Numerical Analysis (MATH-411)
Homework #3: Due Thursday, 11/10/2016

1. **Finite differencing for curved coordinates** : Consider the case for data collected on a set of evenly spaced gridpoints in cylindrical coordinates $\{r, \theta\}$. Since we can convert back and forth with Cartesian coordinates, we can take derivatives in either coordinate system. Let's construct the 4-point differencing operator for $\frac{\partial^2 f}{\partial x \partial y}$ operating near a point $\{R, 0\} = (R, 0)$ without going through the coordinate transformation. Call the operator D_{xy}, and we can assume that

$$D_{xy}[f] = Af\{R - r, \theta\} + Bf\{R + r, \theta\} + Cf\{R - r, -\theta\} + Df\{R + r, -\theta\}$$

To solve for A, B, C, D, we note that we can convert the four evaluation points back to (x, y), and then plug in values for the function

$$f(x, y) = f_0 + gx + hy + kxy$$

We need to determine the coefficients such that $D_{xy}[f] = k$ in this example. Zeroing out the constant terms implies that $A + B + C + D = 0$, for instance. Can you work out analytically the values of the coefficients as a function of R, r, θ?

Confirm your values are correct for the point $(5, 0)$ and the function $f = r^3(\sin\theta + \cos\theta) = (x + y)(x^2 + y^2)$ by evaluating the differencing operator for smaller values of r and θ and confirming the results converge to the analytical expression.

2. **Numerical differentiation and integration in the presence of noise**

The standard rule of thumb is that (uncorrelated) noise is very bad for numerical differentiation, and less so for numerical integration. Consider the function $f(x) = \sin(x)e^{\cos x}$ for $0 \leq x \leq \pi$. You will want to consider n evenly spaced gridpoints, and then add to each one a random number ϵ that ranges from $\pm 10^{-3}$ to produce a function we'll call $g(x)$. If you need help with random numbers, either consult your language's reference manual, or ask me for a list of a few thousand I can provide you in a file.

i.) Using centered 5-point stencils (i.e., fourth-order centered for first and second derivatives, see wikipedia on "five-point stencil"), evaluate the first and second derivatives of $g(x)$, the function plus noise. Vary n from something small, ~ 10 points total, to something large enough to establish where you achieve the maximum absolute accuracy compared to the exact result for the derivative function itself, $f'(x) = (\cos x - \sin^2 x)e^{\cos x}$. For a couple of values of n, plot the absolute and relative accuracy of the measurement.

ii.) Using Simpson's rule on the same data as in part i.) evaluate $\int_0^\pi g(x)dx$ for different values of n, beginning from $n = 4$. Make a plot of the absolute error vs. the number of points you use, and determine how it scales with n for both small and large values of n.

3. **Integration: Cepheid Lightcurve**

Cepheids are stars that exhibit variations in their apparent magnitude due to radial pulsations driven by varying opacity (the κ-Mechanism). Astronomers have taken the following lightcurve data from a quickly varying (1-day period) Cepheid, whose apparent magnitude is equal to its absolute magnitude because it is 10pc away:

time (days)	apparent magnitude (M)
0.0	0.302
0.1	0.264
0.2	0.185
0.3	0.106
0.4	0.093
0.5	0.24
0.6	0.579
0.7	0.561
0.8	0.468
0.9	0.387
1.0	0.302

i.) Noting that

$$L = 85.5 \times 10^{-0.4M} L_\odot$$

where $L_\odot$ is the sun's bolometric luminosity, use Trapezoid's and Simpson's rule to estimate how much energy the Cepheid emitted during the day it was observed.

ii.) While you are at it, use a natural cubic spline to interpolate the data points (in terms of magnitudes), convert this to a function of luminosity over time, and then integrate to find the total emitted energy. Next, convert the given magnitudes to luminosities first, and then cubic spline the luminosities and integrate to find the emitted energy.

4. **Integration: Planck's law**

We've worked with Planck's law for the energy spectrum emitted by a blackbody, but since $E = h\nu$ for photons, we can also consider the number density per unit frequency, in units of photons/volume/frequency interval:

$$n_\nu(\nu, T) = \frac{1}{h\nu} u_\nu(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

The total number density of photons is $\int_0^\infty n_\nu d\nu$. Similarly, we can also define

$$n_\lambda = \frac{8\pi}{\lambda^4} \frac{1}{e^{hc/\lambda kT} - 1}$$

Evaluate this integral numerically, in dimensionless form if you prefer. I recommend Simpson's rule, but any method we discussed is fine.

You will want to rescale the variable somehow, either by a variable transformation $\nu = \nu(\xi)$ where $0 \le \xi \le$ some number corresponds to $0 \le \nu \le \infty$. What are:

1. The median photon energy $\nu_{1/2}$ defined such that

$$\int_0^{\nu_{1/2}} n_\nu d\nu = \frac{1}{2}\int_0^\infty n_\nu d\nu$$

as a function of T? Note that the median wavelength $\lambda_{1/2}$ automatically satisfies $\lambda_{1/2}\nu_{1/2} = c$.

2. The *mean* photon energy $\bar{\nu}$:

$$\bar{\nu} \equiv \frac{\int_0^\infty \nu n_\nu d\nu}{\int_0^\infty n_\nu d\nu}$$

and the corresponding mean wavelength, both as a function of temperature. Do these multiply to c?

3. The standard deviation in wavelength σ_ν:

$$\sigma_\nu \equiv \left[\frac{\int_0^\infty (\nu - \bar{\nu})^2 n_\nu d\nu}{\int_0^\infty n_\nu d\nu}\right]^{1/2}$$

5. **Romberg integration:**

Compute $R_{3,3}$ for the following integrals using Romberg integration, and compare your result to the exact answer:

$$
\begin{aligned}
&a.) \quad \int_{-1}^{1} \cos^2 x dx \\
&b.) \quad \int_{-3/4}^{3/4} x \ln(x+1) dx \\
&c.) \quad \int_{1}^{4} (\sin^2 x - 2x\sin x + 1) dx \\
&d.) \quad \int_{e}^{2e} \frac{dx}{x \ln x}
\end{aligned}
$$

6. **Gaussian quadrature:**

Determine constants a, b, c, d, e such that the following quadrature formula has order 4:

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(0) + cf(1) + df'(-1) + ef'(1)$$