

Numerical Analysis Homework 2

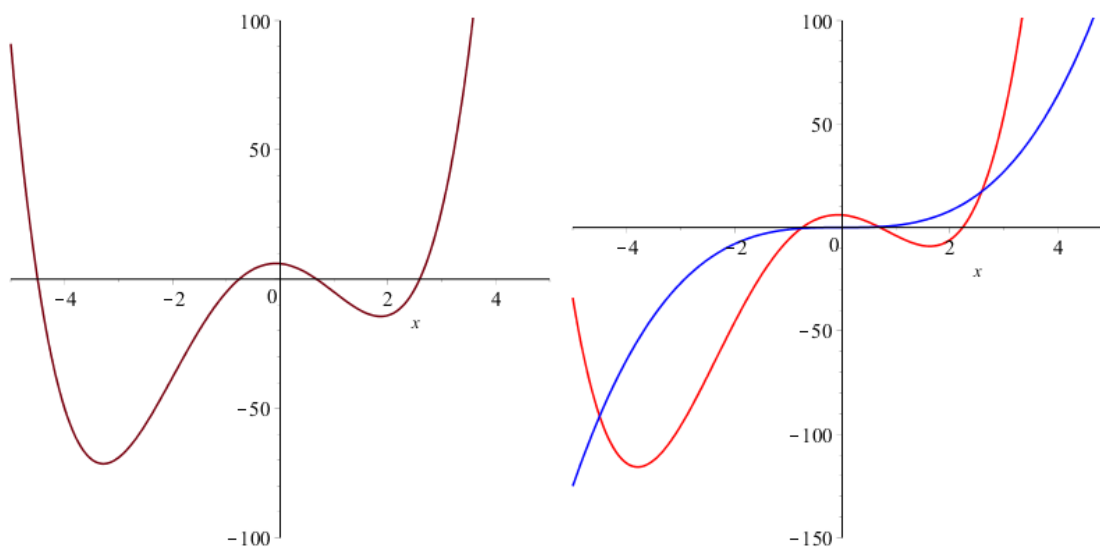
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Problem 1

No, a degree 3 polynomial cannot intersect a degree 4 polynomial in exactly 5 points - let $f(x)$ be a degree 3 polynomial, and $g(x)$ be degree 4. Any intersection of f and g must be a root of $f - g$ and $g - f$, which are degree 4 polynomials, and therefore may have at most 4 real roots.

It is, however, possible for f and g to intersect at exactly four points. Consider $f(x) = x^3$, and $g(x) = x^4 + 3x^3 - 12x^2 - 2x + 6$. Their difference, $x^4 + 2x^3 - 12x^2 - 2x + 6$, is plotted below and clearly has exactly 4 distinct real roots. f and g can intersect only at those roots, and must intersect at them, so thus f and g intersect exactly 4 times, as shown on the second plot.



Problem 2

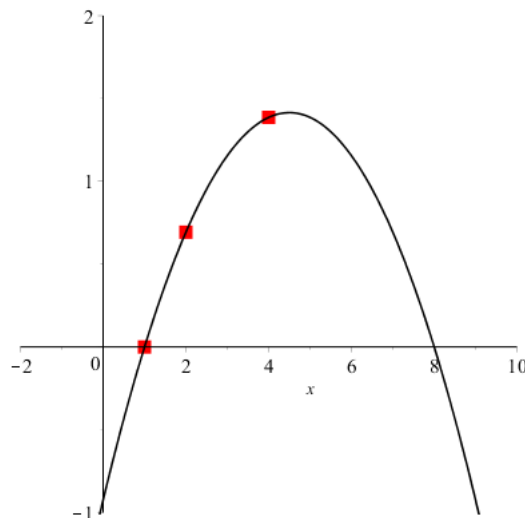
Using the formula

$$P_n(x) = \sum_{j=1}^n y_j \prod_{k=1, k \neq j}^n \frac{x - x_k}{x_j - x_k}$$

we obtain

$$\begin{aligned}
P_2(x) &= \frac{(x-2)(x-4)}{3} \cdot 0 + \frac{(x-1)(x-4)}{-2} \cdot \ln 2 + \frac{(x-1)(x-2)}{6} \cdot \ln 4 \\
P_2(x) &= \frac{-(x-1)(x-4)}{2} \cdot \ln 2 + \frac{(x-1)(x-2)}{3} \cdot \ln 2 \\
P_2(x) &= \frac{-\ln 2}{6} x^2 + \frac{3 \ln 2}{2} x - \frac{4 \ln 2}{3}
\end{aligned}$$

The graph of this polynomial fit is shown below.



Using this approximation, we obtain $\ln 3 \approx P_2(3) = \frac{-9 \ln 2}{6} + \frac{9 \ln 2}{2} - \frac{4 \ln 2}{3} = \frac{5}{3} \ln 2 \approx 1.1552453$. Using the formula

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| \cdot \max_{[2,4]} \left| \prod_{i=0}^n (x - x_i) \right|$$

with $f(x) = \ln x$, $f^{(3)}(x) = \frac{2}{x^3}$, we obtain

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{1}{3(\xi)^3} \right| \cdot \max_{[2,4]} |x^3 - 7x^2 + 14x - 8|$$

Because $f^{(3)}(\xi)$ is strictly decreasing and non-negative on our interval, we know the maximum on $[2, 4]$ is $\frac{1}{24}$ at 2. Using the derivative of $x^3 - 7x^2 + 14x - 8$ with the quadratic formula, we find that the maximum absolute value on $[2, 4]$ is $-\frac{2}{27}(10 + 7\sqrt{7})$ at $x = \frac{7}{3} + \frac{\sqrt{7}}{3}$, or approximately 2.11261.

Multiplying these values, we obtain $|f(x) - P(x)| \leq .0880255$.

Comparing our value of 1.1552453 to the actual value of $\ln 3 \approx 1.0986123$, we have $|f(x) - P_2(x)| = .056630 < .0880255$, as expected.

Problem 3

Using the formulas $B(0) = P_0$ and $B(1) = P_3$, we obtain $P_0 = (1, 1)$ and $P_3 = (9, 1)$. We then calculate $B'(t) = (x'(t), y'(t)) = (12t + 6t^2, 3t^2 - 1)$, and note that because in general $B'(t) = 3(1-t)^2(P_1 - P_0) + 6(1-t)t(P_2 - P_1) + 3t^2(P_3 - P_2)$, $B'(0) = 3(P_1 - P_0)$ and $B'(1) = 3(P_3 - P_2)$. Thus we have $P_1 = (1, \frac{2}{3})$ and $P_2 = (3, \frac{1}{3})$. This gives us the complete set of control points for the curve, namely

$$\{(1, 1), (1, \frac{2}{3}), (3, \frac{1}{3}), (9, 1)\}$$

Problem 4

Natural Endpoint Condition

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 - 0 & 2(2 - 0) & 2 - 1 & 0 \\ 0 & 2 - 1 & 2(3 - 1) & 3 - 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{2-5}{2-1} - \frac{5-3}{1-0} \right) \\ 6 \left(\frac{1-2}{3-2} - \frac{2-5}{2-1} \right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

Curvature-Adjusted Endpoint Condition

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} k_1 \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ k_n \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 - 0 & 2(2 - 0) & 2 - 1 & 0 \\ 0 & 2 - 1 & 2(3 - 1) & 3 - 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} k_n \\ 6 \left(\frac{2-5}{2-1} - \frac{5-3}{1-0} \right) \\ 6 \left(\frac{1-2}{3-2} - \frac{2-5}{2-1} \right) \\ k_n \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} k_1 \\ -30 \\ 12 \\ k_n \end{bmatrix}$$

Clamped Endpoint Condition

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 6 \left(\frac{y_2 - y_1}{x_2 - x_1} - c_n \right) \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ 6 \left(c_n - \frac{y_4 - y_3}{x_4 - x_3} \right) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 - 0 & 2(2 - 0) & 2 - 1 & 0 \\ 0 & 2 - 1 & 2(3 - 1) & 3 - 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 6 \left(\frac{5-3}{1-0} - c_n \right) \\ 6 \left(\frac{2-5}{2-1} - \frac{5-3}{1-0} \right) \\ 6 \left(\frac{1-2}{3-2} - \frac{2-5}{2-1} \right) \\ 6 \left(c_n - \frac{1-2}{3-2} \right) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 12 - 6c_n \\ -30 \\ 12 \\ 6c_n + 1 \end{bmatrix}$$

Parabolically Terminated Endpoint Condition

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

Not-a-Knot Endpoint Condition

$$\begin{bmatrix} x_3 - x_2 & -(x_3 - x_1) & x_2 - x_1 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & x_4 - x_3 & -(x_4 - x_2) & x_3 - x_2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 2 & 0 \\ 1-0 & 2(2-0) & 2-1 & 0 \\ 0 & 2-1 & 2(3-1) & 3-2 \\ 0 & -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{2-5}{2-1} - \frac{5-3}{1-0} \right) \\ 6 \left(\frac{1-2}{3-2} - \frac{2-5}{2-1} \right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 2 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

Problem 5

- i.)
- ii.)
- iii.)

Problem 6

- i.)
- ii.)
- iii.)

Problem 7

Problem 8

- i.)
- ii.)
- iii.)