

REGRESSION MODELS FOR ORDINAL DATA: A CONCISE SUMMARY

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1. PAPER SUMMARY

- The paper proposes two models for ordinal data, namely the proportional odds model and the proportional hazards model.
- The proportional odds model is a generalisation of the logistic regression model for ordinal data. Here, the odds of the response variable $Y \leq j$ are given by

$$\kappa_j = \kappa_j \exp(-\beta^T \mathbf{x})$$

.

- The proportional hazards model considers a hazard function $\lambda(t)$, which expresses the probability of failure at time t , of the form

$$\lambda(t) = \lambda_0(t) \exp(-\beta^T \mathbf{x})$$

.

- The paper proposes a generalised empirical logit transform for the two models.
- The paper also discusses
 - The properties of the two models, proposing a few alternative link functions.
 - Invariances of the models under reversal of the ordering.
 - Asymptotic properties of the two models.
 - Parameter estimation for both models.
 - Application of the models to real data.

2. PARAMETER ESTIMATION

Revising the notation from the paper, we have the probabilities of the k ordered categories of the response variable Y given by $\{\pi_1, \dots, \pi_k\}$, as a function of the covariant vector \mathbf{x} , and their cumulative probabilities given by $\gamma_j = \sum_{i=1}^j \pi_i$.

The cumulative odds are thus $\kappa_j = \frac{\gamma_j}{1-\gamma_j}$.

We then have the likelihood function $\kappa_j = \kappa_j \exp(\beta^T \mathbf{x})$, which we can reframe as

$$\begin{aligned} \frac{\gamma_j}{1-\gamma_j} &= \exp(\theta_j - \beta^T \mathbf{x}) \\ \implies \gamma_{j+1} &= \frac{1}{1 + \exp(\theta_j - \beta^T \mathbf{x})} \end{aligned} \tag{1}$$

where we set $\theta_0 = 0$. We also define $R_j = \sum_{i=1}^j n_i$.

We have the likelihood function:

$$\begin{aligned}
 p(\mathbf{y}|\mathbf{x}, \boldsymbol{\beta}) &= \prod_{j=1}^k \pi_j^{n_j} \\
 &= \pi_1^{n_1} \prod_{j=1}^k (\gamma_{j+1} - \gamma_j)^{n_{j+1}} \\
 &= \pi_1^{n_1} \prod_{j=1}^k (\gamma_{j+1} - \gamma_j)^{R_{j+1} - R_j} \\
 &= \prod_{n=1}^k \left(\frac{\gamma_j}{\gamma_{j+1}} \right)^{R_j} \left(1 - \frac{\gamma_j}{\gamma_{j+1}} \right)^{R_{j+1} - R_j}
 \end{aligned} \tag{2}$$

Taking the logarithm, we have

$$-\ln p(\mathbf{y}|\mathbf{x}, \boldsymbol{\beta}) = \sum_{j=1}^k R_j \ln \left(\frac{1 + \exp(\theta_{j-1} - \boldsymbol{\beta}^T \mathbf{x})}{1 + \exp(\theta_j - \boldsymbol{\beta}^T \mathbf{x})} \right) - (R_{j+1} - R_j) \left(\ln \left(\frac{e^{\theta_j} - e^{\theta_{j-1}}}{1 + \exp(\theta_j - \boldsymbol{\beta}^T \mathbf{x})} \right) - \boldsymbol{\beta}^T \mathbf{x} \right) \tag{3}$$

This does not have a closed form solution. We can use gradient descent to find the optimal $\boldsymbol{\beta}$.

3. CODE

Refer to `wine.ipynb`.