# SET THEORY

1) set - collection of well-defined objects

$A = \{1, 2, 3, 4\}$

$B = \{a, e, i, o, u\}$

2) Cardinality: Number of elements in a set

$|A| = 4$ , $|B| = 5$

3) subset : $[\subseteq]$ Notation $\phi \in A$

↳ Null set is subset of every set.

All elements of A in B.

4) special sets

$\mathbb{N} \Rightarrow$ Natural

$\mathbb{Z} \Rightarrow$ Integer $\qquad N \subseteq Z \subseteq Q \subseteq \mathbb{R}$

$\mathbb{Q} \Rightarrow$ Rational

$\mathbb{R} \Rightarrow$ Real

5) Two ways of set representation

1) Listing method $\qquad A = \{1, 2, \ldots\}$

2) Set Builder Notation $\qquad A = \{2n / n \leq 5\}$

6) power set of a set

Set of all subsets of a set

$A = \{1, 2, 3, \ldots\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}, \{3\}\}$

$|P(A)| = 8$

General Formula as $2^{|A|} = 2^3 = 8$

$|A| \rightarrow$ cardinality of A.

▷ operations performed on a set

1) union - $A \cup B = \{x \mid x \in A$ or $x \in B\}$

$A = \{1, 2, 4\}$ $\qquad A \cup B = \{1, 2, 3, 4\}$

$B = \{1, 2, 3\}$

$A \cap B = \{x \mid x \in A \text{ \& } x \in B\}$

$A \cap B = \{1, 2\}$

3) set difference : $A - B = \{x \mid x \in A \text{ \& } x \notin B\}$

$A - B = 4$ , $B - A = 3$

4) Symmetric Difference

$A \oplus B = \{x \mid (x \in A \text{ or } x \in B) \text{ and } x \notin (A \cap B)\}$

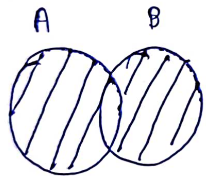
$A \oplus B = (A \cup B) - (A \cap B)$

$= \{3, 4\}$

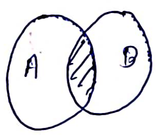
5) Universal set $= U$

6) complement of a set $\bar{A}$.

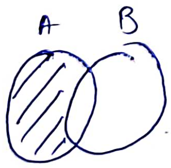
All elements in $U$ but not in $A$ : $\bar{A} = U - A$



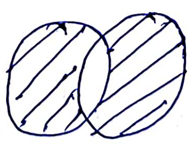

$A \cup B$




$A \cap B$




$A - B$



$A \oplus B$

▷ De-Morgan's law

$\overline{A \cup B} = \bar{A} \cap \bar{B}$ . $\overline{A \cap B} = \bar{A} \cup \bar{B}$

▷ Laws of sets

1) Commutative : $a+b = b+a$

$A \cup B = B \cup A$

$A \cap B = B \cap B$

$A \oplus B = B \oplus A$

$A - B \neq B - A$

2) Associative :

$(a+b)+c = a+(b+c)$

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

$(A \oplus B) \oplus C = A \oplus (B \oplus C)$

3) Distributive laws

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

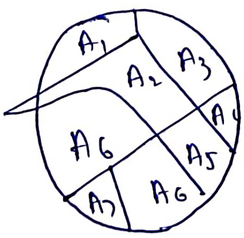
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4) Identity element

$\phi$ is an identity element

▷ Partition of a set

A :




$A = A_1 + A_2 + A_3 + A_4 + A_5 \ldots\ldots$

i.e. $A_1 \cup A_2 \cup A_3 \ldots\ldots$

Intersection is empty $\phi$.

$|A_i|$ or $n(A)$

$\Rightarrow n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

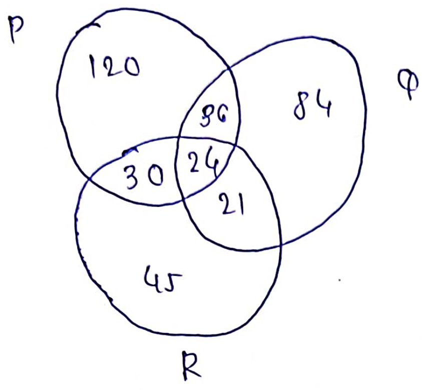
$\Rightarrow n(A \cap B \cap C) = n(A) + n(B) + n(C) - n(A \cup B) - n(B \cup C) - n(A \cup C) + n(A \cup B \cup C)$

Q. P, Q, R are the subsets of universal set U. of $n(U) = 390$

$n(P) = 210$, $n(Q) = 165$; $n(R) = 120$, $n(P \cap Q) = 60$.

Illustrate using Venn Diagram.

$n_o(P)$, $n_o(Q)$, $n_o(R)$, $n_o(P \cap Q)$, $n_o(Q \cap R)$, $n_o(P \cap R)$




$n_o(P) = 120$

$= 210 - 36 - 24 - 30$

$n_o(Q) = 165 - 36 - 24 - 21$

$= 84$

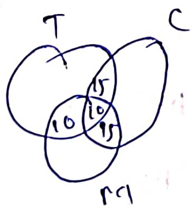
$n_o(R) = 45$

$n_o(P \cap Q) = 36$

$n_o(Q \cap R) = 21$

$n_o(P \cap R) = 30$

eg. In a survey of group of people 60 → Tea, 45 → coffee, 30 → milk, 25 → Tea & coffee, 20 → Tea & milk, 15 → coffee & milk, 10 → coffee tea & milk. How many people were there?




People like only tea = $60 - 25 - 10 - 20$

$= 25$

coffee = $45 - 30$

$= 15$

milk = $30 - 25$

$= 5$

$$\text{Total} = 15 + 10 + 10 + 5 + 5 + 15 + 25$$
$$= 85$$

$$n(T \cup C \cup M) = n(T) + n(C) + n(M) - n(T \cap C) - n(C \cap M)$$
$$- n(T \cap M) + n(T \cap M \cap C).$$
$$= 60 + 45 + 30 - 25 - 15 - 20 + 10$$
$$= \underline{\underline{85}}$$

Mathematical Induction

$p(n)$ : where $n$ is any +ve integer.

It is a technique to prove Formula, theorem

Principle of Mathematical Induction

Two steps

1. Basis steps: $p(n)$ given, show that $p(n)$ is true for $n=0$ or $n=1$
2. Inductive step : Assume that $p(n)$ is true for $n=k$, $k$ is any scalar +ve integer. then show that $p(n)$ is also true for $n=k+1$

eg. $p(n) :- 1+2+3+4+\dots\dots+n = \frac{n(n+1)}{2}$

1. Basis step : $p(n)$ is true for $n=1$

$LHS = 1 \qquad RHS = \frac{1(1+1)}{2} = 1$

2. Inductive step : Assume $p(n)$ is true for $n=k$

$1+2+3+4+\dots\dots+k = \frac{k(k+1)}{2} \Rightarrow$ True $-$ ①

shows it is also true for $n=k+1$

$1+2+3+4+\dots\dots+k+1$

$\underbrace{1+2+3+4+\dots\dots+k}_{\text{From} - ①}+k+1$

$$LHS = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$RHS = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$$

LHS = RHS

Hence p(n) is true.

Q Show that using M.I.

① $1^2 + 3^2 + 5^2 + \ldots (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

1. Basis step :- p(n) For (n=1)

$$1^2 = \frac{1(2-1)(2+1)}{3}$$

$$1 = 1$$

LHS = RHS

true for n=1

2. Inductive step :- Assume p(n) is true for n=k

$$1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \quad —①$$

Showing p(n) is also true for n=k+1

LHS :

$$1^2 + 3^2 + 5^2 + \ldots + [2(k+1)-1]^2$$

$$\underbrace{1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2}_{\text{From ①}} + (2(k+1)-1)^2$$

$$= \frac{k(2k-1)(2k+1)}{3} + (2(k+1)-1)^2$$

$$= \frac{k(2k+1)(2k+1) + 3(2k+2-1)^2}{3}$$

$$= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$$

$$= (2k+1)\left(k(2k-1) + 3(2k+1)\right) / 3$$

$$RHS = \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

$$= \frac{(k+1)(2k+2-1)(2k+2+1)}{3}$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$= (2k+1)(k+1)(2k+3)/3$$

$$= (2k+1)\left(k(2k+3) + 2k+3\right) / 3$$

$$= (2k+1)(2k^2+3k+2k+3)/3$$

$$= (2k+1)(2k^2+5k+3)/3$$

$$= (2k+1)(2k^2+6k-k+3)/3$$

$$= (2k+1)\left(k(2k-1) + 3(2k+1)\right)/3$$

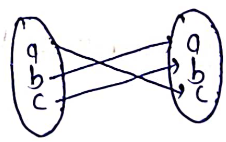
Thus LHS = RHS

Hence p(n) is true.

Functions :-

It is a rule that relates one entity to other it maps from one set to other set

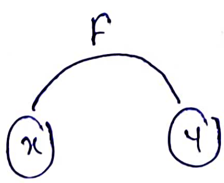
$F : A \to B$




$F(a) = c$

$F(b) = a$

$F(c) = b$




Domain co-domain/Range

① Injection :- i.e. Injective or One to one

- A Function is one to one if it follows.

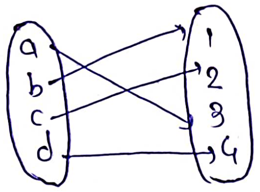
$\forall$ $a, b \in A$, if $F(a) = F(b)$ then $a = b$

2) $\forall$ $a, b \in A$, if $a = b$ the $F(a) = F(b)$

② onto / surjective

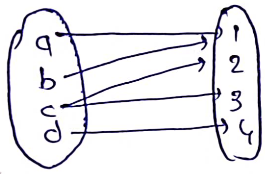
$\forall$ $y \in B$ there exists $x \in A$ $y = F(x)$

①



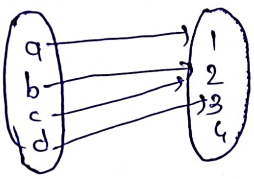

- one to one
- onto

②




- Not a function

③




- many-to-one

eg. ① $F : R \to R$ where $F(x) = x^2$

② $F : R \to R$, where $F(x) = 2x + 3$

③ $F : Z \to Z$, where $F(x) = 2x + 3$.

④ $F : R \to R$, where $F(x) = x^2 + 1$

⑤ $F : Z \to Z$ where $F(x) = x^2 + 1$

Relation:

cartesian product

$A \times B = \{(a,b) \mid a \in A,\ b \in B\}$

$|A| = m$ , $|B| = n$

A cartesian product is a binary operator between two sets and it is set of ordered pair.

eg. $A = \{1, 2\}$ $B = \{a, b, c\}$ $|A|$

$$A \times B = \begin{Bmatrix} (1,a) & (1,b) & (1,c) \\ (2,a) & (2,b) & (2,c) \end{Bmatrix}$$

$A \times B \neq B \times A$

$|A| = 2$, $|B| = 3$ $|A| \times |B| = 2 \times 3 = 6$

Relation: A relation from set A to set B is nothing but any possible subset of a cartesian product $A \times B$.

OR :- A relation on a set 'A' is relation from A to A

in other words relation on A is a subset of $A \times A$.

Types of relation

① Reflexive relation

$A = \{a, b, c\}$

$$A \times A = \{a,b,c\} \times \{a,b,c\} = \begin{Bmatrix} (a,a) & (a,b) & (a,c) \\ (b,a) & (b,b) & (b,c) \\ (c,a) & (c,b) & (c,c) \end{Bmatrix}$$

In matrix form,

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & aa & ab & ac \\ b & ba & bb & bc \\ c & ca & cb & cc \end{array}$$

← Focus on diagonal element

$R = \{(a,b)\ (b,c)\ (a,a), (c,c)\}$

↳ not reflexive

$\forall a \in A, (a, a) \in R$

$R = \{(a,a)\ (b,b), (c,c)\}$

↳ reflexive set

eg. $A = \{(a,a)\ (b,b)\ (c,c), (a,b), (b,c)\}$

↳ reflexive set also.

Defn: A ~~set~~ ~~is~~ relation on a set $A$ is said to be a reflexive $\forall a \in A, (a,a) \in R$.

- The set must contains diagonal elements.

2) Irreflexive Relation:
(contains no diagonal element pair)

$R = \{(a,b), (b,a), (a,c), (c,a)\} \to$ Irreflexive

$R = \{(a,b), (b,a), (a,a), (b,b), (a,c)\} \to$

A relation on any set 'A' is said to be irreflexive

For $\forall a \in A, (a,a) \notin R$.

eg. cartesian product is not irreflexive relation
(i.e. $A \times A$)

3) Symmetric Relation
A relation on any set A is said to be symmetric
$\forall a, b \in A$, $(a,b) \in R$ then $(b,a) \in R$.

eg. $A \times A$ is a symmetric relation.

$R = \{(a,b), (a,c), (c,c)\} \to$ not a symmetric

$R = \{(a,a), (b,b), (c,c)\} \to$ symmetric

4) Antisymmetric Relation
A relation on set 'A' is said to antisymmetric if
$\forall a,b \in A$, $(a,b) \in R$ ; $(b,a) \in R$ then $a = b$

- It allows the diagonal pair but doesn't allow the symmetry.

eg. $R = \{(a,b), (a,a)\}$

$R = \{A \times A\}$ is not a antisymmetric relation

$R = \{(a,b), (b,c), (a,c), (c,a), (a,a), (c,c)\} \to$ not antisymmetric

5) Assymmetric Relation

A relation on any set 'A' is said to be assymmetric if $\forall\ (a,b) \in A$ $(a,b) \in R$, $(b,a) \notin R$

R = Relation (not real no)

eg. $R = \{(a,c), (a,b)\} \rightarrow$ asymmetric

$R = A \times A \rightarrow$ not assymmetric relation

$R = \{(a,a)(b,b)(b,a)(b,c)\} \rightarrow$ Not assymmetric relation

6) transitive Relation

A relation on any set 'A' is said to be transitive if $\forall\ a,b \in A$, $(a,b) \in R$, $b,c \in R$ then $a,c \in R$.

eg. $R = \{(a,b)(b,c)(a,c)\} \rightarrow$ Transitive

$a \rightarrow b \rightarrow c$

$R = \{(a,a), (b,b), (c,c)\} \rightarrow$ Transitive

$R = \{(c,a)(b,c)\} \rightarrow$ Not transitive

$b - c - a$

$R = \{(a,b)(b,a)(a,a)(b,b)\}$

$a \rightarrow b$

$A = (x, y)$

$B = (y, z)$

same

same

2nd of 1st     1st of 2nd (same) i.e. x

$A = (x, y)$    $B = (y, z)$

same → to be transitive

closure Using Warshall Algorithm:

eg. ① set $A = \{1, 2, 3, 4\}$

$R = \{(1,4)\ (2,1)\ (2,3)\ (3,1),\ (3,4)(4,3)\}$

soln →

1st step :

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

IInd step :

Ist column and Ist Row

C R

$\{2, 3\}$ $\{4\}$

$CP = C \cdot R = \{(2,4), (3,4)\}$

$$W_1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

II nd step 2nd row 2nd column

C R

$\phi$ $\{1, 3, 4\}$

$C \cdot P = \phi$

$W_1 = W_2$

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

III<sup></sup>rd step

III rd column $\{2, 4\}$ III rd row $\{1, 4\}$

$$CP = C.R. = \{(2,1)\ (2,4)\ (4,1)\ (4,4)\}$$

$$W_3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{array}$$

4th colm $\{1, 2, 3, 4\}$ 4th row $\{1, 3, 4\}$

$$CP = C.R = \{(1,1)\ (1,3)\ (1,4)\ (2,1)\ (2,3)\ (2,4)\ (3,1)\ (3,3)\ (3,4)\ (4,1)\ (4,3)\ (4,4)\}$$

$$R^c = \{(1,1)\ (1,3),\ (1,4),\ (2,1),\ (2,3),\ (2,4)\ (3,1)(3,3)\ (3,4)\ (4,1)\ (4,3)\ (4,4)\}$$

eg. ② set $A = \{1, 2, 3, 4\}$

$R = \{(1,2), (2,1), (2,3), (3,4)\}$

Step I

$$M_R = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Ist column     Ist row

$\{2\}$     $\{2\}$

$CP = \{(2,2)\}$

$$W_1 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

IInd colum     IInd row

$\{1, 2\}$     $\{1, 2, 3\}$.

$C \times R = CP = \{(1,1)\ (1,2)\ (1,3)$
$(2,1)\ (2,2)\ (2,3)\}$

$$W_2 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

IIIrd colm     IIIrd row

$\{1, 2\}$     $\{4\}$

$CP = C \times R = \{(1,4)\ (2,4)\}$

$$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

IV th colm          IV th row

$\{\cancel{\emptyset}1, 2, 3\}$          $\phi$

$CP = C \times R = \phi$

$R^+ = \cancel{\phi} \{(1,1)\ (1,2)\ (1,3)\ (2,1)\ (2,2)\ (2,3)$
$(1,4)\ (2,4)\ (3,4)\}$

Equivalence Relation

A relation is on set A are reflexive, symmetric and transitive is known as Equivalence

eg. $A = \{1, 2, 3\}$

$R_1 = \{(1,1)\ (2,2)\ (3,3)\} \rightarrow$ equivalence relation

$R_2 = \{(1,1)\ (2,2)\ (3,3)\ (2,1)\}$
$R_3 = \{(1,1), (1,3)\ (2,1)\ (3,1)\}$ } Not

$R = \phi$ (Not equivalence)

eg. ① ~~B =~~

Which of the following is set of equivalence Relation.

① $R_1 = \{(a,b) \mid a-b$ is an integer$\}$

Reflexive ⇒ $(2,2) \rightarrow 2-2=0$          It is an equivalence relation.

Symmetric ⇒ $(1,2)\ (2,1)$
$1-2 \quad 2-1$
$-1 \quad 1$

transitive → $(2,5)\ (5,4)$
$2-5=-3 \quad 5-4=1 \quad 5-5=0$

② $R_2 = \{(a,b) \mid a-b \text{ is divisible by } 3.\}$

Soln 1) Reflexive

$(3, 3)$

$3-3=0 \quad (\because \frac{0}{3}=0)$

Holds.

2) Symmetric

$(a, b)$
$(b, a)$

$\cancel{(2,1)}$ $(1, 2)$

$\because$

~~a 1 is not divisible by 3~~

~~condition II not hold.~~

Thus the ~~given relation~~ is ~~not~~ equivalence.

③ ~~2) Symmetric~~ $\Rightarrow$

if $a-b$ is divisible by 3 then $b-a$ is also divisible by 3.

Let $a=4, \; b=1$

$(4-1)=3 \qquad (1-4)=-3$

Divisible by 3

satisfied.

3) Transitive

because if $a-b$ and $b-c$ are both divisible by 3.

then $a-c$ is also divisible by 3.

eg. $(7\ 1)\ (1\ 4)$

$7-4=3 \quad (\frac{3}{3}=0)$

$1-4=-3 \quad (-\frac{3}{3}=0)$

$1-\varphi=0 \quad (\frac{0}{3}=0)$

satisfied ✓

Thus the relation is equivalence.

(3) $R_3 = \{(a,b) \mid a-b \text{ is an odd}\}$

Reflexive

$(2,2) \Rightarrow 2-2=0$ $(\because 0 \text{ is even})$

$R_3$ is not an equivalence relation..

Equivalence classes

Equivalence classes of $x$ is denoted by $[x]$ member of set

$[x] = \{y \mid y \in A, \text{ and } (x,y) \in R\}$

$A = \{1,2,3,4,5\}$

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (5,5)\ (1,2)\ (2,1)\ (4,5)\ (5,4)\}$

(a) Reflexive → Holds

(b) symmetric → Holds

(c) Transitive → Holds

Thus it is an equivalence relation

Equivalence classes: $[1] = \{1,2\}$
$[2] = \{1,2\}$
$[3] = \{3\}$
$[4] = \{4,5\}$
$[5] = \{4,5\}$

Here $P_1 = \{1,2\}$
$P_2 = \{3\}$
$P_3 = \{4,5\}$

$P_1 \cup P_2 \cup P_3 = A$

$P_1 \cap P_2 \cap P_3 = \phi$

Partition - It is a non-empty set A is collection of set R if it satisfies $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \phi$ then it is cartesian.

Q. let $A = \{1, 2, 3, 4\}$

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (3,1)\ (1,3)\ (4,2)\ (2,4)\}$

It is an equivalence relation.

$[1] = \{1, 3\}$
$[2] = \{2, 4\}$
$[3] = \{3, 1\}$
$[4] = \{4, 2\} = \{2, 4\}$

$P_1 = \{1, 3\}$

$P_2 = \{2, 4\}$

$P_1 \cup P_2 = A$

$P_1 \cap P_2 = \phi$

# Partial Order Relation (poset)

eg. $A = \{1, 2, 3\}$

$R_1 = \{(1,1)$

A relation on a set A is said to be a partial order relation if it is

(i) Reflexive
(ii) Antisymmetric
(iii) Transitive.

eg. $A = \{1, 2, 3, 4\}$

$R_1 = A \times A$ → Not partial

$R_2 = \{(1,1) (2,2) (3,3)\}$ → partial

$R_3 = \{(1,1) (2,2) (3,3) (1,3), (3,2)\}$ → partial

Q. ① $R = \{(a,b) \mid a,b \in Z, a < b\}$

1) Reflexive.

Let $(a,b) = (2,2) \ni 2 < 2$

Not reflexive

Thus R is not a partial order relation.

② $R = \{(a,b) \mid a,b \in Z, a \leq b\}$

1) Reflexive

$(2,2) \ni 2 = 2$ satisfied

ii) Antisymmetric ✓

$(1,2)$ → $a < b$  $(2,1)$ → $a > b$ ✗

iii) Transitive

$(1,2)$ ✓ $(2,1)$ ✗ Not satisfied  Not a partial order relation.

② $R = \{(a,b) \mid a, b \in z, \frac{b}{a} \in z\}$

1) Reflexive

$(2,2)$, $\frac{b}{a} = \frac{2}{2} = 1 \in Z$

2) Antisymmetric

$(1,2)$, $\frac{b}{a} = \frac{2}{1} = 2 \in Z$

3) Transitive

$(2,4)$ $(4,8)$

$\frac{4}{2} = 2 \in Z$ $\frac{8}{4} = 2 \in Z$

All the three conditions are satisfied.

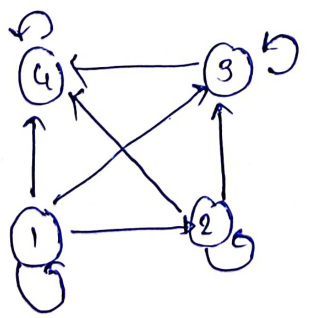
Thus R is partial order relation.

▷ Hase Diagram

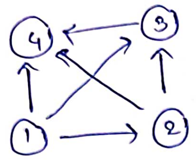
$A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2) (1,3) (1,4), (2,2), (2,3), (2,4)$
$(3,3) (3,4) (4,4)\}$

Step 1 →

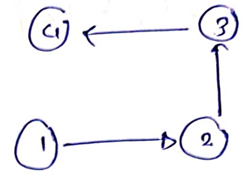


step 2



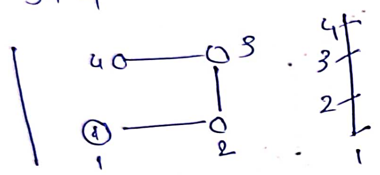

Remove ~~transitive~~
self edges
loops

step 3 :




Remove transitive
edges

step 4 :




Remove direction.

Definition :
A Hose diagram is the graphical representation of the partial order relation (po-set) with implied upward orientation.

Steps : 1) Draw the directed graph for the corresponding relation

2) Directed graph are need to be contain self loop at every vertex because element always greater than equal to itself.

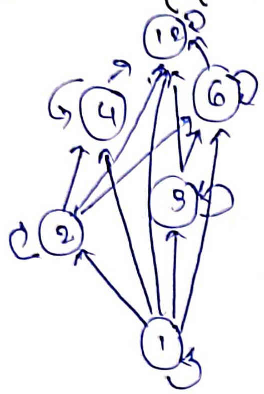
3) Remove all the self loops because it irreflexive so don't need to show that

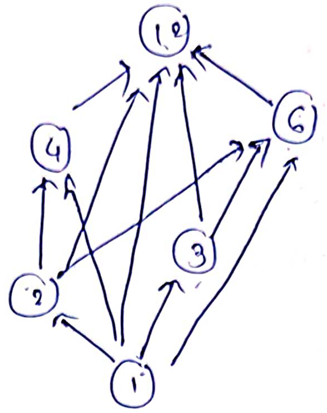
4) Finally remove all the transitive edges and directions.

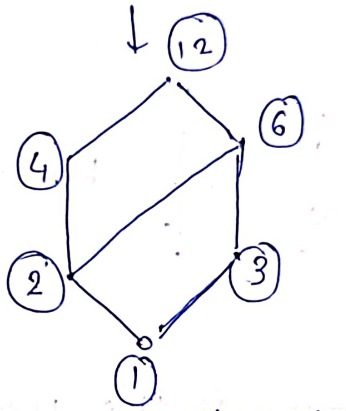
5) Make sure that initial vertex is below the terminal vertex.

Q.

1) $A = [\{1, 2, 3, 4, 6\ 12\}, /]$

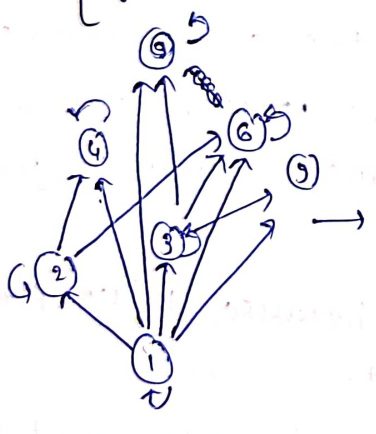


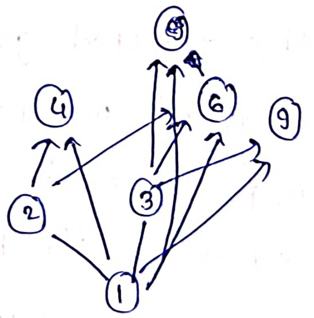



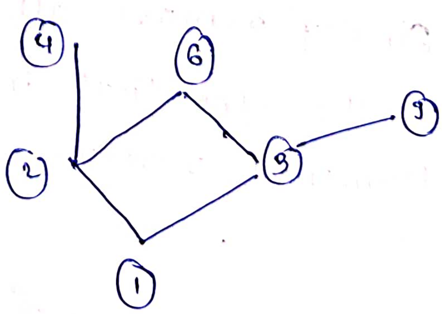




Hose Diagram

2) $A = [\{1, 2, 3, 4, 6, 9\}, /]$



Draw the Hasse diagram for divisibility

① R = {1, 2, 3, 4, 5, 6}

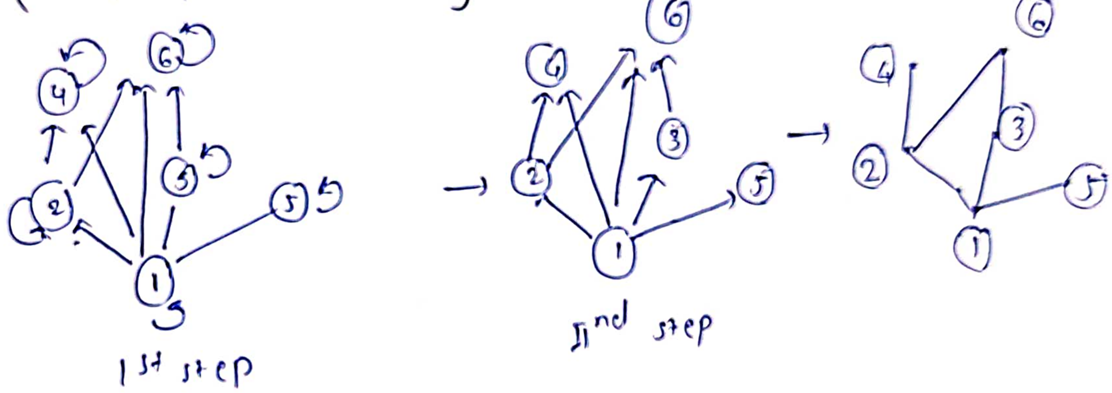
② R = {1, 2, 3, 5, 7, 11, 13}

③ R = {1, 2, 3, 6, 12, 24, 36, 48}

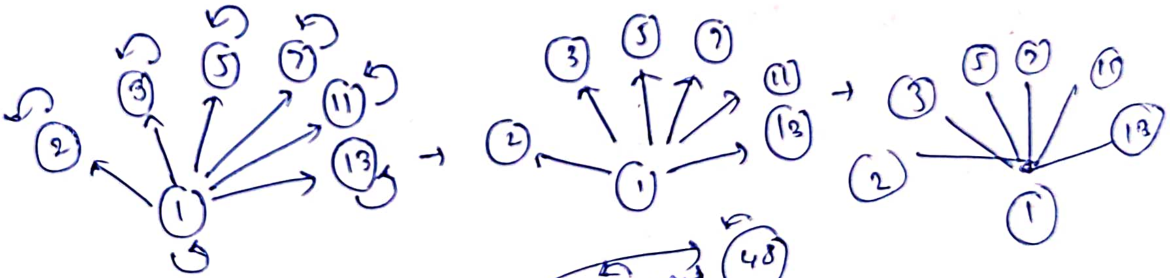
④ R = {3, 5, 7, 11, 13, 16, 17}

⑤ R = {2, 3, 5, 10, 11, 15, 25}

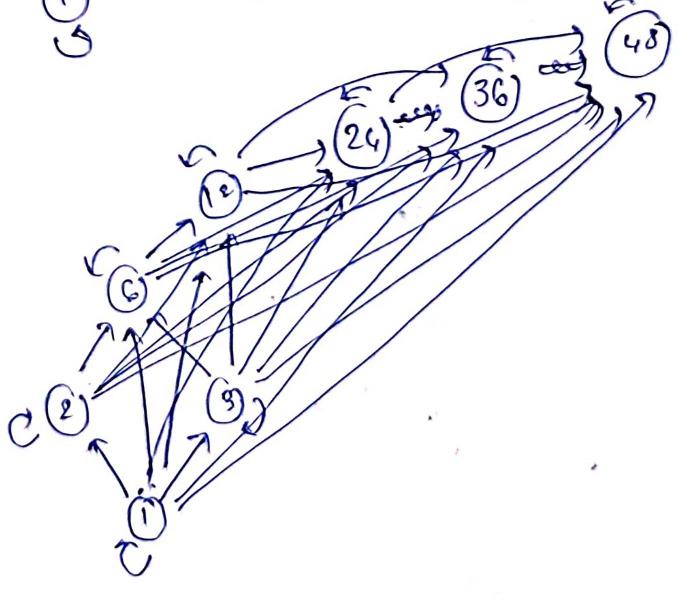
① Soln



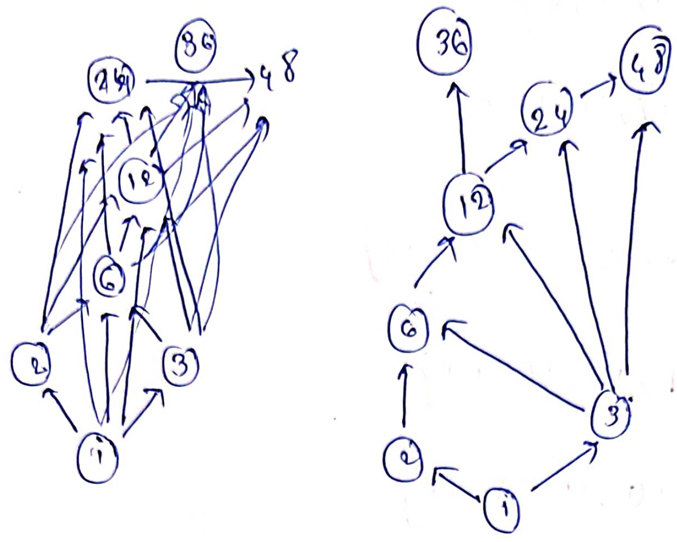

②



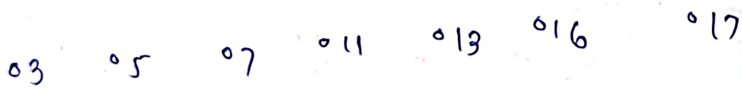
③



{1, 2, 3, 6, 12, 24, 36, 48}

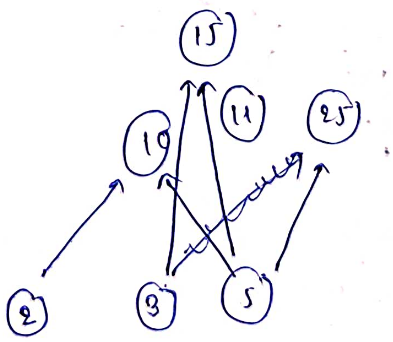


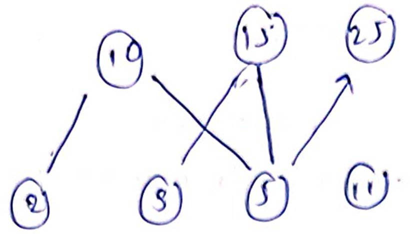

4


5

Soln




10
15
25
2
8
5
11

Maximal Element

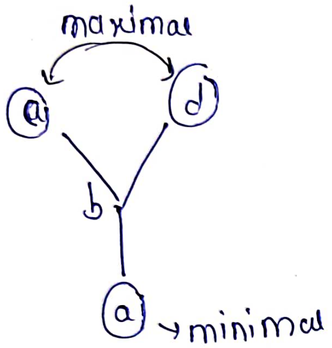
If in a poset, an element is not related to any other elemen (upper)

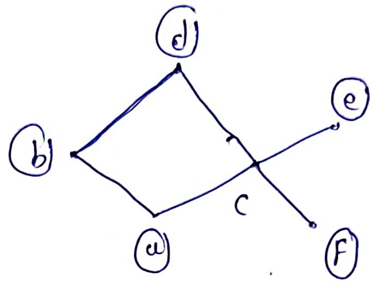
▷ More than one element can be minimal or maximal.

Minimal elemen

If in a poset, no element is related to an element.

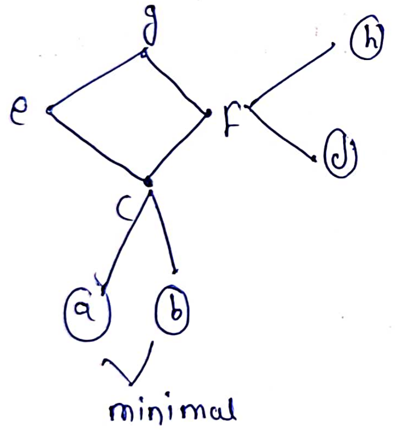
poset → partial order relation.

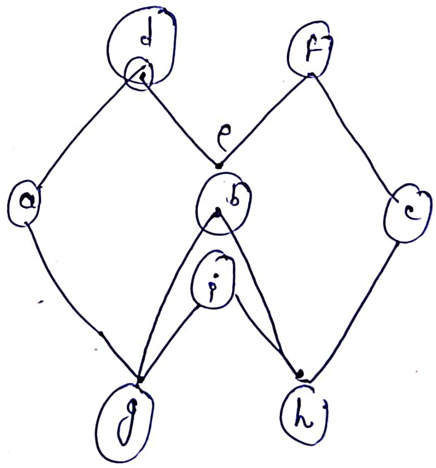



maximal = d, e

minimal = a, f




maximal = d, f, b, i

minimal = e, g, h

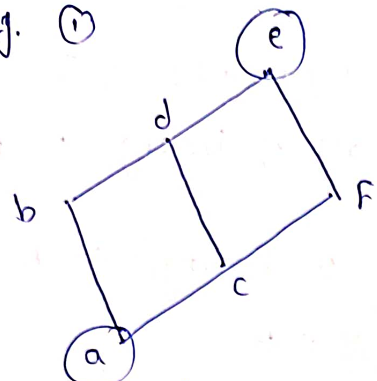
(maximum) (minimum)
Greatest and least Element

Maximum element (Greatest)

If it is a maximal and every element is related to it

Minimum / least element

If it is minimal and it is related to every element in poset eg. (1)



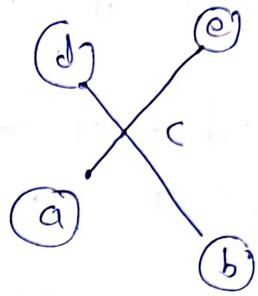

maximal = e

minimal = a

maximum = e

minimum = a

(2)



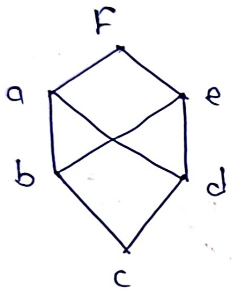

maximal = d, e

Greatest = $\phi$

minimal = a, b

least = $\phi$

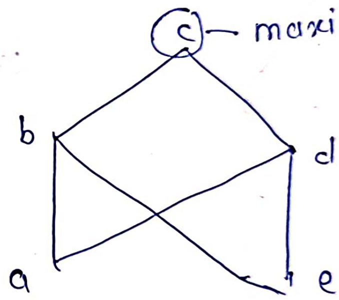
(3)




maximal = maximum = f

minimal = minimum = c

(4)

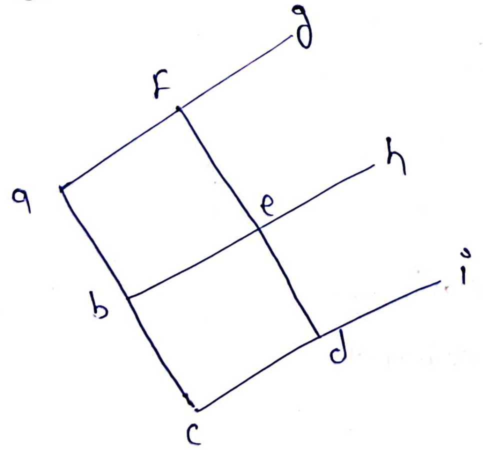



maximal = maximum

minimal = a, e

minimum = $\phi$

(5)




maximal = g, h, i maximum = $\phi$

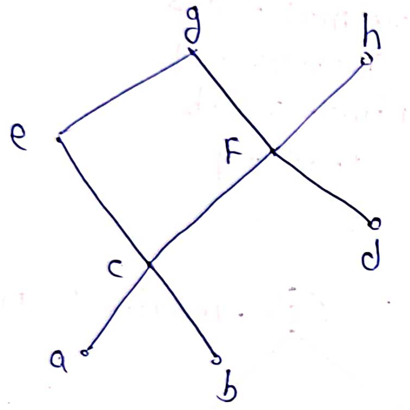
minimal = minimum = least = c

▷ Upper Bond and Lower Bond

① Let B be a seibset of A. An element $x$ belongs to A is in upper bond of B if $(y, x \in \text{poset})$ for all $y$ belongs to B

② Let B be a subset of set A. An element $x$ belongs to A is in lower bond of B if $(x, y \in \text{poset}$ $\forall y \in B$

eg. ①




① $B = \{e, c\}$

upperbond = $\{g, e\}$

Lower bond = $\{a, b\}$

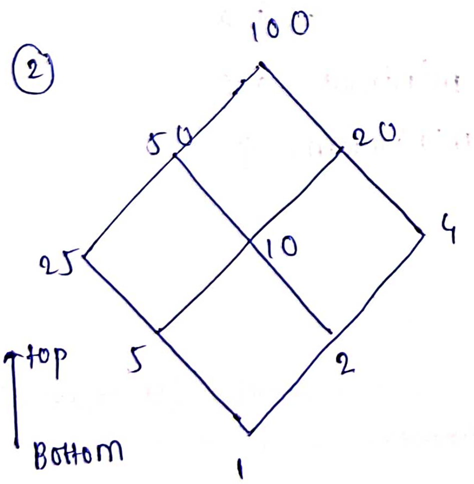
② $B = \{c, F, d\}$

upper bond = $\{g, F, h\}$

lower bond = $\phi$

(d does not have any edge to relate)

②




$B = \{50, 10\}$

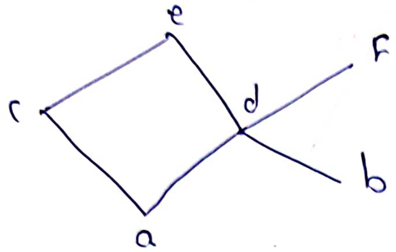
$U.B = \{50, 100\}$

$L.B = \{1, 2, 5, 10\}$

▷ Least upper Bond (supremum / join / $\vee$)

Greatest Lower Bond (Infimum / meet / $\wedge$)

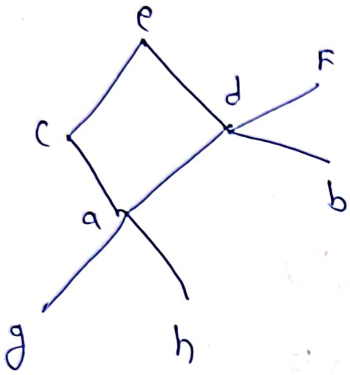
Least means minimum element in upperbond eg. ① $B = \{c, d\}$




$U.B = \{e\}$ ⟹ least upper bond

$L.B = \{a\}$ ⟹ greatest lower bond

②




$B = \{c, a\}$

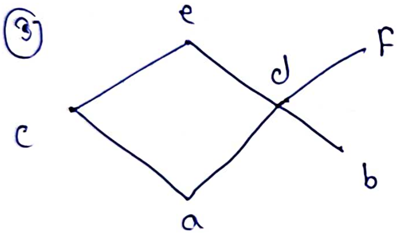
$U.B = \{c, e\}$

$L.U.B = \{c\}$

$L.B. = \{g, h, a\}$

$G.L.B = \{a\}$.

③




$B = \{e, F\}$

$U.B = \phi$

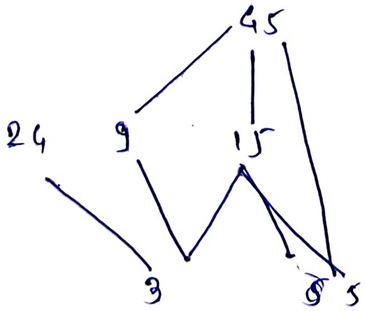
$G.U.B = \phi$

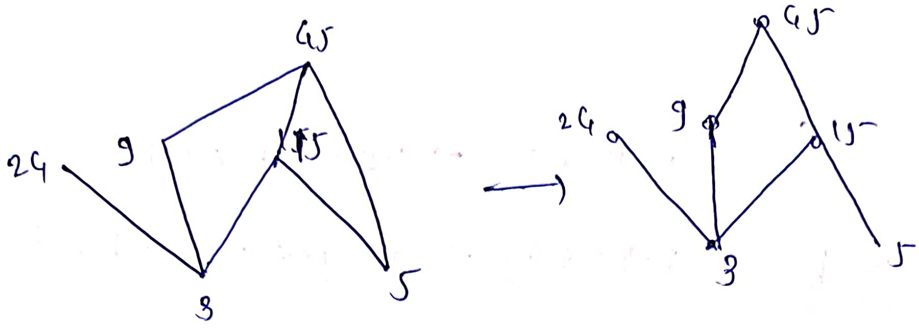
④ poset$[\{3, 5, 9, 15, 24, 45\} /]$

ⓐ Find the maximal

ⓑ minimal ⓒ is their any greatest element ?

ⓓ is their any least element ? ⓔ Find the upper bond of (9,5) ? → LUB if exist ⓔ Find all the lower bond {15, 45} & GLB if (exist)






(a) maximal : 24, 45

(b) minimal = 3, 5

(c) Greatest = $\phi$

(d) ~~m~~ least = $\phi$

(e) $B = \{3, 5\}$ U.B = $\{15, 45\}$ L.U.B = $\{15\}$

(d) $B = \{15, 45\}$ L.B = $\{3, 5, 15\}$ G.L.B = $\{15\}$