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* Binary operation on A

$* : A \times A \to A$

$(a, b) \to c \quad , c \in A$

* Identity element of a set :

$a * e = e * a = a$ where $e \in A$

ex. $A = \mathbb{R}$ , $* = +$

$\therefore \mathbb{R} + 0 = 0 + \mathbb{R} = \mathbb{R}$

$\therefore$ 0 is I.e.

• $A = P(\mathbb{N})$ , $* = \cup$

$\emptyset \cup A = A \cup \emptyset = A$

$\therefore \emptyset$ = I.e.

• $A = P(N)$ , $* = \cap$

$\therefore N \cap A = A \cap N = N$

$\therefore$ N is I.e.

Imp :- $A = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| \begin{array}{l} a \neq 0 \\ a \in R \end{array} \right\}$

* → usual multiplication

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

but $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin A$

$\therefore$ I is not I.e.

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e = Identity element of set.

$$\therefore \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} e & e \\ e & e \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$\begin{bmatrix} 2ae & 2ae \\ 2ae & 2ae \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$2ae = a$

$\therefore e = 1/2$

$\therefore$ I.e. → $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

* Inverse of an elements :

Let $\emptyset \neq A$ any set and $*$ is a binary operation on A , $a \in A$ has inverse say b if. $a \neq 0$.

$a * b = b * a = e$

ex. $A = \mathbb{R}$ , $* = +$

$a \in \mathbb{R}$ , then $b = -a \in \mathbb{R}$

$\therefore a - a = a - a = 0$

as $0 = e$

$\therefore -a$ is inverse.

ex. $A = \mathbb{R}$ , $* = \times$

$\therefore a \in \mathbb{R}$ , then $b = 1/a$

$\therefore a \times \frac{1}{a} = \frac{1}{a} \times a = 1$

$1 = e$

$\therefore 1/a$ is inverse.

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8. $A = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| \begin{matrix} a \neq 0 \\ a \in R \end{matrix} \right\}$

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} x & x \\ x & x \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 2ax & 2ax \\ 2ax & 2ax \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$2ax = 1/2$

$\therefore x = 1/4a$

$\therefore A^{-1} = \begin{bmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{bmatrix}$

* $A = R$, $*$ = binary operation is

$* = a + b + a \cdot b$

then Find identity of A w.r.t $*$ and inverse of identity

$a * \cancel{b}e = \cancel{b}e * a = a$

$a \in R$, $b = 0$

$\therefore a * 0 = 0 * a = a$

as $a + 0 + 0 = 0 + a + 0 = a$

$\therefore e = 0.$

for inverse

$a * b = b * a = e = 0$

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* Field : $(F ; +, \cdot)$

let $\phi \neq F$ be any set with two binary operation addition (+) and multiplication (·) then F is a Field, if it satisfies following properties.

1) $a + (b + c) = (a + b) + c \quad \forall a, b, c \in F$

2) $a + 0 = 0 + a = a \quad \forall a \in F$ (Existance of additive Identity)

3) $a * +(a) = (-a) + a = 0 \quad \forall a \in F$ (additive inverse)

4) $a + b = b + a \quad \forall a, b \in F$

5) $a(b \cdot c) = (a \cdot b)c \quad \forall a, b, c \in F$

6) $a \cdot b = b \cdot a = 1 \quad \forall a \in F$ (Inverse)

7) $a \cdot 1 = 1 \cdot a = a$ (multi Identity)

8) $a(b + c) = ab + ac$, $(a + b)c = ac + bc$ $\} a, b \in F$

9) $ab = ba \quad \forall a, b \in F$

ex. $(Q ; + \cdot) \rightarrow$ field $\}$
$(R ; + \cdot)$
$(C ; + \cdot) \rightarrow$ complex no

grp: set with 1 binary operation
ring: set with 2 — , , — .

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## Alternative defⁿ for Field :

- A set is called fied if conditions :
  1) $a \cdot b = b \cdot a \quad \forall a, b \in \mathbb{R}$
  2) Each non zero element must have multiplicative Inverse $\forall\ 0 \neq a \in \mathbb{F}, \exists\ b \in \mathbb{F}$ such that $a \cdot b = b \cdot a = 1$ where $b = 1/a = a^{-1}$ is called multiplicative inverse. of a.

# $(G, *)$ is a group if it satisfies first 3 prop. of field.

- $a * (b * c) = (a * b) * c$
- $a * e = e * a = a$
- $a * b = b * a = e$

ex. $(\mathbb{R}, +), (\mathbb{Q}, +)$.

Group $\longrightarrow$ Ring $\longrightarrow$ Field

ex. group : $G = \mathbb{R}$, $* = +$

1) $a + (b + c) = (a + b) + c$
2) $a + 0 = 0 + a = a$
3) $a + (-a) = (-a) + a = 0$

all properties satisfied thus it is a group.

* and if 4) $a * b = b * a$ prop. satisfied

ex. $a + b = b + a$

$\Rightarrow$ ~~abelian~~ abelian group.

if 4 prop. satisfies - abelian grp
3 - group ; 1st only - semi group
2 - monoid.

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ex. only group not abelian
- 3 x 3 matrix
  $G = \{ A_n \in M_n(\mathbb{R}) \mid |A| \neq 0 \}$
  * Matrix Multiplication

ex. semi group :
$G = \mathbb{N} \quad * = +$
1) satisfies
2) not as $e \notin \mathbb{N}$

$\therefore (\mathbb{N}, +)$ is a semigroup.

ex. Monoid :
$G = P(A)$ , $* = \cup$
1) $A \cup B \cup C$ ✓
2) $A \cup \phi = \phi \cup A = A$ ✓
3) $A \cup B = B \cup A \neq \phi$

$\therefore (P(N), \cup)$ is a monoid.

Ex. Abelian group :
$G = \mathbb{R} \ : \ * = +$
1) 2) 3) 4) ✓

$\therefore (\mathbb{R}, +)$ is abelian grp.

# Ring $(R ; +, \cdot)$

- Let R be a ~~ring~~ any non-empty set with two binary operation '+' and '·'. is said to be ring if :-
  1) $(\mathbb{R}, +)$ is a grp.
  2) $(R, \cdot)$ is semigrp.
  3) $a(b+c) = ab + ac$
     $(a+b)c = ac + ab$ $\Big\} \forall a, b, c \in R$.

$(Q, R, +)$ ...

Integers

ex. $\mathbb{Z}$ ; $(+, \cdot)$

add, multiply.

→ 1) $(\mathbb{Z}, +)$ is grp.

2) $(\mathbb{Z}, \cdot)$ is s.grp.

3) $a(b+c)$ satisfies

$(a+b)c$

thus it is a ring. but not field.

Group ⟶ Ring ⟶ field

# Finite field :-

$(\mathbb{F}_p ; +_p, \times_p)$

Let us define the set of all reminder. when any integer divide by $n \in N$.

$$\mathbb{Z}_n = \{ \bar{0}, \bar{1}, \bar{2} .. \overline{n-1} \}$$

Now we will define two binary operation on $\mathbb{Z}_n$ namely addition modulas $n(+_n)$ and multiplication Modulus $n(\times_n)$ and are defined as,

$\forall a, b \in \mathbb{Z}_n$

$a +_n b$ = least +ve reminder when we divide $a+b$ by any integer.(n)

$a \times_n b$ = — , — , — , — $a \cdot b$ by $n \in N$.

Ex. $\mathbb{Z}_1 = \{0\}$ → no.

$\mathbb{Z}_2 = \{0, 1\}$ → F.Field

$\mathbb{Z}_3 = \{0, 1, 2\}$

* Field $(\mathbb{F}, +, \cdot)$

1) $ab = ba$

2) $\forall\, 0 \neq a \in \mathbb{F}$, $\exists\, b \in \mathbb{F}$

$(\mathbb{F}, +, \cdot)$

* $\mathbb{Z}_n = \{0, 1, 2, 3 ... n-1\}$

Here the elements 0, 1, --- are class, such that

$[0] = \{ nk \mid nk/x = 0 \}$

$[1]$ = ---- $[2]$ ...

these are classes.

ex. $\mathbb{Z}_2 = \{0, 1\}$

it means on division with 2 the number gives 0 and 1 reminder.

∴ $[0]$ is even no. } as they give that rem.

$[1]$ is odd no.

* $\mathbb{Z}_n = \{0, 1, 2, ... n-1\}$

'$+_n$' = add^n module n.

'$\times_n$' = mult^n module n

are defined as for $a, b \in \mathbb{Z}_n$.

$a +_n b$ := least positive reminder when $a+b$ divided by n.

$a \times_n b$ = — — — —

ex. $Z_1 = \{0\}$
→ it is not field as it doesn't has non zero element

2. $Z_2 = \{1, 0\}$

→ this is field.
becoz $1^{-1} = 1$ as $\frac{1 \times 1}{1} = 1$ which $\in Z_2$

thus its field.

3. $Z_3 = \{0, 1, 2\}$

→ this is a field.

4. $Z_4 = \{0, 1, 2, 3\}$
→ $0 \times_4 0 = 0$
$1 \times_4 1 = 1$ $1^{-1} = 1$
$2 \times_4 2 = 4/4 =$ rem $= 0$
as. $ab = ba = 1$
$\frac{2 \times 2}{4} = \frac{2 \times 2}{4} \% \neq 1$

thus it is not a field.
the inverse must exist. also it should be equal to multiplicative inverse.

5. $Z_5 = \{0, 1, 2, 3, 4\}$
→ $1^{-1} = 1$
$2^{-1} = 3$ as $2 \times 3 / 5 =$
$3^{-1} = 2$
$4^{-1} = 4$
So, $Z_5$ is a field.

inverse,
$a \times_n b = b \times_n a = 1$
then $b$ is inverse of 1
as $\frac{a \cdot b}{n} \% = 1$ then $b = a^{-1}$
$\therefore \frac{2 \times 3}{5} \% = 1$ (modulus)
$\therefore 3 = 2^{-1}$
& $2 = 3^{-1}$

as the inverse $\in F$ thus it is a field.

★ $Z_n$ is a field if and only if $n$ is a prime number.

$Z_{2023}$ as 2023 is not prime.
$Z_{2017}$ as 2017 is prime no. it is field.

Q. $(F, +_n, \times_n)$ is a field ?
$Z_{11} = \{0 ---10\}$

$5 \times_{11} 9 = 1$
as $5 \times 9 / 11 \% = 1$
$\therefore 5^{-1}$ is 9.

Q. ~~What is $(2/11)^{-1}$ in $Z_{11}$.~~

Q. → $(2/5)^{-1}$ in $Z_{11}$
$2/5 = 2 \times 1/5 = 2 \times 5^{-1}$ as $5^{-1} = 9$
$\therefore 2 \times 9 = 18/5 = 3$
$\therefore 3 = 2^{-1}_{11}$
$5/6 = 5 \times 1/6 = 5 \times 6^{-1}$ ( $6^{-1} = \frac{6 \times 2}{11} = \%$
$\therefore 6^{-1} = 2$
$= 5 \times_{11} 2 = \frac{5 \times 2}{11} = \% \boxed{10}$

Q. What is $-2$ in $Z_5$.

$-2 +_n 0$ as $5/5 \% = 0$

$\Rightarrow -2 + 5$

$= 3$

Q. $-2/5$ in $Z_{13}$.

$\rightarrow -2 \times_{13} 5^{-1}$ as $\frac{5 \times x}{13} \% = 1$

$-2 \times_{13} 8$ $\quad = 8$

$\therefore 5^{-1} = 8$

For $-2$, $-2 +_{13} 0$

$-2 + 13$

$= 11$

$\Rightarrow 11 \times_{13} 8$

$\rightarrow \frac{11 \times 8}{13} \% = 10$

H.W. Q. Find additive and multiplicative inverse in $Z_{17}$.

Q. $K_4 = \{ e, a, b, ab \mid a^2 = b^2 = e, a \cdot b = b \cdot a \}$

Forms a group under usual multiplication.

* Vector Space :-

$V(F)$ Let, $\emptyset \neq V$ a set $F$ is a Field for all $u, v, w \in V$ and $k, m \in F$

1) $u + v \in V$

2) $(u+v)+w = u+(v+w)$

3) $u + 0 = 0 + u = u$

4) $u + (-u) = (-u) + u = 0$

5) $u + v = v + u$

1) $k \cdot u \in V$

2) $k \cdot (u+v) = k \cdot u + k \cdot v$

3) $(k+m) \cdot u = ku + mu$

4) $(km) \cdot u = k(mu)$

5) $1 \cdot u = u$

→ $V$ is a vector space over the field $V(F)$.

ex. : $V = \mathbb{C}$ , $F = \mathbb{R}$

→ $\mathbb{C}(\mathbb{R})$ , $\mathbb{R}(\mathbb{Q})$ , $\mathbb{Q}(\mathbb{R})$ , $\mathbb{C}(\mathbb{C})$

X

$M_{n \times n}(\mathbb{R})(\mathbb{R})$ , $P_n(\mathbb{R})(\mathbb{R})$

$\{p(x) = a + a_1 x + a_2 x^2 + a_3 x^3 \dots$ $| a_0 \dots a_n \in \mathbb{R}\}$

**HW** Show that $P_n(\mathbb{R})(\mathbb{R})$ (set of all polynomial of degree $\leq n$) forms a vector space over the field of real no. (prove all 10 props.)

Q. 1 Check the following set is V.S. or not.

⇒ $V = \mathbb{R}^2$ , $F = \mathbb{R}$ the operations defined on $V$ are $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ , $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ and $k \in F$

$u + v = \begin{bmatrix} u_1 + u_2 \\ v_1 + v_2 \end{bmatrix}$ , $k \cdot u = \begin{bmatrix} u_1 \\ 0 \end{bmatrix}$

$\forall u \in \mathbb{R}^2$.

→ 1) $u + v \in \mathbb{R}$ ✓

2) $(u+v)+w = u+(v+w)$ ✓

3) $u + 0 = 0 + u = u$ ✓

4) $u + (-u) = (-u) + u = 0$ ✓

5) $u + v = v + u$ ✓

6) $k \cdot u \in V$ ✓

7) $k \cdot (u+v) = k \cdot u + k \cdot v$ ✓

8) $(k+m) \cdot u = ku + mu$ ✓

9) $(km) \cdot u = k(mu)$ ✓

but $1 \cdot u = \begin{bmatrix} u_1 \\ 0 \end{bmatrix}$ but $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

thus $1 \cdot u \neq u$

thus it is not a vector space.

2. $V = \mathbb{R}^+ = (0, \infty)$ , $F = \mathbb{R}$ the operations on $V$ are defined as $u + v = u \cdot v$ (· usual multiplicat) $k \cdot u = u^k$ is $V$ over $\mathbb{R}$ forms vector space?

1) ✓ 2) ✓ 3) ✓→ $u + 0 = 0 + u$ (0 is aditive inverse here $v + u = v \cdot u$

4) ✓ $u + (-u) = (-u) + u = 0$ ∴ $0 = 1$

∴ $u + 1 = 1 + u = u$

$u \cdot v = v \cdot u = u$

$u = u = u$ as $v = 1$

here $0 = 1$ as aditive in-

$u + (-1)u \to -1 \cdot u = u^{(-1)}$
$= 1/u$

∴ $u + 1/u \to u \cdot 1/u = 1$

as $0 = 1$ ∴ ✓

multiplication:

1) ✓

2) $k(u+v) = k\cdot u + k\cdot v$

$\rightarrow (u+v)^k = u^k + v^k$

$= u^k \cdot v^k$

$u^k + v^k = u^k + v^k$

✓

3) ✓

4) $(km)\cdot u = k\cdot(mu)$

$k\cdot(mu) = k\cdot(m^u)$

$= (m^u)^k$

$= m^{uk}$

$= (uk)\cdot m$ ✓

5) ✓

* Theorem of Vector space:

⇒ Let, V be the vector space over a field f and k be the any scalar then,

1) $0\cdot u = 0$

2) $a\cdot\theta = 0$

3) $(-1)\cdot u = -u$

4) If $k\cdot u = 0$ then $k=0$ or $u=0$

a) $0\cdot u = 0$

$\exists\ 0 \in F$ s.t. $0+0 = 0$.

$\Rightarrow (0+0)u = 0\cdot u$

$\rightarrow 0u + 0\cdot u = 0\cdot u$

for $0\cdot u \in V\ \exists\ -(0\cdot u) \in V$

$0\cdot u + (0\cdot u + (-0\cdot u)) = 0\cdot u + (-0\cdot u)$

$0\cdot u + 0 = 0$

$0\cdot u$ is additive identity of 0.

$0\cdot u = 0$

b] $k\cdot 0 = 0$

$k(0+0) = 0$ Hence proved.

c) $(-1)\cdot u = -u$

Let, $\forall\ u \in V$

$u + (-1)u = 1\cdot u + (-1)u$

$= (1+(-1))\cdot u$

$= 0u$ — [from a]

$u + (-1)u = 0$

$(-1)u$ is additive inverse of u.

$\Rightarrow (-1)\cdot u = -u$.

d) if $k\cdot u = 0$ then $k=0$ or $u=0$

$(p \Rightarrow q)$

→ let $k\cdot u = 0$

or $k \neq 0 \in f$.

$\exists\ k^{-1} \in f$

$\therefore\ k\cdot u \cdot k^{-1} = k^{-1}\cdot 0$

$k^{-1}(k)(u) = 0$

$1\cdot u = 0$

$\boxed{u=0}$

so $u=0$ thus $k=0$ by using prop a.

HW

Q. Let $V = \mathbb{C}$ and $F = \mathbb{C}$ and operations are defined as

$z_1 + z_2$ usual add. of complex no.

$k\cdot z = Re(z)$

is V vector sp ?

$\mathbb{C}$ $\mathbb{C}$

(W)

# Subspace :-

→ let V be the vector space over the field, $\emptyset \neq W \subseteq V$ then W is a subspace of a vector space V if and only if, W itself form a vector space under the operations defined on V.

1> ✗ 6> ✗
2> ✓ 7> ✗
3> ✗ 8> ✗
4> ✗ 9> ✗
5> ✓ 10> ✓

- to prove W as subspace we need to prove 4 prop.

- $(u+v)+w = u+(v+w)$
- $u+v = v+u$
- $1 \cdot u = u$

→ The properties of V are not inherited by W as a subset of V are,

No guarantee
- a) ~~$a \notin$~~ $v+u \notin W$
- b) $u+0 = 0+u = u$
- c) $u+(-u) = (-u)+u = 0$
- d) $k \cdot u \notin W$

rest 6 prop must satisfy by W to be V.S. or sub.S. (They get satisfied automatically) we just need to check above. (4)

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by the thm a,3 get satisfied we need to check only for 1,4 now.

Theorem : ~~Let~~

If W is set of vectors in V and V is a vector space over F, then W is subspace of V iff,

a) $u+v \in W \; \forall \; u,v \in W$
b) $k \cdot u \in W, \forall \; u,v \in W$

proof :- $p \leftrightarrow q \Rightarrow p \rightarrow q$
$q \rightarrow p$

Assume that,
W is a subspace of V,
→ By defn W is vector space.
(a) and (b) must holds

conversly, assume that,
(a), (b) are holds in W.
i.e. $u+v \in W, \forall u,v \in W$
$k * u \in W, \forall k \in F, u \in W$.

As $k \cdot u \in W$
$\Rightarrow 0 \cdot u \neq 0$
$\therefore 0 \cdot u = 0$ $\therefore 0 \in W$
$k \cdot u \in W$

also, $\exists -1 \in F$
$(-1)u = -u$
$-u \in W$.

ex ① $W = \{0\}$ ~~$\&$~~, $F = R$
→ $u, v \in W$
→ $u+v = 0+0 = 0 \in W$
$k \cdot u = k \cdot 0 = 0 \in W$
$\{0\}$ is subspace of V.

we only checked 1,4 not all 10 to check whether subspace or not.

note: ... are the subspaces of $R^2$, $R^3$.

eqn of line: $y = mx + c$ ($R^2$) also plane passing through
$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = t$ ($R^3$) $R^3$ subsp. of $R^3$.
$ax + by + cz = 0$.

Q. Let, $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \mid x = y \right\}$, $V = R^2$

prop 1 $\Rightarrow$ $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$u + v = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$

as $x = y$ $x_3 = y_3$

$\therefore$ verified

as $\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \in W$

$\therefore u + v \in W$

prop 2 $\Rightarrow$ $k.u \in R^2$

$\Rightarrow k \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} kx_1 \\ ky_1 \end{bmatrix} \in R^2 / W$

$\therefore$ verified.

$\therefore$ It forms a subspace.

Q. Show that $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R^3 \mid ax + by + cz = 0 \right\}$

→ forms a subspace.

as $\vec{0} \in \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

also, $u + v \in R^3$

$k.u \in R^3$

hence proved.

Q. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \mid x, y \geq 0 \right\}$

as, $u, v \in W$, $u_1, v_2, u_2, v_1 \in W$

$\therefore$ 1] $u + v \in W$ ✓

but what if.

$k.u \in W$

& $k < 0$

the $k.u \notin W$

$\therefore$ no. subspace.

Q. Let, $V = M_{n \times n}$ is vector space of all $n \times n$ matrices.

$W = \{ A_{n \times n} \mid A^T = A \}$

→ let's take $n = 2$.

$\therefore \begin{bmatrix} u_1 & u_3 \\ u_2 & u_4 \end{bmatrix}$ as $A^T = A$ $u_2 = u_3$

$\therefore \begin{bmatrix} u_1 & u_2 \\ u_2 & u_4 \end{bmatrix} + \begin{bmatrix} v_1 & v_2 \\ v_2 & v_4 \end{bmatrix} \in W$

also,

$k.u \in W$

as,

both are $A^T = A$.

Hence it is a subspace.

Q. $R^2$ is subspace of $R^3$?

→ not.

W is subspace of $R^3$ similar to $R^2$.

$$W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \middle| \; x, y \in R \right\}$$

Q. Let, $f$ denotes the vectors space of real valued functions under the operations.

For $f_1, f_2 \in f$, $k$ be any scalar

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(k.f)(x) = k.f(x)$$

if C denote set of all continuous function and D are set of all differentiable function then show C and d are subspace of f.

→ Given that ;

$F = \{ f \; R \to R \mid f \text{ is real fuction} \}$

$C = \{ f \; R \to R \mid f \text{ is conti. func} \}$

$D = \{ f. \; R \to R \mid f \text{ is diff fuction} \}$

1) let $f, g \in C \Rightarrow f.g$ are cts.

$f + g \in C$ (addn of cts func is cts).

2) let, $k.f \in C$. as $k$ is scalar

$\therefore$ the function C is subspace of f.

$$\underset{\text{subspace}}{C \subseteq f}$$

similarly, $d \subseteq C$.

$$P \subseteq d \subseteq C \subseteq f$$

H.W

1) $W = \{ f : R \to R \mid f$ is not differentiable$\}$ is W is subspace of F or not?

2) Show that $R$. $P_2$ is subspace of $P_3$.

3) $W = \{ P(x) \in P_3 \mid P(1) + (P') = 0 \}$

Q. Show that the solution space of diff$^n$ eqn $y'' + 2y = 0$ forms a subspaace of f.

→ $W = \{ f \in F \mid f \text{ is the soln of } y'' + 2y = 0 \}$

let $f, g \in W$,

$\therefore \; f'' + 2f = 0 \quad ; \quad g'' + 2g = 0$

$\therefore \; f + g = f'' + g'' + 2(f+g) = 0$

$= (f+g)'' + 2(f+g) = 0$

$\therefore \; (f+g) \in W$

also, $kf \in W$

becoz, $kf'' + 2kf = 0$

$\therefore \; kf \in W$.

Hence.

$$W \subseteq f$$

* let, $W = \{ A \in M_{n \times n} \mid |A| = 0 \}$
is $W$ subspace of $V = M_{n \times n}$ ?

→ $V = M_{n \times n}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

as $A + B \notin W$

$W$ is not a subspace of $V$.

* $W = \{ p(x) \in P_3(R) \mid p(1) = 0 \}$
is $W$ forms a subspace of $P_3(R)$?

→ lets take any two vectors in $W$.

$P_1, P_2 \in W$

$\therefore P_1(1) = 0$

$P_2(1) = 0$

$\therefore P_1(1) + P_2(1) = 0$

$\therefore P_1(1) + P_2(1) \in W$

also,

$P_1(1) = 0$

$K P_1(1) = 0 \cdot K$

$\therefore K P_1(1) \in W$

Thus it is a subspace of $W$.

$W \underset{\text{Subspace}}{\subset} P_3(R)$



* let $\mathbb{C}$ be vector space over the field $\mathbb{C}$ and $W = \{ z \in \mathbb{C} \mid |z| = 1 \}$

→ let $z_1$ and $z_2$ be the element of $W$.

$z_1, z_2 \in W$

as $|z_1| = 1$, $|z_2| = 1$

$\therefore z_1 + z_2 = \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$

$\neq 1$

$\therefore z_1 + z_2 \notin W$

$\therefore \mathbb{C} \underset{\text{sub}}{\not\subset} \mathbb{C}$.

* $W = \{ z \in \mathbb{C} \mid Re(z) = 0 \}$

→ if $z_1, z_2 \in W$

→ $Re(z_1) = Re(z_2) = 0$

Also, $Re(z_1 + z_2) = 0$

$z_1 + z_2 \in W$

Also, for any $k$, $Re(kz) = 0$

$kz \in W$.

**HW** :- check wheather the following is a subspace or not.

1> $W = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \,\middle|\, a, b \in R \right\}$, $V = M_{2 \times 2}(R)$

2> $W = \left\{ \begin{bmatrix} a \\ +b \\ a \\ -b \end{bmatrix} \,\middle|\, a, b \in R \right\}$, $V = R^4$

3> $W = \{ x \in R^n \mid AX = 0 \}$ $V = R^n$

## ★ Linear Combination and Coordinates :-

Let $v_1, v_2, \dots v_k \in V$ (Where $V$ is a vector space) and $u \in V$ then we say $u$ is in linear combination of $v_1, v_2, \dots v_k$ if $\exists$, the scalars $c_1, c_2, c_3 \dots c_k$ such that,

$$u = c_1 v_1 + c_2 v_2 \dots c_k v_k$$

and, the vector.

$$[u] = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

is called coordinate vector of $u$ w.r.t $v_1, v_2, v_3 \dots v_k$.

ex. 1) let $v_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $u$ is in L.C. of $v_1, v_2$.

$$u = c_1 v_1 + c_2 v_2$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$1 = 2c_1 - c_2$

$2 = -c_1 + 2c_2$ as $c_2 = 3$

$3 = c_2$ $\quad c_1 = 4$ & $c_1 = 2$

as $c_1$ getting two valus the $u$ can't be a L.C. of $v_1, v_2$.

Q. let $v_1 = 1 - x + x^2$, $v_2 = 3 - x^2$, $v_3 = -1 - x^2$ and are vectors in $P_2$ and $u = 3x^2 - x + 1$ is $u$ is in L.C of $v_1, v_2, v_3$ or not.

$\Rightarrow$ $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$ $u = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

$\Rightarrow$ $u = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$\therefore A|B = \begin{bmatrix} 1 & -1 & -1 & 3 \\ -1 & 0 & 0 & -1 \\ 1 & 3 & -1 & 1 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 = \begin{bmatrix} 1 & -1 & -1 & 3 \\ -1 & 0 & 0 & -1 \\ 0 & 3 & -1 & 0 \end{bmatrix} \to \text{by using calci.}$$

$c_1 = -1$, $c_2 = -1/2$, $c_3 = -3/2$.

thus $v_1, v_2, v_3$ can be written as L.C. gi u

## ★ L.C. of 2 coordinates:

$v_1, v_2 \dots v_k \in V$

$u \in V$ $u$ is a L.C. of $v_1, v_2, v_3 \dots v_k$ if f is scalar $c_1, c_2, \dots c_k$ s.t.

$$u = c_1 v_1 + c_2 v_2 + c_3 v_3 \dots c_k v_k$$

$$[u] = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

Q. Find co-ordinate vector $[u]$

if, $u = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $V_1 = \begin{bmatrix} 7 \\ 6 \\ 9 \end{bmatrix}$, $V_2 = \begin{bmatrix} -6 \\ -6 \\ 7 \end{bmatrix}$, $V_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$u = c_1 V_1 + c_2 V_2 + c_3 V_3$

$7c_1 - 6c_2 + c_3 = 1$

$6c_1 - 6c_2 + 0c_3 = 0$

$9c_1 + 7c_2 + c_3 = 2$

$\therefore c_1 = 1/15$, $c_2 = 1/15$, $c_3 = 14/15$

$\therefore [u] = \begin{bmatrix} 1/15 \\ 1/15 \\ 14/15 \end{bmatrix} = 1/15 \begin{bmatrix} 1 \\ 1 \\ 14 \end{bmatrix}$

Q.2 $u = \cos(2x)$ and $V_1 = \cos^2 x$, $V_2 = \sin^2 x$

Find cordinate Vector.

→ $\cos(2x) = \alpha \cos^2 x + \beta(\sin^2 x)$

$\therefore \alpha = 1$

$\beta = -1$

$\therefore u = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

★ linear dependence & linear independance

Let $V_1, V_2 \dots V_k$ are the vectors in the vector space $V$ we say $V_1, V_2, \dots V_k$ are linearly independent if at least one scalar is non-zero Such that, $c_1V_1 + c_2V_2 + \dots + c_kV_k = 0$.

and if all scalars are zero the $V_1, V_2, V_3 \dots V_k$ are linearly independent Vectors.

EX. $S = \{V_1, V_2 \dots V_k\}$ is L.D.

1) $S = \{0\}$

$K0 = 0$

$K \in R$

so $\{0\}$ is dependent vector.

2) $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

is dependent / independent?

$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\gamma$ can be any no.

thus dependent.

Note: Subset of Vector space V containing zero always linearly dependent.

3) $S = \{u, v, \alpha u + \beta v\} \subset \mathbb{R}^n$

$S = \{u, v_1, v_2 \dots v_k\}$ is L.D. if

$u = c_1 v_1 + c_2 v_2 \dots + c_k v_k$.

4) $S = \left\{ \underset{x}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}, \underset{y}{\begin{bmatrix} 2 & 4 \\ 9 & 7 \end{bmatrix}}, \underset{z}{\begin{bmatrix} 3 & 3 \\ 9 & 8 \end{bmatrix}} \right\}$

we can write $z = x + y$.

So linearly dependent.

5) $S = \{1+x, x+x^2, x^2+1\}$

L.D or I.?

→ $0 = \alpha[1+x] + \beta[x+x^2] + \gamma(x^2+1)$

$= \alpha + \alpha x + \beta x + \beta x^2 + \gamma x^2 + \gamma$

$= 0 + 0 + 0$

$=$ ~~ie~~ $\alpha = 0, \beta = 0, \gamma = 0$.

then it is linearly independent.

6) $S = \{\overset{\cos}{\cancel{\sin}}(2x), \cos^2 x, \sin^2 x, \tan\}$

we can write

$\cos 2x = -1\sin^2 + 1\cos^2 x + 0\tan x$

$\therefore \alpha = -1$

$\beta = 1$

$\gamma = 0$

$\therefore$ L.d.

7) $S = \{1, x, x^2 \dots x^n\} \subseteq P_n(\mathbb{R})$

→ $0 = c_1 1 + c_2 x + c_3 x^2 \dots + c_{n+1} x^n$

as no vectors are dependent.

the vectors are independent.

8) $S = \{1, x, x^2\} \subseteq P_2(\mathbb{R})$

$\alpha + \beta x + \gamma x^2 = 0$

$\beta + 2\gamma x = 0$

$2\gamma = 0$

$\therefore \alpha = \beta = \gamma = 0$

$\therefore$ independent.

9) $S = \{\underset{a}{1-2x+x^2}, \underset{b}{1+2x-x^2}, \underset{c}{2}\}$

$c = a + b$.

thus dependent.

* Spanning set / generating set.

Let $S = \{v_1, v_2, v_3 \dots v_k\}$ is subset of vector space $V$ the span of $S$ is defined as collection of linear combinations of $v_1, v_2 \dots v_k$

i.e. $\text{Span}(S) = \{c_1 v_1 + c_2 v_2 \dots + c_k v_k \mid c_1 c_2 \dots \in$

and if $\text{Span}(S) = V$ then $S$ is called spanning set of $V$.

generating set : $\langle S \rangle = \langle V_1, V_2, \ldots V_k \rangle$

ex. let, $0 \neq u \in \mathbb{R}^2$, $u = \begin{bmatrix} x \\ y \end{bmatrix}$ What is span(u)?

$S = \{u\}$

$\text{span}(S) = \{ cu \mid c \text{ is scalar}, c \in \mathbb{R} \}$

ex. 2] $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \in \mathbb{R}^2$ find span(S)

$S = c_1 V_1 + c_2 V_2$

$S = c_1 \begin{bmatrix} 1 \\ +1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore \text{Span}(S) = \{ c_1 V_1 + c_2 V_2 \mid c_1, c_2 \in \mathbb{R} \}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} -c_2 \\ c_2 \end{bmatrix}$

$\therefore \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$

ex. $\text{span}\left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$

$= \left\{ c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

$= \mathbb{R}^2$

(*) The number of vectors required to generate $\mathbb{R}^n$ is at least $n$.

Q. $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ Find span(S)?

→ $\text{span}(S) = \left\{ \alpha \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$

let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{span} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\alpha - \beta = x$ $\quad\therefore \alpha = y/2$

$2\alpha + \beta 0 = y$ $\quad\&\; \alpha = \frac{x+z}{2}$

$\alpha + \beta = z$

$\therefore$ This system has soln only if $y = x + z$ or $x - y + z = 0$.

$\therefore \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x - y + z = 0 \right\}$

#Ans

$\therefore$ 1 Vector in $\mathbb{R}^2$ generates line, 2 in $\mathbb{R}^3$ generates plane.

Q. $S = \{1, x, x^2\}$ be the subset of a vector space

→ $\text{span}(S) = C_1V_1 + C_2V_2 + C_3V_3$

$\text{span}(1, x, x^2) = \{C_1 + C_2x + C_3x^2 \mid C_1, C_2, C_3 \in R\}$

$\text{span}\{1, x, x^2\} = P_2(R^2)$

* similarly if $S = \{1, x, x^2 \dots x^n\}$

then, $\text{span}(S) = \{a_0 + a_1x + a_2x^2 \dots + a_nx^n \mid a_0 \dots a_n \in R\}$

$\therefore \text{span}(S) = P_n(R)$

* $S = \{1, \sin^2 x, \cos^2 x, \tan^2 x\} \subseteq f$

- linearly dependent.

as,

$1 = \sin^2 x + \cos^2 x + 0\tan^2 x.$

. $S = \{\cos(2x), \cos^2 x, \sin^2 x, \sqrt{100}\}$

$\cos 2x = \sin^2 x - \cos^2 x$

$\therefore$ L.d.

$S = \{\sin x, \cos x\}$

$\alpha\cos x + \beta\sin x = 0$

if $\alpha, \beta = 0$

$\therefore$ L.I.

$\alpha\cos x + \beta\sin x = 0$

$-\alpha\sin x + \beta\cos x = 0$

$\therefore \left[\begin{array}{cc|c} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \end{array}\right]$

$|A| = \cos^2 x + \sin^2 x = 1$

as $|A| \neq 0$

$\therefore \alpha, \beta = 0$

$\therefore$ It is Independent.

ex. $R^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| xy = R \right\}$

$R^2 = \left\{ x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} \middle| x.y = R \right\}$

$R^2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

clearly, $R^2$ is L.I.

$R = \{x.1 \mid x \in R\}$

$= \text{span}(1)$

B is basis for R

similarly,

Q. $P_n(R) = \{ a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \mid a_1, a_2, a_3 \dots \}$

$= span\{1, x, x^2, x^3 \dots x^n\}$

also, $B = \{1, x, x^2 \dots x^n\}$ is L.I.

Hence, B is std basis for $Pn(R)$:

Q. $M_{2\times 2}(R) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in R \right\}$

$\cancel{span}\left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

it is L.I.

$E_{11}$ $E_{12}$ $E_{21}$ $E_{22}$

$\therefore B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is std. basis.

Q. If $M_{n\times n}(R)$ then,

$\begin{bmatrix} a_{11} & \dots \\ \vdots & \dots a_{nn} \end{bmatrix}$

$= span\left\{ a_{11}\begin{bmatrix} 1 & \dots \\ \vdots & \dots 0 \end{bmatrix} + \dots + a_{nn}\begin{bmatrix} 0 & 0 & \dots \\ 0 & & \vdots \\ \vdots & \dots & 1 \end{bmatrix} \right.$

$E_{11}$ $E_{12}$ $E_{13}$ $\dots$ $E_{nn}$

$\therefore$ std basis $\Rightarrow b = \{E_{11} \dots E_{nn}\}$

similarly, std basis for $m \times n$ matrices is $\{E_{11}, E_{12} \dots E_{1n}, \dots E_{mn}\}$

Basis is minimal generating set.

Q. $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is basis for $R^2$

$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} -1 \\ 1 \end{bmatrix} \forall \begin{bmatrix} x \\ y \end{bmatrix} \in R^2$

$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

such basis are known as ordered basis.

ex. $span\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = R^2$

* Thm :- Let B is basis for vector space V then B is minimal generating set and maximal linearly independent set.

- $dim(R^n(R)) = n$.
  $dim(P_n(R)) = n+1$
  $dim(M_{n\times n}(R)) = n^2$
  $dim(M_{m\times n}(R)) = m \cdot n$
  $dim(\mathbb{C}(R)) = 2$

  dimension of $R^n$ over R.

  $dim(\mathbb{C}(\mathbb{C})) = 1$

- $\mathbb{C} = \{x + y \mid x, y \in R\}$
  $\{x + yi \mid x, y \in R\}$

  $\mathbb{C} = span(1, i)$

  1, i are independent

  $\therefore B(1, i)$ is basis for $\mathbb{C}(R)$

Q. 2. $P_2(\mathbb{C})\,\mathbb{R} = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{C}\}$

$\{x_0 + iy_0 + (x_1 + iy_1)x + (x_2 + iy_2)x^2\}$

$\text{span}(1, x, x^2, i, ix, ix^2)$

$\Rightarrow B\ \{1, x, x^2, i, ix, ix^2\}$ is a basis for the vector space $P_2(\mathbb{C})\,\mathbb{R}$

$\dim(P_2, \mathbb{C}(\mathbb{R})) = 6$.

* In general,

1) $\dim(M_{n\times n}(\mathbb{C})(\mathbb{R})) = \boxed{2n^2}$

2) $\dim(\mathbb{P}_n(\mathbb{C})(\mathbb{R})) = 2(n+1)$

3) $\dim(M_{mn}(\mathbb{C})(\mathbb{R})) = 2m\cdot n$.

HW Find $\dim(\mathbb{R}(\mathbb{Q}))$.

Q. $W = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x + y = 0\right\}$ is a subspace of $\mathbb{R}^2$. Find dim of $\mathbb{R}W$?

→ $W = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid -x = y\right\}$

$W = \left\{\begin{bmatrix} x \\ -x \end{bmatrix} \in \mathbb{R}^2 \mid x \in \mathbb{R}\right\} = \left\{x\begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \mathbb{R}^2 \mid x \in \mathbb{R}\right\}$

$\therefore W = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

also the set is L.I.

$\therefore$ Basis for $W$, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore \dim(W) = 1$

$\dim(W) = \dim(\mathbb{R}^n) - \rho(A)$

i.e. $\dim(W) = \dim(\mathbb{R}^3) - \rho(x+y+z=0) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \end{array}\right]$, $Ax = b$

$= 3 - 1 = 2$

Q. $W = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x + y + z = 0\right\}$

$W = \left\{\begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x = -y-z\right\}$

$W = \left\{\begin{bmatrix} -y \\ y \\ z \end{bmatrix} + \begin{bmatrix} -z \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x, y, z \in \mathbb{R}\right\}$

$\Rightarrow W = \left\{y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3 \mid y, z \in \mathbb{R}\right\}$

$\therefore W = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

the set is L.I.

$\therefore$ Basis for $W$, $B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore \dim(W) = 2$.

HW $W = \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_4 = 0 \end{array}\right\}$

Find $\dim(W)$.

* result: The dimension of solution space of homogeneous system of linear eqn is $n - r$, where $A$ is of order $m\times n$ and $x$ belongs to $\mathbb{R}^n$.

in general,

$\dim(W)$ = colm of A − rank

= nulity

= ~~dim~~ number of free variabl

* If $W = \{ p(x) \in P_2(R) \mid p(1) = 0 \}$ is a subspace of $P_2$? Find $\dim(W)$?

$\rightarrow$ $W = \{ a_0 + a_1 x + a_2 x^2 \mid p(1) \neq 0 \}$, $a_0, a_1, a_2 \in R$

for $p(1) = 0 \Rightarrow a_0 + a_1 + a_2 = 0$

thus $\boxed{a_0 = -a_1 - a_2}$

$\therefore x \begin{bmatrix} -a_1 - a_2 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -a_1 \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} -a_2 \\ 0 \\ a_2 \end{bmatrix}$

$= x \cdot a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - a_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Substituting $a_0 = -a_1 - a_2$

$W = \{ -a_1 - a_2 + a_1 x + a_2 x^2 \mid a_1, a_2 \in R \}$

$W = \{ +a_1 (x-1) + a_2 (x^2 - 1) \mid a_1, a_2 \in R \}$

$\therefore \text{span}\{ (x-1), (x^2-1) \}$

as these two are L.I.

Basis $= B = \{ (x-1), (x^2-1) \}$

$\therefore \dim(W) = 2.$

• If $W = \{ p(x) \in P_3(R) \mid x p'(x) = p(x) \}$ is a subspace of $P_3$ where $p'(x) = \frac{d}{dx}(p(x))$ find $\dim(W)$?

$\rightarrow$ $W = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 - a_3 \in R ;\ x p'(x) = p(x) \}$

as $x \cdot \frac{d}{dx}(p(x)) = p(x)$

$\therefore x [ a_1 + 2a_2 x + 3a_3 x^2 ] = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$\therefore \cancel{a_1 x} + 2a_2 x^2 + 3a_3 x^3 = a_0 + \cancel{a_1 x} + a_2 x^2 + a_3 x^3$

$\therefore a_0 = a_2 x^2 + 2a_3 x^3$ also $; -a_0 + a_2 x^2 + 2a_3 x^3 = 0$

thus putting $a_0, a_1, a_2 = 0$.

$\therefore W = \{ \underset{0}{a_2 x^2} + \underset{0}{2a_3 x^2} + a_1 x + \underset{0}{a_2 x^2} + \underset{0}{a_3 x^3} \mid \ldots \}$

$\therefore W = \{ \underset{0x^2}{2a_2 x^2} + \underset{0x^3}{3a_3 x^3} + \underset{a_1 x}{a_1 x} \mid \ldots \}$

$\therefore W = \{ \cancel{x^2}, \cancel{x^3}, x \}$

$\therefore W = \text{span}(x)$

$\therefore \dim(W) = 1$

HW

Find $\dim(W)$ of following

1) $W = \left\{ \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \middle| \begin{matrix} x + y = 0 \\ y - z = 0 \\ z - t = 0 \end{matrix} \right\}, V = R^4$

2) $W = \left\{ \begin{bmatrix} a & -1 \\ b & a \end{bmatrix} \middle| a, b \in R \right\}, V = M_{2\times2}(R)$

3) $W = \{ f \in F \mid f'' + f = 0 \}$ $V, f \to \{ f, R \to R$

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HW

4) $W = \{ A \in M_{n\times n}(R) \mid A^T = A \}$, $V = M_{n\times n}(R)$

* Theorem: Let $B = \{v_1, v_2, v_3 \dots v_n\}$ be the basis for vector space $V$ then,

1) Any set in $V$ with more than 'n' vectors is L.D.

2) Any set in $V$ with less than 'n' vectors cannot span $V$.

ex. $B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

$B' = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$

$B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$\Rightarrow$ Thus, basis is minimal generating set and maximal L.I. set.

ex. $B = \{1, x, x^2\}$ is basis of $P_2$

$B' = \{1, x, x^2, 1+x^2\}$

prop 1

$1 + x^2 = 1 \cdot 1 + 0x + 1x^2$

$B'' = \{1, x^2\}$ $\times$

prop 2

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* Theorem:

Let $B = \{v_1, v_2, v_3 \dots v_n\}$ be the basis for vector space $V$ then for any $u \in V$ $\exists$ a unique scalar $c_1, c_2, c_3 \dots c_n$ such that $u = c_1v_1 + c_2v_2 + c_3v_3 \dots c_nv_n$.

i.e. $u$ can be written uniquely as L.C. of Basis vectors $v_1, v_2 \dots v_n$.

let $u$ can be written in two different ways,

$u = \alpha_1 v_1 + \alpha_2 v_2 \dots \alpha_n v_n$

$u = \beta_1 v_1 + \dots = \beta_n v_n$

$\Rightarrow \alpha_1 v_1 + \dots + \alpha_n v_n = \beta_1 v_1 + \dots \beta_n v_n$

$(\alpha_1 - \beta_1) v_1 \dots (\alpha_n - \beta_n) v_n = 0$

$\therefore \alpha_1 - \beta_1 = 0 \quad \dots \quad \alpha_n - \beta_n = 0$

$\therefore \alpha_1 = \beta_1 \quad \dots \quad \alpha_n = \beta_n$

$u$ has unique linear combination

* Theorem: Let $V$ be a vector space with dimension $n$. i.e. $\dim(V) = n$ then.

i) Any L.I. set in $V$ consist of atmost $n$ vectors

ii) Any spanning set for $V$ contains at least $n$ vectors.

iii) Any L.I. set in $V$ with exactly $n$ vectors is a Basis for $V$.

iv) Any spanning set for $V$ consisting of exactly $n$ vectors is a basis for $V$.

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v) Any L.I. set in V can be extended as a Basis for V.

vi) Any spanning set for V can be reduced to basis for V.

* Extension of a basis :-

Let $V = R^2$

1) $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ , $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

2) $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is L.I. set in $R^3$ we can extend B as basis for $R^3$?

Vector to add in b is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ becoz

let B be the any L.I. set in vector space it can be extended to the basis for is by adding suitable vector from standard vector so that V become a basis for V.

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Q. $B = \{1-x, 1+x\}$ extend it as basis for $P_2$.

$\{1-x, 1+x\} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the vector to be added is $[0 \; 0 \; 1]$

* finite dimension vector space (FDVS)

Vector space V is said to be finite dimensioned if it consist of finite number of basis element

ex. $R^n(R)$ , $\mathbb{C}(R)$ , $M_{n \times n}(R)$, $M_{m \times n}(R)$, $P_n(R)$
$M_{n \times n}(\mathbb{C})(R)$

ex. $R(Q)$ is infinite dim. V.S.

$\rightarrow$ $V = R$ , $F = Q$

as $\sqrt{2}, \sqrt{3} \cdots$ are not in Q but in R.

So to generate R from Q we need to extend the basis upto infinity to generate R.

$B = \{\sqrt{2}, \sqrt{3} \cdots \cdots \sqrt{P}\}$

$\dim(R(Q)) = \infty$

subspace
span, L.C., Basis, dim.

$P = \{ a_0 + a_1 x \dots a_n x^2 \dots \}$
is infinite dim V.S.
$\therefore \dim(P(\mathbb{R})) = \infty$

* **Co-ordinates**: Let, $B = \{V_1, V_2 \dots V_n\}$ be the Basis for a vector space $V$ and $u \in V$ be any vector in the vector space $V$ the coordinate vector of $u$ w.r.t. Basis $B$ is given by,

$$[u]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ such that, } u = c_1 V_1 + c_2 V_2 + \dots + c_n V_n$$

* **Theorem**: Let, $B = \{V_1, V_2, \dots V_n\}$ be a basis for vector space $V$ then,

1) $[u+v]_B = [u]_B + [v]_B, \ \forall u, v \in V$

2) $[ku]_B = k[u]_B$

→ proof:- let $u = \alpha_1 V_1 + \alpha_2 V_2 \dots \alpha_n V_n$, $v = \beta_1 V_1 + \dots + \beta_n V_n$

$\therefore u + v = \alpha_1 V_1 + \alpha_n V_n + \beta_1 V_1 + \beta_n V_n$
$= (\alpha_1 + \beta_1) V_1 + (\alpha_2 + \beta_2) V_2 \dots + (\alpha_n + \beta_n) V_n$

$$\Rightarrow [u+v]_B = \begin{bmatrix} \alpha_1 + \beta_1 \\ \vdots \\ \alpha_n + \beta_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$
$$= [u]_B + [v]_B$$

$ku = k\alpha_1 V_1 + \dots + k\alpha_n V_n$

$$[ku]_B = \begin{bmatrix} k\alpha_1 \\ \vdots \\ k\alpha_n \end{bmatrix} = k \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = k[u]_B$$

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Q. $B = \{1, x, x^2\}$ → std Basis

$x + x^2 \Rightarrow v = x + x^2 \in P_2(\mathbb{R})$
$x + x^2 = 0 \cdot 1 + x + x^2$

$$[v]_B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

2) Ordered of $P_2 = B' = \{x^2 + x + 1\}$

$x + x^2 = 1x + 1x^2 + 0 \cdot 1$

$$[v]_{B'} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

3) $B'' = \{x+1, x-1, x^2\}$

$$\therefore [v]_{B''} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

as, $\alpha(x+1) + \beta(x-1) + x^2 \cdot \gamma$
$\therefore \alpha + \beta = 0$
$\alpha - \beta = 1$
$\gamma = 1$
$\therefore \alpha = 1/2, \ \beta = -1/2, \gamma = 1.$

* **The Basis Theorem**: Let, $V$ be any FDVS then any two Bases for $V$ have same cardinality.

$|B| = |B'|$

ex. $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = B'$

and $B' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are the Bases for $R^2$ that have same coordinality

* dimension theorem:

let $W$ be the subspace of FDVS, $V$ then,

1) $\dim(W) \le \dim(V)$
2) $\dim W = \dim V$ if and only if $V = W$.

Ex. $P_2(R) \underset{\text{subspace}}{\subseteq} P_3(R)$

1) $\dim(P_2) \le \dim(P_3)$
$3 \le 4$

2) $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| y = x \right\}$

$= \dim(W) = 1$
$V = R^2 = \dim(R^2) = 2$
$= \dim(W) \le \dim(V)$

* $V = f$ = set of all real valued function
$e$ = set of cont. funtion
as $e \subset f$

$|Bf| = \infty$ :

also $|Be| = \infty$

$\therefore \dim[W] = \dim(\tau) = \infty$
but $W \ne e$

* coordinates

$B = \{v_1, v_2 ---- v_n\}$ be the Basis for Vector space $V$ and $x \in V$ be any vector then we can write.

$x = \alpha_1 v_1 + \alpha_2 v_2 + --- \alpha_n v_n$

uniquely

$[x]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$

Is the coordinate vector of $x$ w.r.t Basis $B$.

#) 1) $[u_1 + u_2] = [u_1]_B + [u_2]_B \quad \forall u_1, u_2 \in V$

2] $[ku]_B = k[u]_B, \forall k \in R, \forall u \in V$.

or $[ku + mv]_B = k[u]_B + m[v]_B$.

Thm :- let $B = \{v_1, v_2, v_3 -- v_n\}$ be the basis for vector space $V$.
$B' = \{u_1, u_2 -- u_n\}$ be the another basis for $V$ then the vector $\{[u_1]_B, [u_2]_B -- [u_n]_B\}$ is linerly independent if and only if $\{u_1, u_2 -- u_n\}$ is L.I.

Prf: let $\alpha_1, \alpha_2 -- \alpha_n$ be the scalars
s.t. $\alpha_1[u_1]_B + \alpha_2[u_2]_B + --- + \alpha_n[u_n]_B = 0$

$[\alpha_1 u_1]_B + [\alpha_2 u_2]_B -- + [\alpha_n u_n]_B = 0$

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$\Rightarrow [\alpha_1 u_1 + \alpha_n u_n] = 0$

$\therefore$ As $\{u_1, u_2 \ldots u_n\}$ is Basis is L.I.

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 \ldots \alpha_n = 0$

thus,

$\{[u_1]_B, [u_2]_B \ldots [u_n]_B\}$ is L.I. set.

*) $R^2$

$B = \left\{ u_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, B' = \left\{ u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$x = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

$[x]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ $\quad x = \alpha_1 u_1 + \alpha_2 u_2$

$\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \alpha_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$[x]_B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad \alpha_1 = 1, \alpha_2 = 3$

similarly, $[x]_{B'} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

Now,

$[x]_{B'} = [1u_1 + 3u_2]_{B'}$

$= [1u]_{B'} + [3u_2]_{B'}$

$[x]_{B'} = \big[ [u]_{B'} + [u_2]_B \big]_{2\times 2} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$[x]_{B'} = P\,[x]_B$

Here $P = P_{B' \leftarrow B}$ is called change of basis matrix from $B$ to $B'$

$[x]_B = P'[x]_{B'}$

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$\therefore$ change of basis matrix :

let $B = \{u_1, u_2, u_3 \ldots u_n\}$ and

$B' = \{V_1, V_2 \ldots V_n\}$ are the two bases for a vector space V. The $n\times n$ matrix whose columns are coordinate vector

$[u_1]_B, [u_2]_{B'} \ldots [u_n]_{B'}$

of the vectors in B w.r.t. B' is denoted by $P_{B'} = B$ and it is given by

$P_{B' \leftarrow B} = ([u_1]_{B'} \ldots [u_n]_{B'})_{n\times n}$

$[x]_{B'} = P_{B' \leftarrow B}[x]_B$

$P_{B' \leftarrow B} = \big[ [u_1]_{B'} \ldots [u_n]_{B'} \big]_{n\times n}$

* find the change of basis matrices $P_{B' \leftarrow B}$ and $P_{B \leftarrow B'}$ where,

$B = \{1, x, x^2\}$ and $B' = \{1+x, x+x^2, 1+x^2\}$ are the bases of $P_2$.

$P_{B' \leftarrow B} = \big[ [u_1]_{B'}\ [u_2]_{B'}\ [u_3]_{B'} \big]_{3\times 3}$

&

$P_{B \leftarrow B'} = \big[ [V_1]_B\ [V_2]_B\ [V_3]_B \big]_{3\times 3}$

$\{u_1, u_2, u_3\} = \{1, x, x^2\}$

$\{V_1, V_2, V_3\} = \{1+x, x+x^2, 1+x^2\}$

$V_1 = \alpha u_1 + \beta u_2 + \gamma u_3 = 1 + 1x + 0x^2 = 1 + x^2$

$[V_1]_B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

similarly,

$V_2 = \alpha u_1 + \beta V_2 + \gamma u_3$

$\therefore [V_2]_B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\therefore [V_3]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore P_{B \leftarrow B'} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{3\times3}$

thus,

$P_{B' \leftarrow B} = \left[ [u_1]_{B'} \; [u_2]_{B'} \; [u_3]_{B'} \right]$

$[u_1]_{B'} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ such that $u_1 = \alpha V_1 + \beta V_2 + \gamma V_3$

$1 = \alpha(1+x) + \beta(x+x^2) + \gamma(1+x^2)$

$\alpha + \beta = 0$

$\alpha + \gamma = 1 \quad \therefore \begin{bmatrix} 1/2 \\ -1/2 \\ +1/2 \end{bmatrix}$

$\beta + \gamma = 0$

$[u_2]_{B'} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ $u_2 = \alpha V_1 + \beta V_2 + \gamma V_3$

$x = \alpha(1+x) + \beta(x+x^2) + \gamma(1+x^2)$

$\therefore \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

$\therefore [u_3]_{B'} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$ $\therefore P_{B' \leftarrow B} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$

$(P_{B' \leftarrow B})^{-1} = P_{B \leftarrow B'}$

$P_{B \leftarrow B'} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{3\times3}$ $P_{B' \leftarrow B} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

we can observe that,

$(P_{B \leftarrow B'})^{-1} = P_{B' \leftarrow B}$

they are inverses of each other.

Q. Find $[P(x)]_{B'}$ where $P(x) = 1+2x-x^2$

$\rightarrow P(x) = \alpha V_1 + \beta V_2 + \gamma V_3$

$1+2x-x^2 = \alpha(1+x) + \beta(x+x^2) + \gamma(1+x^2)$

$\therefore \alpha + \gamma = 1$

$\alpha + \beta = 2$

$\beta + \gamma = -1$

$\rightarrow \alpha = -\beta + 2 \quad \therefore -\beta + 2 = +1 + \gamma$

$\alpha = -1 + \gamma$

$\therefore \alpha = 2, \; \beta = 0, \; \gamma = -1$

$\therefore [P(x)]_{B'} = [2 \;\; 0 \;\; -1]$

also, $[P(x)]_B$

$\Rightarrow P(x) = \alpha u_1 + \beta u_2 + \gamma u_3$

$\therefore 1+2x-x^2 = \alpha 1 + \beta x + \gamma x^2$

$\therefore \alpha = 1, \; \beta = 2, \; \gamma = -1$

Now, $P_{B' \leftarrow B} \cdot [X]_B$

$\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

**HW**

In $M_{22}$ $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and

$B' = \{A, B, C, D\}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

are two Bases find $P_{B' \leftarrow B}$, $P_{B \leftarrow B'}$, $[P(x)]_B$, $[P(x)]_{B'}$, $P_{B' \leftarrow B} \cdot [x]_B$ for $x = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

→ $P_{B' \leftarrow B} =$

$U_1 = \alpha U_1 + \cancel{\beta} V_2 + \beta V_3 + \gamma V_4$

$U_2 = \quad -1-$

$U_3 = \quad -1-$

$[U_1]_{B'} = E_{11} = \alpha V_1 + 0 \ldots$

$$\therefore [E_{11}]_{B'} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

similarly, $[U_2]_{B'} = \begin{bmatrix} -1 \\ +1 \\ 0 \\ 0 \end{bmatrix}$, $[U_3]_{B'} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $[U_4]_{B'} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$$\therefore P_{B' \leftarrow B} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

also, $P_{B \leftarrow B'} = [V_1]_B = \alpha U_1 + \beta U_2 + \gamma U_3 + \delta U_4$

$\therefore U_1 = V_1, \therefore \beta \gamma \delta = 0$

$$\therefore [V_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$[V_2]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $[V_3]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $[V_4]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\therefore P_{B \leftarrow B'} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$[P(x)]_B = x = \alpha E_{11} + \beta E_{12} + \gamma E_{13} + \delta E_{14}$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \delta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \alpha = 1$, $\gamma = 3$

$\beta = 2$ $\delta = 4$

$$\therefore [P(x)]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$[P(x)]_{B'} = x = \alpha A + \beta B + \gamma C + \delta D$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \delta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$1 = \alpha + \beta + \gamma + \delta$ $\quad \alpha = -1$

$2 = \beta + \gamma + \delta$ $\quad \beta = -1$

$3 = \gamma + \delta$ $\quad \therefore \gamma = -1$

$4 = \delta$

$$\therefore [P(x)]_{B'} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

Verifying $P_{B' \leftarrow B} \cdot [x]_B$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} +1 \\ +2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

Verified as $P_{B' \leftarrow B} [x]_B = [x]_{B'}$

* Linear Transformation : $(T : V \to W)$

- A linear transformation from a vector space $V$ to a vector space $W$ is a mapping / function:

$T : V \to W$.

Such that,

a) $T(u+w) = Tu + Tw \quad \forall\ u, w \in V$

b) $T(\alpha \cdot u) = \alpha T(u) \quad \forall\ \alpha \in R,\ u \in V$

Or $T(\alpha u + \beta v) = T(\alpha u) + T(\beta v)$

$= \alpha(T(u)) + \beta(T(v))$

$\forall\ \alpha, \beta \in R;\ u, v \in V$

ex. The matrix transformation is a linear transformation defined by $T(x) = Ax$, $x \in R^n$ and $A$ is $n \times n$ matrix.

$\rightarrow$ let $x_1, x_2 \in R^n$

$T(x_1 + x_2) = A(x_1 + x_2)$

$= Ax_1 + Ax_2$

$= T(x_1) + T(x_2)$

$\rightarrow$ let $\alpha \in R$, $x \in R^n$

$T(\alpha x) = A(\alpha x)$

$= \alpha A(x)$

$= \alpha T(x)$

Hence proved.

Q. let $T : M_{nn} \to M_{nn}$ defined by $T(A) = -A^T$

for 1) let $x_1, x_2 \in R^n$

$\therefore T(x_1 + x_2) = -(x_1)^T + (-x_2)^T$

$= -(x_1^T + x_2^T)$

$= -(x_1 + x_2)^T = -(x_1 + x_2)^T$

$= T(x_1 + x_2)$

2) $\alpha \in R$, $x \in R^n$

$\Rightarrow T(\alpha x) = -(\alpha x)^T$

$= -\alpha (x)^T$

$= +\alpha T(x)$

Hence proved

3> let, D is differential operator

$D : \mathcal{F} \to \mathcal{F}$ defined by

$$D(f(x)) = \frac{d}{dx}(f(x))$$

1> $T(x_1 + x_2) = D(f(x_1) + g(x_1))$

$$= \frac{d}{dx}(f(x)) + \frac{d}{dx}(gx)$$

$$= T(x_1) + T(x_2).$$

2> $T(\alpha x) = \frac{d}{dx}(\alpha f(x)) = \alpha \frac{d}{dx}(fx)$

$$= \alpha T(x)$$

H.P.

4> let Integral operator $(\int)$ defined by $T(f(x))$

$$= \int_a^b f(x)\,dx$$

$T : C[a,b] \to R$ set of all cont. fun on $[a,b]$

$\therefore T$ is L.T.

5. Let $T : \mathbb{C}(\mathbb{C}) \to \mathbb{C}(\mathbb{C})$ defined by $T(z) = \bar{z}$

$\Rightarrow$ let, $z, \omega \in \mathbb{C}$

$\rightarrow T(z + \omega) = \overline{z + \omega}$

$$= \bar{z} + \bar{\omega}$$

$$= T(z) + T(\omega)$$

$\rightarrow T(\alpha z) = \overline{\alpha z} = \bar{\alpha} . \bar{z}$

$$= \bar{\alpha}\, T(z)$$

$$\neq \alpha T(z)$$

no L.T.

* Check linerity of following Trans.

a> $T : M_{22} \to R$ defined by $T(A) = \det(A)$

b> $T : R \to R$ ⊥ $T(x) = 2^x$

c> $T : R \to R$ ⊥ $T(x) = |x|$

1st failed for all

1> $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$|A| + |B| \neq |A+B|$

2> $2^2 + 2^1 \neq 2^3$

3> $|-5+3| \neq |-5| + |3|$

* LA Place transformation

$L : C[0,\infty] \to R$

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt, \quad s > 0$$

- it is L.T. (used in signal processing)

Remark: If $T : V \to W$ is linear transformation then $T(v)$ is called the image of $v \in V$ under the transformation $T$. and $T(v) = w$ for some $w \in W$

*

$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is the Basis for $\mathbb{R}^2$ $\forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + y\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Note : The most imp property of L.T. is we completly determine the transformation T when the images of Bases Vectors are known.

ex. ① Let, $T : \mathbb{R}^2 \to P_2$ and $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1-x+x^2$ and

$T\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 1-x^2$ find $T\begin{bmatrix} a \\ b \end{bmatrix}$ and $T\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\Rightarrow \cancel{T\begin{bmatrix} a \\ b \end{bmatrix} = T\left(a\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(b\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)}$$

$$\cancel{= a\,T\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\,T\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \alpha - \beta = a$$
$$\alpha + \beta = b$$

$$\therefore \alpha = \frac{a+b}{2}, \quad \beta = \frac{b-a}{2}$$

$$\therefore \begin{bmatrix} a \\ b \end{bmatrix} = \frac{a+b}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-a+b}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T\begin{bmatrix} a \\ b \end{bmatrix} = \frac{a+b}{2}\,T\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b-a}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \left(\frac{a+b}{2}\right)(1-x+x^2) + \left(\frac{b-a}{2}\right)(1-x^2)$$

$$= \left(\frac{a+b}{2} + \frac{b-a}{2}\right) + \left(\frac{-a-b}{2}\right)x + \left(\frac{a+b}{2} + \frac{a-b}{2}\right)x$$

$$T\begin{bmatrix} a \\ b \end{bmatrix} = \left(\frac{a+b}{2}\right) - \left(\frac{a+b}{2}\right)x + (a)x^2$$

$$= b - (a+b/2)x + ax^2$$

* Theorem : Let $T : V \to W$ is a linear transformat and $B = \{v_1, v_2, \dots v_n\}$ is the spanning set of $V$ then $T(B) = \{T(v_1), T(v_2) \dots T(v_n)\}$ is the spanning set for $W$ $T(V) \subseteq W$ subs

→ As, $B = \{v_1, v_2 \dots v_n\}$ is spanning set for $V$ the for every $u \in V$,

$$u = \alpha_1 v_1 + \alpha_2 v_2 \dots \alpha_n v_n$$

$$T(u) = \alpha_1 T(v_1) + \alpha_2 T(v_2) \dots \alpha_n T(v_n)$$

$$= T(B) = \{T(v_1), T(v_2) \dots T(v_n)\}$$

* **Range of a linear transformation ($R(T)$)**

= let, $T : V \to W$ be a linear Transformation the range space $R(T)$ is defined as the collection of all images of $v \in V$ under L.T. 'T'

i.e. $R(T) = \{ T(v) \mid v \in V \} \subseteq W$.

* **kernal of a linear transformation ($\ker(T)$)**

let, $T : V \to W$ be a linear Transformation the kernal of T is defined as $\ker(T)$

$$= \{ v \in V \mid T(v) = 0 \} \subseteq V$$

* Ex. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ y \end{bmatrix}$ be a linear transformation from $R^2 \to R^2$ find $R(T)$ and $\ker(T)$.

→ $R(T) = \{ T(v) \mid v \in V \}$

$$= \left\{ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \Big| \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \right\}$$

$$= \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \Big| y \in R \right\}$$

$$\left\{ y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Big| y \in R \right\} = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \subseteq W$$

Now, $\ker(T) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \Big| T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

$$= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \Big| \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \Big| y = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \Big| x \in R \right\} = \left\{ x \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Big| x \in R \right\}$$

$$= \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$$

* let, $T : D \to F$ defined by $T(f(x)) = f'(x)$

Range $= R(T) = \{ T(v) \mid v \in F \}$

$$= \{ T(f(x)) \mid f(x) \in F \}$$

$$= \{ f'(x) \mid f'(x) \in D \}$$

$$= R(T) = D$$

$\ker(T) = \{ f'(x) \in D \mid f'(x) = 0 \}$

= set of all constant fnᶜ.

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* check linerity of following Trans.

a> $T: M_{22} \to R$ defined by $T(A) = \det(A)$

b> $T: R \to R$ — $T(x) = 2^x$

1st failed for all

c> $T: R \to R$ — $T(x) = |x|$

1> $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$|A| + |B| \neq |A+B|$

2> $2^2 + 2^1 \neq 2^3$

3> $|-5+3| \neq |-5| + |3|$

* LA place transformation

$L: C[0,\infty] \to R$

$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \ , \ s>0$

- it is L.T. (used in signal processing)

Remark: If $T: V \to W$ is linear transformation then $T(v)$ is called the image of $v \in V$ under the transformation T. and $T(v) = w$ for some $w \in W$

*

$$\therefore \begin{bmatrix} a \\ b \end{bmatrix} = \frac{a+b}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-a+b}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T\begin{bmatrix} a \\ b \end{bmatrix} = \frac{a+b}{2} T\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b-a+}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \left(\frac{a+b}{2}\right)(1-x+x^2) + \left(\frac{b-a}{2}\right)(1-x^2)$$

$$= \left(\frac{a+b}{2} + \frac{b-a}{2}\right) + \left(\frac{-a-b}{2}\right)x + \left(\frac{a+b}{2} + \frac{a-b}{2}\right)x^2$$

$$T\begin{bmatrix} a \\ b \end{bmatrix} = \left(\frac{\cancel{a}+b}{\cancel{2}}\right) - \left(\frac{a+b}{2}\right)x + (a)x^2$$

$$= b - (a+b/2)x + ax^2$$

Theorem : Let $T : V \to W$ is a linear transformation and $B = \{v_1, v_2, \dots v_n\}$ is the spanning set of $V$ then $T(B) = \{T(v_1), T(v_2) \dots T(v_n)\}$ is the spanning set for ~~W~~ $T(V)$. $\subseteq W$.
subset

→ As, $B = \{v_1, v_2 \dots v_n\}$ is spanning set for $V$ the for every $u \in V$,

$$u = \alpha_1 v_1 + \alpha_2 v_2 \dots \alpha_n v_n$$

$$T(u) = \alpha_1 T(v_1) + \alpha_2 T(v_2) \dots \alpha_n T(v_n)$$

$$= T(B) = \{T(v_1), T(v_2) \dots T(v_n)\}$$

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- Range of a Linear transformation $(R(T))$

- let, $T : V \to W$ be a linear Transformation the range space $R(T)$ is defined as the collection of all images of $v \in V$ under L.T. 'T'
i.e. $R(T) = \{ T(v) \mid v \in V \} \subseteq W$.

- kernal of a linear transformation $(\ker(T))$

Let, $T : V \to W$ be a linear Transformation the kernal of T is defined as $\ker(T)$

$$= \{ v \in V \mid T(v) = 0 \} \subseteq V$$

- Ex. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ y \end{bmatrix}$ be a linear transformation from $R^2 \to R^2$ find $R(T)$ and $\ker(T)$.

$\to$ $R(T) = \{ T(v) \mid v \in V \}$

$$= \left\{ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \middle| \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \right\}$$

$$= \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \middle| y \in R \right\}$$

$$\left\{ y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \middle| y \in R \right\} = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \subseteq W$$

$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is the Basis for $\mathbb{R}^2$ $\forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + y\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Note : The most imp property of L.T. is we completly determine the transformation T when the images of Bases Vectors are known.

ex. ① Let, $T : \mathbb{R}^2 \to P_2$ and $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 - x + x^2$ and

$T\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 1 - x^2$ find $T\begin{bmatrix} a \\ b \end{bmatrix}$ and $T\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\Rightarrow T\begin{bmatrix} a \\ b \end{bmatrix} = T\left(a\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(b\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= a\,T\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\,T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \alpha - \beta = a$$

$$\alpha + \beta = b$$

$$\therefore \alpha = \frac{a+b}{2}, \quad \beta = \frac{b-a}{2}$$

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Q. The matrix transformation is a linear transformation defined by $T(x) = Ax$, $x \in R^n$ and $A$ is $n \times n$ matrix.

→ let $x_1, x_2 \in R^n$

$$T(x_1 + x_2) = A(x_1 + x_2)$$
$$= Ax_1 + Ax_2$$
$$= T(x_1) + T(x_2)$$

→ let $\alpha \in R$, $x \in R^n$

$$T(\alpha x) = A(\alpha x)$$
$$= \alpha A(x)$$
$$= \alpha T(x)$$

Hence proved.

Q. let $T: M_{nn} \to M_{nn}$ defined by $T(A) = -A^T$

for 1) let $x_1, x_2 \in R^n$

$$\therefore T(x_1 + x_2) = -(x_1)^T + (-x_2)^T$$
$$= -(x_1^T + x_2^T)$$
$$= -(x_1 + x_2)^T$$
$$= -(x_1 + x_2)^T = T(x_1 + x_2)$$

2) $\alpha \in R$, $x \in R^n$

$$\Rightarrow T(\alpha x) = -(\alpha x)^T$$
$$= -\alpha(x)^T$$
$$= +\alpha T(x)$$

Hence proved

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verifying $P_{B' \leftarrow B} \cdot [x]_B$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} +1 \\ +2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

verified as $P_{B' \leftarrow B} [x]_B = [x]_{B'}$

* Linear Transformation : $(T : V \rightarrow W)$

- A linear transformation from a vector space $V$ to a vector space $W$ is a mapping / function.

$T : V \rightarrow W$.

Such that,

a> $T(u+w) = Tu + Tw \quad \forall\, u, w \in V$

b> $T(\alpha \cdot u) = \alpha T(u) \quad \forall\, \alpha \in R,\ u \in V$

or $T(\alpha u + \beta v) = T(\alpha u) + T(\beta v)$

$= \alpha (T(u)) + \beta (T(v))$

$\forall\, \alpha, \beta \in R;\ u, v \in V$

$$[V_2]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad [V_3]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad [V_4]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore P_{B \leftarrow B'} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[P(x)]_B = x = \alpha E_{11} + \beta E_{12} + \gamma E_{13} + \delta E_{14}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \delta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \alpha = 1, \quad \gamma = 3$
$\beta = 2 \quad \delta = 4$

$$\therefore [P(x)]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$[P(x)]_{B'} = x = \alpha A + \beta B + \gamma C + \delta D$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \delta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$1 = \alpha + \beta + \gamma + \delta \quad \alpha = -1$

$2 = \beta + \gamma + \delta \quad \beta = -1$

$3 = \gamma + \delta \quad \therefore \gamma = -1$

$4 = \delta$

$$\therefore [P(x)]_{B'} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

$$(P_{B' \leftarrow B})^{-1} = P_{B \leftarrow B'}$$



$$P_{B \leftarrow B'} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{3\times 3} \qquad P_{B' \leftarrow B} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

we can observe that,

$$(P_{B \leftarrow B'})^{-1} = P_{B' \leftarrow B}$$

they are inverses of each other.

Q. Find $[P(x)]_{B'}$ where $P(x) = 1 + 2x - x^2$

$\rightarrow \quad P(x) = \alpha v_1 + \beta v_2 + \gamma v_3$

$$1 + 2x - x^2 = \alpha(1+x) + \beta(x + x^2) + \gamma(1 + x^2)$$

$\therefore \quad \alpha + \gamma = 1$

$\alpha + \beta = 2$

$\beta + \gamma = -1$

$\rightarrow \quad \alpha = -\beta + 2 \qquad \therefore \; -\beta + 2 = +1 + \gamma$

$\alpha = -1 + \gamma \qquad \therefore \; \cancel{\gamma}$

$\therefore \; \alpha = 2, \; \beta = 0, \; \gamma = -1$

$$\therefore [P(x)]_{B'} = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}$$

also, $[P(x)]_B$

$\Rightarrow \quad P(x) = \alpha u_1 + \beta u_2 + \gamma u_3$

$\therefore \quad 1 + 2x - x^2 = \alpha 1 + \beta x + \gamma x^2$

$\therefore \; \alpha = 1, \; \beta = 2, \; \gamma = -1$

Now, $P_{B' \leftarrow B} \cdot [\cancel{x}]_B$

$$\tfrac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

## Change of basis matrix :

Let $B = \{u_1, u_2, u_3 \ldots u_n\}$ and

$B' = \{v_1, v_2 \ldots v_n\}$ are the two base

for a vector space $V$. The $n \times n$ matrix

whose coloumns are cordinate vector

$[u_1]_{B'}, [u_2]_{B'} \ldots [u_n]_{B'}$

of the vectors in $B$ w.r.t. $B'$ is denoted

by $P_{B' \leftarrow B}$ and it is given by

$$P_{B' \leftarrow B} = ([u_1]_{B'} \ldots [u_n]_{B'})$$

$$[x]_{B'} = P_{B' \leftarrow B}[x]_B$$

$$P_{B' \leftarrow B} = \left[[u_1]_{B'} \ldots [u_n]_{B'}\right]_{n \times n}$$

* find the change of basis matrices $P_{B' \leftarrow B}$ and $P_{B \leftarrow B'}$ where,

$B = \{1, x, x^2\}$ and $B' = \{1+x, x+x^2, 1+x$

are the bases of $P_2$.

$$P_{B' \leftarrow B} = \left[[u_1]_{B'} [u_2]_{B'} [u_3]_{B'}\right]_{3 \times 3}$$

&

$$P_{B \leftarrow B'} = \left[[v_1]_B [v_2]_B [v_3]_B\right]_{3 \times 3}$$

$\{u_1, u_2, u_3\} = \{1, x, x^2\}$

$\{v_1, v_2, v_3\} = \{1+x, x+x^2, 1+x^2\}$

$$v_1 = \alpha u_1 + \beta u_2 + \gamma u_3 = 1\cdot 1 + 1x + 0x^2 =$$

$$[v_1]_B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

similarly,

$$v_3 = \alpha u_1 + \beta v_2 + \gamma u_3$$

$$\therefore [v_2]_B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore [v_3]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore P_{B \leftarrow B'} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{3 \times 3}$$

thus,

$$P_{B' \leftarrow B} = \left[ [u_1]_{B'} \; [u_2]_{B'} \; [u_3]_{B'} \right]$$

$$[u_1]_{B'} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$ such that $u_1 = \alpha v_1 + \beta v_2 + \gamma v_3$

$$1 = \alpha(1+x) + \beta(x+x^2) + \gamma(1+x^2)$$

$$\alpha + \beta = 0$$
$$\alpha + \gamma = 1$$
$$\beta + \gamma = 0$$

$$\therefore \begin{bmatrix} 1/2 \\ -1/2 \\ +1/2 \end{bmatrix}$$

$$[u_2]_{B'} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$ $u_2 = \alpha v_1 + \beta v_2 + \gamma v_3$

$$x = \alpha(1+x) + \beta(x+x^2) + \gamma(1+x^2)$$

$$\therefore \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$\therefore [u_3]_{B'} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\therefore P_{B' \leftarrow B} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

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Now, $\ker(T) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \;\middle|\; T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \;\middle|\; \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \;\middle|\; y = 0 \right\}$

$= \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \;\middle|\; x \in R \right\} = \left\{ x \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\middle|\; x \in R \right\}$

$= \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$

* let, $T : D \to F$ defined by $T(f(x)) = f'(x)$

Range $= R(T) = \{ T(v) \mid v \in F \}$

$= \{ T(f(x) \mid f(x) \in F \}$

$= \{ f'(x) \mid f'(x) \in D \}$

$= R(T) = D$

$\ker(T) = \{ f'(x) \in D \mid f'(x) = 0 \}$

= set of all constant fnⁿ.

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3> let, D is differential operator
$D : \mathcal{F} \to f$ defined by
$D(f(x)) = \frac{d}{dx}(f(x))$

1> $T(x_1 + x_2) = D(f(x_1) + g(x_1))$
$= \frac{d}{dx}(f(x) + \frac{d}{dx}(gx)$
$= T(x_1) + T(x_2)$

2> $T(\alpha x) = \frac{d}{dx}(\alpha f(x)) = \alpha \frac{d}{dx}(fx)$
$= \alpha T(x)$
H. P.

4> let B Integral operator $(\int)$ defined by $T(f(x))$
$= \int_a^b f(x)\, dx$

$T : C[a,b] \to R$ set of all cont fun on $[a,b]$

$\therefore T$ is L T.

5. Let $T : \mathbb{C}(\mathbb{C}) \to \mathbb{C}(\mathbb{C})$ defined by $T(z) = \bar{z}$
$\Rightarrow$ let, $z, \omega \in \mathbb{C}$
$\to T(z + \omega) = \overline{z + \omega}$
$= \bar{z} + \bar{\omega}$
$= T(z) + T(\omega)$
$\to T(\alpha z) = \overline{\alpha z} = \bar{\alpha} \cdot \bar{z}$
$= \bar{\alpha}\, T(z)$
$\neq \alpha T(z)$
no L.T.

In $M_{22}$ $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and

$B' = \{A, B, C, D\}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

are two Bases find $P_{B' \leftarrow B}$, $P_{B \leftarrow B'}$, $[P(x)]_B$, $[P(x)]_{B'}$, $P_{B' \leftarrow B} \cdot [x]_B$ for $x = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

→ $P_{B' \leftarrow B} =$

$U_1 = \alpha U_1 + \cancel{\beta} V_2 + \beta V_3 + \gamma V_4$

$U_2 =$ — 1 —

$U_3 =$ — 1 —

$[U_1]_{B'} = E_{11} = \alpha V_1 + 0$ - - - -

$\therefore [E_{11}]_{B'} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

similarly, $[U_2]_{B'} = \begin{bmatrix} -1 \\ +1 \\ 0 \\ 0 \end{bmatrix}, [U_3]_{B'} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, [U_4]_{B'} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$\therefore P_{B' \leftarrow B} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

also, $P_{B \leftarrow B'} = [V_1]_B = \alpha U_1 + \beta U_2 + \gamma U_3 + \delta U_4$

$\therefore U_1 = V_1$, $\therefore \beta \gamma \delta = 0$

$\therefore [U_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$