Syllabus

1) Vector spaces
2) Linear Transformation
3) Inner product space (IPS)
4) Matrix Norms
5) Diagnolization (Eigen vector & Eigen values)
6) Singular value decomposition (SVD).

▷ Matrix & its types

① Matrix: Arrangement of $m \cdot n$ numbers into rectangular array.

- Arrangement of m-dimensional vector in n-columns in a rectangular Array.

- Vector is a vertical arrangement of scalars.

▷ Types

① Row matrix - The matrix in which only one row is present.

$$A = [a_{11} \; a_{12} \; \cdots \; a_{1n}]_{1\times n}$$

② column matrix - The matrix in which only one column is present.

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}_{m\times 1}$$

③ square matrix: The matrix in which equal no. of rows & columns are present.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{n\times n} = [a_{ij}] \;\; \forall \; i,j = 1,2,\ldots,n$$

④ zero Matrix (Null matrix): All entries are zero

$$A = [0_{ij}]_{n\times m} \quad \forall \; i,j$$

⑤ Diagonal Matrix: It is defined as and

$$A = \begin{cases} d_{ij} \;; & \forall \; i = j \\ 0 \;; & i \neq j \end{cases}$$

$$A = \begin{bmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & \cdots & 0 \\ 0 & \vdots & d_{33} & \vdots \\ 0 & \cdots & \cdots & d_{nn} \end{bmatrix}_{n\times n}$$

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(6) Scalar Matrix : It is diagonal matrix and it is given by,

$A = \begin{cases} d & ; i=j \\ 0 & ; j \neq j \end{cases}$ $A = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & \cdots & \\ 0 & \cdots & \cdots & d \end{bmatrix}_{n \times n}$

(7) Identity matrix

(8) Lower triangular

(9) upper triangular

(10) Symmetric $\boxed{A = A^T}$

(11) Skew symmetric $A = -A^T$

Note : every square matrix can be written as sum of symmetric & skew-symmetric matrix.

(12) conjugate $A = [(x+iy)_{ij}]_{n \times n} \xrightarrow{\text{conjugate}} [(x-iy)_{ij}]_{n \times n}$

(13) Hermition

↳ square matrix, if $A = A^*$ where $A^* = (\bar{A})^T$

(14) Skew Hermition

$A^* = -A$

(15) orthogonal

If A is invertiable → orthogonal ⇒ $A^T = A^{-1}$ i.e. $AA^T = A^TA = I$

(16) Idempotent

if $A^2 = A$

(17) Involuntary

Let A be invertible → involuntary ⇒ $A^2 = I$ $(A = A^{-1})$

(18) Nilpotent $A^n = O_{n \times n}$

↳ index of nilpotent

(19) Unitary matrix

Let A be square complex matrix & invertible

if $A^* = A^{-1}$

(or) $A^*A = AA^* = I$

$\begin{bmatrix} 1 + i/2 & 1 - i/2 \\ 1 - i/2 & 1 + i/2 \end{bmatrix}$ $\begin{bmatrix} 1/\sqrt{3} & (1+i)/\sqrt{3} \\ (1-i)/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$

(20) Normal matrix

A is called normal if

$A^*A = AA^*$

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Vector Space

▷ Binary operation on a Set

Let A be the any non-empty set. we say $*$ is Binary operation on set A if $*$ is mapping / function from

$$A \times A \longrightarrow A$$

$$(* : \underbrace{A \times A}_{} \longrightarrow A)$$

↳ cartesian product of set A.

$$(a,b) \rightarrow c \,,\ c \in A$$

i.e. $a * b \in A$

eg. ① $A = N$, $* = +$
(usual addition)

Soln: If $a,b \in N$

↳ $a * b = a + b \in N \quad \forall\, a,b \in N$

$* = +$ is a binary operation on set N

② $A = Z$, $* = -$

'$-$' is Binary operation on Z, since

$a, b \in Z$

$a * b = a - b \in Z$

③ $A = Q$, $* = \div$

$a = 1$, $b = 0$

$1 * 0 = 1 \div 0$ is not defined

so if $A = Q \setminus \{0\}$ then $* = \div$ is Binary operation on set A.

④ $A = Q^c$, $* = +$

$a = \sqrt{2}$, $b = -\sqrt{2}$

$a * b = (\sqrt{2}) + (-\sqrt{2}) = 0 \notin Q^c$

⑤ $A = \mathbb{R}$, $a * b = a^b \ \forall\, a,b \in A$

$a = -1$, $b = 1/2$

$a * b = a^b = (-1)^{1/2} = i \notin \mathbb{R}$

⑥ $A = P(\mathbb{N})$, $* = \cup$

$M, N \in P(\mathbb{N}) \; \forall \; M, N$

$M \cup N \subseteq \mathbb{N} \in P(\mathbb{N})$

▷ Identity element of a set :

Let, A be the any non-empty set $e \in A$ is said to be an identify element if $*$ is Binary operation on A then e is identify element of A if

$$a * e = e * a = a \quad \forall \; a \in A$$

eg ① $A = \mathbb{R}$, $* = +$

clearly 0 is the identify of $\mathbb{R}$ since

$$a + 0 = 0 + a = a \quad , \forall \; a \in \mathbb{R}$$

Here, 0 is called as additive identity of $\mathbb{R}$.

② $A = \mathbb{R}$, $* = \times$

then 1 is the identity element of $\mathbb{R}$ w.r.t. Binary operation '$\times$'. since $a \times 1 = 1 \times a = a$

(1 is called as multiplicative identity of $\mathbb{R}$)

③ $A = M_n(\mathbb{R})$, $* = +$

↳ Null matrix

$A = M_n(\mathbb{R})$, $* = \times$

(Null matrix, I, 0)

④ $A = P(\mathbb{N})$, $* = \cup$

Here, identity element of a set $P(\mathbb{N})$ is ϕ.

since, $M \cup \phi = \phi \cup M = M \quad \forall \; M \in P(\mathbb{N})$

⑤ $A = P(\mathbb{N})$, $* = \cap$

$\mathbb{N}$ is identity element of $P(\mathbb{N})$ w.r.t. Binary operation '$\cap$'

since , $\mathbb{N} \cap A = A \cap \mathbb{N}$, $\forall \; A \in P(\mathbb{N})$

▷ Associative & commutative Binary operation on set :

Let A be the any non-empty set and $*$ be any Binary operation on A then $*$ is said to be commutative Binary operation.

IF $a*b = b*a$, $\forall\ a,b \in A$ and $*$ is Associative if

$$(a*b)*c = a*(b*c) \ \forall\ a,b,c \in A$$

eg ① $A = M_n(\mathbb{R})$, $* = \times$

It is not commutative but Associative

▷ Inverse of a element :

Let A be the any non-empty set and $*$ is the Binary operation on A, b is said to be inverse of a w.r.t. binary operation $*$ if,

$$a*b = b*a = e$$

(where, e is identity element of A w.r.t. B. O. $*$)

eg. ① $A = \mathbb{R}$, $* = +$

$\forall$, $a \in \mathbb{R}$ then exist an inverse.

w.r.t. B.O. '+', $b = -a$

such that, $a*b = a+(-a) = 0$

$(-a)+a = 0$

② $A = \mathbb{R}$, $* = \times$

$\forall$, $a \in \mathbb{R}$, there exist an inverse

w.r.t. B.O. '$\times$', $b = \frac{1}{a}$

such that, $a*b = a \times \frac{1}{a} = 1$

$\frac{1}{a} \times a = 1$

eg. Q. let $A = \mathbb{R}$, $* \to$ B.O on $A = \mathbb{R}$ and it is defined as

$$a*b = a+b+a.b \ \forall\ a,b \in \mathbb{R}$$

Find ① Identity element w.r.t. $*$

② Inverse of element (possible) in $\mathbb{R}$.

Field : Let $\phi \neq F$ be any set together with binary operations addition (+) and multiplication (·) is said to be Field if following

properties are satisfied:

1) $a+(b+c) = (a+b)+c$, $\forall\, a,b,c \in F$

(Associativity w.r.t. +)

2) $a+0 = 0+a = a$ .... (Existence of additive identity)

3) $a+(-a) = (-a)+a = 0$ .... (Existence of additive inverse)

4) $a+b = b+a$ $\forall\, a,b \in F$...(commutativity)

5) $a\cdot(b\cdot c) = (a\cdot b)\cdot c$ ..... (Associativity w.r.t. (·))

6) $a\cdot 1 = 1\cdot a = a$ ..... (Existence of multiplication identity)

7) $a\cdot b = b\cdot a = 1$ ..... (Existence of multiplicative inverse)

(given $b=1$ or $b=a^{-1}$) $(0 \neq a)$

8) $a\cdot(b+c) = a\cdot b + a\cdot c$ $\forall\, a,b,c \in F \rightarrow$ (left distribution)

$(a+b)\cdot c = a\cdot c + b\cdot c$ $\forall\, a,b,c \in F \rightarrow$ (Right distribution)

9) $a\cdot b = b\cdot a$ $\forall\, a,b \in F$ (commutativity)

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Group $(G, *)$

$\downarrow$ property (i) (ii) (iii) hold

Ring $(R, +, \cdot)$

(i) $(R, +)$ is a group

(ii) $(R, \cdot)$ is a semigroup

3) property holds

ex $(\mathbb{R}, +, \cdot)$

$(\mathbb{Z}, +, \cdot)$

$(\mathbb{Q}, +, \cdot)$

Field

$(\mathbb{R}, +, \cdot)$

skew Field $\leftarrow (\mathbb{Q}, +, \cdot)$

Universal Field $\leftarrow (\mathbb{C}, +, \cdot)$

Group

① $a * (b * c) = (a * b) * c$

② $a * e = e * a = a$

③ $a * b = b * a = e$

④ $a * b = b * a$

ex : $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$

$(\mathbb{Q}, +)$, $(\mathbb{C}, +)$

$(\mathbb{R}^*, \times)$

Alternative definition of Field

Let, $\phi = \mathbb{F}$ be any set with two Binary operation addition $(+)$ and multiplication $(\cdot)$ is Field if Following properties are satisfied.

(i) $a \cdot b = b \cdot a$ $\forall a, b \in \mathbb{F}$ (commutativity)

(ii) Existence of inverse of each non-zero element in $\mathbb{F}$.

i.e.; $\forall\ 0 \neq a \in \mathbb{F}$, $\exists\ b \in \mathbb{F}$

st. $a \cdot b = b \cdot a = 1$

$\Rightarrow$ b is called inverse of a and it is defined by

$b = a^{-1}$ or $b = \frac{1}{a}$

Ex : $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are standard Fields

(infinite)

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▷ Finite Field : $(\mathbb{F}_p, +_n, \times_n)$

The Field in which Finite no. of elements are present.

Z_n = set of all remainder when we divide any integer by a number $n \in Z^+$.

$Z_2 = \{0, 1\}$, $Z_n = \{0, 1, 2, \ldots\ldots, n-1\}$

$Z_3 = \{0, 1, 2\}$

We define B.O called addition module n $(+_n)$ and multiplication module n $(\times_n)$

$\forall$ a, b $\in Z_n$ we have,

$a +_n b$ = least +ve remainder when we divide a+b by n.

$a \times_n b$ = least +ve remainder when we divide a * b by n.

ex : $Z_3 = \{0, 1, 2\}$ $1 +_3 2 = 0 = \text{rem}\left(\frac{1+2}{3}\right)$

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Q H.W. Find all additive inverse and multiplicative inverse for all elements in $\mathbb{Z}_{11}$.

Vector Spaces ($V(\mathbb{F})$)

Let, V be a non-empty set of object on which two binary operations i.e. addition (+) and multiplication ($\cdot$)

We say V is a vector space over a field $\mathbb{F}$ if following properties/conditions are satisfied for all $u, v, w \in V$ and $k, m \in \mathbb{F}$

1) If u, v are the object in V then $u+v$ is in V

$\because (\forall, u, v \Rightarrow u+v \in V)$

2) $u+(v+w) = (u+v)+w$ (vector addition must be associative)

3) There is an object 0 in V called zero vector such that

$0+u = u+0$, $\forall u \in V$ (existence of zero vector)

4) For each 'u' in V, there is an object '$-u$' such that

$u+(-u) = (-u)+u = 0$

▷ $(V, +)$ is a group

5) If k is any scalar in $\mathbb{F}$ and u is any object in V then $k \cdot u$ is in V

(i.e. $\forall k \in \mathbb{F}$, $u \in V \Rightarrow ku \in V$)

6) $k(u+v) = ku + kv$ distributes over vector add^n

7) $(k+m)u = ku + mu$ distributes over scalar addition

8) $k(mu) = (km)u$ Abelian group $\Rightarrow (V, +)$

9) $1 \cdot u = u$, $\forall u \in V$

↳ multiplicative identity in $\mathbb{F}$

10) $u+v = v+u$ $\forall u, v \in V$ (commutative)

▷ Where, objects in the V are called as vectors and the elements in the $\mathbb{F}$ (field) are called as scalars.

eg. ① $V = \{0\}$ forms a vector space over ~~set~~ field of Real Numbers. $\mathbb{R}$.

① $V = \mathbb{R}^n$, $\mathbb{F} = \mathbb{R}$

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \; x_1, \dots, x_n \in \mathbb{R} \right\}$$

$\mathbb{R}^n(\mathbb{R})$ forms a vector space over co-ordinate wise addition (+) scalar multiplication defined by,

If $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $k \in \mathbb{F}$

$$u + v = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}, \quad ku = k \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix}$$

HW. Show that $\mathbb{R}^n(\mathbb{R})$ is vector space

② V = set of 2×2 matrices over $\mathbb{R}$, $\mathbb{F} = \mathbb{R}$

$$u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \; v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}, \; w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}, \; k \in \mathbb{R}$$

Forms vector space w.r.t. operation + and scalar multiplication defined by

$$u + v = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} \text{ and } ku = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

③ $V = \mathbb{R}$, $\mathbb{F} = \mathbb{Q}$

↳ Forms a vector space.

$\mathbb{R}(\mathbb{Q})$ forms a vector space

w.r.t usual addition and usual multiplication

④ $V = \mathbb{Q}$, $\mathbb{F} = \mathbb{R}$

$k = \sqrt{2}$, $v = \frac{1}{2}$

$ku = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \notin \mathbb{Q}$

$\mathbb{Q}(\mathbb{R})$ does not form vector space

⑤ $\mathbb{Q}(\mathbb{Q})$ ~~is~~ forms vector space.

⑥ $V = \mathbb{Z}_p$, $\mathbb{F} = \mathbb{Z}_p$ $(\mathbb{Z}_p; +_n, \times_n)$

↳ forms a vector space

$\mathbb{Z}_p(\mathbb{Z}_p)$

addi^n $(+_n)$ & scalar mul^n $(\times_n)$

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H.W.

V = set of all polynomials of degree $\leq n$ with coefficient from $\mathbb{R}$.

$$V = \{a_0 + a_1x + a_2x^2 + \dots + a_nx^n \mid a_i \in \mathbb{R},\ i = 1, \dots, n\}$$

$\mathbb{P}_n(\mathbb{R})$, $\mathbb{F} = \mathbb{R}$

Prove that $\mathbb{P}_n(\mathbb{R})(\mathbb{R})$ forms a vector space with respect to operation '+' and multiplication (scalar) defined by

For $p_1(x), p_2(x) \in P_n(\mathbb{R})^n$

$\Rightarrow p_1(x) + p_2(x)$

$\Rightarrow k\,p_1(x)$

Here,

let $v_1 = \{a_0 + a_1x + a_2x^2 + \dots + a_nx^n\}$

$v_2 = \{b_0 + b_1x + b_2x^2 + \dots + b_nx^n\}$

$\therefore v_1 + v_2 = \{(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \dots + (a_n+b_n)x^n\} \in R$

closed under vector addition

let k be any scalar

$\therefore kv_1 = \{ka_0 + ka_1x + ka_2x^2 + \dots + ka_nx^n\} \in \mathbb{R}$

closed under scalar multiplication

Thus $\mathbb{P}_n(\mathbb{R})$ forms a vector space.

(7) $V = \mathbb{C}$, $\mathbb{F} = \mathbb{R}$

$\mathbb{C}(\mathbb{R})$ forms a vector space under usual vector addition and usual multiplication

(8) $V = R$; $\mathbb{F} = \mathbb{C}$

$\mathbb{R}(\mathbb{C})$ does not form a vector space.

$k.u = R$

$i \in \mathbb{C}$, $u = 100 \in \mathbb{R}$

$\Rightarrow i.100 = 100i \notin \mathbb{R}$

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* Field is a vectorspace over its subfield

$\mathbb{Q} \in \mathbb{R} \in \mathbb{C}$

- The field F forms a vector space over its subfield.
- $\mathbb{F}$ (F) forms a vector space,
  F is subfield of $\mathbb{F}$.

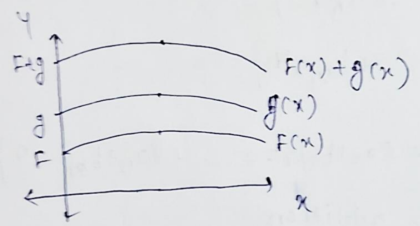
③ V = set of all real valued functions $f: \mathbb{R} \to \mathbb{R}$

$V = \{f: \mathbb{R} \to \mathbb{R} \mid f \text{ is real valued}\}$ and $\mathbb{F} = \mathbb{R}$

$\Rightarrow$ forms a vector space under operations

i) $(f+g)(x) = f(x) + g(x)$, $\forall f, g \in V$

ii) $(kf)(x) = kf(x)$ $\forall$ $k \in \mathbb{R}$, $f \in V$



④. $V = \mathbb{R}^+$, $\mathbb{F} = \mathbb{R}$ In $\mathbb{R}^+$ 1 is the value of 0.

$u + v = uv$

$ku = u^k$

in this $0 = 1$

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eg.① Let $V = \mathbb{R}^2$ and define addition and scalar multiplication as follows

If $u, v \in \mathbb{R}^2$ and $k \in \mathbb{R}$

$u = (u_1, u_2)$

$v = (v_1, v_2)$

$u+v = (u_1+v_1, u_2+v_2)$

$k \cdot u = (k \cdot u_1, 0)$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$u+v = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$$

given, $ku = (ku_1, 0)$

but $1 \cdot u = (u_1, u_2) \neq (u_1, 0)$

Thus, $V = \mathbb{R}^2$ does not form vector space.

② V = set of all positive real numbers ($\mathbb{R}^+$) (unusual vector space)

$\mathbb{F} = \mathbb{R}$

Let $u, v \in V$, $k \in \mathbb{R}$

we define operations on V as follows,

$u + v = u \cdot v$ (usual mult^n)

& $k \cdot u = u^k$ (usual exponential) V, Forms a vector space?

Sol^n

$k(u+v) = ku + kv$

LHS $= k(u+v)$

$= (u+v)^k$

$= (u \cdot v)^k \ldots (u+v = u \cdot v)$

$= u^k \cdot v^k$

$= (ku)(kv)$

$= ku + kv$

$=$ RHS

This forms a vector space.

To find additive inverse and multi. inverse of each element in $(Z_{19}, +_{19}, \times_{19})$

$Z_{19} = \{0, 1, 2, 3, \ldots, 18\}$

The additive inverse of each element in Z_{19}

$a +_{19} b = b +_{19} a = 0$

Let $a = 0$, $b = 0$

$a = 1$, $b = 18$ $\Rightarrow$ $\frac{1+18}{19} = \frac{19}{19}$ $\boxed{rem = 0}$

$a = 2$, $b = 17$

$a = 3$, $b = 16$

$a = 4$, $b = 15$

$a = 5$, $b = 14$

$a = 6$, $b = 13$

$a = 7$, $b = 12$

$a = 8$, $b = 11$

$a = 9$, $b = 10$

$a = 10$, $b = 9$

$a +_n b = r\left(\frac{a+b}{n}\right)$, r = least remainder

* Multiplicative inverse

we want $b \in Z_{19}$ for each non-zero element $a \in Z_{19}$ such that

$a \times_{19} b = 1$

$a = 1$, $1 \times_{19} b = 1$, $b = 1$

$\frac{1 \times 1}{19}$ = rem = 1

$a = 2$, $\frac{2 \times_{19} b}{19} = 1$, $b = 10$

$2 \times_{19} 10$ rem = 1

$a = 3$, $\frac{3 \times_{19} b}{19} = 1$

$\frac{3 \times b}{19} = 1$ $b = 13$ $\frac{3 \times 13}{19} = \frac{39}{19}$ $\Rightarrow$ rem = 1

Theorem (1)

Let V be the vector space over a field $\mathbb{F}$ and k be the any scalar, then

(1) $0 \cdot u = 0$

(2) $k \cdot 0 = 0$

(3) $(-1) \cdot u = -u$

(4) If $\underset{\hookrightarrow \text{ integral property}}{\underline{k \cdot u = 0}}$ then $k = 0$ or $u = 0$

Proof: (1) To prove: $0 \cdot u = 0$

since $0 + 0 = 0$, $0 \in \mathbb{F}$

$(0+0) \cdot u = 0$

$0u + 0u = 0$ (By 8)

Adding $-0u$ on both sides

$0u + 0u + (-0u) = 0 + (-0u)$

$0 \cdot u + (0 \cdot u + (-0 \cdot u)) = -0 \cdot u$

$0 \cdot u + 0 = -0 \cdot u$

LHS $= 0 \cdot u = (0+0)u$

$(0+0)u = 0 \cdot u$

$0 \cdot u + 0 \cdot u = 0 \cdot u$ .... By property 8

$0u + 0 \cdot u + (-0 \cdot u) = 0 \cdot u + (-0 \cdot u)$ .... By property 4

$0 \cdot u + 0 = 0$

$0 \cdot u = 0$

(2) To prove: $k \cdot 0 = 0$

$k \cdot 0 = (0+0)k$

$(0+0)k = 0 \cdot k$

$0 \cdot k + 0 \cdot k = 0 \cdot k$

$0 \cdot k + 0 \cdot k + (-0 \cdot k) = 0 \cdot k + (-0 \cdot k)$

$0 \cdot k = 0$

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(3) To prove : $(-1)u = -u$

let $(-1)u = -u$

i.e. $u + (-1)u = 1.u + (-1)u$

$= (1+(-1))\,u$

$= 0.u$

$u+(-1)u = 0$

$(-1)u$ is additive inverse of u.

$\boxed{(-1)u = -u}$

(4) Here, $k.u = 0$ —— ⊛

suppose $k \neq 0$, $k \in \mathbb{F}$

$\Rightarrow \exists\, k^{-1} \in \mathbb{F}$

$\therefore k^{-1}.k.u = k^{-1}.0$

$(1).u = 0$

$\therefore u = 0$ – by property (2)

If $k = 0$

$k.u = 0$

$\therefore 0.u = 0$

$\boxed{u = 0}$ – by property (1)

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Prob(T/F)

1) Is there a vector space with exactly two elements (distinct)

Yes.

eg. $Z_2(Z_2)$, $\mathbb{R}(\mathbb{R})$ $= \{0,1\}$

Every field is vector space over its subfield.

▷ subspace (W)

1) Let, W be the subset of a vector space V is called a subspace of V if W is itself forms a vector space. over the field $\mathbb{F}$ under the same addition and scalar multiplication defined on V.

Note: $\mathbb{Q}(\mathbb{R})$ → does not form vector space

↙ subset ↘ superset

as Field forms v.s. over subfield

i.e. $\mathbb{R}(\mathbb{Q})$ → form a sub vector space.

Theorem: Let W be the non-empty subset of vector space V then W is a subspace of V if and only if following conditions are holding.

1) If $u, v \in W$ then $u+v \in W$

II) If $k \in \mathbb{F}$ (any scalar) $\forall u \in W$ then $k \cdot w \in W$

Proof: Suppose W is a subspace of V

By defn W is a vector space.

obvisously (I), (II) will hold in W.

conversely

Assume that, (a) & (b) are hold in W

i.e. $u+v \in W$, $\forall u, v \in W$

$k \cdot u \in W$, $\forall u \in W$, $k \in \mathbb{F}$.

As $k \cdot u \in W$

If $k = 0 \in \mathbb{F}$

$0 \cdot u = 0 \in W$

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Let $k = -1 \in \mathbb{F}$

$k \cdot u = (-1) \cdot u = -u \in W$

$\Rightarrow$ W is a vector space $\Rightarrow W \in V$

Thus W is a subspace of V.

properties ①, ③, ④, ⑥ are satisfied.

① $u + v \in W$

③ $u + 0 = 0 + u = u$

④ $u + (-u) = (-u) + (u) = 0$

⑥ $ku \in V$

other properties are inhibited by W from V.

eg.

① Let $W = \{0\} \subseteq V$ over the field $\mathbb{F}$.

W is a subspace of V

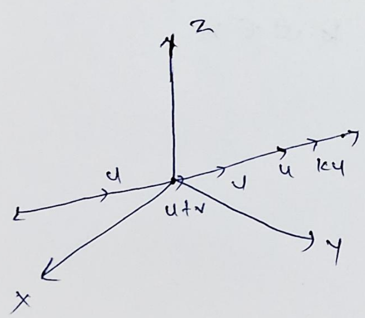
✓ $0 + 0 = 0 \in W$

✓ $k \cdot 0 = 0 \in W$

zero subspace.

② Show that, if W is a line passing through origin in $\mathbb{R}^3$. When W is subspace of $\mathbb{R}^3$.

Let $W = ax + by$



W is closed under vector addition.

$u + v \in W$

closed under scalar multiplication

H.W

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{array}{c} x = at,\ y = bt,\ z = ct \\ c, a, b \in \mathbb{R} \end{array} \right\}$$

Forms subspace in $\mathbb{R}^3$?

It forms it.

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③ If W is a plane passing through origin in $\mathbb{R}^3$ then W forms a subspace of $\mathbb{R}^3$.

$ax+by+cz=0$ eqn of plane passing through origin.

HW

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; ax+by+cz=0 \right\}$$

Yes, it forms.

Let W is a subset of $\mathbb{R}^2$ defined by $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x,y \geq 0 \right\}$

Is W is subspace of $V = \mathbb{R}^2$?

1) If $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in W$

$\Rightarrow u+v = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix} \in W$

2) Let $k \in \mathbb{R}$ and $u = \begin{pmatrix} x \\ 1 \end{pmatrix} \in W$

$k.u \notin W$ take $(k=-1)$ $-u \notin W$

It is not ~~W~~ subspace.

④ Let W be the set of all symmetric matrices. Show that W is a subspace of $M_{n \times n}$. = set of all $n \times n$ matrices.

Given, $W = \left\{ A \in M_{n \times n} \mid A = A^T \right\}$

Let $A, B \in W$

$A = A^T$ & $B = B^T$

$(A+B)^T = A^T + B^T$

$= A + B$

Hence, $(A+B)^T = (A+B)$ is a symmetric matrix $\in W$

Let $k \in \mathbb{R}$, $A \in W \Rightarrow A = A^T$
↳ any scalar

$(k.A)^T = k.A^T = kA$ is a symmetric matrix $\in W$.

Thus, W is a subspace of $M_{n \times n}$.

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Q. (i) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} x = at,\ y = bt,\ z = ct \right\}$ is subspace in $\mathbb{R}^3$?

We have,

let $x = 0$
$y = 0$
$z = 0$

$\therefore \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$

Now,

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} at \\ bt \\ ct \end{bmatrix} = t \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$$

let

$$X_2 = t \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$$

$$X_1 + X_2 = t \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix} \in \mathbb{R}^3$$

let $\alpha \in \mathbb{R}$

$$\therefore \alpha X_1 = \alpha t \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$$

$$= t \begin{bmatrix} \alpha a_1 \\ \alpha b_1 \\ \alpha c_1 \end{bmatrix} \in \mathbb{R}$$

Thus W forms a subspace.

Q.(2)

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid ax+by+cz=0 \right\} \text{ is subspace in } \mathbb{R}^3?$$

Sol^n

let $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

* (1) $V = M_{n\times n}$ = set of all $n\times n$ matrices W = set of all skew-symmetric matrices

It forms a subspace

(2) set of all lower triangular matrices
(3) set of all upper triangular matrices
$\}$ forms a subspace.

(4) set of all matrices with zero determinant.
↳ Not a subspace

eg. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$|A| = 0$ $\quad$ $|B| = 0$

But $|A+B| = 1 \neq 0$

Thus It does not form subspace.

(5) set of all matrices whose trace is zero.

Forms a ~~me~~ subspace.

Trace $(A) = 0$ $\quad$ Trace $(B) = 0$

then

Trace $(A+B) = 0$

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(*) Let C be the set of all continous real valued function.

and defined on $\mathbb{R}$.

D be the set of all differentiable functions defined on $\mathbb{R}$.

Show that, C and D are the subspaces of vector space F, real valued function.

Soln

$C = \{F : \mathbb{R} \to \mathbb{R} \mid F \text{ is continous}\}$

$D = \{F : \mathbb{R} \to \mathbb{R} \mid F \text{ is differentiable function}\}$

$F = \{F : \mathbb{R} \to \mathbb{R}\}$

Let $F_1, F_2 \in C$

F_1 and F_2 are continous

By algebra of functions

$\Rightarrow$ $F_1 + F_2$ is also continous

$F_1 + F_2 \in C$.

~~Th~~ Let $k \in \mathbb{F}$

$\therefore kF_1$ is also continous

$kF_1 \in C$

Thus C forms a subspace of V.

Hence proved.

Let $F_1, F_2 \in D$

F_1 and F_2 are differentiable.

By algebra of functions

$F_1 + F_2$ is also differentiable.

$F_1 + F_2 \in D$.

Let $k \in \mathbb{F}$

$kf_1 \in D$

Thus D is subspace of V. $\mathbb{F}$.

eg. ① set of all non-continous & non-differentiable function ~~are~~ is not subspace.

f_1 and $-f_1$ are not differentiable

then $f_1 + (-f_1) = 0$ and 0 is differentiable thus it does not form subspace.

② P = set of all polynomials forms a subspace.

$$\underbrace{P \subset C}_{\text{subspace}} \quad \underbrace{C \subset D}_{\text{subspace}} \quad \underbrace{\subset \mathbb{F}}_{\text{subspace}}$$

③ Let S be the set of all ~~do~~ solutions of differential equations $y'' + y = 0$ then show that S is a subspace of F. (set of real valued functions).

$$S = \{f \in F \mid f \text{ is solution of } y'' + y = 0\}$$

let $f_1, f_2 \in S$

$\Rightarrow f_1$ and f_2 are solutions of $y'' + y = 0$

$\Rightarrow f_1'' + f_1 = 0$ —① and $f_2'' + f_2 = 0$ —②

Adding eqn ① and ②

$$f_1'' + f_1 + f_2'' + f_2 = 0$$

$$(f_1'' + f_2'') + (f_1 + f_2) = 0$$

$$(f_1 + f_2)'' + (f_1 + f_2) = 0$$

Thus, $f_1 + f_2$ is a solution of $y'' + y = 0$

$f_1 + f_2 \in S$.

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$e^{ix} = \cos x + i\sin x$

$y = c_1 e^{ix} + c_2 e^{-ix}$

$y = c_1 \cos x + c_2 \sin x$

$c_1, c_2 \in \mathbb{C}$

$y'' + y = 0$ — (1)

$r^2 + 1 = 0$ (auxilliary equation)

$r^2 = -1$

$r = \sqrt{-1}$

$r = i$ ⇒ for complex solution

$y = c_1 e^{ix} + c_2 e^{-ix}$

put $y = \cos x$ in (1) $\quad \cos'' x + \cos x = 0$

$-\cos x + \cos x = 0$ (satisfied)

HW Show that following are subspaces of given vector spaces.

a) $W = \left\{ \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \middle| \ x = y,\ z = -t \right\}$ $V = \mathbb{R}^4$

b) $W = \left\{ a + bx - bx^2 + ax^3 \mid a, b \in \mathbb{R} \right\}$

c) (i) $W = \left\{ \begin{bmatrix} a & b \\ -a & a \end{bmatrix} \middle| \ a, b \in \mathbb{R} \right\}$ $V = M_{2\times 2}$

(ii) $W = \left\{ \begin{bmatrix} a & a+1 \\ 0 & b \end{bmatrix} \middle| \ a, b \in \mathbb{R} \right\}$ $V = M_{2\times 2}$

▷ Linear combination and co-ordinate Vector

Let $v_1, v_2, \ldots v_k$ be the vector in V (V is a vector space over field $\mathbb{F}$) then we say $u \in V$ is in linear combination of $v_1, v_2, \ldots v_k$ if there are constant / scalars $c_1, c_2, \ldots c_k \in \mathbb{F}$ such that

$$u = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

eg. ① $W = \{1, x, x^2\}$

$$\text{span}(W) = \{a_0 1 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

↳ set of all 2 degree polynomial.

② $W = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$$\text{span}(W) = \left\{ \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$$

↳ $\mathbb{R}^2$

$u \in V(F)$, $S = \{v_1, v_2, \ldots, v_k\} \in V$ if $\exists$ $c_1, c_2, \ldots, c_k \in \mathbb{F}$ such that $[u = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k]$ and the co-ordinate vector of u with respect to vector $v_1, v_2, \ldots v_k$ is given by,

$$[u] = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

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② $V = M_{2\times 2}(\mathbb{R})$

$u = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

Linear combination = $u = c_1 A + c_2 B + c_3 C + c_4 D$

③ $p(x) = 1 + x^2$

$p_1(x) = 1 + x$

$p_2(x) = 1 - x^2$

$p(x) = c_1 p_1(x) + c_2 p_2(x)$

Do not form linear combination.

$1 + x^2 = (c_1 + c_2) + c_1 x - c_2 x^2$

$c_1 + c_2 = 1$ $\qquad -c_2 = 1$

$c_1 = 2 \; (c_1 = -1)$ $\qquad c_2 = -1$

but $c_1 = 0$ (constant)

▷ Spanning set of a vector space V

If $S = \{v_1, v_2, \dots v_k\}$ is the set of vectors in a vector space V, then the set of all linear combinations of $v_1, v_2, \dots v_k$ is called span of $v_1, v_2, \dots v_k$.

and it is denoted by span of $(v_1, v_2, \dots v_k)$ or $\langle v_1, v_2, \dots v_k \rangle$ or span(s) and if span(s) = V then 's' is called as spanning set of V.

$$\text{span}(s) = \{c_1 v_1 + c_2 v_2 + \dots + c_k v_k \mid c_1, c_2, \dots c_k \in \mathbb{F}\}$$

eg. $\text{span}(1) = \{c_1 \cdot 1 \mid c_1 \in \mathbb{R}\} = \mathbb{R}$

In general,
the spanning set of $\mathbb{R}^n$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0_n \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$

eg. ① Show that the polynomials $1, x, x^2$ spans $P_2(\mathbb{R})$.

$$\underset{(1,x,x^2)}{\text{span}(s)} = \{c_1 \cdot 1 + c_2 x + c_3 x^2 \mid c_1, c_2, c_3 \in \mathbb{R}\} = P_2(\mathbb{R}).$$

In general, the spanning set of $P_n(\mathbb{R})$ is $(1, x, x^2, \ldots\ldots x^n)$

② $V = M_{2\times3}$, what is spanning set of V?

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times3} = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* Elementary matrices

$$E_{ij} := \begin{cases} 1 & \text{if } (i,j)^{th} \text{ entry} \\ 0 & \text{if otherwise.} \end{cases}$$

③ In P_2, Determine whether $r(x) = 1 - 4x + 6x^2$ is in the span of $(p(x), q(x))$. where $p(x) = 1 - x + x^2$, $q(x) = 2 + x - 3x^2$.

$$r(x) = c_1 p(x) + c_2 q(x)$$

$$1 - 4x + 6x^2 = c_1(1 - x + x^2) + c_2(2 + x - 3x^2)$$

$$1 - 4x + 6x^2 = (c_1 + 2c_2) + (-c_1 + c_2)x + (c_1 - 3c_2)x^2$$

$$\therefore \quad c_1 + 2c_2 = 1$$
$$+ \quad -c_1 + c_2 = -4$$
$$3c_2 = \cancel{0}-3$$

Thus, $c_1 = 3$ & $c_2 = -1$

$c_2 = -1$

$\cancel{c_1 = 1 - 2c_2}$

$\cancel{= 1 - 10}$

$\cancel{c_1 = -9}$

$c_1 = 1 + 2$

$c_1 = 3$

$\therefore \cancel{c_1 = -9} \; \cancel{\&} \; c_2 = 5$

(4) In F, is $\sin 2x$ is in span of $(\sin x, \cos x)$

By definition,

$$\sin 2x = a\sin x + b\cos x$$

$$2\sin x \cdot \cos x = a\sin x + b\cos x$$

$x = 0$ put $x = 0$

$x = \pi$

(i) $c_1(0) + c_2(1) = 0$

$x = 3\pi/_2$

$c_2(0)$

(ii) $x = \pi/2$

$$0 = c_1(1) + c_2(0)$$

$$c_1 = 0$$

(iii) $\pi/4 = x$

$$\sin 2\times\pi/4 = a\sin\pi/4 + b\cos\pi/4$$

$$\sin\pi/2 = a\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}b$$

$$1 = \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b$$

$\sin 2x$ is not in span of $(\sin x \cdot \cos x)$

$$\alpha + \beta = a$$

$$\alpha + \gamma = c$$

$$\alpha + \gamma = b$$

$$\beta = d$$

$\beta = d$

$\alpha = a - d$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 0 & 1 & c \\ 1 & 0 & 1 & b \\ 0 & 1 & 0 & d \end{array}\right]$$

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In $M_{2\times2}$ Find the span of $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Here, $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in M_{2\times2}$

Now, let x & y be any scalars.

thus, we can write M in linear combination of B and C.

Thus,

$$M = xB + yC$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} 0 & y \\ y & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$$

comparing both sides,

$x = 1, 0$

$y = 1$

$\therefore$ span $(M_{2\times2}) = \{ \cancel{M = x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mid x, y \in \mathbb{R}} \}$.

M does not form linear combination with B & C thus does not form span.

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in$ span $\{M, B, C\}$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \beta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha+\beta & \alpha+r \\ \alpha+r & \beta \end{bmatrix}$$

$$\text{Span}\{M, B, C\} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right|$$

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Q. Find the span of the vector $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$

Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\exists \ \alpha, \beta$ such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\alpha - \beta = x$

$2\alpha = y$

$3\alpha + \beta = z$

$$\left[\begin{array}{cc|c} 1 & -1 & x \\ 2 & 0 & y \\ 3 & 1 & z \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 2 & y-2x \\ 0 & 4 & z-3x \end{array}\right] \xrightarrow{\frac{R_2}{2}} \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & \frac{y-2x}{2} \\ 0 & \cancel{4}\ 1 & \frac{z-3x}{4} \end{array}\right]$$

$R_3 - R_2$

$$\left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & \frac{y-2x}{2} \\ 0 & 0 & \frac{z-3x}{4} - \frac{y-2x}{2} \end{array}\right]$$

Above system has solution if and only if

$$\frac{z-3x}{4} - \frac{y-2x}{2} = 0$$

$$z - 3x - 2y + 4x = 0$$

$$x + z - 2y = 0$$

$$x - 2y + z = 0$$

$$\operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \ x - 2y + z = 0 \right\}$$

Linear dependence and linear independence

Def^n : let $s = \{v_1, v_2, \dots v_k\}$ be the set of vectors in a vector space V, we say $v_1, v_2, \dots v_k$ are linearly dependent if atleast one scalar is non-zero such that $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$, where c_i's are the scalars.

Otherwise if $c_1 = c_2 = \dots = c_k = 0$ i.e. all scalars are zero such that $c_1v_1 + c_2v_2 + c_kv_k = 0$ (c_i = scalars) then $v_1, v_2, \dots, v_k$ are linearly Independent.

eg. ① $s = \{0\}$ subset of V

$\Rightarrow$ linearly dependent (if we multiplied 0 by any scalar let $k \cdot 0 = 0$ so there exist atleast one scalar such that linear combi- is 0)

② $S = \{u_1, u_2, 0\} \subseteq \mathbb{R}^2$
$u_1, u_2 \neq 0$

$\alpha u_1 + \beta u_2 + 100(0) = 0$

$\gamma \rightarrow$ atleast one constant is non-zero $\Rightarrow$ satisfied

linear dependent.

③ $S = \{u, v, \alpha u + \beta v\} \subseteq \mathbb{R}^3$

$\mathbb{R}^2 \rightarrow$ not field

linearly dependent.

$K \Rightarrow$ forms field.

④ $S = \{1+x, x+x^2, 1+x^2\} \subseteq P_2(\mathbb{R}) \Rightarrow$ linearly independent

$x + x^2 = \alpha(1+x) + \beta(1+x^2)$

$x + x^2 = \alpha + \alpha x + \beta + \beta x^2$

$(\alpha+\beta) + \alpha x + \beta x^2 = x + x^2$

compairing both sides

$\alpha + \beta = 0$, $\alpha = 1$, $\beta = 1$

let a, b, c such that

$a(1+x) + b(x+x^2) + c(1+x^2) = 0$

$a + ax + bx + bx^2 + c + cx^2 = 0$

$(a+c) + (ax+bx) + (bx^2+cx^2) = 0$

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$(a+c) + (a+b)x + (b+c)x^2 = 0$

$a+c=0$

$a+b=0$

$b+c=0$

$b=-c$

$$\begin{array}{r} a+c=0 \\ a-c=0 \\ \hline a=0 \end{array}$$

$a=b=c=0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

as $a=b=c=0$

Thus S is linearly independent.

(5) $S = \phi \subseteq V$

empty set in $V \rightarrow$ linearly independent (always)

(6) $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$ are in $M_{2\times2}$

let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = x\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} + y\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} + z\begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x & -x \\ 2x & 0 \end{bmatrix} + \begin{bmatrix} 2y & -y \\ 2y & 2y \end{bmatrix} + \begin{bmatrix} -z & -2z \\ 0 & -2z \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x+2y-z & -x-y-2z \\ 2x+2y & 2y-2z \end{bmatrix}$$

$a=b=c=d=0$

$\therefore\ x+2y-z=0$ — (1)

$-x-y-2z=0$ — (2)

$2x+2y=0$ — (3)

$2y-2z=0$

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$2y = 2z$

$y = 2$

linearly independent.

eqn (2),

$\cancel{2x}\quad -x-2-2z=0$

$x = -3z$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ -1 & -1 & -2 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \xrightarrow{R_4 \to R_4 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_3 \to \frac{R_3}{-2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = 0 \quad \therefore y - 3z = 0$

$\therefore y = 0$

$\therefore x = 0$

As $x = y = z = 0$

A, B, C are linearly independence

(a) $S = \{1 - x + x^2, -1 + x - x^2, 3 - x^2\}$ in P_2, S is LI or LD

Let a, b, c such that,

$a(1 - x + x^2) + b(-1 + x - x^2) + c(3 - x^2) = 0$

$a - ax + ax^2 - b + bx - bx^2 + 3c - cx^2 = 0$

$(a - b + 3c) + (-a + b)x + (a - b - c)x^2 = 0$

$a - b + 3c = 0 \Rightarrow \cancel{a - b + 3c = 0 \Rightarrow a + 3c = 1}$

$-a + b = 0 \quad \therefore b = a$

$a - b - c = 0 \Rightarrow \cancel{a - 1 - c = 0 \Rightarrow a - c = 1}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$\cancel{a + 3c = 1}$

$\cancel{-a + c = -1}$

$\cancel{2c = 0}$

$\cancel{c = 0}$

$R_2 \to R_2 + R_1$; $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

$\cancel{\therefore a = 1 + c}$

$\cancel{= 1 + 0}$

$\cancel{a = 1}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ a & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c = 0$

$a - b + 3c = 0$

$a - b + 0 = 0$

$a - b = 0$

$a = b$

the linear combination is,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so above vectors in P_2 are linearly dependent.

(7) $S = \{\sin x, \cos x, \sin(2x)\} \subseteq F$

HW

Is S is L.I. or L.D. in F $\left\{ \begin{matrix} \text{set of all} \\ F: \mathbb{R} \to \mathbb{R} \end{matrix} \right\}$

we cannot write $\sin(2x)$ in L.C. of $\sin x$ & $\cos x$

thus they are linearly independent.

(8) $S = \{x, |x|\} \subseteq F$

HW

so S is LI or L.D.

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▷ Basis for a vector space (minimal generating set)

Let B be any non empty set in V. We say B is a Basis for V if and only if it satisfies following conditions.

ⓐ span (B) = V

ⓑ B is L.I.

▷ Diamensions of vector space

If B is Basis for V then dimensions of vector space is number of element in the B.

eg. ① Is $B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ a Basis for $\mathbb{R}^2$?

span (B) = $\mathbb{R}^2$

And B is linearly independent.

(II) $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right\}$ → Not a basis.

(2 eqⁿ & 3 variables)

~~span~~ They are linearly dependent

Note→ If B is a basis for vector space V then it is minimal generating set and maximal linearly independent set.

▷ Dimension of $\mathbb{R}^n$ is n.

HW Let, $S = \{v_1, v_2, \dots, v_k\} \subseteq V$ is a vector space over F.

show that

ⓐ span (S) is subspace of V

ⓑ span (S) is a smallest subspace of V containing $v_1, v_2, \dots v_k$.

ⓐ span (s) $= c_1v_1 + c_2v_2 + \dots + c_kv_k$ where $c_i \in \mathbb{R}$ and are scalars.

as $c_1v_1 \in \mathbb{R}$

and $c_1v_1 + c_2v_2 \in \mathbb{R}$

∴ span(s) is a subspace of V.

ⓑ

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Basis and Dimensions

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| x_1, \ldots, x_n \in \mathbb{R} \right\}$$

$$= \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \middle| x_1, x_2, x_n \in \mathbb{R} \right\}$$

$$= \{ x_1 e_1 + x_2 e_2 + \cdots + e_n x_n \mid x_1, x_2, x_n \in \mathbb{R} \}$$

$$= \text{span}\{e_1, e_2, \ldots, e_n\}$$

$\mathbb{R}^n = \langle e_1, e_2, \ldots, e_n \rangle$ (generating set)

Also $e_1, e_2, \ldots e_n$ are linearly independent.

Standard basis for $\mathbb{R}^n$

↳ $B = \{e_1, e_2, \ldots e_n\}$ is a standard basis of $\mathbb{R}^n$.

eg. Basis for $\mathbb{R}^n = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

→ linearly independent.

- Ordered Basis (Basis other than standard basis)

eg. ① $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ Basis for $\mathbb{R}^2$.

② $B = \left\{ \begin{bmatrix} \pi \\ e \end{bmatrix}, \begin{bmatrix} e \\ \pi \end{bmatrix} \right\}$ Basis for $\mathbb{R}^2$.

③ $P_n(\mathbb{R}_n) = \{ a_0 + a_1 x + \cdots + a_n x^n \mid a_0, a_1, \ldots a_n \in \mathbb{R} \}$

$= \text{span}\{1, x, x^2, x^3 \ldots x^n\}$ ← linearly independent

↓

since

$c_1 \cdot 1 + c_2 x + \cdots c_n x^n = 0$

iff $(c_1 = c_2 = c_n = 0)$

$P_1(\mathbb{R}) = \{ a \cdot 1 + a_1 x \mid a, a_1 \in \mathbb{R} \}$

$= \{1, x\}$

$\alpha \cdot 1 + \beta(x) = 0$

$\frac{d}{dx}(\alpha \cdot 1 + \beta x) = 0$

$0 + \beta = 0$

$\beta = 0$

Wrong scale

$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$

so linearly independent.

$y = c_1 \phi(x) + c_2 \phi_2(x)$ ← solution

$y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1, 1$

General solution,

$y = c_1 e^x + c_2 x e^x$

$\begin{vmatrix} e^x & x.e^x \\ e^x & xe^x + e^x \end{vmatrix} \neq 0$

▷ Dimension of $P_n(\mathbb{R})$ is $n+1$

eg. ① What is the standard basis for $P_4(\mathbb{R})$

$B = \{1, x, x^2, x^3, x^4\}$

Dimension $= n+1 = 4+1 = 5$

② Basis for $M_{n\times n}(\mathbb{R})$ ← vector space of matrices

$M_{n\times n}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \middle| a_{11} \ldots a_{nn} \in \mathbb{R} \right\}$

$M_{n\times n}(\mathbb{R}) = \{ a_{11}E_{11} + a_{12}E_{12} + \ldots + a_{nn}E_{nn} \mid a_{11} \ldots a_{nn} \in \mathbb{R} \}$

where $E_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = i, \beta = j \\ 0 & \text{otherwise} \end{cases} \quad \forall\, i,j = 1,2,n$ eg. $E_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to (3,4)$

↙ elementary matrix

eg. ① → linear combination

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$M_{n\times n}(\mathbb{R}) = \text{span}\{E_{11}, E_{12}, \ldots, E_{nn}\}$

Also $P = \{E_{11}, E_{12}, \ldots E_{nn}\}$ is linearly independent

B is standard basis for $M_{n\times n}(\mathbb{R})$

$\dim(M_{n\times n}(\mathbb{R}))$ is $n\times n = n^2$

Similarly ~~dim~~ Basis of $M_{m\times n}(\mathbb{R}) = \{E_{11} \ldots E_{m\times n}\}$

$\dim(M_{m\times n}(\mathbb{R}))$ is $m\times n$

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(2) what is dimension of vector space $\mathbb{C}(\mathbb{R})$? $z = x + iy$

Basis

$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

$\mathbb{C} = \text{span}\{1, i\}$

$x + iy = 0 \quad , \quad x, y \in \mathbb{R}$

$x = y = 0$

also $\{1, i\}$ is linearly independent in $\mathbb{C}$.

Basis dimension of $\mathbb{C}(\mathbb{R}) = 2$ (as two elements are in span $\mathbb{C}$)

$\hookrightarrow \mathbb{C}(\mathbb{C})$

$\mathbb{C} = \{z \mid z \text{ is of the form } x + iy, \ x, y \in \mathbb{C}\}$

$\{1, i\}$ is linearly dependent in $\mathbb{C}(\mathbb{C})$

$\{1\}$ is a standard basis for $\mathbb{C}(\mathbb{C})$.

what is dimension of $\mathbb{C}^2(\mathbb{R})$

Standard Basis $(\mathbb{C}^2(\mathbb{R})) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix} \right\}$

Dimension of $\mathbb{C}^2(\mathbb{R})$ is 4.

Similarly in general, $\boxed{\dim(\mathbb{C}^n(\mathbb{R})) = 2 \cdot n}$

dimension $(M_{n \times n}(\mathbb{C})(\mathbb{R})) = 2n^2$

dimension $(M_{m \times n}(\mathbb{C})(\mathbb{R})) = 2m \cdot n$

▷ Dimension of vector space $\mathbb{R}(\mathbb{Q})$?

$B = \{1, \sqrt{2}, \sqrt{3}, \ldots \sqrt{p}, \ldots\}$ → linearly independent

$\alpha_1(1) + \alpha_2\sqrt{2} + \ldots + \alpha_n\sqrt{p} = 0$

$\{1, \sqrt{2}\}$ Dimensions $\mathbb{R}(\mathbb{Q})$ = Infinite (∞)

$\alpha \cdot 1 + \beta \cdot \sqrt{2} = 0$

$\alpha = \beta = 0$ L.I.

HW Find the dimension of following subspaces.

1) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{array}{l} x-y=0 \\ y-z=0 \end{array}, V = \mathbb{R}^3 \right\}$

2) $W = \left\{ A \in M_{n\times n}(\mathbb{R}) \mid A^T = A \right\}$ $V = M_{n\times n}(\mathbb{R})$

3) $W = \left\{ p(x) \in P_3(\mathbb{R}) \mid p(3) = 0 \right\}$ $V = P_3(\mathbb{R})$.

4) $W = \left\{ A \in M_{n\times n}(\mathbb{R}) \mid tr(A) = 0 \right\}$, $V = M_{n\times n}(\mathbb{R})$

1) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{array}{l} x-y=0 \\ y-z=0 \end{array} \quad V = \mathbb{R}^3 \right\}$

$x - y = 0$

$y - z = 0$

$\therefore \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

~~Dimensions are~~ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

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eg. ① $V = \mathbb{C}(\mathbb{C})$

$\text{span}(1) = \{\alpha \cdot 1 \mid \alpha \in \mathbb{C}\} = \mathbb{C}$

also $\{1\}$ is linearly independent.

$(\dim(\mathbb{C}(\mathbb{C})) = 1)$

② $V = \mathbb{C}(\mathbb{R})$

$= \{x \cdot 1 + iy \mid x, y \in \mathbb{R}\}$

$= \text{span}(1, i)$ $\quad B = \{1, i\} \Rightarrow \dim(\mathbb{C}(\mathbb{R})) = 2$

also $\{1, i\}$ is linearly independent.

eg. ②

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x+y+z=0 \\ x-y-z=0 \end{matrix} \right\}$$

$V = \mathbb{R}^3(\mathbb{R})$ $\quad \dim(W) = ?$

$x+y+z=0$

$x-y-z=0$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \to \frac{R_2}{-2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 - R_2 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = 0$

$y + z = 0$ $\quad \therefore y = -t$

Let $z = t$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$$

$$W = \left\{ t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Also $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is linearly independent so

$B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is basis for W.

Dimensions for W is 1.

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HW (3)

$V = \mathbb{R}^n(\mathbb{R})$

$W \subseteq V$ subspace

$\dim(W) = n - $ (no. of L.I restrictions & conditions)

$W = \{ p(x) \in P_3(\mathbb{R}) \mid p(1) = 0 \}$

and where $V = P_3(\mathbb{R})$

$W = \{ a + bx + cx^2 + dx^3 \mid p(1) = 0 \}$

also

$\therefore a + b + c + d = 0$

$\therefore a = -b - c - d$

$W = \{ -b - c - d + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R} \}$

$W = \{ b(x-1) + c(x^2-1) + d(x^3-1) \mid b, c, d \in \mathbb{R} \}$

↳ span $((x-1), (x^2-1), (x^3-1))\}$

It forms basis as it is linearly independent.

As the powers are different of each term we cannot write one term into linear combination of other.

Dimension of $W = 3$ (As there are three vectors present in span basis)

HW (2) $W = \{ A \in M_{n \times n}(\mathbb{R}) \mid A^T = A \}$ $V = M_{n \times n}(\mathbb{R})$

For $n = 2$

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

no. of Linearly independent choices are $= 3 = \frac{2(2+1)}{2}$

$n = 3$

$$A = \begin{bmatrix} a & b & c \\ b & \alpha & \beta \\ c & \beta & \gamma \end{bmatrix}$$

No. of L.I. choices are $= 6 = \frac{3(3+1)}{2}$

By induction we can say

$$\dim(W) = \frac{n(n+1)}{2}$$

dimension of lower & upper triangular matrix.

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③ $W = \{A \in M_{n\times n}(\mathbb{R}) \mid \underbrace{tr(A)}_{\text{trace}} = 0\}$

$n = 2$

$$A = \begin{bmatrix} -a & \alpha \\ \beta & a \end{bmatrix} \qquad \begin{aligned} &n^2-1 \\ &4-1 = 3 \end{aligned}$$

$\dim(W) = n^2-1$

$n = 3$

$$A = \begin{bmatrix} a+b & c & d \\ e & -a & h \\ f & g & -b \end{bmatrix}$$

↳ 8 vectors

$$\begin{aligned} &n^2-1 \\ &= 3^2-1 \\ &= 9-1 = 8 \end{aligned}$$

eg. ① $W = \{f \in F \mid f \text{ is differentiable}\}$

$V = F$, $\dim(W) = ?$

↳ infinite

$B = \{1, x, \sin x, \cos x, \log x, e^x, x^2, \sin^2 x \ldots\}$

↳ Infinite elements in basis.

Thus dimensions for W is infinite.

Note:-

If dimension of V and V', V = V' then V and V' are the isomorphic vector spaces.

eg. $M_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4$

$P_n(\mathbb{R}) \cong \mathbb{R}^{n+1}$

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① $W = \{p(x) \in P_2 \mid xp'(x) = p(x)\}$ Find $\dim(W)$, where $p'(x) = \frac{d}{dx}(p(x))$

Soln

$V = P_2(\mathbb{R})$

$p(x) = a_0 + a_1x + a_2x^2$

$p'(x) = a_1 + 2a_2x$

$xp'(x) - p(x) = 0$

$x(a_1 + 2a_2x) - a_0 - a_1x - a_2x^2 = 0$

$a_1x + 2a_2x^2 - a_0 - a_1x - a_2x^2 = 0$

$-a_0 + a_2x^2 = 0$

$a_0 - a_2x^2 = 0$

This implies $\{1, x^2\}$ are linearly independent.

$a_0 = 0 = a_2$

$p(x) = a_1x$

$W = \{a_1x \mid a_1 \text{ is scalar}\} = \text{span}(x)$

also $\{x\}$ is linear Independent

$\Rightarrow (\dim(W) = 1)$

② $W = \{A \in M_{n \times n} \mid A^T = A\}$. Find $\dim(W) = ?$

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▷ Co-ordinates

Let $B = \{v_1, v_2, \ldots v_n\}$ be a basis for vector space V. for

Let V be a vector in V ($V \in V$) and write $V = c_1v_1 + c_2v_2 + \ldots + c_nv_n$ then $c_1, c_2, \ldots c_n$ are called as co-ordinates of V w.r.t. Basis, B. and the column vector

$$[V]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$ is called as co-ordinate vector w.r.t. basis B.

Q.1) $P(x) = 1 - 2x + x^2$ Find co-ordinate w.r.t. the Basis $B = \{1, x, x^2\}$ of P_2

Soln $P(x) = 1 - 2x + x^2 = 1(1) + (-2)x + 1 \cdot x^2$

$$[P(x)]_B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

also consider $B' = \{x^2, x, 1\}$ ordered basis.

$$[P(x)]_{B'} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Q. $B' = \{1+x, x+x^2, 1+x^2\}$ Find co-ordinate vector w.r.t. B'

$P(x) = 1 - 2x + x^2 = (1+1) + [(-2) + (-2)^2] + [1 + 1^2]$

$= 2 \quad -2$

$p(x) = av_1 + bv_2 + cv_3$ (a

$1 - 2x + x^2 = a(1+x) + b(x+x^2) + c(1+x^2)$

$= a + ax + bx + bx^2 + c + cx^2$

$= (a+c) + (a+b)x + (b+c)x^2$

comparing, $a + c = 1$

$a + b = -2$

$b + c = 1$

$c = 1 - b$

$a + 1 - b = 1$

$a - b = 0$

$a + b = -2$

$2a = -2$

$a = -1$

Here a, b, c are scalar in co-ordinate vector.

$\therefore b = -1$

$c = 1 - (-1)$

$c = 2$

$$\therefore [P(x)]_{B'} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \in \mathbb{R}^3$$

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Eg. $P(x) = -1 \in P_n$, Find co-ordinate w.r.t. standard Basis for P_n.

Soln: $-1 = (-1)1 + 0x + 0x^2 + \dots + 0x^n$

$$[-1]_B = \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0_n \end{bmatrix} \in \mathbb{R}^{n+1} \quad \text{Dimensions} = n+1$$

$B = \{100, x, \dots 10x^n\}$, $P(x) = -x+1 \in P_n (\mathbb{R})$

Theorem: Let V be the vector space and B be the basis for V then for every vector v in V there is exactly one way to write v as the linear combination of basis vectors in B.

Proof: Here,

V be the vector space and B be the basis for.

Let $B = \{v_1, v_2, \dots, v_n\}$ be the basis for a vector space V

$v = c_1 v_1 + \cancel{c_2 v_2} + c_3 v_3 + \dots + c_n v_n$

Let us assume we can write $v \in V$ in two different ways as linear combination of basis vectors.

as $v = c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n$

and $v = c_1' v_1 + c_2' v_2 + c_3' v_3 + \dots + c_n' v_n$

$\Rightarrow v = v$

Thus $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = c_1' v_1 + c_2' v_2 + \dots + c_n' v_n$

$(c_1 - c_1') v_1 + (c_2 - c_2') v_2 + \dots + (c_n - c_n') v_n = 0$

as above vector Basis is linearly independent.

$\therefore c_1 - c_1' = 0$ i.e. $c_1 = c_1'$

$c_2 - c_2' = 0$ $\quad c_2 = c_2'$

$\vdots \qquad \vdots$

$c_n - c_n' = 0$ $\quad c_n = c_n'$

Hence proved.

For every vector v in V there is exactly one way to write v as the linear combination of basis vectors in B.

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