Numerical Analysis Homework 3

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Problem 1

Computation

$$
\begin{aligned}
&(A + B + C + D) \cdot f_0 = 0 \\
&A + B + C + D = 0
\end{aligned}
$$

$$
\begin{aligned}
&(A + C)(R - r)\cos\theta + (B + D)(R + r)\cos\theta = 0 \\
&(B + D - A - C) = 0 \\
&B + D = A + C \\
&B + D = 0 \\
&B = -D \\
&A = -C
\end{aligned}
$$

$$
\begin{aligned}
&(B + A - C - D)R\cos(\theta) + (B + C - A - D)r\cos\theta = 0 \\
&(2B + 2A)R + (2B - 2A)r = 0 \\
&(B + A)R + A(R - r) = 0 \\
&B = -\frac{A(R - r)}{R + r}
\end{aligned}
$$

$$(A-C+B-D)R^2+(C-A+B-D)2Rr+(A-C+B-D)r^2=\frac{1}{\sin\theta\cos\theta}$$
$$(A+B)R^2+(B-A)2Rr+(A+B)r^2=\frac{1}{2\sin\theta\cos\theta}$$
$$A(R-r)^2-\frac{A(R-r)}{R+r}(R+r)^2=\frac{1}{2\sin\theta\cos\theta}$$
$$A(R-r)(-2Rr+2r^2)=\frac{1}{2\sin\theta\cos\theta}$$

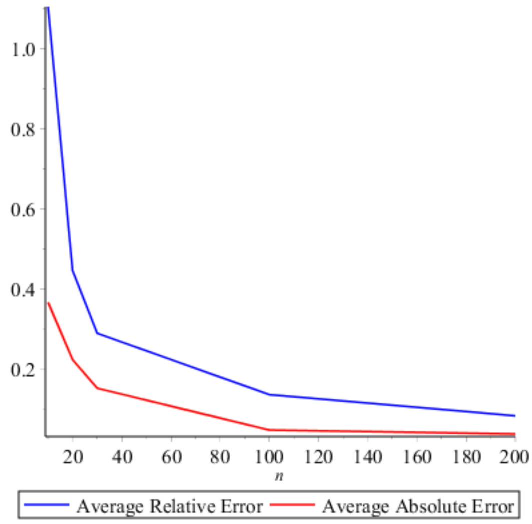
$$A=\frac{1}{2\sin\theta\cos\theta(R-r)(-2Rr+2r^2)}$$
$$B=-\frac{A(R-r)}{R+r}$$
$$C=-A$$
$$D=-B$$

Problem 2

5 Point Stencils

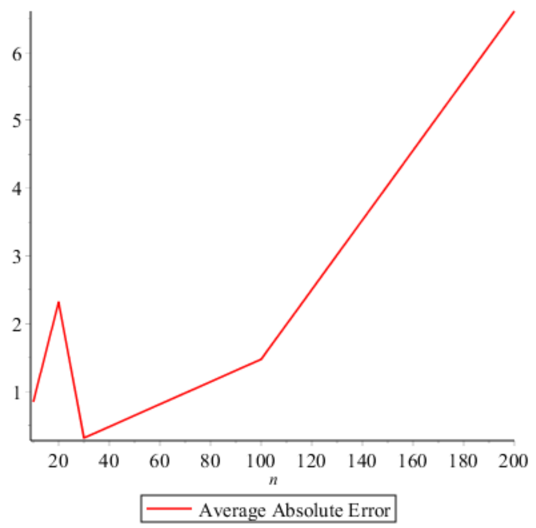
First Derivative

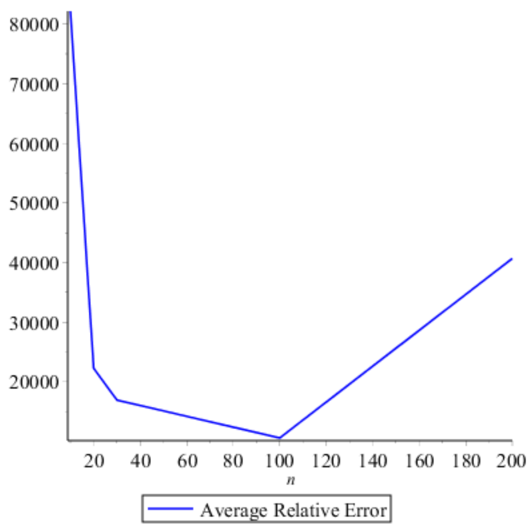
n	Average Absolute Error	Average Relative Error
10	0.367324	1.105306
20	0.223943	0.445905
30	0.152942	0.289964
100	0.048564	0.136987
200	0.038970	0.084145



Second Derivative

n	Average Absolute Error	Average Relative Error
10	0.745691	82151.805057
20	2.315927	22255.208023
30	0.312447	16943.067463
100	1.473741	10594.331845
200	6.611830	40690.086618

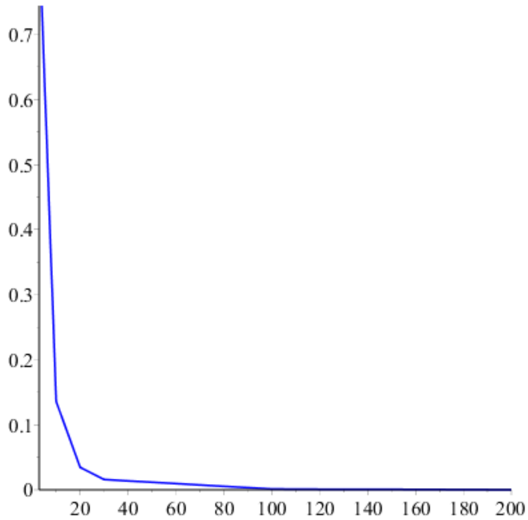




Simpson's Rule

Actual value of $\int_0^\pi \sin x \cdot e^{\cos x} dx$: 2.350402.

n	Simpson's Result	Absolute Error
4	1.606199	.744203
10	2.215001	.135401
20	2.315927	.034475
30	2.334574	.015828
100	2.349178	.001224
200	2.350407	.000005



Analysis

Simpson's method remained fairly stable despite the noise, with the error showing a clear exponential decay as n increased, and achieving 10^{-5} accuracy at $n = 200$.

The first derivative using stencils did a little worse, with the error not only not decreasing as quickly with increasing n, but also seeming to level out in its decay as n becomes large.

The second derivative suffered from huge relative error as the true value of the second derivative became small, and regardless of n, it seems as though the error in the second derivative approximation stayed consistent with (and amplified) the behavior of the noise.

Raw data for this problem can be found in the outputs directory.

Problem 3

Simpson's Method Integration	0.316200
Trapezoid Method Integration	0.318500
Total Emitted Energy from Magnitude Spline	$64.469777 \cdot L_{\odot}$
Total Emitted Energy from Luminosity Spline	$64.476557 \cdot L_{\odot}$

Analysis

Given the number of points, the results of Simpson's Method and the Trapezoid Method are fairly comparable. Converting to luminosity before splining rather than after does not seem to have had a tremendous impact on the approximated result.

Problem 4

Integration

First we define $\hat{\nu} = \frac{h\nu}{kT}$, with $\frac{d\hat{\nu}}{d\nu} = \frac{h}{kT}$. Thus we have

$$\frac{8\pi}{c^3}\int_0^\infty \frac{\nu^2 kT}{(e^{\frac{h\nu}{kT}}-1)h}\cdot\frac{h}{kT}d\nu = \frac{8\pi k^3T^3}{c^3h^3}\int_0^\infty \frac{\hat{\nu}^2}{e^{\hat{\nu}}-1}d\hat{\nu}$$

Then through another coordinate change, with $x = \frac{\hat{\nu}}{\hat{\nu}+1}$ and $\frac{dx}{d\hat{\nu}} = \frac{1}{(\hat{\nu}+1)^2}$, we have

$$\frac{8\pi k^3T^3}{c^3h^3}\int_0^\infty \frac{\hat{\nu}^2(\hat{\nu}+1)^2}{e^{\hat{\nu}}-1}\cdot\frac{1}{(\hat{\nu}+1)^2}d\hat{\nu} = \frac{8\pi k^3T^3}{c^3h^3}\int_0^1 \frac{(\frac{x}{1-x})^2(\frac{x}{1-x}+1)^2}{e^{\frac{x}{1-x}}-1}dx$$

which can be integrated numerically with Simpson's method.
With the bounds set to .0000001 and .9999999 with 40 intervals, Simpson's method yields $\frac{8\pi k^3T^3}{c^3h^3}\cdot 2.404073$ as an approximation of the improper integral.

Median Photon Energy

Using a bisection - like method to test possible values (see problem4.c), the median photon energy is approximately $x_{1/2} = \frac{8\pi k^3T^3}{c^3h^3}\cdot .702091$, $\nu_{1/2} = \frac{8\pi k^3T^3}{c^3h^3}\cdot 2.356729$.

Mean Photon Energy

The mean photon energy is $\bar{x} = \frac{8\pi k^3T^3}{c^3h^3}\cdot .666550$, $\bar{\nu} = \frac{8\pi k^3T^3}{c^3h^3}\cdot 1.998950$.
Through similar changes of variables as above, we find $\int_0^\infty \eta_\lambda d\lambda = -\frac{8\pi h^2c^2}{kT}\int_0^\infty \frac{1}{(e^{\hat{\lambda}}-1)\hat{\lambda}^2}d\hat{\lambda}$, which is equivalent to $-\frac{8\pi h^2c^2}{kT}\int_0^1 \frac{(\frac{y}{1-y}+1)^2}{(e^{\frac{y}{1-y}}-1)\frac{y}{1-y}}dy$ where $y = \frac{\hat{\lambda}}{\hat{\lambda}+1}$.

As y approaches 0, the value of the total wavelength goes to infinity due to the $\frac{1}{\lambda^4}$ term in the original expression. This does not happen nearly as quickly in the $\lambda\eta_\lambda$ expression, and so the mean wavelength tends to 0 as our computational bounds approach 0 and 1. 0 certainly does not multiply to c with any number.

Standard Deviation in Wavelength

The standard deviation in wavelength, $\int_0^1 \frac{(\frac{x}{1-x}-\bar{\nu})^2\eta_x}{\eta_x}dx$, evaluates to $\sigma_x = \frac{8\pi k^3T^3}{c^3h^3}2.680589$, $\sigma_\nu = \frac{8\pi k^3T^3}{c^3h^3}\cdot 1.595029480$.

Problem 5

a.) $\int_{-1}^{1} \cos^2 x dx$
Actual value: 1.4546
Romberg 3,3 Value: 1.452126

b.) $\int_{-\frac{3}{4}}^{\frac{3}{4}} x \ln(x+1) dx$
Actual value: .324332
Romberg 3,3 Value: 0.322879

c.) $\int_{1}^{4} \sin^2 x - 2x \sin x + 1 dx$
Actual value: 1.3668
Romberg 3,3 Value: 1.315255

d.) $\int_{e}^{2e} \frac{1}{x \ln x} dx$
Actual value: .52659
Romberg 3,3 Value: .525648

Problem 6

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(0) + c(1) + df'(-1) + ef'(1)$$

$$\begin{aligned}\int_{-1}^{1} k_4x^4 + k_3x^3 + k_2x^2 + k_1x + k_0 dx = \quad & a(k_4 - k_3 + k_2 - k_1 + k_0) \\ & +bk_0 \\ & +c(k_4 + k_3 + k_2 + k_1 + k_0) \\ & +d(-4k_4 + 3k_3 - 2k_2 + k_1) \\ & +e(4k_4 + 3k_3 + 2k_2 + k_1)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}k_3 + k_1 =& a(k_4 - k_3 + k_2 - k_1 + k_0) + bk_0 + c(k_4 + k_3 + k_2 + k_1 + k_0) \\ & + d(-4k_4 + 3k_3 - 2k_2 + k_1) + e(4k_4 + 3k_3 + 2k_2 + k_1)\end{aligned}$$

$$\begin{aligned}& a + c - 4d + 4e = 0 \\ & -a + c + 3d + 3e = \frac{1}{2} \\ & a + c - 2d + 2e = 0 \\ & -a + c + d + e = 1 \\ & a + b + c = 0\end{aligned}$$

$$
\begin{aligned}
a &= c \\
b &= 0 \\
d &= e
\end{aligned}
$$

$$
\begin{aligned}
2c + 2d &= 1 \\
2c + 6d &= \frac{1}{2} \\
-4d &= \frac{1}{2} \\
d &= \frac{1}{8} \\
c &= \frac{3}{8}
\end{aligned}
$$

$$
\begin{aligned}
a &= -\frac{3}{8} \\
b &= 0 \\
c &= -\frac{3}{8} \\
d &= \frac{1}{8} \\
e &= \frac{1}{8}
\end{aligned}
$$