frac{1}{1+\tan x} dx.$

Solution. $\int \frac{1}{1+\tan x} dx = \int \frac{1}{1 + \frac{\sin x}{\cos x}} dx$

$$= \int \frac{\cos x}{\sin x + \cos x} dx = \frac{1}{2} \int \frac{2\cos x}{\sin x + \cos x} dx$$
$$= \frac{1}{2} \int \frac{\sin x + \cos x + \cos x - \sin x}{\sin x + \cos x} dx$$
$$= \frac{1}{2} \int \left(1 + \frac{\cos x - \sin x}{\sin x + \cos x}\right) dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x}dx$$

$$= \frac{1}{2}x + \frac{1}{2}\int \frac{dt}{t} \qquad t = \sin x + \cos x$$

$$= \frac{1}{2}x + \frac{1}{2}\log(t) + c \qquad dt = (\cos x - \sin x)dx$$

$$= \frac{x}{2} + \frac{1}{2}\log(\sin x + \cos x)) + c.$$

Example 3.32. Evaluate $\int \frac{\tan x}{\sec x + \cos x}dx$.

Solution. $\int \frac{\tan x}{\sec x + \cos x}dx = \int \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \cos x}dx$

$$= \int \frac{\sin x}{1 + \cos^2 x}dx$$

$$= \int \frac{-dt}{1 + t^2} \qquad t = \cos x$$

$$= -\tan^{-1}(t) + c \qquad dt = -\sin x dx$$

$$= -\tan^{-1}(\cos x) + c \qquad -dt = \sin x dx$$

Example 3.33. Evaluate $\int_0^4 \sqrt{2x + 1}dx$.

Solution. Let $2x + 1 = t$.

$$2dx = dt \Rightarrow dx = \frac{dt}{2}.$$

When $x = 0, t = 1$.

When $x = 4, t = 9$.

$$\therefore \int_0^4 \sqrt{2x+1}dx = \int_1^9 \sqrt{t}\frac{dt}{2}$$

$$= \frac{1}{2}\int_1^9 t^{\frac{1}{2}}dt$$

$$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)_1^9 = \frac{1}{2} \times \frac{2}{3}[9^{\frac{3}{2}} - 1] = \frac{1}{3}[27 - 1] = \frac{1}{3} \times 26 = \frac{26}{3}.$$

Example 3.34. Evaluate $\displaystyle\int_1^2 \frac{1}{(3-5x)^2}dx$.

Solution. Let $3-5x=t$.

$$-5dx = dt \Rightarrow dx = -\frac{dt}{5}.$$

When $x=0, t=-2$.

When $x=2, t=-7$.

$$\therefore \int_1^2 \frac{1}{(3-5x)^2} = \int_{-2}^{-7} \frac{1}{t^2}\left(-\frac{dt}{5}\right)$$

$$= -\frac{1}{5}\int_{-2}^{-7} t^{-2}dt$$

$$= -\frac{1}{5}\left(\frac{t^{-1}}{-1}\right)_{-2}^{-7}$$

$$= \frac{1}{5}\left(\frac{1}{t}\right)_{-2}^{-7} = \frac{1}{5}\left(\frac{-1}{7}+\frac{1}{2}\right) = \frac{1}{5}\left[\frac{-2+7}{14}\right] = \frac{1}{5}\cdot\frac{5}{14} = \frac{1}{14}.$$

Example 3.35. Evaluate $\displaystyle\int_1^e \frac{\log x}{x}dx$.

Solution. Let $t=\log x$.

$$dt = \frac{1}{x}dx.$$

When $x=1, t=\log 1 = 0$.

When $x=e, t=\log e = 1$.

$$\therefore \int_1^e \frac{\log x}{x}dx = \int_0^1 tdt = \left(\frac{t^2}{2}\right)_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

Integrals of symmetric functions

Theorem. Suppose f is continuous on $[-a,a]$

(i) If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

(ii) If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.

Proof. We have $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$

$$= -\int_{0}^{-a} f(x)dx + \int_{0}^{a} f(x)dx.$$

In the first integral let us make the substitution

$$t = -x.$$

$$dt = -dx.$$

When $x = 0, \quad t = 0.$

When $x = -a, \quad t = a.$

$$\therefore \int_{-a}^{a} f(x)dx = -\int_{0}^{a} f(-t)(-dt) + \int_{0}^{a} f(x)dx.$$

$$= \int_{0}^{a} f(-t)dt + \int_{0}^{a} f(x)dx.$$

$$= \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx.$$

(i) If f is even, then $f(-x) = f(x)$.

$$\therefore \int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx. = 2\int_{0}^{a} f(x)dx.$$

(ii) If f is odd then $f(-x) = -f(x)$

$$\therefore \int_{-a}^{a} f(x)dx = \int_{0}^{a} -f(x)dx + \int_{0}^{a} f(x)dx = 0.$$

Example 3.36. Evaluate $\int_{-2}^{2} (x^6 + 1)dx$.

Solution. Let $f(x) = x^6 + 1$.

$$f(-x) = (-x)^6 + 1 = x^6 + 1.$$

$\therefore$ f(x) is even.

By the properties of definite integrals

$$\therefore \int_{-2}^{2} f(x)dx = 2\int_{0}^{2} f(x)dx$$

$$= 2\int_{0}^{2} (x^6+1)dx$$

$$= 2\left[\int_{0}^{2} x^6 dx + \int_{0}^{2} dx\right]$$

$$= 2\left[\left(\frac{x^7}{7}\right)_0^2 + (x)_0^2\right] = 2\left[\frac{2^7}{7} - 0 + 2 - 0\right]$$

$$= 2\left[\frac{128}{7} + 2\right] = 2\left[\frac{128+14}{7}\right] = \frac{2\times 142}{7} = \frac{284}{7}.$$

Example 3.37. Evaluate $\displaystyle\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}dx.$

Solution. Let $f(x) = \dfrac{\tan x}{1+x^2+x^4}.$

$$f(-x) = \frac{\tan(-x)}{1+(-x)^2+(-x)^4}$$

$$= \frac{-\tan x}{1+x^2+x^4} = -f(x).$$

$\therefore f(x)$ is odd.

By the properties of definite integrals

$$\int_{-1}^{1} f(x)dx = 0$$

$$\text{i.e,} \int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}dx = 0.$$

3.4 Integration of rational functions by Partial Fractions.

Techniques of resolving a given fraction into partial fractions.

A rational function is generally of the form $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are algebraic expressions.

If the degree of $p(x)$ is less than the degree of $q(x)$ then the fraction $\frac{p(x)}{q(x)}$ is called a proper fraction.

If the degree of $p(x) \geq$ degree of $q(x)$, then the given fraction $\frac{p(x)}{q(x)}$ is improper.

While resolving a fraction $\frac{p(x)}{q(x)}$ into partial fractions, the following technique can be adopted.

Case (i) Let $\frac{p(x)}{q(x)}$ be proper and if $q(x)$ is expressed as linear factors.

For example, if the given fraction is $\frac{1+x}{(x-1)(x+2)(x-3)}$, then we can express the fraction in the following way

$$\frac{1+x}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}.$$

Case (ii) Let $\frac{p(x)}{q(x)}$ be proper and $q(x)$ contains repeated linear factors.

Consider the following example $\frac{x^2+2x-3}{(x-1)^3(2x+3)}$.This must be written as

$$\frac{x^2+2x-3}{(x-1)^3(2x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{2x+3}.$$

Case (iii) Let $\frac{p(x)}{q(x)}$ be proper and $q(x)$ contains nonfactorisable second degree factors.

Consider the following example $\frac{x^2+3x+2}{(x-1)^2(x^2+4)(x^2+9)}$. This can be resolved into partial fractions as follows.

$$\frac{x^2+3x+2}{(x-1)^2(x^2+4)(x^2+9)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{x^2+9}.$$

Methodology for improper fractions.

Case (i) If degree of $p(x)$ = degree of $q(x)$.

Example 1. Consider $\frac{x^2+x+1}{(x-1)(x-2)}$.

This must be resolved as

$$\frac{x^2+x+1}{(x-1)(x-2)} = A + \frac{B}{x-1} + \frac{C}{x-2}.$$

$\left[\dfrac{p(x)}{q(x)}\right.$ = A constant + the usual methods adopted for proper fractions.]

Example 2. $\dfrac{x^3+2x+4}{(x-1)(x^2+4)} = A + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+9}$.

Case (ii) If the degree of $p(x)$ is one more than that of $q(x)$, then

$\dfrac{p(x)}{q(x)}$ = A First degree expression + usual method adopted in proper functions.

Example 1. $\dfrac{x^4+x+4}{(x-1)(x^2+2)} = Ax+B+\dfrac{C}{x-1}+\dfrac{Dx+E}{x^2+2}$.

Example 2. $\dfrac{x^3+2x+3}{(x+2)^2} = Ax+B+\dfrac{C}{x+2}+\dfrac{D}{(x+2)^2}$.

Case (iii) If the degree of $p(x)$ is 2 more than that of $q(x)$, then

$\dfrac{p(x)}{q(x)}$ = A second degree expression + usual method adopted in proper functions.

Example 1. $\dfrac{x^7+2x+4}{(x+2)(x-1)^2(x^2+1)} = Ax^2+Bx+C+\dfrac{D}{x+2}+\dfrac{E}{x-1}+\dfrac{F}{(x-1)^2}+\dfrac{Gx+H}{x^2+1}$.

This procedure can be extended for any improper fraction depending on the nature of the fraction $\dfrac{p(x)}{q(x)}$.

Standard results.

1. Evaluate $\int \dfrac{1}{x^2+a^2}dx$.

   Solution. Let $x = a\tan\theta$.

$$dx = a\sec^2\theta d\theta.$$

$$\begin{aligned}
\therefore \int \frac{1}{x^2+a^2}dx &= \int \frac{1}{a^2\tan^2\theta + a^2}\cdot a\sec^2\theta d\theta \\
&= \int \frac{1}{a^2(\tan^2\theta+1)}\cdot a\sec^2\theta d\theta \\
&= \frac{1}{a}\int \frac{1}{\sec^2\theta}\cdot \sec^2 d\theta \\
&= \frac{1}{a}\int d\theta \\
&= \frac{1}{a}\theta + c = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c.
\end{aligned}$$

2. Evaluate $\int \dfrac{1}{x^2-a^2}dx$.

   Solution. Let $\dfrac{1}{x^2-a^2} = \dfrac{1}{(x-a)(x+a)} = \dfrac{A}{x-a}+\dfrac{B}{x+a}$

$$= \frac{A(x+a)+B(x-a)}{(x-a)(x+a)}$$

$$\therefore 1 = A(x+a)+B(x-a).$$

Put $x = a$.

$$1 = A \cdot 2a \Rightarrow A = \frac{1}{2a}.$$

Put $x = -a$.

$$1 = B \cdot (-2a) \Rightarrow B = -\frac{1}{2a}.$$

$$\therefore \frac{1}{x^2-a^2} = \frac{\frac{1}{2a}}{x-a} + \frac{\frac{-1}{2a}}{x+a} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

$$\therefore \int \frac{1}{x^2-a^2}dx = \int \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$$

$$= \frac{1}{2a}\left(\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx\right)$$

$$= \frac{1}{2a}\left[\log(x-a) - \log(x+a)\right] + c = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c.$$

3. Evaluate $\int \frac{1}{a^2-x^2}dx$.

Solution. Let $\frac{1}{a^2-x^2} = \frac{1}{(a-x)(a+x)} = \frac{A}{a-x} + \frac{B}{a+x} = \frac{A(a+x)+B(a-x)}{(a-x)(a+x)}$

$$\therefore 1 = A(a+x)+B(a-x).$$

Put $x = a$.

$$1 = A \cdot 2a \Rightarrow A = \frac{1}{2a}.$$

Put $x = -a$.

$$1 = B \cdot (2a) \Rightarrow B = \frac{1}{2a}.$$

$$\therefore \frac{1}{a^2-x^2} = \frac{\frac{1}{2a}}{a-x} + \frac{\frac{1}{2a}}{a+x} = \frac{1}{2a}\left[\frac{1}{a-x} + \frac{1}{a+x}\right]$$

$$\therefore \int \frac{1}{a^2-x^2}dx = \int \frac{1}{2a}\left(\frac{1}{a-x} + \frac{1}{a+x}\right)dx$$

$$= \frac{1}{2a}\left(\int \frac{1}{a-x}dx + \int \frac{1}{a+x}dx\right)$$

$$= \frac{1}{2a}\left[\frac{\log(a-x)}{-1} + \log(a+x)\right] + c$$

$$= \frac{1}{2a}\left[\log(a+x) - \log(a-x)\right] + c$$
$$= \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c.$$

Type 1. Evaluation of integrals of the form $\int \frac{1}{ax^2+bx+c}dx$

Method. Express the denominator to any one of the forms x^2-a^2, x^2+a^2 or a^2-x^2 and then apply the correct formula.

Worked Examples

Example 3.38. Evaluate $\int \frac{1}{x^2+8x-7}dx.$

Solution.
$$\int \frac{1}{x^2+8x-7}dx = \int \frac{1}{(x+4)^2-7-16}dx$$
$$= \int \frac{1}{(x+4)^2-23}dx$$
$$= \int \frac{1}{(x+4)^2-(\sqrt{23})^2}dx$$
$$= \frac{1}{2\sqrt{23}}\log\left(\frac{x+4-\sqrt{23}}{x+4+\sqrt{23}}\right) + c.$$

Example 3.39. Evaluate $\int \frac{1}{1+x-x^2}dx.$

Solution.
$$\int \frac{1}{1+x-x^2}dx = \int \frac{1}{1-(x^2-x)}dx$$
$$= \int \frac{1}{1-\left((x-\frac{1}{2})^2-\frac{1}{4}\right)}dx$$
$$= \int \frac{1}{1-(x-\frac{1}{2})^2+\frac{1}{4}}dx$$
$$= \int \frac{1}{\frac{5}{4}-(x-\frac{1}{2})^2}dx$$
$$= \int \frac{1}{(\frac{\sqrt{5}}{2})^2-(x-\frac{1}{2})^2}dx$$
$$= \frac{1}{2\frac{\sqrt{5}}{2}}\log\left(\frac{\frac{\sqrt{5}}{2}+x-\frac{1}{2}}{\frac{\sqrt{5}}{2}-x+\frac{1}{2}}\right) + c$$
$$= \frac{1}{\sqrt{5}}\log\left(\frac{\sqrt{5}+2x-1}{\sqrt{5}-2x+1}\right) + c.$$

Example 3.40. Evaluate $\int \frac{1}{2x^2 - x + 5} dx$.

Solution. $\int \frac{1}{2x^2 - x + 5} dx = \int \frac{1}{2\left(x^2 - \frac{x}{2} + \frac{5}{2}\right)} dx$

$$= \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{4}\right)^2 + \frac{5}{2} - \frac{1}{16}} dx$$

$$= \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{4}\right)^2 + \frac{40 - 1}{16}} dx$$

$$= \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{4}\right)^2 + \frac{39}{16}} dx$$

$$= \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{4}\right)^2 + \left(\frac{\sqrt{39}}{4}\right)^2} dx$$

$$= \frac{1}{2} \frac{1}{\frac{\sqrt{39}}{4}} \tan^{-1}\left(\frac{x - \frac{1}{4}}{\frac{\sqrt{39}}{4}}\right) + c = \frac{2}{\sqrt{39}} \tan^{-1}\left(\frac{4x - 1}{\sqrt{39}}\right) + c$$

Type 2. Evaluation of integrals using partial fractions.

Example 3.41. Evaluate $\int \frac{x + 5}{x^2 + x - 2} dx$.

Solution. Let $\frac{x + 5}{x^2 + x - 2} = \frac{x + 5}{(x + 2)(x - 1)} = \frac{A}{x + 2} + \frac{B}{x - 1}$.

$$= \frac{A(x - 1) + B(x + 2)}{(x + 2)(x - 1)}$$

$$\therefore x + 5 = A(x - 1) + B(x + 2).$$

Put $x = 1$.

$6 = 3B \Rightarrow B = 2.$

Put $x = -2$.

$3 = -3A \Rightarrow A = -1.$

$$\therefore \frac{x+5}{x^2+x-2} = -\frac{1}{x+2} + \frac{2}{x-1}$$

$$\therefore \int \frac{x+5}{x^2+x-2}dx = \int \left(-\frac{1}{x+2} + \frac{2}{x-1}\right)dx$$

$$= -\int \frac{1}{x+2}dx + 2\int \frac{1}{x-1}dx = -\log(x+2) + 2\log(x-1) + c.$$

Example 3.42. Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx.$

Solution. $\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x(2x^2+3x-2)} = \frac{x^2+2x-1}{x(x+2)(2x-1)}.$

$$\text{Let} \frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}$$

$$= \frac{A(x+2)(2x-1) + Bx(2x-1) + Cx(x+2)}{x(x+2)(2x-1)}$$

$$\therefore x^2+2x-1 = A(x+2)(2x-1) + Bx(2x-1) + Cx(x+2).$$

Put $x = 0$.

$-2A = -1 \Rightarrow A = \frac{1}{2}.$

Put $x = -2$.

$B(-2)(-5) = 4-4-1 \Rightarrow 10B = -1 \Rightarrow B = \frac{-1}{10}.$

Put $x = \frac{1}{2}$.

$C \cdot \frac{1}{2} \cdot \frac{5}{2} = \frac{1}{4} + 1 - 1 \Rightarrow \frac{5C}{4} = \frac{1}{4} \Rightarrow C = \frac{1}{5}.$

$$\therefore \frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{\frac{1}{2}}{x} - \frac{\frac{1}{10}}{x+2} + \frac{\frac{1}{5}}{2x-1}.$$

$$\therefore \int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \frac{1}{2}\int \frac{1}{x}dx - \frac{1}{10}\int \frac{1}{x+2}dx + \frac{1}{5}\int \frac{1}{2x-1}dx$$

$$= \frac{1}{2}\log x - \frac{1}{10}\log(x+2) + \frac{1}{5} \cdot \frac{\log(2x-1)}{2} + c$$

$$= \frac{1}{2}\log x - \frac{1}{10}\log(x+2) + \frac{1}{10} \cdot \log(2x-1) + c.$$

Example 3.43. Evalutate $\int \frac{\sec^2 x}{\tan^2 x + 3\tan x + 2}\, dx.$ [A.U. Dec. 2015]

Solution. Let $\tan x = t$.

$\sec^2 x dx = dt.$

$$\begin{aligned}
\therefore \int \frac{\sec^2 x}{\tan^2 x + 3\tan x + 2}\, dx &= \int \frac{1}{t^2 + 3t + 2} dt \\
&= \int \frac{1}{(t+1)(t+2)} dt \\
&= \int \frac{(t+2)-(t+1)}{(t+1)(t+2)} dt \\
&= \int \left(\frac{1}{(t+1)} - \frac{1}{t+2} \right) dt \\
&= \int \frac{1}{(t+1)} dt - \int \frac{1}{t+2} dt + c \\
&= \log(t+1) - \log(t+2) + c \\
&= \log\left(\frac{t+1}{t+2} \right) + c \\
&= \log\left(\frac{\tan x + 1}{\tan x + 2} \right) + c.
\end{aligned}$$

Example 3.44. Evaluate $\int \frac{3x+1}{(x-1)^2(x+3)} dx.$

Solution. Let $\frac{3x+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$

$$= \frac{A(x-1)(x+3) + B(x+3) + C(x-1)^2}{(x-1)^2(x+3)}$$

$$\therefore 3x+1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2.$$

Put $x = 1.$

$4B = 4 \Rightarrow B = 1.$

Put $x = -3.$

$16C = -8 \Rightarrow C = \frac{-1}{2}.$

Equating the coefficients of x^2 on both sides we get,

$A + C = 0 \Rightarrow A = -C = \frac{1}{2}.$

$$\therefore \frac{3x+1}{(x-1)^2(x+3)} = \frac{\frac{1}{2}}{x-1} + \frac{1}{(x-1)^2} - \frac{\frac{1}{2}}{x+3}$$

$$\begin{aligned}
\therefore \int \frac{3x+1}{(x-1)^2(x+3)}dx &= \frac{1}{2}\int \frac{1}{x-1}dx + \int \frac{1}{(x-1)^2}dx - \frac{1}{2}\int \frac{1}{x+3}dx \\
&= \frac{1}{2}\log(x-1) + \int (x-1)^{-2}dx - \frac{1}{2}\cdot\log(x+3) + c \\
&= \frac{1}{2}(\log(x-1) - \log(x+3)) + \frac{(x-1)^{-1}}{-1} + c \\
&= \frac{1}{2}\log\left(\frac{x-1}{x+3}\right) - \frac{1}{x-1} + c.
\end{aligned}$$

Example 3.45. Evaluate $\int \frac{2x^2-x+4}{x^3+4x}dx$.

Solution. Let $\frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+(Bx+C)x}{x(x^2+4)}$.

$\therefore 2x^2 - x + 4 = A(x^2+4) + (Bx+C)x.$

Put $x = 0$.

$4A = 4 \Rightarrow A = 1.$

Equating the coefficients of x^2 on both sides we get,

$A + B = 2 \Rightarrow 1 + B = 2 \Rightarrow B = 1.$

Equating the coefficients of x on both sides we get,

$C = -1.$

$$\therefore \frac{2x^2-x+4}{x^3+4x} = \frac{1}{x} + \frac{x-1}{x^2+4} = \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}.$$

$$\begin{aligned}
\therefore \int \frac{2x^2-x+4}{x^3+4x} &= \int \frac{1}{x}dx + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx \\
&= \log x + \int \frac{1}{t}\frac{dt}{2} - \int \frac{1}{x^2+2^2}dx \\
&= \log x + \frac{1}{2}\log t - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c \\
&= \log x + \frac{1}{2}\log(x^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c.
\end{aligned}$$

In the second integral

let $x^2 + 4 = t$

$2xdx = dt$

$xdx = \frac{dt}{2}.$

Example 3.46. Evaluate $\int \frac{2}{(1-x)(1+x^2)}dx$.

Solution. Let $\dfrac{2}{(1-x)(1+x^2)} = \dfrac{A}{1-x} + \dfrac{Bx+C}{1+x^2}$

$$= \frac{A(1+x^2)+(Bx+C)(1-x)}{(1-x)(1+x^2)}.$$

$$\therefore 2 = A(1+x^2)+(Bx+C)(1-x).$$

Put $x = 1$.

$2A = 2 \Rightarrow A = 1.$

Equating the coefficients of x^2 on both sides we get,

$A - B = 0 \Rightarrow A = B = 1.$

Equating the coefficients of x on both sides we get,

$B - C = 0$

$B = C \Rightarrow C = 1.$

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2} = \frac{1}{1-x} + \frac{x}{1+x^2} + \frac{1}{1+x^2}$$

$$\therefore \int \frac{2}{(1-x)(1+x^2)}dx = \int \frac{1}{1-x}dx + \int \frac{x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$$

$$= \frac{\log(1-x)}{-1} + \int \frac{1}{t}\frac{dt}{2} + \tan^{-1}x + c$$

$$= -\log(1-x) + \frac{1}{2}\log t + \tan^{-1}x + c$$

$$= -\log(1-x) + \frac{1}{2}\log(1+x^2) + \tan^{-1}x + c.$$

In the second integral

let $1 + x^2 = t$

$2xdx = dt$

$xdx = \dfrac{dt}{2}.$

Example 3.47. Evaluate using partial fractions $\displaystyle\int \frac{10}{(x-1)(x^2+9)}dx.$ [A.U. Dec. 2015]

Solution. Let $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9} = \dfrac{A(x^2+9)+(Bx+C)(x-1)}{(x-1)(x^2+9)}.$

$\therefore 10 = A(x^2+9)+(Bx+C)(x-1).$

Put $x = 1$.

$10A = 10 \Rightarrow A = 1.$

Equating the coefficients of x^2 on both sides we get,

$A + B = 0 \Rightarrow B = -A = -1.$

Equating the constants on both sides we get,

$$9A - C = 10 \quad \Rightarrow C = 9A - 10 = 9 - 10 = -1.$$

$$\begin{aligned}
\therefore \frac{10}{(x-1)(x^2+9)} &= \frac{1}{x-1} + \frac{-x-1}{x^2+9} \\
&= \frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}. \\
\therefore \int \frac{10}{(x-1)(x^2+9)}dx &= \int \frac{1}{x-1}dx - \int \frac{x}{x^2+9}dx - \int \frac{1}{x^2+9}dx \\
&= \log(x-1) - \frac{1}{2}\int \frac{2x}{x^2+9}dx - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c \\
&= \log(x-1) - \frac{1}{2}\log(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c \\
&= \log(x-1) - \log(\sqrt{x^2+9}) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c \\
&= \log\left(\frac{(x-1)}{\sqrt{x^2+9}}\right) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c.
\end{aligned}$$

Example 3.48. Evaluate $\int \frac{4x^2-3x+2}{4x^2-4x+3}dx.$

Solution. The integrand $\frac{4x^2-3x+2}{4x^2-4x+3}$ is an improper fraction with the degrees of the Nr and Dr same.

$\therefore$By the method of partial fractions we have

$$\frac{4x^2-3x+2}{4x^2-4x+3} = A + \frac{Bx+C}{4x^2-4x+3} = \frac{A(4x^2-4x+3)+Bx+C}{4x^2-4x+3}$$

[$\because 4x^3 - 4x + 3$ is a nonfactorizable second degree factor]

$\therefore 4x^2 - 3x + 2 = A(4x^2-4x+3) + Bx + C.$

Equating the coefficients of x^2.

$4A = 4 \Rightarrow A = 1.$

Equating the coefficients of x

$-4A + B = -3 \Rightarrow -4 + B = -3 \Rightarrow B = 4 - 3 = 1.$

Equating the constants

$3A + C = 2 \Rightarrow 3 + C = 2 \Rightarrow C = 2 - 3 = -1.$

$$\therefore \frac{4x^2-3x+2}{4x^2-4x+3} = 1 + \frac{x-1}{4x^2-4x+3}$$
$$= 1 + \frac{x-1}{4(x^2-x+3/4)}$$
$$= 1 + \frac{x-1}{4\left((x-\frac{1}{2})^2+\frac{3}{4}-\frac{1}{4}\right)} = 1 + \frac{x-1}{4\left((x-\frac{1}{2})^2+\frac{1}{2}\right)}$$

$$\therefore \int \frac{4x^2-3x+2}{4x^2-4x+3}dx = \int \left[1 + \frac{x-1}{4\left((x-\frac{1}{2})^2+\frac{1}{2}\right)}\right]dx$$
$$= \int dx + \frac{1}{4}\int \frac{x-1}{(x-\frac{1}{2})^2+\frac{1}{2}}dx$$
$$= x + \frac{1}{4}\int \frac{x-\frac{1}{2}-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{1}{2}}dx$$
$$= x + \frac{1}{4}\int \frac{x-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{1}{2}}dx - \frac{1}{8}\int \frac{1}{(x-\frac{1}{2})^2+\frac{1}{2}}dx.$$
$$= x + \frac{1}{4}\int \frac{1}{t+\frac{1}{2}}\frac{dt}{2} - \frac{1}{8}\int \frac{1}{(x-\frac{1}{2})^2+(\frac{1}{\sqrt{2}})^2}dx$$

In the second integral

Put $\left(x-\dfrac{1}{2}\right)^2 = t$

$$2\left(x-\frac{1}{2}\right)dx = dt$$
$$\left(x-\frac{1}{2}\right)dx = \frac{dt}{2}.$$

$$= x + \frac{1}{8}\log(t+\frac{1}{2}) - \frac{1}{8}\frac{1}{\frac{1}{\sqrt{2}}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{\sqrt{2}}}\right) + c$$
$$= x + \frac{1}{8}\log\left((x-\frac{1}{2})^2+\frac{1}{2}\right) - \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{2x-1}{2}\times\sqrt{2}\right) + c$$
$$= x + \frac{1}{8}\log\left(x^2+\frac{1}{4}-x+\frac{1}{2}\right) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + c$$
$$= x + \frac{1}{8}\log\left(\frac{4x^2-4x+3}{4}\right) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + c.$$

Example 3.49. Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx.$ [A.U Nov 2016].

Solution. The integrand is an improper fraction in which the degree of the Nr is 1 more than that of the Dr. Hence, by the method of partial fractions we have

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = \frac{x^4-2x^2+4x+1}{x^2(x-1)-(x-1)} = \frac{x^4-2x^2+4x+1}{(x-1)(x^2-1)}$$
$$= \frac{x^4-2x^2+4x+1}{(x-1)(x-1)(x+1)} = \frac{x^4-2x^2+4x+1}{(x-1)^2(x+1)}.$$

Let $\frac{x^4-2x^2+4x+1}{(x-1)^2(x+1)} = Ax+B+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{x+1}$

$$= \frac{(Ax+B)(x-1)^2(x+1)+C(x-1)(x+1)+D(x+1)+E(x-1)^2}{(x-1)^2(x+1)}$$

$$\therefore x^4-2x^2+4x+1 = (Ax+B)(x-1)^2(x+1)+C(x-1)(x+1)+D(x+1)+E(x-1)^2.$$

Put $x=1: 2D = 1-2+4+1 = 4 \quad \Rightarrow D = 2.$

Put $x=-1: 4E = 1-2+4+1 = 4 \quad \Rightarrow 4E = -4 \Rightarrow E = -1.$

Equating the coefficients of x^4 on both sides we get,

$A = 1.$

Put $x=0: B-C+D+E = 1$

$B-C+2-1 = 1.$

$B-C = 0.$ (1)

Put $x=2: (2A+B)\cdot 3+3C+3D+E = 16-8+8+1$

$6A+3B+3C+3D+E = 17.$

$6+3B+3C+6-1 = 17$

$3B+3C = 17-11 = 6$

$B+C = 2.$ (2)

$(1)+(2) \Rightarrow 2B = 2 \quad \Rightarrow B = 1.$

$(1) \Rightarrow B = C = 1.$

$$\therefore \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}$$

$$\therefore \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx = \int x dx + \int dx + \int \frac{1}{x-1} dx + 2\int \frac{1}{(x-1)^2} dx - \int \frac{1}{x+1} dx$$

$$= \frac{x^2}{2} + x + \log(x-1) + 2\int (x-1)^{-2} dx - \log(x+1) + c$$

$$= \frac{x^2}{2} + x + \log(x-1) + 2\frac{(x-1)^{-1}}{-1} - \log(x+1) + c$$

$$= \frac{x^2}{2} + x + \log\left(\frac{x-1}{x+1}\right) - \frac{2}{x+1} + c.$$

Example 3.50. Evaluate $\int \frac{x^3}{(x-1)(x-2)} dx$.

Solution. The integrand is an improper fraction in which the degree of the Nr is 1 more than that of the Dr.

$\therefore$ By the method of partial fractions we have

$$\frac{x^3}{(x-1)(x-2)} = Ax + B + \frac{C}{x-1} + \frac{D}{x-2}$$

$$= \frac{(Ax+B)(x-1)(x-2) + C(x-2) + D(x-1)}{(x-1)(x-2)}$$

$$\therefore x^3 = (Ax+B)(x-1)(x-2) + C(x-2) + D(x-1).$$

Put $x = 1 : -C = 1 \Rightarrow C = -1.$

Put $x = 2 : D = 8.$

Equating the coefficients of x^3 on both sides we get,

$A = 1.$

Put $x = 0 : 2B - 2C - D = 0$

$2B + 2 - 8 = 0.$

$2B - 6 = 0$

$2B = 6 \quad \Rightarrow B = 3.$

$$\therefore \frac{x^3}{(x-1)(x-2)} = x + 3 - \frac{1}{x-1} + \frac{8}{x-2}$$

$$\therefore \int \frac{x^3}{(x-1)(x-2)} dx = \int x dx + 3\int dx - \int \frac{1}{x-1} dx + 8\int \frac{1}{x-2} dx$$

$$= \frac{x^2}{2} + 3x - \log(x-1) + 8\log(x-2) + c.$$

Example 3.51. Evaluate $\int \frac{x^3+x}{x-1}dx$.

Solution. The integrand is an improper fraction in which the degree of the Nr is 2 more than that of the Dr.

$\therefore$ By the method of partial fractions we have

$$\frac{x^3+x}{x-1} = Ax^2+Bx+C+\frac{D}{x-1} = \frac{(Ax^2+Bx+C)(x-1)+D}{x-1}.$$

$\therefore x^3+x = (Ax^2+Bx+C)(x-1)+D.$

Put $x=1: D=2.$

Equating the coefficients of x^3 on both sides we get,

$A=1.$

Equating the coefficients of x^2 on both sides we get,

$-A+B=0.$

$B=A=1.$

Put $x=0: -C+D=0$

$C=D=2.$

$$\therefore \frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}.$$

$$\therefore \int \frac{x^3+x}{x-1}dx = \int x^2dx + \int xdx + 2\int dx + 2\int \frac{1}{x-1}dx$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+2x+2\log(x-1)+c.$$

3.5 Integration of irrational functions

Standard Results

1. Evaluate $\int \frac{1}{\sqrt{a^2-x^2}}dx$.

   Solution. Let $x = a\sin\theta \quad \Rightarrow \quad dx = a\cos\theta d\theta.$

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{1}{\sqrt{a^2 - a^2 \sin^2 \theta}} \cdot a \cos \theta d\theta$$
$$= \int \frac{1}{\sqrt{a^2(1 - \sin^2 \theta)}} \cdot a \cos \theta d\theta$$
$$= \int \frac{1}{\sqrt{a^2 \cos^2 \theta}} \cdot a \cos \theta d\theta$$
$$= \int \frac{1}{a \cos \theta} \cdot a \cos \theta d\theta = \int d\theta + c = \theta + c$$
$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c.$$

2. Evaluate $\int \frac{1}{\sqrt{a^2 + x^2}} dx$.

Solution. Let $x = a \sin h\theta$

$$dx = a \cos h\theta d\theta.$$

$$\therefore \int \frac{1}{\sqrt{a^2 + x^2}} dx = \int \frac{1}{\sqrt{a^2 + a^2 \sin h^2\theta}} \cdot a \cos h\theta \ d\theta$$
$$= \int \frac{1}{\sqrt{a^2(1 + \sin h^2\theta)}} \cdot a \cos h\theta \ d\theta$$
$$= \int \frac{1}{\sqrt{a^2 \cos h^2\theta}} \cdot a \cos h\theta \ d\theta$$
$$= \int \frac{1}{a \cos h\theta} \cdot a \cos h\theta \ d\theta = \int d\theta + c = \theta + c$$
$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin h^{-1}\left(\frac{x}{a}\right) + c = \log(x + \sqrt{x^2 + a^2}) + c.$$

3. Evaluate $\int \frac{1}{\sqrt{x^2 - a^2}} dx$.

Solution. Let $x = a\cos h\theta$

$$dx = a\sin h\theta d\theta.$$

$$\therefore \int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{1}{\sqrt{a^2\cos h^2\theta - a^2}} \cdot a\sin h\theta\ d\theta$$

$$= \int \frac{1}{\sqrt{a^2(\cos h^2\theta - 1)}} \cdot a\sin h\theta\ d\theta$$

$$= \int \frac{1}{\sqrt{a^2\sin h^2\theta}} \cdot a\sin h\theta\ d\theta$$

$$= \int \frac{1}{a\sin h\theta} \cdot a\sin h\theta\ d\theta$$

$$= \int d\theta + c = \theta + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \cos h^{-1}\left(\frac{x}{a}\right) + c = \log(x + \sqrt{x^2-a^2}) + c.$$

4. Evaluate $\int \sqrt{a^2-x^2}dx$.

Solution. Let $x = a\sin\theta$

$$dx = a\cos\theta d\theta.$$

$$\therefore \int \sqrt{a^2-x^2}dx = \int \sqrt{a^2 - a^2\sin^2\theta} \cdot a\cos\theta\ d\theta$$

$$= \int \sqrt{a^2(1-\sin^2\theta)} \cdot a\cos\theta\ d\theta$$

$$= \int \sqrt{a^2\cos^2\theta} \cdot a\cos\theta\ d\theta$$

$$= a^2 \int \cos^2\theta\ d\theta$$

$$= a^2 \int \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{a^2}{2} \int (1+\cos\theta) d\theta$$

$$= \frac{a^2}{2}\left[\int d\theta + \int \cos 2\theta d\theta\right]$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\cdot 2\sin\theta\cos\theta\right] + c$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\cdot\frac{x}{a}\cdot\ \sqrt{\cos^2\theta} + c$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{ax}{2}\sqrt{1-\sin^2\theta} + c$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{ax}{2}\sqrt{1-\frac{x^2}{a^2}} + c$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{ax}{2}\frac{\sqrt{a^2-x^2}}{a} + c$$

$$\int\sqrt{a^2-x^2}dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + c.$$

5. Evaluate $\int\sqrt{a^2+x^2}dx$.

Solution. Let $x = a\sin h\theta$

$dx = a\cos h\theta d\theta.$

$$\therefore \int\sqrt{a^2+x^2}dx = \int\sqrt{a^2+a^2\sin h^2\theta}\cdot a\cos h\theta\ d\theta$$

$$= \int\sqrt{a^2(1+\sin h^2\theta)}\cdot a\cos h\theta\ d\theta$$

$$= \int\sqrt{a^2\cos h^2\theta}\cdot a\cos h\theta\ d\theta$$

$$= a^2\int\cos h^2\theta\ d\theta$$

$$= a^2\int\frac{1+\cos h2\theta}{2}d\theta$$

$$= \frac{a^2}{2}\int(1+\cos h2\theta)d\theta$$

$$= \frac{a^2}{2}\left[\int d\theta + \int\cos h2\theta d\theta\right]$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin h2\theta}{2}\right] + c$$

$$= \frac{a^2}{2}\left[\sin h^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\cdot 2\sin h\theta\cos h\theta\right] + c$$

$$= \frac{a^2}{2}\sin h^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\cdot\frac{x}{a}\cdot\ \sqrt{\cos h^2\theta} + c$$

$$= \frac{a^2}{2} \sin h^{-1}\left(\frac{x}{a}\right) + \frac{ax}{2}\sqrt{1 + \sin h^2\theta} + c$$
$$= \frac{a^2}{2} \sin h^{-1}\left(\frac{x}{a}\right) + \frac{ax}{2}\sqrt{1 + \frac{x^2}{a^2}} + c$$
$$= \frac{a^2}{2} \sin h^{-1}\left(\frac{x}{a}\right) + \frac{ax}{2}\frac{\sqrt{a^2 + x^2}}{a} + c$$
$$= \frac{a^2}{2} \sin h^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 + x^2} + c.$$
$$\int \sqrt{a^2 + x^2}dx = \frac{a^2}{2}\log(x + \sqrt{x^2 + a^2}) + \frac{x}{2}\sqrt{a^2 + x^2} + c.$$

6. Evaluate $\int \sqrt{x^2 - a^2}dx$.

Solution. Let $x = a\cos h\theta$

$dx = a\sin h\theta d\theta.$

$$\therefore \int \sqrt{x^2 - a^2}dx = \int \sqrt{a^2\cos h^2\theta - a^2} \cdot a\sin h\theta\, d\theta$$
$$= \int \sqrt{a^2(\sin h^2\theta - 1)} \cdot a\sin h\theta\, d\theta$$
$$= \int \sqrt{a^2\sin h^2\theta} \cdot a\sin h\theta\, d\theta$$
$$= a^2 \int \sin h^2\theta\, d\theta$$
$$= a^2 \int \frac{\cos h2\theta - 1}{2} d\theta$$
$$= \frac{a^2}{2}\int (\cos h2\theta - 1)d\theta$$
$$= \frac{a^2}{2}\left[\int \cos h2\theta d\theta - \int d\theta\right]$$
$$= \frac{a^2}{2}\left[\frac{\sin h2\theta}{2} - \theta\right] + c$$
$$= \frac{a^2}{2}\left[\sin h\theta\cos h\theta - \cos h^{-1}\left(\frac{x}{a}\right)\right] + c$$
$$= \frac{a^2}{2}\left[\sqrt{\sin h^2\theta} \cdot \frac{x}{a} - \cos h^{-1}\left(\frac{x}{a}\right)\right] + c$$
$$= \frac{ax}{2}\sqrt{\cosh^2\theta - 1} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)c$$

$$= \frac{ax}{2}\sqrt{\frac{x^2}{a^2} - 1} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)c$$
$$= \frac{ax}{2}\frac{\sqrt{x^2 - a^2}}{a} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)c$$
$$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)c.$$
$$\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log(x + \sqrt{x^2 - a^2}) + c.$$

3.5.1 Type I. Evaluation of integrals of the form $\int \frac{1}{\sqrt{ax^2 + bx + c}}dx$

Evaluation procedure. By the method of completing the square technique, express $\sqrt{ax^2 + bx + c}$ in any one of the forms $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ or $\sqrt{a^2 + x^2}$ and apply the correct formula.

Worked Examples

Example 3.52. Evaluate $\int \frac{1}{\sqrt{x^2 + 16x + 100}}dx.$

Solution.
$$\int \frac{1}{\sqrt{x^2 + 16x + 100}}dx = \int \frac{1}{\sqrt{(x+8)^2 + 100 - 64}}dx$$
$$= \int \frac{1}{\sqrt{(x+8)^2 + 36}}dx$$
$$= \int \frac{1}{\sqrt{(x+8)^2 + 6^2}}dx$$
$$= \log\left((x+8) + \sqrt{(x+8)^2 + 6^2}\right) + c$$
$$= \log\left((x+8) + \sqrt{x^2 + 16x + 100}\right) + c.$$

Example 3.53. Evaluate $\int \frac{1}{\sqrt{9 + 8x - x^2}}dx.$

Solution.
$$\int \frac{1}{\sqrt{9 + 8x - x^2}}dx = \int \frac{1}{\sqrt{9 - (x^2 - 8x)}}dx$$
$$= \int \frac{1}{\sqrt{9 - \{(x-4)^2 - 16\}}}dx$$

$$= \int \frac{1}{\sqrt{9-(x-4)^2+16}}dx$$
$$= \int \frac{1}{\sqrt{25-(x-4)^2}}dx$$
$$= \int \frac{1}{\sqrt{5^2-(x-4)^2}}dx$$
$$= \sin^{-1}\left(\frac{x-4}{5}\right)+c.$$

Example 3.54. Evaluate $\int \frac{1}{\sqrt{9x^2+24x}}dx$.

Solution. $$\int \frac{1}{\sqrt{9x^2+24x}}dx = \int \frac{1}{\sqrt{9\left(x^2+\frac{24x}{9}\right)}}dx$$
$$= \frac{1}{3}\int \frac{1}{\sqrt{x^2+\frac{8x}{3}}}dx$$
$$= \frac{1}{3}\int \frac{1}{\sqrt{\left(x+\frac{4}{3}\right)^2-\frac{16}{9}}}dx$$
$$= \frac{1}{3}\int \frac{1}{\sqrt{\left(x+\frac{4}{3}\right)^2-\left(\frac{4}{3}\right)^2}}dx$$
$$= \frac{1}{3}\log\left(\left(x+\frac{4}{3}\right)+\sqrt{\left(x+\frac{4}{3}\right)^2-\frac{16}{9}}\right)+c$$
$$= \frac{1}{3}\log\left(\left(\frac{3x+4}{3}\right)+\sqrt{x^2+\frac{8}{3}x}\right)+c.$$

3.5.2 Type II. Evaluation of integrals of the form $\int \frac{\ell x+m}{\sqrt{ax^2+bx+c}}dx$

Evaluation procedure. Assume express $\ell x+m = A\cdot\frac{d}{dx}\left(ax^2+bx+c\right)+B$. Find the values of A and B and then substitute for $\ell x+m$ in the integral which will be evaluated easily.

Worked Examples

Example 3.55. Evaluate $\int \frac{3x+1}{\sqrt{2x^2+x+3}}dx.$

Solution. Let $3x+1 = A\frac{d}{dx}(2x^2+x+3)+B$

$$= A(4x+1)+B.$$

Equating the coefficients of x on both sides we get,

$$4A = 3 \Rightarrow A = \frac{3}{4}.$$

Equating the constants

$$A+B = 1 \Rightarrow \frac{3}{4}+B = 1$$

$$B = 1-\frac{3}{4} = \frac{1}{4}.$$

$$\therefore 3x+1 = \frac{3}{4}(4x+1)+\frac{1}{4}.$$

$$\therefore \int \frac{3x+1}{2x^2+x+3}dx = \int \frac{\frac{3}{4}(4x+1)+\frac{1}{4}}{\sqrt{2x^2+x+3}}dx.$$

$$= \frac{3}{4}\int \frac{4x+1}{\sqrt{2x^2+x+3}}dx+\frac{1}{4}\int \frac{1}{\sqrt{2x^2+x+3}}dx.$$

$$= \frac{3}{4}\int (2x^2+x+3)^{\frac{-1}{2}}d(2x^2+x+3)+\frac{1}{4}\int \frac{1}{\sqrt{2\left(x^2+\frac{x}{2}+\frac{3}{2}\right)}}dx$$

$$= \frac{3}{4}\frac{(2x^2+x+3)^{\frac{1}{2}}}{\frac{1}{2}}+\frac{1}{4\sqrt{2}}\int \frac{1}{\sqrt{\left(x+\frac{1}{4}\right)^2+\frac{3}{2}-\frac{1}{16}}}dx$$

$$= \frac{3}{4}\times 2\sqrt{2x^2+x+3}+\frac{1}{4\sqrt{2}}\int \frac{1}{\sqrt{\left(x+\frac{1}{4}\right)^2+\frac{24-1}{16}}}dx$$

$$= \frac{3}{2}\sqrt{2x^2+x+3}+\frac{1}{4\sqrt{2}}\int \frac{1}{\sqrt{\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}}dx$$

$$= \frac{3}{2}\sqrt{2x^2+x+3}+\frac{1}{4\sqrt{2}}\log\left(x+\frac{1}{4}+\sqrt{\left(x+\frac{1}{4}\right)^2+\frac{23}{16}}\right)+c$$

$$= \frac{3}{2}\sqrt{2x^2+x+3} + \frac{1}{4\sqrt{2}}\log\left(\frac{4x+1}{4} + \sqrt{x^2+\frac{x}{2}+\frac{3}{2}}\right) + c.$$

Example 3.56. Evaluate $\int \frac{1}{\sqrt{a^2-x^2}}dx$ by using trignometric substitution. Hence use it in evaluating $\frac{6x+5}{6+x-2x^2}dx$. [A.U Nov 2016]

Solution. For the first part, refer Result(1). Let us evaluate the second part.

$$\text{Let } 6x+5 = A\frac{d}{dx}(6+x-2x^2) + B$$

$$= A(1-4x) + B.$$

Equating the coefficients of x on both sides we get

$$-4A = 6 \Rightarrow A = -\frac{6}{4} = -\frac{3}{2}.$$

Equating the constants,we get

$$A + B = 5 \Rightarrow -\frac{3}{2} + B = 1$$

$$B = 5 + \frac{3}{2} = \frac{13}{2}.$$

$$\therefore 6x+5 = -\frac{3}{2}(1-4x) + \frac{13}{2}.$$

$$\therefore \int \frac{6x+5}{\sqrt{6+x-2x^2}}dx = \int \frac{\frac{-3}{2}(1-4x)+\frac{13}{2}}{\sqrt{6+x-2x^2}}dx.$$

$$= -\frac{3}{2}\int \frac{1-4x}{\sqrt{6+x-2x^2}}dx + \frac{13}{2}\int \frac{1}{\sqrt{6+x-2x^2}}dx.$$

$$= -\frac{3}{2}\int (6+x-2x^2)^{\frac{-1}{2}}d(6+x-2x^2) + \frac{13}{2}\int \frac{1}{\sqrt{2(3+\frac{x}{2}-x^2)}}dx$$

$$= -\frac{3}{2}\frac{(6+x-2x^2)^{\frac{1}{2}}}{\frac{1}{2}} + \frac{13}{2\sqrt{2}}\int \frac{1}{\sqrt{3-(x^2-\frac{x}{2})}}dx$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\int \frac{1}{\sqrt{3-\left\{(x-\frac{1}{4})^2-\frac{1}{16}\right\}}}dx$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\int \frac{1}{\sqrt{3-(x-\frac{1}{4})^2+\frac{1}{16}}}dx$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\int \frac{1}{\sqrt{(\frac{7}{4})^2-(x-\frac{1}{4})^2}}dx$$

$$= -3\sqrt{6 + x - 2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{x - \frac{1}{4}}{\frac{7}{4}}\right) + c$$
$$= -3\sqrt{6 + x - 2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x - 1}{7}\right) + c.$$

Example 3.57. Evaluate $\int \frac{4x-3}{\sqrt{x^2+2x-1}}dx.$

Solution. Let $4x - 3 = A\frac{d}{dx}(x^2 + 2x - 1) + B$

$$= A(2x + 2) + B.$$

Equating the coefficients of x on both sides we get,

$2A = 4 \Rightarrow A = 2.$

Equating the constants

$2A + B = -3 \Rightarrow 4 + B = -3 \quad \Rightarrow B = -3 - 4 = -7.$

$\therefore 4x - 3 = 2(2x + 2) - 7.$

$$\therefore \int \frac{4x-3}{\sqrt{x^2+2x-1}}dx = \int \frac{2(2x+2)-7}{\sqrt{x^2+2x-1}}dx.$$
$$= 2\int \frac{2x+2}{\sqrt{x^2+2x-1}}dx - 7\int \frac{1}{\sqrt{x^2+2x-1}}dx.$$
$$= 2\int (x^2+2x-1)^{\frac{-1}{2}}d(x^2+2x-1) - 7\int \frac{1}{\sqrt{(x+1)^2-1-1}}dx$$
$$= 2\frac{(x^2+2x-1)^{\frac{1}{2}}}{\frac{1}{2}} - 7\int \frac{1}{\sqrt{(x+1)^2-2}}dx$$
$$= 4\sqrt{x^2+2x-1} - 7\int \frac{1}{\sqrt{(x+1)^2-\left(\sqrt{2}\right)^2}}dx$$
$$= 4\sqrt{x^2+2x-1} - 7\log\left(x+1+\sqrt{(x+1)^2-\left(\sqrt{2}\right)^2}\right) + c$$
$$= 4\sqrt{x^2+2x-1} - 7\log\left(x+1+\sqrt{x^2+2x-1}\right) + c.$$

Example 3.58. Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}dx.$

Solution. Let $x = A\frac{d}{dx}(3 - 2x - x^2) + B$

$$= A(-2 - 2x) + B.$$

Equating the coefficients of x on both sides we get,

$-2A = 1 \Rightarrow A = -\frac{1}{2}$.

Equating the constants

$-2A + B = 0 \Rightarrow 1 + B = 0 \quad \Rightarrow B = -1$.

$$\therefore x = -\frac{1}{2}(-2-2x) - 1.$$

$$\begin{aligned}
\therefore \int \frac{x}{\sqrt{3-2x-x^2}}dx &= \int \frac{-\frac{1}{2}(-2-2x)-1}{\sqrt{3-2x-x^2}}dx. \\
&= -\frac{1}{2}\int \frac{-2-2x}{\sqrt{3-2x-x^2}}dx - \int \frac{1}{\sqrt{3-2x-x^2}}dx. \\
&= -\frac{1}{2}\int (3-2x-x^2)^{\frac{-1}{2}} d(3-2x-x^2) - \int \frac{1}{\sqrt{3-(x^2+2x)}}dx \\
&= -\frac{1}{2}\frac{\left(3-2x-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} - \int \frac{1}{\sqrt{3-\{(x+1)^2-1\}}}dx \\
&= -\sqrt{3-2x-x^2} - \int \frac{1}{3-(x+1)^2+1}dx \\
&= -\sqrt{3-2x-x^2} - \int \frac{1}{\sqrt{4-(x+1)^2}}dx \\
&= -\sqrt{3-2x-x^2} - \int \frac{1}{\sqrt{2^2-(x+1)^2}}dx \\
&= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + c.
\end{aligned}$$

3.5.3 Type III. Evaluation of integrals of the form $\int \sqrt{ax^2+bx+c}\,dx$

Evaluation procedure. By completing the square form, express ax^2+bx+c as any of $x^2-a^2, a^2+x^2, a^2-x^2$ and apply the appropriate formula.

Worked Examples

Example 3.59. Evaluate $\int \sqrt{x^2-4x+6}\,dx$.

Solution.
$$\int \sqrt{x^2 - 4x + 6}\, dx = \int \sqrt{(x-2)^2 + 6 - 4}\, dx$$
$$= \int \sqrt{(x-2)^2 + 2}\, dx$$
$$= \int \sqrt{(x-2)^2 + (\sqrt{2})^2}\, dx$$
$$= \left(\frac{x-2}{2}\right)\sqrt{(x-2)^2 + (\sqrt{2})^2} + \frac{(\sqrt{2})^2}{2}\log\left((x-2) + \sqrt{(x-2)^2 + (\sqrt{2})^2}\right) + c$$
$$= \left(\frac{x-2}{2}\right)\sqrt{x^2 - 4x + 6} + \log\left((x-2) + \sqrt{x^2 - 4x + 6}\right) + c.$$

3.5.4 Type IV.Integration by trigonometric substitution.

Certain integrals can be easily evaluated by trigonometric substitution. There are three cases.

(i) Any integral involving quantities of the form $\sqrt{a^2 - x^2}$ can be evaluated by the substitution $x = a\sin\theta$.

(ii) Integrals involving functions of the form $\sqrt{a^2 + x^2}$ can be evaluated by the substitution $x = a\tan\theta$ or $x = a\sin h\theta$.

(iii) Integrals involving functions of the form $\sqrt{x^2 - a^2}$ can be evaluated by the substitution $x = a\sec\theta$ or $x = a\cos h\theta$.

Worked Examples

Example 3.60. Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2}\, dx.$

Solution. Since $\sqrt{9 - x^2}$ is of the form $\sqrt{a^2 - x^2}$, we can evaluate this integral by the substitution.

$$x = 3\sin\theta. \quad \Rightarrow dx = 3\cos\theta d\theta.$$

$$\begin{aligned}
\therefore \int \frac{\sqrt{9-x^2}}{x^2}dx &= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}3\cos\theta d\theta \\
&= \int \frac{\sqrt{9(1-\sin^2\theta)}}{9\sin^2\theta}3\cos\theta d\theta \\
&= \int \frac{\sqrt{\cos^2\theta}}{\sin^2\theta}\cos\theta d\theta \\
&= \int \frac{\cos\theta}{\sin^2\theta}\cos\theta d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta}d\theta \\
&= \int \cot^2\theta d\theta \\
&= \int (\mathrm{cosec}^2\theta - 1)d\theta \\
&= \int \mathrm{cosec}^2\theta d\theta - \int d\theta \\
&= -\cot\theta - \theta + c = -\frac{\cos\theta}{\sin\theta} - \theta + c \\
&= -\frac{\sqrt{\cos^2\theta}}{\sin\theta} - \sin^{-1}\left(\frac{x}{3}\right) + c \\
&= -\frac{\sqrt{1-\sin^2\theta}}{\sin\theta} - \sin^{-1}\left(\frac{x}{3}\right) + c \\
&= -\frac{\sqrt{1-\frac{x^2}{9}}}{\frac{x}{3}} - \sin^{-1}\left(\frac{x}{3}\right) + c \\
&= -\frac{3}{x}\sqrt{\frac{9-x^2}{9}} - \sin^{-1}\left(\frac{x}{3}\right) + c \\
&= -\frac{1}{x}\sqrt{9-x^2} - \sin^{-1}\left(\frac{x}{3}\right) + c.
\end{aligned}$$

Example 3.61. Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}dx$.

Solution. Since the integral involves one factor of the form $\sqrt{x^2+4}$,we can evaluate this integral by the substitution

$$x = 2\tan\theta \quad \Rightarrow dx = 2\sec^2\theta d\theta.$$

$$\begin{aligned}
\therefore \int \frac{1}{x^2\sqrt{x^2+4}}dx &= \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} \cdot 2\sec^2\theta d\theta \\
&= \frac{1}{2}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{4(\tan^2\theta+1)}}d\theta \\
&= \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\sec^2\theta}}d\theta \\
&= \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}d\theta \\
&= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta \\
&= \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}d\theta \\
&= \frac{1}{4}\int \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}d\theta \\
&= \frac{1}{4}\int \text{cosec}\theta\cdot\cot\theta\, d\theta \\
&= -\frac{1}{4}\text{cosec}\theta + c \\
&= -\frac{1}{4}\sqrt{\text{cosec}^2\theta} + c \\
&= -\frac{1}{4}\sqrt{1+\cot^2\theta} + c \\
&= -\frac{1}{4}\sqrt{1+\frac{4}{x^2}} + c \qquad \tan\theta = \frac{x}{2}, \cot\theta = 2/x \\
&= -\frac{1}{4}\sqrt{\frac{x^2+4}{x^2}} + c = -\frac{\sqrt{x^2+4}}{4x} + c.
\end{aligned}$$

Example 3.62. Evaluate $\int \frac{x}{\sqrt{x^2+4}}dx$.

Solution. Since the integrand involves one factor of the form $\sqrt{x^2+4}$, we can evaluate this integral by the substitution

$$x = 2\tan\theta \quad \Rightarrow dx = 2\sec^2\theta d\theta.$$

$$\therefore \int \frac{x}{\sqrt{x^2+4}}dx = \int \frac{2\tan\theta}{\sqrt{4\tan^2\theta+4}} \cdot 2\sec^2\theta d\theta$$

$$= 4\int \frac{\tan\theta\sec^2\theta}{\sqrt{4(\tan^2\theta+1)}}\,d\theta$$
$$= 4\int \frac{\tan\theta\sec^2\theta}{2\sqrt{\sec^2\theta}}\,d\theta$$
$$= 2\int \frac{\tan\theta\sec^2\theta}{\sec\theta}\,d\theta$$
$$= 2\int \tan\theta\cdot\sec\theta\,d\theta$$
$$= 2\sec\theta + c$$
$$= 2\sqrt{\sec^2\theta} + c$$
$$= 2\sqrt{1+\tan^2\theta} + c$$
$$= 2\sqrt{1+\frac{x^2}{4}} + c$$
$$= 2\frac{\sqrt{4+x^2}}{2} + c = \sqrt{4+x^2} + c.$$

Note. This, integral can be evaluated easily by the substitution $4 + x^2 = t$.

Example 3.63. Evaluate $\displaystyle\int \frac{x^3+1}{\sqrt{1-x^2}}dx$.

Solution. Since the integrand contains one factor of the form $\sqrt{a^2 - x^2}$, let us make the substitution

$$x = \sin\theta \quad [a = 1]$$
$$dx = \cos\theta d\theta$$

$$\int \frac{x^3+1}{\sqrt{1-x^2}}dx = \int \frac{\sin^3\theta+1}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta$$
$$= \int \frac{\sin^3\theta+1}{\sqrt{\cos^2\theta}}\cdot\cos\theta d\theta$$
$$= \int \frac{\sin^3\theta+1}{\cos\theta}\cdot\cos\theta d\theta$$
$$= \int (\sin^3\theta+1)\,d\theta$$
$$= \int \sin^3\theta\,d\theta + \int d\theta$$

$$= \int \left(\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right) d\theta + \theta$$
$$= \frac{3}{4}\int \sin\theta d\theta - \frac{1}{4}\int \sin 3\theta d\theta + \theta$$
$$= \frac{3}{4}(-\cos\theta) - \frac{1}{4}\left(\frac{-\cos 3\theta}{3}\right) + \theta + c$$
$$= -\frac{3}{4}\sqrt{\cos^2\theta} + \frac{1}{12}(4\cos^3\theta - 3\cos\theta) + \sin^{-1} x + c$$
$$= -\frac{3}{4}\sqrt{1-\sin^2\theta} + \frac{1}{3}(\cos^2\theta)^{\frac{3}{2}} - \frac{1}{4}\sqrt{\cos^2\theta} + \sin^{-1} x + c$$
$$= -\frac{3}{4}\sqrt{1-x^2} + \frac{1}{3}(1-\sin^2\theta)^{\frac{3}{2}} - \frac{1}{4}\sqrt{1-\sin^2\theta} + \sin^{-1} x + c$$
$$= -\frac{3}{4}\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{\frac{3}{2}} - \frac{1}{4}\sqrt{1-x^2} + \sin^{x} + c$$
$$= \frac{1}{3}(1-x^2)^{\frac{3}{2}} - \sqrt{1-x^2} + \sin^{-1} x + c.$$

Example 3.64. Evaluate $\int \frac{1}{x^3\sqrt{x^2-9}}dx.$

Solution. Since the integrand contains one factor of the form $\sqrt{a^2-x^2}$, we can make the substitution

$$x = 3\sec\theta \quad [\therefore a = 3]$$
$$dx = 3\sec\theta \cdot \tan\theta d\theta$$

$$\int \frac{1}{x^3\sqrt{x^2-9}}dx = \int \frac{1}{3^3\sec^3\theta \cdot \sqrt{9\sec^2\theta - 9}} \cdot 3\sec\theta\tan\theta d\theta$$
$$= \frac{1}{9}\int \frac{1}{\sec^3\theta \cdot \sqrt{9(\sec^2\theta - 1)}} \sec\theta\tan\theta d\theta$$
$$= \frac{1}{9}\int \frac{1}{\sec^3\theta \times 3\sqrt{\tan^2\theta}} \sec\theta\tan\theta d\theta$$
$$= \frac{1}{27}\int \frac{1}{\sec^3\theta\tan\theta} \sec\theta\tan\theta d\theta$$
$$= \frac{1}{27}\int \frac{1}{\sec^2\theta} d\theta$$
$$= \frac{1}{27}\int \cos^2\theta d\theta$$
$$= \frac{1}{27}\int \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{1}{54}\left[\int d\theta + \int \cos 2\theta d\theta\right]$$
$$= \frac{1}{54}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$$
$$= \frac{1}{54}\theta + \frac{1}{54}\sin\theta\cos\theta + c$$
$$= \frac{1}{54}\sec^{-1}\left(\frac{x}{3}\right) + \frac{1}{54}\sqrt{\sin^2\theta}\cdot\frac{1}{\sec\theta} + c$$
$$= \frac{1}{54}\sec^{-1}\left(\frac{x}{3}\right) + \frac{1}{54}\sqrt{1-\cos^2\theta}\cdot\frac{3}{x} + c$$
$$= \frac{1}{54}\sec^{-1}\left(\frac{x}{3}\right) + \frac{1}{54}\sqrt{1-\frac{9}{x^2}}\cdot\frac{3}{x} + c$$
$$= \frac{1}{54}\sec^{-1}\left(\frac{x}{3}\right) + \frac{1}{18}\frac{\sqrt{x^2-9}}{x^2} + c.$$

Example 3.65. Evaluate $\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}}dx$.

Solution. Since the integrand contains a factor of the form $\sqrt{a^2+x^2}$, we can make the substitution

$$x = a\tan\theta \quad \Rightarrow dx = a\sec^2\theta d\theta.$$

$$\therefore \int \frac{1}{(a^2+x^2)^{\frac{3}{2}}}dx = \int \frac{1}{(a^2+a^2\tan^2\theta)^{\frac{3}{2}}}\cdot a\sec^2\theta d\theta$$
$$= \int \frac{1}{(a^2(1+\tan^2\theta))^{\frac{3}{2}}}\cdot a\sec^2\theta d\theta$$
$$= \int \frac{1}{(a^2\sec^2\theta)^{\frac{3}{2}}}\cdot a\sec^2\theta d\theta$$
$$= \int \frac{1}{a^3\sec^3\theta}\cdot a\sec^2\theta d\theta$$
$$= \frac{1}{a^2}\int \frac{1}{\sec\theta}d\theta$$
$$= \frac{1}{a^2}\int \cos\theta\, d\theta$$
$$= \frac{1}{a^2}\sin\theta + c$$
$$= \frac{1}{a^2}\frac{\tan\theta}{\sec\theta} + c$$

$$= \frac{1}{a^2}\frac{x}{a} \cdot \frac{1}{\sqrt{\sec^2\theta}} + c$$
$$= \frac{x}{a^3}\frac{1}{\sqrt{1+\tan^2\theta}} + c$$
$$= \frac{x}{a^3}\frac{1}{\sqrt{1+\frac{x^2}{a^2}}} + c = \frac{x}{a^3}\frac{a}{\sqrt{a^2+x^2}} + c = \frac{x}{a^2\sqrt{a^2+x^2}} + c.$$

3.6 Integration by parts

Every differentiation rule has a corresponding integration rule. The rule that correspond to the product for differentiation is called the integration by parts.

From differential calculus we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to x

$$\int \frac{d}{dx}(uv)dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$
$$uv = \int udv + \int vdu$$
$$\int udv = uv - \int vdu.$$

Priority rules for choosing u.

1. Inverse functions.
2. logarithmic functions.
3. Polynomials in x.
4. Any other function other than the above three.

Worked Examples

Example 3.66. Evaluate $\int \tan^{-1} x dx$ and hence deduce the value of $\int_0^1 \tan^{-1} x dx$ [A.U.Nov 2016]

Solution. Let $u = \tan^{-1} x$ $\qquad dv = dx$

$$du = \frac{1}{1+x^2}\ dx \qquad \int dv = \int dx \Rightarrow v = x.$$

Applying integration by parts

$$\int \tan^{-1} dx = \int udv$$

$$= uv - \int vdu$$

$$= (\tan^{-1} x) \cdot x - \int x \cdot \frac{1}{1+x^2} dx$$

$$= x \cdot \tan^{-1} x - \int \frac{1}{t}\frac{dt}{2}$$

$$= x \cdot \tan^{-1} x - \frac{1}{2} \log t + c$$

$$= x \cdot \tan^{-1} x - \frac{1}{2} \log(1+x^2) + c.$$

In the second integral

put $1 + x^2 = t$

$2xdx = dt$

$xdx = \frac{dt}{2}.$

$$\text{Now, } \int_0^1 \tan^{-1} x\, dx = \left(x \tan^{-1} x\right)_0^1 - \frac{1}{2}\left[\log(1+x^2)\right]_0^1$$

$$= \tan^{-1} 1 - 0 - \frac{1}{2}(\log 2 - \log 1)$$

$$= \frac{\pi}{4} - \frac{1}{2}\log 2.$$

Example 3.67. Evaluate $\int \sin^{-1} x dx$.

Solution. Let $u = \sin^{-1} x$ $\qquad dv = dx$

$$du = \frac{1}{\sqrt{1-x^2}} dx \qquad \int dv = \int dx \Rightarrow v = x.$$

Applying integration by parts

$$\int \sin^{-1} x\ \ dx = \int u\, dv$$

$$= uv - \int v\, du$$

$$= (x\sin^{-1}x) - \int x \cdot \frac{1}{\sqrt{1-x^2}}dx$$

In the second integral

put $1 - x^2 = t$

$-2xdx = dt$

$xdx = -\dfrac{dt}{2}.$

$$= x \cdot \sin^{-1}x - \int \frac{1}{\sqrt{t}}\left(\frac{-dt}{2}\right)$$

$$= x \cdot \sin^{-1}x + \frac{1}{2}\int (t)^{-\frac{1}{2}} + c$$

$$= x \cdot \sin^{-1}x + \frac{1}{2}\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + c$$

$$\therefore \int \sin^{-1}x\,dx = (x\sin^{-1}x) + \sqrt{1-x^2} + c.$$

Example 3.68. Evaluate $\int x\sin xdx$.

Solution. Let $u = x$ $\qquad dv = \sin xdx$

$du = dx \qquad \int dv = \int \sin xdx \Rightarrow v = -\cos x.$

Applying integration by parts

$$\int x\sin xdx = \int udv = uv - \int vdu$$

$$= -x \cdot \cos x - \int -\cos xdx = -x\cos x + \sin x + c.$$

Example 3.69. Evaluate $\int xe^x dx$.

Solution. Let $u = x$ $\qquad dv = e^x dx$

$du = dx \qquad \int dv = \int e^x dx \Rightarrow v = e^x.$

Applying integration by parts

$$\int xe^x dx = \int udv = uv - \int vdu$$

$$= x \cdot e^x - \int e^x dx$$

$$= xe^x - e^x + c = e^x(x-1) + c.$$

Example 3.70. Evaluate $\int \log xdx$.

Solution. Let $u = \log x$ $\qquad dv = dx$

$du = \dfrac{1}{x}dx \qquad \int dv = \int dx \Rightarrow v = x.$

Applying integration by parts

$$\int \log x dx = \int u dv = uv - \int v du$$

$$= x \cdot \log x - \int x \cdot \frac{1}{x} dx$$

$$= x \log x - \int dx + c = x \log x - x + c = x(\log x - 1) + c.$$

Example 3.71. Evaluate $\int (\log x)^2 \, dx$. [A.U. Dec. 2015]

Solution. Let $u = (\log x)^2$ $\qquad dv = dx$

$$du = 2(\log x)\frac{1}{x}dx \qquad \int dv = \int dx \Rightarrow v = x.$$

Applying integration by parts

$$\int (\log x)^2 \, dx = \int u dv = uv - \int v du$$

$$= (\log x)^2 \cdot x - \int x \cdot 2 (\log x) \frac{1}{x} dx$$

$$= x (\log x)^2 - \int \log x dx$$

$$= x (\log x)^2 - 2x(\log x - 1) + c. \text{ [From the provious problem]}$$

Example 3.72. Evaluate $\int x^n \log x dx$.

Solution. Let $u = \log x$ $\qquad dv = x^n dx$

$$du = \frac{1}{x}dx \qquad \int dv = \int x^n dx \Rightarrow v = \frac{x^{n+1}}{n+1}.$$

Applying integration by parts

$$\int x^n \log x dx = \int u dv = uv - \int v du$$

$$= \frac{x^{n+1}}{n+1} \cdot \log x - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} dx$$

$$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \int x^n dx + c$$

$$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \cdot \frac{x^{n+1}}{n+1} + c$$

$$= \frac{x^{n+1}}{n+1} \left(\log x - \frac{1}{n+1} \right) + c.$$

Example 3.73. Evaluate $\int x^2 e^x dx$.

Solution. Let $u = x^2 \qquad dv = e^x dx$

$$du = 2xdx \qquad \int dv = \int e^x dx \Rightarrow v = e^x.$$

Applying integration by parts

$$\int x^2 e^x dx = \int udv = uv - \int vdu$$

$$= x^2 \cdot e^x - \int e^x \cdot 2x\, dx$$

$$= x^2 e^x - 2\int xe^x\, dx$$

$$= x^2 e^x - 2\,[xe^x - e^x] + c \qquad \text{[Refer Example 3.79]}$$

$$= x^2 e^x - 2xe^x + 2e^x + c = e^x(x^2 - 2x + 2) + c$$

Example 3.74. Evaluate $\int x^2 \tan^{-1} xdx$.

Solution. Let $u = \tan^{-1} x \qquad dv = x^2 dx$

$$du = \frac{1}{1+x^2}dx \qquad \int dv = \int x^2 dx \Rightarrow v = \frac{x^3}{3}.$$

Applying integration by parts

$$\int x^2 \tan^{-1} xdx = \int udv = uv - \int vdu$$

$$= \frac{x^3}{3}\tan^{-1} x - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2}dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx \qquad \text{Let } 1 + x^2 = t$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^2}{1+x^2} \cdot xdx \qquad 2xdx = dt$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{t-1}{t} \cdot \frac{dt}{2} \qquad xdx = \frac{dt}{2}.$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{6}\int \left(1 - \frac{1}{t}\right)dt$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{6}\int dt + \frac{1}{6}\int \frac{1}{t}dt$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{6} \cdot t + \frac{1}{6} \cdot \log t + c = \frac{x^3}{3}\tan^{-1} x - \frac{1+x^2}{6} + \frac{1}{6}\log(1+x^2) + c.$$

Example 3.75. Evaluate $\int e^{\tan^{-1}x}\frac{1+x+x^2}{1+x^2}dx.$ [A.U. Dec. 2015]

Solution. Let $u = \tan^{-1}x$ or $x = \tan u dx$

$$du = \frac{1}{1+x^2}dx \qquad dx = \sec^2 u \; du.$$

Applying integration by parts

$$\begin{aligned}
\int e^{\tan^{-1}x}\frac{1+x+x^2}{1+x^2}dx &= \int e^{\tan^{-1}x}\left(\frac{1+x^2}{1+x^2}+\frac{x}{1+x^2}\right)dx\\
&= \int e^{\tan^{-1}x}\left(1+\frac{x}{1+x^2}\right)dx\\
&= \int e^{\tan^{-1}x}\,dx + \int e^{\tan^{-1}x}\left(\frac{x}{1+x^2}\right)dx\\
&= \int e^u \sec^2 u\,du + \int e^u \tan u\,du\\
&= \int e^u \sec^2 u\,du + \int \tan u\,d\left(e^u\right)\\
&= \int e^u \sec^2 u\,du + e^u \tan u - \int e^u \sec^2 u\,du\\
&= e^u \tan u + c = xe^{\tan^{-1}x} + c.
\end{aligned}$$

3.7 Trigonometric Integrals

Theorem 1. Prove the reduction formula
$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x dx$ and deduce that
$\int_0^{\frac{\pi}{2}} \sin^n dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}x dx.$ Find also the general value of $\int_0^{\frac{\pi}{2}} \sin^n x dx.$ Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x dx$ and $\int_0^{\frac{\pi}{2}} \sin^8 x dx.$ [A.U.Dec 2016]

Proof. Write $\int \sin^n x dx = \int \sin^{n-1}x \sin x dx.$

Choose $u = \sin^{n-1}x$ $\qquad dv = \sin x dx$

$$du = (n-1)\sin^{n-2}x \cdot \cos x dx \qquad \int dv = \int \sin x dx \Rightarrow v = -\cos x.$$

Applying integration by parts

$$\int \sin^n x dx = \int u dv = uv - \int v du$$

$$= -\sin^{n-1} x \cdot \cos x - \int -\cos x \cdot (n-1)\sin^{n-2} x \cos x dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x \cos^2 x dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x) dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx$$

$$\int \sin^n x dx + (n-1)\int \sin^n x dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x dx$$

$$n\int \sin^n x dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x dx$$

$$\int \sin^n x dx = \frac{-1}{n}\sin^{n-1} x \cos x + \frac{(n-1)}{n}\int \sin^{n-2} x dx$$

Taking limits from 0 to $\frac{\pi}{2}$ everywhere we get

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \left(\frac{-1}{n}\sin^{n-1} x \cos x\right)_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx.$$

$$= 0 + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx.$$

Applying the same procedure again for the integral on the RHS we get

$$\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx = \frac{n-3}{n-2}\int_0^{\frac{\pi}{2}} \sin^{n-4} x dx$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\int_0^{\frac{\pi}{2}} \sin^{n-4} x dx$$

Repeating this process n times we get

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\text{(ultimate integral)}.$$

If n is even, the ultimate integral is $\int_0^{\frac{\pi}{2}} \sin^{n-n} x dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$.

If n is odd, the ultimate integral is $\int_0^{\frac{\pi}{2}} \sin x dx = \left(-\cos x\right)_0^{\frac{\pi}{2}} = -[\cos\frac{\pi}{2} - \cos 0]$

$= -[0 - 1] = 1.$

$\therefore$ The general value of $\int_0^{\frac{\pi}{2}} \sin^n x dx$ is

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4}, \cdots \dfrac{2}{3} \cdot 1, & \text{if n is odd} \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4}, \cdots \dfrac{1}{2} \cdot \dfrac{\pi}{2}, & \text{if n is even.} \end{cases}$$

Now, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^7 x dx = \frac{7-1}{7} \cdot \frac{7-3}{7-2} \cdot \frac{7-5}{7-4} \cdot 1 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{16}{35}.$

Also, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^8 x dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{35\pi}{256}.$

Theorem 2. Prove the reduction formula

$\int \cos^n x dx = \dfrac{1}{n} \cos^{n-1} x \sin x + \dfrac{n-1}{n} \int \cos^{n-2} x dx$. Hence reduce the value of $\int_0^{\frac{\pi}{2}} \cos^n dx$.

Proof. Write $\displaystyle\int \cos^n x dx = \int \cos^{n-1} x \cos x dx.$

Choose $u = \cos^{n-1} x$ $\qquad$ $dv = \cos x dx$

$du = (n-1)\cos^{n-2} x \cdot (-\sin x) dx$ $\qquad$ $\displaystyle\int dv = \int \cos x dx \Rightarrow v = \sin x.$

Applying integration by parts

$$\int \cos^n x dx = \int u dv = uv - \int v du$$

$$= \cos^{n-1} x \cdot \sin x - \int \sin x \cdot (n-1)\cos^{n-2} x (-\sin x) dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x \sin^2 x dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx$$

$$\int \cos^n x dx + (n-1)\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$n\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{(n-1)}{n}\int \cos^{n-2} x dx.$$

If n is odd, the ultimate integral is $\int \cos^{n-(n-1)} x dx = \int \cos x dx = \sin x$.

If n is even, the ultimate integral is $\int \cos^{n-n} x dx = \int dx = x$.

Now,applying limit from 0 to $\frac{\pi}{2}$ we obtain

$$\int_0^{\frac{\pi}{2}} \cos^n x dx = \left(\frac{1}{n}\cdot \cos^{n-1} x \sin x\right)_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x dx.$$

By repeatedly applying this process we obtain

$$\int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots(\text{ultimate integral})$$

If n is even, the ultimate integral is $\left(x\right)_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$.

If n is odd, the ultimate integral is $\int_0^{\frac{\pi}{2}} \cos x dx = \left(\sin x\right)_0^{\frac{\pi}{2}} = 1$.

$$\therefore \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4},\cdots\frac{2}{3}\cdot 1, & \text{if n is odd} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4},\cdots\frac{1}{2}\cdot\frac{\pi}{2}, & \text{if n is even.} \end{cases}$$

Result. By looking at the above two theorems, we obtain the following result

If n is odd (say $n = 2k+1$),then

$$\int_0^{\frac{\pi}{2}} \sin^{2k+1} x dx = \int_0^{\frac{\pi}{2}} \cos^{2k+1} x dx = \frac{2.4.6....2k}{3.5.7...(2k+1)}.$$

If n is even (say $n = 2k$),then

$$\int_0^{\frac{\pi}{2}} \sin^{2k} x dx = \int_0^{\frac{\pi}{2}} \cos^{2k} x dx = \frac{1.3.5....(2k-1)}{2.4.6.....(2k)}\cdot\frac{\pi}{2}.$$

For example.

$$\int_0^{\frac{\pi}{2}} \sin^7 x dx = \int_0^{\frac{\pi}{2}} \cos^7 dx = \int_0^{\frac{\pi}{2}} \cos^{2.3+1} x dx \qquad [n=3]$$

$$= \frac{2.4.6}{3.5.7} = \frac{16}{35}.$$

$$\int_0^{\frac{\pi}{2}} \sin^8 x dx = \int_0^{\frac{\pi}{2}} \cos^8 dx = \int_0^{\frac{\pi}{2}} \cos^{2.4} x dx \qquad [n=4]$$

$$= \frac{1.3.5.7}{2.4.6.8} \times \frac{\pi}{2} = \frac{35\pi}{256}.$$

Evaluation of integrals of the form $\int \sin^m x \cdot \cos^n x dx$

Evaluation procedure

(i) If the power of cosine is odd (say $n = 2k + 1$)then write

$$\begin{aligned}\int \sin^m x \cos^n x dx &= \int \sin^m x \cdot \cos^{2k+1} x dx \\ &= \int \sin^m x \cdot \cos^{2k} x \cos x dx \\ &= \int \sin^m x \cdot (\cos^2 x)^k \cos x dx \\ &= \int \sin^m x \cdot (1 - \sin^2 x)^k \cos x dx\end{aligned}$$

Now make the substitution $t = \sin x$, and hence the integral is evaluated easily.

(ii) If the power of sine is odd (say $n = 2k + 1$)then write

$$\begin{aligned}\int \sin^m x \cos^n x dx &= \int \sin^{2k+1} x \cdot \cos^n x \cos x dx \\ &= \int \sin^{2k} x \cdot \sin x \cos^n x dx \\ &= \int (1 - \cos^2 x)^k \cdot \cos^n x \sin x dx.\end{aligned}$$

Now make the substitution $t = \cos x$, and hence the integral is evaluated easily.

(iii) If the powers of both sine and cosine are even, then use the relations

$$\sin^2 x = \frac{(1 - \cos 2x)}{2}$$

$$\cos^2 x = \frac{(1 + \cos 2x)}{2}$$

so that the resulting integrals can be evaluated easily.

(iv) If the powers of both sine and cosine are odd, then apply either case(i) or case(ii).

Worked Examples

Example 3.76. Evaluate $\displaystyle\int \sin^5 x \cos^2 x dx$.

Solution. Here the power of sine is odd.

$$\therefore \text{ We write } \int \sin^5 x \cos^2 x dx = \int \sin^4 x \cdot \sin x \cos^2 x dx$$

$$= \int (\sin^2 x)^2 \cdot \cos^2 x \sin x dx$$

$$= \int (1-\cos^2 x)^2 \cdot \cos^2 x \sin x dx \qquad \text{put } \cos x = t$$

$$= \int (1-t^2)^2 \cdot t^2(-dt) \qquad -\sin x dx = dt$$

$$= -\int (1+t^4-2t^2)t^2 dt \qquad \sin x dx = -dt.$$

$$= -\int (t^2+t^6-2t^4)dt$$

$$= -\left[\frac{t^3}{3}+\frac{t^7}{7}-2\cdot\frac{t^5}{5}\right]+c = -\frac{\cos^3 x}{3}-\frac{\cos^7 x}{7}+\frac{2}{5}\cos^5 x + c.$$

Example 3.77. Evaluate $\displaystyle\int \sin^6 x \cos^3 x dx$.

Solution. Here the power of cosine is odd.

$$\therefore \text{ we write } \int \sin^6 x \cos^3 x dx = \int \sin^6 x \cdot \cos^2 x \cdot \cos x dx$$

$$= \int \sin^6 x(1-\sin^2 x) \cdot \cos x dx$$

$$= \int t^6(1-t^2)dt \qquad \text{put } \sin x = t$$

$$= \int (t^6-t^8)dt \qquad \cos x dx = -dt.$$

$$= \frac{t^7}{7}-\frac{t^9}{9}+c = \frac{\sin^7 x}{7}-\frac{\sin^9 x}{9}+c.$$

Example 3.78. Evaluate $\displaystyle\int \sin^9 x \cos^5 x dx$.

Solution. Here, the powers of both sine and cosine are odd.

$$\therefore \text{ we write } \int \sin^9 x \cos^5 x dx = \int \sin^9 x \cdot \cos^4 x \cdot \cos x dx$$

$$= \int \sin^9 x(1-\sin^2 x)^2 \cdot \cos x dx$$

$$= \int t^9(1-t^2)^2 dt \qquad \text{put } \sin x = t$$

$$= \int t^9(1+t^4-2t^2)dt \qquad \cos x dx = dt.$$

$$= \int (t^9 + t^{13} - 2t^{11})dt$$
$$= \frac{t^{10}}{10} + \frac{t^{14}}{14} - 2\frac{t^{12}}{12} + c = \frac{\sin^{10} x}{10} + \frac{\sin^{14} x}{14} - \frac{\sin^{12} x}{6} + c.$$

Evaluation of integrals of the form $\int \tan^m x \cdot \sec^n x dx$

Evaluation procedure.

(i) If the power of secant is even (say $n = 2k$) then write the given integral as

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^m x \cdot \sec^{2k} x dx\\ &= \int \tan^m x \cdot \sec^{2k-2} x \sec^2 x dx\\ &= \int \tan^m x \cdot \sec^{2(k-1)} x \sec^2 x dx\\ &= \int \tan^m x \cdot (\sec^2 x)^{k-1} \sec^2 x dx \qquad \text{put } \tan x = t\\ &= \int \tan^m x \cdot (1 + \tan^2 x)^{k-1} \sec^2 x dx \qquad \sec^2 x dx = dt.\\ &= \int t^m (1 + t^2)^{k-1} dt, \text{ which can be evaluated easily.}\end{aligned}$$

(ii) If the power of tangent is odd (say $m = 2k + 1$)then write the given integral as

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^{2k+1} x \cdot \sec^n x dx\\ &= \int \tan^{2k} x \cdot \tan x \sec^{n-1} x \sec x dx\\ &= \int (\tan^2 x)^k \cdot \sec^{n-1} x \sec x \tan x dx \qquad \text{put } \sec x = t\\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x dx \qquad \sec x \tan x dx = dt.\\ &= \int (t^2 - 1)^k \cdot t^{n-1} dt, \text{ which can be evaluated easily.}\end{aligned}$$

Worked Examples

Example 3.79. Evaluate $\int \tan^6 x \sec^4 x dx$.

Solution. Here, the power of secant is even. Hence we write the given integral as

$$\int \tan^6 x \sec^4 x dx = \int \tan^6 x \sec^2 x \sec^2 x dx$$

$$= \int \tan^6 x(1+\tan^2 x)\sec^2 x dx \qquad \text{put } \tan x = t$$

$$\sec^2 x dx = dt.$$

$$= \int t^6(1+t^2)dt$$

$$= \frac{t^7}{7} + \frac{t^9}{9} + c$$

$$= \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + c.$$

Example 3.80. Evaluate $\int \tan^5 x \sec^7 x dx$.

Solution. Here, the power of tangent is odd. Hence we write the given integral as

$$\int \tan^5 x \sec^7 x dx = \int \tan^4 x \sec^6 x \sec x \tan x dx$$

$$= \int (\sec^2 x - 1)^2 \sec^6 x \sec x \tan x dx \qquad \text{put } \sec x = t$$

$$\sec x \tan x dx = dt.$$

$$= \int (t^2-1)^2 t^6 dt$$

$$= \int (t^4 + 1 - 2t^2)t^6 dt$$

$$= \int (t^{10} + t^6 - 2t^8)dt$$

$$= \frac{t^{11}}{11} + \frac{t^7}{7} - 2\frac{t^9}{9} + c = \frac{\sec^{11} x}{11} + \frac{\sec^7 x}{7} - \frac{2}{9}\sec^9 x + c.$$

Some standard results.

1. Find $\int \tan x dx$

Solution. $\int \tan x dx = \int \frac{\sin x}{\cos x}dx \qquad \cos x = t$

$$= \int \frac{1}{t}(-dt) \qquad -\sin x dx = dt$$

$$= -\log t + c \qquad \sin x dx = -dt.$$

$$= -\log(\cos x) + c \qquad \text{(or)}$$

$$= \log(\cos x)^{-1} + c = \log\left(\frac{1}{\cos x}\right) + c = \log(\sec x) + c.$$

2. Find $\int \sec x dx$

Solution.
$$\int \sec x dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx$$
$$= \int \frac{1}{t}(dt)$$
$$= \log t + c$$
$$= \log(\sec x + \tan x) + c.$$

$$\sec x + \tan x = t$$
$$(\sec x \tan x + \sec^2 x)dx = dt$$
$$\sec x(\sec x + \tan x)dx = dt.$$

3. Find $\int \operatorname{cosec} x dx$

Solution.
$$\int \operatorname{cosec} x dx = \int \operatorname{cosec} x \cdot \frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x} dx$$
$$= \int \frac{1}{t}(-dt)$$
$$= -\log t + c$$
$$= -\log(\operatorname{cosec} x + \cot x) + c$$

$$\operatorname{cosec} x + \cot x = t$$
$$(-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x)dx = dt$$
$$-\operatorname{cosec} x(\operatorname{cosec} x + \cot x)dx = dt.$$

4. Find $\int \cot x \, dx$.

Solution. $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{d(\sin x)}{\sin x} = \log(\sin x) + c.$

Worked Examples

Example 3.81. Evaluate $\int \tan^3 x dx$.

Solution.
$$\int \tan^3 x dx = \int \tan^2 x \tan x dx$$
$$= \int (\sec^2 x - 1) \tan x dx$$
$$= \int \sec^2 x \tan x dx - \int \tan x dx$$
$$= \int \tan x d(\tan x) - \log \sec x$$
$$= \frac{\tan^2 x}{2} - \log(\sec x) + c.$$

Example 3.82. Evaluate $\int \sec^3 x dx$.

Solution. $\int \sec^3 x dx = \int \sec x \cdot \sec^2 x dx$

Let us apply integration by parts

Choose, $u = \sec x \qquad dv = \sec^2 x dx$

$$du = \sec x \tan x dx \quad \int dv = \int \sec^2 x dx \Rightarrow v = \tan x$$

$$\int \sec^3 x dx = \int u dv = uv - \int v du$$

$$= \sec x \cdot \tan x - \int \tan x \cdot \sec x \cdot \tan x dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$\int \sec^3 x dx + \int \sec^3 x dx = \sec x \tan x + \log(\sec x + \tan x) + c$$

$$2 \int \sec^3 x dx = \sec x \tan x + \log(\sec x + \tan x) + c$$

$$\int \sec^3 x dx = \frac{1}{2} [\sec x \tan x + \log(\sec x + \tan x)] + c.$$

3.8 Improper integrals

Definition of an improper integral Type-I

(i) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).

(ii) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

(iii) If both $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{a} f(x)dx$ are convergent then we define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx.$$

Here a is any real number.

Worked Examples

Example 3.83. Discuss the convergence of $\int_0^{\infty} e^{-x}dx$.

Solution. $$\int_0^{\infty} e^{-x}dx = \lim_{t\to\infty} \int_0^t e^{-x}dx$$

$$= \lim_{t\to\infty} \left(\frac{e^{-x}}{-1}\right)_0^t = -\lim_{t\to\infty} \left(e^{-t} - e^0\right) = -\lim_{t\to\infty} \left(\frac{1}{e^t} - 1\right) = -(-1) = 1$$

$$\therefore \int_0^{\infty} e^{-x}dx \text{ is convergent.}$$

Example 3.84. Determine whether the integral $\int_1^{\infty} \frac{1}{x^2}dx$ is convergent or divergent.

Solution. $$\int_1^{\infty} \frac{1}{x^2}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}dx$$

$$= \lim_{t\to\infty} \int_1^t x^{-2}dx$$

$$= \lim_{t\to\infty} \left(\frac{x^{-1}}{-1}\right)_1^t = -\lim_{t\to\infty} \left(\frac{1}{x}\right)_1^t = -\lim_{t\to\infty} \left(\frac{1}{t} - 1\right) = -(-1) = 1.$$

$$\therefore \int_1^{\infty} \frac{1}{x^2}dx \text{ is convergent.}$$

Example 3.85. Determine the convergence of $\int_1^{\infty} \frac{1}{x}dx$.

Solution. $\displaystyle\int_1^{\infty} \frac{1}{x}dx = \lim_{t\to\infty}\int_1^{t} \frac{1}{x}dx$

$$= \lim_{t\to\infty}(\log x)_1^t = \lim_{t\to\infty}(\log t - \log 1) = \lim_{t\to\infty}\log t = \infty$$

i.e the limit does not exist, as a finite number.

$\therefore \int_1^{\infty} \frac{1}{x}dx$ is divergent.

Example 3.86. Discuss the convergence of $\int_0^{\infty} \frac{1}{a^2+x^2}dx$.

Solution. $\displaystyle\int_0^{\infty} \frac{1}{a^2+x^2}dx = \lim_{t\to\infty}\int_0^{t} \frac{1}{a^2+x^2}dx$

$$= \lim_{t\to\infty}\frac{1}{a}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]_0^t$$

$$= \lim_{t\to\infty}\frac{1}{a}\left[\tan^{-1}\frac{t}{a} - \tan^{-1}0\right]$$

$$= \frac{1}{a}(\tan^{-1}\infty - 0)$$

$$= \frac{1}{a}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2a} \qquad \text{which is a finite number.}$$

$\therefore \int_0^{\infty} \frac{1}{a^2+x^2}dx$ is convergent.

Example 3.87. Determine whether the integral $\int_1^{\infty} \frac{dx}{\sqrt{x}}$ convergent or divergent

Solution. $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x}} = \lim_{t\to\infty}\int_1^{t} x^{-\frac{1}{2}}dx = \lim_{t\to\infty}\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)_0^t = 2\lim_{t\to\infty}(\sqrt{t}-1) = 2\cdot\infty = \infty.$

Here the limit does not exist as a finite number.

$\therefore \int_1^{\infty} \frac{1}{\sqrt{x}}dx$ is divergent.

Example 3.88. Discuss the convergence of the integral $\int_1^{\infty} \log x dx$.

Solution. $\displaystyle\int_1^\infty \log x\,dx = \lim_{t\to\infty}\int_1^t \log x dx$

$$= \lim_{t\to\infty}\left[(\log x\cdot x)_0^t - \int_1^t x\cdot\frac{1}{x}dx\right] \qquad \text{[using integration by parts]}$$

$$= 2\lim_{t\to\infty}\left[t\log t-(x)_1^t\right] = 2\lim_{t\to\infty}[t\log t - t + 1] = \infty$$

$\therefore \int_1^\infty \log x dx$ is divergent.

Example 3.89. Determine whether the integral $\displaystyle\int_1^\infty \frac{\log x}{x}dx$ is convergent or divergent. Evaluate if the integral is convergent. [A.U. Dec. 2015]

Solution. $\displaystyle\int_1^\infty \frac{\log x}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\log x}{x}dx = \lim_{t\to\infty}\int_1^t \log x d(\log x)$

$$= \lim_{t\to\infty}\left(\frac{(\log x)^2}{2}\right)_1^t$$

$$= \frac{1}{2}\lim_{t\to\infty}\left((\log t)^2-(\log 1)^2\right)$$

$$= \frac{1}{2}\lim_{t\to\infty}(\log t)^2 = \infty, \qquad \text{[which is not a finite number.]}$$

$\therefore \displaystyle\int_1^\infty \frac{\log x}{x}\,dx$ is divergent.

Example 3.90. For what values of p is the integral $\int_1^\infty \frac{1}{x^p}dx$ convergent?

Solution. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}dx$

$$= \lim_{t\to\infty}\left(\frac{x^{-p+1}}{-p+1}\right)_1^t$$

$$= \frac{1}{1-p}\lim_{t\to\infty}(t^{1-p}-1) = \frac{1}{1-p}\cdot\left(\frac{1}{t^{p-1}}-1\right).$$

If $p>1$, then $p-1>0$ and hence $t^{p-1}\to\infty$ as $t\to\infty$ and hence $\frac{1}{t^{p-1}}\to 0$.

$\therefore \int_1^\infty \frac{1}{x^p} dx = \frac{1}{1-p}(0-1) = \frac{1}{p-1}$, which is a finite number.

$\therefore \int_1^\infty \frac{1}{x^p} dx$ is convergent if $p > 1$.

If $p < 1$, then $p - 1 < 0$ or $1 - p > 0$.

Hence $\frac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t \to \infty$.

$\therefore \int_1^\infty \frac{1}{x^p} dx = \frac{1}{1-p}(\infty - 1) = \infty$.

$\therefore \int_1^\infty \frac{1}{x^p} dx$ diverges, if $p < 1$.

Also, when $p = 1$, $\int_1^\infty \frac{1}{x} dx = \infty$ [Example 3.96]

$\therefore \int_1^\infty \frac{1}{x^p} dx$ is divergent if $p = 1$.

Hence, $\int_1^\infty \frac{1}{x^p} dx$ is convergent if $p > 1$ and divergent if $p \le 1$.

Example 3.91. Discuss the convergence of $\int_{-\infty}^{0} e^x dx$

Solution. $\int_{-\infty}^{0} e^x dx = \lim_{t \to -\infty} \int_t^0 e^x dx$

$$= \lim_{t \to -\infty} [e^x]_t^0 = \lim_{t \to -\infty} [e^0 - e^t] = 1 - 0 = 1, \text{ which is a finite number}$$

$\therefore$ The given integral is convergent.

Example 3.92. Evaluate $\int_{-\infty}^{0} \frac{1}{(1-3x)^2} dx$.

Solution. $\int_{-\infty}^{0} \frac{1}{(1-3x)^2} dx = \lim_{t \to -\infty} \int_t^0 (1-3x)^{-2} dx$

$$= \lim_{t \to -\infty} \left[\frac{(1-3x)^{-1}}{(-1)(-3)} \right]_t^0 = \lim_{t \to -\infty} \frac{1}{3} \left[\frac{1}{1-3x} \right]_t^0 = \frac{1}{3} \lim_{t \to -\infty} [1 - 0] = \frac{1}{3}.$$

Example 3.93. Evaluate $\int_{-\infty}^{0} x e^x dx$.

Solution. $$\int_{-\infty}^{0} xe^x dx = \lim_{t\to-\infty}\int_t^0 x d(e^x)$$
$$= \lim_{t\to-\infty}\left[(x\cdot e^x)_t^0 - \int_t^0 e^x dx\right]$$
$$= \lim_{t\to-\infty}\left[0 - e^t - (e^x)_t^0\right]$$
$$= \lim_{t\to-\infty}\left[-e^t - (1-e^t)\right] = \lim_{t\to-\infty}\left[-e^t - 1 + e^t\right] = \lim_{t\to-\infty}(-1) = -1.$$

Example 3.94. Evaluate $\int_{-\infty}^{\infty} \frac{1}{4+x^2}dx$.

Solution. $$\int_{-\infty}^{\infty} \frac{1}{4+x^2}dx = \int_{-\infty}^{0} \frac{1}{4+x^2}dx + \int_{0}^{\infty} \frac{1}{4+x^2}dx$$
$$= \lim_{t\to-\infty}\int_t^0 \frac{1}{4+x^2}dx + \lim_{t\to-\infty}\int_0^t \frac{1}{4+x^2}dx$$
$$= \lim_{t\to-\infty}\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)_t^0 + \lim_{t\to\infty}\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)_0^t$$
$$= \frac{1}{2}\lim_{t\to-\infty}\left(\tan^{-1}(0) - \tan^{-1}\frac{t}{2}\right)_t^0 + \frac{1}{2}\lim_{t\to\infty}\left(\tan^{-1}\frac{t}{2} - \tan^{-1}(0)\right)_0^t$$
$$= \frac{1}{2}\left(\tan^{-1}0 - \tan^{-1}(-\infty)\right) + \frac{1}{2}\left(\tan^{-1}(\infty) - \tan^{-1}0\right)$$
$$= \frac{1}{2}\left[0 - \left(-\frac{\pi}{2}\right)\right] + \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{1}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = \frac{\pi}{2}.$$

Example 3.95. If $a > 0$, discuss the convergency or divergency of $\int_a^{\infty} \sin x dx$.

Solution. $$\int_a^{\infty} \sin x dx = \lim_{t\to\infty}\int_a^t \sin x dx = \lim_{t\to\infty}\left[-\cos x\right]_a^t = -\lim_{t\to\infty}[\cos t - \cos a].$$

We know that cost is bounded and it lies between -1 and 1.

Hence the integral $\int_a^{\infty} \sin x dx$ oscillates finitely[It neither converges nor diverges].

Example 3.96. Discuss the convergence of $\int_{-\infty}^{0} x\sin x dx$.

Solution.
$$\int_{-\infty}^{0} x\sin x dx = \lim_{t\to-\infty}\int_{t}^{0} xd(-\cos x)$$
$$= \lim_{t\to-\infty}\left[[-x\cos x]_t^0 - \int_t^0 -\cos x dx\right]$$
$$= \lim_{t\to-\infty}[0 - t\cos t + (\sin x)_t^0]$$
$$= \lim_{t\to-\infty}(-t\cos t + 0 - \sin t)$$

Since $\cos x$ and $\sin x$ oscillates between -1 and 1, $\int_{-\infty}^{0} x\sin x dx$ oscillates finitely.

Definition of an improper integral type II

(i) If f is continuous on $[a, b)$ and is discontinuous at b, then
$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$$
$$= \lim_{\in\to 0}\int_a^{b-\in} f(x)dx$$
if the limit exists (as a finite number).

(ii) If f is continuous on $(a, b]$ and is discontinuous at a, then
$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$
$$= \lim_{\in\to 0}\int_{a+\in}^{b} f(x)dx$$
if the limit exists (as a finite number).

The improper integral $\int_a^b f(x)dx$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.

(iii) If f hss a discontinuity at c where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Worked Examples

Example 3.97. Evaluate $\int_0^1 \frac{1}{\sqrt{x}}dx$ if possible.

Solution. The integrand has an infinite discontinuity at the lower limit 0.

$$\therefore \int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{\epsilon\to 0}\int_{0+\epsilon}^1 x^{-\frac{1}{2}}dx = \lim_{\epsilon\to 0}\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)_\epsilon^1 = 2\lim_{\epsilon\to 0}(1-\sqrt{\epsilon}) = 2.$$

Example 3.98. Determine whether the integral $\int_0^3 \frac{dx}{\sqrt{x}}$ is convergent or divergent. Evaluate if it is convergent. [A.U. Dec. 2015]

Solution. The integrand has an infinite discontinuity at the lower limit 0.

$$\therefore \int_0^3 \frac{1}{\sqrt{x}}dx = \lim_{\epsilon\to 0}\int_{0+\epsilon}^3 x^{-\frac{1}{2}}dx = \lim_{\epsilon\to 0}\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)_\epsilon^3 = 2\lim_{\epsilon\to 0}(\sqrt{3}-\sqrt{\epsilon}) = 2\sqrt{3},$$

[which is a finite number.]

Hence, the given integral is convergent and its values is $2\sqrt{3}$.

Example 3.99. Find the value of the improper integral $\int_2^5 \frac{1}{\sqrt{x-2}}dx$ if possible.

Solution. The integrand has an infinite discontinuity at the lower limit 2.

$$\therefore \int_2^5 \frac{1}{\sqrt{x-2}}dx = \lim_{\epsilon\to 0}\int_{2+\epsilon}^5 (x-2)^{-\frac{1}{2}}dx = \lim_{\epsilon\to 0}\left(\frac{(x-2)^{\frac{1}{2}}}{\frac{1}{2}}\right)_{2+\epsilon}^5 = 2\lim_{\epsilon\to 0}(\sqrt{3}-\sqrt{\epsilon}) = 2\sqrt{3}.$$

Example 3.100. Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ if possible.

Solution. The integrand has an infinite discontinuity at the upper limit 1.

$$\therefore \int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \lim_{\epsilon\to 0}\int_0^{1-\epsilon} \frac{1}{\sqrt{1-x^2}}dx$$
$$= \lim_{\epsilon\to 0}(\sin^{-1} x)_0^{1-\epsilon}$$
$$= \lim_{\epsilon\to 0}\left(\sin^{-1}(1-\epsilon) - \sin^{-1} 0\right) = \sin^{-1} 1 - 0 = \frac{\pi}{2}.$$

Example 3.101. Discuss the convergence of $\int_0^{\frac{\pi}{2}} \sec x dx$ if possible.

Solution. The integrand has an infinite discontinuity at the upper limit $\frac{\pi}{2}$.

$$\therefore \int_0^{\frac{\pi}{2}} \sec x dx = \lim_{\epsilon\to 0}\int_0^{\frac{\pi}{2}-\epsilon} \sec x dx$$
$$= \lim_{\epsilon\to 0}(\log(\sec x + \tan x))_0^{\frac{\pi}{2}-\epsilon}$$
$$= \lim_{\epsilon\to 0}\left(\log(\sec(\frac{\pi}{2}-\epsilon) + \tan(\frac{\pi}{2}-\epsilon)) - \log(\sec 0 + \tan 0)\right)$$
$$= \log(\sec(\frac{\pi}{2}) + \tan(\frac{\pi}{2})) - \log 1 = \log\infty = \infty.$$

$\therefore \int_0^{\frac{\pi}{2}} \sec x dx$ diverges.

Example 3.102. Evaluate the improper integral $\int_2^3 \frac{1}{\sqrt{3-x}}dx$ if possible.

[A.U.Nov 2016]

Solution. The integrand has an infinite discontinuity at the upper limit 3.

$$\therefore \int_2^3 \frac{1}{\sqrt{3-x}}dx = \lim_{\epsilon\to 0}\int_2^{3-\epsilon} (3-x)^{-\frac{1}{2}}dx = \lim_{\epsilon\to 0}\left[\frac{(3-x)^{\frac{1}{2}}}{-\frac{1}{2}}\right]_2^{3-\epsilon}$$
$$= \lim_{\epsilon\to 0}(-2)\left(\sqrt{3-(3-\epsilon)} - \sqrt{3-2}\right)$$
$$= -2\lim_{\epsilon\to 0}\left(\sqrt{3-3+\epsilon} - 1\right) = -2(0-1) = 2.$$

Example 3.103. Evaluate $\int_{-1}^1 \frac{1}{x^{\frac{2}{3}}}dx$ if possible.

Solution. The integrand has an infinite discontinuity at $x = 0$,which lies between

-1 and 1.

$$\therefore \int_{-1}^{1} \frac{1}{x^{\frac{2}{3}}} dx = \int_{-1}^{0} \frac{1}{x^{\frac{2}{3}}} dx + \int_{0}^{1} \frac{1}{x^{\frac{2}{3}}} dx$$

$$= \lim_{\epsilon \to 0} \int_{-1}^{0-\epsilon} x^{-\frac{2}{3}} dx + \lim_{\epsilon \to 0} \int_{0+\epsilon}^{1} x^{-\frac{2}{3}} dx$$

$$= \lim_{\epsilon \to 0} \left(\frac{x^{\frac{1}{3}}}{\frac{1}{3}} \right)_{-1}^{-\epsilon} + \lim_{\epsilon \to 0} \left(\frac{x^{\frac{1}{3}}}{\frac{1}{3}} \right)_{\epsilon}^{1}$$

$$= 3 \lim_{\epsilon \to 0} \left((-\epsilon)^{\frac{1}{3}} + 1 \right) + 3 \lim_{\epsilon \to 0} \left(1 - (\epsilon)^{\frac{1}{3}} \right) = 3 \times 1 + 3 \times 1 = 6.$$

Example 3.104. Evaluate the improper integral $\int_{0}^{3} \frac{1}{x-1} dx$ if possible.

Solution. The integrand has an infinite discontinuity at $x = 1$ which lies between 0 and 3.

$$\therefore \int_{0}^{3} \frac{1}{x-1} dx = \int_{0}^{1} \frac{1}{x-1} dx + \int_{1}^{3} \frac{1}{x-1} dx$$

$$= \lim_{\epsilon \to 0} \int_{0}^{1-\epsilon} \frac{1}{x-1} dx + \lim_{\epsilon \to 0} \int_{1+\epsilon}^{3} \frac{1}{x-1} dx$$

$$= \lim_{\epsilon \to 0} [\log |x-1|]_{0}^{1-\epsilon} + \lim_{\epsilon \to 0} [\log |x-1|]_{1+\epsilon}^{3}$$

$$= \lim_{\epsilon \to 0} (\log |\epsilon| - \log |-1| + \log 2 - \log |\epsilon|)$$

$$= \log 0 + \log 2 - \log 0$$

$$= -\infty + \log 2 + \infty \qquad \text{which is an indeterminate}$$

$$\therefore \int_{0}^{3} \frac{1}{x-1} dx \text{ diverges.}$$

Example 3.105. Evaluate $\int_{-1}^{1} \frac{1}{x} dx$ if possible.

Solution. The integrand has an infinite discontinuity at $x = 0$ which lies between

-1 and 1.

$$\therefore \int_{-1}^{1} \frac{1}{x}dx = \int_{-1}^{0} \frac{1}{x}dx + \int_{0}^{1} \frac{1}{x}dx$$

$$= \lim_{\epsilon\to 0} \int_{-1}^{0-\epsilon} \frac{1}{x}dx + \lim_{\epsilon\to 0} \int_{0+\epsilon}^{1} \frac{1}{x}dx$$

$$= \lim_{\epsilon\to 0} \Big[\log|x|\Big]_{-1}^{-\epsilon} + \lim_{\epsilon\to 0} \Big[\log|x|\Big]_{-\epsilon}^{1}$$

$$= \lim_{\epsilon\to 0} (\log|-\epsilon| - \log|-1|) + \lim_{\epsilon\to 0} (\log 1 - \log|\epsilon|)$$

$$= \lim_{\epsilon\to 0} (\log|\epsilon| - \log(\epsilon))$$

$$= \lim_{\epsilon\to 0} \log\left(\frac{|\epsilon|}{|\epsilon|}\right) = \lim_{\epsilon\to 0} \log 1 = 0, \quad \text{which is a finite number}$$

$\therefore \int_{-1}^{1} \frac{1}{x}dx$ converges.

A comparison test for improper integrals.

Comparison theorem. Suppose that f and g are continuous functions with

$f(x) \geq g(x) \geq 0$ for $x \geq a$.

(i) If $\int_{a}^{\infty} f(x)dx$ is convergent, then $\int_{a}^{\infty} g(x)dx$ is convergent.

(ii) If $\int_{a}^{\infty} g(x)dx$ is divergent, then $\int_{a}^{\infty} f(x)dx$ is divergent.

Worked Examples

Example 3.106. Show that $\int_{0}^{\infty} e^{-x^2}dx$ is convergent.

Solution. $\int_{0}^{\infty} e^{-x^2}dx = \int_{0}^{1} e^{-x^2}dx + \int_{1}^{\infty} e^{-x^2}dx$

$\int_{0}^{1} e^{-x^2}dx$ =a definite value and hence it is convergent.

Let us consider the integral $\int_{1}^{\infty} e^{-x^2}dx$.

For all $x \geq 1$,we have $x^2 \geq x$.

$$\Rightarrow -x^2 \leq -x$$

$$\Rightarrow e^{-x^2} \leq e^{-x} \Rightarrow e^{-x} \geq e^{-x^2}.$$

Now $$\int_1^\infty e^{-x}dx = \lim_{t\to\infty}\int_1^t e^{-x}dx = \lim_{t\to\infty}(-e^{-x})_1^t$$

$$= -\lim_{t\to\infty}\left(e^{-t} - e^{-1}\right) = -\left(0 - \frac{1}{e}\right) = \frac{1}{e}, \quad \text{which is finite.}$$

$\therefore \int_1^\infty e^{-x}dx$ converges.

By comparison theorem $\int_0^\infty e^{-x^2}dx$ converges.

$\Rightarrow \int_0^\infty e^{-x^2}dx$ is convergent.

Example 3.107. Prove that the integral $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}dx$ is divergent.

Solution. We have for all $x \geq 1$, $\dfrac{1+e^{-x}}{x} = \dfrac{1}{x} + \dfrac{e^{-x}}{x} > \dfrac{1}{x}$

We know from Example 3.101, $\int_1^\infty \frac{1}{x}dx$ diverges $\quad [\therefore p = 1]$

$\therefore$ By comparison theorem, $\int_1^\infty \dfrac{1+e^{-x}}{x}dx$ is divergent.

Example 3.108. Determine the convergency or divergency of the integral $\displaystyle\int_1^\infty \frac{1}{x^3+1}dx$, using comparison test.

Solution. For all $x \geq 1$, we have, $\dfrac{1}{x^3+1} < \dfrac{1}{x^3} \quad \Rightarrow \dfrac{1}{x^3} > \dfrac{1}{x^3+1}$.

We have from Example 3.101, $\int_1^\infty \frac{1}{x^3}dx$ converges $\quad [p > 1]$.

$\therefore$ By comparison theorem $\int_1^\infty \dfrac{1}{x^3+1}dx$ is convergent.

Example 3.109. Determine the convergency or divergency of the integral $\displaystyle\int_2^\infty \frac{\cos^2 x}{x^2}dx$.

Solution. We know that for all $x \geq 2$ $\cos^2 x \leq 1$.

$$\Rightarrow \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}. \qquad [\because x^2 > 0]$$

$$\text{or } \frac{1}{x^2} \geq \frac{\cos^2 x}{x^2}.$$

By Example 3.101, $\int_2^\infty \frac{1}{x^2} dx$ converges $\qquad [p > 1]$.

$\therefore$ By comparison theorem $\int_2^\infty \frac{\cos^2 x}{x^2} dx$ also converges.

Example 3.110. Discuss the convergency or divergency of the integral $\int_2^\infty \frac{1}{x + e^x} dx$.

Solution. For all $x > 2$, we have $\frac{1}{x + e^x} < \frac{1}{e^x} = e^{-x}$.

$$\Rightarrow e^{-x} > \frac{1}{x + e^x}.$$

$$\text{Now } \int_2^\infty e^{-x} dx = \lim_{t \to \infty} \int_2^t e^{-x} dx = \lim_{t \to \infty} \left(-e^{-x}\right)_2^t$$

$$= -\lim_{t \to \infty} \left(e^{-t} - e^{-2}\right) = -\left(0 - \frac{1}{e^2}\right) = \frac{1}{e^2} \quad \text{which is finite.}$$

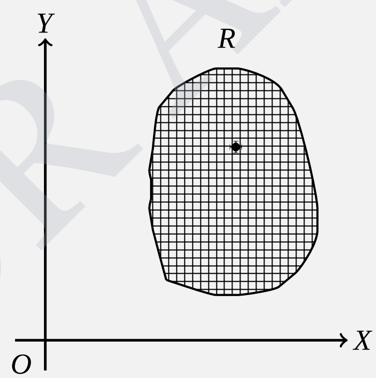
$\therefore$ $\int_2^\infty e^{-x} dx$ is convergent.

$\therefore$ By comparison theorem, $\int_2^\infty \frac{1}{x + e^x} dx$ is convergent.

4 Multiple Integrals

4.1 Evaluation of double integrals

Definition. Let $f(x,y)$ be a single valued and bounded function of two independent variables x and y which is defined in a closed region R of the xy plane. Divide the region R into rectangles by drawing lines parallel to the coordinate axes. Let A_i be the i^{th} rectangle. Let (x_i, y_i) be any point inside the i^{th} rectangle and ΔA_i be its area. Let there be n rectangles $A_1, A_2, \ldots, A_n$ which lie completely inside R.



Let us consider the sum

$$f(x_1,y_1)\Delta A_1 + f(x_2,y_2)\Delta A_2 + \cdots + f(x_n,y_n)\Delta A_n = \sum_{i=1}^{n} f(x_i,y_i)\Delta A_i. \tag{1}$$

As $n \to \infty$, the number of rectangles increase indefinitely such that the largest linear dimension of the rectangle which is the diagonal of ΔA_i tends to zero. The limit of the sum (1) if exists is defined to be the double integral of $f(x,y)$ over the region R and it is denoted by $\iint\limits_R f(x,y)dA$.

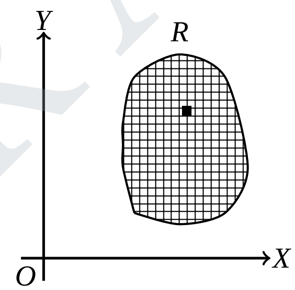
$$\text{i.e.,} \lim_{\substack{n\to\infty \\ \Delta A_i\to 0}} \sum_{i=1}^{n} f(x_i,y_i)\Delta A_i = \iint\limits_R f(x,y)dA.$$

If A is a typical rectangle whose dimensions parallel to the coordinate axes are dx and dy then $dA = dxdy$. Then the above double integral can be written as

$$\iint\limits_R f(x,y)dA = \iint\limits_R f(x,y)dxdy = \iint\limits_R f(x,y)dydx.$$

In cartesian coordinates the following cases are to be taken into account while evaluating the double integrals.

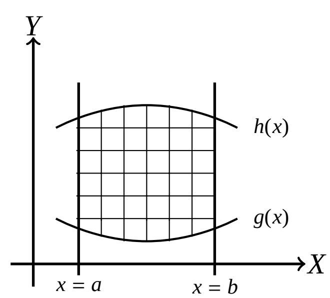
Case (i) Consider the case where the limits are constants. Let the region R be a rectangle given by $R = \{(x,y) : a \le x \le b, c \le y \le d\}$, where a,b,c,d are constants, then, the double integral is given by



$$\int\int_R f(x,y)dxdy = \int_a^b \left[\int_c^d f(x,y)dy\right] dx \text{ or } = \int_c^d \left[\int_a^b f(x,y)dx\right] dy.$$

i.e., if the limits are constants, the order of integration, is immaterial, provided proper limits are taken.

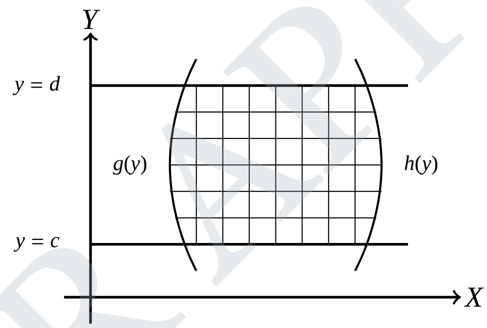
Case (ii) Consider the case when the limits of x are constants and the limits of y are functions of x. In this case the region R is given by $R = \{(x,y) : a \le x \le b, g(x) \le y \le h(x)\}$, where a,b are constants. Now the double integral becomes



$$\iint_R f(x,y)\,dxdy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\,dy\right] dx.$$

i.e., if the limits of the innermost integral are functions of x, then the order of evaluation of the integral is, first with respect to y and finally with respect to x.

Case (iii) Consider the case when the limits of y are constants and the limits of x are functions of y. Now the region R is given by $R = \{(x,y) : g(y) \le x \le h(y), c \le y \le d\}$, where c, d are constants. The double integral now takes the form



$$\iint_R f(x,y)\,dxdy = \int_c^d \left[\int_{g(y)}^{h(y)} f(x,y)\,dx\right] dy.$$

i.e., if the limits of the innermost integral are functions of y, the order of integration is first with respect to x and finally with respect to y.

Worked Examples

Example 4.1. Evaluate $\iint_R dxdy$ over the region R bounded by $x = 0, x = 2, y = 0, y = 2$. [Jan 1996]

Solution. $\iint_R dxdy = \int_0^2 \int_0^2 dxdy.$

Since the limits are constants, the order of integration is immaterial.

$$\therefore \iint_R dxdy = \int_0^2 (x)_0^2 dy = \int_0^2 2dy = 2\int_0^2 dy = 2[y]_0^2 = 4.$$

Example 4.2. Evaluate $\int_0^1 \int_1^2 x(x+y)dydx$. [Jun 1996]

Solution. $\int_0^1 \int_1^2 x(x+y)dydx = \int_0^1 \left(\int_1^2 x(x+y)dy \right) dx$

$$= \int_0^1 \left(x^2(y)_1^2 + x\left(\frac{y^2}{2}\right)_1^2 \right) dx = \int_0^1 \left(x^2 + x\left(2 - \frac{1}{2}\right) \right) dx$$

$$= \int_0^1 \left(x^2 + \frac{3x}{2} \right) dx = \left(\frac{x^3}{3}\right)_0^1 + \frac{3}{2}\left(\frac{x^2}{2}\right)_0^1 = \frac{1}{3} + \frac{3}{4} = \frac{13}{12}.$$

> **Note.** In some cases, the integrand can be expressed as the product of two quantities which are independent. In that case the evaluation of the double integral is equivalent to the product of the integrals involving the independent integrands as the following examples suggest.

Example 4.3. Evaluate $\int_1^a \int_1^b \frac{dxdy}{xy}$.

Solution. $\int_1^a \int_1^b \frac{dx}{x}\frac{dy}{y} = \int_1^a \frac{dx}{x} \int_1^b \frac{dy}{y} = (\log x)_1^a (\log y)_1^b = \log a \log b.$

Example 4.4. Evaluate $\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}}$.

Solution. $\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^1 \frac{dy}{\sqrt{1-y^2}}$

$$= (\sin^{-1} x)_0^1 (\sin^{-1} y)_0^1 = \frac{\pi}{2}\frac{\pi}{2} = \frac{\pi^2}{4}.$$

Example 4.5. Prove that $\int_1^2 \int_3^4 (xy+e^y)dydx = \int_3^4 \int_1^2 (xy+e^y)dxdy$

Solution.
$$\begin{aligned}\int_1^2 \int_3^4 (xy+e^y)dydx &= \int_1^2 \left(x\frac{y^2}{2}+e^y\right)_{y=3}^{y=4} dx\\ &= \int_1^2 \left(\frac{x}{2}(16-9)+(e^4-e^3)\right)dx\\ &= \frac{7}{2}\int_1^2 xdx + \int_1^2 (e^4-e^3)dx\\ &= \frac{7}{2}\left(\frac{x^2}{2}\right)_1^2 + (e^4-e^3)(x)_1^2\\ &= \frac{7}{4}(4-1)+(e^4-e^3)(2-1) = \frac{21}{4}+e^4-e^3. \qquad (1)\end{aligned}$$

$$\begin{aligned}\int_3^4 \int_1^2 (xy+e^y)dxdy &= \int_3^4 \left(y\frac{x^2}{2}+e^y x\right)_{x=1}^{x=2} dy\\ &= \int_3^4 \left(\frac{3}{2}y+e^y\right)dy\\ &= \left(\frac{3}{2}\frac{y^2}{2}+e^y\right)_3^4\\ &= \frac{3}{4}(16-9)+e^4-e^3 = \frac{21}{4}+e^4-e^3. \qquad (2)\end{aligned}$$

From (1) and (2) we have the required result.

Example 4.6. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2}$.

Solution. Since the limits of the innermost integral are functions of x, the order of evaluation must be, first w.r.t. y and finally w.r.t. x.

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2} = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\right] dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} (\tan^{-1} 1) dx = \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$$

$$= \frac{\pi}{4} \log(x + \sqrt{1+x^2})_0^1 = \frac{\pi}{4}(\log(1+\sqrt{2})).$$

Example 4.7. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dx dy.$

Solution. Since the limits of the innermost integral are functions of x, the order of evaluation must be first w.r.t. x and finally w.r.t. y.

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dx dy = \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2 - y^2) - x^2}\, dx dy$$

$$= \int_0^a \left[\frac{x}{2}\sqrt{a^2 - y^2 - x^2} + \frac{a^2 - y^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2 - y^2}}\right]_{x=0}^{x=\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \frac{a^2 - y^2}{2} \sin^{-1}(1) dy$$

$$= \frac{\pi}{4} \int_0^a (a^2 - y^2) dy$$

$$= \frac{\pi}{4} \left(a^2 y - \frac{y^3}{3}\right)_0^a$$

$$= \frac{\pi}{4} \left(a^3 - \frac{a^3}{3}\right) = \frac{\pi}{4} \frac{2a^3}{3} = \frac{\pi a^3}{6}.$$

Example 4.8. Evaluate $\int_0^a \int_0^{\sqrt{a^2+x^2}} \frac{dx dy}{x^2 + y^2 + a^2}.$

Solution. Since the limits of the innermost integral are functions of x, the order of integration must be first w.r.t. y and finally w.r.t. x.

$$\therefore \int_0^a \int_0^{\sqrt{a^2+x^2}} \frac{dxdy}{x^2+y^2+a^2} = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2+x^2}} \frac{dy}{x^2+y^2+a^2} dx.$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2+x^2}} \frac{dy}{\left(\sqrt{x^2+a^2}\right)^2 + y^2} dx.$$

$$= \int_{x=0}^{a} \left(\frac{1}{\sqrt{x^2+a^2}} \tan^{-1} \frac{y}{\sqrt{x^2+a^2}} \right)_0^{\sqrt{a^2+x^2}} dx.$$

$$= \int_{x=0}^{a} \frac{1}{\sqrt{x^2+a^2}} (\tan^{-1} 1 - \tan^{-1} 0) dx$$

$$= \int_{x=0}^{a} \frac{1}{\sqrt{x^2+a^2}} \frac{\pi}{4} dx = \frac{\pi}{4} \cdot \int_{x=0}^{a} \frac{1}{\sqrt{x^2+a^2}} dx$$

$$= \frac{\pi}{4} \left[\log\left(x + \sqrt{x^2+a^2}\right) \right]_0^a$$

$$= \frac{\pi}{4} \left[\log\left(a + \sqrt{2a^2}\right) - \log a \right]$$

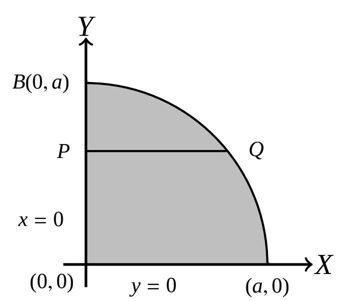
$$= \frac{\pi}{4} \left[\log\left(a + \sqrt{2}a\right) - \log a \right]$$

$$= \frac{\pi}{4} \log \frac{a\left(1+\sqrt{2}\right)}{a} = \frac{\pi}{4} \log\left(1+\sqrt{2}\right).$$

Example 4.9. Evaluate $\iint_R xy dxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. [Jan 2014, Jun 1996]

Solution. The region of integration is bounded by the coordinate axes $y = 0, x = 0$ and the circle $x^2 + y^2 = a^2$.

Since the innermost integration is w.r.t. x, divide the region into strips parallel to the x-axis. Along a typical strip, the y–coordinate is constant but x varies from 0 to $\sqrt{a^2 - y^2}$. Finally y varies from 0 to a.

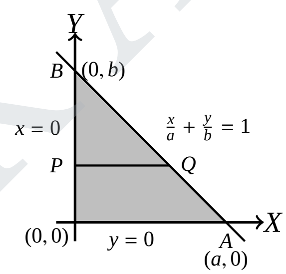


$$\therefore \iint_R xydxdy = \int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} xydxdy = \int_{y=0}^{a} y\left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2}\int_{y=0}^{a} y(a^2-y^2)dy = \frac{1}{2}\left[a^2\frac{y^2}{2} - \frac{y^4}{4}\right]_0^a = \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{a^4}{4}\left[1-\frac{1}{2}\right] = \frac{a^4}{8}.$$

Example 4.10. Evaluate $\iint xydxdy$ over the region in the positive quadrant bounded by $\frac{x}{a} + \frac{y}{b} = 1$.

Solution. The region of integration is bounded by the coordinate axes $y = 0, x = 0$ and the line $\frac{x}{a} + \frac{y}{b} = 1$.



Since the innermost integration is w.r.t. x, divide the region into strips parallel to the x-axis. Along a typical strip PQ, x varies from 0 to $a\left(1-\frac{y}{b}\right)$. y varies from 0 to b.

$$\therefore \iint_R xydxdy = \int_{y=0}^{b}\int_{x=0}^{a(1-\frac{y}{b})} xydxdy$$

$$= \int_{y=0}^{b} y\cdot\left(\frac{x^2}{2}\right)_0^{a(1-\frac{y}{b})} dy$$

$$= \frac{1}{2}\int_{y=0}^{b} y\left[a^2\left(1-\frac{y}{b}\right)^2 - 0\right]dy$$

$$= \frac{a^2}{2}\int_{y=0}^{b} y\left(1+\frac{y^2}{b^2}-\frac{2y}{b}\right)dy$$

$$= \frac{a^2}{2}\int_{y=0}^{b} \left(y+\frac{y^3}{b^2}-\frac{2y^2}{b}\right)dy$$

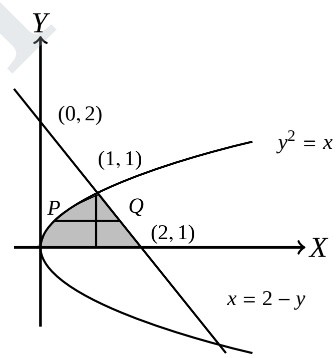
$$= \frac{a^2}{2}\left[\frac{y^2}{2} + \frac{y^4}{4b^2} - \frac{2y^3}{3b}\right]_0^b$$
$$= \frac{a^2}{2}\left[\frac{b^2}{2} + \frac{b^4}{4b^2} - \frac{2b^3}{3b}\right]$$
$$= \frac{a^2}{2}\left[\frac{b^2}{2} + \frac{b^2}{4} - \frac{2b^2}{3}\right]$$
$$= \frac{a^2}{2}\left[\frac{6b^2 + 3b^2 - 8b^2}{12}\right]$$
$$= \frac{a^2}{2}\frac{b^2}{12} = \frac{a^2b^2}{24}$$

Example 4.11. Evaluate $\iint\limits_R xydxdy$ where R is the region bounded by the parabola $y^2 = x$, the x–axis and the line $x + y = 2$ lying on the first quadrant.

Solution. Solving $y^2 = x$ and $x + y = 2$ we get the points of intersection.

$$y^2 + y - 2 = 0 \Rightarrow (y+2)(y-1) = 0 \Rightarrow y = 1, -2. \text{ When } y = 1, x = 1.$$

Divide the region into strips parallel to the x–axis . Along a typical strip, x varies from y^2 to $2-y$ and finally y varies from 0 to 1.



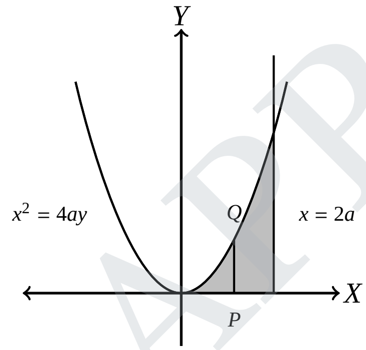
$$\iint\limits_R xydxdy = \int\limits_{y=0}^{1}\int\limits_{x=y^2}^{2-y} xydxdy = \int\limits_{y=0}^{1} y\left(\frac{x^2}{2}\right)_{y^2}^{2-y} dy$$
$$= \int\limits_{y=0}^{1} \frac{y}{2}\left((2-y)^2 - y^4\right)dy = \frac{1}{2}\int\limits_{y=0}^{1} y\left(4 + y^2 - 4y - y^4\right)dy$$
$$= \frac{1}{2}\int\limits_{0}^{1}\left(4y + y^3 - 4y^2 - y^5\right)dy = \frac{1}{2}\left(4\frac{y^2}{2} + \frac{y^4}{4} - 4\frac{y^3}{3} - \frac{y^6}{6}\right)_0^1$$
$$= \frac{1}{2}\left(\frac{4}{2} + \frac{1}{4} - \frac{4}{3} - \frac{1}{6}\right) = \frac{1}{2}\left[\frac{24 + 3 - 16 - 2}{12}\right] = \frac{1}{2} \times \frac{9}{12} = \frac{3}{8}.$$

Example 4.12. Evaluate $\iint\limits_A xydxdy$ where A is the region bounded by $x = 2a$ and the curve $x^2 = 4ay$. [Jan 2006]

Solution.

Let us evaluate this double integral first w.r.t. y and then w.r.t. x. Divide the region into strips parallel to the y–axis. PQ is one such strip.

Along this strip, y varies form 0 to $\frac{x^2}{4a}$ and finally x varies from 0 to $2a$.



$$\iint\limits_A xydxdy = \int\limits_{x=0}^{2a}\int\limits_{y=0}^{\frac{x^2}{4a}} xydydx$$

$$= \int\limits_{x=0}^{2a} x\left[\frac{y^2}{2}\right]_0^{\frac{x^2}{4a}} dx$$

$$= \int\limits_{x=0}^{2a} x\left(\frac{x^4}{32a^2}\right)dx = \int\limits_{x=0}^{2a}\frac{x^5}{32a^2}dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{1}{32a^2\times 6}(64a^6 - 0) = \frac{a^4}{3}.$$

Example 4.13. Evaluate $\iint\limits_R xy(x+y)dxdy$ over the area between $y = x^2$ and $y = x$.

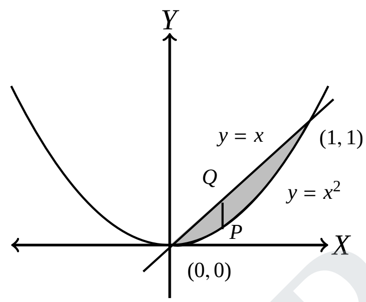
Solution. Solving the two equations we get the points of intersection.

$$y = x^2, y = x.$$

$$\Longrightarrow x^2 = x \Longrightarrow x^2 - x = 0 \Longrightarrow x(x-1) = 0 \Longrightarrow x = 0 \ \&\ x = 1.$$

When $x = 0, y = 0$, when $x = 1, y = 1$.

$\therefore$ The points of intersection are $(0,0)$ and $(1,1)$.



Let us evaluate this double integral, first w.r.t. y and then w.r.t. x. Divide this region into strips parallel to the y–axis. PQ is one such strip. Along this strip, y varies from x^2 to x and x varies from 0 to 1.

$$\iint_R xy(x+y)dxdy = \int_{x=0}^{1}\int_{y=x^2}^{x}(x^2y+xy^2)dydx$$

$$= \int_{x=0}^{1}\left(x\left(\frac{y^3}{3}\right)_{x^2}^{x} + x^2\left(\frac{y^2}{2}\right)_{x^2}^{x}\right)dx$$

$$= \int_{0}^{1}\left(\frac{x}{3}(x^3-x^6)+\frac{x^2}{2}(x^2-x^4)\right)dx$$

$$= \int_{0}^{1}\left(\frac{x^4}{2}-\frac{x^6}{2}+\frac{x^4}{3}-\frac{x^7}{3}\right)dx.$$

$$= \left(\frac{x^5}{10}-\frac{x^7}{14}+\frac{x^5}{15}-\frac{x^8}{24}\right)_0^1 = \frac{1}{10}-\frac{1}{14}+\frac{1}{15}-\frac{1}{24} = \frac{3}{56}.$$

Example 4.14. Evaluate $\iint_R ydxdy$ over the region R bounded by $y = x$ and $y = 4x - x^2$.

Solution. The equation of the parabola is

$$y = -(x^2-4x) = -((x-2)^2-4) = -(x-2)^2+4$$

$$\Rightarrow y-4 = -(x-2)^2$$

vertex is $(2,4)$ and it is open downwards.

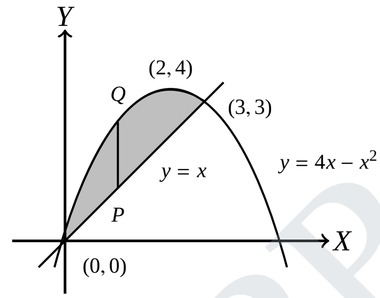
Solving the two equations we get

$x = 4x - x^2 \Rightarrow x^2 + x - 4x = 0 \Rightarrow x^2 - 3x = 0 \Rightarrow x(x-3) = 0 \Rightarrow x = 0, x = 3.$

When $x = 0, y = 0$.

When $x = 3, y = 12 - 9 = 3$.

The points of intersection are $(0,0), (3,3)$. Let us evaluate the double integral first w.r.t. y and then w.r.t. x. Divide the region into strips parallel to the y–axis. PQ is one such strip. Along this strip, y varies from x to $4x - x^2$. Finally x varies from 0 to 3.



$$\iint_R y\,dx\,dy = \int_{x=0}^{x=3}\int_{y=x}^{4x-x^2} y\,dy\,dx = \int_{x=0}^{3}\left(\frac{y^2}{2}\right)_x^{4x-x^2} dx$$

$$= \frac{1}{2}\int_{x=0}^{3}\left((4x-x^2)^2 - x^2\right)dx = \frac{1}{2}\int_0^3\left(16x^2 + x^4 - 8x^3 - x^2\right)dx$$

$$= \frac{1}{2}\left(15\frac{x^3}{3} + \frac{x^5}{5} - 8\frac{x^4}{4}\right)_0^3 = \frac{1}{2}\left[5\times 3^3 + \frac{3^5}{5} - 2\times 3^4\right]$$

$$= \frac{1}{2}27\left(5 + \frac{9}{5} - 6\right) = \frac{27}{2}\left(\frac{25+9-30}{5}\right) = \frac{27}{2}\frac{4}{5} = \frac{54}{5}.$$

4.2 Change of order of integration

Consider the integral $\displaystyle\int_{x=a_1}^{a_2}\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx.$

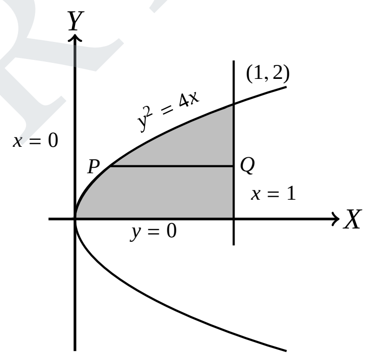
To evaluate this integral, since the innermost integral has functions of x as limits, first we integrate w.r.t. y and then we integrate w.r.t x. Sometimes, it is very difficult to evaluate in this order. By changing the order of integration we can easily evaluate. In this example the original order of integration is first w.r.t. y and then w.r.t. x. By the change of order of integration we need to integrate first w.r.t. x and then w.r.t. y. For this, divide the region into strips parallel to the x-axis. Consider one such strip. The limits for x will be functions of y which are the values of x corresponding to the curves in which the ends of the strip lies. Finally the values of y will be constants according to the movement of this strip along the given region. If the first integration is with respect to y then divide the region into strips parallel to the y–axis and then find the corresponding limits for y and x and then evaluate.

Worked Examples

Example 4.15. Change the order of integration in $\int_0^1 \int_0^{2\sqrt{x}} f(x,y)dydx$. [Jan 2014]

Solution. The region of integration is given by $y = 0, y = 2\sqrt{x}, x = 0, x = 1$.

i.e., $y = 0, y^2 = 4x, x = 0, x = 1$.



By changing the order of integration, the first integration is w.r.t. x. Hence, divide the region into strips parallel to the x–axis. PQ is one such strip. Along this strip, x varies from $\frac{y^2}{4}$ to 1. Finally y varies from 0 to 2.

$$\therefore \int_0^1 \int_0^{2\sqrt{x}} f(x,y)dydx = \int_{y=0}^{2} \int_{x=\frac{y^2}{4}}^{1} f(x,y)dxdy.$$

Example 4.16. Change the order of integration in $\int_0^a \int_y^a \frac{x}{x^2+y^2}dxdy$ and hence evaluate. [Jan 2014, Jun 2013, Jun 2011]

Solution. The region of integration is given by $x = y, x = a, y = 0, y = a$.

i.e., the region of integration is bounded by the lines $x = 0, x = a, y = 0, y = x$ By changing the order of integration, the first integration is w.r.t y. The double integral takes the form

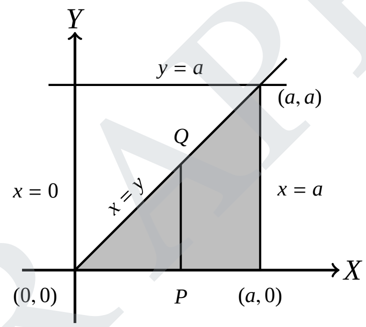
$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dxdy = \iint_R \frac{x}{x^2+y^2}\,dydx.$$

Divide the shaded region into strips parallel to the y-axis. PQ is one such strip. Along this strip, y varies from 0 to x and finally x varies from 0 to a.

$$\therefore \int_0^a \int_y^a \frac{x}{x^2+y^2}\,dxdy = \int_{x=0}^{a} \int_{y=0}^{x} \frac{x}{x^2+y^2}\,dydx$$

$$= \int_{x=0}^{a} x\left(\frac{1}{x}\tan^{-1}\frac{y}{x}\right)_0^x dx$$

$$= \int_{x=0}^{a} \frac{\pi}{4}dx = \frac{\pi}{4}(x)_0^a = \frac{\pi a}{4}.$$



Example 4.17. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dydx$ by changing the order of integration.

[May 2011, Jan 1999]

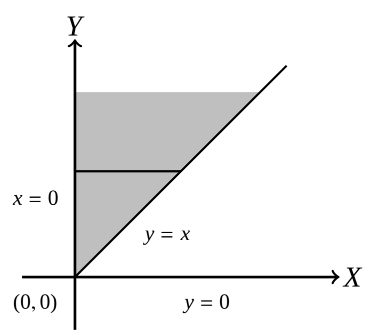
Solution. The region of integration is $y = x, y = \infty, x = 0, x = \infty$.

Here, the first integration is w.r.t y and then we integrate w.r.t x.

By changing the order of integration, the innermost integration must be w.r.t.x.

So, we divide the region of integration into strips parallel to the $x-$axis.

Along a typical strip, x varies from 0 to y and finally y varies from 0 to ∞.



$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dydx = \int_{y=0}^\infty \left(\int_{x=0}^y \frac{e^{-y}}{y} dx \right) dy = \int_{y=0}^\infty \frac{e^{-y}}{y} \left(\int_0^y dx \right) dy$$

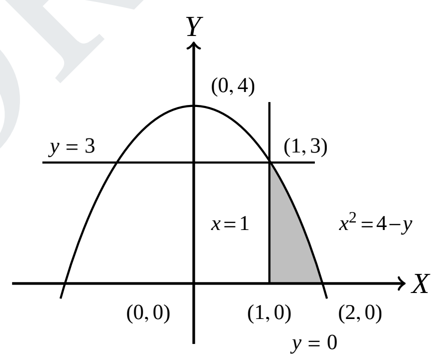
$$= \int_{x=0}^\infty \frac{e^{-y}}{y} (x)_0^y dy = \int_0^\infty e^{-y} dy = \left(\frac{e^{-y}}{-1} \right)_0^\infty = -[e^{-\infty} - e^0] = -[0-1] = 1$$

Example 4.18. Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)dxdy$ by changing the order of integration.

[Jan 2003]

Solution. The region of integration is bounded between $x = 1, x = \sqrt{4-y}$ and $y = 0, y = 3$. i.e.,$x = 1, x^2 = 4 - y$ and $y = 0, y = 3$.
$x = 1, x^2 = -(y-4)$ and $y = 0, y = 3$.

The region is bounded between $x = 1$ and the parabola $x^2 = -(y-4)$ with vertex $(0,4)$ and $y = 0$ and $y = 3$. By changing the order of integration the innermost integration is w.r.t.y. Hence, divide this region into strips parallel to the y-axis. Along a typical strip y varies from 0 to $4 - x^2$ and finally x varies from 1 to 2.



$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y)dxdy = \int_{x=1}^2 \int_{y=0}^{4-x^2} (x+y)dydx = \int_{x=1}^2 \left(xy + \frac{y^2}{2} \right)_0^{4-x^2} dx$$

$$= \int_{x=1}^2 \left(x(4-x^2) + \frac{(4-x^2)^2}{2} \right) dx$$

$$= \int_1^2 \left(4x - x^3 + \frac{16 + x^4 - 8x^2}{2} \right) dx$$

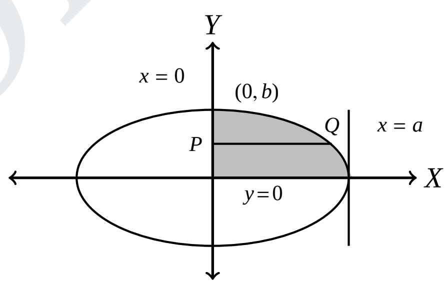
$$= \left(4\frac{x^2}{2} - \frac{x^4}{4} + 8x + \frac{1}{2}\frac{x^5}{5} - 4\frac{x^3}{3}\right)_1^2$$
$$= 2(4-1) - \frac{1}{4}(16-1) + 8(2-1) + \frac{1}{10}(32-1) - \frac{4}{3}(8-1)$$
$$= 6 - \frac{15}{4} + 8 + \frac{31}{10} - \frac{28}{3} = \frac{840-225+186-560}{60} = \frac{241}{60}.$$

Example 4.19. Change the order of integration in $\int_0^a \int_0^{\frac{b\sqrt{a^2-x^2}}{a}} x^2 dydx$ and hence evaluate it. [Jan 2012]

Solution. The region of integration is given by $y = 0, y = \frac{b\sqrt{a^2-x^2}}{a}, x = 0, x = a$.

$$\text{i.e., } y = 0, \quad y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$
$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

After changing the order of integration, the first integration is w.r.t. x and finally w.r.t. y. Divide the region into strips parallel to the $x-$ axis. PQ is one such strip. Along this strip x varies from 0 to $a\sqrt{1-\frac{y^2}{b^2}}$. Finally y varies from 0 to b.



$$\therefore \int_0^a \int_0^{\frac{b\sqrt{a^2-x^2}}{a}} x^2 dydx = \int_{y=0}^{b} \int_{x=0}^{a\sqrt{1-\frac{y^2}{b^2}}} x^2 dxdy$$
$$= \int_{y=0}^{b} \left(\frac{x^3}{3}\right)_0^{a\sqrt{1-\frac{y^2}{b^2}}} dy$$
$$= \frac{1}{3}\int_0^b \frac{a^3}{b^3}(b^2 - y^2)^{3/2} dy$$
$$= \frac{a^3}{3b^3}\int_0^b (b^2 - y^2)^{3/2} dy$$

$$y = b\sin\theta$$
$$dy = b\cos\theta d\theta$$
$$\text{when} \quad y = 0, b\sin\theta = 0$$
$$\Rightarrow \theta = 0$$
$$\text{when} \quad y = b, b\sin\theta = b$$
$$\Rightarrow \sin\theta = 1 \therefore \theta = \frac{\pi}{2}$$

$$= \frac{a^3}{3b^3} \int_0^{\frac{\pi}{2}} (b^2 - b^2 \sin^2\theta)^{3/2} b \cos\theta d\theta$$

$$= \frac{a^3}{3b^3} \int_0^{\frac{\pi}{2}} b^3 (1 - \sin^2\theta)^{3/2} b \cos\theta d\theta$$

$$= \frac{a^3 b}{3} \int_0^{\frac{\pi}{2}} (\cos^2\theta)^{3/2} \cos\theta d\theta$$

$$= \frac{a^3 b}{3} \int_0^{\frac{\pi}{2}} \cos^3\theta \cdot \cos\theta d\theta$$

$$= \frac{a^3 b}{3} \int_0^{\frac{\pi}{2}} \cos^4\theta d\theta$$

$$= \frac{a^3 b}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^3 b}{16}.$$

Example 4.20. Change the order of integration in $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy dx dy$ and then evaluate it. [Jan 2000]

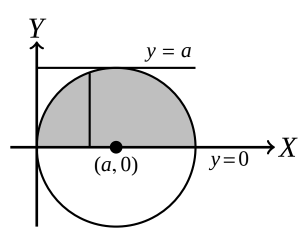
Solution. The region of integration is between

$$x = a - \sqrt{a^2 - y^2}, x = a + \sqrt{a^2 - y^2}, y = 0, y = a.$$

$$\text{i.e.,} (x-a)^2 = a^2 - y^2, (x-a)^2 = a^2 - y^2, y = 0, y = a.$$

$$(x-a)^2 + y^2 = a^2, (x-a)^2 + y^2 = a^2, y = 0, y = a.$$

$\therefore$ The region of integration is the circle with $(a, 0)$ as centre and radius a and between $y = 0$ and $y = a$.



When we change the order of integration, the inner most integration must be w.r.t y.

Hence, divide the region into strips parallel to y–axis. Along a typical strip, y varies from 0 to $\sqrt{a^2-(x-a)^2}$ and finally x varies from 0 to $2a$.

$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\,dx\,dy = \int_{x=0}^{2a} \int_{y=0}^{\sqrt{a^2-(x-a)^2}} xy\,dy\,dx$$

$$= \int_{x=0}^{2a} x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-(x-a)^2}} dx$$

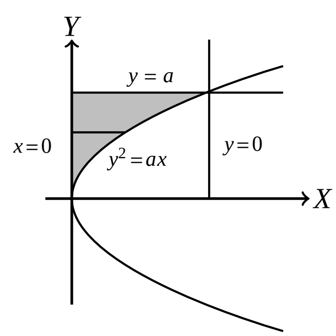
$$= \frac{1}{2}\int_0^{2a} x(a^2-(x-a)^2)dx = \frac{1}{2}\int_0^{2a} (a^2x - x(x^2-2ax+a^2))dx$$

$$= \frac{1}{2}\int_0^{2a} (a^2x - x^3 + 2ax^2 - a^2x)dx = \frac{1}{2}\left(2a\left(\frac{x^3}{3}\right)_0^{2a} - \left(\frac{x^4}{4}\right)_0^{2a}\right)$$

$$= \frac{1}{2}\left[\frac{2a}{3}8a^3 - \frac{16a^4}{4}\right] = \frac{a^4}{2}\left[\frac{16}{3}-4\right] = \frac{1}{2}a^4\frac{4}{3} = \frac{2a^4}{3}.$$

Example 4.21. Change the order of integration and hence evaluate $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4-a^2x^2}}dydx$.

Solution. The region of integration is given by $y = \sqrt{ax}, y = a, x = 0$ and $x = a$. i.e., $y^2 = ax, y = a, x = 0$ and $x = a$. By changing the order of integration, the first integration is w.r.t. x.



Hence, we divide the region into strips parallel to the x-axis. Along a typical strip x varies from 0 to $\frac{y^2}{a}$ and finally y varies from 0 to a.

$$\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4-a^2x^2}}dydx = \int_{y=0}^{a} \int_{x=0}^{\frac{y^2}{a}} \frac{y^2}{\sqrt{y^4-a^2x^2}}dxdy$$

$$= \frac{1}{a}\int_{y=0}^{a}\int_{x=0}^{\frac{y^2}{a}} \frac{y^2}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}} dxdy$$

$$= \frac{1}{a}\int_{y=0}^{a}\left(y^2 \sin^{-1}\left(\frac{x}{\frac{y^2}{a}}\right)\right)_{x=0}^{\frac{y^2}{a}} dy$$

$$= \frac{1}{a}\int_{0}^{a} y^2(\sin^{-1} 1)dy$$

$$= \frac{\pi}{2a}\left(\frac{y^3}{3}\right)_0^a$$

$$= \frac{\pi}{2a}\frac{a^3}{3} = \frac{\pi a^2}{6}.$$

Note. While changing the order of integration, we may come across situations where the region has to be divided into several parts and evaluation has to be done accordingly. The following examples illustrate this aspect.

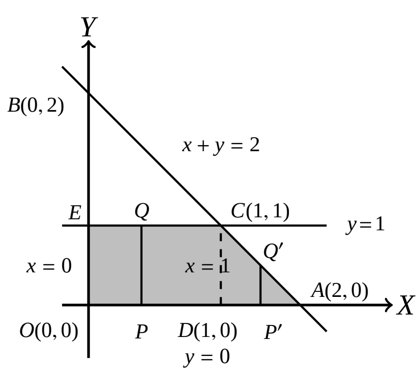
Example 4.22. Change the order of integration and hence evaluate $\int_0^1\int_0^{2-y} xydxdy.$ [Dec 2011]

Solution. The region of integration is given by $x = 0, \quad x = 2 - y, \quad y = 0, y = 1.$

$$x + y = 2$$

The region of integration is $OACE$. After changing the order of integration, the first integration is w.r.t. y and final integration is w.r.t. x.

Divide the region into two regions $ODCE$ and DAC.



$$\therefore \int_0^1\int_0^{2-y} xydxdy = \iint_{ODCE} xydydx + \iint_{DAC} xydydx.$$

Consider the region ODCE

Divide the region into strips parallel to the $y-$axis. PQ is one such strip. Along this strip, y varies from 0 to 1 and finally x varies from 0 to 1.

$$\therefore \iint\limits_{ODCE} xydydx = \int\limits_{x=0}^{1}\int\limits_{y=0}^{1} xydydx.$$

$$= \int\limits_{x=0}^{1} x\cdot\left(\frac{y^2}{2}\right)_0^1 dx$$

$$= \frac{1}{2}\int\limits_{0}^{1} x(1-0)dx$$

$$= \frac{1}{2}\int\limits_{0}^{1} xdx$$

$$= \frac{1}{2}\cdot\left(\frac{x^2}{2}\right)_0^1$$

$$= \frac{1}{4}(1-0)$$

$$= \frac{1}{4}.$$

Consider the region DAC

Divide this region into strips parallel to $y-$axis. $P'Q'$ is one such strip. Along this strip, y varies from 0 to $2-x$ and finally x varies from 1 to 2.

$$\therefore \iint\limits_{DAC} xydydx = \int\limits_{x=1}^{2}\int\limits_{y=0}^{2-x} xydydx.$$

$$= \int\limits_{x=1}^{2} x\cdot\left(\frac{y^2}{2}\right)_0^{2-x} dx.$$

$$= \frac{1}{2}\int\limits_{1}^{2} x(2-x)^2 dx.$$

$$= \frac{1}{2}\int\limits_{0}^{2} x(4+x^2-4x)dx.$$

$$= \frac{1}{2}\int_0^2 (4x + x^3 - 4x^2)dx.$$

$$= \frac{1}{2}\left(4\cdot\left(\frac{x^2}{2}\right)_1^2 + \left(\frac{x^4}{4}\right)_1^2 - 4\cdot\left(\frac{x^3}{3}\right)_1^2\right)$$

$$= \frac{1}{2}\left(2[4-1] + \frac{1}{4}[16-1] - \frac{4}{3}(8-1)\right)$$

$$= \frac{1}{2}\left(6 + \frac{15}{4} - \frac{28}{3}\right)$$

$$= \frac{1}{2}\left(\frac{72+45-112}{12}\right)$$

$$= \frac{1}{24}(5) = \frac{5}{24}$$

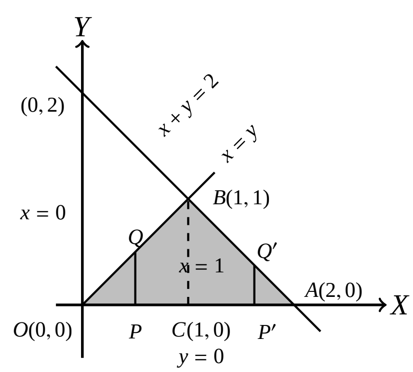
$$\therefore \int_0^1\int_0^{2-y} xydxdy = \frac{1}{4} + \frac{5}{24} = \frac{6+5}{24} = \frac{11}{24}.$$

Example 4.23. Change the order of integration in $\int_0^1\int_y^{2-y} xydxdy$ and hence evaluate. [Jan 2001]

Solution. The region of integration is $x = y, x = 2-y, y = 0, y = 1$.

i.e.,$x = y, x + y = 2, y = 0, y = 1$. Solving we get $2x = 2 \Rightarrow x = 1$.

B is $(1, 1)$, A is $(2, 0)$.



The region of integration is OAB.

After the change of order of integration the innermost integration is w.r.t.y and the next integration is w.r.t.x.

Divide this region into two regions OCB and CBA.

$$\int_0^1\int_y^{2-y} xydxdy = \iint_{OCB} xydydx + \iint_{CBA} xydydx.$$

Consider the region OCB

Divide this region into strips parallel to the y–axis.

One such strip is PQ.

Along this strip y varies from 0 to x.

Finally x varies from 0 to 1.

$$\iint_{OCB} xydydx = \int_{x=0}^{1}\int_{y=0}^{x} xydydx = \int_{x=0}^{1} x\left(\frac{y^2}{2}\right)_0^x dx$$

$$= \frac{1}{2}\int_0^1 x(x^2-0)dx = \frac{1}{2}\int_0^1 x^3 dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4}\right)_0^1$$

$$= \frac{1}{8}.$$

Consider the region CBA

Divide this region into strips parallel to the y–axis.

One such strip is $P'Q'$. Along this strip y varies from 0 to $2-x$ and finally x varies from 1 to 2.

$$\iint_{CBA} xydydx = \int_{x=1}^{2}\int_{y=0}^{2-x} xydydx = \int_{x=1}^{2} x\left(\frac{y^2}{2}\right)_0^{2-x} dx$$

$$= \frac{1}{2}\int_{x=1}^{2} x(2-x)^2 dx = \frac{1}{2}\int_{x=1}^{2} x(4+x^2-4x)dx$$

$$= \frac{1}{2}\int_{x=1}^{2}(4x+x^3-4x^2)dx = \frac{1}{2}\left[4\left(\frac{x^2}{2}\right)_1^2 + \left(\frac{x^4}{4}\right)_1^2 - 4\left(\frac{x^3}{3}\right)_1^2\right]$$

$$= \frac{1}{2}\left[2(4-1)+\frac{1}{4}(16-1)-\frac{4}{3}(8-1)\right] = \frac{1}{2}\left[6+\frac{15}{4}-\frac{28}{3}\right]$$

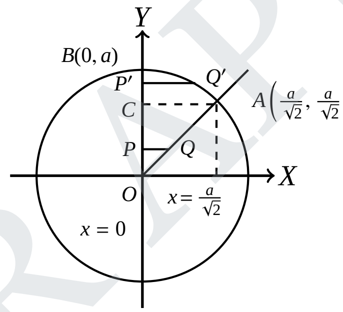
$$= \frac{1}{2}\left(\frac{72+45-112}{12}\right) = \frac{1}{2}\left(\frac{5}{12}\right) = \frac{5}{24}.$$

$$\therefore \int_0^1 \int_y^{2-y} xy\,dx\,dy = \frac{1}{8} + \frac{5}{24} = \frac{3+5}{24} = \frac{8}{24} = \frac{1}{3}.$$

Example 4.24. Evaluate $\int_0^{\frac{a}{\sqrt{2}}} \int_x^{\sqrt{a^2-x^2}} y^2 dy dx$ by changing the order of integration.

Solution. The region of integration is bounded by $y = x, y = \sqrt{a^2 - x^2}, x = 0, x = \dfrac{a}{\sqrt{2}}$.

i.e.,$y = x, x^2 + y^2 = a^2,\ x = 0, x = \dfrac{a}{\sqrt{2}}$.



The required region is OAB. Draw AC perpendicular to OB.

Divide this region into two parts OAC and CAB. After changing the order of integration, the innermost integration must be w.r.t.x.

$$\therefore \int_0^{\frac{a}{\sqrt{2}}} \int_x^{\sqrt{a^2-x^2}} y^2 dy dx = \iint_{OAC} y^2 dx dy + \iint_{CAB} y^2 dx dy.$$

<u>Consider the region OAC</u>

Divide this region into strips parallel to the x−axis.

Let PQ be one such strip.

Along this strip x varies from 0 to y and finally y varies from 0 to $\dfrac{a}{\sqrt{2}}$.

$$\iint_{OAC} y^2 dx dy = \int_{y=0}^{\frac{a}{\sqrt{2}}} \int_0^y y^2 dx dy = \int_{y=0}^{\frac{a}{\sqrt{2}}} y^2 (x)_0^y dy$$

$$= \int_0^{\frac{a}{\sqrt{2}}} y^3 dy = \left(\frac{y^4}{4}\right)_0^{\frac{a}{\sqrt{2}}} = \frac{1}{4}\frac{a^4}{4} = \frac{a^4}{16}.$$

<u>Consider the region CAB</u>

Divide this region into strips parallel to the x−axis.

One such strip is $P'Q'$. Along this strip x varies from 0 to $\sqrt{a^2-y^2}$ and finally y varies from $\frac{a}{\sqrt{2}}$ to a.

$$\iint_{CAB} y^2 dxdy = \int_{y=\frac{a}{\sqrt{2}}}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} y^2 dxdy = \int_{y=\frac{a}{\sqrt{2}}}^{a} y^2 (x)_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_{\frac{a}{\sqrt{2}}}^{a} y^2 \sqrt{a^2-y^2} dy$$

$$\left[y = a\sin\theta, dy = a\cos\theta d\theta, \text{ when } y = \frac{a}{\sqrt{2}} \Rightarrow \sin\theta = \frac{1}{\sqrt{2}}, \theta = \frac{\pi}{4} \right]$$

$$\left[\text{When } y = a \Rightarrow a = a\sin\theta \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2} \right]$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} a^2 \sin^2\theta a\cos\theta a\cos\theta d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} a^4 \sin^2\theta \cos^2\theta d\theta$$

$$= \frac{a^4}{4} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\sin^2\theta\cos^2\theta d\theta = \frac{a^4}{4} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^2 d\theta$$

$$= \frac{a^4}{4} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin^2 2\theta) d\theta = \frac{a^4}{4} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2} d\theta$$

$$= \frac{a^4}{8} \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 4\theta d\theta \right)$$

$$= \frac{a^4}{8} \left((\theta)_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \left(\frac{\sin 4\theta}{4} \right)_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right)$$

$$= \frac{a^4}{8} \left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{4} (\sin 2\pi - \sin\pi) \right)$$

$$= \frac{a^4}{8} \frac{\pi}{4} = \frac{\pi a^4}{32}.$$

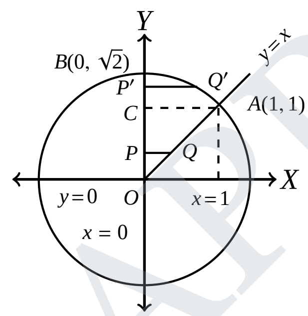
$\therefore$ Value of the integral$= \frac{a^4}{16} + \frac{\pi a^4}{32} = \frac{a^4}{32}[\pi + 2]$.

Example 4.25. Evaluate $\int\limits_0^1 \int\limits_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx$ by changing the order of integration.

Solution. The region of integration is $y = x, y = \sqrt{2-x^2}$, $x = 0, x = 1$.

i.e., $y = x, x^2 + y^2 = 2$, $x = 0, x = 1$.

After changing the order of integration the inner most integration is w.r.t.x and the next integration is w.r.t. y. The given region is OAB. Draw AC perpendicular to OB.



Divide this region into two parts OAC and CAB.

$$\therefore \int\limits_0^1 \int\limits_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx = \iint\limits_{OAC} \frac{x}{\sqrt{x^2+y^2}} dxdy + \iint\limits_{CAB} \frac{x}{\sqrt{x^2+y^2}} dxdy$$

<u>Consider the region OAC</u>

Divide this region into strips parallel to the x–axis.

Let PQ be one such strip. Along this strip x varies from 0 to y and finally y varies from 0 to 1.

$$\iint\limits_{OAC} \frac{x}{\sqrt{x^2+y^2}} dxdy = \int\limits_{y=0}^1 \int\limits_{x=0}^y \frac{x}{\sqrt{x^2+y^2}} dxdy = \frac{1}{2} \int\limits_{y=0}^1 \int\limits_{x=0}^y \frac{2x}{\sqrt{x^2+y^2}} dxdy$$

$$= \frac{1}{2} \int\limits_{y=0}^1 \int\limits_0^y (x^2+y^2)^{-\frac{1}{2}} d(x^2+y^2) dy = \frac{1}{2} \int\limits_{y=0}^1 \left(\frac{(x^2+y^2)^{\frac{1}{2}}}{\frac{1}{2}} \right)_0^y dy$$

$$= \int\limits_{y=0}^1 \left((2y^2)^{\frac{1}{2}} - y \right) dy = \int\limits_{y=0}^1 (\sqrt{2}y - y) dy$$

$$= \sqrt{2} \left(\frac{y^2}{2} \right)_0^1 - \left(\frac{y^2}{2} \right)_0^1 = \frac{\sqrt{2}}{2} - \frac{1}{2} = \frac{1}{2}(\sqrt{2} - 1).$$

Consider the region CAB

Divide this region into strips parallel to the x–axis.

One such strip is $P'Q'$. Along this strip x varies from 0 to $\sqrt{2-y^2}$ and finally y varies from 1 to $\sqrt{2}$.

$$\iint_{CAB} \frac{x}{\sqrt{x^2+y^2}}dxdy = \int_{y=1}^{\sqrt{2}}\int_{x=0}^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}dxdy = \frac{1}{2}\int_{y=1}^{\sqrt{2}}\left(\frac{(x^2+y^2)^{\frac{1}{2}}}{\frac{1}{2}}\right)_0^{\sqrt{2-y^2}} dy$$

$$= \int_{y=1}^{\sqrt{2}} (2^{\frac{1}{2}} - y)dy = \left(\sqrt{2}y - \frac{y^2}{2}\right)_1^{\sqrt{2}}$$

$$= \sqrt{2}(\sqrt{2}-1) - \frac{1}{2}(2-1) = \sqrt{2}(\sqrt{2}-1) - \frac{1}{2}$$

$$\text{Value of the integral} = \frac{\sqrt{2}-1}{2} + \sqrt{2}(\sqrt{2}-1) - \frac{1}{2}$$

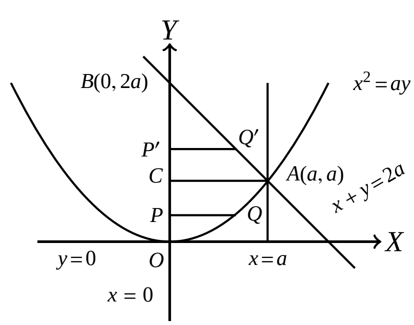
$$= \frac{\sqrt{2}-1+2\sqrt{2}(\sqrt{2}-1)-1}{2}$$

$$= \frac{\sqrt{2}-1+4-2\sqrt{2}-1}{2} = \frac{2-\sqrt{2}}{2}.$$

Example 4.26. Change the order of integration in the integral $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydydx$ and evaluate it. [Jan 2013]

Solution. The region of integration is given by $y = \dfrac{x^2}{a}, y = 2a-x, x = 0, x = a.$

$$x^2 = ay, x+y = 2a, x = 0, x = a.$$



After changing the order of integration, the first integration is w.r.t. y and the final integration is w.r.t. x. OAB is the required region. Divide this region into OAC and CAB.

$$\therefore \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydydx = \iint_{OAC} xydxdy + \iint_{CAB} xydxdy.$$

Consider the region OAC

Divide this region into strips parallel to the x–axis. PQ is one such strip. Along this strip, x varies from 0 to $\sqrt{ay}$ and finally y varies from 0 to a.

$$\therefore \iint_{OAC} xydxdy = \int_{y=0}^{a}\int_{x=0}^{\sqrt{ay}} xydxdy$$

$$= \int_{y=0}^{a} y\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_0^a y\cdot(ay-0)dy$$

$$= \frac{a}{2}\int_0^a y^2dy = \frac{a}{2}\left(\frac{y^3}{3}\right)_0^a = \frac{a}{6}\cdot a^3 = \frac{a^4}{6}.$$

Consider the region CAB

Divide this region into strips parallel to the x–axis. $P'Q'$ is one such strip.

$$\therefore \iint_{CAB} xydxdy = \int_{y=a}^{2a}\int_{x=0}^{2a-y} xydxdy$$

$$= \int_{y=a}^{2a} y\left(\frac{x^2}{2}\right)_0^{2a-y} dy$$

$$= \frac{1}{2}\int_a^{2a} y(2a-y)^2dy$$

$$= \frac{1}{2}\int_a^{2a} y(4a^2+y^2-4ay)dy$$

$$= \frac{1}{2}\int_a^{2a} (4a^2y+y^3-4ay^2)dy$$

$$= \frac{1}{2}\left(4a^2\left(\frac{y^2}{2}\right)_a^{2a} + \left(\frac{y^4}{4}\right)_a^{2a} - 4a\cdot\left(\frac{y^3}{3}\right)_a^{2a}\right)$$

$$= \frac{1}{2}\left(2a^2(4a^2 - a^2) + \frac{1}{4}[16a^4 - a^4] - \frac{4a}{3}[8a^3 - a^3]\right)$$
$$= \frac{1}{2}\left(6a^4 + \frac{15a^4}{4} - \frac{28a^4}{3}\right)$$
$$= \frac{1}{2}\left(\frac{72a^4 + 45a^4 - 112a^4}{12}\right) = \frac{a^4}{24}[72 + 45 - 112] = \frac{5a^4}{24}.$$

$$\therefore \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydydx = \frac{a^4}{6} + \frac{5a^4}{24} = \frac{4a^4 + 5a^4}{24} = \frac{9a^4}{24} = \frac{3a^4}{8}.$$

Example 4.27. Change the order of integration in the integral $\int_0^1 \int_{x^2}^{2-x} xydydx$ and evaluate it.

Solution. In the above Example, put $a = 1$.

$$\therefore \int_0^1 \int_{x^2}^{2-x} xydydx = \frac{3}{8}.$$

4.3 Double integral in polar coordinates

Usually the double integral in polar coordinates (r, θ) is of the form

$$\int_{\theta=\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{f_2(\theta)} f(r,\theta)drd\theta.$$

Result. Reduction formula

$$\int_0^{\frac{\pi}{2}} \cos^n \theta d\theta = \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta$$
$$= \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{2}{3}1 \text{ if } n \text{ is odd}$$
$$= \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{1}{2}\frac{\pi}{2} \text{ if } n \text{ is even.}$$

Worked Examples

Example 4.28. Evaluate $\int_0^{\pi}\int_0^{a} r dr d\theta$. [Jan 2014]

Solution. $\int_0^{\pi}\int_0^{a} r dr d\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^a d\theta = \frac{a^2}{2}\int_0^{\pi} d\theta = \frac{a^2}{2}[\theta]_0^{\pi} = \frac{\pi a^2}{2}$.

Example 4.29. Evaluate $\int_0^{\pi}\int_0^{\sin\theta} r dr d\theta$. [Jan 2014]

Solution. $\int_0^{\pi}\int_0^{\sin\theta} r dr d\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{\sin\theta} d\theta$

$$= \frac{1}{2}\int_0^{\pi}\sin^2\theta d\theta = \frac{1}{2}\frac{1}{2}\frac{\pi}{2} = \frac{\pi}{8}.$$

Example 4.30. Evaluate $\int_0^{\pi}\int_0^{\cos\theta} r dr d\theta$.

Solution. $\int_0^{\pi}\int_0^{\cos\theta} r dr d\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{\cos\theta} d\theta$

$$= \frac{1}{2}\int_0^{\pi}\cos^2\theta d\theta$$

$$= \frac{1}{2}2\int_0^{\frac{\pi}{2}}\cos^2\theta = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}.$$

Example 4.31. Evaluate $\int_0^{2}\int_0^{\pi} r\sin^2\theta dr d\theta$. [Jan 2013]

Solution. $I = \int_0^{2} r\left[\int_0^{\pi}\sin^2\theta d\theta\right] dr$

$$= \int_0^2 r \left[2 \int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta \right] dr$$

$$= 2 \int_0^2 r \cdot \frac{1}{2} \cdot \frac{\pi}{2} dr = \frac{\pi}{2} \cdot \left(\frac{r^2}{2} \right)_0^2 = \frac{\pi}{4} \cdot 4 = 4.$$

Example 4.32. Evaluate $\displaystyle\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 dr d\theta.$

Solution. $I = \displaystyle\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^3}{3} \right)_0^{2\cos\theta} d\theta = \frac{1}{3} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 8\cos^3\theta d\theta = \frac{8}{3} 2 \int_0^{\frac{\pi}{2}} \cos^3\theta d\theta = \frac{16}{3}\frac{2}{3} = \frac{32}{9}.$

Example 4.33. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} r\sqrt{a^2 - r^2} dr d\theta.$

Solution. $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} r\sqrt{a^2 - r^2} dr d\theta = \frac{-1}{2} \int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} (a^2 - r^2)^{\frac{1}{2}} (-2rdr) d\theta$

$$= \frac{-1}{2} \int_0^{\frac{\pi}{2}} \left(\frac{(a^2 - r^2)^{\frac{3}{2}}}{\frac{3}{2}} \right)_0^{a\cos\theta} d\theta$$

$$= \frac{-1}{2}\frac{2}{3} \int_0^{\frac{\pi}{2}} \left((a^2 - a^2\cos^2\theta)^{\frac{3}{2}} - (a^2)^{\frac{3}{2}} \right) d\theta$$

$$= \frac{-1}{3} \int_0^{\frac{\pi}{2}} (a^3\sin^3\theta - a^3) d\theta$$

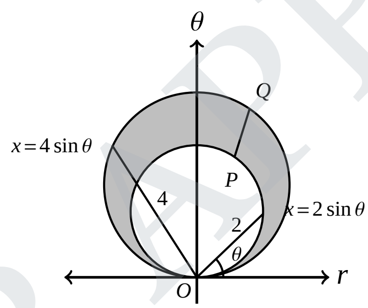
$$= \frac{-1}{3} \left(a^3 \int_0^{\frac{\pi}{2}} \sin^3\theta d\theta - a^3 \int_0^{\frac{\pi}{2}} d\theta \right)$$

$$= \frac{-a^3}{3} \left(\frac{2}{3} 1 - \frac{\pi}{2} \right) = \frac{-a^3}{3} \left(\frac{2}{3} - \frac{\pi}{2} \right)$$

$$= \frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right) = \frac{a^3}{3}\left(\frac{3\pi - 4}{6}\right) = \frac{a^3}{18}(3\pi - 4).$$

Example 4.34. Evaluate $\iint\limits_A r^3 dr d\theta$ where A is the area between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$. [Jan 2006]

Solution. The region of integration is the shaded portion.



The first integration is w.r.t. r.

Consider the radius vector PQ. In the region r varies from $2\sin\theta$ to $4\sin\theta$. θ varies from 0 to π.

$$\iint\limits_A r^3 dr d\theta = \int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3 dr d\theta = \int_{\theta=0}^{\pi}\left(\frac{r^4}{4}\right)_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi}(4^4\sin^4\theta - 2^4\sin^4\theta)d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi}(256\sin^4\theta - 16\sin^4\theta)d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi}240\sin^4\theta d\theta = 60\int_{\theta=0}^{\pi}\sin^4\theta d\theta$$

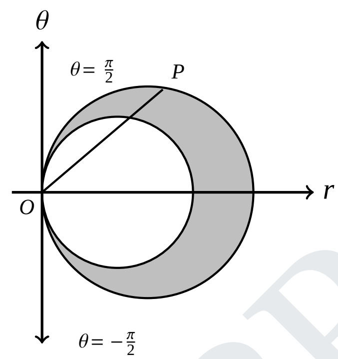
$$= 60 \times 2\int_{\theta=0}^{\frac{\pi}{2}}\sin^4\theta d\theta$$

$$= 120\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{45\pi}{2}.$$

Example 4.35. Evaluate $\iint\limits_R r^3 dr d\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

Solution.

We first integrate w.r.t. r. Consider the radial strip OP. Along the strip r varies from $2\cos\theta$ to $4\cos\theta$. θ varies from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$.



$$\iint_R r^3 dr d\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} r^3 dr d\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^4}{4}\right)_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (4^4\cos^4\theta - 2^4\cos^4\theta)d\theta = \frac{1}{4}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (256\cos^4\theta - 16\cos^4\theta)d\theta$$

$$= \frac{1}{4}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 240\cos^4\theta d\theta = 60(2)\int_0^{\frac{\pi}{2}} \cos^4\theta d\theta = 120\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{45\pi}{2}.$$

4.4 Change of Variables in double integral

The evaluation of the double integral can be made easy in several occasions by changing the variables.

Case (i) Change of variables from (x, y) to u and v.

Let $\iint_R f(x, y)dxdy$ be the given double integral.

Let $x = g(u, v)$ and $y = h(u, v)$. By this transformation the elementary area $dxdy$ is transformed to $dxdy = |J|dudv$ where $J = \frac{\partial(x, y)}{\partial(u, v)}$ is the Jacobian of the transformation.

$$\iint_R f(x, y)dxdy = \iint_R F(u, v)|J|dudv.$$

Case (ii) Change of variables from Cartesian to polar coordinates.

Let $x = r\cos\theta, y = r\sin\theta$.

Then $dxdy = |J|drd\theta$, where $J = \frac{\partial(x, y)}{\partial(u, v)}$.

Now, $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$

$\therefore dxdy = rdrd\theta.$

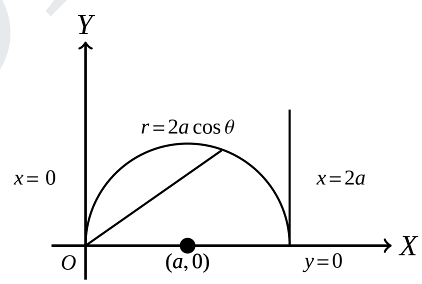
$$\iint_R f(x,y)dxdy = \iint_R F(r,\theta)rdrd\theta.$$

Worked Examples

Example 4.36. By changing into polar coordinates, evaluate the integral
$\displaystyle\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)dydx.$ [Jan 1999]

Solution. The limits for y are $y = 0, y = \sqrt{2ax-x^2} \Rightarrow y^2 = 2ax - x^2$
$x^2+y^2-2ax = 0 \Rightarrow (x-a)^2 - a^2 + y^2 = 0 \Rightarrow (x-a)^2 + y^2 = a^2.$

The limits for x are $x = 0, 2a$.
The region of integration is the upper semi circle, with centre $(a,0)$ and radius a. In polar coordinates
$x = r\cos\theta, y = r\sin\theta, dxdy = rdrd\theta.$
$x^2+y^2-2ax = 0 \Rightarrow r^2 - 2ar\cos\theta = 0$
$\Rightarrow r(r-2a\cos\theta) = 0 \Rightarrow r = 0, r = 2a\cos\theta.$



The limits for r are $r = 0, 2a\cos\theta$.
θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore I = \int_{\theta=0}^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2 rdrd\theta = \int_0^{\frac{\pi}{2}}\left(\frac{r^4}{4}\right)_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 2^4a^4\cos^4\theta d\theta = 4a^4\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{3\pi a^4}{4}.$$

Example 4.37. Transform the integral $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2)dydx$ into polar coordinates and hence evaluate it. [May 2011]

Solution. In the previous problem, put $a = 1$.

Example 4.38. Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}dydx$ by changing into polar coordinates. [Jan 2001]

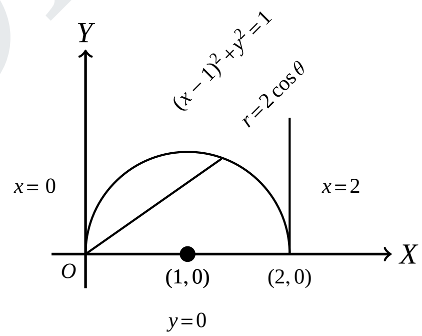
Solution. Let $I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}dydx$.

The limits for y are $y = 0, y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2$. x varies from 0 to 2.

$$x^2 + y^2 - 2x = 0 \Rightarrow (x-1)^2 - 1 + y^2 = 0$$

$$\Rightarrow (x-1)^2 + y^2 = 1.$$

It is a circle with centre $(1, 0)$ and radius $= 1$. The region of integration is the upper semi circle. To change into polar coordinates we have $x = r\cos\theta, y = r\sin\theta$ and $dxdy = rdrd\theta$.



$$\text{Now, } x^2 + y^2 - 2x = 0$$

$$r^2 - 2r\cos\theta = 0 \Rightarrow r(r - 2\cos\theta) = 0$$

$$r = 0, r = 2\cos\theta.$$

$\therefore$ Limits for r are $r = 0, 2\cos\theta$.

θ varies from 0 to $\frac{\pi}{2}$.

$$I = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{r\cos\theta}{\sqrt{r^2}} r dr d\theta = \int_0^{\frac{\pi}{2}} \cos\theta \left(\frac{r^2}{2}\right)_0^{2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos\theta 4\cos^2\theta d\theta = 2\int_0^{\frac{\pi}{2}} \cos^3\theta d\theta$$

$$= 2\frac{2}{3}.1 = \frac{4}{3}.$$

Example 4.39. Evaluate $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} dxdy$ by changing to polar coordinates.

Solution. The limits for x are $x = y, x = a$. Limits for y are $y = 0, y = a$.

The shaded portion is the region of integration.

Changing into polar coordinates we have $x = r\cos\theta, y = r\sin\theta, dxdy = rdrd\theta$.

When $x = a, r\cos\theta = a \Rightarrow r = \frac{a}{\cos\theta}$.

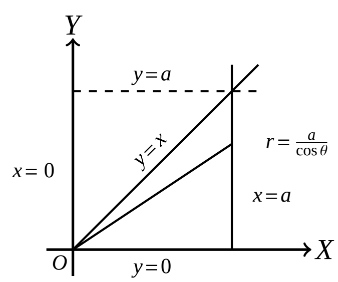
$\therefore r$ varies from 0 to $\frac{a}{\cos\theta}$

θ varies from 0 to $\frac{\pi}{4}$.

$$I = \int_{\theta=0}^{\frac{\pi}{4}} \int_0^{\frac{a}{\cos\theta}} \frac{r^2\cos^2\theta}{r} r dr d\theta = \int_0^{\frac{\pi}{4}} \cos^2\theta \left(\frac{r^3}{3}\right)_0^{\frac{a}{\cos\theta}} d\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} \cos^2\theta \frac{a^3}{\cos^3\theta} d\theta = \frac{1}{3}\int_0^{\frac{\pi}{4}} \sec\theta d\theta$$

$$= \frac{a^3}{3}\left(\log(\sec\theta + \tan\theta)\right)_0^{\frac{\pi}{4}} = \frac{a^3}{3}\left(\log(\sqrt{2}+1)\right).$$



Example 4.40. Transform the integral into polar coordinates and hence evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2} dydx$. [Jun 2012]

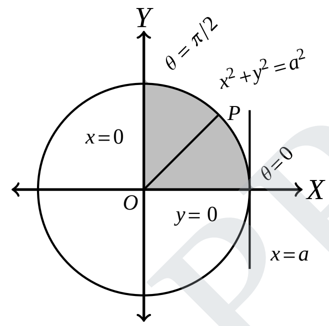
Solution.

The limits for y are $y = 0, y = \sqrt{a^2 - x^2}$

$$y^2 = a^2 - x^2$$

$$x^2 + y^2 = a^2.$$

The limits for x are $x = 0, x = a$.



By changing into polar coordinates we have $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$. In the region of integration, r varies from 0 to a and θ varies from 0 to $\frac{\pi}{2}$.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}dydx = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} \sqrt{r^2}rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^a r^2drd\theta = \int_0^{\frac{\pi}{2}} \left(\frac{r^3}{3}\right)_0^a d\theta = \int_0^{\frac{\pi}{2}} \frac{a^3}{3}d\theta = \frac{a^3}{3}\cdot[\theta]_0^{\frac{\pi}{2}} = \frac{a^3}{3}\frac{\pi}{2} = \frac{\pi a^3}{6}.$$

Example 4.41. Transform the integral $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dxdy}{\sqrt{a^2-x^2-y^2}}$ into polar coordinates and then evaluate it. [Jan 2012]

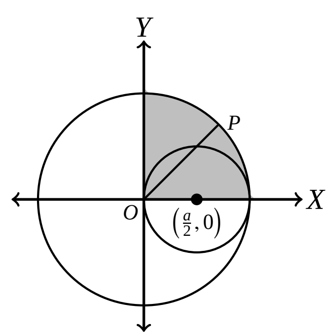
Solution.

The limits for y are

$$y = \sqrt{ax - x^2} \qquad y = \sqrt{a^2 - x^2}$$

$$y^2 = ax - x^2 \qquad y^2 = a^2 - x^2$$

$$x^2 + y^2 - ax = 0 \qquad x^2 + y^2 = a^2,$$



$x^2 + y^2 = a^2$ is a circle with centre $(0, 0)$, radius$= a$. $x^2 + y^2 - ax = 0$ is a circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

The limits for x are $x = 0, x = a$.

Consider the curve $x^2 + y^2 - ax = 0$

By changing into polar coordinates we have,

$$r^2 - a \cdot r\cos\theta = 0$$
$$r(r - a\cos\theta) = 0$$
$$r \neq 0 \Rightarrow \ r = a\cos\theta.$$

Consider the equation $x^2 + y^2 = a^2$. We have $r = a$.

In the region of integration we have r varies from $a\cos\theta$ to a and θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dxdy}{\sqrt{a^2 - x^2 - y^2}} = \int_{\theta=0}^{\frac{\pi}{2}} \int_{a\cos\theta}^{a} \frac{rdrd\theta}{\sqrt{a^2 - r^2}}$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=a\sin\theta}^{0} \frac{-tdt}{t} d\theta$$

$$t^2 = a^2 - r^2$$
$$2tdt = -2rdr.$$
$$rdr = -tdt$$

when $r = a\cos\theta$,

$$t^2 = a^2 - a^2\cos^2\theta$$
$$= a^2(1 - \cos^2\theta)$$
$$= a^2\sin^2\theta.$$
$$t = a\sin\theta.$$

when $r = a$. $\quad t = 0$.

$$= -\int_0^{\frac{\pi}{2}} (t)_{a\sin\theta}^{0} d\theta$$

$$= -\int_0^{\frac{\pi}{2}} (0 - a\sin\theta)d\theta$$

$$= a\int_0^{\frac{\pi}{2}} \sin\theta d\theta$$

$$= a(-\cos\theta)_0^{\frac{\pi}{2}}$$

$$= -a\left(\cos\frac{\pi}{2} - \cos 0\right) = -a(0 - 1) = a.$$

Example 4.42. Evaluate $\iint\limits_R \sqrt{a^2 - x^2 - y^2}dxdy$ where R is the semicircle $x^2 + y^2 = ax$ in the first quadrant by changing to polar coordinates.

Solution. $x^2+y^2-ax=0 \Rightarrow \left(x-\frac{a}{2}\right)^2+y^2-\frac{a^2}{4}=0 \Rightarrow \left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$.
By changing into polar coordinates we get $x=r\cos\theta, y=r\sin\theta, dxdy=rdrd\theta$.
$r^2-ar\cos\theta=0 \Rightarrow r(r-a\cos\theta)=0 \Rightarrow r=0, r=a\cos\theta$.
Limits for r are $0, a\cos\theta$.
Limits for θ are $0, \frac{\pi}{2}$.

$$\text{Now } I=\iint_R \sqrt{a^2-x^2-y^2}\,dxdy=\int_0^{\frac{\pi}{2}}\int_{r=0}^{a\cos\theta}\sqrt{a^2-r^2}\,rdrd\theta$$

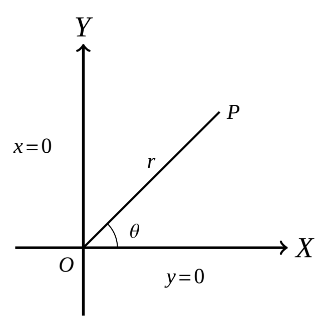
Let $t^2=a^2-r^2 \Rightarrow 2tdt=-2rdr \Rightarrow tdt=-rdr$

when $r=0, t=a$ and when $r=a\cos\theta, t=a\sin\theta$

$$I=\int_0^{\frac{\pi}{2}}\int_a^{a\sin\theta}-ttdtd\theta=-\int_0^{\frac{\pi}{2}}\left(\frac{t^3}{3}\right)_a^{a\sin\theta}d\theta=\frac{-1}{3}\int_0^{\frac{\pi}{2}}(a^3\sin^3\theta-a^3)d\theta$$
$$=\frac{-1}{3}\left(a^3\frac{2}{3}.1-a^3\frac{\pi}{2}\right)=\frac{-a^3}{3}\left(\frac{2}{3}.1-\frac{\pi}{2}\right)=\frac{-a^3}{3}\left(\frac{4-3\pi}{6}\right)=\frac{a^3}{18}(3\pi-4).$$

Example 4.43. Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$ and hence evaluate $\int_0^\infty e^{-x^2}dx$.
[Jan 2014, Jun 2011, Jan 2010, Jan 2004]

Solution. Let $I=\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$.



Since x varies from 0 to ∞ and y varies from 0 to ∞, the region of integration is the entire first quadrant.
Changing to polar coordinates we get $x=r\cos\theta, y=r\sin\theta$ and $dxdy=rdrd\theta$.
Here, r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore I = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy = \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2} rdrd\theta \qquad \text{Let } r^2 = t, 2rdr = dt \Rightarrow rdr = \frac{dt}{2}$$

$$= \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-t}\frac{dt}{2}d\theta \qquad \text{When } r = 0, t = 0, \text{ when } r = \infty, t = \infty$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\frac{e^{-t}}{-1}\right)_0^\infty d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}}(0-1)d\theta = \frac{1}{2}(\theta)_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

To find $\int_0^\infty e^{-x^2}dx$.

We have, $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy = \frac{\pi}{4}$

$$\int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}dxdy = \frac{\pi}{4}$$

$$\int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = \frac{\pi}{4}$$

$$\left(\int_0^\infty e^{-x^2}dx\right)^2 = \frac{\pi}{4}$$

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

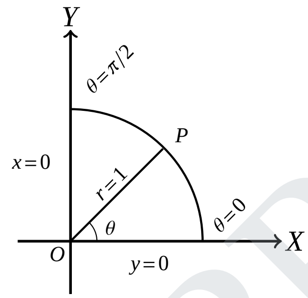
Example 4.44. Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}dxdy$ over the positive quadrant of the circle $x^2+y^2=1$.

Solution.

By changing into polar coordinates we get $x = r\cos\theta, y = r\sin\theta$ and $dxdy = rdrd\theta$.
Now $x^2 + y^2 = 1$

$\implies r^2 = 1$

i.e., $r = 1$. For the region of integration, we have r varies from 0 to 1 and θ varies from 0 to $\frac{\pi}{2}$.



$$\text{Let } I = \iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dxdy$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\,rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 \frac{r(1-r^2)}{\sqrt{1-r^4}}\,drd\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(\int_0^1 \frac{r}{\sqrt{1-r^4}} - \frac{r^3}{\sqrt{1-r^4}}\right)drd\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(\frac{1}{2}\int_0^1 \frac{2r}{\sqrt{1-r^4}}dr + \frac{1}{4}\int_0^1 \frac{-4r^3}{\sqrt{1-r^4}}dr\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(\frac{1}{2}\int_0^1 \frac{dt}{\sqrt{1-t^2}}dr + \frac{1}{4}\int_0^1 \frac{d(1-r^4)}{\sqrt{1-r^4}}\right)d\theta \qquad [t = r^2]$$

$$= \int_0^{\frac{\pi}{2}}\left(\frac{1}{2}(\sin^{-1} t)_0^1 + \frac{1}{4}\left(\frac{(1-r^4)^{\frac{1}{2}}}{\frac{1}{2}}\right)_0^1\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(\frac{1}{2}\frac{\pi}{2} + \frac{1}{4}(0-2)\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{4} - \frac{1}{2}\right) d\theta$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\right) (\theta)_0^{\frac{\pi}{2}}$$

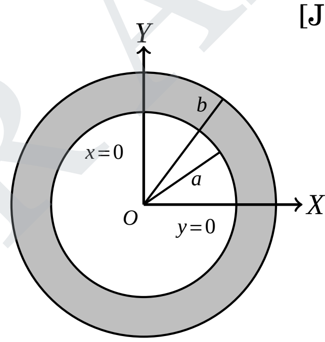
$$= \left(\frac{\pi}{4} - \frac{1}{2}\right) \frac{\pi}{2}$$

$$= \frac{\pi^2}{8} - \frac{\pi}{4}.$$

Example 4.45. By transforming into polar coordinates evaluate $\iint \frac{x^2y^2}{x^2+y^2} dxdy$ over the annular region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$, $(b>a)$ [Jan 2013]

Solution. By changing into polar coordinates we get $x = r\cos\theta, y = r\sin\theta$ and $dxdy = rdrd\theta, x^2+y^2=r^2$.

In the region of integration we have r varies from a to b and θ varies from 0 to 2π.



$$\iint \frac{x^2y^2}{x^2+y^2} dxdy = \int_{\theta=0}^{2\pi} \int_{r=a}^{b} \frac{r^2\cos^2\theta \cdot r^2\sin^2\theta}{r^2} \cdot rdrd\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=a}^{b} r^3\cos^2\theta\sin^2\theta drd\theta$$

$$= \int_0^{2\pi} \cos^2\theta\sin^2\theta \cdot \left(\frac{r^4}{4}\right)_{r=a}^{b} d\theta.$$

$$= \frac{1}{4}\int_0^{2\pi} \cos^2\theta\sin^2\theta(b^4-a^4)d\theta$$

$$= \frac{b^4 - a^4}{4} \int_0^{2\pi} (\sin\theta\cos\theta)^2 d\theta$$

$$= \frac{b^4 - a^4}{4} \int_0^{2\pi} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= \frac{b^4 - a^4}{16} \int_0^{2\pi} \sin^2 2\theta d\theta.$$

$$= \frac{b^4 - a^4}{16} \int_0^{2\pi} \frac{1 - \cos 4\theta}{2} d\theta$$

$$= \frac{b^4 - a^4}{32} \left[\int_0^{2\pi} d\theta - \int_0^{2\pi} \cos 4\theta d\theta\right]$$

$$= \frac{b^4 - a^4}{32} \left([\theta]_0^{2\pi} - \left[\frac{\sin 4\theta}{4}\right]_0^{2\pi}\right) = \frac{b^4 - a^4}{32}(2\pi) = \frac{(b^4 - a^4)\pi}{16}$$

4.5 Area enclosed by plane curves

Result. Area of a region R is given by $\iint_R dxdy$.

Worked Examples

Example 4.46. Find the area of the circle of radius a by double integration.

[Jan 2006]

Solution. The circle is $x^2 + y^2 = a^2$.

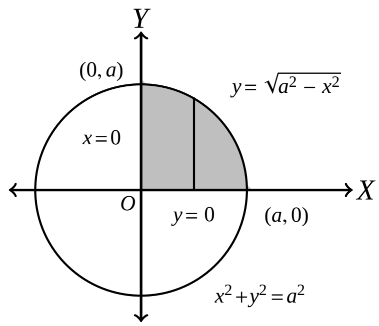
By symmetry Area $= 4\times$ area in the first quadrant.

Divide the region in the first quadrant into strips parallel to the y-axis. Along a typical strip, y varies from 0 to $\sqrt{a^2 - x^2}$ and finally x varies from 0 to a.

$$\therefore \text{ Area } = 4 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} dydx = 4 \int_{x=0}^{a} (y)_0^{\sqrt{a^2-x^2}} dx$$

$$= 4 \int_0^a \sqrt{a^2 - x^2} dx = 4\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)_0^a$$

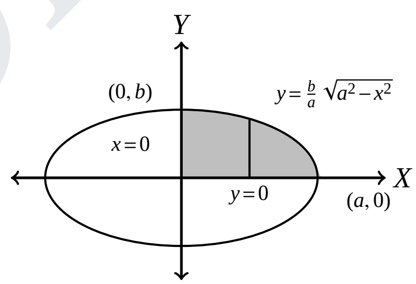
$$= 4\left(\frac{a^2}{2}\sin^{-1}(1)\right) = 4\frac{a^2}{2}\frac{\pi}{2} = \pi a^2.$$



Example 4.47. Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ using double integration. [Jun 2013]

Solution.

Consider the area in the first quadrant. Divide this area into strips parallel to the y-axis. Along a typical strip, y varies from 0 to $\dfrac{b}{a}\sqrt{a^2 - x^2}$ and finally x varies from 0 to a.



By symmetry, Area of the ellipse = 4× Area in the first quadrant.

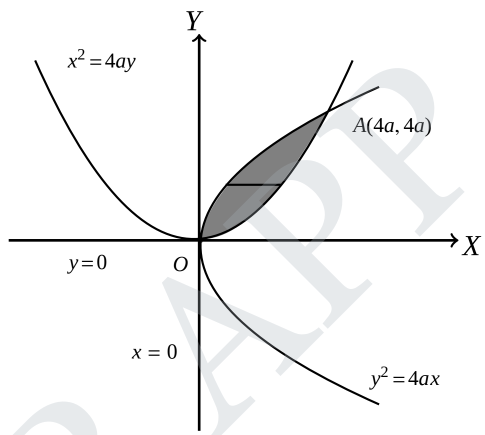
$$\text{Area} = 4\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dydx = 4\int_{x=0}^{a} (y)_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}dx = \frac{4b}{a}\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)_0^a$$

$$= \frac{4b}{a}\frac{a^2}{2}\sin^{-1}(1) = 2ab\frac{\pi}{2} = \pi ab.$$

Example 4.48. Using double integral, find the area bounded by the parabola $y^2 = 4ax$ and $x^2 = 4ay$ [Jun 2013, May 2011, Jan 2010]

Solution.



To find the points of intersection, solve the two equations.

$$y^2 = 4ax \quad (1)$$

$$x^2 = 4ay \quad (2)$$

$$(2)^2 \Rightarrow x^4 = 16a^2y^2$$
$$= 16a^2 \cdot 4ax$$
$$= 64a^3x$$
$$x^4 - 64a^3x = 0$$
$$x(x^3 - 64a^3) = 0$$
$$x = 0, \; x^3 - 64a^3 = 0$$
$$x^3 = 64a^3 \Rightarrow x = 4a.$$

When $x = 0, y = 0$. One point of intersection is O(0, 0).

When $x = 4a, 4ay = 16a^2$

$$y = \frac{16a^2}{4a} = 4a.$$

$\therefore$ The other point of intersection is $A(4a, 4a)$.

The shaded portion is the required region. Divide the region into strips parallel to the $x-$axis. Along one such strip, x varies form $\frac{y^2}{4a}$ to $\sqrt{4ay}$ and finally y varies form 0 to $4a$.

$$\therefore \text{ Required area } = \int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{\sqrt{4ay}} dxdy$$

$$= \int_{y=0}^{4a} [x]_{\frac{y^2}{4a}}^{\sqrt{4ay}} dy$$

$$= \int_{0}^{4a} \left[\sqrt{4ay} - \frac{y^2}{4a}\right] dy$$

$$= 2\sqrt{a}\cdot\int_0^{4a} y^{1/2}dy - \frac{1}{4a}\int_0^{4a} y^2 dy$$

$$= 2\sqrt{a}\cdot\left[\frac{y^{3/2}}{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\cdot\left(\frac{y^3}{3}\right)_0^{4a}$$

$$= 2\sqrt{a}\times\frac{2}{3}\cdot(4a)^{3/2} - \frac{1}{12a}64a^3$$

$$= \frac{4\sqrt{a}}{3}8a\sqrt{a} - \frac{16a^2}{3}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}\text{ Sq.units}$$

Example 4.49. Find the area common to $y^2 = 4x$ and $x^2 = 4y$ using double integration. [Dec 2011]

Solution. Put $a = 1$ in the previous example.

Example 4.50. Find by double integration, the area between the parabolas $3y^2 = 25x$ and $5x^2 = 9y$. [Jun 2012]

Solution. To find the points of intersection, solve the two equations.

$$y^2 = \frac{25}{3}x \quad (1)$$

$$x^2 = \frac{9}{5}y \quad (2)$$

$$(2)^2 \Rightarrow \quad x^4 = \frac{81}{25}y^2$$

$$\cancel{25}x^4 = \frac{81\times\cancel{25}}{3}x$$

$$x^4 = 27x$$

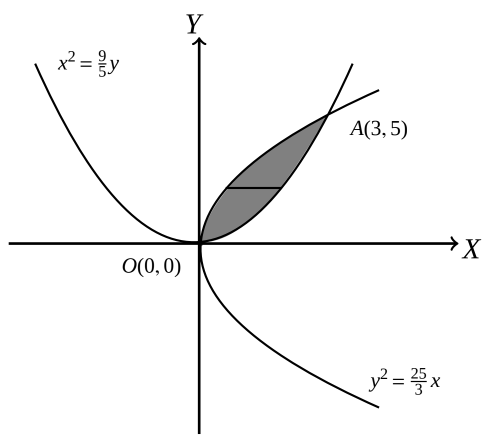
$$x^4 - 27x = 0$$

$$x(x^3 - 27) = 0$$

$$x = 0, x^3 - 27 = 0$$

$$x^3 = 27$$

$$x = 3.$$



when $x = 0, 9y = 0 \Rightarrow y = 0$.

One point of intersection is O$(0, 0)$.

where $x = 3, 9y = 5 \times 9$

$y = 5$.

$\therefore$ The other point of intersection is $(3, 5)$.

The shaded portion is the region of integration. Divide the region into strips parallel to the x–axis. Along one such strip, x varies from $\frac{3y^2}{25}$ to $\frac{3\sqrt{y}}{\sqrt{5}}$ and finally y varies from 0 to 5.

$$\therefore \text{ Area } = \int_0^5 \int_{\frac{3y^2}{25}}^{\frac{3\sqrt{y}}{\sqrt{5}}} dxdy = \int_0^5 (x)_{\frac{3y^2}{25}}^{\frac{3\sqrt{y}}{\sqrt{5}}} dy$$

$$= \int_0^5 \left(\frac{3}{\sqrt{5}} \sqrt{y} - \frac{3y^2}{25} \right) dy$$

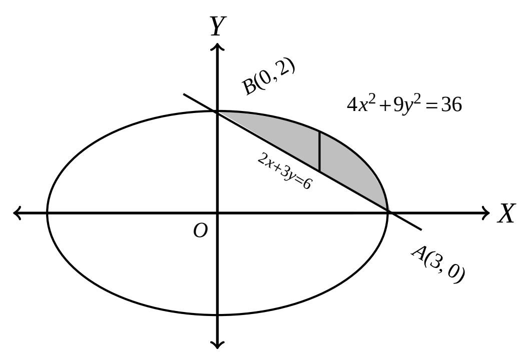
$$= \frac{3}{\sqrt{5}} \left(\frac{y^{3/2}}{\frac{3}{2}} \right)_0^5 - \frac{3}{25} \cdot \left(\frac{y^3}{3} \right)_0^5 = \frac{2}{\sqrt{5}} [5\sqrt{5}] - \frac{1}{25} \cdot 5^3 = 10 - 5 = 5 \text{ Sq.units.}$$

Example 4.51. Find the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x + 3y = 6$. [Jan 2012]

Solution. Solving the given two equations, we get the points of intersection.

$$4x^2 + 9y^2 = 36 \quad (1)$$

$$2x + 3y = 6 \quad (2)$$



From (2), $3y = 6 - 2x$

i.e., $y = \frac{6 - 2x}{3}$.

Substituting in (1) we get

$$4x^2 + \frac{9(6 - 2x)^2}{9} = 36$$

$$4x^2 + 36 + 4x^2 - 24x = 36$$
$$8x^2 - 24x = 0$$
$$x^2 - 3x = 0$$
$$x(x-3) = 0 \quad \Rightarrow x = 0 \text{ or } x = 3.$$

When $x = 0, (2) \Rightarrow \; y = \frac{6}{3} = 2.$

One point of intersection is $(0, 2)$

When $x = 3, y = \frac{6-6}{3} = 0.$

The other point of intersection is $(3, 0)$.

The shaded portion is the smaller area. Divide the region into strips parallel to the y–axis. Along one such strip, y varies from $\frac{6-2x}{3}$ to $\sqrt{4 - \frac{4}{9}x^2}$ and finally x varies from 3 to 0.

$$\therefore \text{ Area } = \int_{x=3}^{0} \int_{y=\frac{6-2x}{3}}^{\sqrt{4-\frac{4}{9}x^2}} dxdy = \int_{3}^{0} [y]_{\frac{6-2x}{3}}^{\sqrt{4-\frac{4}{9}x^2}} dx$$
$$= \int_{3}^{0} \left[\sqrt{4 - \frac{4}{9}x^2} - \left(\frac{6-2x}{3}\right)\right] dx$$
$$= \int_{3}^{0} \sqrt{\frac{4}{9}(9 - x^2)}dx - \int_{3}^{0} 2dx + \frac{2}{3}\int_{3}^{0} xdx$$
$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_3^0 - 2[x]_3^0 + \frac{2}{3}\left[\frac{x^2}{2}\right]_3^0$$
$$= \frac{2}{3}\left[0 - \frac{9}{2}\frac{\pi}{2}\right] - 2(0-3) + \frac{1}{3}[0-9] = \frac{-3\pi}{2} + 6 - 3 = 3 - \frac{3\pi}{2}$$
$$= 3\left(1 - \frac{\pi}{2}\right) = \frac{3(2-\pi)}{2} \text{ Sq.units.}$$

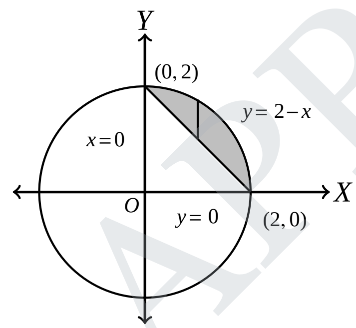
Example 4.52. Find the smaller of the areas bounded by $y = 2 - x$ and $x^2 + y^2 = 4$ using double integral.

Solution. Solving the given two equations, we get the points of intersection.

We have $x^2 + y^2 = 4 \Rightarrow x^2 + (2 - x)^2 = 4 \Rightarrow x^2 + 4 + x^2 - 4x = 4 \Rightarrow 2x^2 - 4x = 0$

$$\Rightarrow x^2 - 2x = 0 \Rightarrow x(x - 2) = 0 \Rightarrow x = 0, x = 2.$$

When $x = 0, y = 2$, when $x = 2, y = 0$.
The points of intersection are $(0, 2)$ and $(2, 0)$. The shaded portion is the required region. Divide this region into strips parallel to the y-axis. Along one such typical strip, y varies from $2-x$ to $\sqrt{4 - x^2}$ and finally x varies from 0 to 2.



$$\text{Now area } = \iint_R dxdy. = \int_{x=0}^{2} \int_{y=2-x}^{\sqrt{4-x^2}} dydx = \int_{x=0}^{2} (y)_{2-x}^{\sqrt{4-x^2}} dx = \int_{x=0}^{2} (\sqrt{4 - x^2} - (2 - x))dx$$

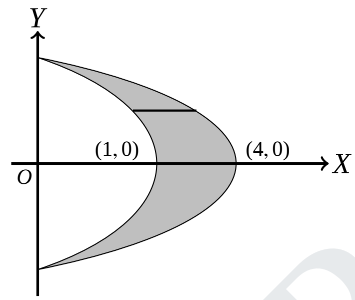
$$= \left(\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right)_0^2 - 2(x)_0^2 + \left(\frac{x^2}{2}\right)_0^2$$

$$= 2\sin^{-1}(1) - 2(2 - 0) + \frac{4}{2} = 2\frac{\pi}{2} - 4 + 2 = \pi - 2.$$

Example 4.53. Find the area bounded by the parabola $y^2 = 4 - x$ and $y^2 = 4 - 4x$ as a double integral and evaluate it. [Jan 2001]

Solution. The parabola $y^2 = 4 - x = -(x - 4)$. Its vertex is at $(4, 0)$.
Consider the parabola $y^2 = 4 - 4x = -4(x - 1)$. Its vertex is at $(1, 0)$.
Both the parabolas are open to the left. Let us solve the two equations to find the points of intersection, $y^2 = 4 - x$ and $y^2 = 4 - 4x$.
$\therefore 4 - x = 4 - 4x \Rightarrow 4x - x = 0 \Rightarrow 3x = 0 \Rightarrow x = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2.$

The points of intersection are $(2,0),(-2,0)$. The shaded portion is the required region. Divide this region into strips parallel to the x-axis. Along one typical strip, x varies from $1-\frac{y^2}{4}$ to $4-y^2$. Finally y varies from -2 to 2.



$$\text{Now area } = \iint_R dxdy. = \int_{y=-2}^{2}\int_{x=1-\frac{y^2}{4}}^{x=4-y^2} dydx = \int_{y=-2}^{2} (x)_{1-\frac{y^2}{4}}^{4-y^2} dy = \int_{y=-2}^{2}\left(4-y^2-\left(1-\frac{y^2}{4}\right)\right)dy$$

$$= \int_{y=-2}^{2}\left(4-y^2-1+\frac{y^2}{4}\right)dy = 4(y)_{-2}^{2} - \left(\frac{y^3}{3}\right)_{-2}^{2} - (y)_{-2}^{2} + \frac{1}{4}\left(\frac{y^3}{3}\right)_{-2}^{2}$$

$$= 4(2+2) - \frac{1}{3}(8+8) - (2+2) + \frac{1}{12}(8+8) = 16 - \frac{16}{3} - 4 + \frac{16}{12}$$

$$= 16 - 4 - \frac{16}{3} + \frac{4}{3} = 12 - \frac{12}{3} = 12 - 4 = 8.$$

Example 4.54. Find the area of the region bounded by $y = x-2$ and $y^2 = 2x+4$.

Solution. The parabola is $y^2 = 2x+4 = 2(x+2)$.

The vertex is $(-2,0)$.

To find the points of intersection, let us solve the two equations.

The curves are $y = x-2, y^2 = 2x+4$.

$$(x-2)^2 = 2x+4.$$

$$x^2 - 4x + 4 - 2x - 4 = 0$$

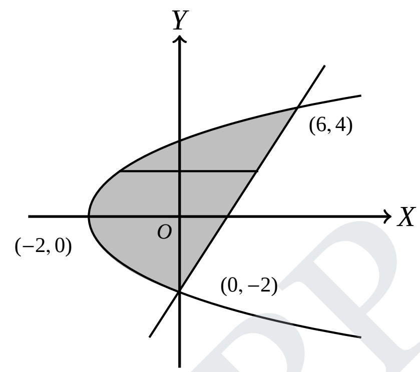
$$x^2 - 6x = 0$$

$$x(x-6) = 0 \quad \Rightarrow x = 0, \quad x = 6.$$

When $x = 0, y = -2$.

When $x = 6, y = 4$.

The points of intersection are $(0, -2)$ and $(6, 4)$. The shaded portion is the required region. Divide the region into strips parallel to the x-axis. Along one such strip, x varies from $\frac{y^2-4}{2}$ to $y+2$ and finally y varies from -2 to 4.



$$\text{Now, Area } = \iint_R dxdy = \int_{y=-2}^{4} \int_{x=\frac{y^2-4}{2}}^{y+2} dxdy = \int_{y=-2}^{4} (x)_{x=\frac{y^2-4}{2}}^{y+2} dy$$

$$= \int_{-2}^{4} \left(y + 2 - \left(\frac{y^2}{2} - 2\right)\right) dy = \int_{-2}^{4} \left(y + 2 - \frac{y^2}{2} + 2\right) dy = \int_{-2}^{4} \left(y + 4 - \frac{y^2}{2}\right) dy$$

$$= \left(\frac{y^2}{2} + 4y - \frac{y^3}{6}\right)_{-2}^{4} = \frac{1}{2}(16-4) + 4(4+2) - \frac{1}{6}(64+8) = 6 + 24 - 12 = 18.$$

Area as double integral in polar coordinates

Result. Area in polar coordinates is given by $\iint_R r dr d\theta$.

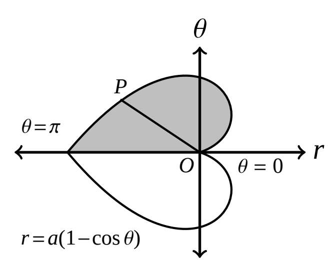
Worked Examples

Example 4.55. Find the area of the cardioid $r = a(1 - \cos\theta)$.

Solution.

Since the curve is symmetric about the initial line, Area $= 2\times$ area above the initial line.

Along this region, r varies from 0 to $a(1-\cos\theta)$ and θ varies from 0 to π.



$$\text{Area} = 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1-\cos\theta)} r\,dr\,d\theta = 2\int_{0}^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta = \int_0^{\pi} a^2(1-\cos\theta)^2 d\theta$$

$$= a^2\int_0^{\pi}(1-\cos^2\theta - 2\cos\theta)d\theta = a^2\int_0^{\pi}(1+\frac{1+\cos 2\theta}{2} - 2\cos\theta)d\theta$$

$$= a^2\int_0^{\pi}(\frac{3}{2}+\frac{\cos 2\theta}{2} - 2\cos\theta)d\theta = a^2\left(\frac{3}{2}(\theta)_0^{\pi} + \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)_0^{\pi} - 2(\sin\theta)_0^{\pi}\right)$$

$$= a^2\left(\frac{3\pi}{2}\right) = \frac{3\pi a^2}{2}.$$

Example 4.56. Find the area of one loop of the lemniscate of Bernoulli $r^2 = a^2\cos 2\theta$.

Solution. The curve is symmetric about both the axes. One loop lies to the right of y-axis and symmetric about the x-axis.

$\therefore$ Area = $2\times$ area of the portion which lies above the axis.

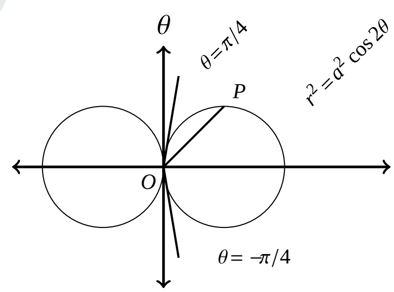
Along the region r varies from 0 to $a\sqrt{\cos 2\theta}$ and θ varies from 0 to $\frac{\pi}{4}$.

$$\therefore\ \text{Area} = 2\int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\frac{\pi}{4}}\left(\frac{r^2}{2}\right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_0^{\frac{\pi}{4}} a^2\cos 2\theta\, d\theta$$

$$= a^2\left(\frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{4}} = \frac{a^2}{2}\left(\sin\frac{\pi}{2} - 0\right) = \frac{a^2}{2}.$$



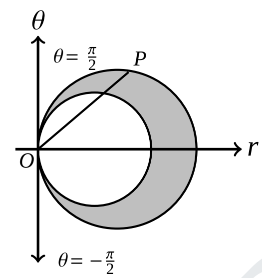
Example 4.57. Find the area between $r = 2\cos\theta$ and $r = 4\cos\theta$.

Solution.

$$\text{Area} = \int_{\theta=\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_{r=2\cos\theta}^{r=4\cos\theta} r\,dr\,d\theta = \int_{\theta=\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(\frac{r^2}{2}\right)_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(16\cos^2\theta - 4\cos^2\theta)d\theta = 6\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\, d\theta$$

$$= 6 \times 2 \int_{0}^{\frac{\pi}{2}} \cos^2 \theta d\theta = 12 \times \frac{1}{2}\frac{\pi}{2} = 3\pi.$$



Example 4.58. Find the area inside the circle $r = a\sin\theta$ but lying outside the cardioid $r = a(1 - \cos\theta)$. [Jan 2009]

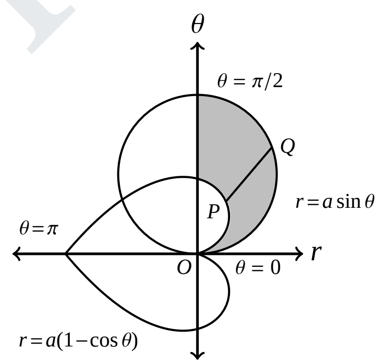
Solution. The equation of the given curves are $r = a\sin\theta, r = a(1 - \cos\theta)$. Solving, we get $a\sin\theta = a(1 - \cos\theta) \Rightarrow \sin\theta = 1 - \cos\theta$.

$\Rightarrow \sin\theta + \cos\theta = 1 \Rightarrow (\sin\theta + \cos\theta)^2 = 1$.

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1 \Rightarrow 1 + \sin 2\theta = 1$

$\Rightarrow \sin 2\theta = 0 \Rightarrow 2\theta = 0$ or $2\theta = \pi \Rightarrow \theta = 0, \theta = \frac{\pi}{2}$.

The shaded portion is the required region. Consider the radius vector OP. The limits for r and θ as OP traverses the required region is as follows. r varies from $a(1 - \cos\theta)$ to $a\sin\theta$ and θ varies from 0 to $\frac{\pi}{2}$.



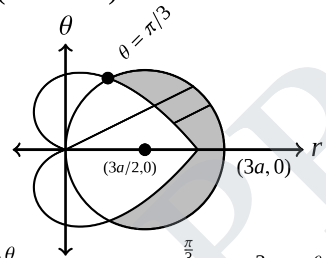
$$\text{Area} = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=a(1-\cos\theta)}^{r=a\sin\theta} r dr d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{2}} \left(a^2\sin^2\theta - a^2(1-\cos\theta)^2\right) d\theta = \frac{a^2}{2}\int_{0}^{\frac{\pi}{2}} \left(\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta\right) d\theta$$

$$= \frac{a^2}{2}\left(\frac{1}{2}\frac{\pi}{2} - \frac{\pi}{2} - \frac{1}{2}\frac{\pi}{2} + 2.1\right) = \frac{a^2}{2}\left(2 - \frac{\pi}{2}\right) = \frac{a^2}{4}(4 - \pi).$$

Example 4.59. Find the area which is inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1+\cos\theta)$. [Jan 2013]

Solution. The shaded portion is the required area. Solving the two equations we get the points of intersection, $r = 3a\cos\theta$ and $r = a(1+\cos\theta)$.



$$\therefore\ 3a\cos\theta = a(1+\cos\theta)$$

$$3\cos\theta = 1+\cos\theta$$

$$2\cos\theta = 1$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Required area $= 2 \times$ area above $\theta = 2\int\limits_{\theta=0}^{\frac{\pi}{3}}\int\limits_{r=a(1+\cos\theta)}^{3a\cos\theta} r\,dr\,d\theta. = 2\int\limits_{\theta=0}^{\frac{\pi}{3}}\left[\frac{r^2}{2}\right]_{r=a(1+\cos\theta)}^{3a\cos\theta} d\theta$

$$= 2\int\limits_0^{\frac{\pi}{3}}[9a^2\cos^2\theta - a^2(1+\cos\theta)^2]d\theta$$

$$= 2\int\limits_0^{\frac{\pi}{3}}(9a^2\cos^2\theta - a^2 - a^2\cos^2\theta - 2a^2\cos\theta)d\theta$$

$$= 2a^2\int\limits_0^{\frac{\pi}{3}}(9\cos^2\theta - 1 - \cos^2\theta - 2\cos\theta)d\theta$$

$$= 2a^2\int\limits_0^{\frac{\pi}{3}}(8\cos^2\theta - 2\cos\theta - 1)d\theta$$

$$= 2a^2\int\limits_0^{\frac{\pi}{3}}\left(\frac{8(1+\cos 2\theta)}{2} - 2\cos\theta - 1\right)d\theta$$

$$= 2a^2\int\limits_0^{\frac{\pi}{3}}(4 + 4\cos 2\theta - 2\cos\theta - 1)d\theta$$

$$= 2a^2\int\limits_0^{\frac{\pi}{3}}(3 + 4\cos 2\theta - 2\cos\theta)d\theta$$

$$= 2a^2\left(3\int_0^{\frac{\pi}{3}} d\theta + 4\int_0^{\frac{\pi}{3}} \cos 2\theta d\theta - 2\int_0^{\frac{\pi}{3}} \cos\theta d\theta\right)$$

$$= 2a^2\left(3(\theta)_0^{\frac{\pi}{3}} + 4\left(\frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{3}} - 2(\sin\theta)_0^{\frac{\pi}{3}}\right)$$

$$= 2a^2\left(3\left(\frac{\pi}{3} - 0\right) + 2\left(\sin\frac{2\pi}{3} - \sin 0\right) - 2\left(\sin\frac{\pi}{3} - \sin 0\right)\right)$$

$$= 2a^2\left(\pi + 2\cdot\frac{\sqrt{3}}{2} - 2\cdot\frac{\sqrt{3}}{2}\right) = 2\pi a^2 \text{ Sq.units.}$$

4.6 Triple integral in Cartesian coordinates

Let the function $f(x,y,z)$ be defined in a specific region V in space for all x,y and z with proper limits. The triple integral of $f(x,y,z)$ w.r.t. x,y and z is defined as $\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)dxdydz.$

The evaluation of the triple integral is as follows.

(i) If all the limits are constants, then the integration can be performed in any order with proper limits.

(ii) Consider $\int_a^b \int_{y=g_1(z)}^{g_2(z)} \int_{x=f_1(y,z)}^{f_2(y,z)} f(x,y,z)dxdydz.$

First we integrate with respect to x treating y and z as constants. Next we integrate the resulting function of y and z w.r.t.y treating z as constant. Finally integrate the resulting function of z w.r.t.z.

Worked Examples

Example 4.60. Evaluate $\int_0^1\int_0^2\int_1^2 x^2yzdxdydz.$

Solution.

$$\int_0^1\int_0^2\int_1^2 x^2yzdxdydz = \int_0^1\int_0^2 yz\left(\frac{x^3}{3}\right)_1^2 dydz$$

$$= \frac{1}{3}\int_0^1\int_0^2 yz(8-1)dydz$$

$$= \frac{7}{3}\int_0^1 z\left(\frac{y^2}{2}\right)_0^2 dz = \frac{7}{6}\int_0^1 z4dz$$

$$= \frac{28}{6}\left(\frac{z^2}{2}\right)_0^1$$

$$= \frac{28}{6}\frac{1}{2} = \frac{7}{3}.$$

Example 4.61. Find $\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)dxdydz.$ [Jun 2009]

Solution.

$$\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)dxdydz = \int_0^a\int_0^b \left(\frac{x^3}{3}+y^2x+z^2x\right)_0^c dydz$$

$$= \int_0^a\int_0^b \left(\frac{c^3}{3}+cy^2+cz^2\right)dydz$$

$$= \int_0^a \left(\frac{c^3}{3}(y)_0^b + c\left(\frac{y^3}{3}\right)_0^b + cz^2(y)_0^b\right)dz$$

$$= \int_0^a \left(\frac{c^3b}{3}+\frac{cb^3}{3}+cz^2b\right)dz$$

$$= \frac{bc^3}{3}(z)_0^a + \frac{cb^3}{3}(z)_0^a + bc\left(\frac{z^3}{3}\right)_0^a$$

$$= \frac{abc^3}{3}+\frac{ab^3c}{3}+\frac{a^3bc}{3} = \frac{abc}{3}(a^2+b^2+c^2).$$

Example 4.62. Evaluate $\int_0^{\log 2}\int_0^x\int_0^{x+y} e^{x+y+z}dxdydz.$ [Jan 2009]

Solution.

$$\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dxdydz = \int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^x e^y e^z\,dxdydz = \int_0^{\log 2}\int_0^{x} e^x e^y\left(\int_0^{x+y} e^z dz\right)dxdy$$

$$= \int_0^{\log 2}\int_0^{x} e^x e^y (e^z)_0^{x+y}\,dxdy = \int_0^{\log 2}\int_0^{x} e^x e^y (e^{x+y}-1)\,dxdy$$

$$= \int_0^{\log 2}\int_0^{x} (e^{2x+2y}-e^x e^y)\,dxdy$$

$$= \int_0^{\log 2}\left(\int_0^{x} e^{2x}e^{2y}dy - \int_0^{x} e^x e^y dy\right)dx$$

$$= \int_0^{\log 2}\left(e^{2x}\left(\frac{e^{2y}}{2}\right)_0^x - e^x(e^y)_0^x\right)dx$$

$$= \int_0^{\log 2}\left(e^{2x}\left(\frac{e^{2x}}{2}-\frac{1}{2}\right) - e^x(e^x-1)\right)dx$$

$$= \int_0^{\log 2}\left(\frac{e^{4x}}{2}-\frac{e^{2x}}{2}-e^{2x}+e^x\right)dx$$

$$= \int_0^{\log 2}\left(\frac{e^{4x}}{2}-\frac{3e^{2x}}{2}+e^x\right)dx$$

$$= \frac{1}{2}\left(\frac{e^{4x}}{4}\right)_0^{\log 2} - \frac{3}{2}\left(\frac{e^{2x}}{2}\right)_0^{\log 2} + (e^x)_0^{\log 2}$$

$$= \frac{1}{8}[16-1] - \frac{3}{4}(4-1) + (2-1)$$

$$= \frac{15}{8} - \frac{9}{4} + 1$$

$$= \frac{15-18+8}{8} = \frac{5}{8}.$$

Example 4.63. Evaluate $\int_0^1\int_0^{1-x}\int_0^{x+y} e^z\,dzdydx.$ [Jan 1996]

Solution.

$$\int_0^1\int_0^{1-x}\int_0^{x+y} e^z dzdydx = \int_0^1\int_0^{1-x}(e^z)_0^{x+y}\,dydx = \int_0^1\int_0^{1-x}(e^{x+y}-1)\,dydx$$

$$= \int_0^1\int_0^{1-x}(e^xe^y-1)\,dydx = \int_0^1\left(e^x\int_0^{1-x}e^y - \int_0^{1-x}dy\right)dx$$

$$= \int_0^1\left(e^x\,(e^y)_0^{1-x} - (y)_0^{1-x}\right)dx$$

$$= \int_0^1\left(e^x(e^{1-x}-1)-(1-x)\right)dx = \int_0^1(e-e^x-1+x)dx$$

$$= e(x)_0^1 - (e^x)_0^1 - (x)_0^1 + \left(\frac{x^2}{2}\right)_0^1 = e-(e-1)-1+\frac{1}{2}$$

$$= e-e+1-1+\frac{1}{2} = \frac{1}{2}.$$

Example 4.64. Evaluate $\displaystyle\int_1^3\int_{\frac{1}{x}}^1\int_0^{\sqrt{xy}} xydzdydx.$ [Jan 2001]

Solution.

$$\int_1^3\int_{\frac{1}{x}}^1\int_0^{\sqrt{xy}} xydzdydx = \int_1^3\int_{\frac{1}{x}}^1 xy(z)_0^{\sqrt{xy}}dydx = \int_1^3\int_{\frac{1}{x}}^1 xy\,\sqrt{xy}dydx = \int_1^3\int_{\frac{1}{x}}^1 x^{\frac{3}{2}}y^{\frac{3}{2}}dydx$$

$$= \int_1^3 x^{\frac{3}{2}}\left(\frac{y^{\frac{5}{2}}}{\frac{5}{2}}\right)_{\frac{1}{x}}^1 dx = \frac{2}{5}\int_1^3 x^{\frac{3}{2}}\left(1-\left(\frac{1}{x}\right)^{\frac{5}{2}}\right)dx$$

$$= \frac{2}{5}\int_1^3\left(x^{\frac{3}{2}}-\frac{1}{x}\right)dx = \frac{2}{5}\left(\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)_1^3 - (\log x)_1^3\right)$$

$$= \frac{2}{5}\left(\frac{2}{5}(3^{\frac{5}{2}}-1)-\log 3\right) = \frac{2}{5}\left(\frac{2}{5}(9\sqrt{3}-1)-\log 3\right).$$

Example 4.65. Evaluate $\displaystyle\int_0^1\int_0^{1-x}\int_0^{(x+y)^2} xdzdydx.$ [Jan 1996]

Solution.

$$\int_0^1\int_0^{1-x}\int_0^{(x+y)^2} xdzdydx = \int_0^1\int_0^{1-x} x(z)_0^{(x+y)^2}dydx = \int_0^1\int_0^{1-x} x(x+y)^2dydx$$

$$= \int_0^1 x\left(\frac{(x+y)^3}{3}\right)_0^{1-x} dx = \frac{1}{3}\int_0^1 x((x+1-x)^3 - x^3)dx$$

$$= \frac{1}{3}\int_0^1 x(1-x^3)dx = \frac{1}{3}\int_0^1 (x-x^4)dx = \frac{1}{3}\left(\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^5}{5}\right)_0^1\right)$$

$$= \frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right) = \frac{1}{3}\left(\frac{5-2}{10}\right) = \frac{1}{10}.$$

Example 4.66. Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}dzdydx.$ [Jun 2013]

Solution.

$$\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}dzdydx = \int_0^{\log 2}\int_0^{x} [\ e^{x+y+z}\ ]_0^{x+\log y} dydx$$

$$= \int_0^{\log 2}\int_0^{x} \left(e^{x+y+x+\log y} - e^{x+y}\right) dydx.$$

$$= \int_0^{\log 2}\int_0^{x} \left(e^{2x}e^{y+\log y} - e^x\cdot e^y\right) dydx$$

$$= \int_0^{\log 2}\int_0^{x} \left(e^{2x}\cdot e^y\cdot e^{\log y} - e^xe^y\right) dydx$$

$$= \int_0^{\log 2}\int_0^{x} \left(e^{2x}ye^y - e^xe^y\right) dydx$$

$$= \int_0^{\log 2}\left(e^{2x}\int_0^{x} ye^ydy - e^x\int_0^{x} e^ydy\right) dx$$

$$= \int_0^{\log 2} \left(e^{2x} \int_0^x y d(e^y) - e^x (e^y)_0^x \right) dx$$

$$= \int_0^{\log 2} \left\{ e^{2x} \left([ye^y]_0^x - \int_0^x e^y dy \right) - e^x (e^x - 1) \right\} dx$$

$$= \int_0^{\log 2} \left(e^{2x} \left(xe^x - (e^y)_0^x \right) - e^{2x} + e^x \right) dx$$

$$= \int_0^{\log 2} \{ e^{2x} (xe^x - e^x + 1) - e^{2x} + e^x \} dx$$

$$= \int_0^{\log 2} \left(xe^{3x} - e^{3x} + e^{2x} - e^{2x} + e^x \right) dx$$

$$= \int_0^{\log 2} \left(xe^{3x} - e^{3x} + e^x \right) dx$$

$$= \int_0^{\log 2} xe^{3x} dx - \int_0^{\log 2} e^{3x} dx + \int_0^{\log 2} e^x dx.$$

$$= \int_0^{\log 2} xd\left(\frac{e^{3x}}{3}\right) - \left(\frac{e^{3x}}{3}\right)_0^{\log 2} + (e^x)_0^{\log 2}$$

$$= \left(\frac{xe^{3x}}{3}\right)_0^{\log 2} - \int_0^{\log 2} \frac{e^{3x}}{3} dx - \frac{1}{3}\left(e^{3\log 2} - 1\right) + (e^{\log 2} - 1)$$

$$= \log 2 \cdot \frac{8}{3} - \left(\frac{e^{3x}}{9}\right)_0^{\log 2} - \frac{1}{3}[8 - 1] + (2 - 1)$$

$$= \frac{8}{3}\log 2 - \left(\frac{8}{9} - \frac{1}{9}\right) - \frac{7}{3} + 1$$

$$= \frac{8}{3}\log 2 - \left(\frac{7}{9} + \frac{7}{3} - 1\right)$$

$$= \frac{8}{3}\log 2 - \left(\frac{7 + 21 - 9}{9}\right)$$

$$= \frac{8}{3}\log 2 - \frac{19}{9}.$$

Example 4.67. Evaluate $\iiint \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$ for all positive values of x, y, z for which the integral is real. [Jan 2013, Jan 2012]

Solution. The integral is real for all values of $x^2+y^2+z^2<1$.

$$\therefore \iiint \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}} = \int_0^1 \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} dzdydx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dydx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} (\sin^{-1}1 - \sin^{-1}0) dydx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2} dydx$$

$$= \frac{\pi}{2}\int_0^1 [y]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\sin^{-1}1 - 0\right]$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.$$

Example 4.68. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} dzdydx.$ [Dec 2011]

Solution.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} dzdydx = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{a^2-x^2-y^2}}\right]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(\sin^{-1} 1 - \sin^{-1} 0\right) dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{\pi}{2} dydx.$$

$$= \frac{\pi}{2} \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^a \sqrt{a^2-x^2} dx$$

$$= \frac{\pi}{2} \left[\frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2-x^2}\right]_0^a$$

$$= \frac{\pi}{2} \left[\frac{a^2}{2} \sin^{-1} 1 - 0\right]$$

$$= \frac{\pi a^2}{4} \cdot \frac{\pi}{2} = \frac{\pi^2 a^2}{8}.$$

Example 4.69. Evaluate $\iiint_V \frac{dxdydz}{(x+y+z+1)^3}$, where V is the region bounded by $x = 0, y = 0, z = 0$, and $x + y + z = 1$. [Jan 2014, Dec 2011]

Solution. The limits for x, y and z are as follows.

$z : 0$ to $1 - x - y$

$y : 0$ to $1 - x$

$x : 0$ to 1

$$\therefore \iiint_V \frac{dzdydz}{(1+x+y+z)^3} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1+x+y+z)^{-3} dzdydx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(1+x+y+z)^{-2}}{-2} \right]_0^{1-x-y} dydx$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left(2^{-2} - (1+x+y)^{-2}\right) dydx$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left(\frac{1}{4} - (1+x+y)^{-2}\right) dydx$$

$$= -\frac{1}{2} \int_0^1 \left(\frac{1}{4}[y]_0^{1-x} - \left[\frac{(1+x+y)^{-1}}{-1} \right]_0^{1-x} \right) dx$$

$$= -\frac{1}{2} \int_0^1 \left(\frac{1}{4}(1-x) + 2^{-1} - (1+x)^{-1} \right) dx$$

$$= -\frac{1}{2} \left(\frac{1}{4}(x)_0^1 - \frac{1}{4} \cdot \left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}[x]_0^1 - (\log(1+x))_0^1 \right)$$

$$= -\frac{1}{2} \left(\frac{1}{4} - \frac{1}{8} + \frac{1}{2} - \log 2 \right)$$

$$= -\frac{1}{2} \left(\frac{2-1+4}{8} - \log 2 \right)$$

$$= -\frac{1}{2} \left(\frac{5}{8} - \log 2 \right) = \frac{1}{2} \log 2 - \frac{5}{16}.$$

Example 4.70. Find the value of $\iiint xyzdxdydz$ through the positive spherical octant for which $x^2 + y^2 + z^2 \leq a^2$. [Jan 2014, Jun 2010]

Solution. The limits for x, y, z are as follows.

$z : 0$ to $\sqrt{a^2 - x^2 - y^2}$

$y : 0$ to $\sqrt{a^2 - x^2}$

$x : 0$ to a.

$$\therefore \iiint xyzdxdydz = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyzdzdydx.$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy \cdot \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dydx.$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} xy \cdot (a^2 - x^2 - y^2) dydx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} \left(a^2xy - x^3y - xy^3\right) dydx$$

$$= \frac{1}{2}\int_0^a \left[a^2x \cdot \frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4}\right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{a^2x}{2}(a^2-x^2) - \frac{x^3}{2}(a^2-x^2) - \frac{x}{4}(a^2-x^2)^2\right) dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{a^4x}{2} - \frac{a^2x^3}{2} - \frac{a^2x^3}{2} + \frac{x^5}{2} - \frac{x}{4}(a^4 + x^4 - 2a^2x^2)\right) dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{a^4}{2}x - a^2x^3 + \frac{x^5}{2} - \frac{a^4}{4}x - \frac{x^5}{4} + \frac{a^2}{2}x^3\right) dx$$

$$= \frac{1}{2}\left(\frac{a^4}{2}\left(\frac{x^2}{2}\right)_0^a - a^2\left(\frac{x^4}{4}\right)_0^a + \frac{1}{2}\left(\frac{x^6}{6}\right)_0^a - \frac{a^4}{4}\left(\frac{x^2}{2}\right)_0^a - \frac{1}{4}\left(\frac{x^6}{6}\right)_0^a + \frac{a^2}{2}\left(\frac{x^4}{4}\right)_0^a\right)$$

$$= \frac{1}{2}\left(\cancel{\frac{a^6}{4}} - \cancel{\frac{a^6}{4}} + \frac{a^6}{12} - \cancel{\frac{a^6}{8}} - \frac{a^6}{24} + \cancel{\frac{a^6}{8}}\right)$$

$$= \frac{1}{2} \times a^6\left(\frac{1}{12} - \frac{1}{24}\right)$$

$$= \frac{a^6}{2}\left(\frac{2-1}{24}\right)$$

$$= \frac{a^6}{48}.$$

4.7 Volume as triple integral

Result. If V is the volume bounded by a definite region D in space, then the volume is calculated by the triple integral $V = \iiint_D dxdydz$ with proper limits.

Example 4.71. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using triple integral.

Solution. We know that the ellipsoid is symmetric about the coordinate planes.

$\therefore$ Volume of the ellipsoid = 8×Volume in the first octant = $8\iiint_V dxdydz$.

The ellipsoid is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

In the first octant, z varies from 0 to $c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

y varies from 0 to $b\sqrt{1 - \frac{x^2}{a^2}}$

and x varies from 0 to a.

$$\therefore \text{Volume} = 8 \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} (z)_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dydx$$

$$= 8c \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dydx$$

$$= 8c \int_{x=0}^{a} \int_{y=0}^{b\sqrt{(1-\frac{x^2}{a^2})}} \frac{1}{b} \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right) - y^2} dydx$$

$$= \frac{8c}{b}\int_{x=0}^{a}\left(\frac{b^2(1-\frac{x^2}{a^2})}{2}\sin^{-1}\frac{y}{\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}}+\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\right)_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \frac{8c}{b}\int_{0}^{a}\frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}1dx$$

$$= 4bc\frac{\pi}{2}\int_0^a\left(1-\frac{x^2}{a^2}\right)dx$$

$$= 2\pi bc\left(x-\frac{x^3}{3a^2}\right)_0^a$$

$$= 2\pi bc\left(a-\frac{a}{3}\right)=\frac{4\pi abc}{3}.$$

Note. The volume of the portion of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which lies in the first octant is $\frac{1}{8}\times$ Volume of the ellipsoid $=\frac{1}{8}\times\frac{4\pi abc}{3}=\frac{\pi abc}{6}$.

Example 4.72. Find the volume of the sphere $x^2+y^2+z^2=a^2$ using triple integrals. [Jan 2006]

Solution. We know that the sphere is symmetric about the coordinate planes.

$\therefore$ Volume of the sphere $= 8\times$ Volume of the portion of the sphere that lies in the first octant.

The sphere is given by $x^2+y^2+z^2=a^2$.

In the first octant z varies from 0 to $\sqrt{a^2-x^2-y^2}$.

y varies from 0 to $\sqrt{a^2-x^2}$.

and x varies from 0 to a.

$$\text{Volume of the sphere} = 8\times\iiint_V dzdydx.$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} dzdydx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (z)_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2} dydx$$

$$= 8 \int_{x=0}^{a} \left(\frac{a^2 - x^2}{2} \sin^{-1} \frac{y}{\sqrt{a^2 - x^2}} + \frac{y}{2} \sqrt{a^2 - x^2 - y^2} \right)_0^{\sqrt{a^2-x^2}} dx$$

$$= 8 \int_{x=0}^{a} \frac{a^2 - x^2}{2} \sin^{-1}(1) dx$$

$$= 4\frac{\pi}{2} \int_{x=0}^{a} (a^2 - x^2) dx$$

$$= 2\pi \left(a^2 x - \frac{x^3}{3} \right)_0^a$$

$$= 2\pi \left(a^3 - \frac{a^3}{3} \right) = \frac{4\pi a^3}{3}.$$

Note

The volume of the portion of the sphere that lies in the first octant is $\frac{1}{8} \times$ Volume of the sphere $= \frac{1}{8} \times \frac{4\pi a^3}{3} = \frac{\pi a^3}{6}$.

Example 4.73. Find the volume of the tetrahedron bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the coordinate planes. [Jun 2006]

Solution. Let D be the region in space bounded by the tetrahedron.

The region of integration is given by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, x = 0, y = 0, z = 0.$

Along the region of integration z varies from 0 to $c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$.

y varies from 0 to $b\left(1 - \frac{x}{a}\right)$.

and x varies from 0 to a.

$$\therefore \text{ Volume } = \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dzdydx.$$

$$= \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} (z)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx.$$

$$= \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dydx.$$

$$= c\int_{x=0}^{a}\left(y-\frac{xy}{a}-\frac{y^2}{2b}\right)_0^{b\left(1-\frac{x}{a}\right)} dx.$$

$$= c\int_{x=0}^{a}\left\{b\left(1-\frac{x}{a}\right)-\frac{x}{a}b\left(1-\frac{x}{a}\right)-\frac{1}{2b}b^2\left(1-\frac{x}{a}\right)^2\right\}dx$$

$$= \frac{bc}{2}\int_{x=0}^{a}\left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{3\left(-\frac{1}{a}\right)}\right]_0^a$$

$$= \frac{-abc}{6}[0-1]$$

$$= \frac{abc}{6}.$$

Example 4.74. Find the volume of the portion of the cylinder $x^2+y^2=1$ intercepted between the plane $x=0$ and the paraboloid $x^2+y^2=4-z$. [Jun 2012]

Solution. The surfaces are $x^2+y^2=1$, $x=0$ and $x^2+y^2=4-z$.

For the required volume integral

x varies from 0 to 1

y varies from 0 to $\sqrt{1-x^2}$

z varies from 0 to $4-x^2-y^2$.

$$\text{Volume} = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{4-x^2-y^2} dzdydx$$

$$= 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} [z]_0^{4-x^2-y^2} dydx$$

$$= 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} (4-x^2-y^2)dydx$$

$$= 4\int_{x=0}^{1}\left(4(y)_0^{\sqrt{1-x^2}} - x^2(y)_0^{\sqrt{1-x^2}} - \left(\frac{y^3}{3}\right)_0^{\sqrt{1-x^2}}\right)dx.$$

$$x = \sin\theta.$$

$$dx = \cos\theta d\theta.$$

When $x = 0,\ \sin\theta = 0 \ \Rightarrow\ \theta = 0.$

When $x = 1\ \sin\theta = 1 \ \Rightarrow\ \theta = \frac{\pi}{2}.$

$$= 4\int_{0}^{1}\left((4-x^2)\sqrt{1-x^2} - \frac{(1-x^2)^{3/2}}{3}\right)dx.$$

$$= 4\int_{0}^{\pi/2}\left((4-\sin^2\theta)\cos\theta - \frac{(\cos^2\theta)^{3/2}}{3}\right)\cos\theta d\theta$$

$$= 4\int_{0}^{\pi/2}\left(4\cos\theta - \sin^2\theta\cos\theta - \frac{\cos^3\theta}{3}\right)\cos\theta d\theta$$

$$= 4\int_{0}^{\pi/2}\left(4\cos^2\theta - \sin^2\theta\cos^2\theta - \frac{\cos^4\theta}{3}\right)d\theta$$

$$= 4\int_{0}^{\pi/2}\left(4\cos^2\theta - \sin^2\theta(1-\sin^2\theta) - \frac{\cos^4\theta}{3}\right)d\theta$$

$$= 4\int_{0}^{\pi/2}\left(4\cos^2\theta - \sin^2\theta + \sin^4\theta - \frac{\cos^4\theta}{3}\right)d\theta$$

$$= 4\left[4\int_{0}^{\pi/2}\cos^2\theta d\theta - \int_{0}^{\pi/2}\sin^2\theta d\theta + \int_{0}^{\pi/2}\sin^4\theta d\theta - \frac{1}{3}\int_{0}^{\pi/2}\cos^4\theta d\theta\right]$$

$$= 4\left[4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \cdot \frac{\pi}{2} + \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right]$$

$$= 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2}\left[4 - 1 + \frac{3}{4} - \frac{1}{4}\right]$$

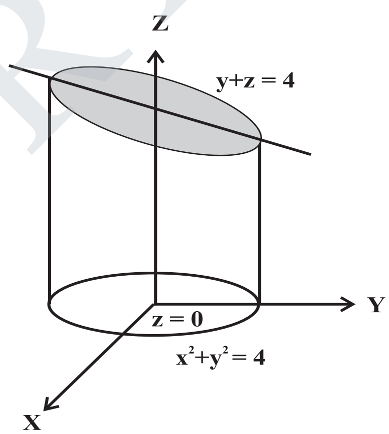
$$7 = \pi\left[3 + \frac{3}{4} - \frac{1}{4}\right]$$

$$= \pi\left[3 + \frac{1}{2}\right]$$

$$= \frac{7\pi}{2} \text{ Cubic units.}$$

Example 4.75. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. [May 2011]

Solution.



The given surface is $x^2 + y^2 = 4$, $z = 0$ and $y + z = 4$.

In the region, x varies from -2 to 2, y varies from $-\sqrt{4 - x^2}$ to $\sqrt{4 - x^2}$, z varies from 0 to $4 - y$.

$$\therefore \text{ Volume} = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{4-y} dz dy dx$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]_0^{4-y} dy dx.$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y) dy dx.$$

$$= \int_{x=-2}^{2} \left[4(y)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} - \left[\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\right] dx.$$

$$= \int_{-2}^{2} \left[4\left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) - \frac{1}{2}\left(4 - x^2 - (4-x^2)\right)\right] dx.$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\,dx.$$

$$= 8 \times 2 \int_{0}^{2} \sqrt{4-x^2}\,dx.$$

$$= 16\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

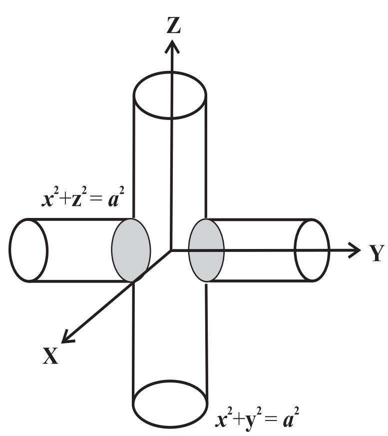
$$= 16\left[2\sin^{-1} 1\right]$$

$$= 16 \times 2 \times \frac{\pi}{2}$$

$$= 16\pi \text{ Cubic units.}$$

Example 4.76. Find the volume of the solid bounded by the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Solution. The surfaces are the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.



In the given region,

x varies from $-a$ to a,

y varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$ and

z varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$.

$$\text{Volume} = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx.$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} [z]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\,dy\,dx.$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2} + \sqrt{a^2-x^2}\right) dy dx.$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2} dy dx$$

$$= 2 \int_{x=-a}^{a} \sqrt{a^2-x^2} (y)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= 2 \int_{x=-a}^{a} \sqrt{a^2-x^2} \left[\sqrt{a^2-x^2} + \sqrt{a^2-x^2}\right] dx$$

$$= 2 \int_{x=-a}^{a} \sqrt{a^2-x^2} \cdot 2\sqrt{a^2-x^2} dx.$$

$$= 4 \int_{x=-a}^{a} (a^2 - x^2) dx. = 4\left[a^2 \cdot (x)_{-a}^{a} - \left(\frac{x^3}{3}\right)_{-a}^{a}\right]$$

$$= 4\left[a^2(a+a) - \frac{1}{3}(a^3 + a^3)\right] = 4\left[2a^3 - \frac{2a^3}{3}\right] = 4 \times \frac{4a^3}{3} = \frac{16a^3}{3} \text{ Cubic units.}$$

Example 4.77. Find the volume of the region bounded by the surfaces $y^2 = 4ax$ and $x^2 = 4ay$ and the plane $z = 0$ and $z = 3$.

Solution. Let us find the common points of the region bounded by

$$y^2 = 4ax \quad (1)$$

$$\text{and } x^2 = 4ay. \quad (2)$$

$$(2) \Rightarrow y = \frac{x^2}{4a}.$$

Substituting in (1) we get

$$\frac{x^4}{16a^2} = 4ax.$$

$$x^4 = 64a^3 x.$$

$$x^4 - 64a^3 x = 0$$

$$x(x^3 - 64a^3) = 0$$
$$x = 0, x^3 = 64a^3$$
$$x = 4a.$$

For the given region,

x varies from 0 to $4a$,
y varies from $\frac{x^2}{4a}$ to $\sqrt{4ax}$,
z varies from 0 to 3.

$$\therefore \text{ Volume} = \int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}}\int_{z=0}^{3} dzdydx$$
$$= \int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} [z]_0^3 dydx. = \int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} 3dydx.$$
$$= 3\int_{x=0}^{4a} [y]_{\frac{x^2}{4a}}^{\sqrt{4ax}} dx.$$
$$= 3\int_{0}^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a}\right) dx.$$
$$= 3\left(2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\cdot\left[\frac{x^3}{3}\right]_0^{4a}\right)$$
$$= 3\left(2\sqrt{a}\frac{2}{3}(4a)^{3/2} - \frac{1}{12a}\cdot 64a^3\right)$$
$$= 3\left(\frac{4\sqrt{a}}{3}\cdot 4a\sqrt{4a} - \frac{16a^2}{3}\right)$$
$$= 3\left(\frac{16a}{3}\sqrt{a}\cdot 2\sqrt{a} - \frac{16a^2}{3}\right)$$
$$= 3\left(\frac{32a^2}{3} - \frac{16a^2}{3}\right) = 3\times\frac{16a^2}{3} = 16a^2 \text{ Cubic units.}$$

Example 4.78. Find the volume of the solid bounded by the surfaces $y^2 = x$ and the planes $x + y + z = 2$ and $z = 0$.

Solution. The given region is bounded by

$$y^2 = x \quad (1)$$

$$x + y + z = 2 \quad (2)$$

$$z = 0. \quad (3)$$

$$\text{When} z = 0, (2) \Rightarrow \; x + y = 2$$

$$x = 2 - y.$$

Substituting in (1) we get

$$y^2 = 2 - y$$

$$y^2 + y - 2 = 0$$

$$(y - 1)(y + 2) = 0$$

$$\therefore y = -2, 1.$$

For the given region we have,

y varies from -2 to 1,

x varies from y^2 to $2 - y$ and

z varies from 0 to $2 - x - y$.

$$\therefore \text{Volume} = \int_{y=-2}^{1} \int_{x=y^2}^{2-y} \int_{z=0}^{2-x-y} dz\,dx\,dy$$

$$= \int_{y=-2}^{1} \int_{x=y^2}^{2-y} [z]_0^{2-x-y} dx\,dy$$

$$= \int_{y=-2}^{1} \int_{x=y^2}^{2-y} (2 - x - y)\,dx\,dy$$

$$= \int_{y=-2}^{1} \left(2 \cdot [x]_{y^2}^{2-y} - \left[\frac{x^2}{2}\right]_{y^2}^{2-y} - y[x]_{y^2}^{2-y} \right) dy$$

$$= \int_{-2}^{1} \left(2\{2 - y - y^2\} - \frac{1}{2}\{(2-y)^2 - y^4\} - y\{2 - y - y^2\} \right) dy.$$

$$= \int_{-2}^{1} \left(4 - 2y - 2y^2 - \frac{1}{2}[4 + y^2 - 4y - y^4] - [2y - y^2 - y^3]\right) dy$$

$$= \int_{-2}^{1} \left(4 - \cancel{2y} - 2y^2 - 2 - \frac{y^2}{2} + \cancel{2y} + \frac{y^4}{2} - 2y + y^2 + y^3\right) dy$$

$$= \int_{-2}^{1} \left(2 - 2y - \frac{3y^2}{2} + y^3 + \frac{y^4}{2}\right) dy$$

$$= 2[y]_{-2}^{1} - 2\left[\frac{y^2}{2}\right]_{-2}^{1} - \frac{3}{2}\left[\frac{y^3}{3}\right]_{-2}^{1} + \left[\frac{y^4}{4}\right]_{-2}^{1} + \frac{1}{2}\left[\frac{y^5}{5}\right]_{-2}^{1}$$

$$= 2(1 + 2) - (1 - 4) - \frac{1}{2}(1 + 8) + \frac{1}{4}(1 - 16) + \frac{1}{10}(1 + 32)$$

$$= 6 + 3 - \frac{9}{2} - \frac{15}{4} + \frac{33}{10}$$

$$= 9 - \frac{9}{2} - \frac{15}{4} + \frac{33}{10} = \frac{180 - 90 - 75 + 66}{20} = \frac{81}{20} \text{ Cubic units.}$$

5 Differential Equations

5.1 Introduction

A differential equation is an equation involving one dependent variable and its derivatives with respect to one or more independent variables.

Ordinary Differential Equation

An ordinary differential equation is an equation in which there is only one independent variable and so the derivatives involved in it are ordinary derivatives.

Partial differential equation

A partial differential equation is an equation in which there are two or more independent variables and partial differential coefficients with respect to any one of them.

Examples

(i) $\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2}$.

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$.

(iii) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 5u$.

Order of a differential equation

The order of a differential equation is the order of the highest derivative that occurs in it.

Degree of a differential equation

The degree of a differential equation is the degree of the highest derivative occurring in it after the equation has been reduced to a form free from radicals and fractions as far as the derivatives are concerned.

Examples

(i) For the differential equation $\frac{d^2x}{dt^2} + a^2x = 0$, the order is 2 and the degree is 1

(ii) For the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = c\frac{d^2y}{dx^2}$, the order is 2 and degree is 2, since it can be reduced to the form $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = c^2\left(\frac{d^2y}{dx^2}\right)^2$.

Formation of a differential equation

An ordinary differential equation can be formed by eliminating certain arbitrary constants from a relation involving variables and constants. In the study of applied mathematics, every geometrical and physical problems when translated into a mathematical model will always give rise to a differential equation.

Worked Examples

Example 5.1. Form the differential equation of the family of straight lines passing through the origin.

Solution. The equation of the family of straight lines passing through the origin is given by

$$y = mx. \tag{1}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = m.$$

Substituting the value of m in (1) we obtain

$$y = \left(\frac{dy}{dx}\right)x$$

which is the required differential equation.

Example 5.2. Obtain the differential equation of the family of circles $x^2 + y^2 + 2ax + r^2 = 0$ by eliminating the arbitrary constant a.

Solution. The given equation is

$$x^2 + y^2 + 2ax + r^2 = 0. \tag{1}$$

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} + 2a = 0$$

$$x + y\frac{dy}{dx} = -a$$

$$a = -\left(x + y\frac{dy}{dx}\right).$$

Substituting the value of a in (1) we obtain

$$x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) + r^2 = 0$$

$$x^2 + y^2 - 2x^2 - 2xy\frac{dy}{dx} + r^2 = 0$$

$$2xy\frac{dy}{dx} = y^2 - x^2 + r^2,$$

which is the required differential equation.

Example 5.3. Find the differential equation of the family of circles with centre (a, b) and radius r.

Solution. The equation of the circle is

$$(x-a)^2 + (y-b)^2 = r^2. \tag{1}$$

Differentiating w.r.t. x, we get $2(x-a) + 2(y-b)\dfrac{dy}{dx} = 0$

$$x - a + (y-b)\frac{dy}{dx} = 0. \tag{2}$$

Again differentiating we get

$$1 + (y-b)\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{dy}{dx} = 0$$

$$(y-b)\frac{d^2y}{dx^2} = -\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$y - b = -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.$$

Substituting this in (2) we get

$$x - a - \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \frac{dy}{dx} = 0$$

$$x - a = \frac{dy}{dx} \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.$$

Substituting the values of $x - a$ and $y - b$ in (1) we obtain

$$\left(\frac{dy}{dx}\right)^2 \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2} + \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2} = r^2$$

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2} \left[1 + \left(\frac{dy}{dx}\right)^2\right] = r^2$$

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3}{\left(\frac{d^2y}{dx^2}\right)^2} = r^2$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = r\frac{d^2y}{dx^2},$$

which is the required differential equation.

Example 5.4. Form the differential equation of the simple harmonic motion $x = a\cos nt$.

Solution. $x = a\cos nt$.

Differentiate w.r.t. t we get

$$\frac{dx}{dt} = -an\sin nt.$$

Again differentiating w.r.t. t we obtain

$$\frac{d^2x}{dt^2} = an^2(-\cos nt) = -n^2 a\cos nt = -n^2 x$$

$$\frac{d^2x}{dt^2} + n^2 x = 0,$$

which is the required differential equation.

Solution of a differential equation

A solution of a differential equation is a relation between the variables that satisfies the given differential equation. A solution of a differential equation is also called as the integral of the equation.

Examples

1. $x = a\cos nt$ is a solution of the differential equation $\frac{d^2x}{dt^2} + n^2x = 0$.

2. For the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$, $y = e^{-x}$ and $y = e^{-2x}$ are solutions.

General Solution

A general or complete solution of a differential equation is the one in which the number of arbitrary constants is equal to the order of the differential equation.

Example. Consider

$$y = A\cos\alpha x + B\sin\alpha x \tag{1}$$

where A and B are the arbitrary constants.

Differentiating w.r.t. x we get

$$\frac{dy}{dx} = -A\alpha\sin\alpha x + B\alpha\cos\alpha x.$$

$$\frac{d^2y}{dx^2} = -A\alpha^2\cos\alpha x - B\alpha^2\sin\alpha x.$$

$$= -\alpha^2(A\cos\alpha x + B\sin\alpha x) = -\alpha^2 y.$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0. \tag{2}$$

Hence, (1) is a general solution of (2) as the number of arbitrary constants A, B is the same as the order of (2).

Particular Solution

A particular solution is a solution that can be obtained by giving particular values to the arbitrary constants in the general solution.

Example. $y = 2\cos\alpha x + \sin\alpha x$ is a particular solution of (2), since it can be derived from the general solution by giving initial values of $0, 2$ and α to x, y and $\dfrac{dy}{dx}$ respectively.

Linear independent solutions

Two solutions y_1 and y_2 of the differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{1}$$

are said to be linearly independent, if there exist constants c_1 and c_2 such that $c_1y_1 + c_2y_2 = 0$ implies $c_1 = 0$ and $c_2 = 0$.

If c_1 and c_2 are not both zero, then the two solutions y_1 and y_2 are said to be linearly dependent.

If y_1 and y_2 are any two solutions of (1) then the linear combination $c_1y_1 + c_2y_2$ where c_1 and c_2 are constants is also a solution of (1).

Example. $y_1 = e^{-x}$ and $y_2 = e^{-2x}$ are two independent solutions of the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0.$$

Then, the general solution is $y = c_1e^{-x} + c_2e^{-2x}$.

Linear Differential Equation

A linear differential equation is an equation in which the dependent variable and its derivatives occur only in the first degree and there is no product of dependent variable and derivative or product of derivatives.

Note. A differential equation which is not linear is called a non linear differential equation.

Examples

1. $x\dfrac{d^2y}{dx^2} + y = x^2$ is linear.
2. $y\dfrac{dy}{dx} + x^2 = 0$ is nonlinear.

5.2 Linear differential equation with constant coefficients

The general form of the n^{th} order linear ordinary differential equation with constant coefficients is

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = Q(x) \tag{1}$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants with $a_0 \neq 0$. If $Q(x) = 0$ then

$$(1) \Rightarrow a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0 \tag{2}$$

which is the homogeneous equation corresponding to (1).

The general solution of (2) is called the complementary function of (1) and it is denoted by y_c. The general solution of (2) contains n arbitrary constants. A solution which contains no arbitrary constants is a particular solution.

If y_p is a particular solution of (1), then the general solution of (1) is $y = y_c + y_p$. This is also called as the complete solution of the ordinary differential equation.

Computation of Complementary function

Let $\frac{d}{dx} = D, \frac{d^2}{dx^2} = D^2, \ldots, \frac{d^n}{dx^n} = D^n$.

Then (1) becomes

$$(a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = Q(x) \tag{3}$$

The auxiliary equation is

$$a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_{n-1}m + a_n = 0.$$

Let $m_1, m_2, \ldots, m_n$ be its roots.

Case (i) If $m_1, m_2, \ldots, m_n$ are real and different then

$$y_c = CF = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}.$$

Case (ii) If some of the roots (say r) are equal, where $r < n$

i.e $m_1 = m_2 = \cdots = m_r = m$, then

$$C.F = y_c = (c_1 + c_2x + c_3x^2 + \cdots + c_rx^{r-1})e^{mx} + c_{r+1}e^{m_{r+1}x} + \cdots + c_ne^{m_nx}.$$

In particular, if two roots are equal say $m_1 = m_2 = m$ then,

$$y_c = (c_1 + c_2 x)e^{mx} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

Case (iii) If two roots are complex, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ and the other roots are real and different, then

$$y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

Case (iv) If $m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta$ and the other roots are real and different, then

$$y_c = [(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]e^{\alpha x} + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}.$$

Computation of particular integral (y_p)

Let $f(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$.

Then (3) can be written as $f(D)y = Q(x)$.

$$\text{Now } y_p = P.I. = \frac{1}{f(D)} Q(x)$$

5.2.1 Type I

Let $Q(x) = e^{\alpha x}$.

Case (i) If α is not equal to any of the $m_i (i = 1, 2, \ldots, n)$, then $f(\alpha) \neq 0$. In this case $y_p = PI = \dfrac{e^{\alpha x}}{f(\alpha)}$ (Replace D by α).

Case (ii) If α is equal to some of the $m_i' s$(say $m_1 = m_2 = \cdots = m_r$) then $f(D) = (D - \alpha)^r g(D)$ where $g(\alpha) \neq 0$.

$$\therefore y_p = PI = \frac{1}{(D-\alpha)^r g(D)} e^{\alpha x} = \frac{e^{\alpha x} x^r}{g(\alpha) r!}.$$

In particular, if $r = 1$, then

$$y_p = \frac{1}{(D-\alpha) g(D)} e^{\alpha x} = \frac{e^{\alpha x}}{g(\alpha)} x.$$

When $r = 2, y_p = \dfrac{1}{(D-\alpha)^2 g(D)} e^{\alpha x} = \dfrac{e^{\alpha x}}{g(\alpha)} \dfrac{x^2}{2!}$.

When $r = 3, y_p = \dfrac{1}{(D-\alpha)^3 g(D)} e^{\alpha x} = \dfrac{e^{\alpha x}}{g(\alpha)} \dfrac{x^3}{3!}$ etc.

Worked Examples

Example 5.5. Solve $(D^2 - 4)y = 1$. [Dec 2013]

Solution. The auxiliary equation is

$$m^2 - 4 = 0 \quad \Rightarrow \quad m^2 = 4 \quad \Rightarrow \quad m = \pm 2.$$

$$\therefore y_c = c_1 e^{2x} + c_2 e^{-2x}.$$

$$y_p = \frac{1}{D^2-4} = \frac{1}{D^2-4} e^{0x} = \frac{e^{0x}}{0-4} = -\frac{1}{4}.$$

Solution is $y = y_c + y_p = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{4}$.

Example 5.6. Solve $(4D^2 - 4D + 1)y = 4$. [Jun 1996]

Solution. A.E is $4m^2 - 4m + 1 = 0 \Rightarrow (2m-1)^2 = 0 \Rightarrow m = \dfrac{1}{2}, \dfrac{1}{2}$.

$$y_c = e^{\frac{1}{2}x}(c_1 + c_2 x).$$

$$y_p = \frac{1}{4D^2 - 4D + 1}(4) = \frac{1}{4D^2 - 4D + 1} e^{0x} = \frac{4}{1} = 4.$$

Solution is $y = y_c + y_p = (c_1 + c_2 x)e^{\frac{x}{2}} + 4$.

Example 5.7. Find the particular integral of $(D^2 - 2D + 1)y = \cos hx$.

[Jun 2013, Jun 2005]

Solution.

$$\begin{aligned}
PI &= \frac{1}{D^2 - 2D + 1} \cos hx = \frac{1}{(D-1)^2} \left(\frac{e^x + e^{-x}}{2} \right) \\
&= \frac{1}{2} \left[\frac{1}{(D-1)^2} e^x + \frac{1}{(D-1)^2} e^{-x} \right] \\
&= \frac{1}{2} \left[e^x \frac{x^2}{2} + \frac{e^{-x}}{(-1-1)^2} \right] \\
&= \frac{1}{2} \left[\frac{x^2 e^x}{2} + \frac{e^{-x}}{4} \right] = \frac{x^2 e^x}{4} + \frac{e^{-x}}{8}.
\end{aligned}$$

Example 5.8. Find the particular integral of $(D^2 - 4)y = \cosh 2x$. [May 2011]

Solution.

$$\begin{aligned}
PI &= \frac{1}{D^2-4}\cosh 2x \\
&= \frac{1}{(D+2)(D-2)}\left[\frac{e^{2x}+e^{-2x}}{2}\right] \\
&= \frac{1}{2}\left[\frac{1}{(D+2)(D-2)}e^{2x} + \frac{1}{(D+2)(D-2)}e^{-2x}\right] \\
&= \frac{1}{2}\left[\frac{x}{4}e^{2x} + \frac{x}{-4}e^{-2x}\right] \\
&= \frac{x}{8}[e^{2x} - e^{-2x}] \\
&= \frac{x}{4}\sinh 2x.
\end{aligned}$$

Example 5.9. Find the particular integral of $(D^3 - 1)y = e^{2x}$. [May 2005]

Solution. $PI = \dfrac{1}{D^3-1}e^{2x} = \dfrac{e^{2x}}{2^3-1} = \dfrac{e^{2x}}{7}$.

Example 5.10. Find the particular integral of $(D-1)^2 = \sinh x$. [Nov 2003]

Solution.

$$\begin{aligned}
PI &= \frac{1}{(D-1)^2}\sinh x = \frac{1}{(D-1)^2}\left[\frac{e^x - e^{-x}}{2}\right] \\
&= \frac{1}{2}\left[\frac{1}{(D-1)^2}e^x - \frac{1}{(D-1)^2}e^{-x}\right] \\
&= \frac{1}{2}\left[\frac{x^2}{2}e^x - \frac{1}{4}e^{-x}\right] \\
&= \frac{x^2}{4}e^x - \frac{e^{-x}}{8}.
\end{aligned}$$

Example 5.11. Solve $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = -2\cosh x$.

Solution. The A.E is

$$m^2 + 4m + 5 = 0 \Rightarrow (m+2)^2 + 5 - 4 = 0$$

$$\Rightarrow (m+2)^2 = -1 \Rightarrow m + 2 = \pm i \Rightarrow m = -2 \pm i.$$

$$y_c = e^{-2x}(c_1 \cos x + c_2 \sin x).$$

$$\begin{aligned}
y_p &= \frac{2}{2(D^2+4D+5)}[-(e^x+e^{-x})] \\
&= \frac{-1}{D^2+4D+5}e^x - \frac{1}{D^2+4D+5}e^{-x} \\
&= \frac{-e^x}{1+4+5} - \frac{e^{-x}}{1-4+5} \\
&= \frac{-e^x}{10} - \frac{e^{-x}}{2}.
\end{aligned}$$

The general solution is $y = y_c + y_p$

$$y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2}.$$

Example 5.12. Find the particular integral of $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$.

[May 2006]

Solution.

$$\begin{aligned}
PI &= \frac{1}{(D+2)(D-1)^2}[e^{-2x} + 2\sinh x] \\
&= \frac{1}{(D+2)(D-1)^2}[e^{-2x} + e^x - e^{-x}] \\
&= \frac{1}{(D+2)(D-1)^2}e^{-2x} + \frac{1}{(D+2)(D-1)^2}e^x - \frac{1}{(D+2)(D-1)^2}e^{-x} \\
&= \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^x - \frac{e^{-x}}{1(4)} \\
&= \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^x - \frac{e^{-x}}{4}.
\end{aligned}$$

Example 5.13. Solve $(D^2+1)^2 y = 0$. [May 2008]

Solution. A.E is $(m^2+1)^2 = 0$.

$m^2 = -1, m^2 = -1 \implies m = \pm i, m = \pm i.$

The complex roots are repeated.

$\therefore y_c = CF = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x.$

$y_p = PI = 0.$

Solution is $y = y_c + y_p$

$$y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x.$$

Example 5.14. Solve $(D^2+1)y = 0$ given that $y(0) = 0, y'(0) = 1$. [May 1996]

Solution. A.E is $m^2 + 1 = 0 \Longrightarrow m^2 = -1 \Longrightarrow m = \pm i$.

$$CF = y_c = e^{0x}[c_1 \cos x + c_2 \sin x] = c_1 \cos x + c_2 \sin x.$$

$$y_p = PI = 0.$$

Solution is $y = y_c + y_p$

$$y = c_1 \cos x + c_2 \sin x.$$

Given: $y(0) = 0$.

$\Longrightarrow c_1 = 0$

$y' = -c_1 \sin x + c_2 \cos x$.

Given: $y'(0) = 1 \Longrightarrow c_2 = 1$.

Hence, solution is, $y = \sin x$.

Example 5.15. Find the particular integral of $(D^2 - 4D + 4)y = 2^x$. [Dec 2012]

Solution.
$$P.I = \frac{1}{D^2 - 4D + 4} 2^x = \frac{1}{(D-2)^2} e^{x \log 2}$$
$$= \frac{2^x}{(\log 2 - 2)^2}.$$

Example 5.16. Find the particular integral of $(D^2 - 4)y = 3^x$. [May 2007]

Solution. $PI = \dfrac{1}{D^2-4} 3^x = \dfrac{1}{D^2-4} e^{x \log 3} = \dfrac{1}{(\log 3)^2 - 4} e^{x \log 3} = \dfrac{3^x}{(\log 3)^2 - 4}$.

Example 5.17. Solve $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 5y = e^{2x}$.

Solution. $(D^2 + 6D + 5)y = e^{2x}$.

A.E is, $m^2 + 6m + 5 = 0 \Rightarrow (m+1)(m+5) = 0 \Rightarrow m = -1, -5$.

$$y_c = c_1 e^{-x} + c_2 e^{-5x}.$$

$$y_p = \frac{1}{D^2 + 6D + 5} e^{2x} = \frac{1}{(D+1)(D+5)} e^{2x} = \frac{e^{2x}}{3(7)} = \frac{e^{2x}}{21}.$$

Solution is, $y = y_c + y_p = c_1 e^{-x} + c_2 e^{-5x} + \dfrac{e^{2x}}{21}$.

Example 5.18. Solve $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{2x}$.

Solution. The A.E is

$$m^3 + 2m^2 + m = 0 \Rightarrow m(m^2 + 2m + 1) = 0$$

$$\Rightarrow m(m+1)^2 = 0 \Rightarrow m = 0, m = -1, -1.$$

$$y_c = c_1 e^{0x} + (c_2 + c_3 x)e^{-x}$$

$$y_c = c_1 + e^{-x}(c_2 + c_3 x).$$

$$\begin{aligned} y_p &= \frac{1}{D^3 + 2D^2 + D} e^{2x} \\ &= \frac{1}{D(D+1)^2} e^{2x} \\ &= \frac{e^{2x}}{2(3)^2} = \frac{e^{2x}}{18}. \end{aligned}$$

The general solution is $y = y_c + y_p$

$$y = c_1 + e^{-x}(c_2 + c_3 x) + \frac{e^{2x}}{18}.$$

Example 5.19. Solve $(D^2 - 2D + 1)y = (e^x + 1)^2$. [Dec 2010]

Solution. AE is $m^2 - 2m + 1 = 0 \Longrightarrow (m-1)^2 = 0 \Longrightarrow m = 1, 1$.

$$\begin{aligned} y_c &= e^x(c_1 + c_2 x) \\ y_p &= \frac{1}{D^2 - 2D + 1}(e^x + 1)^2 \\ &= \frac{1}{(D-1)^2}(e^{2x} + 1 + 2e^x) \\ &= \frac{1}{(D-1)^2} e^{2x} + \frac{1}{(D-1)^2} e^{0x} + 2\frac{1}{(D-1)^2} e^x \\ &= \frac{e^{2x}}{1} + 1 + 2\frac{x^2}{2} e^x \\ &= e^{2x} + 1 + x^2 e^x. \end{aligned}$$

Solution is $y = y_c + y_p$

$$y = e^x(c_1 + c_2 x) + e^{2x} + 1 + x^2 e^x.$$

5.2.2 Type II

$Q(x) = \sin ax$ or $\cos ax$ where a is a constant.

$$y_p = P.I. = \frac{1}{f(D)} \sin ax \text{ or } \frac{1}{f(D)} \cos ax.$$

Working rule. Replace D^2 by $-a^2$ and evaluate till D is eliminated.

If $f(D) = D^2 + a^2$, then $f(D) = 0$ when D^2 is replaced by $-a^2$. In this case

$$P.I. = \frac{1}{D^2 + a^2} \sin ax = \frac{x}{2} \int \sin ax dx = \frac{-x}{2a} \cos ax.$$

and

$$\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2} \int \cos ax dx = \frac{x}{2a} \sin ax.$$

Another method

$$P.I. = \frac{1}{f(D)} \sin ax = \text{Imaginary Part of } \frac{1}{f(D)} e^{iax}$$

and

$$P.I. = \frac{1}{f(D)} \cos ax = \text{Real Part of } \frac{1}{f(D)} e^{iax}$$

which can be evaluated using Type I.

Note

Suppose $f(D) = 0$ when D^2 is replaced by $-a^2$, then $f(D)$ is of the form $f(D^2)$. Then, $P.I. = \frac{1}{f(D^2)} \cos ax = \frac{\cos ax}{f(-a^2)}$. If $f(-a^2) = 0$ then $P.I. = \frac{x \cos ax}{f'(-a^2)}$ if $f'(-a^2) \neq 0$. If $f'(-a^2) = 0$, then $P.I. = \frac{x^2 \cos ax}{f''(-a^2)}$ if $f''(-a^2) \neq 0$, and so on. In a similar way the particular integral for $\sin ax$ can be evaluated replacing $\cos ax$.

Worked Examples

Example 5.20. Find the particular integral of $(D^2 + 4)y = \sin 2x$.

[Dec 2009, Dec 2011]

Solution.

$$PI = \frac{1}{D^2+4}\sin 2x = \frac{x}{2}\int \sin 2x dx \quad [\because f(-a^2)=0]$$
$$= \frac{x}{2}\left(\frac{-\cos 2x}{2}\right)$$
$$= -\frac{x}{4}\cos 2x.$$

Example 5.21. Find the particular integral of $(D^2+1)y = \sin x \sin 2x$. [May 1997]

Solution.
$$PI = \frac{1}{D^2+1}(\sin x \sin 2x)$$
$$= \frac{-1}{D^2+1}\left[\frac{\cos 3x - \cos x}{2}\right]$$
$$= \frac{-1}{2}\left[\frac{1}{D^2+1}\cos 3x - \frac{1}{D^2+1}\cos x\right]$$
$$= \frac{-1}{2}\left[\frac{\cos 3x}{-3^2+1} - \frac{x}{2}\int \cos x dx\right]$$
$$= -\frac{1}{2}\left[\frac{\cos 3x}{-8} - \frac{x}{2}\sin x\right]$$
$$= \frac{\cos 3x}{16} + \frac{x\sin x}{4}.$$

Example 5.22. Find the particular integral of $(D^2+1)y = \sin x$. [Jun 2010]

Solution. P.I $= \frac{1}{D^2+1}\sin x = \frac{x}{2}\int \sin x dx = \frac{-x}{2}\cos x.$

Example 5.23. Find the particular integral of $(D^2+4D+2)y = \sin 3x$. [May 1998]

Solution.
$$PI = \frac{1}{D^2+4D+2}\sin 3x = \frac{1}{-9+4D+2}\sin 3x = \frac{1}{4D-7}\sin 3x$$
$$= \frac{(4D+7)}{(4D+7)(4D-7)}\sin 3x = \frac{4\cos 3x(3) + 7\sin 3x}{16D^2-49}$$
$$= \frac{12\cos 3x + 7\sin 3x}{16(-9)-49}$$
$$= \frac{12\cos 3x + 7\sin 3x}{-144-49}$$
$$= -\frac{1}{193}[12\cos 3x + 7\sin 3x].$$

Example 5.24. Find the particular integral of $(D^2+4)y = \cos 2x$. [May 2001]

Solution. $PI = \frac{1}{D^2+4}\cos 2x = \frac{x}{2}\int \cos 2x dx = \frac{x}{2}\frac{\sin 2x}{2} = \frac{x\sin 2x}{4}.$

Example 5.25. Find the particular integral of $(D^2+1)^2 y = \sin 2x$.

Solution. $PI = \dfrac{1}{(D^2+1)^2}\sin 2x = \dfrac{\sin 2x}{(-4+1)^2} = \dfrac{\sin 2x}{9}$.

Example 5.26. Find the particular integral of $(D^2+1)y = \sin^2 x$.

Solution.

$$\begin{aligned}
PI &= \frac{1}{D^2+1}\sin^2 x = \frac{1}{D^2+1}\left(\frac{1-\cos 2x}{2}\right)\\
&= \frac{1}{2}\left[\frac{1}{D^2+1} - \frac{1}{D^2+1}\cos 2x\right]\\
&= \frac{1}{2}\left[\frac{1}{D^2+1}e^{0x} - \frac{\cos 2x}{-4+1}\right]\\
&= \frac{1}{2}\left[1 + \frac{\cos 2x}{3}\right]\\
&= \frac{1}{2} + \frac{\cos 2x}{6}.
\end{aligned}$$

Example 5.27. Find the particular integral of $(D^2+1)y = \sin(2x+5)$.

Solution. $PI = \dfrac{1}{D^2+1}\sin(2x+5) = \dfrac{\sin(2x+5)}{-4+1} = -\left[\dfrac{\sin(2x+5)}{3}\right]$.

Example 5.28. Find the particular integral of $(D^2+4D+8)y = \cos(2x+3)$.

Solution.

$$\begin{aligned}
PI &= \frac{1}{D^2+4D+8}\cos(2x+3) = \frac{1}{-4+4D+8}\cos(2x+3)\\
&= \frac{1}{4D+4}\cos(2x+3) = \frac{1}{4}\frac{1}{D+1}\cos(2x+3)\\
&= \frac{1}{4}\left[\frac{(D-1)}{(D-1)(D+1)}\cos(2x+3)\right]\\
&= \frac{1}{4}\left[\frac{-2\sin(2x+3)-\cos(2x+3)}{D^2-1}\right]\\
&= -\frac{1}{4}\left[\frac{2\sin(2x+3)+\cos(2x+3)}{-4-1}\right]\\
&= \frac{1}{20}\left[2\sin(2x+3)+\cos(2x+3)\right].
\end{aligned}$$

Example 5.29. Solve $(D^2-4D+3)y = \sin 3x\cos 2x$. [May 2007]

Solution. The A.E is, $m^2-4m+3=0 \Rightarrow (m-1)(m-3)=0 \Rightarrow m=1,3$.

$$y_c = c_1e^x + c_2e^{3x}.$$

$$\begin{aligned}
y_p &= \frac{1}{D^2-4D+3}\sin 3x\cos 2x \\
&= \frac{1}{2}\frac{1}{D^2-4D+3}(\sin 5x + \sin x) \\
&= \frac{1}{2}\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{2}\frac{1}{D^2-4D+3}\sin x \\
&= PI_1 + PI_2.
\end{aligned}$$

$$\begin{aligned}
P.I_1 &= \frac{1}{2}\frac{1}{D^2-4D+3}\sin 5x = \frac{1}{2}\frac{1}{(-25-4D+3)}\sin 5x \\
&= \frac{1}{2}\frac{1}{(-22-4D)}\sin 5x = \frac{-1}{4}\frac{1}{(11+2D)}\sin 5x \\
&= -\frac{1}{4}\frac{(2D-11)}{(2D+11)(2D-11)}\sin 5x \\
&= -\frac{1}{4}\frac{2\times 5\cos 5x - 11\sin 5x}{4D^2-121} \\
&= -\frac{1}{4}\frac{10\cos 5x - 11\sin 5x}{4(-25)-121} \\
&= -\frac{1}{4}\frac{10\cos 5x - 11\sin 5x}{-221} \\
&= \frac{10\cos 5x - 11\sin 5x}{884}.
\end{aligned}$$

$$\begin{aligned}
P.I_2 &= \frac{1}{2}\frac{1}{D^2-4D+3}\sin x = \frac{1}{2}\left[\frac{1}{(-1)-4D+3}\right]\sin x \\
&= \frac{1}{2}\frac{1}{2-4D}\sin x = \frac{1}{4}\frac{1}{1-2D}\sin x \\
&= \frac{1}{4}\frac{(1+2D)}{(1-4D^2)}\sin x \\
&= \frac{1}{4}\frac{\sin x + 2\cos x}{5} \\
&= \frac{\sin x + 2\cos x}{20}.
\end{aligned}$$

The general solution is $y = y_c + y_p$

$$y = c_1e^x + c_2e^{3x} + \frac{10\cos 5x - 11\sin 5x}{884} + \frac{\sin x + 2\cos x}{20}.$$

Example 5.30. Solve $(D^2+1)y = \sin^2 x$. [May 2006]

Solution. $(D^2+1)y = \dfrac{1-\cos 2x}{2}.$

The A.E is $m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$

$$
\begin{aligned}
y_c &= e^{0x}(c_1 \cos x + c_2 \sin x) = c_1 \cos x + c_2 \sin x \\
y_p &= \frac{1}{2(D^2+1)}(1 - \cos 2x) = \frac{1}{2(D^2+1)} - \frac{1}{2(D^2+1)} \cos 2x \\
&= \frac{1}{2(D^2+1)} e^{0x} - \frac{\cos 2x}{2(-4+1)} = \frac{1}{2} + \frac{1}{6} \cos 2x.
\end{aligned}
$$

The general solution is $y = y_c + y_p$

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} + \frac{1}{6} \cos 2x.$$

Example 5.31. Solve the equation $(D^2 + 16)y = \cos^3 x$. [Dec. 2010]

Solution. A.E.is

$$
\begin{aligned}
m^2 + 16 = 0 \Rightarrow m^2 &= -16 \Rightarrow m = \pm 4i \\
y_c &= e^{0x}(c_1 \cos 4x + c_2 \sin 4x) \\
y_c &= c_1 \cos 4x + c_2 \sin 4x.
\end{aligned}
$$

$$
\begin{aligned}
y_p &= \frac{1}{D^2+16} \cos^3 x \\
&= \frac{1}{D^2+16} \left[\frac{1}{4} (\cos 3x + 3 \cos x) \right] \\
&= \frac{1}{4} \left[\frac{1}{D^2+16} \cos 3x + 3 \frac{1}{D^2+16} \cos x \right] \\
&= \frac{1}{4} \left[\frac{\cos 3x}{-9+16} + \frac{3 \cos x}{-1+16} \right] = \frac{1}{4} \left[\frac{\cos 3x}{7} + \frac{3 \cos x}{15} \right]
\end{aligned}
$$

$$
\begin{aligned}
\cos 3\theta &= 4 \cos^3 \theta - 3 \cos \theta \\
4 \cos^3 \theta &= \cos 3\theta + 3 \cos \theta \\
\cos^3 \theta &= \frac{1}{4}(\cos 3\theta + 3 \cos \theta)
\end{aligned}
$$

$$y_p = \frac{\cos 3x}{28} + \frac{3 \cos x}{20}.$$

Solution is $y = y_c + y_p = c_1 \cos 4x + c_2 \sin 4x + \dfrac{\cos 3x}{28} + \dfrac{3 \cos x}{20}$

Example 5.32. Solve $(D^2 - 3D + 2)y = 2\cos(2x+3) + 2e^x$. [Dec 2011, May 2005]

Solution. The A.E is

$$m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2.$$

$$y_c = c_1 e^x + c_2 e^{2x}.$$

$$\begin{aligned}
y_p &= \frac{1}{D^2 - 3D + 2} 2\cos(2x+3) + \frac{1}{D^2 - 3D + 2} 2e^x \\
&= PI_1 + PI_2.
\end{aligned}$$

$$\begin{aligned}
PI_1 &= 2\frac{1}{D^2 - 3D + 2}\cos(2x+3) = 2\frac{1}{-4 - 3D + 2}\cos(2x+3) \\
&= 2\frac{1}{-2 - 3D}\cos(2x+3) = -2\frac{1}{2 + 3D}\cos(2x+3) \\
&= -2\frac{(2 - 3D)}{4 - 9D^2}\cos(2x+3) \\
&= -2\frac{2\cos(2x+3) + 6\sin(2x+3)}{4 + 36} \\
&= -\frac{1}{20}2[\cos(2x+3) + 3\sin(2x+3)] \\
&= -\frac{1}{10}[\cos(2x+3) + 3\sin(2x+3)].
\end{aligned}$$

$$PI_2 = 2\frac{1}{(D-1)(D-2)}e^x = 2x\frac{e^x}{-1} = -2xe^x.$$

The general solution is $y = y_c + y_p$

$$y = c_1 e^x + c_2 e^{2x} - \frac{1}{10}[\cos(2x+3) + 3\sin(2x+3)] - 2xe^x.$$

Example 5.33. Solve $(D^2 + 16)y = e^{-3x} + \cos 4x$.

Solution. The A.E is

$$m^2 + 16 = 0 \Rightarrow m^2 = -16 \Rightarrow m = \pm 4i.$$

$$y_c = e^{0x}(c_1\cos 4x + c_2\sin 4x) = c_1\cos 4x + c_2\sin 4x.$$

$$\begin{aligned}
y_p &= \frac{1}{D^2 + 16}e^{-3x} + \frac{1}{D^2 + 16}\cos 4x = \frac{e^{-3x}}{9 + 16} + \frac{x}{2}\int \cos 4x dx \\
&= \frac{e^{-3x}}{25} + \frac{x}{2}\frac{\sin 4x}{4} = \frac{e^{-3x}}{25} + \frac{x\sin 4x}{8}.
\end{aligned}$$

The general solution is $y = y_c + y_p$.

$$y = c_1\cos 4x + c_2\sin 4x + \frac{e^{-3x}}{25} + \frac{x\sin 4x}{8}.$$

5.2.3 Type III

$Q(x) = x^m$ where m is a positive integer.
$PI = \frac{x^m}{f(D)} = \frac{x^m}{k[1 \pm g(D)]} = \frac{1}{k}[1 \pm g(D)]^{-1}x^m$.
Expand $[1 \pm g(D)]^{-1}$ using binomial series expansion upto D^m and operate on x^m term by term, we obtain the required particular integral.

Note. It is important to note that $\frac{1}{D}f(x) = \int f(x)dx$. $\frac{1}{D^2}f(x)$ means integrate $f(x)$ w.r.t x twice and so on.

The following results will be useful for our discussion.

(i) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$

(ii) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

(iii) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$

(iv) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$.

Example 5.34. Find the particular integral of $(D^2 + D)y = x^2 + 2x + 4$. [Jun 2000]

Solution.
$$\begin{aligned} PI &= \frac{1}{D^2 + D}(x^2 + 2x + 4) = \frac{1}{D(D + 1)}(x^2 + 2x + 4) \\ &= \frac{1}{D}(1 + D)^{-1}(x^2 + 2x + 4) = \frac{1}{D}(1 - D + D^2 - \cdots)(x^2 + 2x + 4) \\ &= \frac{1}{D}(x^2 + 2x + 4 - 2x - 2 + 2) = \frac{1}{D}(x^2 + 4) \\ &= \int (x^2 + 4)dx \qquad [\because \frac{1}{D} \text{ stands for integration}] \\ &= \frac{x^3}{3} + 4x. \end{aligned}$$

Example 5.35. Find the particular integral of $(D^2 + 5D + 1)y = x^2$. [May 2002]

Solution. $PI = \frac{1}{D^2 + 5D + 1}x^2 = (1 + (D^2 + 5D))^{-1}x^2$

$$\begin{aligned} &= (1 - (D^2 + 5D) + (D^2 + 5D)^2 + \cdots)x^2 = (1 - D^2 - 5D + 25D^2 + \cdots)x^2 \\ &= (1 - 5D + 24D^2)x^2 = x^2 - 5(2x) + 24(2) \\ &= x^2 - 10x + 48. \end{aligned}$$

Example 5.36. Solve $(D^3 - 3D^2 - 6D + 8)y = x$. [Dec 2006]

Solution. A.E is $m^3 - 3m^2 - 6m + 8 = 0$

$$(m-1)(m+2)(m-4) = 0 \Rightarrow m = 1, -2, 4.$$

$$y_c = c_1e^x + c_2e^{-2x} + c_3e^{4x}.$$

$$\begin{aligned}
y_p &= \frac{1}{D^3 - 3D^2 - 6D + 8}x \\
&= \frac{1}{8\left[1 + \frac{D^3-3D^2-6D}{8}\right]}x \\
&= \frac{1}{8}\left[1 + \frac{D^3 - 3D^2 - 6D}{8}\right]^{-1} x \\
&= \frac{1}{8}\left(1 - \left(\frac{D^3 - 3D^2 - 6D}{8}\right) + \cdots\right)x \\
&= \frac{1}{8}\left[1 + \frac{6}{8}D + \cdots\right]x = \frac{1}{8}\left[x + \frac{6}{8}\right] \\
&= \frac{1}{8}\left[x + \frac{3}{4}\right].
\end{aligned}$$

The general solution is $y = y_c + y_p$

$$y = c_1e^x + c_2e^{-2x} + c_3e^{4x} + \frac{1}{8}\left[x + \frac{3}{4}\right].$$

Example 5.37. Solve $(D^3 - D^2 - D + 1)y = 1 + x^2$. [Apr 2007]

Solution. The A.E is $m^3 - m^2 - m + 1 = 0 \Rightarrow m^2(m-1) - (m-1) = 0$

$$\Rightarrow (m-1)(m^2-1) = 0 \Rightarrow m = 1, 1, -1.$$

$$y_c = c_1e^{-x} + e^x(c_2 + c_3x).$$

$$\begin{aligned}
y_p &= \frac{1}{D^3 - D^2 - D + 1}(1 + x^2) \\
&= \frac{1}{D^3 - D^2 - D + 1}e^{0x} + \frac{1}{D^3 - D^2 - D + 1}x^2 \\
&= 1 + \frac{x^2}{1 - (D + D^2 - D^3)} = 1 + [1 - (D + D^2 - D^3)]^{-1}x^2 \\
&= 1 + [1 + (D + D^2 - D^3) + (D + D^2 - D^3)^2 + \cdots]x^2 \\
&= 1 + [1 + D + D^2 + D^2 + \cdots]x^2 \\
&= 1 + [x^2 + 2x + 4] = x^2 + 2x + 5.
\end{aligned}$$

The general solution is $y = y_c + y_p$

$$y = c_1 e^{-x} + e^{x}(c_2 + c_3 x) + x^2 + 2x + 5.$$

Example 5.38. Solve $(D^2 + D)y = x^2 + 2x + 4$. [Dec 2008]

Solution. The A.E is

$$m^2 + m = 0 \Rightarrow m(m+1) = 0 \Rightarrow m = 0, -1.$$

$$y_c = c_1 + c_2 e^{-x}.$$

$$\begin{aligned} y_p &= \frac{1}{D^2 + D}(x^2 + 2x + 4) = \frac{1}{D(1+D)}(x^2 + 2x + 4) \\ &= \frac{1}{D}(1+D)^{-1}(x^2 + 2x + 4) \\ &= \frac{1}{D}[1 - D + D^2 - \cdots](x^2 + 2x + 4) \\ &= \frac{1}{D}[x^2 + 2x + 4 - (2x+2) + 2] \\ &= \frac{1}{D}[x^2 + 4] \\ &= \int (x^2 + 4)dx \\ &= \frac{x^3}{3} + 4x. \end{aligned}$$

The Solution is $y = y_c + y_p$

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x.$$

5.2.4 Type IV

$Q(x) = e^{ax} g(x)$ where $g(x)$ may be x^m or $\sin ax$ or $\cos ax$. In this case $P.I = \frac{1}{f(D)} e^{ax} g(x) = e^{ax} \frac{g(x)}{f(D+a)}$. This will fall into any of the previous three types which can be evaluated by the known methods.

5.2.5 Type V

$Q(x) = x^m \cos ax$ or $x^m \sin ax$. In this case

$$\begin{aligned} P.I &= \frac{1}{f(D)} x^m \cos ax \text{ or } \frac{1}{f(D)} x^m \sin ax. \\ &= \text{R.P } \frac{1}{f(D)} e^{iax} x^m \text{ or I.P } \frac{1}{f(D)} x^m e^{iax}. \\ &= \text{R.P } e^{iax} \frac{1}{f(D+ia)} x^m \text{ or I.P } e^{iax} \frac{1}{f(D+ia)} x^m \end{aligned}$$

which can be evaluated by earlier methods.

Worked Examples

Example 5.39. Find the particular integral of $(D^2 + 4D + 4)y = xe^{-2x}$. [May 2005]

Solution.

$$\begin{aligned} PI &= \frac{1}{D^2 + 4D + 4} xe^{-2x} = \frac{1}{(D+2)^2} xe^{-2x} \\ &= e^{-2x} \frac{1}{(D-2+2)^2} x = e^{-2x} \frac{1}{D^2} x \\ &= e^{-2x} \frac{1}{D} \frac{1}{D} x = e^{-2x} \frac{1}{D} \int x dx \\ &= e^{-2x} \frac{1}{D} \left(\frac{x^2}{2} \right) = \frac{e^{-2x}}{2} \int x^2 dx \\ &= \frac{e^{-2x}}{2} \frac{x^3}{3} = \frac{x^3 e^{-2x}}{6}. \end{aligned}$$

Example 5.40. Find the particular integral of $(D^2 + 1)y = xe^x$. [Jun 2003]

Solution.

$$\begin{aligned} P.I &= \frac{1}{D^2+1} xe^x = e^x \frac{1}{(D+1)^2 + 1} x = e^x \frac{1}{D^2 + 2D + 1 + 1} x \\ &= e^x \frac{1}{D^2 + 2D + 2} x = e^x \frac{1}{2\left[1 + \frac{D^2+2D}{2}\right]} x \\ &= \frac{e^x}{2} \left[1 + \frac{D^2 + 2D}{2} \right]^{-1} x \\ &= \frac{e^x}{2} \left[1 - \frac{D^2 + 2D}{2} + \cdots \right] x = \frac{e^x}{2} [1 - D \cdots] x \\ &= \frac{e^x}{2} [x - 1]. \end{aligned}$$

Example 5.41. Find the particular integral of $y'' + 2y' + 5y = e^{-x}\cos 2x$.[May 2005]

Solution.
$$P.I = \frac{1}{D^2+2D+5}e^{-x}\cos 2x = e^{-x}\frac{1}{(D-1)^2+2(D-1)+5}\cos 2x$$
$$= e^{-x}\frac{1}{D^2-2D+1+2D-2+5}\cos 2x = e^{-x}\frac{1}{D^2+4}\cos 2x$$
$$= e^{-x}\frac{x}{2}\int \cos 2x dx = e^{-x}\frac{x}{2}\frac{\sin 2x}{2}$$
$$= \frac{e^{-x}x\sin 2x}{4}.$$

Example 5.42. Find the particular integral of $(D^2 - 2D + 2)y = e^x\cos x$. [Dec 2010]

Solution.
$$P.I = \frac{1}{D^2-2D+2}e^x\cos x.$$
$$= e^x\cdot\frac{1}{(D+1)^2-2(D+1)+2}\cos x.$$
$$= e^x\cdot\frac{1}{D^2+\cancel{2D}+1-\cancel{2D}-\cancel{2}+\cancel{2}}\cos x.$$
$$= e^x\cdot\frac{1}{D^2+1}\cos x. = e^x\cdot x\frac{\cos x}{2D}$$
$$= \frac{xe^x}{2}\int \cos x dx = \frac{xe^x\sin x}{2}.$$

Example 5.43. Find the particular integral of $(D^2 - 2D + 4)y = e^x\cos x$. [Dec 2001]

Solution.
$$PI = \frac{1}{D^2-2D+4}e^x\cos x = e^x\frac{1}{(D+1)^2-2(D+1)+4}\cos x$$
$$= e^x\frac{1}{D^2+2D+1-2D-2+4}\cos x = e^x\frac{1}{D^2+3}\cos x$$
$$= e^x\frac{\cos x}{-1+3} = \frac{e^x\cos x}{2}.$$

Example 5.44. Find the particular integral of $(D-1)^2y = e^x\sin x$.

[May 2003, Jun 2012]

Solution.
$$PI = \frac{1}{(D-1)^2}e^x\sin x = e^x\frac{1}{(D+1-1)^2}\sin x$$
$$= e^x\frac{1}{D^2}\sin x = e^x\frac{1}{D}\frac{1}{D}\sin x$$
$$= e^x\frac{1}{D}\int \sin x dx = e^x\frac{1}{D}(-\cos x)$$
$$= -e^x\int \cos x dx = -e^x\sin x.$$

Example 5.45. Solve $(D^2+5D+4)y = e^{-x}\sin 2x$. [Dec 2012, May 2011]

Solution. A.E is $m^2+5m+4=0$

$$(m+1)(m+4)=0 \Rightarrow m=-1,-4.$$

$$y_c = c_1e^{-x}+c_2e^{-4x}.$$

$$\begin{aligned}
y_p &= \frac{1}{D^2+5D+4}e^{-x}\sin 2x.\\
&= e^{-x}\cdot\frac{1}{(D-1)^2+5(D-1)+4}\sin 2x.\\
&= e^{-x}\frac{1}{D^2-2D+1+5D-5+4}\sin 2x.\\
&= e^{-x}\cdot\frac{1}{D^2+3D}\sin 2x.\\
&= e^{-x}\cdot\frac{1}{-4+3D}\sin 2x.\\
&= e^{-x}\cdot\frac{3D+4}{(3D+4)(3D-4)}\sin 2x.\\
&= e^{-x}\cdot\frac{3D+4}{9D^2-16}\sin 2x.\\
&= e^{-x}\frac{3\times 2\cos 2x+4\sin 2x}{9\times(-4)-16}\\
&= e^{-x}\frac{6\cos 2x+4\sin 2x}{-52}\\
&= -\frac{e^{-x}}{-52}(6\cos 2x+4\sin 2x).
\end{aligned}$$

Solution is $y = y_c+y_p$

$$y = c_1e^{-x}+c_2e^{-4x}-\frac{e^{-x}}{52}(6\cos 2x+4\sin 2x)$$

Example 5.46. Solve $(D^2-2D+2)y = e^{2x}x^2+5+e^{-2x}$. [Jun 2009]

Solution. A.E. is

$$m^2-2m+2=0 \Rightarrow (m-1)^2+2-1=0$$

$$(m-1)^2+1=0 \Rightarrow (m-1)^2=-1$$

$$(m-1)=\pm i \Rightarrow m=1\pm i.$$

$$y_c = e^x(c_1 \cos x + c_2 \sin x).$$

$$\begin{aligned}
y_p &= \frac{1}{D^2 - 2D + 2}(e^{2x}x^2 + 5 + e^{-2x}) \\
&= \frac{1}{D^2 - 2D + 2}e^{2x}x^2 + \frac{5}{D^2 - 2D + 2} + \frac{1}{D^2 - 2D + 2}e^{-2x} \\
&= PI_1 + PI_2 + PI_3.
\end{aligned}$$

$$\begin{aligned}
PI_1 &= \frac{1}{D^2 - 2D + 2}e^{2x}x^2 = e^{2x}\frac{1}{(D+2)^2 - 2(D+2) + 2}x^2 \\
&= e^{2x}\frac{1}{D^2 + 4 + 4D - 2D - 4 + 2}x^2 = e^{2x}\frac{1}{D^2 + 2D + 2}x^2 \\
&= e^{2x}\frac{1}{2(1 + \frac{D^2+2D}{2})}x^2 = \frac{e^{2x}}{2}\left(1 + \frac{D^2 + 2D}{2}\right)^{-1}x^2 \\
&= \frac{e^{2x}}{2}[1 - \frac{D^2 + 2D}{2} + \left(\frac{D^2 + 2D}{2}\right)^2 + \dots]x^2 \\
&= \frac{e^{2x}}{2}[x^2 - \frac{1}{2}.2 - 2x + 2] \\
&= \frac{e^{2x}}{2}[x^2 - 1 - 2x + 2] = \frac{e^{2x}}{2}(x^2 - 2x + 1).
\end{aligned}$$

$$PI_2 = \frac{5}{D^2 - 2D + 2}e^{0x} = \frac{5}{2}.$$

$$PI_3 = \frac{1}{D^2 - 2D + 2}e^{-2x} = \frac{e^{-2x}}{4 + 4 + 2} = \frac{e^{-2x}}{10}.$$

The solution is $y = y_c + y_p$

$$y = e^x(c_1 \cos x + c_2 \sin x) + \frac{e^{2x}}{2}(x^2 - 2x + 1) + \frac{5}{2} + \frac{e^{-2x}}{10}.$$

Example 5.47. Solve $\dfrac{d^2y}{dx^2} - 4y = x \sinh x.$ [Apr 2006]

Solution. $(D^2 - 4)y = x \sinh x$

A.E. is

$m^2 - 4 = 0 \Rightarrow m^2 = 4 \Rightarrow m = \pm 2.$

$y_c = c_1e^{2x} + c_2e^{-2x}$

$$\begin{aligned}
y_p = PI &= \frac{1}{D^2 - 4}x \sinh x = \frac{1}{2(D^2 - 4)}x(e^x - e^{-x}) \\
&= \frac{1}{2}\frac{1}{D^2 - 4}xe^x - \frac{1}{2}\frac{1}{D^2 - 4}xe^{-x}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^x}{2}\frac{1}{(D+1)^2-4}x - \frac{e^{-x}}{2}\frac{1}{(D-1)^2-4}x \\
&= \frac{e^x}{2}\frac{1}{D^2+2D+1-4}x - \frac{e^{-x}}{2}\frac{1}{D^2-2D+1-4}x \\
&= \frac{e^x}{2}\frac{1}{D^2+2D-3}x - \frac{e^{-x}}{2}\frac{1}{D^2-2D-3}x \\
&= \frac{e^x}{2}\frac{1}{(-3)(1-\frac{D^2+2D}{3})}x - \frac{e^{-x}}{2}\frac{1}{(-3)(1-\frac{D^2-2D}{3})}x \\
&= \frac{e^x}{-6}\left(1-\frac{D^2+2D}{3}\right)^{-1}x + \frac{e^{-x}}{6}\left(1-\frac{D^2-2D}{3}\right)^{-1}x \\
&= \frac{-e^x}{6}\left[1+\frac{D^2+2D}{3}+\cdots\right]x + \frac{e^{-x}}{6}\left[1+\frac{D^2-2D}{3}+\cdots\right]x \\
P.I. &= \frac{-e^x}{6}\left[x+\frac{2}{3}\right] + \frac{e^{-x}}{6}\left[x-\frac{2}{3}\right].
\end{aligned}
$$

The solution is $y = y_c + y_p$

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{e^x}{6}\left[x+\frac{2}{3}\right] + \frac{e^{-x}}{6}\left[x-\frac{2}{3}\right]$$

Example 5.48. Solve $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = \dfrac{e^{-x}}{x^2}$. [Jun 2013, May 1989]

Solution. A.E is $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$.

$$
\begin{aligned}
y_c &= e^{-x}(c_1 + c_2x). \\
y_p &= \frac{1}{(D+1)^2}\frac{e^{-x}}{x^2} = e^{-x}\frac{1}{(D-1+1)^2}\frac{1}{x^2} \\
&= e^{-x}\frac{1}{D^2}\left(\frac{1}{x^2}\right) = e^{-x}\frac{1}{D}\int x^{-2}dx \\
&= e^{-x}\frac{1}{D}\left[\frac{x^{-1}}{-1}\right] = -e^{-x}\int \frac{1}{x}dx \\
y_p &= -e^{-x}\log x.
\end{aligned}
$$

The general solution is $y = y_c + y_p$

$$y = e^{-x}(c_1 + c_2x) - e^{-x}\log x.$$

Example 5.49. Solve $(D^4 - 1)y = e^x \cos x$. [Apr 2009]

Solution. A.E is $m^4 - 1 = 0 \Rightarrow (m^2 - 1)(m^2 + 1) = 0$

$$\Rightarrow (m-1)(m+1)(m^2+1) = 0 \Rightarrow m = 1, -1, m^2 = -1 \Rightarrow m = \pm i.$$

$$y_c = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x.$$

$$\begin{aligned}
y_p &= \frac{1}{D^4 - 1} e^x \cos x = e^x \frac{1}{(D+1)^4 - 1} \cos x = e^x \frac{1}{(D^2+2D+1)^2 - 1} \cos x \\
&= e^x \frac{1}{D^4 + 4D^2 + 1 + 4D^3 + 2D^2 + 4D - 1} \cos x \\
&= e^x \frac{1}{D^4 + 4D^3 + 6D^2 + 4D} \cos x \\
&= e^x \frac{1}{1 - 4D - 6 + 4D} \cos x = e^x \frac{\cos x}{-5} \\
y_p &= -\frac{e^x \cos x}{5}.
\end{aligned}$$

The general solution is $y = y_c + y_p$.

$$y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x - \frac{e^x \cos x}{5}.$$

Example 5.50. Solve $(D+2)^2 y = e^{-2x} \sin x$. [Jun 2009]

Solution. $(D+2)^2 y = e^{-2x} \sin x$

$(D^2 + 4D + 4)y = e^{-2x} \sin x.$

A.E. is $(m+2)^2 = 0 \Rightarrow m = -2, -2.$

$$\begin{aligned}
y_c &= e^{-2x}(c_1 + c_2 x). \\
y_p &= \frac{1}{(D+2)^2} e^{-2x} \sin x \\
&= e^{-2x} \frac{1}{(D-2+2)^2} \sin x \\
&= e^{-2x} \frac{1}{D^2} \sin x \\
&= e^{-2x} \frac{\sin x}{-1} \\
&= -e^{-2x} \sin x.
\end{aligned}$$

The general solution is $y = y_c + y_p$

$y = e^{-2x}(c_1 + c_2 x) - e^{-2x} \sin x.$

Example 5.51. Solve $(D^2+4D+3)y = e^{-x}\sin x + xe^{3x}$. [Jun 2010, Jan 2002]

Solution. A.E is $m^2+4m+3=0 \Rightarrow (m+1)(m+3)=0 \Rightarrow m=-1,-3$.

$$y_c = c_1e^{-x}+c_2e^{-3x}.$$

$$\begin{aligned}
y_p &= \frac{1}{D^2+4D+3}(e^{-x}\sin x + xe^{3x})\\
&= \frac{1}{(D+1)(D+3)}e^{-x}\sin x + \frac{1}{(D+1)(D+3)}xe^{3x}\\
&= \frac{e^{-x}}{(D-1+1)(D-1+3)}\sin x + \frac{e^{3x}}{(D+3+1)(D+3+3)}x\\
&= \frac{e^{-x}}{D(D+2)}\sin x + \frac{e^{3x}}{(D+4)(D+6)}x\\
&= PI_1 + PI_2.
\end{aligned}$$

$$\begin{aligned}
PI_1 &= e^{-x}\frac{1}{D^2+2D}\sin x = e^{-x}\frac{1}{2D-1}\sin x\\
&= e^{-x}\frac{(2D+1)}{4D^2-1}\sin x = e^{-x}\frac{2\cos x+\sin x}{-5}\\
&= \frac{-e^{-x}}{5}(2\cos x+\sin x).
\end{aligned}$$

$$\begin{aligned}
PI_2 &= e^{3x}\frac{1}{(D+4)(D+6)}x\\
&= \frac{1}{D^2+10D+24}x\\
&= \frac{e^{3x}}{24}\frac{1}{\left(1+\frac{D^2+10D}{24}\right)}x\\
&= \frac{e^{3x}}{24}\left(1+\frac{D^2+10D}{24}\right)^{-1}x\\
&= \frac{e^{3x}}{24}\left[1-\frac{D^2+10D}{24}+\left(\frac{D^2+10D}{24}\right)^2-\cdots\right]x\\
&= \frac{e^{3x}}{24}\left[x-\frac{10}{24}\right]\\
&= \frac{e^{3x}}{24}\left[x-\frac{5}{12}\right].
\end{aligned}$$

The general solution is $y = y_c + y_p$

$$y = c_1e^{-x}+c_2e^{-3x} - \frac{e^{-x}}{5}(2\cos x+\sin x) + \frac{e^{3x}}{24}\left[x-\frac{5}{12}\right].$$

Example 5.52. Solve $(D^2 - 4D + 3)y = e^x \cos 2x$. [Jun 2012]

Solution. A.E. is

$$m^2 - 4m + 3 = 0$$
$$(m-1)(m-3) = 0$$
$$m = 1, 3.$$
$$y_c = c_1 e^x + c_2 e^{3x}.$$
$$\begin{aligned} y_p &= \frac{1}{D^2 - 4D + 3} e^x \cos 2x. \\ &= e^x \cdot \frac{1}{(D+1)^2 - 4(D+1) + 3} \cos 2x \\ &= e^x \cdot \frac{1}{D^2 + 2D + 1 - 4D - 4 + 3} \cos 2x \\ &= e^x \frac{1}{D^2 - 2D} \cos 2x. \\ &= e^x \frac{1}{-4 - 2D} \cos 2x \\ &= -\frac{e^x}{2} \frac{1}{D+2} \cos 2x \\ &= -\frac{e^x}{2} \frac{D-2}{(D-2)(D+2)} \cos 2x \\ &= -\frac{e^x}{2} \left(\frac{-2\sin 2x - 2\cos 2x}{-4-4} \right) \\ &= -\frac{e^x}{16} 2(\sin 2x + \cos 2x) \\ &= -\frac{e^x}{8} (\sin 2x + \cos 2x). \end{aligned}$$

Solution is $y = y_c + y_p$

$$= c_1 e^x + c_2 e^{3x} - \frac{e^x}{8} (\sin 2x + \cos 2x).$$

Example 5.53. Solve $(D^2 - 3D + 2)y = x \cos x$. [Jan 2007]

Solution. The A.E. is $m^2 - 3m + 2 = 0 \Rightarrow (m-2)(m-1) = 0 \Rightarrow m = 1, 2$.

$$y_c = C_1 e^x + C_2 e^{2x}.$$

$$y_p = \frac{1}{D^2 - 3D + 2} x \cos x = \frac{1}{D^2 - 3D + 2} x (\text{ R.P. } e^{ix})$$

$$= \text{R.P. } e^{ix} \frac{1}{(D+i)^2 - 3(D+i) + 2} x$$
$$= \text{R.P. } e^{ix} \frac{1}{D^2 - 1 + 2Di - 3D - 3i + 2} x$$
$$= \text{R.P. } e^{ix} \frac{1}{D^2 - 3D + 2Di + 1 - 3i} x$$
$$= \text{R.P. } e^{ix} \frac{1}{(1-3i)\left[1 + \frac{D^2}{1-3i} + \frac{D(2i-3)}{1-3i}\right]} x$$
$$= \text{R.P. } \frac{e^{ix}}{1-3i}\left[1 + \frac{D(2i-3)}{1-3i} + \frac{D^2}{1-3i}\right]^{-1} x$$
$$= \text{R.P. } \frac{e^{ix}}{1-3i}\left[1 - \frac{D(2i-3)}{1-3i}\right] x$$
$$= \text{R.P. } \frac{e^{ix}}{1-3i}\left[x - \frac{2i-3}{1-3i}\right]$$
$$= \text{R.P. } \frac{e^{ix}}{(1-3i)^2}\left[x(1-3i) - (2i-3)\right]$$
$$= \text{R.P. } \frac{e^{ix}}{1+9-6i}\left[x - 3ix - 2i + 3\right]$$
$$= \text{R.P. } \frac{e^{ix}}{10-6i}\left[x + 3 - i(2+3x)\right]$$
$$= \text{R.P. } \frac{(10+6i)e^{ix}}{136}\left[x + 3 - i(2+3x)\right]$$
$$= \text{R.P. } \frac{(10+6i)(\cos x + i\sin x)(x + 3 - i(2+3x))}{136}$$
$$= \text{R.P. } \frac{(10\cos x - 6\sin x + i(10\sin x + 6\cos x))(x + 3 - i(2+3x))}{136}$$
$$= \frac{(10\cos x - 6\sin x)(x+3) + (10\sin x + 6\cos x)(2+3x))}{136}.$$

The solution is $y = y_c + y_p$.

$$y = C_1e^x + C_2e^{2x} + \frac{(10\cos x - 6\sin x)(x+3) + (10\sin x + 6\cos x)(2+3x))}{136}.$$

Example 5.54. Solve $(D^2+1)^2 y = x^2 \cos x$. [May 2002]

Solution. A.E. is $(m^2+1)^2 = 0 \Rightarrow m^2 = -1, -1 \Rightarrow m = \pm i, \pm i$.

$$y_c = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x.$$

$$\begin{aligned}
y_p &= \frac{1}{(D^2+1)^2} x^2 \cos x = \frac{1}{(D^2+1)^2} x^2 (\text{R.P. } e^{ix}) \\
&= \text{R.P. } e^{ix} \frac{1}{((D+i)^2+1)^2} x^2 \\
&= \text{R.P. } e^{ix} \frac{1}{(D^2-1+2Di+1)^2} x^2 \\
&= \text{R.P. } e^{ix} \frac{1}{(D^2+2Di)^2} x^2 \\
&= \text{R.P. } e^{ix} \frac{1}{(2Di)^2\left(1+\frac{D}{2i}\right)^2} x^2 \\
&= \text{R.P. } e^{ix} \frac{\left(1+\frac{D}{2i}\right)^{-2}}{-4D^2} x^2 \\
&= \text{R.P. } \frac{-e^{ix}}{4D^2}\left[1 - 2\frac{D}{2i} + 3\frac{D^2}{-4} + \cdots\right] x^2 \\
&= \text{R.P. } \frac{-e^{ix}}{4D^2}\left[1 + iD - \frac{3D^2}{4} + \cdots\right] x^2 \\
&= -\text{R.P. } \frac{e^{ix}}{4D^2}\left[x^2 + i2x - \frac{3}{4}2\right] \\
&= -\text{R.P. } \frac{e^{ix}}{4D^2}\left[x^2 + 2ix - \frac{3}{2}\right] \\
&= -\text{R.P. } \frac{e^{ix}}{4} \frac{1}{D} \int \left(x^2 + 2ix - \frac{3}{2}\right) dx \\
&= -\text{R.P. } \frac{e^{ix}}{4} \frac{1}{D} \left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right) \\
&= -\text{R.P. } \frac{e^{ix}}{4} \int \left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right) dx \\
&= -\text{R.P. } \frac{e^{ix}}{4} \left[\frac{x^4}{12} + i\frac{x^3}{3} - \frac{3x^2}{4}\right] \\
&= -\text{R.P. } \left(\frac{\cos x + i \sin x}{4}\right)\left[\frac{x^4}{12} + i\frac{x^3}{3} - \frac{3x^2}{4}\right] \\
y_p &= -\frac{1}{4}\left[\left[\frac{x^4}{12} - \frac{3x^2}{4}\right]\cos x - \frac{x^3}{3}\sin x\right].
\end{aligned}$$

The solution is $y = y_c + y_p$

$$y = (C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x - \frac{1}{4}\left[\left[\frac{x^4}{12} - \frac{3x^2}{4}\right]\cos x - \frac{x^3}{3}\sin x\right].$$

Example 5.55. Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \sin x.$ [Dec 2013, May 2010]

Solution. A.E. is $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$

$$
\begin{aligned}
y_c &= e^x(C_1 + C_2 x).\\
y_p &= \frac{1}{(D-1)^2} xe^x \sin x\\
&= e^x \frac{1}{(D+1-1)^2} x \sin x\\
&= e^x \frac{1}{D^2} x \sin x\\
&= e^x \frac{1}{D} \int x \sin x dx\\
&= e^x \frac{1}{D} \int x d(-\cos x)\\
&= e^x \frac{1}{D}\Big[-x \cos x - \int -\cos x dx\Big]\\
&= e^x \frac{1}{D}\Big[-x \cos x + \sin x\Big]\\
&= e^x \Big[-\int x \cos x dx + \int \sin x dx\Big]\\
&= e^x \Big[-\int x d(\sin x) - \cos x\Big]\\
&= e^x \Big[-\Big\{x \sin x - \int \sin x dx\Big\} - \cos x\Big]\\
&= e^x \Big[-x \sin x - \cos x - \cos x\Big]\\
&= -e^x \Big[x \sin x + 2 \cos x\Big].
\end{aligned}
$$

The solution is $y = y_c + y_p$

$$y = e^x(C_1 + C_2 x) - e^x\Big[x \sin x + 2 \cos x\Big].$$

Example 5.56. Solve $(D^2 - 4D + 4)y = x^2 e^{2x} \cos 2x.$ [Jan 2007]

Solution. A.E. is $(m-2)^2 = 0 \Rightarrow m = 2, 2.$

$$y_c = e^{2x}(C_1 + C_2 x).$$

$$y_p = \frac{1}{(D-2)^2}x^2e^{2x}\cos 2x = e^{2x}\frac{1}{(D+2-2)^2}x^2\cos 2x$$
$$= e^{2x}\frac{1}{D^2}x^2\cos 2x = e^{2x}\frac{1}{D}\int x^2\cos 2x dx$$
$$= e^{2x}\frac{1}{D}\int x^2 d\Big(\frac{\sin 2x}{2}\Big)$$
$$= e^{2x}\frac{1}{D}\Big[\frac{x^2\sin 2x}{2} - \int \frac{\sin 2x}{2}2xdx\Big]$$
$$= e^{2x}\frac{1}{D}\Big[\frac{x^2\sin 2x}{2} - \int x\sin 2xdx\Big]$$
$$= e^{2x}\frac{1}{D}\Big[\frac{x^2\sin 2x}{2} - \int xd\Big(\frac{-\cos 2x}{2}\Big)\Big]$$
$$= e^{2x}\frac{1}{D}\Big[\frac{x^2\sin 2x}{2} - \Big(\frac{-x\cos 2x}{2} - \int\frac{-\cos 2x}{2}dx\Big)\Big]$$
$$= e^{2x}\frac{1}{D}\Big[\frac{x^2\sin 2x}{2} + \frac{x\cos 2x}{2} - \frac{1}{2}\frac{\sin 2x}{2}\Big]$$
$$= e^{2x}\frac{1}{D}\Big[\frac{x^2\sin 2x}{2} + \frac{x\cos 2x}{2} - \frac{\sin 2x}{4}\Big]$$
$$= \frac{e^{2x}}{2}\Big[\int x^2\sin 2xdx + \int x\cos 2xdx - \frac{1}{2}\int\sin 2xdx\Big]$$
$$= \frac{e^{2x}}{2}\Big[\int x^2 d\Big(\frac{-\cos 2x}{2}\Big) + \int xd\Big(\frac{\sin 2x}{2}\Big) - \frac{1}{2}\Big(\frac{-\cos 2x}{2}\Big)\Big]$$
$$= \frac{e^{2x}}{4}\Big[-x^2\cos 2x + \int\cos 2x(2x)dx + x\sin 2x - \int\sin 2xdx + \frac{\cos 2x}{2}\Big]$$
$$= \frac{e^{2x}}{4}\Big[-x^2\cos 2x + 2\int xd\Big(\frac{\sin 2x}{2}\Big) + x\sin 2x + \frac{\cos 2x}{2} + \frac{\cos 2x}{2}\Big]$$
$$= \frac{e^{2x}}{4}\Big[-x^2\cos 2x + x\sin 2x - \int\sin 2xdx + x\sin 2x + \cos 2x\Big]$$
$$= \frac{e^{2x}}{4}\Big[-x^2\cos 2x + x\sin 2x + \frac{\cos 2x}{2} + x\sin 2x + \cos 2x\Big]$$
$$= \frac{e^{2x}}{4}\Big[-x^2\cos 2x + 2x\sin 2x + \frac{3}{2}\cos 2x\Big]$$
$$y_p = \frac{e^{2x}}{8}\Big[4x\sin 2x + (3-2x^2)\cos 2x\Big].$$

The solution is $y = y_c + y_p$

$$y = e^{2x}(C_1 + C_2x) + \frac{e^{2x}}{8}\Big[4x\sin 2x + (3-2x^2)\cos 2x\Big].$$

Example 5.57. Solve $(D^2 - 2D)y = x^2 e^x \cos x$. [Jan 2002]

Solution. A.E. is $m^2 - 2m = 0 \Rightarrow m(m-2) = 0 \Rightarrow m = 0, 2$.

$$y_c = C_1 + C_2 e^{2x}.$$

$$\begin{aligned}
y_p &= \frac{1}{D^2 - 2D} x^2 e^x \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1)} x^2 \cos x \\
&= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2} x^2 \cos x = e^x \frac{1}{D^2 - 1} x^2 \cos x \\
&= e^x \frac{}{D^2 - 1} (\text{R.P. } e^{ix}) x^2 \\
&= \text{R.P. } e^x e^{ix} \frac{1}{(D+i)^2 - 1} x^2 \\
&= \text{R.P. } e^x e^{ix} \frac{1}{D^2 - 1 + 2Di - 1} x^2 \\
&= \text{R.P. } e^x e^{ix} \frac{1}{D^2 - 2 + 2Di} x^2 \\
&= \text{R.P. } e^x e^{ix} \frac{1}{(-2)\left(1 - \frac{D^2 + 2Di}{2}\right)} x^2 \\
&= \text{-R.P. } \frac{e^x e^{ix}}{2} \left(1 - \frac{D^2 + 2Di}{2}\right)^{-1} x^2 \\
&= \text{-R.P. } \frac{e^x e^{ix}}{2} \left(1 + \frac{D^2 + 2Di}{2} + \left(\frac{D^2 + 2Di}{2}\right)^2 + \cdots\right) x^2 \\
&= \text{-R.P. } \frac{e^x e^{ix}}{2} \left(x^2 + \frac{1}{2}(2 + 4ix) + (-2)\right) \\
&= \text{-R.P. } \frac{e^x e^{ix}}{2} \left(x^2 + 1 + 2ix - 2\right) \\
&= \text{-R.P. } \frac{e^x e^{ix}}{2} \left(x^2 - 1 + 2ix\right) \\
&= \text{-R.P. } \frac{e^x}{2} (\cos x + i \sin x)\left(x^2 - 1 + 2ix\right) \\
&= \frac{-e^x}{2} [\cos x (x^2 - 1) - 2x \sin x] \\
&= \frac{e^x}{2} [(1 - x^2) \cos x + 2x \sin x].
\end{aligned}$$

The solution is $y = y_c + y_p$

$$y = C_1 + C_2 e^{2x} + \frac{e^x}{2} [(1 - x^2) \cos x + 2x \sin x].$$

Example 5.58. Solve $(D^2 - 4D + 4)y = 8(e^{2x} + \sin 2x + x^2)$.

Solution. A.E is $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$.

$$y_c = e^{2x}(c_1 + c_2 x).$$

$$\begin{aligned}
P.I &= \frac{1}{(D-2)^2}(8e^{2x} + 8\sin 2x + 8x^2) \\
&= \frac{1}{(D-2)^2}8e^{2x} + \frac{1}{(D-2)^2}8\sin 2x + \frac{1}{(D-2)^2}8x^2 \\
&= PI_1 + PI_2 + PI_3.
\end{aligned}$$

$$PI_1 = \frac{1}{(D-2)^2}8e^{2x} = 8\frac{x^2}{2}e^{2x} = 4x^2e^{2x}.$$

$$\begin{aligned}
PI_2 &= \frac{1}{D^2 - 4D + 4}8\sin 2x = \frac{1}{-4 - 4D + 4}8\sin 2x \\
&= -\frac{1}{4D}8\sin 2x \\
&= -2\int \sin 2x dx = \frac{2\cos 2x}{2} = \cos 2x.
\end{aligned}$$

$$\begin{aligned}
PI_3 &= \frac{1}{(D-2)^2}8x^2 = \frac{8}{4}\frac{1}{(1-\frac{D}{2})^2}x^2 = 2(1-\frac{D}{2})^{-2}x^2 \\
&= 2(1 + 2\frac{D}{2} + 3\frac{D^2}{4} + \dots)x^2 \\
&= 2(x^2 + 2x + \frac{3}{4}2) \\
&= 2x^2 + 4x + 3.
\end{aligned}$$

The general solution is $y = y_c + y_p$

$$y = e^{2x}(c_1 + c_2 x) + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3.$$

Example 5.59. Solve $(D^2 + 4)^2 y = \cos 2x$.

Solution. The A.E is $(m^2 + 4)^2 = 0 \Rightarrow m = \pm 2i, \pm 2i$.

$$y_c = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x.$$

$$\begin{aligned}
y_p &= \frac{1}{(D^2 + 4)^2}\cos 2x \\
&= x\frac{1}{2(D^2 + 4)2D}\cos 2x \text{[See note on Page 134]}
\end{aligned}$$

$$= \frac{x}{4}\frac{1}{D(D^2+4)}\cos 2x$$
$$= \frac{x}{4}\frac{1}{D^3+4D}\cos 2x$$
$$= \frac{x}{4}\frac{x}{(3D^2+4)}\cos 2x$$
$$= \frac{x^2}{4}\left(\frac{\cos 2x}{-12+4}\right)$$
$$= \frac{-x^2}{32}\cos 2x.$$

The general solution is $y = y_c + y_p$

$y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x - \frac{x^2}{32}\cos 2x.$

An important result to remember

Let us evaluate $\frac{1}{D-a} f(x)$.

Let $\frac{1}{D-a} f(x) = y$.

Operating on both sides by $D - a$ we obtain

$$\Rightarrow (D-a)y = f(x) \Rightarrow Dy - ay = f(x).$$

This is a linear equation of the first order, where $P = -a, Q = f(x)$.

$$I.F. = e^{\int pdx} = e^{\int -adx} = e^{-ax}.$$

The solution is

$$y \times I.F = \int Q \times I.F dx$$
$$ye^{-ax} = \int f(x)e^{-ax}dx$$
$$y = e^{ax}\int e^{-ax} f(x)dx.$$

Example 5.60. Solve $\frac{d^2y}{dx^2} + a^2 y = \tan ax$.

Solution. A.E is $m^2 + a^2 = 0 \Rightarrow m = \pm ai$.

$$y_c = c_1 \cos ax + c_2 \sin ax.$$
$$y_p = \frac{1}{D^2+a^2}\tan ax = \frac{1}{(D+ai)(D-ai)}\tan ax$$

$$= \frac{1}{2ai}\Big[\frac{1}{D-ai} - \frac{1}{D+ai}\Big]\tan ax$$
$$= \frac{1}{2ai}\Big[\frac{1}{D-ai}\tan ax - \frac{1}{D+ai}\tan ax\Big].$$
$$PI_1 = \frac{1}{D-ai}\tan ax$$
$$= e^{aix}\int \tan ax e^{-iax}dx \qquad \Big[\because \frac{1}{D-a}f(x) = e^{ax}\int f(x)e^{-ax}dx\Big]$$
$$= e^{aix}\int \tan ax(\cos ax - i\sin ax)dx$$
$$= e^{iax}\int (\sin ax - i\frac{\sin^2 ax}{\cos ax})dx$$
$$= e^{iax}\Big(\frac{-\cos ax}{a} - i\int \frac{1-\cos^2 ax}{\cos ax}dx\Big)$$
$$= e^{iax}\Big(\frac{-\cos ax}{a} - \frac{i}{a}\log(\sec ax + \tan ax) + i\frac{\sin ax}{a}\Big)$$
$$= \frac{-e^{iax}}{a}\Big(\cos ax - i\sin ax + i\log(\sec ax + \tan ax)\Big)$$
$$= \frac{-1}{a}\Big(e^{iax}e^{-iax} + ie^{iax}\log(\sec ax + \tan ax)\Big)$$
$$= \frac{-1}{a}\Big(1 + ie^{iax}\log(\sec ax + \tan ax)\Big).$$

changing i into $-i$ we get

$$\frac{1}{D+ai}\tan ax = \frac{-1}{a}\Big(1 - ie^{-iax}\log(\sec ax + \tan ax)\Big)$$

$$y_p = \frac{-1}{2ai}\frac{1}{a}(1 + ie^{iax}\log(\sec ax + \tan ax)) + \frac{1}{2ai}\frac{1}{a}(1 - ie^{-iax}\log(\sec ax + \tan ax))$$
$$= \frac{-1}{2a^2}(-i + e^{iax}\log(\sec ax + \tan ax)) + \frac{1}{2a^2}(-i - e^{-iax}\log(\sec ax + \tan ax))$$
$$= \frac{1}{2a^2}\Big[(i - e^{iax}\log(\sec ax + \tan ax) - i - e^{-iax}\log(\sec ax + \tan ax))\Big]$$
$$= -\frac{1}{2a^2}[(\cos ax + i\sin ax)\log(\sec ax + \tan ax)$$
$$+ (\cos ax - i\sin ax)\log(\sec ax + \tan ax)]$$
$$= \frac{-1}{2a^2}\log(\sec ax + \tan ax)2\cos ax$$
$$y_p = \frac{-1}{a^2}\cos ax\log(\sec ax + \tan ax).$$

Solution is $y = y_c + y_p$.

$$y = c_1 \cos ax + c_2 \sin ax + \frac{-1}{a^2} \cos ax \log(\sec ax + \tan ax).$$

5.3 Linear differential equations with variable coefficients

5.3.1 Cauchy's homogeneous linear differential equation

The general form of the n^{th} order linear D.E with variable coefficients is

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x)$$

where $a_0, a_1, \ldots, a_n$ are constants with $a_0 \neq 0$.

This can be reduced to a linear D.E with constant coefficients as follows.

Let $x = e^z, \Rightarrow z = \log x$.

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x}$$

$$x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\Big(\frac{dy}{dx}\Big) = \frac{d}{dx}\Big(\frac{1}{x}\frac{dy}{dz}\Big) \\
&= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\Big(\frac{dy}{dz}\Big) \\
&= -\frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\Big(\frac{dy}{dz}\Big)\frac{dz}{dx} \\
&= -\frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{1}{x} \\
&= -\frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} \\
x^2\frac{d^2y}{dx^2} &= \frac{d^2y}{dz^2} - \frac{dy}{dz}.
\end{aligned}$$

Let $\theta = \frac{d}{dz}, \theta^2 = \frac{d^2}{dz^2}$.

$$\text{Now, } \theta y = \frac{dy}{dz} = x\frac{dy}{dx} = xDy$$

Then $xD = \theta$.

$$x^2 \frac{d^2y}{dx^2} = \theta^2 y - \theta y = \theta(\theta - 1)y$$
$$x^2 D^2 y = \theta(\theta - 1)y.$$
$$\text{Similarly } x^3 D^3 = \theta(\theta - 1)(\theta - 2)y \text{ etc.}$$

On substitution in (1), it will be reduced to a D.E with constant coefficients, which can be evaluated as usual.

Worked Examples

Example 5.61. Reduce $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = x$ into a differential equation with constant coefficients. [May 2007]

Solution. The given equation is

$$(x^2D^2 - 3xD + 3)y = x. \tag{1}$$

Let $x = e^z$ or $z = \log x$.

Then, $xD = \theta$ and $x^2D^2 = \theta(\theta - 1)$ where $\theta = \frac{d}{dz}$.

Now (1) gives $(\theta(\theta - 1) - 3\theta + 3)y = e^z$.

$$\text{i.e, } (\theta^2 - \theta - 3\theta + 3)y = e^z$$

$$(\theta^2 - 4\theta + 3)y = e^z$$
$$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 3y = e^z,$$

which is a linear second order differential equation with constant coefficients.

Example 5.62. Solve $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$ [Jun 2006]

Solution. The given equation is $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$

Multiplying by x, we get

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

Let $x = e^z$ or $z = \log x$.

Now $x\dfrac{d}{dx} = \theta$, then $x^2\dfrac{d^2}{dx^2} = \theta(\theta - 1)$ where $\theta = \dfrac{d}{dz}$.

The given equation becomes

$$\begin{aligned}(\theta(\theta - 1) + \theta)\, y &= 0\\ (\theta^2 - \theta + \theta)y &= 0\\ \theta^2 y &= 0.\end{aligned}$$

The A.E is $m^2 = 0 \Longrightarrow m = 0, 0$

$$\begin{aligned}\therefore y_c &= e^{0z}(c_1 + c_2 z) = c_1 + c_2 z\\ y_c &= c_1 + c_2 \log x.\\ y_p &= 0.\end{aligned}$$

The general solution is $y = y_c + y_p$

$y = c_1 + c_2 \log x$.

Example 5.63. Solve $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = 0$. [Jun 2013, May 2008]

Solution. Let $x = e^z$ or $z = \log x$.

Let $x\dfrac{d}{dx} = \theta$. Then, $x^2\dfrac{d^2}{dx^2} = \theta(\theta - 1)$ where $\theta = \dfrac{d}{dz}$.

The given equation is reduced to

$(\theta(\theta - 1) + 4\theta + 2)y = 0$

$(\theta^2 - \theta + 4\theta + 2)y = 0$

$(\theta^2 + 3\theta + 2)y = 0$.

A.E is $m^2 + 3m + 2 = 0$

$(m + 1)(m + 2) = 0, \quad m = -1, m = -2$.

$$\begin{aligned}y_c &= c_1 e^{-z} + c_2 e^{-2z}\\ &= c_1 e^{-\log x} + c_2 e^{-2\log x}\\ &= c_1 e^{\log(x^{-1})} + c_2 e^{\log x^{-2}} = c_1 x^{-1} + c_2 x^{-2} = \frac{c_1}{x} + \frac{c_2}{x^2}.\\ y_p &= 0.\end{aligned}$$

The general solution is $y = y_c + y_p$

$$y = \frac{c_1}{x} + \frac{c_2}{x^2}.$$

Example 5.64. Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0.$ [Jun 2009, May 2003]

Solution. Let $x = e^z$ or $z = \log x$.

Define $x\frac{dy}{dx} = \theta y$ where $\theta = \frac{d}{dz}$.

Then, $x^2\frac{d^2y}{dx^2} = \theta(\theta - 1)y$.

$\therefore$ The given equation becomes

$\theta(\theta - 1)y - \theta y + y = 0$

$(\theta^2 - \theta - \theta + 1)y = 0$

$(\theta^2 - 2\theta + 1)y = 0$

A.E is $m^2 - 2m + 1 = 0 \Rightarrow (m - 1)(m - 1) = 0 \Rightarrow m = 1, 1.$

$$y_c = (c_1 + c_2 z)e^z.$$
$$= (c_1 + c_2 \log x)x.$$
$$y_p = 0.$$

The solution $y = y_c + y_p$

i.e., $y = (c_1 + c_2 \log x)x$.

Example 5.65. Transform the equation $x^2y'' + xy' = x$ into a linear differential equation with constant coefficients. [May 2011, Dec 2011, Jun 2010]

Solution. Let $x = e^z$ or $z = \log x$.

Let $x\frac{d}{dx} = \theta$. Then, $x^2\frac{d^2}{dx^2} = \theta(\theta - 1)$ where $\theta = \frac{d}{dz}$.

The given equation is reduced to

$$(\theta(\theta - 1) + \theta)y = e^z$$
$$(\theta^2 - \theta + \theta)y = e^z$$
$$\theta^2 y = e^z$$
$$\frac{d^2y}{dz^2} = e^z,$$

which is the required linear equation.

Example 5.66. Reduce the equation $(x^2D^2 + xD + 1)y = \log x$ into an ordinary differential equation with constant coefficients. [Dec 2010]

Solution. Let $x = e^z$ or $z = \log x$.

Let $x\dfrac{d}{dx} = \theta$. Then, $x^2\dfrac{d^2}{dx^2} = \theta(\theta - 1)$ where $\theta = \frac{d}{dz}$.

The given equation is reduced to

$$(\theta(\theta - 1) + \theta + 1)y = z$$
$$(\theta^2 - \theta + \theta + 1)y = z$$
$$(\theta^2 + 1)y = z$$
$$\frac{d^2y}{dz^2} + y = z,$$

which is the required linear equation.

Example 5.67. Convert $(x^2D^2 + xD + 7)y = \dfrac{2}{x}$ into an equation with constant coefficients. [Dec 2009]

Solution. Let $x = e^z$ or $z = \log x$.

Define $D = \dfrac{d}{dx}, \theta = \dfrac{d}{dz}$.

Then $x\dfrac{d}{dx} = \theta$. $x^2\dfrac{d^2}{dx^2} = \theta(\theta - 1)$.

The given equation is reduced to

$$(\theta(\theta - 1) + \theta + 7)y = \frac{2}{e^z} = 2e^{-z}$$
$$(\theta^2 - \theta + \theta + 7)y = 2e^{-z}$$
$$(\theta^2 + 7)y = 2e^{-z}$$
$$\frac{d^2y}{dz^2} + 7y = 2e^{-z},$$

which is the required differential equation with constant coefficients.

Example 5.68. Convert $(3x^2D^2 + 5xD + 7)y = \frac{2}{x}\log x$ into an equation with constant coefficients [Dec 2013]

Solution. Let $x = e^z$ or $z = \log x$.

Let $D = \frac{d}{dx}, \theta = \frac{d}{dz}$.

Then, $x\frac{d}{dx} = \theta, x^2\frac{d^2}{dx^2} = \theta(\theta - 1)$.

The given equation is reduced to

$$(3 \cdot \theta(\theta - 1) + 5\theta + 7)y = \frac{2}{e^z} \cdot z$$

$$(3\theta^2 - 3\theta + 5\theta + 7)y = 2e^{-z}z$$

$$(3\theta^2 + 2\theta + 7)y = 2e^{-z}z$$

$$3\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + 7y = 2e^{-z}z,$$

which is the required equation.

Example 5.69. Solve $(x^2D^2 + xD + 1)y = 0$. [Jun 2005, May 2002]

Solution. Let $x = e^z$ or $z = \log x$.

Define $D = \frac{d}{dx}, \theta = \frac{d}{dz}$.

Then, $x\frac{d}{dx} = \theta.\ x^2\frac{d^2}{dx^2} = \theta(\theta - 1)$

The given equation is reduced to

$$(\theta(\theta - 1) + \theta + 1)y = 0$$

$$(\theta^2 - \theta + \theta + 1)y = 0$$

$$(\theta^2 + 1)y = 0.$$

$$\text{A.E is } m^2 + 1 = 0.$$

$$m^2 = -1$$

$$m = \pm i.$$

$$y_c = e^{0z}(c_1 \cos z + c_2 \sin z) = c_1 \cos(\log x) + c_2 \sin(\log x)$$

$$y_p = 0.$$

The general solution is $y = y_c + y_p$

$$y = c_1 \cos(\log x) + c_2 \sin(\log x).$$

Example 5.70. Solve $x^2y'' - xy' + y = x$.

Solution. $(x^2D^2 - xD + 1)y = x$.

Let $x = e^z$ or $z = \log x$.

Define $D = \dfrac{d}{dx}, \theta = \dfrac{d}{dz}$.

Then, $xD = \theta, \quad x^2D^2 = \theta(\theta - 1)$.

Now, the given equation is reduced to

$$(\theta(\theta - 1) - \theta + 1)y = e^z$$
$$(\theta^2 - \theta - \theta + 1)y = e^z$$
$$(\theta^2 - 2\theta + 1)y = e^z.$$

A.E is $m^2 - 2m + 1 = 0 \Rightarrow (m - 1)^2 = 0 \Rightarrow m = 1, 1$.

$$y_c = e^z(c_1 + c_2z) = x(c_1 + c_2 \log x).$$
$$y_p = \frac{1}{\theta^2 - 2\theta + 1}e^z = \frac{1}{(\theta - 1)^2}e^z = \frac{z^2}{2}e^z$$
$$= \frac{x(\log x)^2}{2}$$

The general solution is $y = y_c + y_p$

i.e., $y = x(c_1 + c_2 \log x) + \dfrac{x}{2}(\log x)^2$.

Example 5.71. Solve $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x \log x$. [Dec 2006, May 2003]

Solution. Let $x = e^z$, or $z = \log x$.

Define $D = \dfrac{d}{dx}, \theta = \dfrac{d}{dz}$.

Then, $xD = \theta, \quad x^2D^2 = \theta(\theta - 1)$.

Now, the given equation is reduced to

$$(\theta(\theta - 1) + 4\theta + 2)y = e^z z$$
$$(\theta^2 - \theta + 4\theta + 2)y = e^z z$$
$$(\theta^2 + 3\theta + 2)y = e^z z.$$

A.E is, $m^2 + 3m + 2 = 0 \Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2.$

$$y_c = c_1e^{-z} + c_2e^{-2z} = \frac{c_1}{x} + \frac{c_2}{x^2}.$$

$$\begin{aligned}
y_p &= \frac{1}{\theta^2 + 3\theta + 2}e^z z = e^z \frac{1}{(\theta+1)^2 + 3(\theta+1) + 2}z \\
&= e^z \frac{1}{\theta^2 + 2\theta + 1 + 3\theta + 3 + 2}z = e^z \frac{1}{\theta^2 + 5\theta + 6}z \\
&= e^z \frac{1}{6\left(1 + \frac{5\theta+\theta^2}{6}\right)}z = \frac{e^z}{6}\left(1 + \frac{5\theta+\theta^2}{6}\right)^{-1}z \\
&= \frac{e^z}{6}\left(1 - \frac{5\theta+\theta^2}{6} + \cdots\right)z = \frac{e^z}{6}\left(1 - \frac{5\theta}{6}\right)z \\
&= \frac{e^z}{6}\left(z - \frac{5\theta}{6}z\right) = \frac{e^z}{6}\left(z - \frac{5}{6}.1\right) = \frac{x}{6}\left(\log x - \frac{5}{6}\right).
\end{aligned}$$

Solution is $y = y_c + y_p$

$$\text{i.e., } y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{x}{6}\left(\log x - \frac{5}{6}\right).$$

Example 5.72. Solve $(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2.$ [Jun 2005]

Solution. Let $x = e^z$, or $z = \log x$.

Define $D = \frac{d}{dx}, \theta = \frac{d}{dz}.$

Then, $xD = \theta, \quad x^2D^2 = \theta(\theta - 1).$

Now, the given equation is reduced to

$$\begin{aligned}
(\theta(\theta-1) - \theta + 1)y &= (ze^{-z})^2 \\
(\theta^2 - \theta - \theta + 1)y &= z^2e^{-2z} \\
(\theta^2 - 2\theta + 1)y &= z^2e^{-2z} \\
(\theta - 1)^2 y &= z^2e^{-2z}.
\end{aligned}$$

A.E is $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$

$$y_c = e^z(c_1 + c_2z) = x(c_1 + c_2\log x).$$

$$y_p = \frac{1}{(1-\theta)^2}z^2e^{-2z} = e^{-2z}\frac{1}{(1-(\theta-2))^2}z^2 = e^{-2z}\frac{1}{(1-\theta+2)^2}z^2$$

$$= e^{-2z}\frac{1}{(3-\theta)^2}z^2 = \frac{e^{-2z}}{9}\frac{z^2}{\left(1-\frac{\theta}{3}\right)^2} = \frac{e^{-2z}}{9}\left(1-\frac{\theta}{3}\right)^{-2}z^2$$

$$= \frac{e^{-2z}}{9}\left(1+\frac{2}{3}\theta+3\frac{\theta^2}{9}+\cdots\right)z^2 = \frac{e^{-2z}}{9}\left(z^2+\frac{2}{3}2z+2\frac{1}{3}\right)$$

$$= \frac{e^{-2z}}{9}\left(z^2+\frac{4}{3}z+\frac{2}{3}\right)$$

$$y_p = \frac{1}{9x^2}\left((\log x)^2+\frac{4}{3}\log x+\frac{2}{3}\right).$$

The solution is $y = y_c + y_p$.

$$y = x(c_1 + c_2\log x) + \frac{1}{9x^2}\left((\log x)^2+\frac{4}{3}\log x+\frac{2}{3}\right).$$

Example 5.73. Solve $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x^2 + \dfrac{1}{x^2}$. [Jun 2013, May 2008]

Solution. Let $x = e^z$, or $z = \log x$.

Define $D = \dfrac{d}{dx}, \theta = \dfrac{d}{dz}$.

Then, $xD = \theta, \quad x^2D^2 = \theta(\theta-1)$.

Now, the given equation is reduced to

$$(\theta(\theta-1)+4\theta+2)y = e^{2z}+e^{-2z}$$

$$(\theta^2-\theta+4\theta+2)y = e^{2z}+e^{-2z}$$

$$(\theta^2+3\theta+2)y = e^{2z}+e^{-2z}.$$

A.E is $m^2+3m+2=0 \Rightarrow (m+1)(m+2)=0 \Rightarrow m=-1,-2$.

$$y_c = c_1e^{-z}+c_2e^{-2z} = \frac{c_1}{x}+\frac{c_2}{x^2}.$$

$$y_p = \frac{1}{\theta^2+3\theta+2}(e^{2z}+e^{-2z})$$

$$= \frac{1}{(\theta+1)(\theta+2)}e^{2z} + \frac{1}{(\theta+1)(\theta+2)}e^{-2z}$$

$$= \frac{e^{2z}}{12} + \frac{ze^{-2z}}{-1}$$

$$y_p = \frac{x^2}{12} - \frac{\log x}{x^2}.$$

Solution is $y = y_c + y_p = \dfrac{c_1}{x} + \dfrac{c_2}{x^2} + \dfrac{x^2}{12} - \dfrac{\log x}{x^2}$.

Example 5.74. Solve $(x^2D^2 - 3xD + 4)y = x^2$ given that $y(1) = 1, y'(1) = 0$.

Solution. Let $x = e^z$, or $z = \log x$.

Define $D = \frac{d}{dx}, \theta = \frac{d}{dz}$.

Then, $xD = \theta, \quad x^2D^2 = \theta(\theta - 1)$.

Now, the given equation is reduced to

$$(\theta(\theta - 1) - 3\theta + 4)y = e^{2z}$$
$$(\theta^2 - \theta - 3\theta + 4)y = e^{2z}$$
$$(\theta^2 - 4\theta + 4)y = e^{2z}$$
$$(\theta - 2)^2 y = e^{2z}.$$

A.E is $m^2 - 4m + 4 = 0 \Rightarrow (m - 2)^2 = 0 \Rightarrow m = 2, 2$.

$$y_c = e^{2z}(c_1 + c_2 z)$$
$$= x^2(c_1 + c_2 \log x).$$
$$y_p = \frac{1}{(\theta - 2)^2} e^{2z} = \frac{z^2}{2} e^{2z} = \frac{(\log x)^2}{2} x^2.$$

Solution is $y = y_c + y_p$.

$$y = x^2(c_1 + c_2 \log x) + \frac{x^2}{2}(\log x)^2.$$

When $x = 1, y = 1$ which gives $c_1 = 1$

$$y' = x^2 c_2 \frac{1}{x} + (c_1 + c_2 \log x)2x + (\log x)^2 x + \frac{x^2}{2} 2 \log x \frac{1}{x}.$$

When $x = 1, y' = 0$. which implies

$$0 = c_2 + 2$$
$$c_2 = -2$$

Solution is

$$y = x^2(1 - 2\log x) + \frac{x^2}{2}(\log x)^2.$$

Example 5.75. Solve the differential equation $(x^2D^2 - xD + 4)y = x^2\sin(\log x)$

[Jun 2012]

Solution. Let $x = e^z$ or $z = \log x$.

Let $D = \dfrac{d}{dx}, \theta = \dfrac{d}{dz}$.

Then, $x\dfrac{d}{dx} = \theta, x^2\dfrac{d^2}{dx^2} = \theta(\theta - 1)$.

The given equation is reduced to

$$(\theta(\theta-1) - \theta + 4)y = e^{2z}\sin z$$

$$(\theta^2 - \theta - \theta + 4)y = e^{2z}\sin z$$

$$(\theta^2 - 2\theta + 4)y = e^{2z}\sin z.$$

The A.E. is

$$m^2 - 2m + 4 = 0$$

$$(m-1)^2 + 4 - 1 = 0$$

$$(m-1)^2 + 3 = 0$$

$$(m-1)^2 = -3$$

$$m - 1 = \pm\sqrt{3}i$$

$$m = 1 \pm \sqrt{3}i.$$

$$y_c = e^z(c_1\cos\sqrt{3}z + c_2\sin\sqrt{3}z)$$

$$y_c = x(c_1\cos\sqrt{3}\log x + c_2\sin\sqrt{3}\log x).$$

$$\begin{aligned} y_p &= \frac{1}{\theta^2 - 2\theta + 4}e^{2z}\sin z \\ &= e^{2z}\frac{1}{(\theta+2)^2 - 2(\theta+2) + 4}\sin z \\ &= e^{2z}\frac{1}{\theta^2 + 4\theta + 4 - 2\theta - 4 + 4}\sin z \\ &= e^{2z}\frac{1}{\theta^2 + 2\theta + 4}\sin z \\ &= e^{2z}\frac{1}{-1 + 2\theta + 4}\sin z \end{aligned}$$

$$= e^{2z}\frac{1}{2\theta+3}\sin z$$
$$= e^{2z}\frac{2\theta-3}{(2\theta-3)(2\theta+3)}\sin z$$
$$= e^{2z}\frac{2\theta-3}{4\theta^2-9}\sin z$$
$$= e^{2z}\frac{2\cos z-3\sin z}{-4-9}$$
$$= \frac{-e^{2z}}{13}(2\cos z-3\sin z) = \frac{-x^2}{13}(2\cos(\log x)-3\sin(\log x)).$$

Solution is $y = y_c + y_p$

$$= x(c_1\cos\sqrt{3}\log x + c_2\sin\sqrt{3}\log x) - \frac{x^2}{13}(2\cos(\log x)-3\sin(\log x)).$$

Example 5.76. Solve $(x^2D^2 - 3xD + 4)y = x^2\cos\log x$ [Dec 2010]

Solution. Let $x = e^z$ or $z = \log x$.

Let $D = \frac{d}{dx}$, $\theta = \frac{d}{dz}$.

Then, $x\frac{d}{dx} = \theta$, $x^2\frac{d^2}{dx^2} = \theta(\theta-1)$.

The given equation is reduced to

$$(\theta(\theta-1) - 3\theta + 4)y = e^{2z}\cos z$$
$$(\theta^2 - \theta - 3\theta + 4)y = e^{2z}\cos z$$
$$(\theta^2 - 4\theta + 4)y = e^{2z}\cos z$$
$$(\theta-2)^2 y = e^{2z}\cos z.$$

The A.E. is $(m-2)^2 = 0$

$$m = 2, 2.$$
$$y_c = e^{2z}(c_1 + c_2 z)$$
$$= x^2(c_1 + c_2\log x).$$
$$y_p = \frac{1}{(\theta-2)^2}e^{2z}\cos z$$
$$= e^{2z}\frac{1}{(\theta+2-2)^2}\cos z$$

$$= e^{2z}\frac{1}{\theta^2}\cos z$$
$$= e^{2z}\frac{1}{\theta}\int \cos z dz$$
$$= e^{2z}\frac{1}{\theta}\sin z$$
$$= e^{2z}\int \sin z dz$$
$$= -e^{2z}\cos z = -x^2\cos(\log x).$$

Solution is $y = y_c + y_p$

$$= x^2(c_1 + c_2\log x) - x^2\cos(\log x).$$

Example 5.77. Solve $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x.$ [Jun 2010]

Solution. Let $x = e^z$ or $z = \log x$.

Define $D = \frac{d}{dx}, \theta = \frac{d}{dz}$.

Then, $x\frac{d}{dx} = \theta, x^2\frac{d^2}{dx^2} = \theta(\theta-1)$.

The given equation is now reduced to

$$(\theta(\theta-1) - 2\theta - 4)y = e^{2z} + 2z.$$
$$(\theta^2 - \theta - 2\theta - 4)y = e^{2z} + 2z.$$
$$(\theta^2 - 3\theta - 4)y = e^{2z} + 2z.$$

The A.E. is

$$m^2 - 3m - 4 = 0$$
$$(m-4)(m+1) = 0.$$
$$m = -1, 4.$$
$$y_c = c_1e^{-z} + c_2e^{4z}$$
$$= \frac{c_1}{x} + c_2x^4.$$
$$y_p = \frac{1}{\theta^2 - 3\theta - 4}(e^{2z} + 2z)$$

$$
\begin{aligned}
&= \frac{1}{\theta^2 - 3\theta - 4} e^{2z} + \frac{2}{\theta^2 - 3\theta - 4} z \\
&= \frac{e^{2z}}{4 - 6 - 4} + \frac{2}{-4\left(1 - \frac{\theta^2 - 3\theta}{4}\right)} z \\
&= \frac{e^{2z}}{-6} - \frac{1}{2}\left(1 - \frac{\theta^2 - 3\theta}{4}\right)^{-1} z \\
&= \frac{-e^{2z}}{6} - \frac{1}{2}\left(1 + \frac{\theta^2 - 3\theta}{4} + \cdots\right) z \\
&= \frac{-e^{2z}}{6} - \frac{1}{2}\left[z - \frac{3}{4}\right] \\
&= \frac{-x^2}{6} - \frac{1}{2}\left[\log x - \frac{3}{4}\right].
\end{aligned}
$$

Solution is $y = y_c + y_p$

$$
= \frac{c_1}{x} + c_2 x^4 - \frac{x^2}{6} - \frac{1}{2}\left[\log x - \frac{3}{4}\right].
$$

Example 5.78. Solve $(x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y) = e^x$

(or)

$$
x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^{e^{\log x}}.
$$

[Dec 2013]

Solution. Let $x = e^z$, or $z = \log x$.

Define $D = \frac{d}{dx}, \theta = \frac{d}{dz}$.

Then, $xD = \theta, \quad x^2D^2 = \theta(\theta - 1)$.

Now, the given equation is reduced to

$(\theta(\theta - 1) + 4\theta + 2)y = e^{e^z}$

$(\theta^2 - \theta + 4\theta + 2)y = e^{e^z}$

$(\theta^2 + 3\theta + 2)y = e^{e^z}$.

A.E is $m^2 + 3m + 2 = 0 \Rightarrow (m + 1)(m + 2) = 0 \Rightarrow m = -1, -2$.

$$
\begin{aligned}
y_c &= (c_1 e^{-z} + c_2 e^{-2z}) \\
&= \frac{c_1}{x} + \frac{c_2}{x^2}.
\end{aligned}
$$

$$y_p = \frac{1}{(\theta+1)(\theta+2)}e^{e^z} = \frac{(\theta+2)-(\theta+1)}{(\theta+1)(\theta+2)}e^{e^z} = \left(\frac{1}{\theta+1} - \frac{1}{\theta+2}\right)e^{e^z}.$$

$$= \frac{1}{\theta+1}e^{e^z} - \frac{1}{\theta+2}e^{e^z}$$

$$y_p = PI_1 - PI_2.$$

$$PI_1 = \frac{1}{\theta+1}e^{e^z} = \frac{1}{\theta-(-1)}e^{e^z}$$

$$= e^{-z}\int e^z e^{e^z}\,dz = e^{-z}\int e^{e^z}\,d(e^z)$$

$$= e^{-z}e^{e^z} = \frac{e^x}{x}.$$

$$PI_2 = \frac{1}{\theta+2}e^{e^z} = \frac{1}{\theta-(-2)}e^{e^z}$$

$$= e^{-2z}\int e^{2z}e^{e^z}\,dz = e^{-2z}\int e^z e^{e^z} e^z\,dz.$$

Let $e^z = t$.

$e^z dz = dt$.

$$PI_2 = e^{-2z}\int te^t dt = e^{-2z}\int \left[td(e^t)\right]$$

$$= e^{-2z}\left[te^t - \int e^t dt\right] = e^{-2z}[te^t - e^t]$$

$$= e^{-2z}[e^z e^{e^z} - e^{e^z}] = \frac{1}{x^2}[xe^x - e^x]$$

$$= \frac{x-1}{x^2}e^x = \frac{e^x}{x} - \frac{e^x}{x^2}.$$

Now $P.I = PI_1 - PI_2 = \frac{e^x}{x} - \frac{e^x}{x} + \frac{e^x}{x^2} = \frac{e^x}{x^2}$.

Solution is $y = y_c + y_p$.

$y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2}$.

5.3.2 Legendre's linear differential equation

The general form of the Legendre's linear differential equation is

$$(ax+b)^n\frac{d^ny}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = Q(x)$$

where a, b, a_i's are constants.

It can be reduced to the previous type by the substitution $e^z = ax + b$ and $\theta = \frac{d}{dz}$.

$$\Rightarrow z = \log(ax + b)$$

$$(ax + b)\frac{dy}{dx} = a\theta y \Rightarrow (ax + b)D = a\theta.$$

$$\text{Then, } (ax + b)^2\frac{d^2y}{dx^2} = a^2\theta(\theta - 1)y \Rightarrow (ax + b)^2D^2 = a^2\theta(\theta - 1) \text{ etc.}$$

Worked Examples

Example 5.79. Transform the equation $(2x + 3)^2y'' - (2x + 3)y' + 2y = 6x$ into a differential equation with constant coefficients. [May 2005]

Solution. Let $2x + 3 = e^z$ or $z = \log(2x + 3)$.

Define $\theta = \frac{d}{dz}$.

Then, $(2x + 3)D = 2\theta$ and $(2x + 3)^2D^2 = 4\theta(\theta - 1)$.

Now, the given differential equation is reduced to

$$(4\theta(\theta - 1) - 2\theta + 2)y = 6\frac{(e^z - 3)}{2}$$

$$(4\theta^2 - 4\theta - 2\theta + 2)y = 3(e^z - 3)$$

$$(4\theta^2 - 6\theta + 2)y = 3(e^z - 3)$$

$$4\frac{d^2y}{dz^2} - 6\frac{dy}{dz} + 2y = 3(e^z - 3),$$

which is the required equation.

Example 5.80. Transform the equation $(2x + 3)^2\frac{d^2y}{dx^2} - 2(2x + 3)\frac{dy}{dx} - 12y = 6x$ into a differential equation with constant coefficients. [May 2015, Jun 2012]

Solution. Let $2x + 3 = e^z$ or $z = \log(2x + 3)$.

Define $\theta = \frac{d}{dz}$.

Then, $(2x + 3)\frac{d}{dx} = 2\theta$ and $(2x + 3)^2\frac{d^2}{dx^2} = 4\theta(\theta - 1)$.

Now, the given differential equation is reduced to

$$(4\theta(\theta-1)-2(2)\theta-12)y = 6\frac{(e^z-3)}{2}$$
$$(4\theta^2-4\theta-4\theta-12)y = 3(e^z-3)$$
$$(4\theta^2-8\theta-12)y = 3(e^z-3)$$
$$4\frac{d^2y}{dz^2}-8\frac{dy}{dz}-12y = 3(e^z-3),$$

which is the required equation.

Example 5.81. Solve $(x+2)^2\frac{d^2y}{dx^2}-(x+2)\frac{dy}{dx}+y=3x+4.$ [Jun 2008]

Solution. Let $x+2=e^z \Rightarrow z=\log(x+2)$.

Define $\theta = \frac{d}{dz}$.

Now $(x+2)\frac{d}{dx} = \theta$

and $(x+2)^2\frac{d^2}{dx^2} = \theta(\theta-1)$.

The given equation is now reduced to

$$(\theta(\theta-1)-\theta+1)y = 3(e^z-2)+4 = 3e^z-2$$
$$(\theta^2-\theta-\theta+1)y = 3e^z-2$$
$$(\theta^2-2\theta+1)y = 3e^z-2$$
$$(\theta-1)^2y = 3e^z-2.$$

A.E is, $(m-1)^2=0 \Rightarrow m=1,1.$

$$y_c = e^z(c_1+c_2z) = (x+2)(c_1+c_2\log(x+2)).$$
$$y_p = \frac{1}{(\theta-1)^2}(3e^z-2)$$
$$= 3\frac{1}{(\theta-1)^2}e^z - 2\frac{1}{(\theta-1)^2}e^{0z}$$
$$= 3\frac{z^2}{2}e^z-2$$
$$y_p = \frac{3}{2}(\log(x+2))^2(x+2)-2.$$

The general solution is $y = y_c + y_p$.

$$y = (x+2)(c_1 + c_2 \log(x+2)) + \frac{3}{2}(\log(x+2))^2(x+2) - 2.$$

Example 5.82. Solve $(2x+3)^2 y'' - (2x+3)y' - 12y = 6x$. [Dec 2009]

Solution. Let $2x+3 = e^z$ and $\theta = \frac{d}{dz}$

Now $(2x+3)D = 2\theta$

and $(2x+3)^2 D^2 = 4\theta(\theta - 1)$.

The given equation is now reduced to

$$(4\theta(\theta-1) - 2\theta - 12)y = 6\frac{1}{2}(e^z - 3)$$

$$(4\theta^2 - 4\theta - 2\theta - 12)y = 3(e^z - 3)$$

$$(4\theta^2 - 6\theta - 12)y = 3(e^z - 3)$$

$$2(2\theta^2 - 3\theta - 6)y = 3(e^z - 3)$$

$$(2\theta^2 - 3\theta - 6)y = \frac{3}{2}(e^z - 3).$$

A.E is, $2m^2 - 3m - 6 = 0$

$$2(m^2 - \frac{3}{2}m - 3) = 0 \Rightarrow (m - \frac{3}{4})^2 - 3 - \frac{9}{16} = 0$$

$$(m - \frac{3}{4})^2 = \frac{57}{16} \Rightarrow m - \frac{3}{4} = \pm\frac{\sqrt{57}}{4}$$

$$m = \frac{3}{4} \pm \frac{\sqrt{57}}{4} \Rightarrow m = \frac{3+\sqrt{57}}{4}, \frac{3-\sqrt{57}}{4}.$$

$$y_c = c_1 e^{\left(\frac{3+\sqrt{57}}{4}\right)z} + c_2 e^{\left(\frac{3-\sqrt{57}}{4}\right)z}$$

$$y_c = c_1(2x+3)^{\left(\frac{3+\sqrt{57}}{4}\right)} + c_2(2x+3)^{\left(\frac{3-\sqrt{57}}{4}\right)}.$$

$$PI = \frac{1}{2\theta^2 - 3\theta - 6}\frac{3}{2}(e^z - 3) = \frac{1}{2\theta^2 - 3\theta - 6}\left(\frac{3}{2}e^z - \frac{9}{2}\right)$$

$$= \frac{3}{2}\frac{1}{2\theta^2 - 3\theta - 6}e^z - \frac{\frac{9}{2}}{2\theta^2 - 3\theta - 6}e^{0z}$$

$$= \frac{3}{2}\frac{e^z}{2 - 3 - 6} - \frac{\frac{9}{2}}{0 - 0 - 6}$$

$$= \frac{3}{2}\left(\frac{e^z}{-7}\right) + \frac{9}{2(6)} = -\frac{3}{14}(2x + 3) + \frac{3}{4}.$$

Solution is $y = y_c + y_p$.

$$y = c_1(2x + 3)^{\left(\frac{3+\sqrt{57}}{4}\right)} + c_2(2x + 3)^{\left(\frac{3-\sqrt{57}}{4}\right)} - \frac{3}{14}(2x + 3) + \frac{3}{4}.$$

Example 5.83. Solve $(3x + 2)^2\frac{d^2y}{dx^2} + 3(3x + 2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1.$ [Jun 2013]

Solution. Let $3x + 2 = e^z$ or $z = \log(3x + 2)$.

Define $D = \frac{d}{dx}, \theta = \frac{d}{dz}$.

Then, $(3x + 2)\frac{d}{dx} = 3\theta, (3x + 2)^2\frac{d^2}{dx^2} = 9\theta(\theta - 1)$

The given equation is now reduced to

$$(9\theta(\theta - 1) + 3 \cdot 3\theta - 36)y = 3\left(\frac{e^z - 2}{3}\right)^2 + 4\left(\frac{e^z - 2}{3}\right) + 1$$

$$(9\theta^2 - 9\theta + 9\theta - 36)y = \frac{3}{9}(e^{2z} + 4 - 4e^z) + \frac{4}{3}(e^z - 2) + 1$$

$$(9\theta^2 - 36)y = \frac{1}{3}[e^{2z} + 4 - 4e^z + 4e^z - 8] + 1$$

$$9(\theta^2 - 4)y = \frac{1}{3}[e^{2z} - 4] + 1$$

$$(\theta^2 - 4)y = \frac{1}{27}e^{2z} - \frac{4}{27} + \frac{1}{9}$$

$$= \frac{1}{27}e^{2z} - \frac{1}{27}.$$

The A.E. is $m^2 - 4 = 0$

$$m^2 = 4$$

$$m = \pm 2.$$

$$y_c = c_1e^{2z} + c_2e^{-2z}$$

$$= c_1(3x+2)^2 + \frac{c_2}{(3x+2)^2}.$$

$$y_p = \frac{1}{27(\theta^2-4)}(e^{2z}-1)$$

$$= \frac{1}{27}\left[\frac{1}{\theta^2-4}\cdot e^{2z} - \frac{1}{\theta^2-4}\cdot e^{0z}\right]$$

$$= \frac{1}{27}\left[\frac{1}{(\theta-2)(\theta+2)}\cdot e^{2z} - \frac{1}{-4}\right]$$

$$= \frac{1}{27}\left[\frac{ze^{2z}}{4} + \frac{1}{4}\right]$$

$$= \frac{(3x+2)^2\log(3x+2)}{108} + \frac{1}{108}$$

Solution is $y = y_c + y_p$

$$y = c_1(3x+2)^2 + \frac{c_2}{(3x+2)^2} + \frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right]$$

Example 5.84. Solve $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos[\log(1+x)]$.

[Dec 2014, Dec 2011]

Solution. Let $1+x = e^z$ or $z = \log(1+x)$.

Define $D = \dfrac{d}{dx}$, $\theta = \dfrac{d}{dz}$.

Then, $(1+x)\dfrac{d}{dx} = \theta$, $(1+x)^2\dfrac{d^2}{dx^2} = \theta(\theta-1)$.

The given equation is now reduced to

$$(\theta(\theta-1)+\theta+1)y = 4\cos z.$$

$$(\theta^2-\theta+\theta+1)y = 4\cos z$$

$$(\theta^2+1)y = 4\cos z.$$

The A.E. is $m^2+1=0$

$$m^2 = -1$$

$$m = \pm i.$$

$$y_c = e^{0z}(c_1\cos z + c_2\sin z)$$

$$= c_1 \cos\log(1+x) + c_2 \sin\log(1+x).$$

$$y_p = \frac{4}{\theta^2+1}\cos z$$
$$= 4\frac{z}{2\theta}\cos z$$
$$= 2z\int \cos z dz$$
$$= 2z\sin z$$
$$= 2\log(1+x)\sin\log(1+x).$$

Solution is $y = y_c + y_p$

$$y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) + 2\log(1+x)\sin\log(1+x).$$

5.4 Method of variation of parameters

Consider the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x). \tag{1}$$

Let $C.F = c_1y_1(x) + c_2y_2(x)$ where c_1 and c_2 are arbitrary constants, then $y_1(x)$ and $y_2(x)$ are two independent solutions of

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0. \tag{2}$$

By the method of variation of parameters, y_p is evaluated by $y_p = u(x)y_1 + v(x)y_2$ where $u(x)$ and $v(x)$ are evaluated by the following way.

Define the Wronskian of y_1 and y_2 by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0,$$

Then, $u(x) = -\int \frac{y_2R(x)}{W}dx$ and $v(x) = \int \frac{y_1R(x)}{W}dx$.

Then, the general solution is given by $y = y_c + y_p$.

Worked Examples

Example 5.85. Find the Wronskian of y_1, y_2 of $y'' - 2y' + y = e^x \log x$. [Dec 2012]

Solution. The A.E. is

$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0$$
$$m = 1, 1.$$

$$y_c = e^x(c_1 + c_2 x).$$
$$\therefore\ y_1 = e^x, y_2 = xe^x.$$
$$y_1' = e^x \qquad y_2' = xe^x + e^x = e^x(x+1).$$
$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x(x+1) \end{vmatrix}$$
$$= e^x \times e^x \begin{vmatrix} 1 & x \\ 1 & x+1 \end{vmatrix}$$
$$= e^{2x}(x + 1 - x) = e^{2x}$$

Example 5.86. Solve $\dfrac{d^2y}{dx^2} + a^2 y = \sec ax$. [Jun 2012]

Solution. A.E is $m^2 + a^2 = 0 \Rightarrow m^2 = -a^2 \Rightarrow m = \pm ai$.

$$CF = y_c = C_1 \cos ax + C_2 \sin ax.$$
$$y_1 = \cos ax, \qquad y_2 = \sin ax.$$
$$R(x) = \sec ax.$$
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix}$$
$$= a \cos^2 ax + a \sin^2 ax = a.$$

Now, $y_p = u(x) y_1 + v(x) y_2$.

$$u(x) = -\int \frac{y_2 R(x)}{W} dx = -\int \frac{\sin ax \sec ax}{a} dx$$
$$= -\frac{1}{a} \int \tan ax dx = -\frac{1}{a} \frac{\log(\sec ax)}{a}$$

$$= -\frac{1}{a^2}\log\sec ax.$$

$$v(x) = \int \frac{y_1 R(x)}{W}dx = \int \frac{\cos ax \sec ax}{a}dx = \frac{1}{a}x.$$

$$\text{Hence, } y_p = u(x)y_1 + v(x)y_2 = \frac{-1}{a^2}\log\sec ax\cos ax + \frac{x}{a}\sin ax.$$

The solution is $y = y_c + y_p$

$$y = C_1\cos ax + C_2\sin ax - \frac{\cos ax}{a^2}\log\sec ax + \frac{x}{a}\sin ax.$$

Example 5.87. Solve by the method of variation of parameters, $\frac{d^2y}{dx^2} + y = x\sin x$.

[Jun 2010, May 2002]

Solution. A.E is $m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$.

$$y_c = c_1\cos x + c_2\sin x.$$

$$y_1 = \cos x, y_2 = \sin x.$$

$$\text{Now, } y_p = u(x)y_1 + v(x)y_2.$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

$$\begin{aligned}
u(x) &= -\int \frac{y_2 R(x)}{W}dx \\
&= -\int \frac{\sin x \times x\sin x dx}{1} = -\int x\sin^2 x dx \\
&= -\int \frac{x(1-\cos 2x)}{2}dx = -\frac{1}{2}[\int xdx - \int x\cos 2xdx] \\
&= -\frac{1}{2}(\frac{x^2}{2} - \int xd(\frac{\sin 2x}{2})) \\
&= -\frac{1}{2}(\frac{x^2}{2} - (\frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2}dx)) \\
&= -\frac{1}{2}(\frac{x^2}{2} - \frac{x\sin 2x}{2} - \frac{\cos 2x}{4}).
\end{aligned}$$

$$v(x) = \int \frac{y_1 R(x)}{W}dx = \int \cos x \times x\sin x dx$$

$$= \int x \frac{\sin 2x}{2} dx = \frac{1}{2} \int x d(-\frac{\cos 2x}{2})$$
$$= -\frac{1}{2}(\frac{x \cos 2x}{2} - \int \frac{\cos 2x}{2} dx) = -\frac{1}{2}(\frac{x \cos 2x}{2} - \frac{\sin 2x}{4}).$$

Hence, $y_p = u(x)y_1 + v(x)y_2$

$$= -\frac{1}{2} \cos x(\frac{x^2}{2} - \frac{x \sin 2x}{2} - \frac{\cos 2x}{4}) - \frac{1}{2} \sin x(\frac{x \cos 2x}{2} - \frac{\sin 2x}{4}).$$

Solution is $y = y_c + y_p$.

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{2} \cos x\left(\frac{x^2}{2} - \frac{x \sin 2x}{2} - \frac{\cos 2x}{4}\right) - \frac{1}{2} \sin x\left(\frac{x \cos 2x}{2} - \frac{\sin 2x}{4}\right).$$

Example 5.88. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + y = cosecx \cot x$. [Dec 2007]

Solution. A.E is $m^2 + 1 = 0 \Rightarrow m^2 = \pm - 1 \Rightarrow m = \pm i$.

$$y_c = c_1 \cos x + c_2 \sin x.$$

$$y_1 = \cos x, y_2 = \sin x.$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

$$y_p = u(x)y_1 + v(x)y_2, R(x) = cosecx \cot x.$$

$$u(x) = -\int \frac{y_2 R(x)}{W} dx = -\int \sin x cosecx \cot x dx = -\log(\sin x).$$
$$v(x) = \int \frac{y_1 R(x)}{W} dx = \int \cos x cosecx \cot x dx$$
$$= \int \frac{\cos^2 x}{\sin^2 x} dx = \int \cot^2 x dx$$
$$= \int (cosec^2 x - 1) dx = \int cosec^2 x dx - \int dx = -\cot x - x.$$
$$y_p = u(x)y_1 + v(x)y_2 = -(\log \sin x) \cos x + (-\cot x - x) \sin x.$$

Solution is, $y = y_c + y_p$.

$$y = c_1 \cos x + c_2 \sin x - (\log \sin x) \cos x + (-\cot x - x) \sin x.$$

Example 5.89. Solve by the method of variation of parameters $(D^2 + a^2)y = \tan ax$.

[Dec 2014, Jun 2009]

Solution. A.E is $m^2 + a^2 = 0 \Rightarrow m = \pm ai$.

$$y_c = c_1 \cos ax + c_2 \sin ax.$$

$$y_1 = \cos ax, y_2 = \sin ax.$$

$$R(x) = \tan ax.$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a.$$

$$y_p = u(x)y_1 + v(x)y_2.$$

$$\begin{aligned} u(x) &= -\int \frac{y_2 R(x)}{W} dx = -\int \frac{\sin ax \tan ax}{a} dx \\ &= -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax} dx = -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax} dx \\ &= -\frac{1}{a}\int \sec ax dx + \frac{1}{a}\int \cos ax dx \\ &= -\frac{1}{a}\frac{\log(\sec ax + \tan ax)}{a} + \frac{1}{a^2}\sin ax. \\ v(x) &= \int \frac{y_1 R(x)}{W} dx = \int \frac{\cos ax \tan ax}{a} dx \\ &= \frac{1}{a}\int \sin ax dx = -\frac{1}{a^2}\cos ax. \end{aligned}$$

$$\begin{aligned} y_p &= u(x)y_1 + v(x)y_2 \\ &= -\frac{1}{a^2}(\log(\sec ax + \tan ax) - \sin ax)\cos ax - \frac{\cos ax \sin ax}{a^2}. \end{aligned}$$

Solution is $y = y_c + y_p$.

$$y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}(\log(\sec ax + \tan ax) - \sin ax)\cos ax - \frac{\cos ax \sin ax}{a^2}.$$

Example 5.90. Solve by the method of variation of parameters $2\frac{d^2y}{dx^2} + 8y = \tan 2x$.

[Dec 2013]

Solution. The given equation can be written as

$$2\left(\frac{d^2y}{dx^2} + 4y\right) = \tan 2x$$

$$\frac{d^2y}{dx^2} + 4y = \frac{\tan 2x}{2}.$$

The A.E. is $m^2 + 4 = 0$

$$m^2 = -4$$

$$m = \pm 2i.$$

$$y_c = c_1 \cos 2x + c_2 \sin 2x.$$

$$\therefore y_1 = \cos 2x, y_2 = \sin 2x.$$

$$R(x) = \frac{\tan 2x}{2}.$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$= 2.$$

$$y_p = u(x)y_1 + v(x)y_2.$$

$$u(x) = -\int \frac{y_2 R(x)}{W} dx$$

$$= -\int \frac{\sin 2x \cdot \tan 2x}{2 \times 2} dx.$$

$$= -\frac{1}{4}\int \sin 2x \cdot \frac{\sin 2x}{\cos 2x} dx$$

$$= -\frac{1}{4}\int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{4}\int \frac{1 - \cos^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{4}\int (\sec 2x - \cos 2x) dx$$

$$= -\frac{1}{4}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right]$$

$$= \frac{-1}{8}[\log(\sec 2x + \tan 2x) - \sin 2x]$$

$$v(x) = \int \frac{y_1 R(x)}{W} dx = \int \cos 2x \cdot \frac{\tan 2x}{2 \times 2} dx.$$

$$= \frac{1}{4}\int \cos 2x \cdot \frac{\sin 2x}{\cos 2x} dx$$

$$= \frac{1}{4}\int \sin 2x dx.$$

$$= \frac{1}{4}\left(-\frac{\cos 2x}{2}\right) = -\frac{\cos 2x}{8}$$

$$\therefore y_p = u(x)y_1 + v(x)y_2$$

$$= -\frac{1}{8}[\log(\sec 2x + \tan 2x) - \sin 2x]\cos 2x - \frac{\cos 2x \sin 2x}{8}$$

Solution is $y = y_c + y_p$

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{8}[\log(\sec 2x + \tan 2x) - \sin 2x]\cos 2x - \frac{\sin 4x}{16}.$$

Example 5.91. Solve the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{e^{-x}}{x^2}$ by the method of variation of parameters. [Jun 2013]

Solution. The A.E. is

$$m^2 + 2m + 1 = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1.$$

$$y_c = e^{-x}(c_1 + c_2 x)$$

$$y_1 = e^{-x}, y_2 = xe^{-x}.$$

$$R(x) = \frac{e^{-x}}{x^2}.$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x} + e^{-x} \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x}(1-x) \end{vmatrix}$$

$$= e^{-x} \times e^{-x} \begin{vmatrix} 1 & x \\ -1 & 1-x \end{vmatrix}$$

$$= e^{-2x}(1 - x + x) = e^{-2x}.$$

$$y_p = u(x)y_1 + v(x)y_2.$$

$$u(x) = -\int \frac{y_2 R(x)}{W} dx$$

$$= -\int \frac{xe^{-x} \cdot e^{-x}}{x^2 e^{-2x}} dx.$$

$$= -\int \frac{1}{x} dx = -\log x.$$

$$v(x) = \int \frac{y_1 R(x)}{W} dx = \int \frac{e^{-x} e^{-x}}{x^2 e^{-2x}} dx$$

$$= \int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} = -\frac{1}{x}.$$

$$\text{Now } y_p = u(x)y_1 + v(x)y_2$$

$$= -\log x(e^{-x}) - \frac{1}{x} xe^{-x}$$

$$= -e^{-x}(\log x + 1)$$

Solution is $y = y_c + y_p$

$$y = e^{-x}(c_1 + c_2 x) - e^{-x}(\log x + 1).$$

Example 5.92. Solve by the method of variation of parameters the differential equation $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x.$ [Dec 2012, May 2011]

Solution. The auxiliary equation is

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm i.$$

$$y_c = c_1 \cos x + c_2 \sin x.$$

$$
\begin{aligned}
y_1 &= \cos x, y_2 = \sin x. \\
R(x) &= \operatorname{cosec} x. \\
W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\
&= \cos^2 x + \sin^2 x = 1. \\
y_p &= u(x)y_1 + v(x)y_2. \\
\text{Now, } u(x) &= -\int \frac{y_2 R(x)}{W} dx \\
&= -\int \sin x \cdot \operatorname{cosec} x dx. \\
&= -\int dx = -x. \\
v(x) &= \int \frac{y_1 R(x)}{W} dx \\
&= \int \cos x \cdot \operatorname{cosec} x dx \\
&= \int \cot x dx = \log \sin x. \\
\therefore y_p &= u(x)y_1 + v(x)y_2 \\
&= -x \cos x + \log(\sin x) \cdot \sin x.
\end{aligned}
$$

Solution is $y = y_c + y_p$

$$y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log \sin x.$$

5.5 Simultaneous linear differential equations with constant coefficients

We have seen so far, the method of solving a single differential equation involving one independent variable x and one dependent variable y. Quite often we come across linear differential equations in which there will be two or more dependent variables and a single independent variable. Such equations are known as simultaneous linear equations. In this section, we consider linear differential equations with one independent variable t and two dependent variables x and y. We need two differential equations to solve for x and y. Hence, we will be given a system of two linear differential equations which need not be of the same order. We shall consider here only first order linear differential equations with constant coefficients and we consider three types of equations.

5.5.1 Type I

We consider simultaneous equations of the form

$$a_1\frac{dx}{dt} + b_1 y = f(t), \qquad a_2\frac{dy}{dt} + b_2 x = g(t).$$

First we eliminate one of the dependent variables from the two equations which results in a second order linear differential equation with constant coefficients in the other dependent variable and the independent variable t.

Worked Examples

Example 5.93. Solve for x and y if $\frac{dy}{dt} = x, \frac{dx}{dt} = y.$ [May 2004]

Solution. The given equations are

$$\frac{dy}{dt} = x \tag{1}$$

$$\frac{dx}{dt} = y. \tag{2}$$

Differentiating (2) w. r. t. x we get

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} = x\text{[from (1)]}$$

$$\frac{d^2x}{dt^2} - x = 0$$

$$(D^2 - 1)x = 0 \text{where } D = \frac{d}{dt}.$$

A.E is $m^2 - 1 = 0$

$$m^2 = 1$$

$$m = \pm 1$$

$$\therefore x = c_1 e^t + c_2 e^{-t}.$$

Now from (2) we have $y = \frac{dx}{dt} = c_1 e^t - c_2 e^{-t}$.

Example 5.94. Eliminate y from the system $\frac{dx}{dt} + 2y = -\sin t$ and $\frac{dy}{dt} - 2x = \cos t$.

Solution. Differentiating the first equation w.r.t. t we get

$$\frac{d^2x}{dt^2} + 2\frac{dy}{dt} = -\cos t.$$

$$\frac{d^2x}{dt^2} + 2[2x + \cos t] = -\cos t.$$

$$\frac{d^2x}{dt^2} + 4x + 2\cos t + \cos t = 0$$

$$\frac{d^2x}{dt^2} + 4x = -3\cos t,$$

which is the required equation.

Example 5.95. Eliminate y from the simultaneous equations $\frac{dx}{dt} + y = \sin t$ and $\frac{dy}{dt} + x = \cos t$.

Solution. Differentiating the first equation w.r.t. t we get

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} = \cos t.$$

$$\frac{d^2x}{dt^2} + \cos t - x = \cos t.$$

$$\frac{d^2x}{dt^2} - x = 0,$$

which is the required equation.

Example 5.96. Solve $\frac{dx}{dt} - y = t$, $\frac{dy}{dt} + x = t^2$. [May 2011, Jun 2006]

Solution. The equations are

$$\frac{dx}{dt} - y = t. \tag{1}$$

and

$$\frac{dy}{dt} + x = t^2. \quad (2)$$

Differentiating (1) w. r. t. t we get

$$\frac{d^2x}{dt^2} - \frac{dy}{dt} = 1$$

$$\frac{d^2x}{dt^2} - [-x + t^2] = 1 \qquad [\text{ from } (2)]$$

$$\frac{d^2x}{dt^2} + x = 1 + t^2.$$

This is a second order differential equation.

Now, A.E is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$x_c = CF = c_1 \cos t + c_2 \sin t.$$

$$x_p = \frac{1}{1 + D^2} 1 + t^2 = (1 + D^2)^{-1}(1 + t^2)$$

$$= (1 - D^2 + D^4 - \cdots)(1 + t^2) = 1 + t^2 - 2 = t^2 - 1.$$

Solution is $x = x_c + x_p = c_1 \cos t + c_2 \sin t + t^2 - 1$.

From (1) we get

$$y = \frac{dx}{dt} - t = -c_1 \sin t + c_2 \cos t + 2t - t$$

$$y = c_2 \cos t - c_1 \sin t + t.$$

The required solution is $x = c_1 \cos t + c_2 \sin t + t^2 - 1$.

$$y = c_2 \cos t - c_1 \sin t + t.$$

Example 5.97. Solve $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$ given that $x = 2, y = 0$ when $t = 0$.
[Dec 2009]

Solution. The given equations are

$$\frac{dx}{dt} + y = \sin t. \quad (1)$$

$$\frac{dy}{dt} + x = \cos t. \quad (2)$$

Differentiating (1) w.r.t t we get

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} = \cos t$$

$$\frac{d^2x}{dt^2} - x + \cos t = \cos t \quad [\text{from } (2)]$$

$$\frac{d^2x}{dt^2} - x = 0.$$

This is a second order linear differential equation with constant coefficients.

A.E is $m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1.$

The solution is $x = c_1e^t + c_2e^{-t}$.

From (1) we obtain $y = -\dfrac{dx}{dt} + \sin t$

i.e., $y = c_1e^t - c_2e^{-t} + \sin t.$

Given, $x = 2$ when $t = 0$

$$\Longrightarrow c_1 + c_2 = 2. \tag{3}$$

$$y = 0 \text{ when } t = 0.$$

$$\Longrightarrow c_1 - c_2 = 0. \tag{4}$$

$$(3) + (4) \Longrightarrow 2c_1 = 2 \Rightarrow c_1 = 1 \Rightarrow c_2 = 1.$$

The solution is $x = e^t + e^{-t}$

$$y = e^t - e^{-t} + \sin t.$$

Example 5.98. Solve $\dfrac{dx}{dt} + y = e^t, x - \dfrac{dy}{dt} = t.$ [Dec 2012]

Solution. The equations are

$$\frac{dx}{dt} + y = e^t. \tag{1}$$

$$x - \frac{dy}{dt} = t. \tag{2}$$

Differentiating(1) w.r.t. t we get

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} = e^t$$

$$\frac{d^2x}{dt^2} + x - t = e^t \quad \text{[from (2)]}$$

$$\frac{d^2x}{dt^2} + x = t + e^t.$$

The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m^2 = -1$$

$$m = \pm i.$$

$$x_c = c_1 \cos t + c_2 \sin t$$

$$\begin{aligned}
x_p &= \frac{1}{D^2+1}(t + e^t) \\
&= \frac{1}{D^2+1}t + \frac{1}{D^2+1}e^t \\
&= (1 + D^2)^{-1}t + \frac{1}{2}e^t \\
&= (1 - D^2 + D^4 \cdots)t + \frac{e^t}{2} \\
&= t + \frac{e^t}{2}.
\end{aligned}$$

The solution for x is

$$x = x_c + x_p$$

$$x = c_1 \cos t + c_2 \sin t + t + \frac{e^t}{2}.$$

Substituting in (1) we get

$$\begin{aligned}
y &= e^t - \frac{dx}{dt} \\
&= e^t - \left[-c_1 \sin t + c_2 \cos t + 1 + \frac{e^t}{2}\right] \\
&= e^t + c_1 \sin t - c_2 \cos t - 1 - \frac{e^t}{2} \\
y &= \frac{e^t}{2} + c_1 \sin t - c_2 \cos t - 1.
\end{aligned}$$

Example 5.99. Solve the simultaneous equations $\frac{dx}{dt} + 2y = \sin 2t$, $\frac{dy}{dt} - 2x = \cos 2t$.
[Jun 2012]

Solution. The given equations are

$$\frac{dx}{dt} + 2y = \sin 2t \tag{1}$$

$$\frac{dy}{dt} - 2x = \cos 2t. \tag{2}$$

Differentiating (1) w.r.t.t we get

$$\frac{d^2x}{dt^2} + 2\frac{dy}{dt} = 2\cos 2t.$$

$$\frac{d^2x}{dt^2} + 2[2x + \cos 2t] = 2\cos 2t. \quad [from(2)]$$

$$\frac{d^2x}{dt^2} + 4x + 2\cos 2t = 2\cos 2t.$$

$$\frac{d^2x}{dt^2} + 4x = 0.$$

The auxiliary equation is

$$m^2 + 4 = 0 \Rightarrow m^2 = -4$$

$$m = \pm 2i.$$

$$x_c = c_1 \cos 2t + c_2 \sin 2t.$$

$$x_p = 0.$$

The solution for x is

$$x = x_c + x_p = c_1 \cos 2t + c_2 \sin 2t.$$

Substituting in (1) we get

$$2y = \sin 2t - \frac{dx}{dt}$$

$$= \sin 2t - [-2c_1 \sin 2t + 2c_2 \cos 2t]$$

$$= \sin 2t + 2c_1 \sin 2t - 2c_2 \cos 2t.$$

$$y = \frac{\sin 2t}{2} + c_1 \sin 2t - c_2 \cos 2t.$$

Example 5.100. Solve $\frac{dx}{dt} - y = t$ and $\frac{dy}{dt} + x = t^2$ given that $x(0) = y(0) = 2$.

[Dec 2011]

Solution. Refer Example 2.104.

The solution is

$$x = c_1 \cos t + c_2 \sin t + t^2 - 1$$

$$y = c_2 \cos t - c_1 \sin t + t.$$

Given: when $t = 0, x = 2$

$\therefore c_1 - 1 = 2 \Rightarrow c_1 = 3.$

Also, when $t = 0, y = 2.$

$\therefore c_2 = 2.$

The solutions are

$$x = 3\cos t + 2\sin t + t^2 - 1$$
$$y = 2\cos t - 3\sin t + t.$$

Example 5.101. Solve $\frac{dx}{dt} + 2y = -\sin t$, $\frac{dy}{dt} - 2x = \cos t$ given, $x = 1$ and $y = 0$ at $t = 0$.

[Dec 2010]

Solution. The given equations are

$$\frac{dx}{dt} + 2y = -\sin t \quad (1)$$
$$\frac{dy}{dt} - 2x = \cos t. \quad (2)$$

Differentiating (1) w.r.t.t we get

$$\frac{d^2x}{dt^2} + 2\frac{dy}{dt} = -\cos t$$
$$\frac{d^2x}{dt^2} + 2[2x + \cos t] = -\cos t \quad \text{[from (2)].}$$
$$\frac{d^2x}{dt^2} + 4x + 2\cos t = -\cos t.$$
$$\frac{d^2x}{dt^2} + 4x = -3\cos t.$$

The auxiliary equation is

$$m^2 + 4 = 0 \Rightarrow m^2 = -4$$
$$m = \pm 2i.$$
$$x_c = c_1 \cos 2t + c_2 \sin 2t.$$

$$
\begin{aligned}
x_p &= \frac{1}{D^2+4}(-3\cos t)\\
&= -3 \cdot \frac{\cos t}{-1+4}\\
&= -3\frac{\cos t}{3} = -\cos t.
\end{aligned}
$$

$\therefore$ The solution for x is

$x = x_c + x_p$

$x = c_1 \cos 2t + c_2 \sin 2t - \cos t.$

Substituting in (1) we get

$$
\begin{aligned}
2y &= -\sin t - \frac{dx}{dt}\\
&= -\sin t - [-c_1 2\sin 2t + 2c_2 \cos 2t + \sin t]\\
&= -\sin t + 2c_1 \sin 2t - 2c_2 \cos 2t - \sin t\\
&= 2c_1 \sin 2t - 2c_2 \cos 2t - 2\sin t.\\
y &= c_1 \sin 2t - c_2 \cos 2t - \sin t.
\end{aligned}
$$

Given: when $t = 0, x = 1$.

$\therefore c_1 - 1 = 1 \Rightarrow c_1 = 2.$

When $t = 0, y = 0$.

$-c_2 = 0 \Rightarrow c_2 = 0.$

$\therefore$ The solutions are

$x = 2\cos 2t - \cos t$

$y = 2\sin 2t - \sin t.$

5.5.2 Type II

We consider a system of first order linear differential equations of the form

$$a_1\frac{dx}{dt} + b_1 x + c_1 y = f(t) \text{ and } a_2\frac{dx}{dt} + b_2 x + c_2 y = g(t).$$

Procedure. Replace $\frac{d}{dt}$ by D and rewrite the given equations involving D. Eliminating either x or y from the two equations we obtain a second order differential equation in the other variable for which the solution can be obtained by earlier methods. Substituting this value in any one of the given equations, we obtain the solution for the other variable.

Worked Examples

Example 5.102. Solve $\frac{dx}{dt} = 3x + 8y, \frac{dy}{dt} = -x - 3y, x(0) = 6, y(0) = -2.$ [May 2007]

Solution. Let $\frac{d}{dt} = D$. The given equations are reduced to

$$\frac{dx}{dt} = 3x + 8y$$

$$(D-3)x - 8y = 0. \tag{1}$$

$$\frac{dy}{dt} = -x - 3y$$

$$(D+3)y + x = 0. \tag{2}$$

Operating (1) by $D + 3$ we obtain

$$(D+3)(D-3)x - 8(D+3)y = 0$$

$$(D^2 - 9)x + 8x = 0$$

$$(D^2 - 9 + 8)x = 0$$

$$(D^2 - 1)x = 0$$

$$m^2 - 1 = 0.$$

$$m = \pm 1.$$

$$x = c_1 e^t + c_2 e^{-t}.$$

From (1) we have $8y = \frac{dx}{dt} - 3x \Longrightarrow y = \frac{1}{8}\left[\frac{dx}{dt} - 3x\right]$

$$y = \frac{1}{8}\left\{[c_1 e^t - c_2 e^{-t}] - 3[c_1 e^t + c_2 e^{-t}]\right\} = \frac{1}{8}[-2c_1 e^t - 4c_2 e^{-t}] = \frac{-1}{4}[c_1 e^t + 2c_2 e^{-t}].$$

When $t = 0, x = 6.$

$$\Longrightarrow c_1 + c_2 = 6. \tag{3}$$

When $t = 0, y = -2$.

$\Rightarrow \dfrac{c_1 + 2c_2}{-4} = -2$

$c_1 + 2c_2 = 8.$ (4)

$(4) - (3) \Rightarrow c_2 = 2 \Rightarrow c_1 = 4.$

Solution is $x = 4e^t + 2e^{-t}$

$y = -\dfrac{1}{4}[4e^t + 4e^{-t}] = -[e^t + e^{-t}].$

Example 5.103. Solve $\dfrac{dx}{dt} + 2x - 3y = t, \dfrac{dy}{dt} - 3x + 2y = e^{2t}.$ [Dec 2014, Jun 2006]

Solution. Let $\dfrac{d}{dt} = D$. The given equations are reduced to

$$(D+2)x - 3y = t. \tag{1}$$

$$-3x + (D+2)y = e^{2t}. \tag{2}$$

$$(1) \times 3 \Longrightarrow 3(D+2)x - 9y = 3t \tag{3}$$

Operating (2) by $D + 2$ we obtain

$$-3(D+2)x + (D+2)^2 y = (D+2)e^{2t} = 2e^{2t} + 2e^{2t} = 4e^{2t}. \tag{4}$$

$$\begin{aligned}(3) + (4) \Longrightarrow ((D+2)^2 - 9)y &= 4e^{2t} + 3t.\\ (D^2 + 4D + 4 - 9)y &= 4e^{2t} + 3t\\ (D^2 + 4D - 5)y &= 4e^{2t} + 3t.\end{aligned}$$

This is a second order linear differential equation.

A.E is, $m^2 + 4m - 5 = 0 \Rightarrow (m+5)(m-1) = 0 \Rightarrow m = -5, m = 1.$

$$\text{C.F.} = y_c = c_1 e^{-5t} + c_2 e^t.$$

$$\begin{aligned}y_p &= \frac{1}{D^2 + 4D - 5} 4e^{2t} + \frac{1}{-5(1 - \frac{D^2+4D}{5})} 3t\\ &= 4\frac{e^{2t}}{7} - \frac{3}{5}\Big(1 - \frac{D^2 + 4D}{5}\Big)^{-1} t\\ &= \frac{4}{7}e^{2t} - \frac{3}{5}\Big(1 + \frac{D^2}{5} + \frac{4}{5}D + \cdots\Big)t\end{aligned}$$

$$PI = \frac{4}{7}e^{2t} - \frac{3}{5}(t + \frac{4}{5})$$
$$y = c_1 e^{-5t} + c_2 e^t + \frac{4}{7}e^{2t} - \frac{3}{5}(t + \frac{4}{5}).$$

From (2) we have

$$3x = \frac{dy}{dt} + 2y - e^{2t}$$
$$= -5c_1 e^{-5t} + c_2 e^t + \frac{4}{7}2e^{2t} - \frac{3}{5} + 2\left[c_1 e^{-5t} + c_2 e^t + \frac{4}{7}e^{2t} - \frac{3}{5}(t + \frac{4}{5})\right] - e^{2t}$$
$$3x = -3c_1 e^{-5t} + 3c_2 e^t + \frac{9}{7}e^{2t} - \frac{6}{5}t - \frac{39}{25}.$$
$$x = -c_1 e^{-5t} + c_2 e^t + \frac{3}{7}e^{2t} - \frac{2}{5}t - \frac{9}{5}.$$

Example 5.104. Solve $(D+2)x + 3y = 0, 3x + (D+2)y = 2e^{2t}$. [May 2003]

Solution. The given equations are

$$(D+2)x + 3y = 0. \quad (1)$$

$$3x + (D+2)y = 2e^{2t}. \quad (2)$$

Operate (1) by $(D+2)$ we get

$$(D+2)^2 x + 3(D+2)y = 0. \quad (3)$$
$$(2) \times 3 \Rightarrow 9x + 3(D+2)y = 6e^{2t}. \quad (4)$$
$$(3) - (4) \Rightarrow ((D+2)^2 - 9)x = -6e^{2t}$$
$$(D^2 + 4D + 4 - 9)x = -6e^{2t}$$
$$(D^2 + 4D - 5)x = -6e^{2t}.$$

A.E is, $m^2 + 4m - 5 = 0 \Rightarrow (m+5)(m-1) = 0.$

$$x_c = c_1 e^{-5t} + c_2 e^t.$$
$$x_p = -6\frac{e^{2t}}{D^2 + 4D - 5}$$
$$= -6\frac{e^{2t}}{7} = \frac{-6}{7}e^{2t}.$$

The solution for x is

$$x = x_c + x_p.$$

$$x = c_1 e^{-5t} + c_2 e^{t} - \frac{6}{7} e^{2t}.$$

From (1) we have

$$3y = -(D+2)x = -Dx - 2x$$

$$= 5c_1 e^{-5t} - c_2 e^{t} + \frac{12}{7} e^{2t} - 2c_1 e^{-5t} - 2c_2 e^{t} + \frac{12}{7} e^{2t}$$

$$3y = 3c_1 e^{-5t} - 3c_2 e^{t} + \frac{24}{7} e^{2t}$$

$$y = \frac{1}{3}(3c_1 e^{-5t} - 3c_2 e^{t} + \frac{24}{7} e^{2t}) = c_1 e^{-5t} - c_2 e^{t} + \frac{8}{7} e^{2t}.$$

Example 5.105. Solve $\frac{dx}{dt} + 4x + 3y = t;\ \frac{dy}{dt} + 2x + 5y = e^{2t}$. [Dec 2013]

Solution. The given equations are

$$(D+4)x + 3y = t. \tag{1}$$

$$2x + (D+5)y = e^{2t}. \tag{2}$$

Operate (1) by $D+5$, we get

$$(D+4)(D+5)x + 3(D+5)y = (D+5)t = 1 + 5t.$$

$(2) \times 3 \Rightarrow$

$$6x + 3(D+5)y = 3e^{2t}.$$

Subtracting, we get

$$(D+4)(D+5)x - 6x = 1 + 5t - 3e^{2t}$$

$$(D^2 + 9D + 20 - 6)x = 1 + 5t - 3e^{2t}$$

$$(D^2 + 9D + 14)x = 1 + 5t - 3e^{2t}.$$

The auxiliary equation is

$$m^2 + 9m + 14 = 0$$
$$(m+2)(m+7) = 0$$
$$m = -2, -7.$$
$$x_c = c_1 e^{-2t} + c_2 e^{-7t}.$$
$$\begin{aligned}
x_p &= \frac{1}{D^2+9D+14}(1+5t-3e^{2t}) \\
&= \frac{1}{D^2+9D+14}(1+5t) - \frac{3}{D^2+9D+14}e^{2t} \\
&= \frac{1}{14\left(1+\frac{9D+D^2}{14}\right)}(1+5t) - \frac{3e^{2t}}{4+18+14} \\
&= \frac{1}{14}\cdot\left(1+\frac{9D+D^2}{14}\right)^{-1}(1+5t) - \frac{3e^{2t}}{36} \\
&= \frac{1}{14}\left(1-\frac{9D+D^2}{14}\cdots\right)(1+5t) - \frac{e^{2t}}{12} \\
&= \frac{1}{14}\left[1+5t-\frac{9}{14}\cdot 5\right] - \frac{e^{2t}}{12} \\
&= \frac{1}{14}\left[1+5t-\frac{45}{14}\right] - \frac{e^{2t}}{12} \\
&= \frac{1}{14}\left[\frac{14+70t-45}{14}\right] - \frac{e^{2t}}{12} \\
&= \frac{70t-31}{196} - \frac{e^{2t}}{12}.
\end{aligned}$$

The solution for x is

$$x = x_c + x_p$$
$$x = c_1 e^{-2t} + c_2 e^{-7t} + \frac{70t-31}{196} - \frac{e^{2t}}{12}.$$

Substituting in (1) we get

$$3y = t - (D+4)x.$$
$$= t - (D+4)\left(c_1 e^{-2t} + c_2 e^{-7t} + \frac{70t-31}{196} - \frac{e^{2t}}{12}\right)$$

$$= t - \left(-2c_1e^{-2t} - 7c_2e^{-7t} + \frac{70}{196} - \frac{e^{2t}}{6} + 4c_1e^{-2t} + 4c_2e^{-7t} + \frac{70t-31}{49} - \frac{e^{2t}}{3}\right)$$
$$= t - 2c_1e^{-2t} + 3c_2e^{-7t} - \frac{70}{196} + \frac{e^{2t}}{6} - \frac{70t-31}{49} + \frac{e^{2t}}{3}.$$
$$= t - \frac{70t}{49} + \frac{31}{49} - \frac{70}{196} - 2c_1e^{-2t} + 3c_2e^{-7t} + \frac{e^{2t}}{2}.$$
$$3y = -\frac{21t}{49} + \frac{54}{196} - 2c_1e^{-2t} + 3c_2e^{-7t} + \frac{e^{2t}}{2}.$$
$$y = \frac{1}{3}\left(-\frac{21t}{49} + \frac{54}{196} - 2c_1e^{-2t} + 3c_2e^{-7t} + \frac{e^{2t}}{2}\right).$$

5.5.3 Type III

Here, we consider a system of linear first order differential equations of the form

$$a_1\frac{dx}{dt} + b_1\frac{dy}{dt} + c_1x = f(t)$$
$$a_2\frac{dx}{dt} + b_2\frac{dy}{dt} + c_2y = g(t).$$

As in Type II, replace $\frac{d}{dt}$ by D and eliminate any one variable and solve for the other variable by obtaining a second order linear differential equation and finally on substitution of this value in any one of the equations, we obtain the solution for the other variable.

Example 5.106. Solve $\frac{dx}{dt} - \frac{dy}{dt} + 2y = \cos 2t$, $\frac{dx}{dt} + \frac{dy}{dt} - 2x = \sin 2t$. [Dec 2008]

Solution. Let $\frac{d}{dt} = D$. The given equations can be written as

$$Dx - (D-2)y = \cos 2t \tag{1}$$

$$(D-2)x + Dy = \sin 2t \tag{2}$$

Operating (1) by D we get

$$D^2x - D(D-2)y = -2\sin 2t \tag{3}$$

Operating (2) by $(D-2)$ we obtain

$$(D-2)^2x + D(D-2)y = (D-2)\sin 2t = 2\cos 2t - 2\sin 2t \tag{4}$$

(3)+(4)$\Longrightarrow$

$$\begin{aligned}
(D^2 + (D-2)^2)x &= 2\cos 2t - 4\sin 2t \\
(D^2 + D^2 - 4D + 4)x &= 2\cos 2t - 4\sin 2t \\
(2D^2 - 4D + 4)x &= 2\cos 2t - 4\sin 2t \\
(D^2 - 2D + 2)x &= \cos 2t - 2\sin 2t.
\end{aligned}$$

$$\begin{aligned}
\text{A.E is, } m^2 - 2m + 2 &= 0 \\
(m-1)^2 + 2 - 1 &= 0 \\
(m-1)^2 &= -1 \\
m - 1 &= \pm i \quad \Rightarrow m = 1 \pm i.
\end{aligned}$$

$$\begin{aligned}
x_c &= e^t(c_1\cos t + c_2\sin t). \\
x_p &= \frac{1}{D^2 - 2D + 2}(\cos 2t - 2\sin 2t) \\
&= \frac{1}{D^2 - 2D + 2}\cos 2t - 2\frac{1}{D^2 - 2D + 2}\sin 2t \\
&= \frac{1}{-4 - 2D + 2}\cos 2t - 2\frac{1}{-4 - 2D + 2}\sin 2t \\
&= \frac{1}{-2 - 2D}\cos 2t - 2\frac{1}{-2 - 2D}\sin 2t \\
&= \frac{-1}{2}\frac{1}{D+1}\cos 2t + \frac{1}{D+1}\sin 2t \\
&= \frac{-1}{2}\frac{(D-1)}{D^2 - 1}\cos 2t + \frac{(D-1)}{D^2 - 1}\sin 2t \\
&= \frac{-1}{2}\left(\frac{-2\sin 2t - \cos 2t}{-5}\right) + \frac{2\cos 2t - \sin 2t}{-5} \\
&= \left(\frac{2\sin 2t + \cos 2t}{-10}\right) + \frac{2\cos 2t - \sin 2t}{-5} \\
&= \frac{-1}{10}(2\sin 2t + \cos 2t + 4\cos 2t - 2\sin 2t) = \frac{-1}{10}(5\cos 2t). \\
x_p &= \frac{-1}{2}(\cos 2t)
\end{aligned}$$

The solution is $x = x_c + x_p = e^t(c_1\cos t + c_2\sin t) - \frac{1}{2}(\cos 2t)$.

$$(1) + (2) \Rightarrow 2\frac{dx}{dt} + 2y - 2x = \cos 2t + \sin 2t$$

$$2y = 2x - 2\frac{dx}{dt} + \cos 2t + \sin 2t.$$

$$= 2\left[e^t(c_1 \cos t + c_2 \sin t) - \frac{\cos 2t}{2}\right]$$

$$- 2\left[e^t(-c_1 \sin t + c_2 \cos t) + e^t(c_1 \cos t + c_2 \sin t) - \frac{1}{2}\left(\frac{-\sin 2t}{2}\right)\right] + \cos 2t + \sin 2t$$

$$= 2e^t((c_1 \sin t - c_2 \cos t) + \frac{\sin 2t}{2}$$

$$y = e^t((c_1 \sin t - c_2 \cos t) + \frac{\sin 2t}{4}.$$