

Numerical Analysis Homework 2

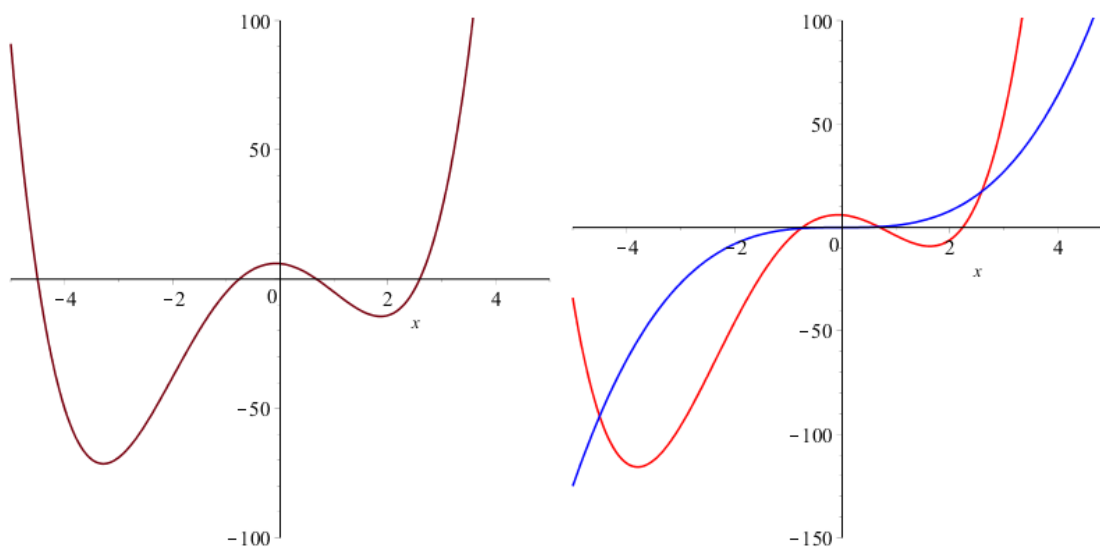
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Problem 1

No, a degree 3 polynomial cannot intersect a degree 4 polynomial in exactly 5 points - let $f(x)$ be a degree 3 polynomial, and $g(x)$ be degree 4. Any intersection of f and g must be a root of $f - g$ and $g - f$, which are degree 4 polynomials, and therefore may have at most 4 real roots.

It is, however, possible for f and g to intersect at exactly four points. Consider $f(x) = x^3$, and $g(x) = x^4 + 3x^3 - 12x^2 - 2x + 6$. Their difference, $x^4 + 2x^3 - 12x^2 - 2x + 6$, is plotted below and clearly has exactly 4 distinct real roots. f and g can intersect only at those roots, and must intersect at them, so thus f and g intersect exactly 4 times, as shown on the second plot.



Problem 2

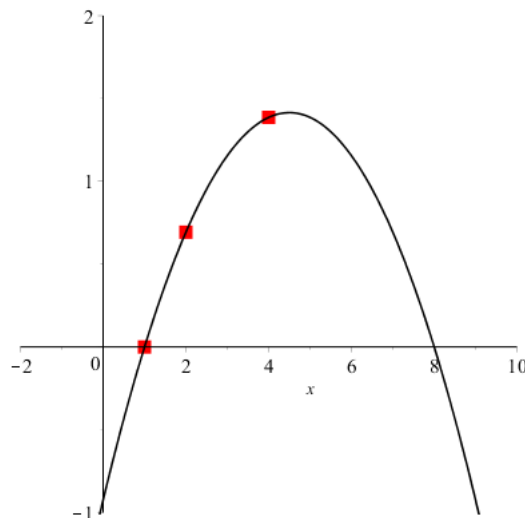
Using the formula

$$P_n(x) = \sum_{j=1}^n y_j \prod_{k=1, k \neq j}^n \frac{x - x_k}{x_j - x_k}$$

we obtain

$$\begin{aligned}
P_2(x) &= \frac{(x-2)(x-4)}{3} \cdot 0 + \frac{(x-1)(x-4)}{-2} \cdot \ln 2 + \frac{(x-1)(x-2)}{6} \cdot \ln 4 \\
P_2(x) &= \frac{-(x-1)(x-4)}{2} \cdot \ln 2 + \frac{(x-1)(x-2)}{3} \cdot \ln 2 \\
P_2(x) &= \frac{-\ln 2}{6}x^2 + \frac{3\ln 2}{2}x - \frac{4\ln 2}{3}
\end{aligned}$$

The graph of this polynomial fit is shown below.



Using this approximation, we obtain $\ln 3 \approx P_2(3) = \frac{-9\ln 2}{6} + \frac{9\ln 2}{2} - \frac{4\ln 2}{3} = \frac{5}{3} \ln 2 \approx 1.1552453$. Using the formula

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| \cdot \max_{[2,4]} \left| \prod_{i=0}^n (x - x_i) \right|$$

with $f(x) = \ln x$, $f^{(3)}(x) = \frac{2}{x^3}$, we obtain

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{1}{3(\xi)^3} \right| \cdot \max_{[2,4]} |x^3 - 7x^2 + 14x - 8|$$

Because $f^{(3)}(\xi)$ is strictly decreasing and non-negative on our interval, we know the maximum on $[2, 4]$ is $\frac{1}{24}$ at 2. Using the derivative of $x^3 - 7x^2 + 14x - 8$ with the quadratic formula, we find that the maximum absolute value on $[2, 4]$ is $-\frac{2}{27}(10 + 7\sqrt{7})$ at $x = \frac{7}{3} + \frac{\sqrt{7}}{3}$, or approximately 2.11261.

Multiplying these values, we obtain $|f(x) - P(x)| \leq .0880255$.

Comparing our value of 1.1552453 to the actual value of $\ln 3 \approx 1.0986123$, we have $|f(x) - P_2(x)| = .056630 < .0880255$, as expected.

Problem 3

Using the formulas $B(0) = P_0$ and $B(1) = P_3$, we obtain $P_0 = (1, 1)$ and $P_3 = (9, 1)$. We then calculate $B'(t) = (x'(t), y'(t)) = (12t + 6t^2, 3t^2 - 1)$, and note that because in general $B'(t) = 3(1-t)^2(P_1 - P_0) + 6(1-t)t(P_2 - P_1) + 3t^2(P_3 - P_2)$, $B'(0) = 3(P_1 - P_0)$ and $B'(1) = 3(P_3 - P_2)$. Thus we have $P_1 = (1, \frac{2}{3})$ and $P_2 = (3, \frac{1}{3})$. This gives us the complete set of control points for the curve, namely

$$\{(1, 1), (1, \frac{2}{3}), (3, \frac{1}{3}), (9, 1)\}$$

Problem 4

We first note that

$$\begin{aligned} A_1 &= \frac{x_2 - x_1}{x_2 - x_1} = 1 - x \\ A_2 &= \frac{x_3 - x_1}{x_3 - x_2} = 2 - x \\ A_3 &= \frac{x_4 - x_1}{x_4 - x_3} = 3 - x \\ B_1 &= \frac{x - x_1}{x_2 - x_1} = x - 0 \\ B_2 &= \frac{x - x_2}{x_3 - x_2} = x - 1 \\ B_3 &= \frac{x - x_3}{x_4 - x_3} = x - 2 \\ C_1 &= \frac{(x_2 - x_1)^2}{6} [A_1^3 - A_1] = \frac{1}{6} (-x^3 + 3x^2 - 2x) \\ C_2 &= \frac{(x_3 - x_2)^2}{6} [A_2^3 - A_2] = \frac{1}{6} (-x^3 + 6x^2 - 11x + 6) \\ C_3 &= \frac{(x_4 - x_3)^2}{6} [A_3^3 - A_3] = \frac{1}{6} (-x^3 + 9x^2 - 26x + 24) \\ D_1 &= \frac{(x_2 - x_1)^2}{6} [B_1^3 - B_1] = \frac{1}{6} (x^3 - x) \\ D_2 &= \frac{(x_3 - x_2)^2}{6} [B_2^3 - B_2] = \frac{1}{6} (x^3 - 3x^2 + 2x) \\ D_3 &= \frac{(x_4 - x_3)^2}{6} [B_3^3 - B_3] = \frac{1}{6} (x^3 - 6x^2 + 11x - 6) \end{aligned}$$

Natural Endpoint Condition

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 - 0 & 2(2 - 0) & 2 - 1 & 0 \\ 0 & 2 - 1 & 2(3 - 1) & 3 - 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{2-5}{2-1} - \frac{5-3}{1-0} \right) \\ 6 \left(\frac{1-2}{3-2} - \frac{2-5}{2-1} \right) \\ 0 \end{bmatrix}$$

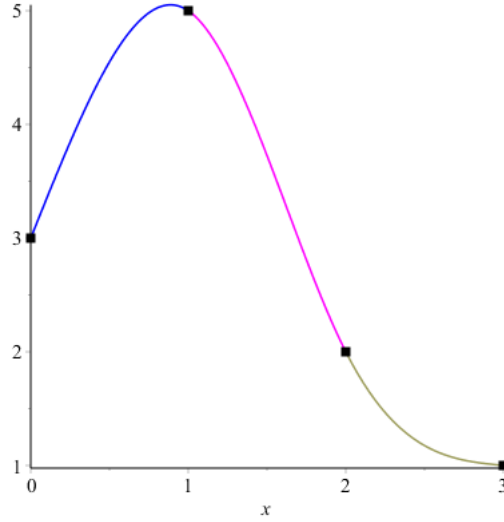
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{4}{15} & \frac{4}{15} & -\frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & -\frac{1}{15} & \frac{4}{15} & -\frac{4}{15} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{44}{5} \\ \frac{26}{5} \\ 0 \end{bmatrix}$$

Using these derived values of y_i'' with our A, B, C , and D equations above, we have

$$\begin{aligned} y_1^{(cubic)} &= y_1 A_1 + y_2 B_1 + y_1'' C_1 + y_2'' D_1 = -\frac{22}{15}x^3 + \frac{52}{15}x + 3 \\ y_2^{(cubic)} &= y_2 A_2 + y_3 B_2 + y_2'' C_2 + y_3'' D_2 = \frac{7}{3}x^3 - \frac{57}{5}x^2 + \frac{223}{15}x - \frac{4}{5} \\ y_3^{(cubic)} &= y_3 A_3 + y_4 B_3 + y_3'' C_3 + y_4'' D_3 = -\frac{13}{15}x^3 + \frac{39}{5}x^2 - \frac{353}{15}x + \frac{124}{5} \end{aligned}$$



Curvature-Adjusted Endpoint Condition

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} k_1 \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ k_n \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-0 & 2(2-0) & 2-1 & 0 \\ 0 & 2-1 & 2(3-1) & 3-2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} k_n \\ 6 \left(\frac{2-5}{2-1} - \frac{5-3}{1-0} \right) \\ 6 \left(\frac{1-2}{3-2} - \frac{2-5}{2-1} \right) \\ k_n \end{bmatrix}$$

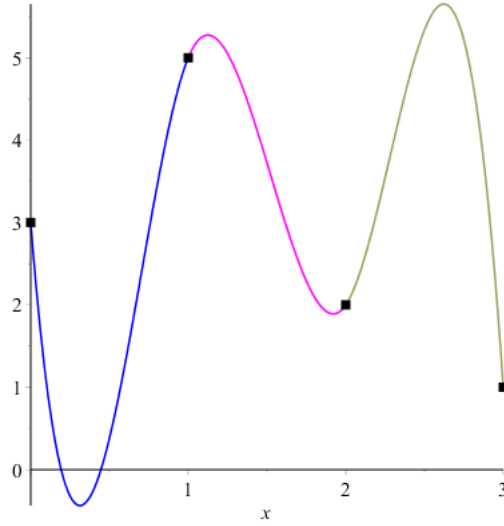
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} k_1 \\ -30 \\ 12 \\ k_n \end{bmatrix}$$

Choosing $k_1 = 100$, $k_2 = -100$, we have

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{4}{15} & \frac{4}{15} & -\frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & -\frac{1}{15} & \frac{4}{15} & -\frac{4}{15} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ -30 \\ 12 \\ -100 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 100 \\ -\frac{632}{15} \\ \frac{578}{15} \\ -100 \end{bmatrix}$$

$$\begin{aligned} y_1^{(cubic)} &= -\frac{1066}{45}x^3 + 50x^2 - \frac{1094}{45}x + 3 \\ y_2^{(cubic)} &= \frac{121}{9}x^3 - \frac{307}{5}x^2 + \frac{3919}{45}x - \frac{512}{15} \\ y_3^{(cubic)} &= -\frac{1039}{45}x^3 + \frac{789}{5}x^2 - \frac{15809}{45}x + \frac{3872}{15} \end{aligned}$$



Clamped Endpoint Condition

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 6 \left(\frac{y_2 - y_1}{x_2 - x_1} - c_n \right) \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ 6 \left(c_n - \frac{y_4 - y_3}{x_4 - x_3} \right) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 - 0 & 2(2 - 0) & 2 - 1 & 0 \\ 0 & 2 - 1 & 2(3 - 1) & 3 - 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 6 \left(\frac{5-3}{1-0} - c_n \right) \\ 6 \left(\frac{2-5}{2-1} - \frac{5-3}{1-0} \right) \\ 6 \left(\frac{1-2}{3-2} - \frac{2-5}{2-1} \right) \\ 6 \left(c_n - \frac{1-2}{3-2} \right) \end{bmatrix}$$

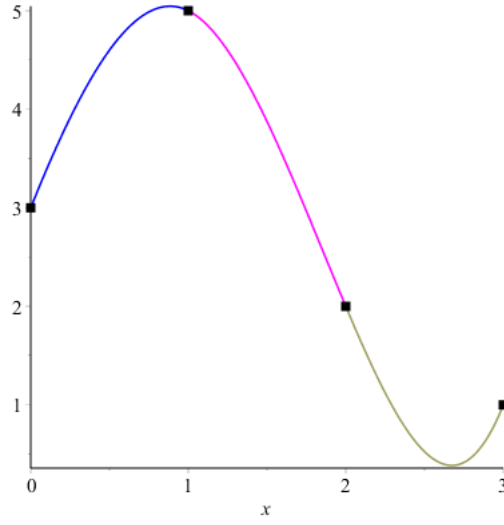
$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 12 - 6c_n \\ -30 \\ 12 \\ 6(c_n + 1) \end{bmatrix}$$

Choosing $c_n = 4$, we have

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \frac{1}{45} \begin{bmatrix} 26 & -7 & 2 & -1 \\ -7 & 14 & -4 & 2 \\ 2 & -4 & 14 & -7 \\ -1 & 2 & -7 & 26 \end{bmatrix} \begin{bmatrix} -12 \\ -30 \\ 12 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} \\ -\frac{36}{5} \\ \frac{6}{5} \\ \frac{72}{5} \end{bmatrix}$$

$$\begin{aligned} y_1^{(cubic)} &= -\frac{4}{5}x^3 - \frac{6}{5}x^2 + 4x + 3 \\ y_2^{(cubic)} &= \frac{7}{5}x^3 - \frac{39}{5}x^2 + \frac{53}{5}x + \frac{4}{5} \\ y_3^{(cubic)} &= -\frac{11}{5}x^3 - \frac{63}{5}x^2 + \frac{101}{5}x - \frac{28}{5} \end{aligned}$$



Parabolically Terminated Endpoint Condition

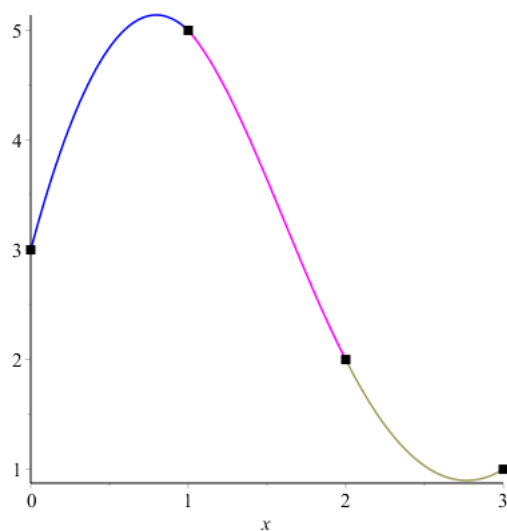
$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 19 & 5 & -1 & 1 \\ -5 & 5 & -1 & 1 \\ 1 & -1 & 5 & -5 \\ 1 & -1 & 5 & 19 \end{bmatrix} \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} -\frac{27}{4} \\ -\frac{27}{4} \\ \frac{15}{4} \\ \frac{15}{4} \end{bmatrix}$$

$$\begin{aligned} y_1^{(cubic)} &= -\frac{27}{8}x^2 + \frac{43}{8}x + 3 \\ y_2^{(cubic)} &= \frac{13}{8}x^3 - \frac{63}{8}x^2 + \frac{37}{4}x + 2 \\ y_3^{(cubic)} &= \frac{15}{8}x^2 - \frac{83}{8}x - \frac{61}{4} \end{aligned}$$



Not-a-Knot Endpoint Condition

$$\begin{bmatrix} x_3 - x_2 & -(x_3 - x_1) & x_2 - x_1 & 0 \\ x_2 - x_1 & 2(x_3 - x_1) & x_3 - x_2 & 0 \\ 0 & x_3 - x_2 & 2(x_4 - x_2) & x_4 - x_3 \\ 0 & x_4 - x_3 & -(x_4 - x_2) & x_3 - x_2 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1} \right) \\ 6 \left(\frac{y_4 - y_3}{x_4 - x_3} - \frac{y_3 - y_2}{x_3 - x_2} \right) \\ 0 \end{bmatrix}$$

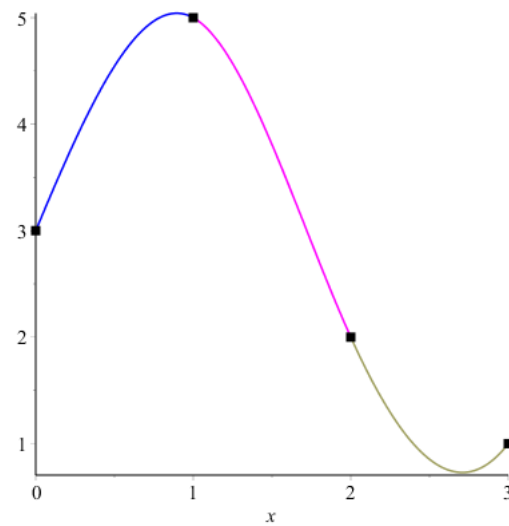
$$\begin{bmatrix} -3 & 1 & 2 & 0 \\ 1 - 0 & 2(2 - 0) & 2 - 1 & 0 \\ 0 & 2 - 1 & 2(3 - 1) & 3 - 2 \\ 0 & -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \left(\frac{2-5}{2-1} - \frac{5-3}{1-0} \right) \\ 6 \left(\frac{1-2}{3-2} - \frac{2-5}{2-1} \right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 2 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \frac{1}{198} \begin{bmatrix} -62 & 12 & 21 & 7 \\ 16 & 48 & -15 & -5 \\ -2 & -6 & 39 & 13 \\ -8 & -24 & 57 & -47 \end{bmatrix} \begin{bmatrix} 0 \\ -30 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \\ y_4'' \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} \\ -\frac{90}{11} \\ \frac{36}{11} \\ \frac{78}{11} \end{bmatrix}$$

$$\begin{aligned}
y_1^{(cubic)} &= -\frac{14}{11}x^3 - \frac{3}{11}x^2 + \frac{39}{11}x + 3 \\
y_2^{(cubic)} &= \frac{21}{11}x^3 - \frac{108}{11}x^2 + \frac{144}{11}x - \frac{2}{11} \\
y_3^{(cubic)} &= \frac{7}{11}x^3 - \frac{24}{11}x^2 - \frac{24}{11}x + 10
\end{aligned}$$



Problem 5

- i.)
- ii.)
- iii.)

Problem 6

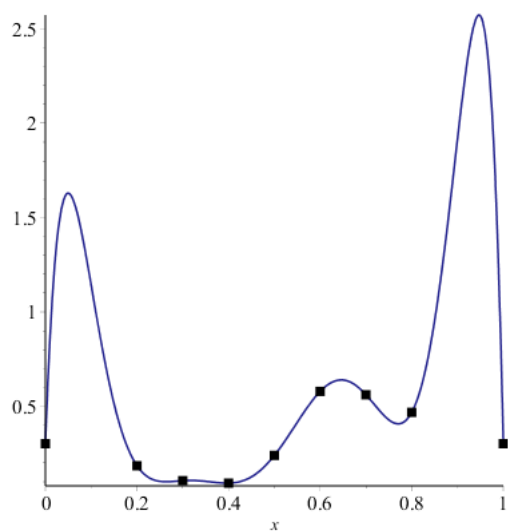
- i.)
- ii.)
- iii.)

Problem 7

Problem 8

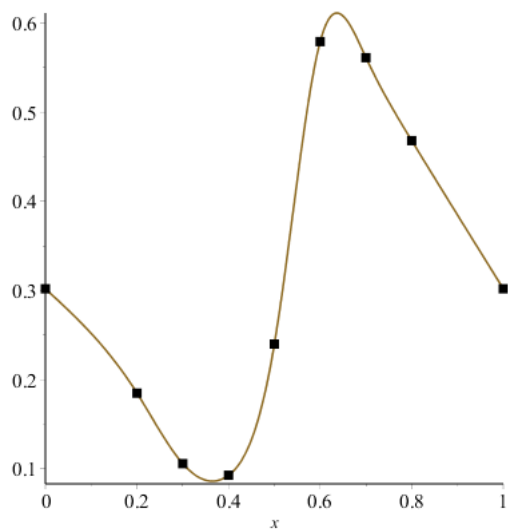
Lagrange Interpolation

$$L_8(x) = -10717.01390x^8 + 41523.43743x^7 - 66130.60764x^6 + 55973.20311x^5 - 27150.02639x^4 + 7547.542702x^3 - 1112.916584x^2 + 66.38116670x + .302$$



Cubic Spline Interpolation

$$f(x) = \begin{cases} .302 - .473x - 2.802x^3 & x < 0.2 \\ .12974 + 2.11093x - 12.91926x^2 + 18.72996x^3 & x < 0.3 \\ .47410379 - 1.3326681x - 1.44057x^2 + 5.975866x^3 & x < 0.4 \\ -2.430901 + 20.45487x - 55.90942x^2 + 51.36657x^3 & x < 0.5 \\ 26.420189 - 152.651673x + 290.3036659x^2 - 179.442153x^3 & x < 0.6 \\ -37.6981566 + 167.940056x - 244.015883x^2 + 117.40204x^3 & x < 0.7 \\ 5.37168 - 16.644978x + 19.67702x^2 - 8.1660098x^3 & x < 0.8 \\ 1.257759 - 1.2177579x + .39299789x^2 - .130999x^3 & x \geq .8 \end{cases}$$



Analysis

The cubic spline interpolation produces a much more convincing interpolation between the data points - the Lagrange polynomial suffers from very erratic behavior at the endpoints, and indeed $f(.1) = 0.251906420711722$ is a much more consistent value with the data than $L_8(.1) = 1.14113749$.