Numerical Analysis Homework 2

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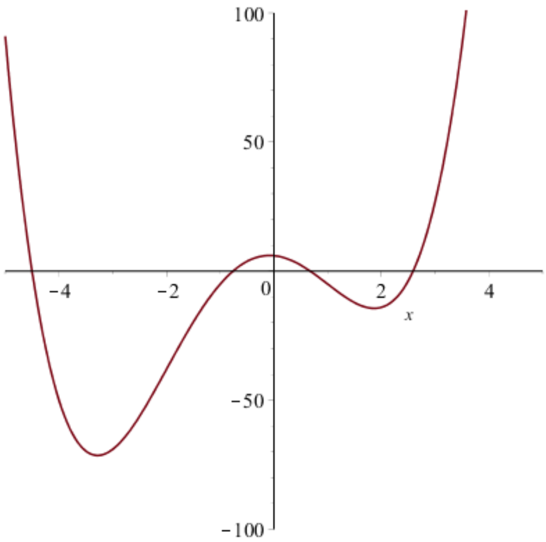
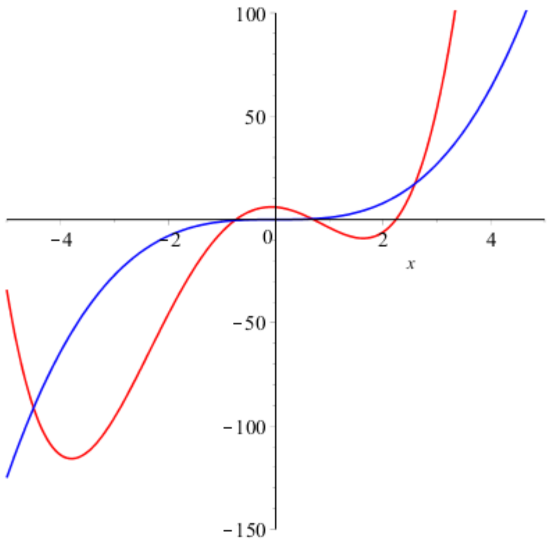
Problem 1

No, a degree 3 polynomial cannot intersect a degree 4 polynomial in exactly 5 points - let $f(x)$ be a degree 3 polynomial, and $g(x)$ be degree 4. Any intersection of f and g must be a root of $f-g$ and $g-f$, which are degree 4 polynomials, and therefore may have at most 4 real roots.

It is, however, possible for f and g to intersect at exactly four points. Consider $f(x) = x^3$, and $g(x) = x^4+3x^3-12x^2-2x+6$. Their difference, $x^4+2x^3-12x^2-2x+6$, is plotted below and clearly has exactly 4 distinct real roots. f and g can intersect only at those roots, and must intersect at them, so thus f and g intersect exactly 4 times, as shown on the second plot.





Problem 2

Using the formula

$$P_n(x) - \sum_{j=1}^{n} y_j \prod_{k=1,k\neq j}^{n} \frac{x-x_k}{x_j-x_k}$$

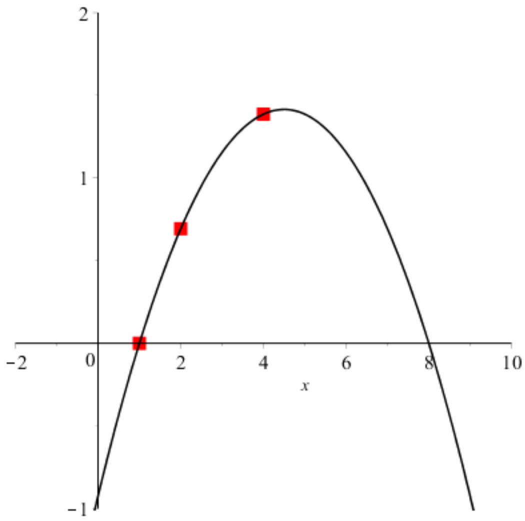
we obtain

$$P_2(x) = \frac{(x-2)(x-4)}{3} \cdot 0 + \frac{(x-1)(x-4)}{-2} \cdot \ln 2 + \frac{(x-1)(x-2)}{6} \cdot \ln 4$$

$$P_2(x) = \frac{-(x-1)(x-4)}{2} \cdot \ln 2 + \frac{(x-1)(x-2)}{3} \cdot \ln 2$$

$$P_2(x) = \frac{-\ln 2}{6}x^2 + \frac{3\ln 2}{2}x - \frac{4\ln 2}{3}$$

The graph of this polynomial fit is shown below.



Using this approximation, we obtain $\ln 3 \approx P_2(3) = \frac{-9\ln 2}{6} + \frac{9\ln 2}{2} - \frac{4\ln 2}{3} = \frac{5}{3}\ln 2 \approx 1.1552453$. Using the formula

$$|f(x) - P(x)| \le \max_{[2,4]} \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| \cdot \max_{[2,4]} \left| \prod_{i=0}^{n} (x - x_i) \right|$$

with $f(x) = \ln x$, $f^{(3)}(x) = \frac{2}{x^3}$, we obtain

$$|f(x) - P(x)| \le \max_{[2,4]} \left| \frac{1}{3(\xi)^3} \right| \cdot \max_{[2,4]} \left| x^3 - 7x^2 + 14x - 8 \right|$$

Because $f^{(3)}(\xi)$ is strictly decreasing and non-negative on our interval, we know the maximum on $[2,4]$ is $\frac{1}{24}$ at 2. Using the derivative of $x^3 - 7x^2 + 14x - 8$ with the quadratic formula, we find that the maximum absolute value on $[2,4]$ is $-\frac{2}{27}(10 + 7\sqrt{7})$ at $x = \frac{7}{3} + \frac{\sqrt{7}}{3}$, or approximately 2.11261.

Multiplying these values, we obtain $|f(x) - P(x)| \le .0880255$.

Comparing our value of 1.1552453 to the actual value of $\ln 3 \approx 1.0986123$, we have $|f(x) - P_2(x)| = .056630 < .0880255$, as expected.

Problem 3

Using the formulas $B(0) = P_0$ and $B(1) = P_3$, we obtain $P_0 = (1, 1)$ and $P_3 = (9, 1)$. We then calculate $B'(t) = (x'(t), y'(t)) = (12t + 6t^2, 3t^2 - 1)$, and note that because in general $B'(t) = 3(1-t)^2(P_1 - P_0) + 6(1-t)t(P_2 - P_1) + 3t^2(P_3 - P_2)$, $B'(0) = 3(P_1 - P_0)$ and $B'(1) = 3(P_3 - P_2)$. Thus we have $P_1 = (1, \frac{2}{3})$ and $P_2 = (3, \frac{1}{3})$. This gives us the complete set of control points for the curve, namely

$$\{(1,1), (1, \frac{2}{3}), (3, \frac{1}{3}), (9, 1)\}$$

Problem 4

We first note that

$$\begin{array}{rcccl} A_1 &=& \frac{x_2 - x}{x_2 - x_1} &=& 1 - x \\ A_2 &=& \frac{x_3 - x}{x_3 - x_2} &=& 2 - x \\ A_3 &=& \frac{x_4 - x}{x_4 - x_3} &=& 3 - x \end{array}$$

$$\begin{array}{rcccl} B_1 &=& \frac{x - x_2}{x_2 - x_1} &=& x - 1 \\ B_2 &=& \frac{x - x_3}{x_3 - x_2} &=& x - 2 \\ B_3 &=& \frac{x - x_4}{x_4 - x_3} &=& x - 3 \end{array}$$

$$\begin{array}{rcccl} C_1 &=& \frac{(x_2 - x_1)^2}{6}[A_1^3 - A_1] &=& \frac{1}{6}(-x^3 + 3x^2 - 2x) \\ C_2 &=& \frac{(x_3 - x_2)^2}{6}[A_2^3 - A_2] &=& \frac{1}{6}(-x^3 + 6x^2 - 11x + 6) \\ C_3 &=& \frac{(x_4 - x_3)^2}{6}[A_3^3 - A_3] &=& \frac{1}{6}(-x^3 + 9x^2 - 26x + 24) \end{array}$$

$$\begin{array}{rcccl} D_1 &=& \frac{(x_2 - x_1)^2}{6}[B_1^3 - B_1] &=& \frac{1}{6}(x^3 - 3x^2 + 2x) \\ D_2 &=& \frac{(x_3 - x_2)^2}{6}[B_2^3 - B_2] &=& \frac{1}{6}(x^3 - 6x^2 + 11x - 6) \\ D_3 &=& \frac{(x_4 - x_3)^2}{6}[B_3^3 - B_3] &=& \frac{1}{6}(x^3 - 9x^2 + 26x - 24) \end{array}$$

Natural Endpoint Condition

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6\left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1}\right) \\ 6\left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2}\right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 - 0 & 2(2 - 0) & 2 - 1 & 0 \\ 0 & 2 - 1 & 2(3 - 1) & 3 - 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6\left(\frac{2-5}{2-1} - \frac{5-3}{1-0}\right) \\ 6\left(\frac{1-2}{3-2} - \frac{2-5}{2-1}\right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\1&4&1&0\\0&1&4&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_1''\\y_2''\\y_3''\\y_4''\end{bmatrix}=\begin{bmatrix}0\\-30\\12\\0\end{bmatrix}$$

$$\begin{bmatrix}y_1''\\y_2''\\y_3''\\y_4''\end{bmatrix}=\begin{bmatrix}1&-\frac{4}{15}&\frac{1}{15}&0\\0&\frac{4}{15}&-\frac{1}{15}&0\\0&-\frac{1}{15}&\frac{4}{15}&0\\0&\frac{1}{15}&-\frac{4}{15}&1\end{bmatrix}\begin{bmatrix}0\\-30\\12\\0\end{bmatrix}$$

$$\begin{bmatrix}y_1''\\y_2''\\y_3''\\y_4''\end{bmatrix}=\begin{bmatrix}\frac{44}{5}\\-\frac{44}{5}\\\frac{26}{5}\\-\frac{26}{5}\end{bmatrix}$$

Using these derived values of y_i'' with our A,B,C, and D equations above, we have

$$\begin{aligned}y_1^{(cubic)} &= y_1A_1+y_2B_1+y_1''C_1+y_2''D_1 &= \\ y_2^{(cubic)} &= y_2A_2+y_3B_2+y_2''C_2+y_3''D_2 &= \\ y_3^{(cubic)} &= y_3A_3+y_4B_3+y_3''C_3+y_4''D_3 &= \end{aligned}$$

Curvature-Adjusted Endpoint Condition

$$\begin{bmatrix}1&0&0&0\\x_2-x_1&2(x_3-x_1)&x_3-x_2&0\\0&x_3-x_2&2(x_4-x_2)&x_4-x_3\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_1''\\y_2''\\y_3''\\y_4''\end{bmatrix}=\begin{bmatrix}k_1\\6\left(\frac{y_3-y_2}{x_3-x_2}-\frac{y_2-y_1}{x_2-x_1}\right)\\6\left(\frac{y_4-y_3}{x_4-x_3}-\frac{y_3-y_2}{x_3-x_2}\right)\\k_n\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\1-0&2(2-0)&2-1&0\\0&2-1&2(3-1)&3-2\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_1''\\y_2''\\y_3''\\y_4''\end{bmatrix}=\begin{bmatrix}k_n\\6\left(\frac{2-5}{2-1}-\frac{5-3}{1-0}\right)\\6\left(\frac{1-2}{3-2}-\frac{2-5}{2-1}\right)\\k_n\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\1&4&1&0\\0&1&4&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_1''\\y_2''\\y_3''\\y_4''\end{bmatrix}=\begin{bmatrix}k_1\\-30\\12\\k_n\end{bmatrix}$$

$$\begin{aligned}y_1^{(cubic)} &= \\ y_2^{(cubic)} &= \\ y_3^{(cubic)} &= \end{aligned}$$

Clamped Endpoint Condition

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 6\left(\frac{y_2 - y_1}{x_2 - x_1} - c_n\right) \\ 6\left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1}\right) \\ 6\left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2}\right) \\ 6\left(c_n - \frac{y_4 - y_3}{x_4 - x_3}\right) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 - 0 & 2(2 - 0) & 2 - 1 & 0 \\ 0 & 2 - 1 & 2(3 - 1) & 3 - 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 6\left(\frac{5-3}{1-0} - c_n\right) \\ 6\left(\frac{2-5}{2-1} - \frac{5-3}{1-0}\right) \\ 6\left(\frac{1-2}{3-2} - \frac{2-5}{2-1}\right) \\ 6\left(c_n - \frac{1-2}{3-2}\right) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 12 - 6c_n \\ -30 \\ 12 \\ 6c_n + 1 \end{bmatrix}$$

$$\begin{aligned} y_1^{(cubic)} &= \\ y_2^{(cubic)} &= \\ y_3^{(cubic)} &= \end{aligned}$$

Parabolically Terminated Endpoint Condition

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6\left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1}\right) \\ 6\left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2}\right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{aligned} y_1^{(cubic)} &= \\ y_2^{(cubic)} &= \\ y_3^{(cubic)} &= \end{aligned}$$

Not-a-Knot Endpoint Condition

$$\begin{bmatrix} x_3-x_2 & -(x_3-x_1) & x_2-x_1 & 0 \\ x_2-x_1 & 2(x_3-x_1) & x_3-x_2 & 0 \\ 0 & x_3-x_2 & 2(x_4-x_2) & x_4-x_3 \\ 0 & x_4-x_3 & -(x_4-x_2) & x_3-x_2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6\left(\frac{y_3-y_2}{x_3-x_2}-\frac{y_2-y_1}{x_2-x_1}\right) \\ 6\left(\frac{y_4-y_3}{x_4-x_3}-\frac{y_3-y_2}{x_3-x_2}\right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 2 & 0 \\ 1-0 & 2(2-0) & 2-1 & 0 \\ 0 & 2-1 & 2(3-1) & 3-2 \\ 0 & -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6\left(\frac{2-5}{2-1}-\frac{5-3}{1-0}\right) \\ 6\left(\frac{1-2}{3-2}-\frac{2-5}{2-1}\right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 2 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

Problem 5

i.)

ii.)

iii.)

Problem 6

i.)

ii.)

iii.)

Problem 7

Problem 8

i.)

ii.)

iii.)