Assignment 12

Abhay Shankar K: CS21BTECH11001

Outline

Question

Given a Wiener process $w(t)$ with parameter α, we form the processes

(i) $x(t) = w\left(t^2\right)$

(ii) $y(t) = w^2(t)$

(iii) $z(t) = |w(t)|$

Show that $x(t)$ is normal with zero mean. Furthermore, if $t_1 < t_2$,

(1) $R_x(t_1, t_2) = \alpha t_1^2$

(2) $R_y(t_1, t_2) = \alpha^2 t_1 (2t_1 + t_2)$

(3) $R_z(t_1, t_2) = \frac{2\alpha}{\pi}\sqrt{t_1 t_2}(\cos\theta + \pi \sin\theta)$

where $\theta = \sin^{-1}(\sqrt{\frac{t_1}{t_2}})$

Finding $x(t), y(t), \& z(t)$

$$
\begin{aligned}
p_w(t) &= \frac{1}{\sqrt{2\pi\alpha t}} e^{-\frac{x^2}{2\alpha t}} \\
\implies p_x(t) &= \frac{1}{\sqrt{2\pi\alpha t}} e^{-\frac{x^2}{2\alpha t^2}} \\
\text{Putting } \sigma &= t\sqrt{\alpha}, \\
p_x(t) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}
\end{aligned}
\tag{1}
$$

which is clearly Gaussian with variance αt^2. Also,

$$
\begin{aligned}
E[x(t)] &= 0 \\
E[x^2(t)] &= Var[x(t)] + E^2[x(t)] \\
&= \alpha t^2
\end{aligned}
\tag{2}
$$

Due to the standard formula for the distribution of the square of a given RV,

$$
\begin{aligned}
p_y\left(t,x\right) &= \frac{1}{\sqrt{x}} p_w\left(t,\sqrt{x}\right) \\
&= \frac{1}{\sqrt{2\pi\alpha t x}} e^{-\frac{x}{2\alpha t}}
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\therefore E[y(t)] &= \int_0^\infty x \frac{1}{\sqrt{2\pi\alpha t x}} e^{-\frac{x}{2\alpha t}} dx \\
&= \alpha t
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
E[y^2(t)] &= \int_0^\infty x^2 \frac{1}{\sqrt{2\pi\alpha t x}} e^{-\frac{x}{2\alpha t}} dx \\
&= 3(\alpha t^2)
\end{aligned} \tag{5}
$$

We know the half normal distribution, by definition,

$$p_{z=|w|}(x,t) = 2u(x)\frac{1}{\sqrt{2\pi\alpha t}}e^{-\frac{x^2}{2\alpha t}} \tag{6}$$

$$\begin{aligned}
\therefore E[z(t)] &= 2\int_0^\infty x\frac{1}{\sqrt{2\pi\alpha tx}}e^{-\frac{x^2}{2\alpha t}}dx \\
&= \sqrt{\frac{2\alpha t}{\pi}} \qquad (7) \\
E[z^2(t)] &= \int_0^\infty x^2\frac{1}{\sqrt{2\pi\alpha tx}}e^{-\frac{x^2}{2\alpha t}}dx \\
&= \alpha t \qquad (8)
\end{aligned}$$

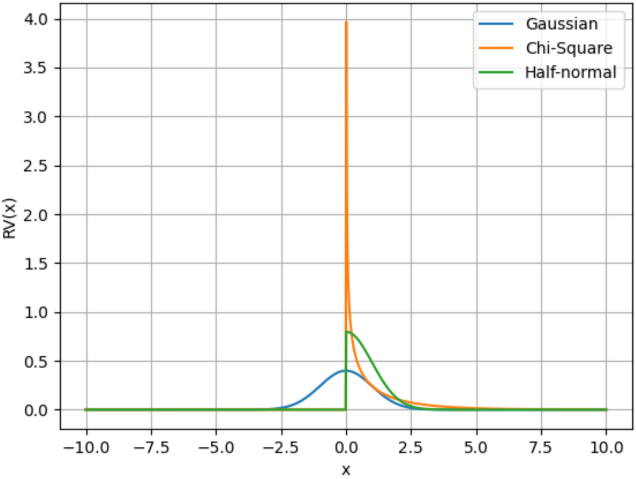
Finding Autocorrelation

$$\begin{aligned} R_x(t_1, t_2) &= E[x(t_1)x(t_2)] \\ &= E[x(t_1)]E[x(t_2) - x(t_1)] + E[x^2(t_1)] \\ &= E[x^2(t_1)] \\ &= \alpha t_1^2 \end{aligned} \tag{9}$$

$$\begin{aligned} R_y(t_1, t_2) &= E[y(t_1)y(t_2)] \\ &= E[y(t_1)]E[y(t_2) - y(t_1)] + E[y^2(t_1)] \\ &= \alpha t_1(\alpha t_2 - \alpha t_1) + 3(\alpha t_1)^2 \\ &= \alpha^2 t_1(2t_1 + t_2) \end{aligned} \tag{10}$$

$$
\begin{aligned}
R_z(t_1, t_2) &= E[z(t_1)z(t_2)] \\
&= E[z(t_1)]E[z(t_2) - z(t_1)] + E[z^2(t_1)] \\
&= \frac{2\alpha}{\pi}\sqrt{t_1 t_2}\left(\sqrt{1 - \frac{t_1}{t_2}} + \pi\sqrt{\frac{t_1}{t_2}}\right) \\
&= \frac{2\alpha}{\pi}\sqrt{t_1 t_2}(\cos\theta + \pi\sin\theta)
\end{aligned} \tag{11}
$$

Graph


Gaussian
Chi-Square
Half-normal
RV(x)
x
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0