Numerical Analysis Homework 2

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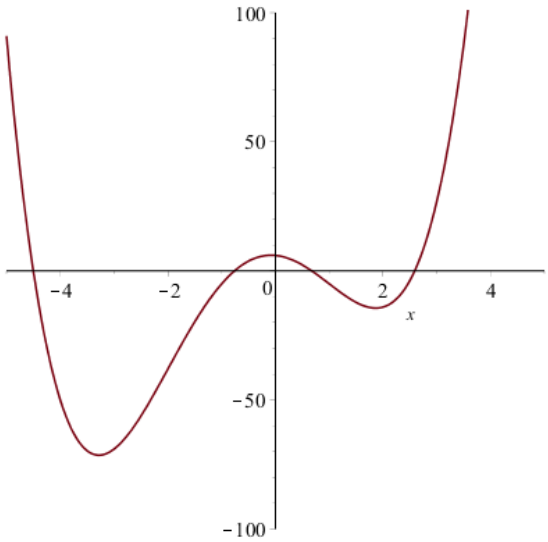
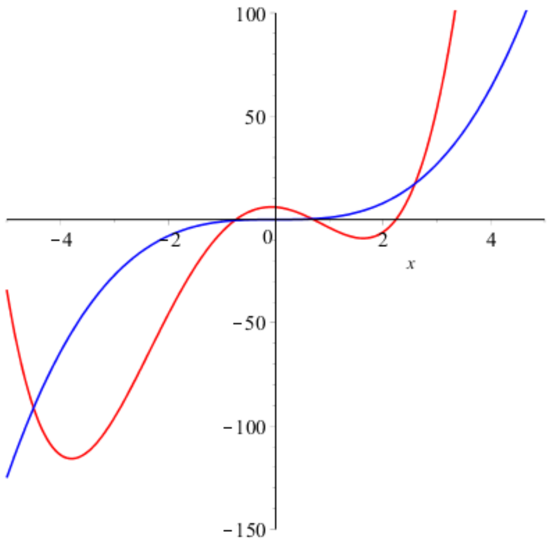
Problem 1

No, a degree 3 polynomial cannot intersect a degree 4 polynomial in exactly 5 points - let $f(x)$ be a degree 3 polynomial, and $g(x)$ be degree 4. Any intersection of f and g must be a root of $f - g$ and $g - f$, which are degree 4 polynomials, and therefore may have at most 4 real roots.

It is, however, possible for f and g to intersect at exactly four points. Consider $f(x) = x^3$, and $g(x) = x^4 + 3x^3 - 12x^2 - 2x + 6$. Their difference, $x^4 + 2x^3 - 12x^2 - 2x + 6$, is plotted below and clearly has exactly 4 distinct real roots. f and g can intersect only at those roots, and must intersect at them, so thus f and g intersect exactly 4 times, as shown on the second plot.





Problem 2

Using the formula

$$P_n(x) - \sum_{j=1}^{n} y_j \prod_{k=1, k\neq j}^{n} \frac{x - x_k}{x_j - x_k}$$

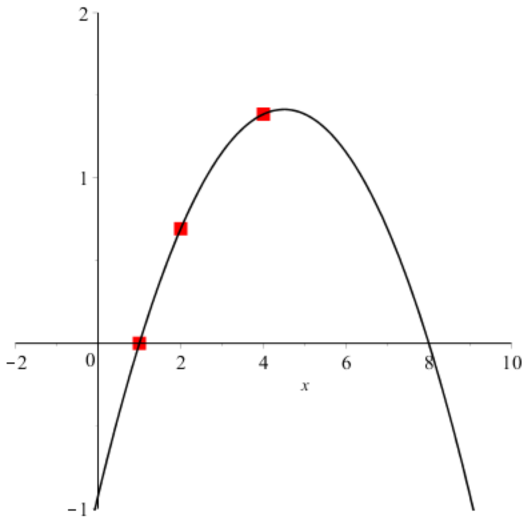
we obtain

$$P_2(x) = \frac{(x-2)(x-4)}{3} \cdot 0 + \frac{(x-1)(x-4)}{-2} \cdot \ln 2 + \frac{(x-1)(x-2)}{6} \cdot \ln 4$$

$$P_2(x) = \frac{-(x-1)(x-4)}{2} \cdot \ln 2 + \frac{(x-1)(x-2)}{3} \cdot \ln 2$$

$$P_2(x) = \frac{-\ln 2}{6}x^2 + \frac{3\ln 2}{2}x - \frac{4\ln 2}{3}$$

The graph of this polynomial fit is shown below.



Using this approximation, we obtain $\ln 3 \approx P_2(3) = \frac{-9\ln 2}{6} + \frac{9\ln 2}{2} - \frac{4\ln 2}{3} = \frac{5}{3}\ln 2 \approx 1.1552453$. Using the formula

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| \cdot \max_{[2,4]} \left| \prod_{i=0}^{n} (x - x_i) \right|$$

with $f(x) = \ln x$, $f^{(3)}(x) = \frac{2}{x^3}$, we obtain

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{1}{3(\xi)^3} \right| \cdot \max_{[2,4]} \left| x^3 - 7x^2 + 14x - 8 \right|$$

Because $f^{(3)}(\xi)$ is strictly decreasing and non-negative on our interval, we know the maximum on $[2,4]$ is $\frac{1}{24}$ at 2. Using the derivative of $x^3 - 7x^2 + 14x - 8$ with the quadratic formula, we find that the maximum absolute value on $[2,4]$ is $-\frac{2}{27}(10 + 7\sqrt{7})$ at $x = \frac{7}{3} + \frac{\sqrt{7}}{3}$, or approximately 2.11261.

Multiplying these values, we obtain $|f(x) - P(x)| \leq .0880255$.

Comparing our value of 1.1552453 to the actual value of $\ln 3 \approx 1.0986123$, we have $|f(x) - P_2(x)| = .056630 < .0880255$, as expected.

Problem 3

Problem 4

Problem 5

i.)

ii.)

iii.)

Problem 6

i.)

ii.)

iii.)

Problem 7

Problem 8

i.)

ii.)

iii.)