Numerical Analysis Homework 1

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Problems 1 and 2

Problems 1 and 2 can be found as problem1.c / problem1.exe and problem2.c / problem2.exe.

Problem 3

Raw Output

```
Please enter the dimension of the matrix: 4
Enter your values, row major:
2.000000
3.000000
1.000000
5.000000
1.000000
0.000000
3.000000
1.000000
0.000000
2.000000
-3.000000
2.000000
0.000000
2.000000
3.000000
1.000000

18.000000
-35.000000
-28.000000
1.000000
9.000000
```

```
-18.000000
-14.000000
1.000000
-2.000000
4.000000
3.000000
0.000000
-12.000000
24.000000
19.000000
-1.000000
```

Matrix Form

$$\begin{bmatrix} 2 & 3 & 1 & 5 \\ 1 & 0 & 3 & 1 \\ 0 & 2 & -3 & 2 \\ 0 & 2 & 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 18 & -35 & -28 & 1 \\ 9 & -18 & -14 & 1 \\ -2 & 4 & 3 & 0 \\ -12 & 24 & 19 & -1 \end{bmatrix}$$

Problem 4

a.) $10.5d = 1010.1b = 01000000001001010\ldots0$

$10d = 1010d, .5d * 2d = 1.0d = 1.0b = .1b * 2b.$ So $(10 + .5)d = (1010 + .1)b.$

$1010.1 = 1.0101x2^3 = 1.0101 \times 2^{1026-1023}$

b.) $\frac{1}{3}d = \overline{.01}b = 001111111101010101\ldots01$

$\frac{1}{3} * 2 = \frac{2}{3}, \frac{2}{3} * 2 = 1 + \frac{1}{3} \to \frac{1}{3} * 2 = \ldots = \overline{.01}b$

$\overline{.01} = 1.\overline{01}x2^{-2} = 1.\overline{01} \times 2^{1021-1023}$

c.) $\frac{22}{7}d = 11.\overline{001}b = 010000000000100100100\ldots001$

$\frac{22}{7} = 3 + \frac{1}{7} \to \frac{1}{7} * 2 = \frac{2}{7} \to \frac{4}{7} \to 1 + 17\ldots = (11.\overline{001})b$

$11.\overline{001} = 1.1\overline{001} \times 2 = 1001001\ldots001$

Problem 5

Bisection

Raw Output

```
i: 0 a: 1.000000000 b: 2.000000000 value: -1.000000000
i: 1 a: 1.000000000 b: 1.500000000 value: 1.375000000
i: 2 a: 1.250000000 b: 1.500000000 value: -0.046875000
i: 3 a: 1.250000000 b: 1.375000000 value: 0.599609375
```

```
i: 4 a: 1.250000000 b: 1.312500000 value: 0.260986328
i: 5 a: 1.250000000 b: 1.281250000 value: 0.103302002
i: 6 a: 1.250000000 b: 1.265625000 value: 0.027286530
i: 7 a: 1.257812500 b: 1.265625000 value: -0.010024548
i: 8 a: 1.257812500 b: 1.261718750 value: 0.008573234
i: 9 a: 1.259765625 b: 1.261718750 value: -0.000740074
i: 10 a: 1.259765625 b: 1.260742188 value: 0.003912973
i: 11 a: 1.259765625 b: 1.260253906 value: 0.001585548
i: 12 a: 1.259765625 b: 1.260009766 value: 0.000422512
i: 13 a: 1.259887695 b: 1.260009766 value: -0.000158837
i: 14 a: 1.259887695 b: 1.259948730 value: 0.000131823
i: 15 a: 1.259918213 b: 1.259948730 value: -0.000013510
i: 16 a: 1.259918213 b: 1.259933472 value: 0.000059156
i: 17 a: 1.259918213 b: 1.259925842 value: 0.000022822
i: 18 a: 1.259918213 b: 1.259922028 value: 0.000004656
i: 19 a: 1.259920120 b: 1.259922028 value: -0.000004427
i: 20 a: 1.259920120 b: 1.259921074 value: 0.000000114
i: 21 a: 1.259920597 b: 1.259921074 value: -0.000002156
i: 22 a: 1.259920835 b: 1.259921074 value: -0.000001021
i: 23 a: 1.259920955 b: 1.259921074 value: -0.000000453
i: 24 a: 1.259921014 b: 1.259921074 value: -0.000000169
i: 25 a: 1.259921044 b: 1.259921074 value: -0.000000028
i: 26 a: 1.259921044 b: 1.259921059 value: 0.000000043
i: 27 a: 1.259921044 b: 1.259921052 value: 0.000000008
i: 28 a: 1.259921048 b: 1.259921052 value: -0.000000010
i: 29 a: 1.259921050 b: 1.259921052 value: -0.000000001
```

Analysis

The bisection method took 29 iterations to successfully find the root within an error of 10^{-8}, which agrees with the error formula for bisection method, $n \geq \frac{\log(2)-\log(10^{-8})}{\log 2} \approx 28$. The method converges slowly, but very consistently, as expected.

Secant

Raw Output

```
i: 0 a: 2.000000000 b: 0.500000000 value: -1.875000000
i: 1 a: 0.500000000 b: 0.857142857 value: -1.370262391
i: 2 a: 0.857142857 b: 1.826714801 value: 4.095540811
i: 3 a: 1.826714801 b: 1.100211976 value: -0.668230377
i: 4 a: 1.100211976 b: 1.202120997 value: -0.262821088
i: 5 a: 1.202120997 b: 1.268187169 value: 0.039623768
i: 6 a: 1.268187169 b: 1.259531737 value: -0.001853412
```

```
i: 7 a: 1.259531737 b: 1.259918506 value: -0.000012113
i: 8 a: 1.259918506 b: 1.259921051 value: 0.000000004
i: 9 a: 1.259921051 b: 1.259921050 value: -0.000000000
```

Analysis

The secant method took only 9 iterations to successfully find the root within 8 digits of accuracy, which considering the method is superlinearly convergent when appropriate initial values are chosen, it is not surprising that it was the the second fastest method to converge.

False Position

Raw Output

```
i: 0 a: 0.500000000 b: 2.000000000 value: -1.875000000
i: 1 a: 0.857142857 b: 2.000000000 value: -1.370262391
i: 2 a: 1.069620253 b: 2.000000000 value: -0.776260854
i: 3 a: 1.176200769 b: 2.000000000 value: -0.372787106
i: 4 a: 1.224390317 b: 2.000000000 value: -0.164477725
i: 5 a: 1.245084774 b: 2.000000000 value: -0.069824644
i: 6 a: 1.253768993 b: 2.000000000 value: -0.029154523
i: 7 a: 1.257377460 b: 2.000000000 value: -0.012088652
i: 8 a: 1.258870669 b: 2.000000000 value: -0.004997956
i: 9 a: 1.259487511 b: 2.000000000 value: -0.002063890
i: 10 a: 1.259742146 b: 2.000000000 value: -0.000851855
i: 11 a: 1.259847230 b: 2.000000000 value: -0.000351525
i: 12 a: 1.259890591 b: 2.000000000 value: -0.000145047
i: 13 a: 1.259908483 b: 2.000000000 value: -0.000059848
i: 14 a: 1.259915865 b: 2.000000000 value: -0.000024693
i: 15 a: 1.259918910 b: 2.000000000 value: -0.000010189
i: 16 a: 1.259920167 b: 2.000000000 value: -0.000004204
i: 17 a: 1.259920686 b: 2.000000000 value: -0.000001734
i: 18 a: 1.259920900 b: 2.000000000 value: -0.000000716
i: 19 a: 1.259920988 b: 2.000000000 value: -0.000000295
i: 20 a: 1.259921024 b: 2.000000000 value: -0.000000122
i: 21 a: 1.259921039 b: 2.000000000 value: -0.000000050
i: 22 a: 1.259921046 b: 2.000000000 value: -0.000000021
i: 23 a: 1.259921048 b: 2.000000000 value: -0.000000009
i: 24 a: 1.259921049 b: 2.000000000 value: -0.000000004
i: 25 a: 1.259921050 b: 2.000000000 value: -0.000000001
i: 26 a: 1.259921050 b: 2.000000000 value: -0.000000001
```

Analysis

False position performed almost as poorly as bisection method, although the reason for this can be easily seen in the output - the right endpoint remains at the initial value for every iteration to maintain bracketing, which stunts it significantly compared to its' sibling, the secant method.

FPI

1. $g(x) = \frac{x}{2} + \frac{1}{x^2}$

   Raw Output

```
i: 0 x: 1.000000000 value: 1.500000000
i: 1 x: 1.500000000 value: 1.194444444
i: 2 x: 1.194444444 value: 1.298141638
i: 3 x: 1.298141638 value: 1.242482157
i: 4 x: 1.242482157 value: 1.269009360
i: 5 x: 1.269009360 value: 1.255474294
i: 6 x: 1.255474294 value: 1.262168081
i: 7 x: 1.262168081 value: 1.258803531
i: 8 x: 1.258803531 value: 1.260481298
i: 9 x: 1.260481298 value: 1.259641299
i: 10 x: 1.259641299 value: 1.260061018
i: 11 x: 1.260061018 value: 1.259851089
i: 12 x: 1.259851089 value: 1.259956036
i: 13 x: 1.259956036 value: 1.259903558
i: 14 x: 1.259903558 value: 1.259929796
i: 15 x: 1.259929796 value: 1.259916677
i: 16 x: 1.259916677 value: 1.259923236
i: 17 x: 1.259923236 value: 1.259919957
i: 18 x: 1.259919957 value: 1.259921597
i: 19 x: 1.259921597 value: 1.259920777
i: 20 x: 1.259920777 value: 1.259921187
i: 21 x: 1.259921187 value: 1.259920982
i: 22 x: 1.259920982 value: 1.259921084
i: 23 x: 1.259921084 value: 1.259921033
i: 24 x: 1.259921033 value: 1.259921058
i: 25 x: 1.259921058 value: 1.259921046
i: 26 x: 1.259921046 value: 1.259921052
i: 27 x: 1.259921052 value: 1.259921049
i: 28 x: 1.259921049 value: 1.259921050
```

Analysis

This choice of $g(x)$ did converge, but very slowly. This makes intuitive sense, considering that the derivative of g, $\frac{1}{2} - \frac{2}{x^3}$, is $-.524$ at 1.25, near the root. This is fairly large in absolute value, considering we would like it to be as close to 0 as possible for fast convergence, and thus our rate of convergence is mediocre at best.

2. $g(x) = \frac{2x}{3} + \frac{2}{3x^2}$

   Raw Output

```
i: 0 x: 1.000000000 value: 1.333333333
i: 1 x: 1.333333333 value: 1.263888889
i: 2 x: 1.263888889 value: 1.259933493
i: 3 x: 1.259933493 value: 1.259921050
```

   Analysis

   This was by far the fastest convergence of any of the methods, and by examining the value of $g'(x)$ at 1.25 (near the root), we see why - the value is only -0.016, which is relatively close to 0, which would give us quadratic convergence. Thus it managed to outdo even the superlinearly convergent secant method, although it required some lucky choices on our part.

3. $g(x) = x - .25(x^3 - 2)$

   Raw Output

```
i: 0 x: 1.000000000 value: 1.250000000
i: 1 x: 1.250000000 value: 1.261718750
i: 2 x: 1.261718750 value: 1.259575441
i: 3 x: 1.259575441 value: 1.259986793
i: 4 x: 1.259986793 value: 1.259908518
i: 5 x: 1.259908518 value: 1.259923438
i: 6 x: 1.259923438 value: 1.259920595
i: 7 x: 1.259920595 value: 1.259921137
i: 8 x: 1.259921137 value: 1.259921033
i: 9 x: 1.259921033 value: 1.259921053
i: 10 x: 1.259921053 value: 1.259921049
```

Analysis

This choice of $g(x)$ was superlinear, comparable to the secant method, but clearly not as good as our second choice of g. This is again explained by the absolute value of $g'(x)$ at 1.25 - a respectable but not particularly small 0.1719.

Problem 6

The functions $g_i(x)$ that I constructed are

$$g_1(x) = \frac{-1}{14.5} f(x) + x$$

$$g_2(x) = \frac{1}{8} f(x) + x$$

$$g_3(x) = \frac{1}{16} f(x) + x$$

where constants were obtained by taking the negative reciprocal of the approximate derivative of $f(x) = 2x^3 - 8x - 1$ at the roots.

Root 1: $x \approx -1.9343$

Raw Output - Initial Guess -2

```
--------------------------
 FALSE POSITION METHOD G_1
 -----------------------



i: 0 x: -2.000000000 value: -1.931034483
i: 1 x: -1.931034483 value: -1.934277890
i: 2 x: -1.934277890 value: -1.934297805
i: 3 x: -1.934297805 value: -1.934297876


--------------------------
 FALSE POSITION METHOD G_2
 -----------------------



i: 0 x: -2.000000000 value: -2.125000000
i: 1 x: -2.125000000 value: -2.523925781
i: 2 x: -2.523925781 value: -4.144478854
i: 3 x: -4.144478854 value: -17.922122637
```

```
i: 4 x: -17.922122637 value: -1439.282558733
i: 5 x: -1439.282558733 value: -745380791.274546146
i: 6 x: -745380791.274546146 value: -1035319987915346794376O6912.00
⋮

i: 9 x:-533864966103796174913924902921888412OO44021
   0977416127945813409938556185753597183064238200
   7503886479476610437469144491876717383499128532 58
   8445361228425038771670435763924914615948002743191 4962
   774780796808037723725177713273312943013888.000000000
value: -inf

--------------------------
 FALSE POSITION METHOD G_3
 -----------------------



i: 0 x: -2.000000000 value: -2.062500000
i: 1 x: -2.062500000 value: -2.190460205
i: 2 x: -2.190460205 value: -2.471490348
i: 3 x: -2.471490348 value: -3.185309780
i: 4 x: -3.185309780 value: -5.695003010
i: 5 x: -5.695003010 value: -25.998297789
i: 6 x: -25.998297789 value: -2209.630166633
i: 7 x: -2209.630166633 value: -1348556479.545314789
i: 8 x: -1348556479.545314789 value: -3065613735123560538570752OO.000000000
⋮

i: 11 x: -5838441740431293736846375112159843378479548958498592641
3267139188404152884583196169193655770678545975858432676567 3067
518356227057007531078079164907760009465965159137875532576323O7
5232593770129865669977434970153274210972182072220712960O.000000000
value: -inf
```

Root 2: $x \approx -0.12549$

Raw Output - Initial Guess 0

```
--------------------------
 FALSE POSITION METHOD G_1
 -----------------------
```

```
i: 0 x: 0.000000000 value: 0.068965517
i: 1 x: 0.068965517 value: 0.175935731
i: 2 x: 0.175935731 value: 0.341218093
i: 3 x: 0.341218093 value: 0.592962151
i: 4 x: 0.592962151 value: 0.960322243
i: 5 x: 0.960322243 value: 1.436965242
i: 6 x: 1.436965242 value: 1.889477770
i: 7 x: 1.889477770 value: 2.070475964
i: 8 x: 2.070475964 value: 2.057516112
i: 9 x: 2.057516112 value: 2.060251590
i: 10 x: 2.060251590 value: 2.059698076
i: 11 x: 2.059698076 value: 2.059811107
i: 12 x: 2.059811107 value: 2.059788068
i: 13 x: 2.059788068 value: 2.059792766
i: 14 x: 2.059792766 value: 2.059791808
i: 15 x: 2.059791808 value: 2.059792003
i: 16 x: 2.059792003 value: 2.059791963
i: 17 x: 2.059791963 value: 2.059791971
i: 18 x: 2.059791971 value: 2.059791970


--------------------------
 FALSE POSITION METHOD G_2
 -----------------------


i: 0 x: 0.000000000 value: -0.125000000
i: 1 x: -0.125000000 value: -0.125488281
i: 2 x: -0.125488281 value: -0.125494026
i: 3 x: -0.125494026 value: -0.125494094


--------------------------
 FALSE POSITION METHOD G_3
 -----------------------


i: 0 x: 0.000000000 value: -0.062500000
i: 1 x: -0.062500000 value: -0.093780518
i: 2 x: -0.093780518 value: -0.109493356
i: 3 x: -0.109493356 value: -0.117410765
```
⋮

```
i: 22 x: -0.125494056 value: -0.125494075
i: 23 x: -0.125494075 value: -0.125494085
i: 24 x: -0.125494085 value: -0.125494089
i: 25 x: -0.125494089 value: -0.125494092
i: 26 x: -0.125494092 value: -0.125494093
```

Root 3: $x \approx 2.0598$

Raw Output - Initial Guess 2

```
--------------------------
 FALSE POSITION METHOD G_1
 -----------------------


i: 0 x: 2.000000000 value: 2.068965517
i: 1 x: 2.068965517 value: 2.057849709
i: 2 x: 2.057849709 value: 2.060184771
i: 3 x: 2.060184771 value: 2.059711749
i: 4 x: 2.059711749 value: 2.059808321
i: 5 x: 2.059808321 value: 2.059788636
i: 6 x: 2.059788636 value: 2.059792650
i: 7 x: 2.059792650 value: 2.059791831
i: 8 x: 2.059791831 value: 2.059791998
i: 9 x: 2.059791998 value: 2.059791964
i: 10 x: 2.059791964 value: 2.059791971
i: 11 x: 2.059791971 value: 2.059791970


--------------------------
 FALSE POSITION METHOD G_2
 -----------------------


i: 0 x: 2.000000000 value: 1.875000000
i: 1 x: 1.875000000 value: 1.522949219
i: 2 x: 1.522949219 value: 0.758072328
i: 3 x: 0.758072328 value: -0.016088951
i: 4 x: -0.016088951 value: -0.125001041
i: 5 x: -0.125001041 value: -0.125488293
```

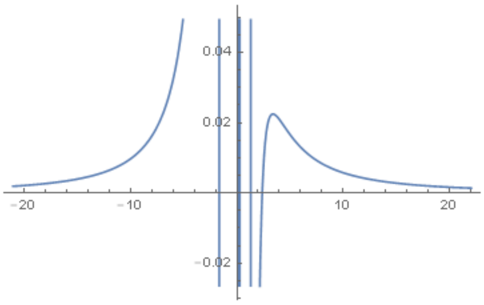
```
i: 6 x: -0.125488293 value: -0.125494026
i: 7 x: -0.125494026 value: -0.125494094


-------------------------
 FALSE POSITION METHOD G_3
 ----------------------


i: 0 x: 2.000000000 value: 1.937500000
i: 1 x: 1.937500000 value: 1.815399170
i: 2 x: 1.815399170 value: 1.593070099
i: 3 x: 1.593070099 value: 1.239411117
i: 4 x: 1.239411117 value: 0.795194169
i: 5 x: 0.795194169 value: 0.397950600
i: 6 x: 0.397950600 value: 0.144352965
i: 7 x: 0.144352965 value: 0.010052482
⋮

i: 30 x: -0.125494053 value: -0.125494074
i: 31 x: -0.125494074 value: -0.125494084
i: 32 x: -0.125494084 value: -0.125494089
i: 33 x: -0.125494089 value: -0.125494092
i: 34 x: -0.125494092 value: -0.125494093
```

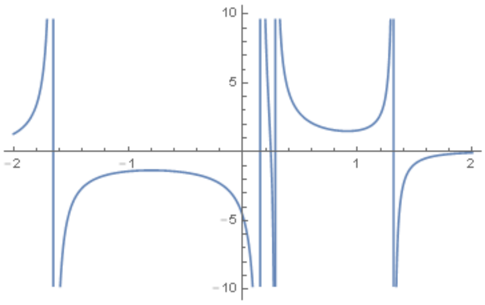
Analysis

$g_1(x)$ is surprisingly stable and useful, converging to two different roots, and converging for all 3 initial values. $g_2(x)$ does a good job of finding the second root, although it diverged for initial value -2, it converged to the root it was chosen for in both other cases. Perplexingly, g_3, while it does sometimes converge, does not seem to converge towards the third root even when nearby, despite being chosen explicitly to find that root. Fortunately, g_1 can find the third root, and thus we do have a function $g_i(x)$ for each root, just not as expected.

Ultimately, g_1 and g_2 together form a good pair of functions for finding the roots of this equation - g_3 is fairly slow to converge in all cases, and the root that it finds can be found by g_2. The functions are all fairly stable considering the usual instability of fixed point iteration, with g_2 and g_3 both diverging relatively quickly for the negative initial value, but converging eventually otherwise, and g_1 converging when started near any of the 3 real roots.

Problem 7





Considering the above graphs of the function $\Omega_1 - \Omega_2$, it is not surprising that the secant method fails miserably. As the method approaches a root, the secant lines can become near vertical, leading to unpredictable results. In addition, the asymptotic behavior at larger values can trick a numerical method into believing it has found a root at very large positive and very large negative values of x. The simplest solution to the problem is to shrink our bounds until we avoid the asymptotic behavior - for example, bounds of .2 and .25 are sufficient to find the root quickly, and this would be easily found by experimentation by simply trying each .05 length interval in the original interval, even if we didn't know where the root was exactly.

Raw Output of Secant Method with [-1,1]

```
i: 0 a: 1.000000000 b: -0.046482412 value: -3.702651461
i: 1 a: -0.046482412 b: 0.690415093 value: 1.780384308
i: 2 a: 0.690415093 b: 0.451138749 value: 3.331354064
i: 3 a: 0.451138749 b: 0.965084439 value: 1.521238272
i: 4 a: 0.965084439 b: 1.397009213 value: -2.471275853
i: 5 a: 1.397009213 b: 1.129657557 value: 1.966787495
i: 6 a: 1.129657557 b: 1.248138041 value: 4.210222009
i: 7 a: 1.248138041 b: 1.025787382 value: 1.595297137
```

$\vdots$

```
i: 42 a: -3937.724726937 b: -5215.963537543 value: 0.000000028
i: 43 a: -5215.963537543 b: -6909.269396967 value: 0.000000016
i: 44 a: -6909.269396967 b: -9152.422040822 value: 0.000000009
i: 45 a: -9152.422040822 b: -12123.966602370 value: 0.000000005
i: 46 a: -12123.966602370 b: -16060.425023712 value: 0.000000003
i: 47 a: -16060.425023712 b: -21275.122167236 value: 0.000000002
i: 48 a: -21275.122167236 b: -28183.125103398 value: 0.000000001
```

Raw Output of Secant Method with [.2,.25]

```
i: 0 a: 0.200000000 b: 0.243206198 value: 0.195326626
i: 1 a: 0.243206198 b: 0.244224922 value: -0.014224914
i: 2 a: 0.244224922 b: 0.244155768 value: 0.000179400
i: 3 a: 0.244155768 b: 0.244156630 value: 0.000000166
i: 4 a: 0.244156630 b: 0.244156630 value: -0.000000000
```

Problem 8

Raw Output

First Method

```
Enter the x value to approximate at: 2
Actual value: 403.428793492735
n: 1 value: 1.000000000000
n: 2 value: 7.000000000000
n: 3 value: 25.000000000000
n: 4 value: 61.000000000000
n: 5 value: 115.000000000000
```

⋮

```
n: 25 value: 403.428791117540
n: 26 value: 403.428792950427
n: 27 value: 403.428793373400

Enter the x value to approximate at: -2
Actual value: 0.002478752177
Enter the x value to approximate at: Actual value: 0.002478752177
n: 1 value: 1.000000000000
n: 2 value: -5.000000000000
n: 3 value: 13.000000000000
n: 4 value: -23.000000000000
```

⋮

```
n: 33 value: 0.002478752181
n: 34 value: 0.002478752176

Enter the x value to approximate at: -12
```

```
Actual value: 0.000000000000
n: 1 value: 1.000000000000
n: 2 value: -35.000000000000
n: 3 value: 613.000000000000
```

⋮

```
n: 98 value: -0.058785404183
n: 99 value: -0.023849127767
n: 100 value: -0.036553228282
```

```
Enter the x value to approximate at: Actual value: 114200738981568423454048256.0000000000
n: 1 value: 1.000000000000
n: 2 value: 61.000000000000
n: 3 value: 1861.000000000000
n: 4 value: 37861.000000000000
n: 5 value: 577861.000000000000
```

⋮

```
n: 98 value: 114200260606732911767453696.000000000000
n: 99 value: 114200453117080195064397824.000000000000
n: 100 value: 114200569790017939393478656.000000000000
```

```
Enter the x value to approximate at: -20
Actual value: 0.000000000000
n: 1 value: 1.000000000000
n: 2 value: -59.000000000000
n: 3 value: 1741.000000000000
n: 4 value: -34259.000000000000
n: 5 value: 505741.000000000000
```

⋮

```
n: 98 value: -1196923832936048230400.000000000000
n: 99 value: 728179639950760673280.000000000000
n: 100 value: -438549737556396277760.000000000000
```

Second Method

```
Enter the x value to approximate at: 2
Actual value: 403.428793492735
n: 1 value: 1.000000000000
```

```
n: 2 value: 7.000000000000
n: 3 value: 25.000000000000
n: 4 value: 61.000000000000
n: 5 value: 115.000000000000
n: 6 value: 179.800000000000
n: 7 value: 244.600000000000
n: 8 value: 300.142857142857
```

⋮

```
n: 32 value: 403.428793492698
n: 33 value: 403.428793492728
n: 34 value: 403.428793492734
n: 35 value: 403.428793492735
n: 36 value: 403.428793492735
n: 37 value: 403.428793492735

Enter the x value to approximate at: -2
Actual value: 0.002478752177
n: 1 value: 1.000000000000
n: 2 value: -5.000000000000
n: 3 value: 13.000000000000
n: 4 value: -23.000000000000
n: 5 value: 31.000000000000
n: 6 value: -33.800000000000
```

⋮

```
n: 37 value: 0.002478752177
n: 38 value: 0.002478752177
n: 39 value: 0.002478752177

Enter the x value to approximate at: -12
Actual value: 0.000000000000
n: 1 value: 1.000000000000
n: 2 value: -35.000000000000
n: 3 value: 613.000000000000
n: 4 value: -7163.000000000000
n: 5 value: 62821.000000000000
n: 6 value: -441063.799999999988
```

⋮

```
n: 125  value: -0.033183842925
n: 126  value: -0.033183842925
n: 127  value: -0.033183842925


Enter the x value to approximate at: 20
Actual value: 114200738981568423454048256.000000000000
n: 1 value: 1.000000000000
n: 2 value: 61.000000000000
n: 3 value: 1861.000000000000
n: 4 value: 37861.000000000000
```

$\vdots$

```
n: 133 value: 114200738981568389094309888.000000000000
n: 134 value: 114200738981568406274179072.000000000000
n: 135 value: 114200738981568423454048256.000000000000
n: 136 value: 114200738981568423454048256.000000000000


Enter the x value to approximate at: Actual value: 0.000000000000
n: 1 value: 1.000000000000
n: 2 value: -59.000000000000
n: 3 value: 1741.000000000000
n: 4 value: -34259.000000000000
```

$\vdots$

```
n: 174 value: 722745700.930318236351
n: 175 value: 722745700.930318593979
n: 176 value: 722745700.930318474770
n: 177 value: 722745700.930318474770
```

Analysis

For positive x and small (in absolute value) negative x, the second method is much more accurate - obviously due to finding the answer to 10^{-15} precision instead of a smaller relative precision, and also because it is not constrained to only 100 iterations. However, for large negative x, neither method converges - likely because the series is not absolutely convergent, and the extremely large (in absolute value) terms that appear late in the series overflow even 64 bit floating point numbers. Because the second method is only looking for the distance between sums to become small, this floating point overflow actually leads to an incorrect convergence - the first method simply runs out of iterations before it can return an answer at all.

Problem 9

1D Newton's Method Raw Output

```
Enter the initial guess for the 1D solver: 2

--------------------------
 1D Newton's Method
 -----------------------


i: 0 x: 2.048570661 value: 0.089009392
i: 1 x: 2.086422203 value: 0.070220115
i: 2 x: 2.116060683 value: 0.055507273
i: 3 x: 2.139352605 value: 0.043944567
⋮

i: 79 x: 2.227867384 value: 0.000000001
i: 80 x: 2.227867384 value: 0.000000001
i: 81 x: 2.227867385 value: 0.000000001
v: 0.016930 w: 2.227867
```

2D Newton's Method Raw Output

```
Enter the initial guess for the 2D system, as "v w": .5 2

--------------------------
 2D Newton's Method
 -----------------------


i:0 x: 0.500000000 y: 2.000000000
f(x,y): 1.923562789 g(x,y): -0.184608616

i:1 x: -0.059149012 y: 2.160628330
f(x,y): -0.361511993 g(x,y): 0.092238340

i:2 x: 0.021509188 y: 2.224876010
f(x,y): 0.022522550 g(x,y): -0.002090010

i:3 x: 0.016917514 y: 2.227869548
f(x,y): -0.000061124 g(x,y): 0.000008626

v: 0.016929818  w: 2.227867386
```

Analysis

The 2D solver is much faster at finding the answer than the 1D solver, provided that $v < 1$ for the initial guess (or else the second equation will return NAN). Considering that choosing $v < 1$ and positive and using the formula for $v(w)$ to approximate a corresponding w will lead naturally to a reasonable guess for the 2D solver, it seems like the clear cut winner between the two methods despite the 1D method seeming like a simplified problem.

Problem 10

i.) $e^{\frac{hv}{kT}} \approx 1 + \frac{hv}{kT}$ for small frequency v. This gives us

$$\frac{8\pi hv^3}{c^3} \cdot \frac{1}{1 + \frac{hv}{kT} - 1}$$

which simplifies to

$$\frac{8\pi v^2 kT}{c^3}$$

which is the Rayleigh-Jeans law.

The second degree approximation is $e^{\frac{hv}{kT}} \approx 1 + \frac{hv}{kT} + \frac{h^2v^2}{2k^2T^2}$, and so we have

$$\begin{aligned}
&\frac{8\pi hv^3}{c^3}\left(\frac{1}{\frac{hv}{kT} + \frac{h^2v^2}{2k^2T^2}}\right) \\
&= \frac{8\pi hv^3}{c^3}\left(\frac{2k^2T^2}{2kThv + h^2v^2}\right) \\
&= \frac{16\pi k^2T^2v^2}{c^3}\left(\frac{1}{2kT + hv}\right) \\
&= \frac{8\pi kTv^2}{c^3}\left(\frac{1}{1 + \frac{hv}{2kt}}\right)
\end{aligned}$$

we can approximate $\frac{1}{1+\frac{hv}{2kt}}$, giving us

$$\frac{4\pi(2kTv^2 - hv^3)}{c^3}$$

ii.) We have

$$\frac{8\pi h}{\lambda^3} \cdot \frac{1}{e^{\frac{hc}{\lambda Tk}} - 1}$$

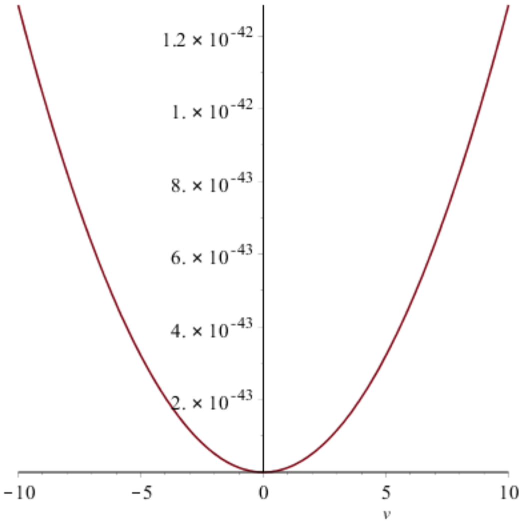
and

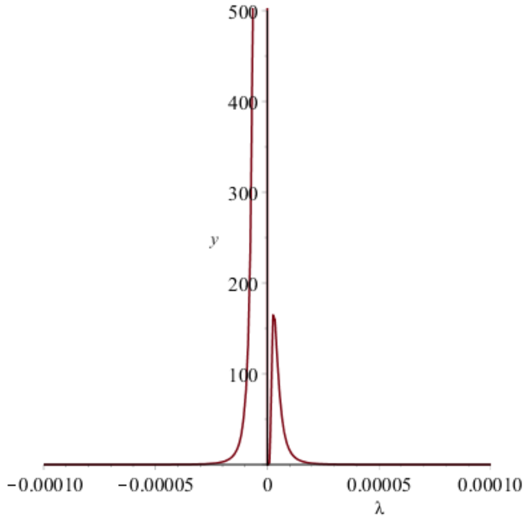
$$\mathrm{d}v = \mathrm{d}\frac{c}{\lambda} = c\,\mathrm{d}\frac{1}{\lambda} = -\frac{c}{\lambda^2}\,\mathrm{d}\lambda$$

which then gives us

$$\frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{\frac{8\pi hc}{\lambda Tk}} - 1}$$

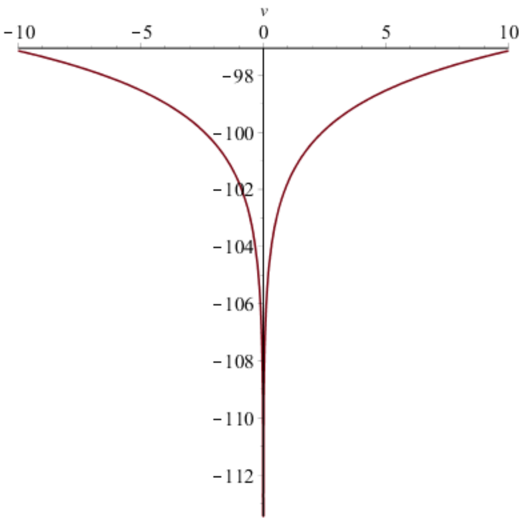
iii.) With linear comparison, $v(\lambda, T)$ had very irregular, steep behavior near $\lambda = 0$, while u was parabolic.

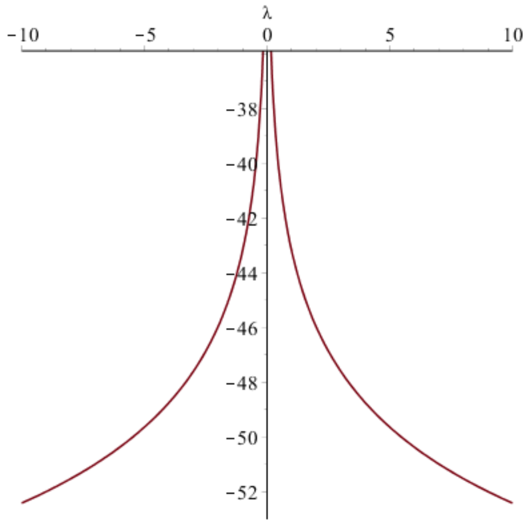




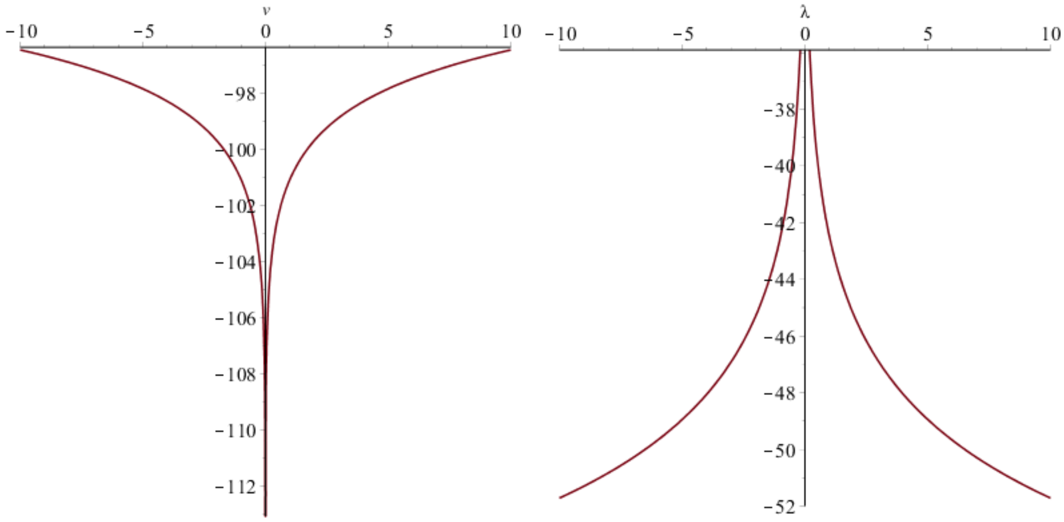
However, the logarithmic function smoothed out v considerably, and thus I chose to use a log-log comparison, because it more accurately depicted the symmetry between the two functions (that comes with λ being inverse to v).

T=500K

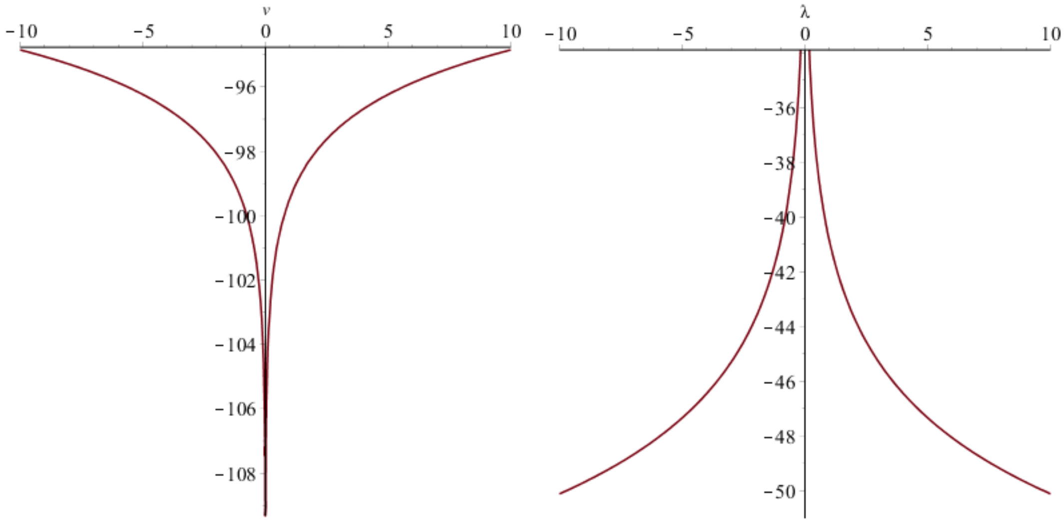




T=1000K



T=5000K



iv.)

$$x_{max} = 2.821439372$$

Calculating the derivative of $\frac{x^3}{e^x-1}$ (the rest of the expression is a constant scaling factor and can be ignored when finding a root of the derivative), we have $-\frac{e^x*(x-3)+3)*x^2}{(e^x-1)^2}$ as the equation for which we must find a suitable root. Noting that this function exhibits asymptotic behavior towards 0 as positive x increases, it is worth noting that many root finding methods will take the false root generated by $f(x)$ becoming very small as x approaches ∞.

For example, when started at an initial guess of 2, Newton's method successfully finds the root very quickly,

```
--------------------------
 1D Newton's Method
 -----------------------


i: 0 x: 2.761594156 value: 0.026815491
i: 1 x: 2.820394918 value: 0.000459740
i: 2 x: 2.821439035 value: 0.000000148
i: 3 x: 2.821439372 value: 0.000000000
```

while if started from a value of greater distance (even to the left!) it will often find that $x \approx 31$ is a root, despite that not being true - the value of $f(x)$ simply is smaller than 10^{-11} for sufficiently large x.

This, along with the concave down nature of the original function surrounding its maximum, makes it seem like a bracketed method would be more stable for finding the true root reliably. However, because Newton's method worked and worked well, I did not personally use another method - I was lucky enough to stumble upon the initial guess of 2 through inspection of the function.

v.) Luckily, the constants in the ratio cancel, leaving us with

$$\frac{x^3}{(e^x - 1)1.4214354727477367351093694 33} - 0.5$$

as our root finding equation.

Outside of x small in absolute value, the function is approximately a scaling of e^{-x} (which will not have a root), and so it was fairly easy to make fairly accurate initial guesses just by trying integers in the range $[-10, 10]$. Given that for x small in absolute value the function is very close to the cubic function, and that it is monotone decreasing for other x, it is fairly safe to say that it cannot have more than 3 roots and thus we have found all FWHM values.

```
--------------------------
 Root 1 - Newton
 -----------------------


i: 0 x: -0.839764202 value: 0.233251768
```

```
i: 1 x: -0.760192263 value: 0.080478297
i: 2 x: -0.728402536 value: 0.025566235
i: 3 x: -0.717659530 value: 0.007770370
  ⋮
i: 16 x: -0.712895519 value: 0.000000001
i: 17 x: -0.712895519 value: 0.000000000


--------------------------
 Root 2 - Bisection
 -----------------------


i: 0 a: 1.000000000 b: 1.500000000 value: 0.181956454
i: 1 a: 1.000000000 b: 1.250000000 value: 0.051751763
i: 2 a: 1.125000000 b: 1.250000000 value: -0.018471709
i: 3 a: 1.125000000 b: 1.187500000 value: 0.016955829
i: 4 a: 1.156250000 b: 1.187500000 value: -0.000688552
i: 5 a: 1.156250000 b: 1.171875000 value: 0.008152217
i: 6 a: 1.156250000 b: 1.164062500 value: 0.003736325
  ⋮
i: 24 a: 1.157464653 b: 1.157464683 value: -0.000000014
i: 25 a: 1.157464668 b: 1.157464683 value: -0.000000006
i: 26 a: 1.157464676 b: 1.157464683 value: -0.000000002
i: 27 a: 1.157464676 b: 1.157464679 value: 0.000000000


--------------------------
 Root 3 - Newton
 -----------------------


i: 0 x: 5.279906861 value: 0.030071791
i: 1 x: 5.371252629 value: 0.009117060
i: 2 x: 5.399490882 value: 0.002723474
i: 3 x: 5.407980476 value: 0.000809345
  ⋮
i: 14 x: 5.411575120 value: 0.000000001
i: 15 x: 5.411575124 value: 0.000000000
```