Numerical Analysis (MATH-411)
Projects #3 and #4: Due 12/1 and 12/8

1 Project 3: The Lane-Emden Equation

1.1 Background

In astrophysics, the standard way to model approximate stellar structures is via the **Lane-Emden equation**. While it only represents an approximation to real stellar structure, about which many books have been written, it does capture nicely virtually all of the physical effects with which one might be concerned. Here is how it is derived:

For a spherically symmetric object in hydrostatic equilibrium, we must have force balance, i.e., pressure gradients in the star pushing outward must balance the inwardly directed force of gravity:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \tag{1}$$

where $M(r)$ is the *enclosed* mass as a function of radius, i.e., $M(0) = 0$ by definition and $M(R_*) = M_*$, where M_8 and R_* are the mass and radius of the star, respectively. Solving for $M(r)$ in the equation above, we find

$$\begin{aligned} M &= -\frac{r^2}{G\rho(r)}\frac{dP}{dr} \\ \frac{dM}{dr} &= -\frac{1}{G}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP}{dr}\right) \end{aligned} \tag{2}$$

We can also relate the enclosed mass to the density as a function of radius:

$$\frac{dM}{dr} = 4\pi r^2 \rho(r) \tag{3}$$

and combine the two expressions for dM/dr to find

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \tag{4}$$

For *polytropic* equations of state where the pressure can be written as a power-law dependent on the density alone, i.e., those where $P = \kappa\rho^\gamma$, with κ and γ constant, we can simplify further. First, define

$$\theta^n = \frac{\rho(r)}{\rho(0)} = \frac{\rho}{\rho_c}; \quad n = \frac{1}{\gamma - 1} \leftrightarrow \gamma = 1 + \frac{1}{n} \tag{5}$$

where $\rho_c = \rho(0)$ is the central density of the star. It follows that the pressure can be rewritten as

$$\frac{P}{P(0)} \equiv \frac{P}{P_c} = \theta^{n+1} \tag{6}$$

By defining

$$\xi \equiv \frac{r}{\alpha}; \quad \alpha = \left(\frac{n+1}{4\pi G}\kappa\rho_c^{(1-n)/n}\right)^{1/2} \tag{7}$$

we can rewrite Eq. 4 as

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{8}$$

a second-order ODE for the variable $\theta(\xi)$, which must satisfy boundary conditions

$$\theta(0) = 1; \quad \left(\frac{d\theta}{d\xi}\right)_{\xi=0} = 0 \tag{9}$$

the latter a consequence of the fact that $M(0) = 0$ by definition, so there is no radial density/pressure gradient at the origin. These conditions define the **Lane-Emden** differential equation.

a.) Finite solutions of the Lane-Emden equations exist for values $0 \leq n \leq 5$. Determine the solutions for $n = 0.5, 1, 2, 3$. Your integration should stop when you reach the surface of the star Ξ, defined by the condition $\theta = 0$. *It may help to note that for very small values of ξ, the solution satisfies the relation $\theta(\xi) \approx 1 - \xi^2/6$, since if you try to integrate at $\xi = 0$ the equation becomes singular and can crash your code!*

To confirm the proper behavior, please plot the dimensionless density $\theta^n(\xi)$ vs. ξ and the dimensionless temperature $\theta(\xi)$ vs. ξ.

b.) For each of the four cases above, compute the dimensionless mass $\hat{M}$, the dimensionless potential energy $\hat{\Omega}$ (in units of $-GM^2/R$), and the dimensionless moment of inertia $\hat{I}$ (in units of MR^2). For each of the four models, please list the values $\hat{M}$, Ξ, $-\left(\frac{d\theta}{d\xi}\right)_{\xi=\Xi}$, $\hat{\Omega}$, and $\hat{I}$.

Note that dimensionless quantities are defined as follows:

$$
\begin{aligned}
\hat{M} &\equiv \int 4\pi\theta^n\xi^2 d\xi \\
I &= \int \frac{2}{3} r^2 dm = \int \frac{2}{3} r^2 \frac{dm}{dr} dr \\
&= \int \frac{8\pi}{3} \theta^n \xi^4 d\xi \\
\hat{I} &\equiv \frac{I}{\hat{M}\Xi^2} \\
\Omega &= -\int \frac{Gm}{r} dm = -\int \frac{Gm}{r}\frac{dm}{dr} dr \\
&= \int 4\pi\hat{M}\theta^n\xi d\xi \\
\hat{\Omega} &= \frac{\Omega}{\hat{M}^2/\Xi} = \frac{\Omega\Xi}{\hat{M}^2}
\end{aligned}
$$

Furthermore, note that if $F \equiv \int f(\xi)d\xi$, it implies that $\frac{dF}{d\xi} = f$.

You should find that

$$
\hat{M} = K\Xi^2 \left(\frac{d\theta}{d\xi}\right)_{\xi=\Xi}
$$

for some constant K. What is this value?

c.) Use an $n = 3$ model to represent the sun. Fix the parameters to physical units by setting $M = M_\odot$ and $R = R_\odot$. Compute the central pressure using the polytropic model and compare with the standard textbook value, 2×10^{17} in cgs units.

1.2 White Dwarfs

For a white dwarf, we may begin from Eq. 4 but modify the equation of state such that it takes the form

$$
P = K_n \rho_0^{5/3} \theta^{5/3} (1 + \theta^{2/3})^{-1/2} \tag{10}
$$

where $\rho = \rho_0\theta$, $\rho_0 = 3.789 \times 10^6$ g/cm^3, and $K_n = 3.166 \times 10^{12}$ in cgs units. We can define a dimensionless distance $s = r/a$, where

$$
a = \left(\frac{K_n \rho_0^{-1/3}}{4\pi G}\right)^{1/2} = 1.557 \times 10^8 \text{ cm} \tag{11}
$$

We begin from the equation of pressure balance:

$$
\begin{aligned}
\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) &= -4\pi G\rho \\
\frac{1}{4\pi G}\left[\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d}{dr}\left\{K_n\rho_0^{5/3}\theta^{5/3}(1+\theta^{2/3})^{-1/2}\right\}\right)\right] &= -\rho
\end{aligned}
$$

We use $r = as$ and $\rho = \rho_0\theta$ to pull out dimensional constants from inside the differential equation:

$$\begin{aligned}
\frac{1}{4\pi G}\left[\frac{1}{(as)^2}\left[\frac{1}{a}\frac{d}{ds}\left(\frac{(as)^2}{\rho_0\theta}\left[\frac{1}{a}\frac{d}{ds}\left\{K_n\rho_0^{5/3}\theta^{5/3}(1+\theta^{2/3})^{-1/2}\right\}\right]\right)\right]\right] &= -\rho_0\theta \\
\frac{K_n\rho_0^{-1/3}}{4\pi G a^2}\left[\frac{1}{s^2}\frac{d}{ds}\left(\frac{s^2}{\theta}\frac{d}{ds}\left\{\theta^{5/3}(1+\theta^{2/3})^{-1/2}\right\}\right)\right] &= -\theta
\end{aligned}$$

But we note that $a^2 = \frac{K_n\rho_0^{-1/3}}{4\pi G}$, so the constant in front of the equation is just 1. Thus, we must solve

$$\left[\frac{1}{s^2}\frac{d}{ds}\left(\frac{s^2}{\theta}\frac{d}{ds}\left\{\theta^{5/3}(1+\theta^{2/3})^{-1/2}\right\}\right)\right] = -\theta$$

If we define

$$F(\theta) \equiv \theta^{5/3}(1+\theta^{2/3})^{-1/2}$$

you can easily show that

$$\begin{aligned}
\frac{d}{d\theta}F(\theta) = F'(\theta) &= \frac{\theta^{2/3}(4\theta^{2/3}+5)}{3(\theta^{2/3}+1)^{3/2}} \\
\frac{d}{ds}F(\theta) &= F'(\theta)\frac{d\theta}{ds}
\end{aligned}$$

Define thr proper form for f in the expression $V \equiv \frac{d\theta}{ds}$, and our equation becomes

$$\begin{aligned}
\left[\frac{1}{s^2}\frac{d}{ds}\left(\frac{s^2}{\theta}\frac{d}{ds}F(\theta)\right)\right] &= -\theta \\
\left[\frac{1}{s^2}\frac{d}{ds}\left(\frac{s^2}{\theta}F'(\theta)V\right)\right] &= -\theta \\
\left[\frac{1}{s^2}\frac{d}{ds}\left(s^2V\frac{F'(\theta)}{\theta}\right)\right] &= -\theta
\end{aligned}$$

We define

$$\begin{aligned}
G(\theta) \equiv \frac{F'(\theta)}{\theta} &= \frac{\theta^{-1/3}(4\theta^{2/3}+5)}{3(\theta^{2/3}+1)^{3/2}} \\
G'(\theta) &= -\frac{8+16\theta^{-2/3}+5\theta^{-4/3}}{9(1+\theta^{2/3})^{5/2}}
\end{aligned}$$

Simplifying the differential equation a final time, we find

$$\left[\frac{1}{s^2}\frac{d}{ds}\left(s^2VG(\theta)\right)\right] = -\theta$$

This is the analogue of the Lane-Emden equation for the white dwarf problem.

a.) Determine $\frac{dV}{ds} = f(V,\theta,s)$. Feel free to bounce this expression off of me whenever you would like, and I will confirm that you are correct. Implement the pair of first-order equations describing white dwarfs into a code.

b.) Pick central values of $\theta(0)$ from 0.01 up to 10^5, one per decade. Your boundary conditions will be $\theta(0) =$ that value, and $\left(\frac{d\theta}{ds}\right)_{s=0} = 0$. Integrate the coupled equations, and plot $\theta(s)$ in each case. You may terminate the calculation, when $\theta/\theta_c = 0.001$.

c.) Calculate the dimensionless mass

$$\hat{M} = \int 4\pi\theta s^2 ds \tag{12}$$

By dimensional analysis, the mass of the star is $M = \rho_0 a^3\hat{M} = (0.09/4\pi)\hat{M}M_\odot$ and the radius is $R = s_{max}a$. Plot the mass-radius relation for white dwarfs on a log-log scale, indicating the units to me somehow.

2 Project 4: Do Cool Stuff Numerically!

Write a code that does cool stuff numerically, using either methods we've learned in class or other techniques. Write up the final result, and submit it to me. That's pretty much it!

You'll need to clear the ideas you have with me ahead of time, to ensure I think they are of appropriate scope for the final project.

Tips for Project #3

Converting second-order equations to systems of first-order equations

There is a standard trick for converting a second-order differential equation into a pair of first-order equations. If you have

$$\theta'' = f(\xi,\ \theta,\ \theta')$$

then we define $v \equiv \theta'$, and we find

$$\begin{aligned}\theta'' \equiv v' &= f(\xi,\ \theta,\ v)\\ \theta' &= v\end{aligned}$$

To evolve a first order system using RK2, RK4, or any other method, you evaluate each equation for a given stage, and then move on to the next one. As an example, here is RK2 explicitly for the system above, starting from $\xi(i), \theta(i),\ v(i)$:

$$\begin{aligned}(\theta')\quad k_1 &= v(i)\\ (v')\quad l_1 &= f(\xi(i),\ \theta(i),\ v(i))\\ k_2 &= v(i) + \frac{dt}{2}\times l_1\\ l_2 &= f\left(\xi(i) + \frac{dt}{2},\ \theta(i) + \frac{dt}{2}\times k_1,\ v(i) + \frac{dt}{2}\times l_1\right)\\ \xi(i+1) &= \xi(i) + dt\\ \theta(i+1) &= \theta(i) + dt\times k_2\\ v(i+1) &= v(i) + dt\times l_2\end{aligned}$$

Initial data

For the Lane-Emden equations, $f(\xi,\theta,v)$ includes a term in the form $-\frac{2v}{\xi}$, which is ill-defined at $\xi = 0$. If you assume for small ξ that $\theta(\xi) \approx 1 - \xi^2/6$, this implies that $v = \theta' \approx -\frac{\xi}{3}$, and the term in question becomes $\frac{2}{3}$, which is not ill-defined, and is approximately equal to $\frac{2}{3}$.

Units

Adding physical units back to a dimensionless calculation can be tricky. Let's see what is involved here. For a given value of n, you will determine a dimensionless value Ξ, the value of ξ where $\theta = 0$, that defines the surface of the star. From our scaling, we know that

$$\xi \equiv \frac{r}{\alpha} \quad\rightarrow\quad \Xi \equiv \frac{R}{\alpha} \quad\rightarrow\quad \alpha = \frac{R}{\Xi}$$

where R is the radius of the star in physical units. Given a value of R and n, we can determine a value of α in physical units (distance in this case).

Next, the mass of the star is given by

$$\begin{aligned}M &\equiv \int_0^R 4\pi\rho r^2\ dr = \alpha^3\int_0^\Xi 4\pi\rho\xi^2\ d\xi = \rho_c\alpha^3\int_0^\Xi 4\pi\theta^n\xi^2\ d\xi\\ &= \rho_c\alpha^3\hat{M}\end{aligned}$$

Using the dimensionless value of $\hat{M}$ that you calculate from the ODE, the value of α we just determined, and the known mass of the star in physical units, you may determine the proper physical value of ρ_c.

Once you know α and ρ_c, you can solve for κ in the equation that originally defined α. Once you know κ, you can calculate $P_c = \kappa\rho_c^{\left(1+\frac{1}{n}\right)}$, and part c.) is complete.

The FTC!

According to the fundamental theorem of calculus, the following two equations are equivalent:

$$\begin{aligned} \hat{M} &= \int 4\pi\theta^n\xi^2 \; d\xi \\ \frac{d\hat{M}}{d\xi} &= 4\pi\theta^n\xi^2 \end{aligned}$$

Starting the white dwarf solution

The white dwarf structure equations have the same kind of singularity at $s = 0$ that we find for the Lane-Emden equations. There are techniques that can be used to generate the Taylor series for the solution $\theta(s)$ for small values of s, but for our purposes, it is simplest to just assume that if we define $V = \frac{d\theta}{ds}$, that the boundary condition is given by $\frac{dV}{ds}(s = 0) = 0$. So long as the stepsize for the ODE solver is small, RK4 will start getting non-zero values halfway through the first full step, and the results will be extremely accurate.