

Numerical Analysis Homework 2

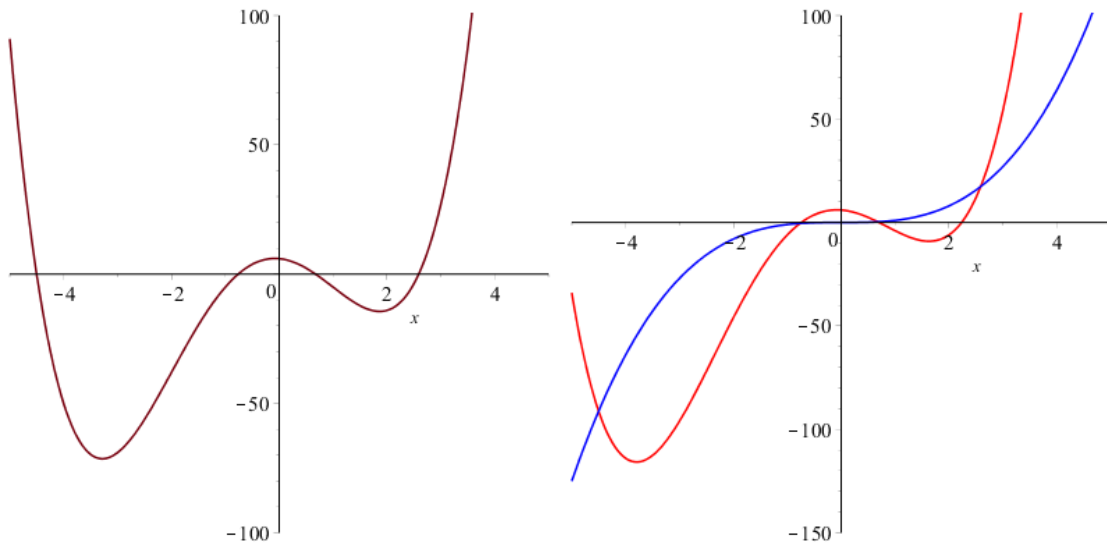
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Problem 1

No, a degree 3 polynomial cannot intersect a degree 4 polynomial in exactly 5 points - let $f(x)$ be a degree 3 polynomial, and $g(x)$ be degree 4. Any intersection of f and g must be a root of $f - g$ and $g - f$, which are degree 4 polynomials, and therefore may have at most 4 real roots.

It is, however, possible for f and g to intersect at exactly four points. Consider $f(x) = x^3$, and $g(x) = x^4 + 3x^3 - 12x^2 - 2x + 6$. Their difference, $x^4 + 2x^3 - 12x^2 - 2x + 6$, is plotted below and clearly has exactly 4 distinct real roots. f and g can intersect only at those roots, and must intersect at them, so thus f and g intersect exactly 4 times, as shown on the second plot.



Problem 2

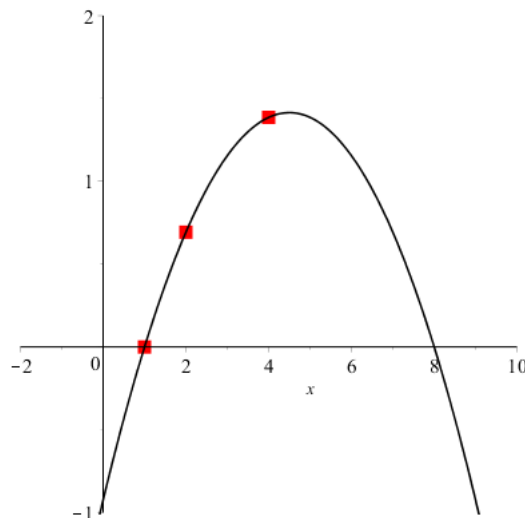
Using the formula

$$P_n(x) = \sum_{j=1}^n y_j \prod_{k=1, k \neq j}^n \frac{x - x_k}{x_j - x_k}$$

we obtain

$$\begin{aligned}
 P_2(x) &= \frac{(x-2)(x-4)}{3} \cdot 0 + \frac{(x-1)(x-4)}{-2} \cdot \ln 2 + \frac{(x-1)(x-2)}{6} \cdot \ln 4 \\
 P_2(x) &= \frac{-(x-1)(x-4)}{2} \cdot \ln 2 + \frac{(x-1)(x-2)}{3} \cdot \ln 2 \\
 P_2(x) &= \frac{-\ln 2}{6}x^2 + \frac{3\ln 2}{2}x - \frac{4\ln 2}{3}
 \end{aligned}$$

The graph of this polynomial fit is shown below.



Using this approximation, we obtain $\ln 3 \approx P_2(3) = \frac{-9\ln 2}{6} + \frac{9\ln 2}{2} - \frac{4\ln 2}{3} = \frac{5}{3} \ln 2 \approx 1.1552453$. Using the formula

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| \cdot \max_{[2,4]} \left| \prod_{i=0}^n (x - x_i) \right|$$

with $f(x) = \ln x$, $f^{(3)}(x) = \frac{2}{x^3}$, we obtain

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{1}{3(\xi)^3} \right| \cdot \max_{[2,4]} |x^3 - 7x^2 + 14x - 8|$$

Because $f^{(3)}(\xi)$ is strictly decreasing and non-negative on our interval, we know the maximum on $[2, 4]$ is $\frac{1}{24}$ at 2. Using the derivative of $x^3 - 7x^2 + 14x - 8$ with the quadratic formula, we find that the maximum absolute value on $[2, 4]$ is $-\frac{2}{27}(10 + 7\sqrt{7})$ at $x = \frac{7}{3} + \frac{\sqrt{7}}{3}$, or approximately 2.11261.

Multiplying these values, we obtain $|f(x) - P(x)| \leq .0880255$.

Comparing our value of 1.1552453 to the actual value of $\ln 3 \approx 1.0986123$, we have $|f(x) - P_2(x)| = .056630 < .0880255$, as expected.

Problem 3

Using the formulas $B(0) = P_0$ and $B(1) = P_3$, we obtain $P_0 = (1, 1)$ and $P_3 = (9, 1)$. We then calculate $B'(t) = (x'(t), y'(t)) = (12t + 6t^2, 3t^2 - 1)$, and note that because in general $B'(t) = 3(1-t)^2(P_1 - P_0) + 6(1-t)t(P_2 - P_1) + 3t^2(P_3 - P_2)$, $B'(0) = 3(P_1 - P_0)$ and $B'(1) = 3(P_3 - P_2)$. Thus we have $P_1 = (1, \frac{2}{3})$ and $P_2 = (3, \frac{1}{3})$. This gives us the complete set of control points for the curve, namely

$$\{(1, 1), (1, \frac{2}{3}), (3, \frac{1}{3}), (9, 1)\}$$

Problem 4

We first note that

$$\begin{aligned} A_1 &= \frac{x_2 - x_1}{x_2 - x_1} = 1 - x \\ A_2 &= \frac{x_3 - x_1}{x_3 - x_2} = 2 - x \\ A_3 &= \frac{x_4 - x_1}{x_4 - x_3} = 3 - x \\ B_1 &= \frac{x - x_1}{x_2 - x_1} = x - 0 \\ B_2 &= \frac{x - x_2}{x_3 - x_2} = x - 1 \\ B_3 &= \frac{x - x_3}{x_4 - x_3} = x - 2 \\ C_1 &= \frac{(x_2 - x_1)^2}{6} [A_1^3 - A_1] = \frac{1}{6}(-x^3 + 3x^2 - 2x) \\ C_2 &= \frac{(x_3 - x_2)^2}{6} [A_2^3 - A_2] = \frac{1}{6}(-x^3 + 6x^2 - 11x + 6) \\ C_3 &= \frac{(x_4 - x_3)^2}{6} [A_3^3 - A_3] = \frac{1}{6}(-x^3 + 9x^2 - 26x + 24) \\ D_1 &= \frac{(x_2 - x_1)^2}{6} [B_1^3 - B_1] = \frac{1}{6}(x^3 - x) \\ D_2 &= \frac{(x_3 - x_2)^2}{6} [B_2^3 - B_2] = \frac{1}{6}(x^3 - 3x^2 + 2x) \\ D_3 &= \frac{(x_4 - x_3)^2}{6} [B_3^3 - B_3] = \frac{1}{6}(x^3 - 6x^2 + 11x - 6) \end{aligned}$$

Natural Endpoint Condition

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 - 0 & 2(2 - 0) & 2 - 1 & 0 \\ 0 & 2 - 1 & 2(3 - 1) & 3 - 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{2-5}{2-1} - \frac{5-3}{1-0} \right) \\ 6 \left(\frac{1-2}{3-2} - \frac{2-5}{2-1} \right) \\ 0 \end{bmatrix}$$

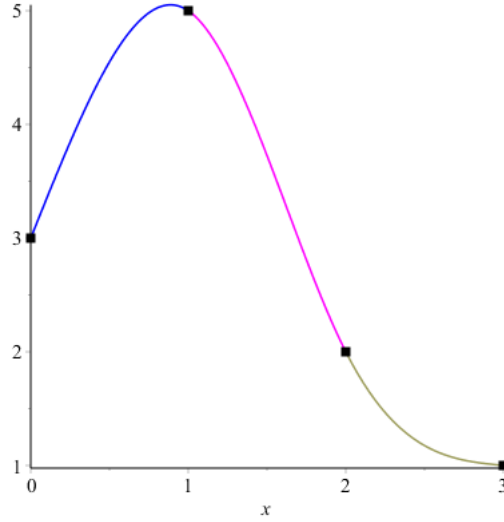
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{4}{15} & \frac{4}{15} & -\frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & -\frac{1}{15} & \frac{4}{15} & -\frac{4}{15} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{44}{5} \\ \frac{26}{5} \\ 0 \end{bmatrix}$$

Using these derived values of y_i'' with our A, B, C , and D equations above, we have

$$\begin{aligned} y_1^{(cubic)} &= y_1 A_1 + y_2 B_1 + y_1'' C_1 + y_2'' D_1 = -\frac{22}{15}x^3 + \frac{52}{15}x + 3 \\ y_2^{(cubic)} &= y_2 A_2 + y_3 B_2 + y_2'' C_2 + y_3'' D_2 = \frac{7}{3}x^3 - \frac{57}{5}x^2 + \frac{223}{15}x - \frac{4}{5} \\ y_3^{(cubic)} &= y_3 A_3 + y_4 B_3 + y_3'' C_3 + y_4'' D_3 = -\frac{13}{15}x^3 + \frac{39}{5}x^2 - \frac{353}{15}x + \frac{124}{5} \end{aligned}$$



Curvature-Adjusted Endpoint Condition

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} k_1 \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ k_n \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-0 & 2(2-0) & 2-1 & 0 \\ 0 & 2-1 & 2(3-1) & 3-2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} k_n \\ 6 \left(\frac{2-5}{2-1} - \frac{5-3}{1-0} \right) \\ 6 \left(\frac{1-2}{3-2} - \frac{2-5}{2-1} \right) \\ k_n \end{bmatrix}$$

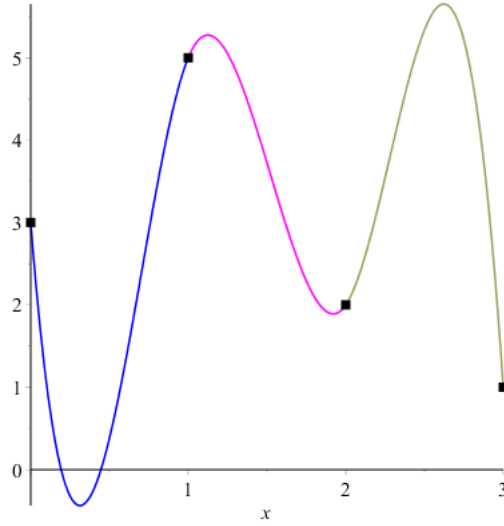
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} k_1 \\ -30 \\ 12 \\ k_n \end{bmatrix}$$

Choosing $k_1 = 100$, $k_2 = -100$, we have

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{4}{15} & \frac{4}{15} & -\frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & -\frac{1}{15} & \frac{4}{15} & -\frac{4}{15} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ -30 \\ 12 \\ -100 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 100 \\ -\frac{632}{15} \\ \frac{578}{15} \\ -100 \end{bmatrix}$$

$$\begin{aligned} y_1^{(cubic)} &= -\frac{1066}{45} + 50x^2 - \frac{1094}{45}x + 3 \\ y_2^{(cubic)} &= \frac{121}{9}x^3 - \frac{307}{5}x^2 + \frac{3919}{45}x - \frac{512}{15} \\ y_3^{(cubic)} &= -\frac{1039}{45} + \frac{789}{5}x^2 - \frac{15809}{45}x + \frac{3872}{15} \end{aligned}$$



Clamped Endpoint Condition

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 6 \left(\frac{y_2 - y_1}{x_2 - x_1} - c_n \right) \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ 6 \left(c_n - \frac{y_4 - y_3}{x_4 - x_3} \right) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 - 0 & 2(2 - 0) & 2 - 1 & 0 \\ 0 & 2 - 1 & 2(3 - 1) & 3 - 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 6 \left(\frac{5-3}{1-0} - c_n \right) \\ 6 \left(\frac{2-5}{2-1} - \frac{5-3}{1-0} \right) \\ 6 \left(\frac{1-2}{3-2} - \frac{2-5}{2-1} \right) \\ 6 \left(c_n - \frac{1-2}{3-2} \right) \end{bmatrix}$$

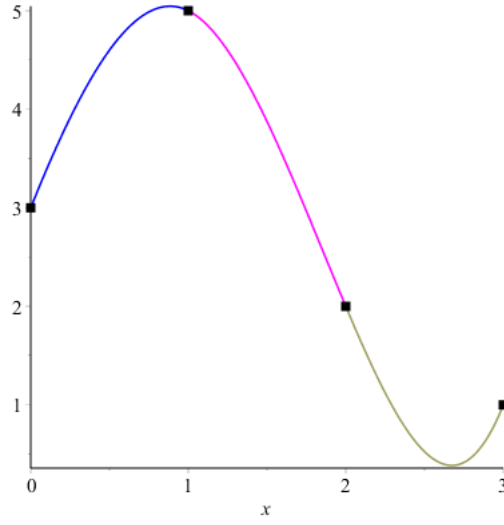
$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 12 - 6c_n \\ -30 \\ 12 \\ 6(c_n + 1) \end{bmatrix}$$

Choosing $c_n = 4$, we have

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \frac{1}{45} \begin{bmatrix} 26 & -7 & 2 & -1 \\ -7 & 14 & -4 & 2 \\ 2 & -4 & 14 & -7 \\ -1 & 2 & -7 & 26 \end{bmatrix} \begin{bmatrix} -12 \\ -30 \\ 12 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} \\ -\frac{36}{5} \\ \frac{6}{5} \\ \frac{72}{5} \end{bmatrix}$$

$$\begin{aligned} y_1^{(cubic)} &= -\frac{4}{5}x^3 - \frac{6}{5}x^2 + 4x + 3 \\ y_2^{(cubic)} &= \frac{7}{5}x^3 - \frac{39}{5}x^2 + \frac{53}{5}x + \frac{4}{5} \\ y_3^{(cubic)} &= -\frac{11}{5}x^3 - \frac{63}{5}x^2 + \frac{101}{5}x - \frac{28}{5} \end{aligned}$$



Parabolically Terminated Endpoint Condition

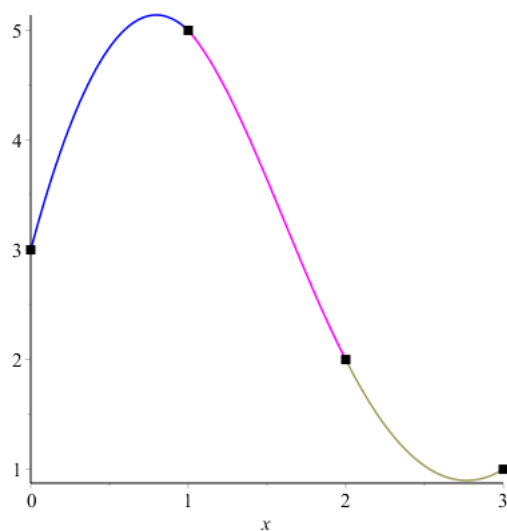
$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 19 & 5 & -1 & 1 \\ -5 & 5 & -1 & 1 \\ 1 & -1 & 5 & -5 \\ 1 & -1 & 5 & 19 \end{bmatrix} \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} -\frac{27}{4} \\ -\frac{27}{4} \\ \frac{15}{4} \\ \frac{15}{4} \end{bmatrix}$$

$$\begin{aligned} y_1^{(cubic)} &= -\frac{27}{8}x^2 + \frac{43}{8}x + 3 \\ y_2^{(cubic)} &= \frac{13}{8}x^3 - \frac{63}{8}x^2 + \frac{37}{4}x + 2 \\ y_3^{(cubic)} &= \frac{15}{8}x^2 - \frac{83}{8}x - \frac{61}{4} \end{aligned}$$



Not-a-Knot Endpoint Condition

$$\begin{bmatrix} x_3 - x_2 & -(x_3 - x_1) & x_2 - x_1 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & x_4 - x_3 & -(x_4 - x_2) & x_3 - x_2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ 0 \end{bmatrix}$$

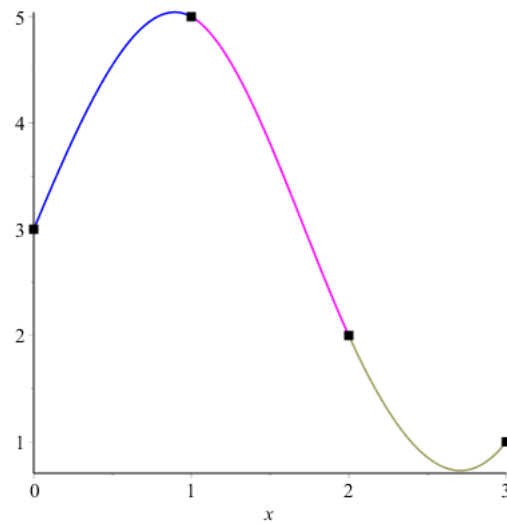
$$\begin{bmatrix} -3 & 1 & 2 & 0 \\ 1 - 0 & 2(2 - 0) & 2 - 1 & 0 \\ 0 & 2 - 1 & 2(3 - 1) & 3 - 2 \\ 0 & -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{2-5}{2-1} - \frac{5-3}{1-0} \right) \\ 6 \left(\frac{1-2}{3-2} - \frac{2-5}{2-1} \right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 2 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \frac{1}{198} \begin{bmatrix} -62 & 12 & 21 & 7 \\ 16 & 48 & -15 & -5 \\ -2 & -6 & 39 & 13 \\ -8 & -24 & 57 & -47 \end{bmatrix} \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} \\ -\frac{90}{11} \\ \frac{36}{11} \\ \frac{78}{11} \end{bmatrix}$$

$$\begin{aligned}
y_1^{(cubic)} &= -\frac{14}{11}x^3 - \frac{3}{11}x^2 + \frac{39}{11}x + 3 \\
y_2^{(cubic)} &= \frac{21}{11}x^3 - \frac{108}{11}x^2 + \frac{144}{11}x - \frac{2}{11} \\
y_3^{(cubic)} &= \frac{7}{11}x^3 - \frac{24}{11}x^2 - \frac{24}{11}x + 10
\end{aligned}$$



Problem 5

Raw Output

5 DATA POINTS

Initial Points:

(-5.000000,0.038462) (-2.500000,0.137931) (0.000000,1.000000)
(2.500000,0.137931) (5.000000,0.038462)

x: -3.750000
lagrange: -0.162550
cubic: -0.039693
curveadj: -0.042043
actual: 0.066390

x: -1.250000
lagrange: 0.629062
cubic: 0.666659
curveadj: 0.667129

actual: 0.390244

x: 1.250000

lagrange: 0.939904

cubic: 0.666659

curveadj: 0.667129

actual: 0.390244

x: 3.750000

lagrange: -1.716761

cubic: -0.039693

curveadj: -0.042043

actual: 0.066390

10 DATA POINTS

Initial Points:

(-5.000000,0.038462) (-3.888889,0.062021) (-2.777778,0.114731)
(-1.666667,0.264706) (-0.555556,0.764151) (0.555556,0.764151)
(1.666667,0.264706) (2.777778,0.114731) (3.888889,0.062021)
(5.000000,0.038462)

x: -4.444444

lagrange: -0.209264

cubic: 0.046749

curveadj: 0.046273

actual: 0.048186

x: -3.333333

lagrange: 0.160420

cubic: 0.087924

curveadj: 0.088051

actual: 0.082569

x: -2.222222

lagrange: 0.115474

cubic: 0.147614

curveadj: 0.147580

actual: 0.168399

x:	-1.111111
lagrange:	0.523247
cubic:	0.515771
curveadj:	0.515781
actual:	0.447514

x:	0.000000
lagrange:	0.861538
cubic:	0.857126
curveadj:	0.857121
actual:	1.000000

x:	1.111111
lagrange:	0.523247
cubic:	0.515771
curveadj:	0.515781
actual:	0.447514

x:	2.222222
lagrange:	0.115474
cubic:	0.147614
curveadj:	0.147580
actual:	0.168399

x:	3.333333
lagrange:	0.160420
cubic:	0.087924
curveadj:	0.088051
actual:	0.082569

x:	4.444444
lagrange:	-0.209264
cubic:	0.046749
curveadj:	0.046273
actual:	0.048186

15 DATA POINTS

Initial Points:

(-5.000000,0.038462) (-4.285714,0.051633) (-3.571429,0.072700)
 (-2.857143,0.109131) (-2.142857,0.178832) (-1.428571,0.328859)
 (-0.714286,0.662162) (0.000000,1.000000) (0.714286,0.662162)
 (1.428571,0.328859) (2.142857,0.178832) (2.857143,0.109131)
 (3.571429,0.072700) (4.285714,0.051633) (5.000000,0.038462)

x: -4.642857
 lagrange: 2.668349
 cubic: 0.044513
 curveadj: 0.044317
 actual: 0.044334

x: -3.928571
 lagrange: -0.301858
 cubic: 0.060808
 curveadj: 0.060861
 actual: 0.060851

x: -3.214286
 lagrange: 0.180694
 cubic: 0.088163
 curveadj: 0.088149
 actual: 0.088249

x: -2.500000
 lagrange: 0.101181
 cubic: 0.138115
 curveadj: 0.138119
 actual: 0.137931

x: -1.785714
 lagrange: 0.259506
 cubic: 0.237467
 curveadj: 0.237466
 actual: 0.238733

x: -1.071429
 lagrange: 0.451153
 cubic: 0.468041
 curveadj: 0.468041
 actual: 0.465558

x:	-0.357143
lagrange:	0.891166
cubic:	0.886911
curveadj:	0.886911
actual:	0.886878
x:	0.357143
lagrange:	0.904422
cubic:	0.886911
curveadj:	0.886911
actual:	0.886878
x:	1.071429
lagrange:	0.433077
cubic:	0.468041
curveadj:	0.468041
actual:	0.465558
x:	1.785714
lagrange:	0.293649
cubic:	0.237467
curveadj:	0.237466
actual:	0.238733
x:	2.500000
lagrange:	0.008506
cubic:	0.138115
curveadj:	0.138119
actual:	0.137931
x:	3.214286
lagrange:	0.569927
cubic:	0.088163
curveadj:	0.088149
actual:	0.088249
x:	3.928571
lagrange:	-3.285979
cubic:	0.060808
curveadj:	0.060861
actual:	0.060851
x:	4.642857

lagrange:	77.271383
cubic:	0.044513
curveadj:	0.044317
actual:	0.044334

Chebyshev Approximation 5 Nodes

x:	-3.847104
chebyshev:	-0.134899
actual:	0.063290

x:	-1.469463
chebyshev:	0.714472
actual:	0.316524

x:	1.469463
chebyshev:	0.714472
actual:	0.316524

x:	3.847104
chebyshev:	-0.134899
actual:	0.063290

Chebyshev Approximation 10 Nodes

x:	-4.696737
chebyshev:	0.032438
actual:	0.043366

x:	-3.995283
chebyshev:	0.074601
actual:	0.058954

x:	-2.902743
chebyshev:	0.077647
actual:	0.106090

x:	-1.526062
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chebyshev:	0.381200
actual:	0.300403
x:	0.000000
chebyshev:	0.730822
actual:	1.000000
x:	1.526062
chebyshev:	0.381200
actual:	0.300403
x:	2.902743
chebyshev:	0.077647
actual:	0.106090
x:	3.995283
chebyshev:	0.074601
actual:	0.058954
x:	4.696737
chebyshev:	0.032438
actual:	0.043366

Chebyshev Approximation 15 Nodes

x:	-4.696737
chebyshev:	-1.346627
actual:	0.043366
x:	-3.995283
chebyshev:	1.079891
actual:	0.058954
x:	-2.902743
chebyshev:	1.866100
actual:	0.106090
x:	-1.526062
chebyshev:	-4.361026
actual:	0.300403

x:	0.000000
chebyshev:	4.266356
actual:	1.000000
x:	1.526062
chebyshev:	2.565579
actual:	0.300403
x:	2.902743
chebyshev:	-6.869444
actual:	0.106090
x:	3.995283
chebyshev:	5.599214
actual:	0.058954
x:	4.696737
chebyshev:	0.085492
actual:	0.043366
x:	0.000000
chebyshev:	4.266356
actual:	1.000000
x:	0.008421
chebyshev:	4.167831
actual:	0.999929
x:	0.324991
chebyshev:	-0.455012
actual:	0.904471
x:	-0.576289
chebyshev:	4.949485
actual:	0.750690
x:	0.324991
chebyshev:	-0.455012
actual:	0.904471

RMS Errors

n	lagrange	natural	curve adjusted	chebyshev
5	.89780	.04383	.04421	
10	.02010	.00341	.00341	
15	427.32136	.000001	.000001	

Analysis

As n increases, the error in the Lagrange approximation explodes due to Runge's phenomenon, despite the error at the interior points improving at a rate consistent with with the spline approximation - the magnitude of the error near the endpoints more than overcomes the reduction in error everywhere else. The splines behave consistently better overall as n increases due to their piecewise construction which allows them to avoid erratic endpoint behavior.

Fixing the second derivatives of the splines marginally improves accuracy near the endpoints, but makes no marked difference in overall error, at least at these small values of n . This makes sense - because splines are piecewise, fixing the endpoint behavior will only greatly effect the points near the endpoints, and thus any improvement does not propagate well through the rest of the approximation, where the majority of the error data will be determined.

Problem 6

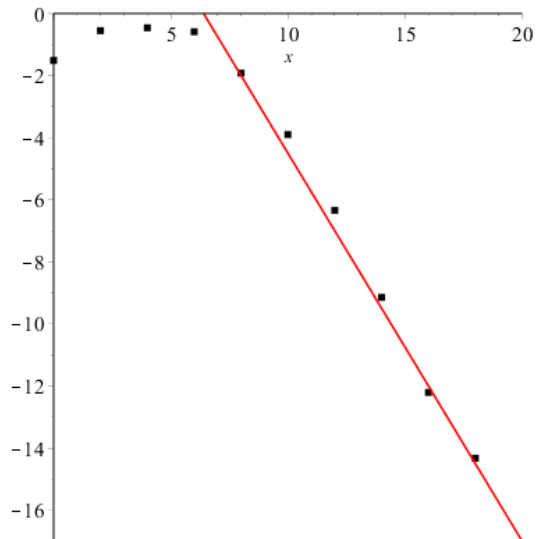
Coefficient Table

Note: The first 10 values of a_n are the coefficients for 10 nodes, so I have not duplicated the data.

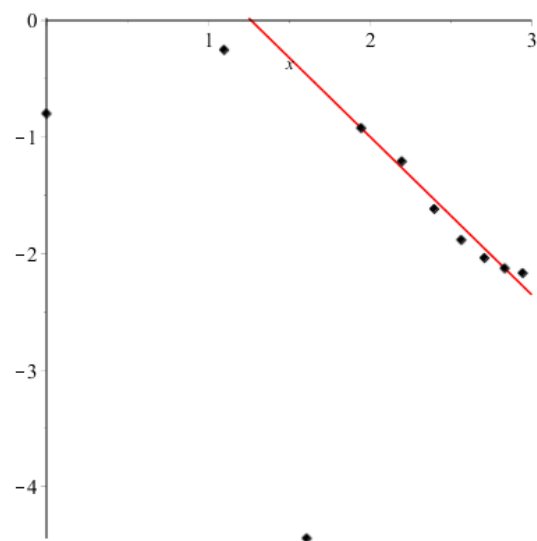
a_n	Function 1	Function 2	Function 3	a_n	Function 1	Function 2	Function 3
a_0	0.220277	0	-0.125724	a_{10}	-0.020277	0	0.135478
a_1	0	0.449218	0	a_{11}	0	-0.198222	0
a_2	0.575761	0	-0.531218	a_{12}	0.001767	0	-0.069607
a_3	0	0.776220	0	a_{13}	0	0.151939	0
a_4	0.631361	0	0.636758	a_{14}	-0.000107	0	0.039138
a_5	0	-0.011739	0	a_{15}	0	-0.130133	0
a_6	-0.555377	0	0.183205	a_{16}	0.000005	0	-0.021989
a_7	0	-0.396909	0	a_{17}	0	0.119113	0
a_8	0.146591	0	-0.253667	a_{18}	0	0	0.010027
a_9	0	0.298239	0	a_{19}	0	-0.114332	0

Plots

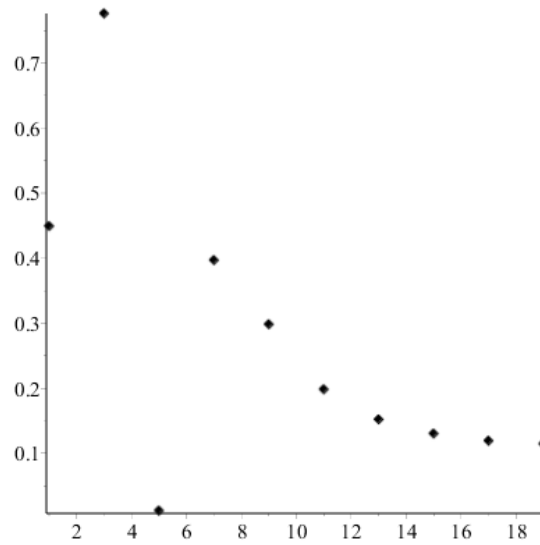
Semi-log plot of the magnitude of Function 1's Chebyshev Coefficients



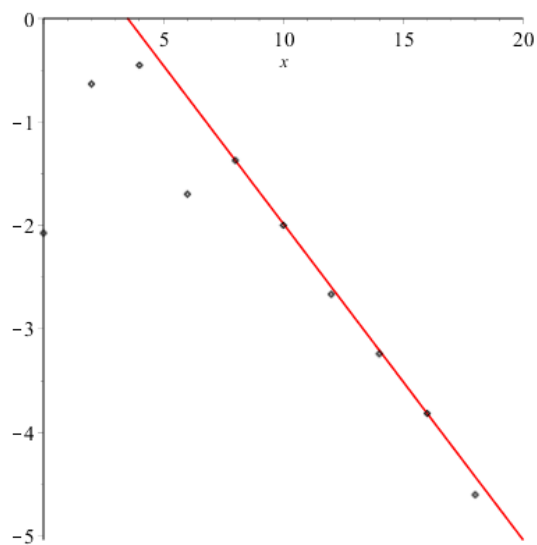
Log-log plot of the magnitude of Function 2's Chebyshev Coefficients



Plot of the magnitude of Function 2's Chebyshev Coefficients



Semi logarithmic plot of the magnitude of Function 3's Chebyshev Coefficients



Analysis

Given the rather obvious linearity of function 1 and 3's semi-log plots, the behavior appears to be exponential, and without even plotting the coefficients, it is clear from the data that the coefficients a_n are asymptotic to 0 as n increases.

Function 2, however, has power-law behavior, as evidenced by the linearity of its log-log plot, and asymptotic to a value around .1, as shown in the plot of the data.

Problem 7

Data Tables

$$T_{10}(x)$$

n	$a_{T'_{10}}$	$a_{T''_{10}}$	$a_{\int T_{10}}$
0	0	500	0
1	20	0	0
2	0	960	0
3	20	0	0
4	0	840	0
5	20	0	0
6	0	640	0
7	20	0	0
8	0	360	0
9	20	0	$\frac{1}{18}$
10	0	0	0
11	0	0	$\frac{1}{22}$

$$T_{15}(x)$$

n	$a_{T'_{15}}$	$a_{T''_{15}}$	$a_{\int T_{15}}$
0	15	0	0
1	0	3360	0
2	30	0	0
3	0	3240	0
4	30	0	0
5	0	3000	0
6	30	0	0
7	0	2640	0
8	30	0	0
9	0	2160	0
10	30	0	0
11	0	1560	0
12	30	0	0
13	0	840	0
14	30	0	$\frac{1}{28}$
15	0	0	0
16	0	0	$\frac{1}{32}$

$$T_{20}(x)$$

n	$a_{T'_{20}}$	$a_{T''_{20}}$	$a_{\int T_{20}}$
0	0	4000	0
1	40	0	0
2	0	7920	0
3	40	0	0
4	0	7680	0
5	40	0	0
6	0	7280	0
7	40	0	0
8	0	6720	0
9	40	0	0
10	0	6000	0
11	40	0	0
12	0	5120	0
13	40	0	0
14	0	4080	0
15	40	0	0
16	0	2880	0
17	40	0	0
18	0	1520	0
19	40	0	$\frac{1}{38}$
20	0	0	0
21	0	0	$\frac{1}{42}$

Analysis

The tables above list the coefficients a_n of T_n for all relevant n for the first two derivatives and integral of each T_n . The patterns in the data can be explained by

$$T_n(x) = \frac{1}{2} \left(\frac{1}{n+1} T'_{n+1}(x) - \frac{1}{n-1} T'_{n-1}(x) \right)$$

which gives us

$$\int T_n(x) = \frac{1}{2} \left(\frac{1}{n+1} T_{n+1}(x) - \frac{1}{n-1} T_{n-1}(x) \right)$$

and

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

which gives

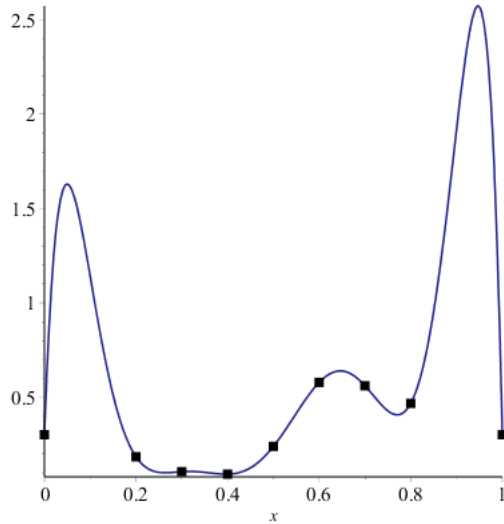
$$T'_n(x) = \frac{1}{1-x^2} (-nxT_n(x) + nT_{n-1}(x))$$

which we can of course take the derivative of to obtain higher order derivatives of T_n .

Problem 8

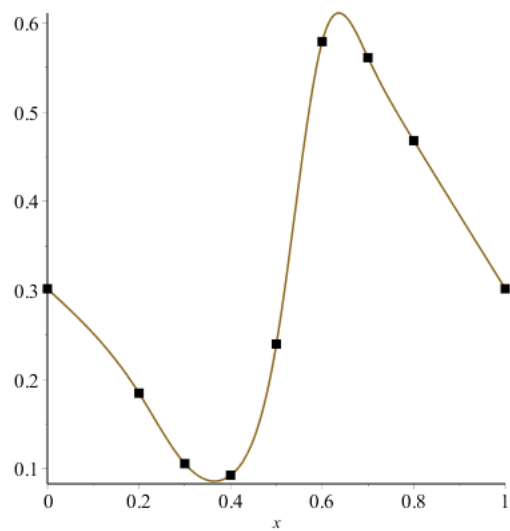
Lagrange Interpolation

$$L_8(x) = -10717.01390x^8 + 41523.43743x^7 - 66130.60764x^6 + 55973.20311x^5 - 27150.02639x^4 + 7547.542702x^3 - 1112.916584x^2 + 66.38116670x + .302$$



Cubic Spline Interpolation

$$f(x) = \begin{cases} .302 - .473x - 2.802x^3 & x < 0.2 \\ .12974 + 2.11093x - 12.91926x^2 + 18.72996x^3 & x < 0.3 \\ .47410379 - 1.3326681x - 1.44057x^2 + 5.975866x^3 & x < 0.4 \\ -2.430901 + 20.45487x - 55.90942x^2 + 51.36657x^3 & x < 0.5 \\ 26.420189 - 152.651673x + 290.3036659x^2 - 179.442153x^3 & x < 0.6 \\ -37.6981566 + 167.940056x - 244.015883x^2 + 117.40204x^3 & x < 0.7 \\ 5.37168 - 16.644978x + 19.67702x^2 - 8.1660098x^3 & x < 0.8 \\ 1.257759 - 1.2177579x + .39299789x^2 - .130999x^3 & x \geq .8 \end{cases}$$



Analysis

The cubic spline interpolation produces a much more convincing interpolation between the data points - the Lagrange polynomial suffers from very erratic behavior at the endpoints (similar to Runge's phenomenon), and indeed $f(.1) = 0.251906420711722$ is a much more consistent value with the data than $L_8(.1) = 1.14113749$.