Sr. No	TITLE	Date	Teacher's Sign / Remark

M-III.

★ Applied Linear Algebra.

Reference book – ① Anton & Rorres (Elementary linear Algebra

② David polley – A Morden Introduction to L.A

③ Gilbert strang – linear Algebra & its Appⁿ.

Unit1 ⇒ Vector Spaces
- Subspace

Imp for ML / AI

Unit2 ⇒ Linear Transformation

($M_n(\mathbb{R})$ – Set of square matrices

Unit3 ⇒ Inner product Space (IPS)

Unit4 ⇒ Matrix Norms

Unit5 ⇒ Diagonalization (eigen Value & eigen Vector)

Unit6 ⇒ Singular Value decomposition (SVD).

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☆ Matrix

- If B is inverse of A then $AB = I$ & $BA = I$

$A^{-1} = \frac{1}{|A|} adj(A)$

$adj(A) = [C_{ij}]^T$

$C_{ij} = (-1)^{i+j} M_{ij}$ M_{ij} = elele. (Submatrix obtained by del i^{th} row & j^{th} column)

$A = [\ \ ]_{m\times n}$ or

$A = (\ \ )_{m\times n}$

→ Arrangement of $m \cdot n$ numbers into rectangular array

Vectors → $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ or $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ x_1, x_2 are scalars

→ Arrangement of m dimention vectors into n column in a rectangular array.

* Types of Matrices :-

1) Row matrix :- The Matrix in which only 1 row is present.

$A[a_{ij}]_{m\times n}$ $i = 1, 2, \ldots m$ (row)
$j = 1, 2, \ldots n$ (column)

$A = [a_{11}\ \ a_{12}\ \ldots\ a_{1n}]_{1\times n}$

2) Column matrix :- The matrix in which only one column is present.

$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}_{m\times 1}$

3) Square Matrix :- A matrix is said to be square matrix if it has no. of rows = no. of column.

$$A = [a_{ij}], \quad i=j, \quad i,j = 1,2,\ldots n.$$

4) Zero matrix :- (or Null matrix) It is defined as

$$A = [0_{ij}]_{m \times n} \quad \forall\, i,j$$
for all

5) Digonal Matrix :- It is Defined as $A = \begin{cases} d_{ij}, & \forall\, i,j \\ 0, & i \neq j \end{cases}$ Square Matrix.

$$A = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{nm} \end{bmatrix}_{m \times n}$$

6) Scalar Matrix :- It is defined as digonal Matrix it is given by

$$A = \begin{cases} d & i=j \\ 0 & i \neq j \end{cases} \qquad A = \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & & d \end{bmatrix}_{m \times n}$$

7) Identity Matrix :- It is Scalar Matrix it is given by

$$A = \begin{cases} 1, & i=j \\ 0 & i \neq 0 \end{cases}$$

i.e $\begin{bmatrix} 1 & 0 \cdot & 0 \\ \vdots & 1 \cdots & 0 \\ 0 \cdots & \ddots & 1 \end{bmatrix}_{m \times n}$

8) Lower triangular Matrix (LTM) :- It is square matrix it is defined as

$$A \begin{cases} a_{ij}, & i \leq j \\ 0, & i > j \end{cases}$$

9) Upper triangular Matrix :- It is Square Matrix It is defined as

$$A = \begin{cases} a_{ij}, & i \geq j \\ 0, & i < j \end{cases}$$

10) Symmetric Matrix :- A square matrix is said to be symmetric if $A^T = A$

or $A = [a_{ij}]_{n \times m}$ is said to be Symmetric matrix is $a_{ij} = a_{ji} \quad \forall\, i,j$.

Ex:- $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\quad A = \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & i \end{bmatrix}_{3 \times 3}$

11) Hermitian Matrix

if $A^{*} = A$

$$A = \begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$$

$$\boxed{A^{*} = (\bar{A^{T}})^{T} = A}$$

$A = \begin{bmatrix} 1 & -1 \\ 1-i & 1+i \end{bmatrix}$ $\quad \bar{A} = \begin{bmatrix} 1 & i \\ 1+i & 1-i \end{bmatrix}$

$$A^{*} = (\bar{A})^{T} = \begin{bmatrix} 1 & 1+i \\ i & 1-i \end{bmatrix}$$

12) Skew Hermitian Matrix :- A is said to be SHM $A^{\dagger} = -A$

Ex → $A = \begin{bmatrix} 3i & 2+4i \\ -2+4i & -7i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} -3i & 2-4i \\ 2-4i & 7i \end{bmatrix}$

13) Orthogonal Matrix :- let, A be the Invertible square matrix it is said to be orthogonal if $A^T = A^{-1}$

i.e. $A \cdot A^T = A^T A = I$

Ex:- check $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$A \cdot A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$A^T = \begin{bmatrix} -\cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$

$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

dot product - 0

1,0 → Idempotent element $1^2 = 1$, $0^2 =$

14) Idempotent matrix :- A square matrix is said to be Idempotent if $A^2 = A$ (or $A^{-1} = A$)

ex → check $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

defⁿ & example

15) Involuntary matrix :- let A be the Invertible matrix is said to be involuntary matrix if $A^2 = I$ $(A = A^{-1})$

Ex:- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

16) Nilpotent Matrix :- let A be the square matrix is said to be Nilpotent iff $\exists$ Natural No. n such that $A^n = O_{n\times n}$ matrix

$A^n = \underbrace{O_{n \times n}}_{\text{Matrix}}$

[n - is index of Nilpotency]

Ex:- 1) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Check power $A^n = 0$ yetekinahi

1) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

H.W. Write down two examples for each
1) Gauss Elimination
2) Gauss Jordan Elimation
3) L-U decomosition & factorization.

1) Gauss Elimination & 2) Gauss Jordan Elimination

Ex:- ①
$x + y + 2z = 9$
$2x + 4y - 3z = 1$
$3x + 6y - 5z = 0$

→ By Gauss Elimination Method :-

$\rightarrow A = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$

$R_2 = R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$

$A = \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$

$R_2 = R_2 \times \frac{1}{2}$

$A = \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{bmatrix}$

$R_3 \rightarrow R_3 - 3R_2$

$A = \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{bmatrix}$

$R_3 \to R_3(-2)$

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \text{—— eq}^n \text{ A}$$

System $\to$ $x + y + 2z = 9$ —(1)

$y - \frac{7}{2}z = \frac{-17}{2}$ —(2)

$z = 3$ —(3)

from eq^n (1), (2), (3) we get

$x = 1,\ y = 2$ & $z = 3$.

By Gauss Jordan Elimination Method

up to eq^n (A) same

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 9 & 26 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 + \frac{7}{2}R_3$

$$\begin{bmatrix} 1 & 0 & 9 & 26 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_1 \to R_1 - 9R_3$

$$\begin{bmatrix} 1 & 0 & 9 & 26 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_1 \to R_1 - 9R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This is RREF form

$\therefore x = -1,\ y = 2,\ z = 3$

(2) $x + y = 0$

$2x + y = 1$

$3x = 1$

By Gauss Elimination

$$\rightarrow A|b = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \text{ — A}$$

$\therefore x = 1,\ y = 1$

By Gauss ~~Eli~~ Jordan Elimination Method

up to eq^n (A) same

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right]$$

It is already in RREF

$\therefore x = 1,\ y = 1.$

* LU Decomposition Method

$$AX = B \;\rightarrow\; LUX = B \;\rightarrow\; UX = Y \;\rightarrow\; LY = B$$

Ex:- $3x + y + z = 4$
$x + 2y + 2z = 3$
$2x + y + 3z = 4$

→ System written as $Ax = b$ —(1)

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

where, $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$

Now $A = LU$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{21} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{21} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ L_{21}u_{11} & L_{21}u_{12}+u_{22} & L_{21}u_{13}+u_{23} \\ L_{31}u_{11} & L_{31}u_{12}+L_{32}u_{22} & L_{31}u_{13}+L_{32}u_{23}+u_{33} \end{bmatrix}$$

$u_{11} = 3$ $\quad$ $u_{12} = 1$ $\quad$ $u_{13} = 1$

$L_{21}u_{11} = 1$ $\quad$ $L_{21}u_{12} + u_{22} = 2$ $\quad$ $L_{21}u_{13} + u_{23} = 2$

$\therefore L_{21} = \frac{1}{3}$ $\quad$ $\therefore u_{22} = \frac{5}{2}$ $\quad$ $\therefore u_{23} = \frac{5}{3}$

$L_{31}u_{11} = 2$ $\quad$ $L_{31}u_{12} + L_{32}u_{22} = 1$ $\quad$ $L_{31}u_{13} + L_{32}u_{23} + u_{33} = 3$

$\therefore L_{31} = \frac{2}{3}$ $\quad$ $\therefore L_{32} = \frac{1}{5}$ $\quad$ $\therefore L_{33} = 1$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/5 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 5/3 & 5/3 \\ 0 & 0 & 2 \end{bmatrix}$$

$LUX = B$ —(2)

Now, let $UX = Y$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

eq^n (2) Becomes
$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$y_1 = 4$, $y_2 = 5/3$, $y_3 = 1$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5/3 \\ 1 \end{bmatrix} \text{ —(3)}$$

eq^n (3) becomes

$\therefore Ux = y$

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 5/3 & 5/3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5/3 \\ 1 \end{bmatrix}$$

$z = 1/2,\ y = 1/2,\ x = 1.$

$$\therefore \begin{bmatrix} 3 & 1 & 1 \\ 0 & 5/3 & 5/3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5/3 \\ 1 \end{bmatrix}$$

✱ LU Factorization

Q) $\rightarrow$ $A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.

$$\longrightarrow A = \underbrace{\begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}}_{L}$$

Multiplier -ve, other side

diagonal - reciprocal element

$R_1 \to R_1/6$

$$\begin{bmatrix} 1 & -1/3 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \quad 1/6 \quad \begin{bmatrix} 6 & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

diagonal reciprocal

multiply

$R_2 \to -9$

$R_3 \to -3$

$$\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 5 \end{bmatrix} \text{ multiply } 1/2 \begin{bmatrix} 6 & 0 & 0 \\ 9 & * & 0 \\ 3 & * & * \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 8 & 5 \end{bmatrix} \text{ multiply } -8. \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & * \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \text{ multiply } 1 \underbrace{\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}}_{L}$$

$LUX = b$, let $UX = Y$, $LY = b$

$$\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$6y_1 = 1$, $-9y_1 + 2y_2 = 1$, $3y_1 + 8y_2 + y_3 = 1$

$y_1 = 1/6$, $y_2 = 3$, $y_3 = -47/2$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 3 \\ -47/2 \end{bmatrix}$$

$UX = Y$

$$\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/6 \\ 3 \\ -47/2 \end{bmatrix}$$

$\therefore z = -47/2$

$y + z/2 = 3;\ \therefore y = \frac{4}{47} + 3$

$x - \frac{1}{3}y = 1/6$

$\therefore x = 1/2 - \frac{4}{47} + 3.$

Revisim

$SM \Rightarrow A^T = A$
$HM = A^\theta = A$
$SSM \Rightarrow A^T = -A$
$SHM \Rightarrow A^\theta = -A$, $A^T = A^{-1}$, $|A| \neq 0$.

17) Unitary Matrix :- let A be square complex matrix which is invertable is said to be unitary matrix, if

$$A^\dagger = A^{-1}$$

or $\underbrace{A^\dagger A = AA^\dagger}$ $= I_n$

Ex :- $\begin{bmatrix} 1+i/2 & (1-i)/2 \\ (1-i)/2 & i+i \end{bmatrix}$

$\begin{bmatrix} 1/\sqrt{3} & (1+i)/\sqrt{3} \\ (1-i)/\sqrt{3} & i/\sqrt{3} \end{bmatrix}$

18) Normal Matrix :- A is called a Normal matrix if $A^\dagger A = AA^\dagger$

★ Vector Spaces

• Binary operation on a set :- let, A be the any non-empty set we say $*$ is binary operation on a set A if $*$ is a mapping or a function from $A \times A \to A$

$\left(* : (\text{is}) \ \underbrace{A \times A}_{\text{domain}} \to \underbrace{A}_{\text{codomain}} \right)$

Cartesian product of A

$(a \cdot b) \to c \ (\in A$
i.e $a * b \in A$.

$f : \underset{\text{domain}}{R} \to \underset{\text{codomain}}{R}$

$R \times R = R^2$
$= \{(x,y) \mid x,y \in R\}$

{$*$ - is a function}

Ex:- ① $A = N$ (Natural no.) ; $* = +$ (usual Add^n)

$\to \exists \ a,b \in N$
$\Rightarrow a * b = a+b \in N$, $\forall \ a,b \in N$
$* = +$ is a binary operation on set N.

$\boxed{N \subset W \subseteq Z \subset Q \subseteq \Phi}$

② $A = Z$, $* = -$
$\to$ '$-$' is binary operation on Z,
sin ϕ $\forall$ $a,b \in Z$
$a * b = a-b \in Z$

③ $A = Q$, $* = \div$
$\to a = 1$, $b = 0$
$1 * 0 = 1 \div 0 = \frac{1}{0}$ is D.N.E.
So, if $A = Q \setminus \{0\}$ then $* = \div$ is Binary operation on set A.

complimenl - Irrational

4) $A = Q^c$, $* = +$

$a = \sqrt{2}$, $b = -\sqrt{2}$

$a * b = \sqrt{2} + (-\sqrt{2}) = 0 \in Q^c$

exeption $\frac{a}{b} = \frac{22}{7}$ - App rim

$+$ is not a binary operation on a set of Irrational no.

5) $A = \mathbb{R}$

$\rightarrow \forall a, b \in A$, $a * b = a^b$

$a = -1$, $b = 1/2$

$a * b = a^b = (-1)^{1/2} = i \notin \mathbb{R}$

Powerset of Natural No.

6) $A = P(N)$, $* = \cup$ — Union

$\rightarrow M, N \in P(N)$

$\Rightarrow \forall M, N$

$\Rightarrow M \cup N \subseteq N \in P(N)$

Identity Element of set :-

Let A be the any non-empty set $e \in A$, is said to be an identity element if $*$ is binary operation on A then e is identity element of A if

$a * e = e * a = a$, $\forall a \in A$

Ex - ① $A = \mathbb{R}$, $* = +$

$\rightarrow$ clearly 0 is the identity of $\mathbb{R}$

Since, $a + 0 = 0 + a = a$, $\forall a \in \mathbb{R}$

Here, 0 is called as additive identity of $\mathbb{R}$

② $A = \mathbb{R}$, $* = \times$

then, 1 is the identity element of $\mathbb{R}$ w.r. binary operation '$\times$'

Since $a \times 1 = 1 \times a = a$

(1 is called as Multiplicative identity of $\mathbb{R}$)

③ $A = M_n(\mathbb{R})$, $* = + \rightarrow *$ (Identity Matrix I)

Matrices

$\rightarrow$ Null matrix

⑥ $A = P(N)$ & $* = \cup$ — union

$\rightarrow$ Here, identity of a set $P(N)$ is ϕ

Since $M \cup \phi = \phi \cup M = M$ $\forall M \in P(N)$

⑤ $A = P(N)$, $* = \cap$ — Intersection

$\rightarrow$ N is identity element of $P(N)$ w.r.t Binary operation '$\cap$'

Since, $N \cap A = A \cap N = A$, $\forall A \in P(N)$.

* Associative & commutative Binary operation on set :-

let A be the any non-empty set & $*$ be the any Binary operation on A then $*$ operation on A then $*$ is said to be commutative Binary operation if,

$A * b = b * a$, $\forall a, b \in A$ &

$*$ is associative if $(a * b) * c = a * (b * c)$ $\forall a, b, c \in A$.

not commutative but Associative example

$(a \times b) \neq (b \times a)$ commutative ✗

$A = M_n(\mathbb{R})$ $* = \times$

Associative ✓

* Inverse of a element :-

let A be the any non-empty set & $*$ is the Binary operation on A, b is said to be inverse of a, w.r.t Binary operation $*$ if

$\boxed{a * b = b * a = e}$ where e is identity element of A w.r.t Binary operation $*$

ex:- 1) $A = \mathbb{R}$, $* = +$

→ $\forall a \in \mathbb{R}$, there exist an inverse w.r.t Binary operation '+' $b = -a$.

such that $a * b = a + (-a) = 0 = (-a) + (a) = 0$

2) $A = \mathbb{R}$, $* = \times$

→ $\forall a \in \mathbb{R}$, there exist an inverse w.r.t. binary operation.

Revision $* : A \times A \to A$

$(a, b) \to c$, $c \in A$

$a * b \in A \Rightarrow *$ is B.O on A.

- Identity element of set A

$e \in A$ is an id. element if

$a * e = e * a = a$ $\forall a \in A$

Ex:- $\left\{ A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}_{2\times2} \middle| a \in \mathbb{R} \right\}$

* usually Multiplication of matrix on A

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = e \in A$$

$0 \neq a \in A$, has inverse b if $a * b = b * a = e$

$b = a^{-1}$

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} x & x \\ x & x \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$ax + ax = 1/2$

$2ax = 1/2$

$x = 1/4a$

$a \neq 0$

$$= \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

★ let $A=\mathbb{R}$ & $*$ is a binary operation on $\mathbb{R}$ & it is defined as $a*b = a+b+a.b$ $\forall a,b \in \mathbb{R}$

find ① Identity element of $\mathbb{R}$ w.r.t $*$

② Find inverse of possible element in $\mathbb{R}$

$\rightarrow 1*1 = 1+1+1 = 3.$

★ Field $(\mathbb{F}, \cdot, +)$

$\rightarrow$ let $\phi \neq \mathbb{F}$ be any set together with binary operation addition $(+)$ & multiplication $(\cdot)$ is said to be field if following properties are satisfied.

Satify 1,2,3 then G.P.

① $a+(b+c) = (a+b)+c$, $\forall a,b,c \in \mathbb{F}$ (Associative w.r.t $+$)

② $a+0 = 0+a = a$ ... (Existance of additive identity)

③ $a+(-a) = (-a)+a = 0$ ... (Existance of additive inverse)

④ $a+b = b+a$, $\forall a,b \in \mathbb{F}$ ... (commutative)

⑤ $a\cdot(b\cdot c) = (ab)\cdot c$ ... (associative w.r.t $\cdot$)

⑥ $a\cdot 1 = 1\cdot a = a$ .... (Existance of Multiplicative Identity)

⑦ $a\cdot b = b\cdot a = 1$ ... [existance of multiplicative of a, inverse of a it is denoted by $b = \frac{1}{a}$ or $b = a^{-1}$]

⑧ $a(b+c) = a.b + a.c$ $\forall a,b,c \in \mathbb{F}$ ... [left distribution law]

$(a+b).c = ac + bc$, $\forall a,b,c \in \mathbb{F}$ ... (Right distribution law)

⑨ $a.b = b.a$, $\forall a,b \in \mathbb{F}$ (Commutative w.r.t $\cdot$)

semigroup — i) $a+(b+c)=(a+b)+c \ \forall a,b,c$
Monoid — ii) $a+e=e+a=a, \ \forall a \in F$
abelian group — iii) $a+b=b+a=e$
Group — iv) $a+b=b+a$

Group $(G,+)$ ✓

Ex ⇒ $(\mathbb{R},+)$ ✓, $(\mathbb{Z},+)$ ✓, $(\mathbb{Q},+)$ ✓, $(\mathbb{N},+)$,
$(\mathbb{C},+)$ ✓, $(\mathbb{R},\times)$ ✗, $(\mathbb{R}^*,\times)$ ✓
$\hookrightarrow \mathbb{R}^* = \mathbb{R}-\{0\}$

↓

Ring i) $(R,+,\cdot)$ - gp
ii) $(R,\cdot)$ is semi gp
(iii) must be hold (Distributive law)
ex:- $(\mathbb{R},+)$ $(\mathbb{R},+,\cdot)$, $(\mathbb{Q},+,\cdot)$

↓

Field - Big structure
ex:- $(\mathbb{R},+)$
$(\mathbb{Q},+,\cdot)$ - ~~skew~~ smallest field
$(\mathbb{C},+,\cdot)$ - Universal field
complex $\quad$ ↳ big

- Alternative def^n of field $(\mathbb{F};+,\cdot)$
  let $\phi \neq F$ be be any set with two binary operations addition $(+)$ & multiplication $(\cdot)$ is said to field if following properties are satisfied

(i) $a\cdot b = b\cdot a \quad \forall a,b \in \mathbb{F}$ ...(commutativity)

(ii) Existance of inverse of each non-zero element in field.

i.e $\forall 0 \neq a \in \mathbb{F}$, $\exists\ b \in \mathbb{F}$
such that $ab = ba = 1$
b is called inverse of a & it denoted by
$b = a^{-1}$ or $b = \frac{1}{a}$.

Ex:- $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are standard fields (Infinite)

- Finite field :- The field in which ~~field~~ finite no. of element

$(\mathbb{F}_p, +_n, \times_n)$ - let define.

$\mathbb{Z}_n$ = set of remainder when we divide any integer by a no. $n \in \mathbb{Z}^+$

$n = 2$ ... divide 16, 17

$\mathbb{Z}_2 = \{0,1\}$
$\mathbb{Z}_3 = \{0,1,2\}$

$\mathbb{Z}_n = \{0,1,2,\dots n-1\}$

We defined an binary operation called addition modula in $(+_n)$ & multiplication mudula $n (\times_n)$ $\forall a,b \in \mathbb{Z}_n$ we have

$a +_n b$ = least +ve remainder when we divide $a+b$ by n

$a \times_n b$ = least +ve remainder when we ~~multiply~~ divide $a\cdot b$ by n

Ex:- $\mathbb{Z}_3 = \{0,1,2\}$

$1 +_3 2 = r\left(\frac{1+2}{3}\right) = r\left(\frac{3}{3}\right) = 0$

$2 \times_3 1 = r\left(\frac{2\times 1}{3}\right) = r\left(\frac{2}{3}\right) = 2$

Group :- let $\phi \neq G$ be any set and $*$ is binary operation on G is said to be group if.

Semigp → 1) $a*(b*c) = (a*b)*c \quad \forall\, a,b,c \in G$

Monoid → 2) $a*e = e*a = a \quad , \forall\, a \in G$

(e is called identity element of G w.r.t $*$)

Gp → 3) $a*b = b*a = e \quad , \forall\, a \in G \; (b = a^{-1})$

Abelian gp → 4) $a*b = b*a, \quad \forall\, a,b \in G$

Ex :- 1) $G = \mathbb{R}$ (real no.) , $* = +$

$(\mathbb{R},+)$ is group?

⟹ Yes, since it satisfy all properties of gp

2) $G = \mathbb{R} \; , * = \times$

$(\mathbb{R},\times)$ is group?

⟹ $(\mathbb{R},\times)$ is a monoid

3) $G = \mathbb{R}^* = \mathbb{R}/\{0\} \; , * = \cdot$

⟹ $(\mathbb{R}^*,\times)$ is a group

4) $G = \mathbb{N} \; , * = +$

⟹ $(\mathbb{N},+)$ is a semi group

5) $(\mathbb{Q},+)$, $(\mathbb{R},+)$ $(\mathbb{C},+)$, $(\mathbb{C}^*,\times)$ ....etc

⟶ Monoid.

Every field is a Ring

Ring :- let, $\phi \neq R$ be any set with two binary operation addition $(+)$ & multiplication $(\cdot)$ is said to be ring if

1) $(R,+)$ is a gp

2) $(R,\cdot)$ is a semi group

3) Distributive law must be hold

i.e $\left.\begin{array}{l} a(b+c) = ab+ac \\ (a+b)c = ac+bc \end{array}\right\} \forall\, a,b,c \in R$

- It is denoted by $(R,+,\cdot)$

Ex :- $(\mathbb{C},+,\cdot)$ forms a ring

$(\mathbb{R},+,\cdot)$ forms a ring

$(\mathbb{Q},+,\cdot)$ forms a ring

$(\mathbb{Z},+,\cdot)$ forms a ring

set of matrices $(M_n(\mathbb{R}),+,\cdot)$ forms a ring.

...etc

HW Let $K_4 = \{e, a, b, ab \,/\, ab = ba \quad a^2 = b^2 = e\}$ forms a group under usual multiplication (e is identity element)

- Finite fields $(\mathbb{F}_p; +_n, \times_n)$

→ $\mathbb{Z}_n = \{0, 1, 2, \dots n-1\}$

$+_n$ = addition modulo n
$\times_n$ = multiplication modulo n
is defined as $\forall$, $a, b \in \mathbb{Z}_n$

$a +_n b$ = least the remainder when we divide $a + b$ by n
$a \times_n$ = least the remainder when we divide $a \cdot b$ by n.

Ex: ① $\mathbb{Z}_1 = \{0\}$ is not a field. $\begin{cases} \text{rem} = \text{even} = 0 \\ \text{rem} = \text{odd} = 1 \end{cases}$

② $\mathbb{Z}_2 = \{[0], [1]\}$.
⇒ clearly, $\forall$ $a, b \in \mathbb{Z}_2$
we have $a \times_2 b = b \times_2 a$
since $1 \times_2 0 = 0 \times_2 1 = 0$
also $1^{-1} = 1$
since, $1 \times_2 x = 1$, $x = 1$
⇒ is field

Class - $[0] = \{\dots -2, 0, 2, 4, \dots\}$
$[1] = \{-3, -1, 0, 1, 3 \dots\}$

③ $\mathbb{Z}_3 = \{0, 1, 2\}$
$\{3k+1 / k \in \mathbb{Z}\}$ $(3k+2 / k \in \mathbb{Z})$
set $\{3k / k \in \mathbb{Z}\}$

is field?

⇒ Clearly 1) $a \times_3 b = b \times_3 a$, $\forall$ $a, b, \in \mathbb{Z}_3$

2) Inverse of Non-zero element
$1^{-1} = 1$ $\quad 2^{-1} = 2$

$\frac{2 \times 2}{3} = \frac{4}{3} = r = 1$, ∴ $b = 2$ $\quad$ Since $\exists$ $b \in \mathbb{Z}$ is such that $2 \times_2 b = 1 \rightarrow b = 2$

So, $\mathbb{Z}_3$ is field

So, $\mathbb{Z}_3$ is field

3) $\mathbb{Z}_4 = \{0, 1, 2, 3\}$
→ clearly $a \times_4 b = b \times_4 a$, $\forall a, b \in \mathbb{Z}_3$
$1^{-1} = 1$

2^{-1}

$2 \times_4 2 = r\left(\frac{2 \times 2}{4}\right) = r(0)$
$0 \neq 1$
$\nexists$ 2^{-1} in $\mathbb{Z}_4$

⇒ $\mathbb{Z}_4$ is not a field

$\boxed{\text{Note}}$ ⇒ $\mathbb{Z}_n$ is a field if n is prime number
2, 3, ~~4~~ $\rightarrow \times$.

$\mathbb{Z}_{127}$ is field
→ Not a field $\begin{cases} \text{close } \sqrt{} \\ \sqrt{121} \\ \mathbb{Z}_{2023} \end{cases}$

Ex → What $\frac{2}{3}$ is $\mathbb{Z}_5$

$\frac{a}{b} = a \times \frac{1}{b} = a b^{-1}$ ✓

revision

$\mathbb{Z}_n$ = set of all remainder when we divide integer by $n \in \mathbb{N}$

$\mathbb{Z}_n = \{\bar{0}, \bar{1}, \bar{2}, \dots \overline{n-1}\}$

$\mathbb{Z}_n \ +_n \ \times_n \quad \forall \ a, b \in \mathbb{Z}_n$

$a +_n b = r\left(\frac{a+b}{n}\right)$

$a \times_n b = r\left(\frac{ab}{n}\right)$

Q $(G, *)$, $(\mathbb{Z}_n, \dots +_n)$ is group?

$\rightarrow 1^{-1} = 1 +_2 1 = r\left(\frac{1+1}{2}\right) = r\left(\frac{2}{2}\right) = r(0) \dots$ additive identity

$(\mathbb{Z}_n, +_n, \times_n)$ has 0 as additive identity & 1 is multiplicative identity

1) $a +_2 (b +_2 c) = (a +_2 b) +_2 c$

2) $a +_2 0 = 0 +_2 a = a$, $\forall a \in \mathbb{Z}_2$

3) $\exists \ a^{-1} \ \forall \ a \in \mathbb{Z}_2$

Thus, it satisfy three properties so It is group.

$2 \times_{19} b = 1$

$2 \times_{19} 10 = 1$

$\cancel{1 \times_{19}}$

Q What is value of $\frac{3}{5}$ in $\mathbb{Z}_{11}$

$\rightarrow \frac{3}{5} = 3 \times_{11} \frac{1}{5}$

$= 3 \times_{11} 5^{-1}$

$= 3 \times_{11} 9$

$= r\left(\frac{3 \times 9}{11}\right)$

$= r\left(\frac{27}{11}\right)$

$\boxed{\mathbb{Z}_{11}}$

11, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Inverse

$5 \times_{11} x = 1$ ← remaind should be 1

$\frac{5 \times (9)}{11} = \frac{45}{11}$

$= r\left(\frac{45}{11}\right)$

$= r(1)$

$\left\{\begin{array}{l} -24 \text{ in } \mathbb{Z}_{11} \rightarrow 2 \\ 53 \text{ in } \mathbb{Z}_{11} \rightarrow 9 \end{array}\right\}$

HW Q Find all aditive inverse & multiplicative inverse for all $\mathbb{Z}_{19}$ $(\mathbb{Z}_{19}, +_{19}, \times_{19})$ ~~why~~ prime - forms a field.

$\rightarrow \mathbb{Z}_{19} = \{0, 1, 2, 3 \dots 18\}$

The additive Inverse of each element in $\mathbb{Z}_{19}$

$a +_{19} b = b +_{19} a = 0$

$\Rightarrow$ let $a = 0$, $b = 0$

$a = 1$, $b = 18 \Rightarrow a +_{19} b = 19 = r(0)$

$a = 2$, $b = 17 \Rightarrow a +_{19} b = 19 = r(0)$

$a = 3$, $b = 16$.

$a = 18$, $b = 1 \dots \Rightarrow a +_{19} b = 19 = r(0)$

We want $b \in \mathbb{Z}_{19}$ ~~such~~ for each non-zero element $a \in \mathbb{Z}_{19}$, such that $a \times_{19} b = 1$, a^{-1}, let $a = 1$, to find $b \in \mathbb{Z}_{19}$ such that $1 \times_{19} b = 1 \Rightarrow b = 1$, $a = 2$ to find $b \in$

$(V, +)$ is group.

★ Vector space $(V(F))$

→ Let, V be the non-empty set of objects with two binary operation addition $(+)$ & multiplication $(\cdot)$, We say V is a vector space over a field F. If it following conditions are ~~satified~~ satisfied for all $u, v, w \in V$ & $k, m \in F$

1) If u, v are the objects in V then $u+v$ is in V (property of closer)
 $\therefore (\forall\ u, v \to u+v \in V)$

2) $u + (v+w) = (u+v)+w$

3) There is an object zero in V called zero vector for V such that $0+u = u+0$, $\forall\ u \in V$.

4) For each u in V there is an object $-u$ such that $u+(-u) = (-u)+u = 0$ $\forall\ v \in V$)

5) If k is any scalar in F & u is any object in V then $k.u$ is in V
 (i.e $\forall\ k \in F$, $u \in V$)
 $\Rightarrow ku \in V$

6) $(k+m)u = ku + mu$ $\forall\ v \in V$, $k, m \in F$

7) $k(u+v) = ku + kv$ $\forall\ k \in F$, & $u, v \in V$)

8) $k(mu) = (km)u$

$F = \mathbb{R}, \mathbb{C}, \mathbb{Q}$

9) $1.u = u$ $\forall\ u \in V$ → (vector space V)
 (1 is multiplicative identity in F)
Where objects in V called as vector
 Elements of fields F is called as scalars

10) $u+v = v+u$. $\forall\ u, v \in V$ (1, 2, 3, 10 → Abelian group, $(V, +)$ →

Example ⇒ 1) $V = \{0\}$ forms a vector space over the field real numbers $\mathbb{R}$?
→ Yes, trivial form of vector space

2) $V = \mathbb{R}^n$, $F = \mathbb{R}$
 $\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \ x_1 \ldots x_n \in \mathbb{R} \right\}$

→ $\mathbb{R}^n(\mathbb{R})$ forms a vector space coordinate wise addition $(+)$ & scalar multiplication defined by

If $\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ & $k \in F$

⇒ $u + v$ (coordinate wise add^n) $= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$

$ku = k\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix}$

$\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ in } \mathbb{R}^2 \quad u+(-u)=0, \quad -u = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

HW Q) Show that $\mathbb{R}^n(\mathbb{R})$ a vector space.

③ V = set of 2×2 matrices over $\mathbb{R}$, $\mathbb{F}=\mathbb{R}$

$M_2(\mathbb{R})$ $[M_2(\mathbb{R})](\mathbb{R})$

$\longrightarrow u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \quad v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$

for all

$w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$ & $k \in \mathbb{R}$

form a vector space w.r.t. operations addition & scalar multiplication of matrices

$u+v = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$

& $ku = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$

Q1) $V = \mathbb{R}$, $\mathbb{F} = \mathbb{Q}$ — rational

$\Longrightarrow$ $\mathbb{R}(\mathbb{Q})$ forms a vector space w.r.t. usual addition & usual multiplication

Q2) $V = \mathbb{Q}$, $\mathbb{F} = \mathbb{R}$

$\longrightarrow$ $\mathbb{Q}(\mathbb{R})$ does not forms a vector space

$k = \sqrt{2}$, $v = \frac{1}{2}$,

$k \cdot u = \frac{\sqrt{2}}{2} = \frac{1}{2} \notin \mathbb{Q}$

Q3) $V = \mathbb{Q}$, $\mathbb{F} = \mathbb{Q}$

$\longrightarrow$ $\mathbb{Q}(\mathbb{Q})$ is form a vector space w.r.t usual addition & usual multiplication.

Q4) $V = \mathbb{Z}_p$, $\mathbb{F} = \mathbb{Z}_p$ — set of prime no.

$\Longrightarrow (\mathbb{Z}_p; +_n, \times_n)$

$\mathbb{Z}_p(\mathbb{Z}_p)$ forms a vector space.

w.r.t. addition $(+_n)$ & scalar multiplication $(\times_n)$

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HW Q) V = set of all polynomials of degree $\le n$ with coefficient from $\mathbb{R}$.

$V = \{ a_0 + a_1 x + a_2 x + a_2 x^2 + \dots + a_n x^n \mid a_i \in \mathbb{R} \}$ $i = 1, \dots, n$

$= \mathbb{P}_n(\mathbb{R})$

$F = \mathbb{R}$ prove that $P_n(\mathbb{R})(\mathbb{R})$ form a vector space with w.r.t. operation addition & scalar multiplication defined by

for $P_1(x), P_2(x) \in P_n(\mathbb{R})_n$

$\Rightarrow P_1(x) + P_2(x)$

$\Rightarrow k P_1(x)$

Ex $P_1(x) = x^2 + x + 1$, $P_2(x) = x^2 + 1$

$100\, P_1(x) = 100^2 + 100 + \dots$

→ $V = P_n(\mathbb{R})$ = set of all polynomials of degree $\le n$ over $\mathbb{R}$

$V = \{ P(x) = a_0 + a_1 x + \dots + a_n x^n \mid a_0, a_1, \dots a_n \in \mathbb{R} \}$

$F = \mathbb{R}$

Sol^n ⇒ $P_1(x) + P_2(x) = x^2 + x + 1$
$\quad + x^2 \quad + 1$

$2x^2 + x + 2$

w.r.t. B.O '+' :

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$u + v = v + u$

$P_1(x) + P_2(x) = 2x^2 + x + 2$

$v + u = 2x^2 + x + 2$

(ii) $u + (v + w) = (u + v)\, w$

Vector Space (under usual add^n & ~~scar~~ scalar Multiplication)

→ $\mathbb{Z}_p(\mathbb{Z}_p)$ ✓

$\mathbb{C}(\mathbb{C})$ ✓

$\mathbb{R}^2(\mathbb{C})$ ✗

$\mathbb{Q}(\mathbb{R})$ ✗

$\mathbb{Q}(\mathbb{Z})$ ✗

$\mathbb{Z}(\mathbb{Z})$ ✗

$\mathbb{C}(\mathbb{R})$ ✓

$\mathbb{Z}_2(\mathbb{Z}_3)$ ✗

$\mathbb{R}^2(\mathbb{R})$ ✓

$\mathbb{C}(\mathbb{Z})$ ✗

$\mathbb{C}(\mathbb{Q})$ ✓

$\mathbb{R}(\mathbb{Q})$ ✓

$\mathbb{R}^\infty = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} \right\}$ = ✓

Vector space

Example 1) $V = \mathbb{C}$, $\mathbb{F} = \mathbb{R}$

$\mathbb{C}(\mathbb{R})$ forms a vector space over usual addition & usually multiplication.

2) $V = \mathbb{R}$, $\mathbb{F} = \mathbb{C}$

$\rightarrow \mathbb{R}(\mathbb{C})$ is not a vector space

$K.u \notin \mathbb{R}$

$K = i \in \mathbb{C}$, $u = 100 \in \mathbb{R}$

$\Rightarrow i \cdot 100 = 100i \notin \mathbb{R}$

$\rightarrow$ ★ Standard Field $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ {scalar come from K & m}

$\rightarrow$ The field $\mathbb{F}$ forms a vector space over it's subfield.

$\mathbb{F}(\mathbb{f})$ forms a vector space, where $\mathbb{f}$ is subfield of $\mathbb{F}$.

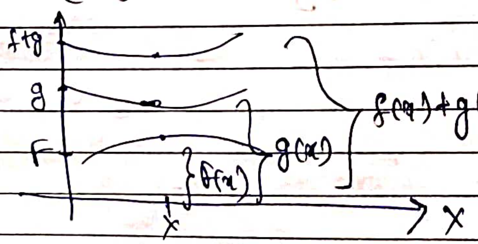
Ex:- V = set of all real valued function $f: \mathbb{R} \to \mathbb{R}$

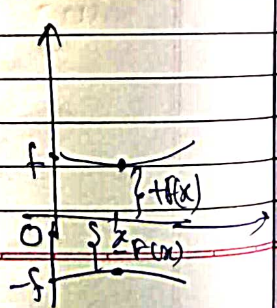
$V = \{f : \mathbb{R} \to \mathbb{R} \,/\, f \text{ is real valued}\}$ & $\mathbb{F} = \mathbb{R}$

$\rightarrow$ form a vector space under operations

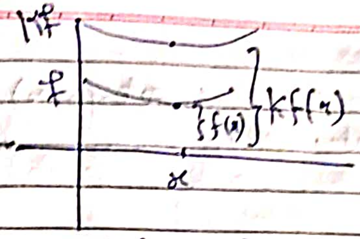
(i) $(f+g)(x) = f(x) + g(x)$ $\forall f, g \in V$

(ii) $(Kf)x = K \cdot f(x)$, $\forall K \in \mathbb{R}$, $f \in V$.





Q) $V = M_{m \times n}(\mathbb{R})$, $F = \mathbb{R}$ $\rightarrow$ forms vector space.



$\rightarrow V(\mathbb{R})$ forms a vector space.

problem 1)) Let $V = \mathbb{R}^2$ & define addition & scalar multiplication as follows

If $u, v \in \mathbb{R}^2$ & $K \in \mathbb{R}$

(u_1, u_2) (v_1, v_2)

$u + v = (u_1 + v_1, u_2 + v_2)$

$K.u = (K.u_1, 0)$ under this binary operation, It is vector space or not?

$\Longleftrightarrow$ No, it is not vector space.

Since, $1.u = (1.u_1, 0) \neq (u_1, u_2)$

problem 2)) V = set of all +ve Real numbers ($\mathbb{R}^+$)

$\mathbb{F} = \mathbb{R}$ let $u, v \in V$, $K \in \mathbb{R}$

Unusual vector space

we defined operation V as follows

$u + v = u \cdot v$ & (usual Multiplication)

$K.u = u^K$ (usual exponential) $\forall K \in \mathbb{F}$, $u, v \in V$

$i^2 = 2x$

Is V is vector space?

$\Rightarrow$ $u + 0 = u \cdot 0 = 0$

$K(u+v) = Ku + Kv$ (Accordance with '+' we define).

LHS $= K(u+v)$

$= (u+v)K$

$= (uv)^K = u^K \cdot v^K = (Ku)(Kv) = Ku + Kv$

$= ku + kv$

$= RHS.$

$\mathbb{R}^+ = (0, \infty)$

Zero element is one

$0' = 1$

Trivalenteal $\rightarrow N \subseteq N$

$\phi \subseteq N$

Theorem $\Rightarrow$ Let V be the vector space over a Field $\mathbb{F}$ & $k \in \mathbb{F}$ be any scalar then

(1) $0 . u = 0$

(2) $k . 0 = 0$

(3) $(-1) u = -u$

(4) If $k . u = 0$ then $k = 0$ or $u = 0$

(1) To show that

$0 . u = 0$, $0 \in \mathbb{F}$

wrong

Since $0 + 0 = 0$

$\Rightarrow (0 + 0) . u = 0$

$\rightarrow 0u + 0u = 0$ (By ... 8 property) ... $(k+m) u = ku + mu$

Adding both side $-0.u$

$\Rightarrow 0.u + 0u + (-0.u) = 0 + (-0.u)$

$\Rightarrow 0.u + (0.u + (-0.u)) = -0.u$

$\Rightarrow 0.u + 0 = -0.u$ (By pro. 4 $\rightarrow u + (-u) = (-u) + u = 0$

LHS $= 0.u = (0+0)u$

$= (0+0)u = 0.u$

$= 0.u + 0.u = 0.u$ ..... (By 8)

$= 0.u + (0.u + (-0.u)) = 0.u (+(-0.u))$ ... (By 4)

$= 0.u + 0 = 0$ ..... (By 4)

$= 0.u = 0$ ...... (By 3)

Hence, prove

(2) To show that

$k . 0 = 0$, $0 \in \mathbb{F}$

LHS $= 0\cdot k = (0+0)k$

$= (0+0)k = 0\cdot k$

$= 0\cdot k + 0\cdot k = 0\cdot k$ ... (By 8)

$= 0\cdot k + (0\cdot k) + (-0\cdot k) = 0\cdot k\,(+(-0\cdot k))$ ... (By 4)

$= 0\cdot k + 0 = 0$ ... (By 4)

$= 0\cdot k = 0$ ... (by 3)

Hence, prove

(3) To prove $(-1)u = -u$

$=$ let $u + (-1)u = 1u + (-1)u$ ... (By 8)

$= (1 + (-1))u$ ... (By 3)

$= 0\cdot u$ ..... (By proof (1))

$= 0$

$u + (-1)u = 0$

$\Rightarrow u$ is additive inverse of $(-1)u$

$\Rightarrow (-1)u = -u$

Hence prove.

(4) If $\underbrace{k\cdot u = 0}_{p}$ then $\underbrace{k = 0 \text{ or } u = 0}_{q}$ $\qquad p \to q$.

let, $k\cdot u = 0$ —— (*)

also, suppose $k \neq 0 \in \mathbb{F}$

$\Rightarrow \exists\ k^{-1} \in \mathbb{F}$

$\Rightarrow \underbrace{k^{-1}\cdot k}u = k^{-1}\cdot 0$ ... (by proof(2))

$\Rightarrow 1\cdot u = 0$

$\Rightarrow u = 0$

$\therefore$ We prove that $k \neq 0$, then $u = 0$

If $k = 0 \Rightarrow 0\cdot u = 0$

Hence, proved.

~~Q~~ $\mathbb{Z}_6 = \{0, 1, \dots 5\}$, $\mathbb{Z}_4$ also not a field.

$2, 3 \in \mathbb{Z}_6$

$\boxed{2 \times_6 3 = 0}$

It is not a field, it is special type of Ring

Problem (T/F)

→ 1) Is there is a vector space with exactly two distinct elements. ?

→ True, coz $\mathbb{Z}_2(\mathbb{Z}_2)$ form a vector space. $= \{0,1\}$

$\mathbb{R}(\mathbb{R})$, $\mathbb{R}(\mathbb{Q})$, $\mathbb{C}(\mathbb{R})$ is also a vector space

★ Subspace (W)

Defn ⟹ (1) let, W be the subset of a vector space V is called a subspace of V if W is itself forms a vector space over the field $\mathbb{F}$ under the same addition & scalar multiplication defined on V.

$\{\phi \neq W \subseteq V(\mathbb{F})$ vector space$\}$

→ let W be the non empty ~~subspace~~ subset of a vector V then W be the subspace of V if only following condition are hold

a) If $u, v \in W$ then $u + v \in W$

b) If $k \in \mathbb{F}$ (any scalar) $\forall\, u \in W$ then $k\cdot w = W$

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LHS $= 0 \cdot k = (0+0)k$

$(0+0)k = 0 \cdot k$

$= 0 \cdot k + 0 \cdot k = 0 \cdot k$ ... (By 8)

$= 0 \cdot k + (0 \cdot k) + (-0 \cdot k)) = 0 \cdot k\ (+(-0 \cdot k))$ ... (By 4)

$= 0 \cdot k + 0 = 0$ ... (By 4)

$= 0 \cdot k = 0$ ... (by 3)

Hence, prove

(3) To prove $(-1)u = -u$

$=$ let $u + (-1)u = 1u + (-1)u$ ... (By 8)

$= (1 + (-1))u$ ... (By 3)

$= 0 \cdot u$ ..... (By proof (1))

$= 0$

$u + (-1)u = 0$

$\Rightarrow u$ is additive inverse of $(-1)u$

$\Rightarrow (-1)u = -u$

Hence prove.

(4) If $\underbrace{k \cdot u = 0}_{p}$ then $\underbrace{k = 0 \text{ or } u = 0}_{q}$ $\quad p \rightarrow q$.

let, $k \cdot u = 0$ —— (*)

also, suppose $k \neq 0 \in F$

$\Rightarrow \exists\ k^{-1} \in F$

$\Rightarrow \underbrace{k^{-1} \cdot k}u = k^{-1} \cdot 0$ ... (by proof (2))

$\Rightarrow 1 \cdot u = 0$

$\Rightarrow u = 0$

$\therefore$ We prove that $k \neq 0$, then $u = 0$

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If $k = 0 \Rightarrow 0 \cdot u = 0$

Hence, proved.

Q. $\mathbb{Z}_6 = \{0, 1, \dots 5\}$, $\mathbb{Z}_4$ also not a field.

$2, 3 \in \mathbb{Z}_6$

$\boxed{2 \times_6 3 = 0}$

It is not a field, it is special type of Ring

Problem (T/F)

→ 1) Is there is a vector space with exactly ~~two~~ distinct elements. ?

→ True, as $\mathbb{Z}_2\ (\mathbb{Z}_2) = \{0,1\}$ form a vector space.

$\mathbb{R}(\mathbb{R})$, $\mathbb{R}(\mathbb{Q})$, $\mathbb{C}(\mathbb{R})$ is also a vector space

★ Subspace (W)

Defn (1) let, W be the subset of a vector space V is called a subspace of V if W is itself forms a vector space over the field F under the same addition & scalar multiplication defined on V.

$\{\phi \neq W \subseteq V(F)$ vector space$\}$

→ let W be the non empty ~~subspace~~ subset ~~of a vector V~~ then W be the subspace of V if only following condition are hold

a) If $u, v \in W$ then $u + v \in W$

b) If $k \in F$ (any scalar) $\forall, u \in W$ then $k \cdot W = W$

Suppose W is a subspace of V

proof $\Rightarrow$ Suppose W is a vector space

$\Rightarrow$ By def^n W is a vectorspace obviously (a), (b) will be hold in W

Conversely

~~stp~~ Assume that (a) & (b) are hold in W

i.e $u+v \in W$, $\forall\ u, v \in W$

& $k.u \in W$ $\forall\ k \in F$, $u \in W$

As $k.u \in W$

$\Rightarrow$ If $k=0 \in F$

$\Rightarrow 0.u = 0 \in W$

$\Rightarrow$ let $k=-1 \in F$ — field

$\Rightarrow k.u \Rightarrow (-1)u = -u \in W$

$\Rightarrow$ W is a Vector space $\Rightarrow W \subseteq V \Rightarrow$

W is subspace of V

Prop. of Vector space

1) $u+v \in W$
2) $u+(v+w) = (u+v)+w$
3) $u+0 = 0+u = u$
4) $u+(-u) = (-u)+u = 0$
5) $u+v = v+u$
6) $ku \in V$
7) $k(u+v) = ku+kv$
8) $(k+m)u = ku+mu$
9) $k(mu) = (km)u$
10) $1u = u$

except This property are inherited property

i.e (2) (5) (7) (8) (9) (10).

Example:- ① Let, $W = \{0\} \subset V$ over the field F. Is W is a subspace of V

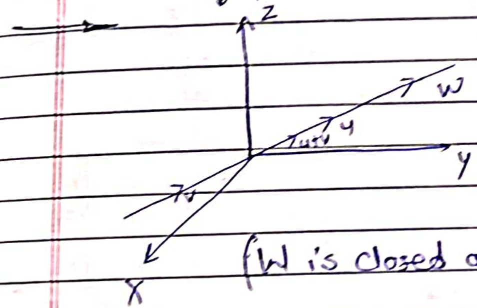
$\rightarrow$ ① $0+0 = 0 \in W$

② $k.0 = 0 \in W$, $\forall\ k \in F$.

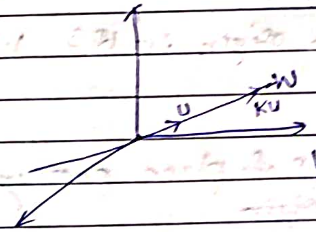
$\Rightarrow$ W is a subspace of V.

{known as Zero subspace}

② Show that If W is a line in $\mathbb{R}^3$ passing through origin then W is subspace of $\mathbb{R}^3$.



(W is closed addition of Vector $u+v \in W$)



W is closed under scalar multiplication

$ku \in W$.

line $\mathbb{R}^3$ forms a subspace.

eq^n of line passing through origin

$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t$ $\{(x_0, y_0, z_0) = (0,0,0)$

$\Rightarrow x = at,\ y = bt,\ z = ct$

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HW $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{array}{c} x = ab,\ y = bt,\ z = ct \\ a, b, c \in \mathbb{R} \end{array} \right\} \subseteq \mathbb{R}^3$

$\rightarrow$

Ex-2 W is a plane passing through origin in $\mathbb{R}^3$. W form a subspace of $\mathbb{R}^3$.

$\longrightarrow$ $ax + by + cz = 0$ - eqn of plane passing through origin

HW $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| ax + by + cz = 0 \right\} \subseteq \mathbb{R}^3$

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Ex-4 Let W is subset of $\mathbb{R}^2$ defined by $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \middle| x, y \geq 0 \right\}$ Is W is subspace of $V = \mathbb{R}^2$?

$\Rightarrow$ pass 1) If $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in W$

$\Rightarrow u + v = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \in W$

2) Let $k \in \mathbb{R}$ & $u = \begin{pmatrix} x \\ 1 \end{pmatrix} \in W$

$k \cdot u \notin W$ take $(k = -1)$

$\Rightarrow \therefore W \underset{\text{subspace}}{\not\subseteq} \mathbb{R}^2$.

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(5) Let W be the set of All symmetric matrices. Show that W is subspace of $M_{n\times n}$ = set of all $n\times n$ matrices.

$\Rightarrow$ Given $W = \{ A \in M_{n\times n} \mid A = A^T \}$

(1) Let, $A, B \in W$

$\Rightarrow A = A^T \;\&\; B = B^T$

$\Rightarrow (A+B)^T = A^T + B^T$

$= (A+B)$

$\Rightarrow A+B$ is a symmetric matrix

satisfied prop(1) $\Rightarrow A+B \in W$

(2) let, K be the any scalar $\in$ F & $A \in W$

$\Rightarrow A = A^T$

$\Rightarrow (K \cdot A^T) = K \cdot A^T$

$= K \cdot A$

$\Rightarrow K \cdot A$ is a symmetric matrix

$\Rightarrow K \cdot A \in W$

From (1) & (2)

$\Rightarrow W \underset{\text{subspace}}{\subset} M_{n\times n}$

Subspace

$V = M_{n\times n}$ = set of all $n\times n$ matrices = Vector space

W = set of skew symmetric matrices

V = set of all LTM - Lower triangular mat.

V = set of all UTM - Upper triangular mat.

W_1 = Set of all matrices with zero determinant

Ex :- $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\xrightarrow{A+B} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\det|A+B| \Rightarrow 1$.

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W_2 = set of all matrices trace(A) = 0

$tr(A+B) = tr(A) + tr(B)$ $= tr(A+B) = tr(0) + tr(0)$

$tr(A+B) = 0$

$tr(K \cdot A) = K\,tr(A)$ $= tr(K \cdot A) = K\,tr(0)$

$tr(K \cdot A) = K \cdot 0 = 0$

it is subspace.

$\Rightarrow \boxed{W_2, V, V, W \subset M_{n\times n}}$

$\Rightarrow \boxed{W_1 \underset{\text{subspace}}{\not\subset} M_{n\times n}}$

(6) Let C be the set of All continuous real value function defined on R, ($C = \{f : R \to R \mid f$ is cont ...) Let D be the set of All differentiable functions defined on R. Show that, C, D are subspace of vector space F, real value function

($D = \{f : R \to R \mid f$ is differentiable function$\}$)

$C = \{ f : R \to R \mid f \text{ is continuous} \}$

$D = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable function} \}$

$F = \{ f : \mathbb{R} \to \mathbb{R} \}$

To prove, $C \underset{\text{subspace}}{\subseteq} F$

(1) Let $f_1, f_2 \in C$

~~$\Rightarrow f_1 + f_2$~~

$\Rightarrow f_1 \;\&\; f_2$ are continuous

$\Rightarrow$ By Algebra of functions

$\Rightarrow f_1 + f_2$ is also continuous.

$\Rightarrow f_1 + f_2 \in C$

also

2) $K.f \in C$ (obvious) $\left\{ \begin{array}{l} K = \text{any continuous func}^n \\ f = \text{continuous func}^n \end{array} \right.$

Hence prove C is ~~prop~~ subset of F

To prove → $D \subseteq C \subseteq F$

W = set of all not diff. func^n

$f_1 = f$ - is not diff.

$f_2 = -f$

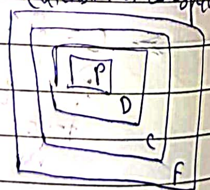
$\Rightarrow f_1 + f_2 \notin W$

So, set of all diff. func^n $\underline{\subset}$ W

P = set of all polynomials { Always continuous

$\underset{\text{subspace}}{P \subseteq C} \subseteq \underset{\text{subspace.}}{D \subseteq F}$

extension of subspace.



7) Let S be the set of all solutions of differential equation ~~b~~ $y'' + y = 0$, then show that S is subspace of F ~~set real~~ = set of all real valued function.

→ $S = \{ f \in F \mid f \text{ is sol}^n \text{ of } y'' + y = 0 \}$

let, $f_1, f_2 \in S$

$\Rightarrow f_1$ and f_2 are sol^n of $y'' + y = 0$

$\Rightarrow f_1'' + f_1 = 0$ ① & $f_2'' + f_2 = 0$ ②

$\Rightarrow$ Add^n of ① & ②

$\Rightarrow (f_1'' + f_2'') + (f_1 + f_2) = 0$

$\Rightarrow (f_1 + f_2)'' + (f_1 + f_2) = 0$

$\Rightarrow f_1 + f_2$ is a sol^n of $y'' + y = 0$

✓(i) $\Rightarrow f_1 + f_2 \in S$

ii) $K.f \in S$

$y'' + y = 0 \qquad t^2 + 1 = 0$

$e^{ix} = \cos x + i\sin x$

$y = C_1 e^{ix} + C_2 e^{ix}$

$y = C_1 \cos x + C_2 \sin x$ ... (linear combination)

$\boxed{C_1, C_2 \in \mathbb{C}}$

$S = \{ C_1 \cos x + C_2 \sin x \mid C_1, C_2 \in \mathbb{C} \}$

Linear Combination :-

HW Show that following are subspace of given vector space

a) $W = \left\{ \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \middle| \; x = y, z = -t \right\}$, $V = R^4$

b) $W = \{ a + bx - bx^2 + ax^3 / a, b \in \mathbb{R} \}$, $V = P_3(\mathbb{R})$

collection of all 3° polynomials

c) $W = \left\{ \begin{bmatrix} a & b \\ -a & a \end{bmatrix} \middle| \; a, b \in \mathbb{R} \right\}$ $V = M_{2\times2}$

d) $W = \left\{ \begin{bmatrix} a & a+1 \\ 0 & b \end{bmatrix} \middle| \; a, b \in \mathbb{R} \right\}$, $V = M_{2\times2}$

* Linear Combination & Co-ordinate Vector :-

- Let $v_1, v_2, \ldots, v_k$ be the vectors in V { V is and vector space over field F } then we say $u \in V$ is in linear combination of $v_1, v_2, \ldots, v_k$ if there are ~~constants~~ scalars $c_1, c_2, \ldots, c_k \in F$ such that $\boxed{u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k}$

$W = \{1, x, x^2\}$

$\text{span}(W) = \{ a_0 1 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$

$= P_2(\mathbb{R})$

$W = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$\text{span}(W) = \left\{ \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$

$= \mathbb{R}^2$

→ and the coordinate vector of u with respect to vectors $v_1, v_2, \ldots, v_k$ is given by $[u] = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$

Ex:- ① if $u = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $v_1 = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}$

$v_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad V \in \mathbb{R}^3(\mathbb{R})$

Find coordinate vector of u w.r.t. v_1, v_2, v_3

→ $u = c_1 v_1 + c_2 v_2 + c_3 v_3$

$\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ -3 & 2 & 0 & 3 \\ 1 & -1 & 1 & -2 \end{array}\right]$

$R_1 \leftrightarrow R_3$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ -3 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right]$

$R_2 \to R_2 + 3R_1$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -1 & 3 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right]$

$R_3 \to R_3 + R_2$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -1 & 3 & -3 \\ 0 & 0 & -2 & 1 \end{array}\right]$

$c_1 - c_2 + c_3 = -2$

$-c_2 + 3c_3 = -3$

$-2c_3 = 1$

$\boxed{\text{so } c_3 = -1/2}$ $\therefore \boxed{c_2 = 3/2}$ $\therefore \boxed{c_1 = 0}$

$c_1 = 0,\ c_2 = \frac{3}{2},\ c_3 = \frac{-1}{2}$. and the co-ordinate vector is $[u] = \begin{bmatrix} 0 \\ 3/2 \\ -1/2 \end{bmatrix}$

Ex:- $V = M_{2\times 2}(\mathbb{R})$

$$U = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$\longrightarrow U = C_1 A + C_2 B + C_3 C + C_4 D.$

Ex:- $P(x) = 1+x^2$

$P_1(x) = 1+x$

$P_2(x) = 1-x^2$

$\longrightarrow P(x) = C_1 P_1(x) + C_2 P_2(x)$

$1+x^2 = C_1(1+x) + C_2(1-x^2)$

$1+x^2 = (C_1+C_2) + C_1 x - C_2 x^2$

$C_2 = -1$ | $\boxed{C_1 = 0}$

$C_1 + C_2 = 1$

$\boxed{C_1 = 2}$

getting two diff. value for C_1.

⊛ Spanning set of a vector space V.

span (S) $= \mathbb{R}^n$

span(S) = V

Defn: If $S = \{V_1, V_2, \dots V_k\}$ is the set of vectors in a vector space V, then the set of all linear combinations of $V_1, V_2 \dots V_k$ is called span of $V_1, V_2, \dots V_k$ & it is denoted by span ~~of~~ $(V_1, V_2 \dots V_k)$ or $\langle V_1, V_2, \dots V_k \rangle$ or generative set/span Span(S) & if span (S) = V then S is called as spanning set of V.

$\text{span}(S) = \{ C_1 V_1 + C_2 V_2 \dots C_k V_k \,/\, C_1, C_2 \dots C_k \in \mathbb{F} \}$

Q) $V = \mathbb{R}(\mathbb{R})$ what is S such that span(S) = $\mathbb{R}$?

$\longrightarrow$ span(S) = $\{c_1 = 1 \mid c_1 \in \mathbb{R}\} = \mathbb{R}$

In general the spanning set of $\mathbb{R}^n$ is

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

Q) Problem :- Show that the polynomial $1, x, x^2$ spans / generates ~~of~~ $P_2(\mathbb{R})$

$\longrightarrow$ $\text{span}(1, x, x^2) = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$

$= P_2(\mathbb{R})$

.... By defn

$\Rightarrow$ Every two degree polynomial / (of degree ≤ 2) can write as a linear combination of $1, x, x^2$

$\Rightarrow$ In general the spanning set of $P_n(\mathbb{R})$ is $(1, x, x^2, \dots x^n)$

Q) $V = M_{2\times3}$ what is spanning set of V

$$\Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cdots + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{ij} = \begin{cases} 1 & \text{if } (i,j)^{th} \text{ entry} \\ 0 & \text{if otherwise} \end{cases}$$

Q) In P_2, determin weather $r(x) = 1 - 4x + 6x^2$ is in the span ~~of~~ $(p(x), q(x))$ where $p(x) = 1 - x + x^2$, $q(x) = 2 + x - 3x^2$.

$\longrightarrow$ Given, $r(x) = 1 - 4x + 6x^2$

$p(x) = 1 - x + x^2$

$q(x) = 2 + x - 3x^2$

To show $\Rightarrow$ $r(x)$ is span $(p(x), q(x))$

Soln $\Rightarrow$ By defn

$$\text{span}(p(x), q(x)) = \{c_1(1 - x + x^2) + c_2(2 + x - 3x^2) \mid c_1, c_2 \in \mathbb{R}\}$$

$\Rightarrow$ let c_1, c_2 such that

$$r(x) = c_1 p(x) + c_2 q(x)$$

$$1 - 4x + 6x^2 = c_1(1 - x + x^2) + c_2(2 + x - 3x^2)$$

$$1 - 4x + 6x^2 = (c_1 + 2c_2) + x(-c_1 + c_2) + x^2(c_1 - 3c_2)$$

$c_1 + 2c_2 = 1$ — (1)

$-c_1 + c_2 = -4$ — (2)

$c_1 - 3c_2 = 6$ — (3)

$$\begin{aligned} -\cancel{c_1} + c_2 &= -4 \\ + \cancel{c_1} - 3c_2 &= 6 \\ \hline -2c_2 &= 2 \end{aligned}$$

$\therefore c_2 = -1$ put in eqn (2)

$c_1 = 3$.

Clearly, $c_1 = 3$, $c_2 = -1$

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Q) In F, is $\sin(2x)$ is in span of $\sin x, \cos x$

$\longrightarrow$ $\sin(2x) = C_1(\sin x) + C_2(\cos x)$

- let $x = 0$

$0 = C_1(0) + C_2(1)$

$C_1 = 0$, $\boxed{C_2 = 0}$

let $x = \pi/2$

$\sin(2\pi/2) = C_1(\sin \pi/2) + C_2(\cos \pi/2)$

$0 = C_1(1) + C_2(0)$

$\boxed{C_1 = 0}$

C_2 - Not have unique value

linear combination does not form.

$\therefore$ The $\sin(2x) \notin \text{span}(\sin x, \cos x)$

HW Q) In M_{22}, Find the span of A

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\rightarrow$ $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} C_1 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} C_2 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} C_3 =$

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$C_1 = C_2 = C_3 = 0$.

L.I.D.

Q. Find the span of $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}_3$

$\longrightarrow$ span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\alpha - \beta = x$
$2\alpha = y$
$3\alpha + \beta = z$

$\left[\begin{array}{cc|c} 1 & -1 & x \\ 2 & 0 & y \\ 3 & 1 & z \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 2 & y-2x \\ 0 & 4 & z-3x \end{array}\right]$

$\begin{matrix} R_2 \to R_2/2 \\ R_3 - R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & (y-2x)/2 \\ 0 & 0 & \frac{(z-3x)}{4} - \left(\frac{y-2x}{2}\right) \end{array}\right]$

$\frac{z-3x}{4} - \frac{y-2x}{2} = 0$

$z - 3x - 2y + 4x = 0$

$x - 2y + z = 0$

Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Big/ x - 2y + z = 0$

★ linear dependance & independance.

let, $S = \{v_1, v_2, \dots v_k\}$ be the set of vectors in a vector space V. We say $v_1, v_2, v_3 \dots v_k$ are linearly dependent if atleast one scalar is non-zero such that $c_1v_1 + c_2v_2 + c_3v_3 + \dots + c_kv_k = 0$ where c_i are the scalars & otherwise if $c_1 = c_2 = c_3 = c_k = 0$ i.e all scalars are zero such that $c_1v_1 + c_2v_2 + \dots c_kv_k = 0$ where c_i's are scalars then $v_1, v_2, \dots v_k$ are linearly independent.

Q/ $S = \{0\} \subseteq V$

$\Rightarrow$ It is linearly dependent

$u = 0$

$k = 100$

$k \cdot u = 0$ $\quad \left\{ \begin{array}{l} ku = 0 \\ \text{if } u = 0 \end{array} \right.$

$100 \cdot u = 0$

k = constant exist $\quad \therefore$ L.D.

Q/ $S = \{u_1, u_2, 0\} \subseteq \mathbb{R}^2$

$u_1, u_2 \neq 0$

$\Rightarrow \alpha u_1 + \beta u_2 + 100 \cdot 0 = 0$

$\underset{0}{\alpha}, \underset{0}{\beta}, \underset{0}{100 \ne}$

$\{\alpha u_1 + \beta u_2 \neq 0$

$\therefore \Rightarrow$ L.D.

Q/ $S = \{u, v, \alpha u + \beta v\} \subseteq \mathbb{R}^3$

$\rightarrow$ L.D.

Q/ $S = \{1 + x, x + x^2, 1 + x^2\} \subseteq P_2(\mathbb{R})$

$\rightarrow -\gamma(x + x^2) = \alpha(1 + x) + \beta(1 + x^2)$

$-\gamma(x + x^2) = \alpha + \alpha x + \beta + \beta x^2$

$-\gamma(x + x^2) = \beta x^2 + \alpha x + \alpha + \beta$

let α, β, γ such that

$\alpha(1 + x) + \beta(x + x^2) + \gamma(1 + x^2) = 0$

$\alpha + \alpha x + \beta x + \beta x^2 + \gamma + \gamma x^2 = 0$

$\alpha(\alpha + \beta)x + (\beta + \gamma)x^2 \Rightarrow 0 \cdot 1 + 0x + 0x^2$

$(\alpha + \gamma) = 0$

$(\alpha + \beta) = 0$

$(\beta + \gamma) = 0$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$

$R_3/2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $\quad \begin{array}{ll} \alpha + \gamma = 0 & \Rightarrow \alpha = 0 \\ \beta - \gamma = 0 & \Rightarrow \beta = 0 \\ \gamma = 0 & \end{array}$

$\therefore \alpha = \beta = \gamma = 0$

HW Q/ $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3 \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \beta$

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$$\left[\begin{array}{cc|c} 1 & 2 & x \\ -1 & 1 & y \\ 1 & 1 & z \end{array}\right]$$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 - R_1$

$$\left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 3 & y+x \\ 0 & -1 & z-x \end{array}\right] \xrightarrow{R_2 \to R_2/3} \left[\begin{array}{ccc} 1 & 2 & x \\ 0 & 1 & \frac{y+x}{3} \\ 0 & -1 & z-x \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 1 & \frac{y+x}{3} \\ 0 & 0 & z-x+\frac{y+x}{3} \end{array}\right]$$

$\cancel{z}$
$z - x + \frac{y+x}{3} = 0$

$-\frac{2x}{3} + \frac{y}{3} + z = 0$

$-2x + y + 3z = 0$

$$\text{Span}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}\right\} = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \,\middle|\, -2x+y+3z=0\right\}$$

Rank < order of matrix
∴ L. Dependent.

L.I.D – All scalar = 0
L.D – At least 1 scalar is ≠ 0.

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Q. $S = \phi \subset V$ → vector space
→ empty set in V
always Linearly Independent

Q// $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$ $C = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$ are in $M_{2\times 2}$

Are A, B, C are L.I or L.D.

→ $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} c_1 + \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} c_2 + \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{cccccc|cc} 1 & -1 & 2 & -1 & -1 & -2 & 0 & 0 \\ 2 & 0 & 2 & 2 & 0 & -2 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{cccccc|cc} 1 & -1 & 2 & -1 & -1 & -2 & 0 & 0 \\ 0 & 2 & -2 & 4 & 2 & +2 & 0 & 0 \end{array}\right]$$

~~A – B = C~~
~~C = (1)A + (-1)B~~
~~(1) A + (-1)B + (-1)C = 0~~

$c_1 = 0$
$c_2 = 0$

∴ L. Independent

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Q. $S = \{\overset{P_1}{1-x+x^2}, \overset{P_2}{-1+x-x^2}, \overset{P_3}{3-x^2}\}$

In P_2, S in L.I or L.D

$\alpha(1-x+x^2) + \beta(-1+x-x^2) + \gamma(3-x^2) = 0$

$[\alpha - \alpha x + \alpha x^2] + [-\beta + \beta x - \beta x^2] + [3\gamma - \gamma x^2] = 0$

$0.1 + 0.x + 0.x^2$

$P_1 = -P_2 + 0\, P_3$

$(1) P_1 + (1) P_2 + 0\, P_3 = 0$

$\Rightarrow$ Linear Dependent

$\alpha - \beta + 3\gamma = 0$ (1) $\quad -\alpha + \beta + 0\gamma = 0$ (2) $\quad \alpha - \beta - \gamma = 0$ (3)

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

$R_2 \xrightarrow{} R_2/3$

$R_3 \to R_3/4$

$R_3 \to R_3 - R_2$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\alpha - \beta + 3\gamma = 0 \qquad , \qquad \alpha - \beta + 3\gamma = 0$

$\gamma = 0 \qquad\qquad \alpha = \beta$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ \gamma \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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$\Rightarrow$ So, above vectors in P_2 are L.D.

HW Q. $S = \{\sin x, \cos x, \sin(2x)\} \subseteq F$

$\to$ Is S is L.I or L.D in $F = \{$set all of $f: \mathbb{R} \to \mathbb{R}\}$

$\to (\sin x)\alpha + (\cos x)\beta + (\sin(2x))\gamma = 0$

$\Rightarrow$ L.I.

$\sin(2x) = C_1(\sin x) + C_2(\cos x)$

let $x = 0$

$C_1 = 0, \; C_2 = 0$

let $x = \pi/2$, $\sin(2\pi/2) = C_1(\sin \pi/2) + C_2(\cos \pi/2)$

$\boxed{C_1 = 0}$

C_2 - Not have unique value

$\therefore$ It is L.I.

|| - Absolute valued func^n (Mod)

Q. $S = \{x, |x|\} \subseteq F$

HW So, S is L.I or L.D.

$\longrightarrow$

$[n \cdot 1] - \mathbb{R}^n \to$ L.D.

✱ Basis for a Vector Space :-

Let, B be any non-empty set in ~~which~~ V we say B is a Basis for V if it satisfy following conditions

a) Span (B) = V

b) B is Linearly Independent

✱ Dimension of a Vector Spaces :-

If B is Basis for V then dimension of Vector space is no. of element in B.

Dimension = |B|

Ex :- ① is $B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ a Basis for $\mathbb{R}^2$?

→ $\begin{bmatrix} 1 \\ -1 \end{bmatrix} c_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} c_2 = 0$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 + R_1$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

Yes, since, Span (B) = $\mathbb{R}^2$

& B is L.I.

② $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right\}$

→ No, it is L.D.

Note ⇒ If B is Basis for Vectorspace V, then it is Minimal Generating set & Maximal linearly Independent set.

HW Let $S = \{v_1, v_2 \ldots v_k\} \subset V$, V is a vector space over field $\mathbb{F}$ then show that

ⓐ Span (S) is subspace of V

ⓑ Span (S) is a smallest subspace of V containing $v_1, v_2 \ldots v_k$

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* Basis & Dimension

$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| x_1, \ldots, x_n \in \mathbb{R} \right\}$

$= \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \middle/ x_1, x_2, x_n \in \mathbb{R} \right\}$

$= \{ x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \mid x_1, \ldots x_n \in \mathbb{R} \}$

$= \text{Span}\{e_1, e_2, \ldots e_n\}$

$\mathbb{R}^n = \langle e_1, e_2, \ldots e_n \rangle$

Also, $\{e_1, e_2, \ldots e_n\}$ are Linearly independent

$\Rightarrow B = \{e_1, e_2, \ldots, e_n\}$ is a standard basis of $\mathbb{R}^n$.

$\Rightarrow$ Dimension of $\mathbb{R}^n \Rightarrow \dim(\mathbb{R}^n) = n$

Ex, Basis for $\mathbb{R}^2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\rightarrow \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\rightarrow$ Linearly Independent

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* Ordered Basis $\Rightarrow$

Basis other than standard basis is called ordered Basis.

Ex:- (1) $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is Basis for $\mathbb{R}^2$.

(2) $B = \left\{ \begin{bmatrix} \pi \\ e \end{bmatrix}, \begin{bmatrix} e \\ \pi \end{bmatrix} \right\}$ is a Basis for $\mathbb{R}^2$

- L.I.

(3) $P_n(\mathbb{R}) = \{ a_0 1 + a_1 x + \cdots + a_n x^n \mid a_0, a_1 \ldots a_n \in \mathbb{R} \}$

$\Rightarrow$ span $\{1, x, x^2, x^3 \ldots x^n\}$

$P_1(\mathbb{R}) = \{ a_0 1 + a_1 x \mid a_0, a_1 \in \mathbb{R} \}$

$= \{1, x\}$ — Is L.I, L.D.

$\alpha \cdot 1 + \beta x = 0$

$0 + \beta = 0$

$y = c_1 \phi_1(x) + c_2 \phi_2(x)$

$W(\phi_1 \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$

det $\rightarrow \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} \neq 0$

If Det. $= 0$ – L.D , Det $\neq 0$

$\therefore$ It is L.I.

$\Rightarrow P_n(\mathbb{R}) = \text{span}\{1, x, x^2, x^3 \ldots x^n\}$

also clearly $B = \{1, x, x^2, \ldots x^n\}$ is L.I. set

Since $c_1 \cdot 1 + c_2 x + \dots + c_n x^n = 0$

if $[c_1 = c_2 = \dots = c_n = 0]$

$\Rightarrow \dim(P_n(\mathbb{R})) = n+1$

Ex:- What is the standard basis for $P_4(\mathbb{R})$

$B = \{1, x, x^2, x^3, x^4\}$.

$\rightarrow \operatorname{Dim}(B) = 5$.

Ex:- $M_{n\times n}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \middle| \; a_{11} \dots a_{nn} \in \mathbb{R} \right\}$

set of matrices

vector space of square Matrices

$= \{ a_{11} E_{11} + a_{12} E_{12} + \dots + a_{nn} E_{nn} \mid a_{11} \dots a_{nn} \in \mathbb{R} \}$

where $E_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = i, \beta = j \\ 0 & \text{otherwise} \end{cases} \quad \forall \, i, j = 1, 2, \dots, n$

$M_{4\times 4}$ $E_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rightarrow \operatorname{Dim}(M_{n\times n}(\mathbb{R})) = n^2$

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{3} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

~~also~~ $= \operatorname{span}\{E_{11}, E_{12}, \dots, E_{nn}\}$

also $B = \{E_{11}, E_{12}, \dots, E_{nn}\}$ is linearly Independent.

$\Rightarrow B$ is standard Basis for $M_{n\times n}(\mathbb{R})$

Ex:- Basis of $M_{m\times n}(\mathbb{R})$

$\rightarrow \dim(M_{m\times n}(\mathbb{R})) = m \cdot n$.

Similarly

$\Rightarrow$ Basis of $M_{m\times n}(\mathbb{R}) = \{E_{11}, \dots, E_{mn}\}$

Ex.- What is dimension of vector space $\mathbb{C}(\mathbb{R})$?

complex No., Real no.

$\longrightarrow \mathbb{C} = \{x \cdot 1 + iy \mid x, y \in \mathbb{R}\}$

$= \operatorname{Span}\{1, i\}$

$\Rightarrow x + iy = 0, \; x, y \in \mathbb{R}$

$\Rightarrow x = y = 0$

also, $\{1, i\}$ is linearly independent in $\mathbb{C}$

$\Rightarrow \boxed{\dim(\mathbb{C}(\mathbb{R})) = 2}$

Q:- What is dimension of vector space $\mathbb{C}(\mathbb{C})$?

→ $\mathbb{C} = \{ z \mid z \text{ is of the form } x+iy \}$
$x, y \in \mathbb{R}$

$\Rightarrow x + iy = 0$
$x = 1,\ i = y$

$\Rightarrow \{1, i\}$ are L.D in $\mathbb{C}(\mathbb{C})$

$\Rightarrow \{1\}$ is a Basis for $\mathbb{C}(\mathbb{C})$, standard basis.

$\Rightarrow \dim \mathbb{C}(\mathbb{C}) = 1$

Q/ $\dim(\mathbb{C}^2(\mathbb{R}))$

$\Rightarrow \dim(\mathbb{C}(\mathbb{R})) = 2$

$\Rightarrow \dim(\mathbb{C}^2(\mathbb{R})) = 4 \rightarrow 2 \times 2 = 4$

$\Rightarrow$ since, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix} \right\}$ is the

standard basis for $\mathbb{C}^2(\mathbb{R})$

Q Similarly In general

$$\Rightarrow \boxed{\dim(\mathbb{C}^n(\mathbb{R})) = 2.n}$$

* $\dim(M_{nn}(\mathbb{C})(\mathbb{R})) = 2n^2$

* $\dim(M_{n\times m}(\mathbb{C})(\mathbb{R})) = 2m\times n$

Q Find the dimension of the vector space ~~$\mathbb{R}(\mathbb{Q})$~~ & $\mathbb{R}(\mathbb{Q}) = ?$

→ $B = \{1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{p}, \ldots\}$
$\alpha_1(1) + \alpha_2(\sqrt{2}) + \cdots + \alpha_n\sqrt{p} + \cdots = 0$

$\operatorname{Dim} \mathbb{R}(\mathbb{Q}) = \infty$

H.W Find the Dimension of following subspaces:

1) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{matrix} x - y = 0 \\ y - z = 0 \end{matrix} \right\}$ $V = \mathbb{R}^3$

2) $W = \{ A \in M_{n\times n}(\mathbb{R}) \mid A^T = A \}$ $V = M_{n\times n}(\mathbb{R})$

3) $W = \{ p(x) \in P_3(\mathbb{R}) \mid p(3) = 0 \}$; $V = P_3(\mathbb{R})$

4) $W = \{ A \in M_{nn}(\mathbb{R}) \mid tr(A) = 0 \}$, $V = M_{nn}(\mathbb{R})$

→ (2) $W = \{ A \in M_{n\times n}(\mathbb{R}) \mid A^T = A \}$

$\Rightarrow n = 2$ $\quad \Rightarrow V = M_{n\times n}(\mathbb{R})$
$\dim(V) = n^2$

symmetric matrix → $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

no. of L. I choices are $= 3$
$= \frac{2(2+1)}{2}$

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put $n=3$

Symmetric Matrix $A = \begin{bmatrix} a & b & c \\ b & \alpha & \beta \\ c & \beta & \gamma \end{bmatrix}$ no. of L.I choice entries $= 6$

$= \frac{3(3+1)}{2}$

By induction we can say that
$\dim(W) = \frac{n(n+1)}{2}$

By formula $\rightarrow$ $\dim(W) = \dim(V) - \left(\begin{matrix}\text{no. of L.I} \\ \text{restriction}\end{matrix}\right)$

$= n^2 - \left(\frac{n(n+1)}{2} - n\right)$

$= 2n^2 - \left(\frac{n(n+1)}{2} - n\right)$

$= 2n^2 - n^2 + n / 2$

$= \frac{n^2+n}{2}$

$\boxed{\dim(W) = \frac{n(n+1)}{2}}$

$n=4 = 8 = \frac{4(4+1)}{2}$

$n=5 : 15$ entries

$3, 6, 10, 15, \cdots \frac{n(n+1)}{2}$
$n=2 \quad n=3 \quad n=4$

$\frac{n(n-1)}{2}$

$\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ & a_{22} & a_{2n} \\ & & \vdots \\ & & a_{nn} \end{bmatrix}$

$n + (n-1) + (n-2) + \cdots + 1$
$1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$

$\boxed{\dim(W) = \frac{n(n+1)}{2}}$

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$n=2$

$\begin{bmatrix} -a & \alpha \\ \beta & a \end{bmatrix} = 3 = 2^2 - 1$

$\begin{bmatrix} -a-b & c & d \\ \alpha & a & \beta \\ \gamma & \delta & b \end{bmatrix} = 3^2 - 1$

$\dim(W) = n^2 - 1$

$a_{11} + a_{22} + \cdots + a_{nn} = 0$

digonal entries are $n-1$

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* Subspace, Basis & dimension :-

* $\dim(V) = |B|$
$= $ no. of vectors in B.

ex :- $\mathbb{C}(\mathbb{R})$
$\Rightarrow \{x \cdot 1 + iy \mid x, y \in \mathbb{R}\}$

$\Rightarrow \text{Span}(1, i)$ $\quad B = \{1, i\} = \dim(\mathbb{C}(\mathbb{R})) = 2$
also $\{1, i\}$ is L.I.

Q) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x+y+z=0 \\ x-y-z=0 \end{matrix} \right\}$

$V = \mathbb{R}^3(\mathbb{R})$, $\dim(W) = ?$

$\rightarrow$ $x+y+z=0$
$x-y-z=0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right] \xrightarrow{R_2 = R_2/-2} \underset{\text{REF}}{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]}$$

$R_1 \to R_1 - R_2$

$$\underset{\text{RREF}}{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]}$$

$x = 0$
$y + z = 0$
let $z = t$

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$x = 0$
$y = -t$
$z = t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$$

$$W = \left\{ t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Also, $B = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ is L.I.

$B = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ is the basis for W

$\Rightarrow (\dim(W) = 1)$

General formula

$V = \mathbb{R}^n(\mathbb{R})$
$W \underset{\text{subspace}}{\subset} V$

$$\boxed{\dim(W) = n - \underbrace{(\text{no. of L.I restrictions})}_{\text{eq}^n/\text{cond}^n}}$$

$\Rightarrow \dim(W) = \dim(V) - (\text{no. of L.I restrictions})$

Q// $W = \{ P(x) \in P_3(\mathbb{R}) \mid P(1) = 0 \}$

→ $\{1, x, x^2, \dots x^n\}$

$\dim(P_n(\mathbb{R})) = n+1$

& where $V = P_3(\mathbb{R})$

⟹ $W = \{ P(x) = a + bx + cx^2 + dx^3 \mid P(1) = 0 \}$

also $P(1) = 0$ $\{x = 1\}$

$a + b + c + d = 0$

$a = -b - c - d$

$W = \{ -b - c - d + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R} \}$

$W = \{ b(x-1) + c(x^2-1) + d(x^3-1) \mid b, c, d \in \mathbb{R} \}$

$W = \text{Span}\,((x-1), (x^2-1), (x^3-1) \dots \}$

$\dim(W) = 3$

also set $B = \{(x-1)\ (x^2-1)\ (x^3-1)\}$ is L.I

B is the Basis for W

⟹ $\boxed{\dim(W) = 3}$

Q// $W = \{ f \in F \mid f$ is differentiable$\}$

$V = F$, $\dim(W) = ?$

⟶ $B = \{1, \sin x, \cos x, e^x, p(x), \log(x^2), x, \sin(x), \sin^2 x, \cos(ex), \sin(2x), e^x, \dots\}$

Basis

all differentiable func^n.

⟹ $|B| = \infty$

⟹ $\dim(W) = \infty$.

Q/ Dimension $\mathbb{R}(\mathbb{Q})$

⟶ $\{1, \sqrt{2}, \dots \sqrt{p} \dots\}$

$\dim(\mathbb{R}(\mathbb{Q})) = \infty$

Result :- If $\dim(V) = \dim(V')$ then V & V' are the isomorphic vector spaces.

ex :- $M_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4$ {both dim = 4 same}

$P_n(\mathbb{R}) \cong \mathbb{R}^{n+1}$

$1 + x + x^2 \longleftrightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$P_2 \cong \mathbb{R}^3$ are isomorphic.

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Q. $W = \{ p(x) \in P_2 \mid x p'(x) = p(x) \}$, where $p'(x) = \frac{d}{dx}(p(x))$
find dim (W)

→ $W = \{ p(x) = a + bx + cx^2 \}$

$\Rightarrow p'(x) = a_1 + a_2 x$

$\Rightarrow x p'(x) - p(x) = 0$

$\Rightarrow x(a_1 + 2a_2 x) - a_0 - a_1 x - a_2 x^2 = 0$

$\Rightarrow x a_1 + 2a_2 x^2 - a_0 - a_1 x - a_2 x^2 = 0$

$\Rightarrow -a_0 + a_2 x^2 = 0$

$\Rightarrow$ as, $\{1, x^2\}$ are L.I

$\Rightarrow a_0 = 0 = a_2$

$\Rightarrow p(x) = a_1 x$

$W = \{ p(x) = a_1 x \mid a_1 \text{ is scalar} \}$

$W = \text{Span}\{x\}$

also $B = \{x\}$ is L.I

$\Rightarrow \dim(W) = 1$

HW Q) $W = \{ A \in M_{n \times n}(\mathbb{R}) \mid A^T = -A \}$
find dim (W) ?

Note: dim. digonal matrix = n
dim. Scalar mat. = 1
dim. Upper tri. matrix = $\frac{n(n+1)}{2}$
dim. Lower tri. matrix = —"—

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★ Co-ordinates :- let $B = \{ v_1, v_2, \dots v_n \}$ be a basis for Vector space V ~~for every~~ let $v \in V$ and write $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ then $c_1, c_2, \dots c_n$ are called as co-ordinates of V w.r.t basis B and the ~~vect~~ column vector

$$[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$ is called co-ordinate Vector w.r.t. ~~Bases~~ B.

Q) $p(x) = 1 - 2x + x^2$ find coordinate vector w.r.t the base $B = \{1, x, x^2\}$ standard bases of P_2

→ $1 - 2x + x^2 = 1(1) + (-2)x + 1 \cdot x^2$

$$[p(x)]_B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$ ordered basis.

also consider $B' = \{x^2, x, 1\}$

$$[p(x)]_{B'} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

→ $B' = \{ \underset{v_1}{1+x}, \underset{v_2}{x+x^2}, \underset{v_3}{1+x^2} \}$

find co-ordinate Vector p(x) w.r.t B'

$p(x) = \cancel{1 - 2x + x^2}\ \alpha v_1 + \beta v_2 + \gamma v_3$

$1 - 2x + x^2 = \alpha(1+x) + \beta(x + x^2) + \gamma(1 + x^2)$

$1 - 2x + x^2 = (\alpha + \gamma)1 + (\alpha + \beta)x + (\beta + \gamma)x^2$

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$\alpha+\gamma=1$ — ①
$\alpha+\beta=-2$ — ②
$\beta+\gamma=1$ — ③

$\therefore \alpha=-1$
$\therefore \beta=-1$
$\therefore \gamma=2$

$$[p(x)]_{B'} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Ex:- $p(x)=-1 \in P_n$ w.r.t standard Basis, find Co-ordinate vector w.r.t standard Basis for P_n

$\Rightarrow p(x)=-1$

$-1 = (-1)1 + 0x + 0x^2 + \dots + 0x^n$

$$\Rightarrow [-1]_B = \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n+1}$$

$B = \{-1, x, \dots x^n\}$ is basis for $p(n)(\mathbb{R}) \to$ Yes

$B = \{100, x, \dots x^n\}$, $-1 \longrightarrow$ Yes

$p(x) = -x+1$

For $\longrightarrow$ cordinate $\begin{bmatrix} -100 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

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Imp Theorem:- Let V be the vector space & B be the basis for V for every vector v in V there is exactly one way to write v as the linear combination of basis vectors in B

$\Rightarrow$ Proof $\Rightarrow$ let $B = \{v_1, v_2, \dots v_n\}$ be the basis for a vector space V &
let us assume we can write $v \in V$ in two different way are L.C (linear combination) of basis vector

$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

also $v = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$

$\Rightarrow \alpha_1 v_1 + \dots + \alpha_n v_n = \beta_1 v_1 + \dots + \beta_n v_n$

$\Rightarrow (\alpha_1 - \beta_1) v_1 + \dots + (\alpha_n - \beta_n) v_n = 0$

$\Rightarrow$ as B is L.I

$\Rightarrow \alpha_1 - \beta_1 = 0, \dots, \alpha_n - \beta_n = 0$

$\Rightarrow \alpha_1 = \beta_1 \dots \alpha_n = \beta_n$

Hence, proved

$1 : [v_1\ v_2 \dots v_n / v]$
$Ax = v$
$x = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ is unique soln of $Ax = v$