# Numerical Analysis Homework 2

Margaret Dorsey

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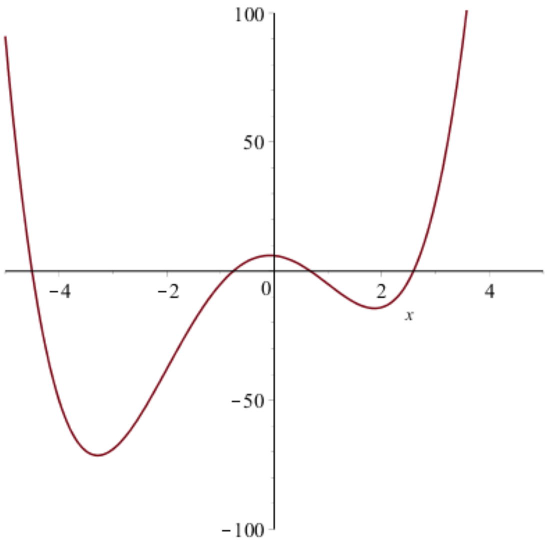
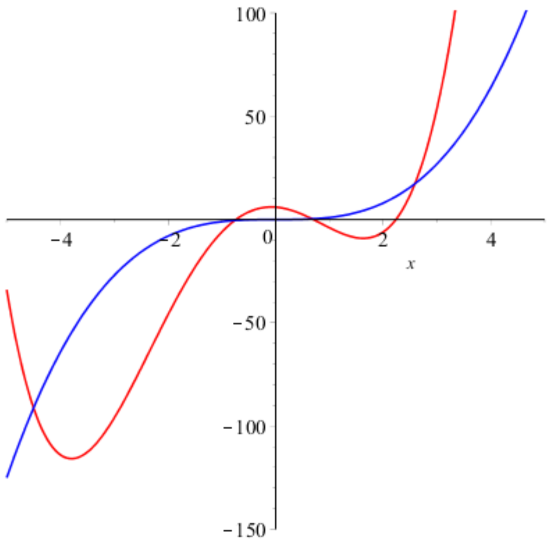
## Problem 1

No, a degree 3 polynomial cannot intersect a degree 4 polynomial in exactly 5 points - let $f(x)$ be a degree 3 polynomial, and $g(x)$ be degree 4. Any intersection of $f$ and $g$ must be a root of $f - g$ and $g - f$, which are degree 4 polynomials, and therefore may have at most 4 real roots.

It is, however, possible for $f$ and $g$ to intersect at exactly four points. Consider $f(x) = x^3$, and $g(x) = x^4 + 3x^3 - 12x^2 - 2x + 6$. Their difference, $x^4 + 2x^3 - 12x^2 - 2x + 6$, is plotted below and clearly has exactly 4 distinct real roots. $f$ and $g$ can intersect only at those roots, and must intersect at them, so thus $f$ and $g$ intersect exactly 4 times, as shown on the second plot.




## Problem 2

Using the formula

$$P_n(x) - \sum_{j=1}^{n} y_j \prod_{k=1, k\neq j}^{n} \frac{x - x_k}{x_j - x_k}$$

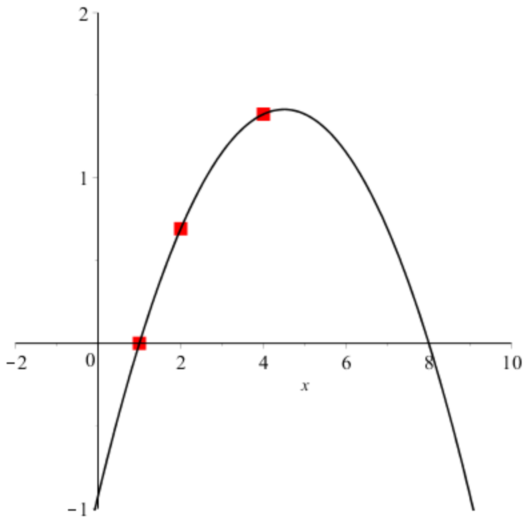
we obtain

$$P_2(x) = \frac{(x-2)(x-4)}{3} \cdot 0 + \frac{(x-1)(x-4)}{-2} \cdot \ln 2 + \frac{(x-1)(x-2)}{6} \cdot \ln 4$$

$$P_2(x) = \frac{-(x-1)(x-4)}{2} \cdot \ln 2 + \frac{(x-1)(x-2)}{3} \cdot \ln 2$$

$$P_2(x) = \frac{-\ln 2}{6} x^2 + \frac{3 \ln 2}{2} x - \frac{4 \ln 2}{3}$$

The graph of this polynomial fit is shown below.




Using this approximation, we obtain $\ln 3 \approx P_2(3) = \frac{-9 \ln 2}{6} + \frac{9 \ln 2}{2} - \frac{4 \ln 2}{3} = \frac{5}{3} \ln 2 \approx 1.1552453$. Using the formula

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| \cdot \max_{[2,4]} \left| \prod_{i=0}^{n} (x - x_i) \right|$$

with $f(x) = \ln x$, $f^{(3)}(x) = \frac{2}{x^3}$, we obtain

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{1}{3(\xi)^3} \right| \cdot \max_{[2,4]} \left| x^3 - 7x^2 + 14x - 8 \right|$$

Because $f^{(3)}(\xi)$ is strictly decreasing and non-negative on our interval, we know the maximum on $[2, 4]$ is $\frac{1}{24}$ at 2. Using the derivative of $x^3 - 7x^2 + 14x - 8$ with the quadratic formula, we find that the maximum absolute value on $[2, 4]$ is $-\frac{2}{27}(10 + 7\sqrt{7})$ at $x = \frac{7}{3} + \frac{\sqrt{7}}{3}$, or approximately 2.11261.

Multiplying these values, we obtain $|f(x) - P(x)| \leq .0880255$.

Comparing our value of 1.1552453 to the actual value of $\ln 3 \approx 1.0986123$, we have $|f(x) - P_2(x)| = .056630 < .0880255$, as expected.

## Problem 3

Using the formulas $B(0) = P_0$ and $B(1) = P_3$, we obtain $P_0 = (1,1)$ and $P_3 = (9,1)$. We then calculate $B'(t) = (x'(t), y'(t)) = (12t + 6t^2, 3t^2 - 1)$, and note that because in general $B'(t) = 3(1-t)^2(P_1 - P_0) + 6(1-t)t(P_2 - P_1) + 3t^2(P_3 - P_2)$, $B'(0) = 3(P_1 - P_0)$ and $B'(1) = 3(P_3 - P_2)$. Thus we have $P_1 = (1, \frac{2}{3})$ and $P_2 = (3, \frac{1}{3})$. This gives us the complete set of control points for the curve, namely

$$\{(1,1), (1, \frac{2}{3}), (3, \frac{1}{3}), (9,1)\}$$

## Problem 4

We first note that

$$\begin{aligned}
A_1 &= \tfrac{x_2 - x}{x_2 - x_1} &&= 1 - x \\
A_2 &= \tfrac{x_3 - x}{x_3 - x_2} &&= 2 - x \\
A_3 &= \tfrac{x_4 - x}{x_4 - x_3} &&= 3 - x
\end{aligned}$$

$$\begin{aligned}
B_1 &= \tfrac{x - x_1}{x_2 - x_1} &&= x - 0 \\
B_2 &= \tfrac{x - x_2}{x_3 - x_2} &&= x - 1 \\
B_3 &= \tfrac{x - x_3}{x_4 - x_3} &&= x - 2
\end{aligned}$$

$$\begin{aligned}
C_1 &= \tfrac{(x_2 - x_1)^2}{6}[A_1^3 - A_1] &&= \tfrac{1}{6}(-x^3 + 3x^2 - 2x) \\
C_2 &= \tfrac{(x_3 - x_2)^2}{6}[A_2^3 - A_2] &&= \tfrac{1}{6}(-x^3 + 6x^2 - 11x + 6) \\
C_3 &= \tfrac{(x_4 - x_3)^2}{6}[A_3^3 - A_3] &&= \tfrac{1}{6}(-x^3 + 9x^2 - 26x + 24)
\end{aligned}$$

$$\begin{aligned}
D_1 &= \tfrac{(x_2 - x_1)^2}{6}[B_1^3 - B_1] &&= \tfrac{1}{6}(x^3 - x) \\
D_2 &= \tfrac{(x_3 - x_2)^2}{6}[B_2^3 - B_2] &&= \tfrac{1}{6}(x^3 - 3x^2 + 2x) \\
D_3 &= \tfrac{(x_4 - x_3)^2}{6}[B_3^3 - B_3] &&= \tfrac{1}{6}(x^3 - 6x^2 + 11x - 6)
\end{aligned}$$

### Natural Endpoint Condition

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6\left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1}\right) \\ 6\left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2}\right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 - 0 & 2(2 - 0) & 2 - 1 & 0 \\ 0 & 2 - 1 & 2(3 - 1) & 3 - 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6\left(\frac{2-5}{2-1} - \frac{5-3}{1-0}\right) \\ 6\left(\frac{1-2}{3-2} - \frac{2-5}{2-1}\right) \\ 0 \end{bmatrix}$$

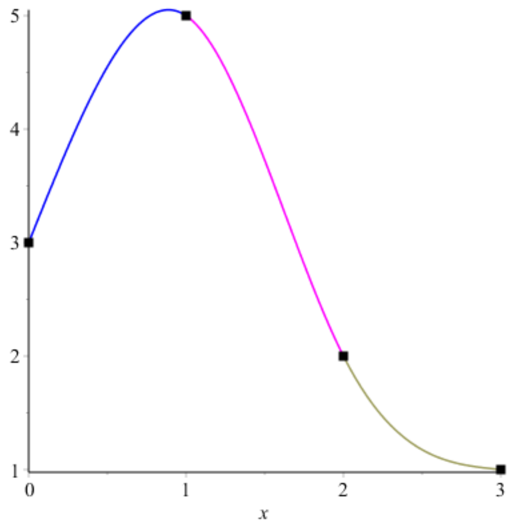
$$\begin{bmatrix}1&0&0&0\\1&4&1&0\\0&1&4&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_1''\\y_2''\\y_3''\\y_4''\end{bmatrix}=\begin{bmatrix}0\\-30\\12\\0\end{bmatrix}$$

$$\begin{bmatrix}y_1''\\y_2''\\y_3''\\y_4''\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-\frac{4}{15}&\frac{4}{15}&-\frac{1}{15}&\frac{1}{15}\\\frac{1}{15}&-\frac{1}{15}&\frac{4}{15}&-\frac{4}{15}\\0&0&0&1\end{bmatrix}\begin{bmatrix}0\\-30\\12\\0\end{bmatrix}$$

$$\begin{bmatrix}y_1''\\y_2''\\y_3''\\y_4''\end{bmatrix}=\begin{bmatrix}0\\-\frac{44}{5}\\\frac{26}{5}\\0\end{bmatrix}$$

Using these derived values of $y_i''$ with our $A$,$B$,$C$, and $D$ equations above, we have

$$\begin{aligned}
y_1^{(cubic)} &= y_1A_1+y_2B_1+y_1''C_1+y_2''D_1 &&= -\tfrac{22}{15}x^3+\tfrac{52}{15}x+3\\
y_2^{(cubic)} &= y_2A_2+y_3B_2+y_2''C_2+y_3''D_2 &&= \tfrac{7}{3}x^3-\tfrac{57}{5}x^2+\tfrac{223}{15}x-\tfrac{4}{5}\\
y_3^{(cubic)} &= y_3A_3+y_4B_3+y_3''C_3+y_4''D_3 &&= -\tfrac{13}{15}x^3+\tfrac{39}{5}x^2-\tfrac{353}{15}x+\tfrac{124}{5}
\end{aligned}$$




## Curvature-Adjusted Endpoint Condition

$$\begin{bmatrix}1&0&0&0\\x_2-x_1&2(x_3-x_1)&x_3-x_2&0\\0&x_3-x_2&2(x_4-x_2)&x_4-x_3\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_1''\\y_2''\\y_3''\\y_4''\end{bmatrix}=\begin{bmatrix}k_1\\6\left(\frac{y_3-y_2}{x_3-x_2}-\frac{y_2-y_1}{x_2-x_1}\right)\\6\left(\frac{y_4-y_3}{x_4-x_3}-\frac{y_3-y_2}{x_3-x_2}\right)\\k_n\end{bmatrix}$$

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-0 & 2(2-0) & 2-1 & 0 \\ 0 & 2-1 & 2(3-1) & 3-2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} k_n \\ 6\left(\frac{2-5}{2-1} - \frac{5-3}{1-0}\right) \\ 6\left(\frac{1-2}{3-2} - \frac{2-5}{2-1}\right) \\ k_n \end{bmatrix}
$$

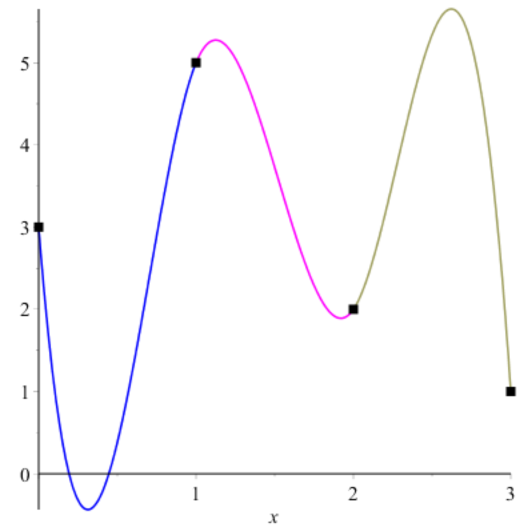
$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} k_1 \\ -30 \\ 12 \\ k_n \end{bmatrix}
$$

Choosing $k_1 = 100$, $k_2 = -100$, we have

$$
\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{4}{15} & \frac{4}{15} & -\frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & -\frac{1}{15} & \frac{4}{15} & -\frac{4}{15} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ -30 \\ 12 \\ -100 \end{bmatrix}
$$

$$
\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 100 \\ -\frac{632}{15} \\ \frac{578}{15} \\ -100 \end{bmatrix}
$$

$$
\begin{aligned}
y_1^{(cubic)} &= -\frac{1066}{45} + 50x^2 - \frac{1094}{45}x + 3 \\
y_2^{(cubic)} &= \frac{121}{9}x^3 - \frac{307}{5}x^2 + \frac{3919}{45}x - \frac{512}{15} \\
y_3^{(cubic)} &= -\frac{1039}{45} + \frac{789}{5}x^2 - \frac{15809}{45}x + \frac{3872}{15}
\end{aligned}
$$



Clamped Endpoint Condition

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 6\left(\frac{y_2-y_1}{x_2-x_1} - c_n\right) \\ 6\left(\frac{y_3-y_2}{x_3-x_2} - \frac{y_2-y_1}{x_2-x_1}\right) \\ 6\left(\frac{y_4-y_3}{x_4-x_3} - \frac{y_3-y_2}{x_3-x_2}\right) \\ 6\left(c_n - \frac{y_4-y_3}{x_4-x_3}\right) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1-0 & 2(2-0) & 2-1 & 0 \\ 0 & 2-1 & 2(3-1) & 3-2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 6\left(\frac{5-3}{1-0} - c_n\right) \\ 6\left(\frac{2-5}{2-1} - \frac{5-3}{1-0}\right) \\ 6\left(\frac{1-2}{3-2} - \frac{2-5}{2-1}\right) \\ 6\left(c_n - \frac{1-2}{3-2}\right) \end{bmatrix}$$

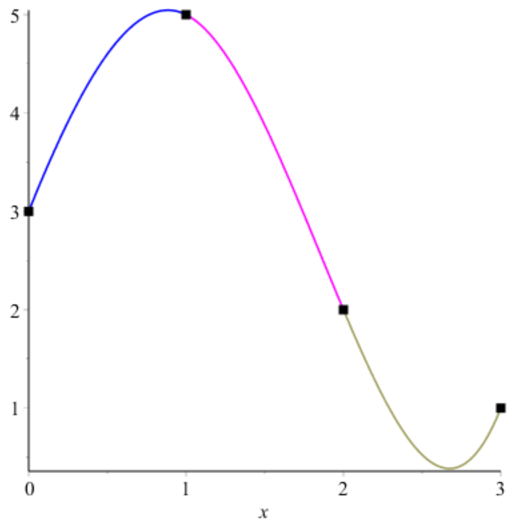
$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 12 - 6c_n \\ -30 \\ 12 \\ 6(c_n + 1) \end{bmatrix}$$

Choosing $c_n = 4$, we have

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \frac{1}{45} \begin{bmatrix} 26 & -7 & 2 & -1 \\ -7 & 14 & -4 & 2 \\ 2 & -4 & 14 & -7 \\ -1 & 2 & -7 & 26 \end{bmatrix} \begin{bmatrix} -12 \\ -30 \\ 12 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} \\ -\frac{36}{5} \\ \frac{6}{5} \\ \frac{72}{5} \end{bmatrix}$$

$$\begin{aligned} y_1^{(cubic)} &= -\tfrac{4}{5}x^3 - \tfrac{6}{5}x^2 + 4x + 3 \\ y_2^{(cubic)} &= \tfrac{7}{5}x^3 - \tfrac{39}{5}x^2 + \tfrac{53}{5}x + \tfrac{4}{5} \\ y_3^{(cubic)} &= -\tfrac{11}{5}x^3 - \tfrac{63}{5}x^2 + \tfrac{101}{5}x - \tfrac{28}{5} \end{aligned}$$




Parabolically Terminated Endpoint Condition

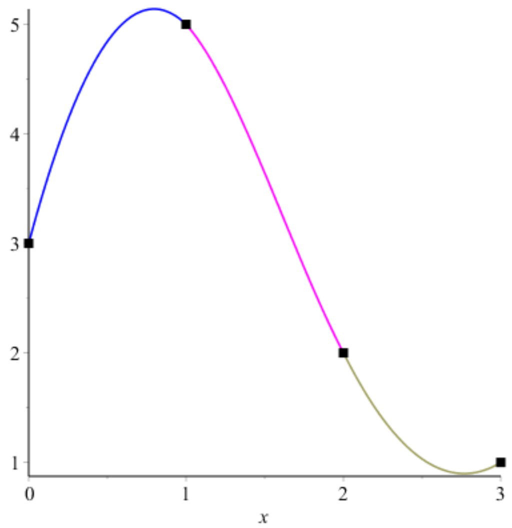
$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6\left(\frac{y_3-y_2}{x_3-x_2} - \frac{y_2-y_1}{x_2-x_1}\right) \\ 6\left(\frac{y_4-y_3}{x_4-x_3} - \frac{y_3-y_2}{x_3-x_2}\right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 19 & 5 & -1 & 1 \\ -5 & 5 & -1 & 1 \\ 1 & -1 & 5 & -5 \\ 1 & -1 & 5 & 19 \end{bmatrix} \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} -\frac{27}{4} \\ -\frac{27}{4} \\ \frac{15}{4} \\ \frac{15}{4} \end{bmatrix}$$

$$\begin{aligned} y_1^{(cubic)} &= -\tfrac{27}{8}x^2 + \tfrac{43}{8}x + 3 \\ y_2^{(cubic)} &= \tfrac{13}{8}x^3 - \tfrac{63}{8}x^2 + \tfrac{37}{4}x + 2 \\ y_3^{(cubic)} &= \tfrac{15}{8}x^2 - \tfrac{83}{8}x - \tfrac{61}{4} \end{aligned}$$




## Not-a-Knot Endpoint Condition

$$\begin{bmatrix} x_3-x_2 & -(x_3-x_1) & x_2-x_1 & 0 \\ x_2-x_1 & 2(x_3-x_1) & x_3-x_2 & 0 \\ 0 & x_3-x_2 & 2(x_4-x_2) & x_4-x_3 \\ 0 & x_4-x_3 & -(x_4-x_2) & x_3-x_2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6\left(\frac{y_3-y_2}{x_3-x_2}-\frac{y_2-y_1}{x_2-x_1}\right) \\ 6\left(\frac{y_4-y_3}{x_4-x_3}-\frac{y_3-y_2}{x_3-x_2}\right) \\ 0 \end{bmatrix}$$

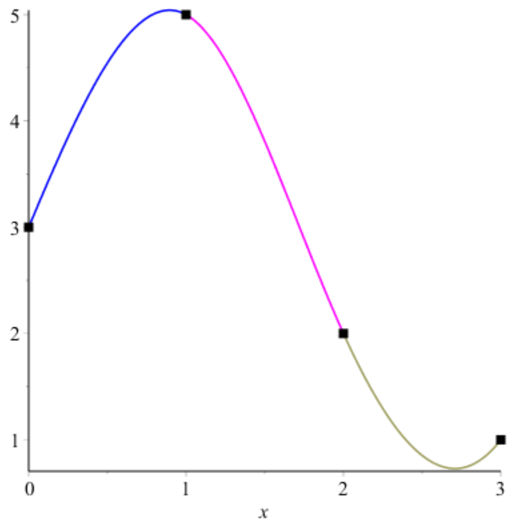
$$\begin{bmatrix} -3 & 1 & 2 & 0 \\ 1-0 & 2(2-0) & 2-1 & 0 \\ 0 & 2-1 & 2(3-1) & 3-2 \\ 0 & -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6\left(\frac{2-5}{2-1}-\frac{5-3}{1-0}\right) \\ 6\left(\frac{1-2}{3-2}-\frac{2-5}{2-1}\right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 2 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \frac{1}{198} \begin{bmatrix} -62 & 12 & 21 & 7 \\ 16 & 48 & -15 & -5 \\ -2 & -6 & 39 & 13 \\ -8 & -24 & 57 & -47 \end{bmatrix} \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} \\ -\frac{90}{11} \\ \frac{36}{11} \\ \frac{78}{11} \end{bmatrix}$$

$$
\begin{aligned}
y_1^{(cubic)} &= -\tfrac{14}{11}x^3 - \tfrac{3}{11}x^2 + \tfrac{39}{11}x + 3 \\
y_2^{(cubic)} &= \tfrac{21}{11}x^3 - \tfrac{108}{11}x^2 + \tfrac{144}{11}x - \tfrac{2}{11} \\
y_3^{(cubic)} &= \tfrac{7}{11}x^3 - \tfrac{24}{11}x^2 - \tfrac{24}{11}x + 10
\end{aligned}
$$




# Problem 5

## Raw Output

5 DATA POINTS

```
Initial Points:

(-5.000000,0.038462) (-2.500000,0.137931) (0.000000,1.000000)
(2.500000,0.137931) (5.000000,0.038462)

x:            -3.750000
lagrange:     -0.162550
cubic:        -0.039693
curveadj:     -0.042043
actual:       0.066390

x:            -1.250000
lagrange:     0.629062
cubic:        0.666659
curveadj:     0.667129
```

```
actual:          0.390244

x:               1.250000
lagrange:        0.939904
cubic:           0.666659
curveadj:        0.667129
actual:          0.390244

x:               3.750000
lagrange:        -1.716761
cubic:           -0.039693
curveadj:        -0.042043
actual:          0.066390
```

10 DATA POINTS

```
Initial Points:

(-5.000000,0.038462) (-3.888889,0.062021) (-2.777778,0.114731)
(-1.666667,0.264706) (-0.555556,0.764151) (0.555556,0.764151)
(1.666667,0.264706) (2.777778,0.114731) (3.888889,0.062021)
(5.000000,0.038462)

x:               -4.444444
lagrange:        -0.209264
cubic:           0.046749
curveadj:        0.046273
actual:          0.048186

x:               -3.333333
lagrange:        0.160420
cubic:           0.087924
curveadj:        0.088051
actual:          0.082569

x:               -2.222222
lagrange:        0.115474
cubic:           0.147614
curveadj:        0.147580
actual:          0.168399
```

```
x:              −1.111111
lagrange:        0.523247
cubic:           0.515771
curveadj:        0.515781
actual:          0.447514

x:               0.000000
lagrange:        0.861538
cubic:           0.857126
curveadj:        0.857121
actual:          1.000000

x:               1.111111
lagrange:        0.523247
cubic:           0.515771
curveadj:        0.515781
actual:          0.447514

x:               2.222222
lagrange:        0.115474
cubic:           0.147614
curveadj:        0.147580
actual:          0.168399

x:               3.333333
lagrange:        0.160420
cubic:           0.087924
curveadj:        0.088051
actual:          0.082569

x:               4.444444
lagrange:       −0.209264
cubic:           0.046749
curveadj:        0.046273
actual:          0.048186
```

15 DATA POINTS

```
Initial Points:
```

```
(−5.000000,0.038462) (−4.285714,0.051633) (−3.571429,0.072700)
(−2.857143,0.109131) (−2.142857,0.178832) (−1.428571,0.328859)
(−0.714286,0.662162) (0.000000,1.000000) (0.714286,0.662162)
(1.428571,0.328859) (2.142857,0.178832) (2.857143,0.109131)
(3.571429,0.072700) (4.285714,0.051633) (5.000000,0.038462)

x:              −4.642857
lagrange:       2.668349
cubic:          0.044513
curveadj:       0.044317
actual:         0.044334

x:              −3.928571
lagrange:       −0.301858
cubic:          0.060808
curveadj:       0.060861
actual:         0.060851

x:              −3.214286
lagrange:       0.180694
cubic:          0.088163
curveadj:       0.088149
actual:         0.088249

x:              −2.500000
lagrange:       0.101181
cubic:          0.138115
curveadj:       0.138119
actual:         0.137931

x:              −1.785714
lagrange:       0.259506
cubic:          0.237467
curveadj:       0.237466
actual:         0.238733

x:              −1.071429
lagrange:       0.451153
cubic:          0.468041
curveadj:       0.468041
actual:         0.465558
```

| | |
|---|---|
| x: | $-0.357143$ |
| lagrange: | 0.891166 |
| cubic: | 0.886911 |
| curveadj: | 0.886911 |
| actual: | 0.886878 |
| | |
| x: | 0.357143 |
| lagrange: | 0.904422 |
| cubic: | 0.886911 |
| curveadj: | 0.886911 |
| actual: | 0.886878 |
| | |
| x: | 1.071429 |
| lagrange: | 0.433077 |
| cubic: | 0.468041 |
| curveadj: | 0.468041 |
| actual: | 0.465558 |
| | |
| x: | 1.785714 |
| lagrange: | 0.293649 |
| cubic: | 0.237467 |
| curveadj: | 0.237466 |
| actual: | 0.238733 |
| | |
| x: | 2.500000 |
| lagrange: | 0.008506 |
| cubic: | 0.138115 |
| curveadj: | 0.138119 |
| actual: | 0.137931 |
| | |
| x: | 3.214286 |
| lagrange: | 0.569927 |
| cubic: | 0.088163 |
| curveadj: | 0.088149 |
| actual: | 0.088249 |
| | |
| x: | 3.928571 |
| lagrange: | $-3.285979$ |
| cubic: | 0.060808 |
| curveadj: | 0.060861 |
| actual: | 0.060851 |
| | |
| x: | 4.642857 |

```
lagrange:        77.271383
cubic:           0.044513
curveadj:        0.044317
actual:          0.044334


 Chebyshev Approximation 5 Nodes

x:               −3.847104
chebyshev:       −0.134899
actual:          0.063290

x:               −1.469463
chebyshev:       0.714472
actual:          0.316524

x:               1.469463
chebyshev:       0.714472
actual:          0.316524

x:               3.847104
chebyshev:       −0.134899
actual:          0.063290



 Chebyshev Approximation 10 Nodes

x:               −4.696737
chebyshev:       0.032438
actual:          0.043366

x:               −3.995283
chebyshev:       0.074601
actual:          0.058954

x:               −2.902743
chebyshev:       0.077647
actual:          0.106090

x:               −1.526062
```

```
chebyshev:        0.381200
actual:           0.300403

x:                0.000000
chebyshev:        0.730822
actual:           1.000000

x:                1.526062
chebyshev:        0.381200
actual:           0.300403

x:                2.902743
chebyshev:        0.077647
actual:           0.106090

x:                3.995283
chebyshev:        0.074601
actual:           0.058954

x:                4.696737
chebyshev:        0.032438
actual:           0.043366



Chebyshev Approximation 15 Nodes

x:                -4.863946
chebyshev:        0.059310
actual:           0.040555

x:                -4.542705
chebyshev:        0.025196
actual:           0.046219

x:                -4.022926
chebyshev:        0.081878
actual:           0.058194

x:                -3.327325
chebyshev:        0.054855
actual:           0.082843
```

```
x:              −2.486305
chebyshev:      0.174450
actual:         0.139243

x:              −1.536621
chebyshev:      0.250984
actual:         0.297512

x:              −0.519779
chebyshev:      0.828958
actual:         0.787296

x:              0.519779
chebyshev:      0.828958
actual:         0.787296

x:              1.536621
chebyshev:      0.250984
actual:         0.297512

x:              2.486305
chebyshev:      0.174450
actual:         0.139243

x:              3.327325
chebyshev:      0.054855
actual:         0.082843

x:              4.022926
chebyshev:      0.081878
actual:         0.058194

x:              4.542705
chebyshev:      0.025196
actual:         0.046219

x:              4.863946
chebyshev:      0.059310
actual:         0.040555
```

### RMS Errors

| n | lagrange | natural | curve adjusted | chebyshev |
|---|---|---|---|---|
| 5 | .89780 | .04383 | .04421 | .09882 |
| 10 | .02010 | .00341 | .00341 | .00976 |
| 15 | 427.32136 | .000001 | .000001 | .00364 |

### Analysis

As $n$ increases, the error in the Lagrange approximation explodes due to Runge's phenomenon, despite the error at the interior points improving at a rate consistent with with the spline approximation - the magnitude of the error near the endpoints more than overcomes the reduction in error everywhere else. The splines behave consistently better overall as $n$ increases due to their piecewise construction which allows them to avoid erratic endpoint behavior.

Fixing the second derivatives of the splines marginally improves accuracy near the endpoints, but makes no marked difference in overall error, at least at these small values of $n$. This makes sense - because splines are piecewise, fixing the endpoint behavior will only greatly effect the points near the endpoints, and thus any improvement does not propagate well through the rest of the approximation, where the majority of the error data will be determined.

The Chebyshev polynomials did not perform as well as the splines, but as far as approximation by a single (non-piecewise) polynomial goes, they clearly won out over the Lagrange interpolation, with error steadily decreasing as $n$ increased despite having most points appearing "farther away" than the Lagrange interpolation, because they do not suffer from the extremely large error at the endpoints.

## Problem 6

### Coefficient Table

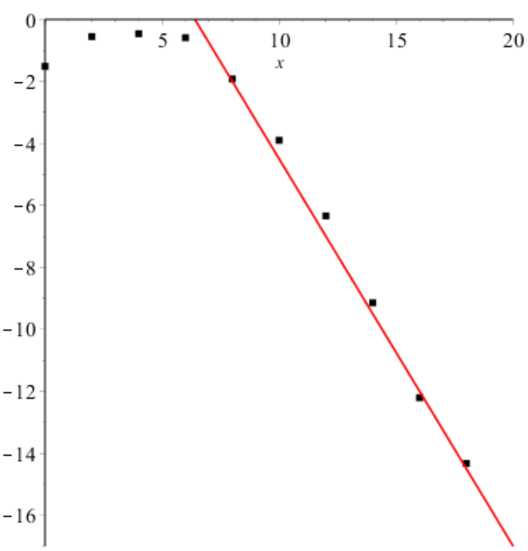
**Note:** The first 10 values of $a_n$ are the coefficients for 10 nodes, so I have not duplicated the data.

| $a_n$ | Function 1 | Function 2 | Function 3 |
|---|---|---|---|
| $a_0$ | 0.220277 | 0 | -0.125724 |
| $a_1$ | 0 | 0.449218 | 0 |
| $a_2$ | 0.575761 | 0 | -0.531218 |
| $a_3$ | 0 | 0.776220 | 0 |
| $a_4$ | 0.631361 | 0 | 0.636758 |
| $a_5$ | 0 | -0.011739 | 0 |
| $a_6$ | -0.555377 | 0 | 0.183205 |
| $a_7$ | 0 | -0.396909 | 0 |
| $a_8$ | 0.146591 | 0 | -0.253667 |
| $a_9$ | 0 | 0.298239 | 0 |

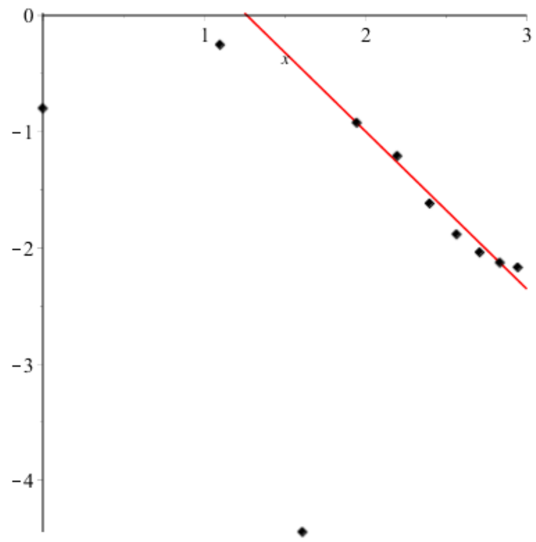
| $a_n$ | Function 1 | Function 2 | Function 3 |
|---|---|---|---|
| $a_{10}$ | -0.020277 | 0 | 0.135478 |
| $a_{11}$ | 0 | -0.198222 | 0 |
| $a_{12}$ | 0.001767 | 0 | -0.069607 |
| $a_{13}$ | 0 | 0.151939 | 0 |
| $a_{14}$ | -0.000107 | 0 | 0.039138 |
| $a_{15}$ | 0 | -0.130133 | 0 |
| $a_{16}$ | 0.000005 | 0 | -0.021989 |
| $a_{17}$ | 0 | 0.119113 | 0 |
| $a_{18}$ | 0 | 0 | 0.010027 |
| $a_{19}$ | 0 | -0.114332 | 0 |

### Plots

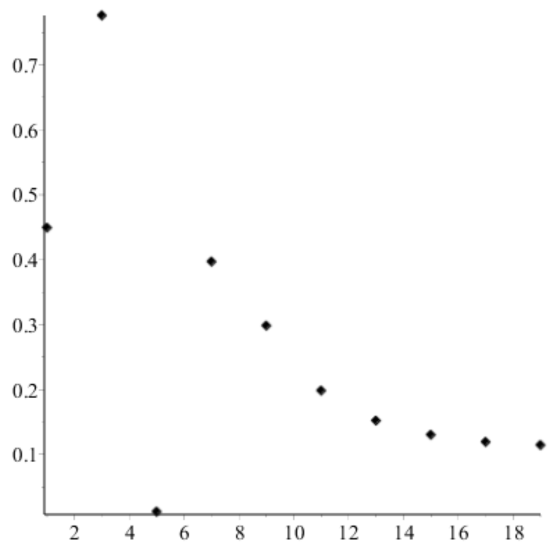
Semi-log plot of the magnitude of Function 1's Chebyshev Coefficients



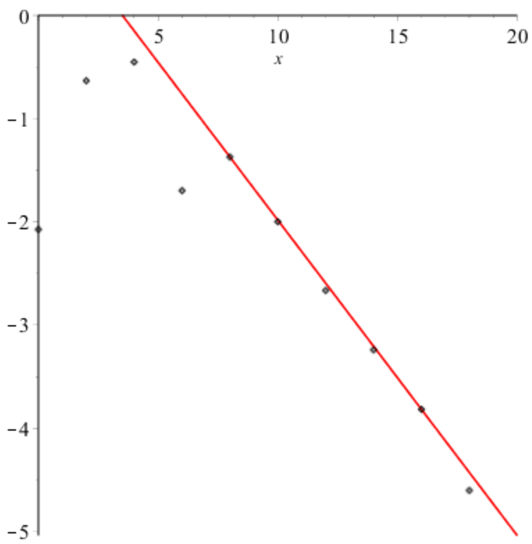
Log-log plot of the magnitude of Function 2's Chebyshev Coefficients




Plot of the magnitude of Function 2's Chebyshev Coefficients



Semi logarithmic plot of the magnitude of Function 3's Chebyshev Coefficients




## Analysis

Given the rather obvious linearity of function 1 and 3's semi-log plots, the behavior appears to be exponential, and without even plotting the coefficients, it is clear from the data that the coefficients $a_n$ are asymptotic to 0 as $n$ increases.

Function 2, however, has power-law behavior, as evidenced by the linearity of its log-log plot, and asymptotic to a value around .1, as shown in the plot of the data.

## Problem 7

### Data Tables

$T_{10}(x)$

| $n$ | $a_{T'_{10}}$ | $a_{T''_{10}}$ | $a_{\int T_{10}}$ |
|---|---|---|---|
| 0 | 0 | 500 | 0 |
| 1 | 20 | 0 | 0 |
| 2 | 0 | 960 | 0 |
| 3 | 20 | 0 | 0 |
| 4 | 0 | 840 | 0 |
| 5 | 20 | 0 | 0 |
| 6 | 0 | 640 | 0 |
| 7 | 20 | 0 | 0 |
| 8 | 0 | 360 | 0 |
| 9 | 20 | 0 | $\frac{1}{18}$ |
| 10 | 0 | 0 | 0 |
| 11 | 0 | 0 | $\frac{1}{22}$ |

$T_{15}(x)$

| $n$ | $a_{T'_{15}}$ | $a_{T''_{15}}$ | $a_{\int T_{15}}$ |
|---|---|---|---|
| 0 | 15 | 0 | 0 |
| 1 | 0 | 3360 | 0 |
| 2 | 30 | 0 | 0 |
| 3 | 0 | 3240 | 0 |
| 4 | 30 | 0 | 0 |
| 5 | 0 | 3000 | 0 |
| 6 | 30 | 0 | 0 |
| 7 | 0 | 2640 | 0 |
| 8 | 30 | 0 | 0 |
| 9 | 0 | 2160 | 0 |
| 10 | 30 | 0 | 0 |
| 11 | 0 | 1560 | 0 |
| 12 | 30 | 0 | 0 |
| 13 | 0 | 840 | 0 |
| 14 | 30 | 0 | $\frac{1}{28}$ |
| 15 | 0 | 0 | 0 |
| 16 | 0 | 0 | $\frac{1}{32}$ |

$T_{20}(x)$

| $n$ | $a_{T'_{20}}$ | $a_{T''_{20}}$ | $a_{\int T_{20}}$ |
|---|---|---|---|
| 0 | 0 | 4000 | 0 |
| 1 | 40 | 0 | 0 |
| 2 | 0 | 7920 | 0 |
| 3 | 40 | 0 | 0 |
| 4 | 0 | 7680 | 0 |
| 5 | 40 | 0 | 0 |
| 6 | 0 | 7280 | 0 |
| 7 | 40 | 0 | 0 |
| 8 | 0 | 6720 | 0 |
| 9 | 40 | 0 | 0 |
| 10 | 0 | 6000 | 0 |
| 11 | 40 | 0 | 0 |
| 12 | 0 | 5120 | 0 |
| 13 | 40 | 0 | 0 |
| 14 | 0 | 4080 | 0 |
| 15 | 40 | 0 | 0 |
| 16 | 0 | 2880 | 0 |
| 17 | 40 | 0 | 0 |
| 18 | 0 | 1520 | 0 |
| 19 | 40 | 0 | $\frac{1}{38}$ |
| 20 | 0 | 0 | 0 |
| 21 | 0 | 0 | $\frac{1}{42}$ |

## Analysis

The tables above list the coefficients $a_n$ of $T_n$ for all relevant $n$ for the first two derivatives and integral of each $T_n$. The patterns in the data can be explained by

$$T_n(x) = \frac{1}{2}\left(\frac{1}{n+1}T'_{n+1}(x) - \frac{1}{n-1}T'_{n-1}(x)\right)$$

which gives us

$$\int T_n(x) = \frac{1}{2}\left(\frac{1}{n+1}T_{n+1}(x) - \frac{1}{n-1}T_{n-1}(x)\right)$$

and

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

which gives

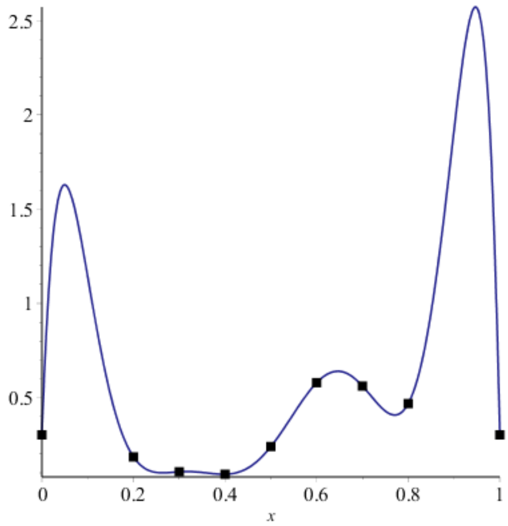
$$T'_n(x) = \frac{1}{1-x^2}\left(-nxT_n(x) + nT_{n-1}(x)\right)$$

which we can of course take the derivative of to obtain higher order derivatives of $T_n$.

## Problem 8

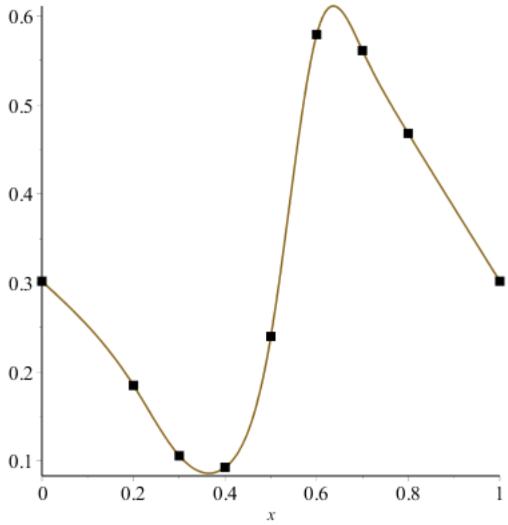
### Lagrange Interpolation

$L_8(x) = -10717.01390x^8 + 41523.43743x^7 - 66130.60764x^6 + 55973.20311x^5 - 27150.02639x^4 + 7547.542702x^3 - 1112.916584x^2 + 66.38116670x + .302$




### Cubic Spline Interpolation

$$f(x) = \begin{cases} .302 - .473x - 2.802x^3 & x < 0.2 \\ .12974 + 2.11093x - 12.91926x^2 + 18.72996x^3 & x < 0.3 \\ .47410379 - 1.3326681x - 1.44057x^2 + 5.975866x^3 & x < 0.4 \\ -2.430901 + 20.45487x - 55.90942x^2 + 51.36657x^3 & x < 0.5 \\ 26.420189 - 152.651673x + 290.3036659x^2 - 179.442153x^3 & x < 0.6 \\ -37.6981566 + 167.940056x - 244.015883x^2 + 117.40204x^3 & x < 0.7 \\ 5.37168 - 16.644978x + 19.67702x^2 - 8.1660098x^3 & x < 0.8 \\ 1.257759 - 1.2177579x + .39299789x^2 - .130999x^3 & x \geq .8 \end{cases}$$




## Analysis

The cubic spline interpolation produces a much more convincing interpolation between the data points - the Lagrange polynomial suffers from very erratic behavior at the endpoints (similar to Runge's phenomenon), and indeed $f(.1) = 0.251906420711722$ is a much more consistent value with the data than $L_8(.1) = 1.14113749$.