CS112: Data Structures

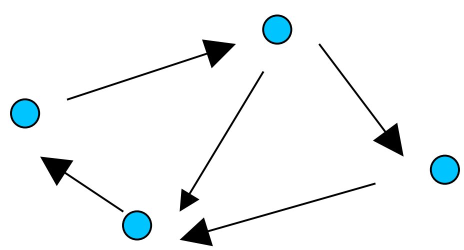
Lecture 12

More Graphs

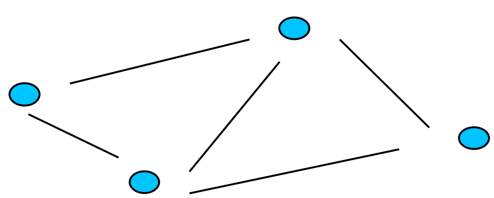
Review: Graphs

Generalization of trees

- **Digraph (Directed Graph)**
  - **Like a tree but any vertex can point to any other**



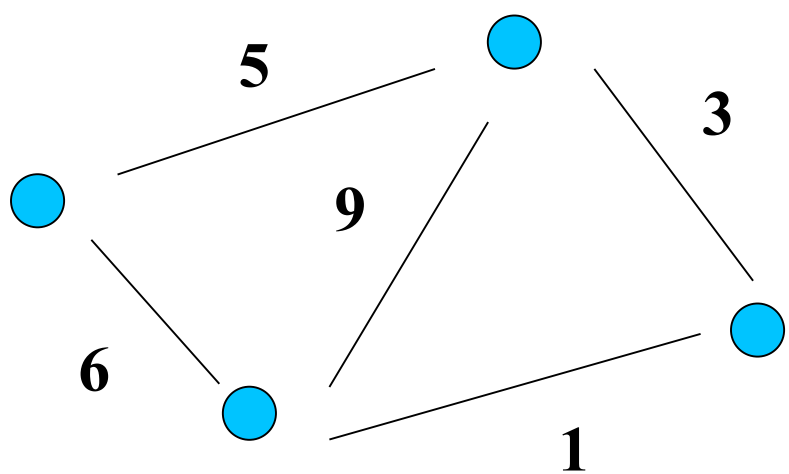
- **Graph**
  - **like digraph but arcs have no direction**



Graphs

Generalization of trees

- **Weighted Graph**
  - **Positive integer weights on each edge**



Applications

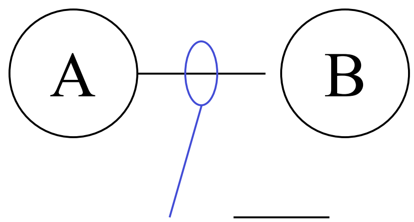
- Paths
  - On streets (eg mapquest)
- Electrical networks
  - On circuit boards
  - Power lines
- Constraints
  - Ordering constraints on building steps
  - Sudoku

Applications

- Relationships
  - Web page references
  - Friendships (online and real world)
- Etc, etc, etc

Notation

- **Arcs are named by the vertices they connect**



(B,A) or $\overline{\mathbf{BA}}$

Representing Graphs

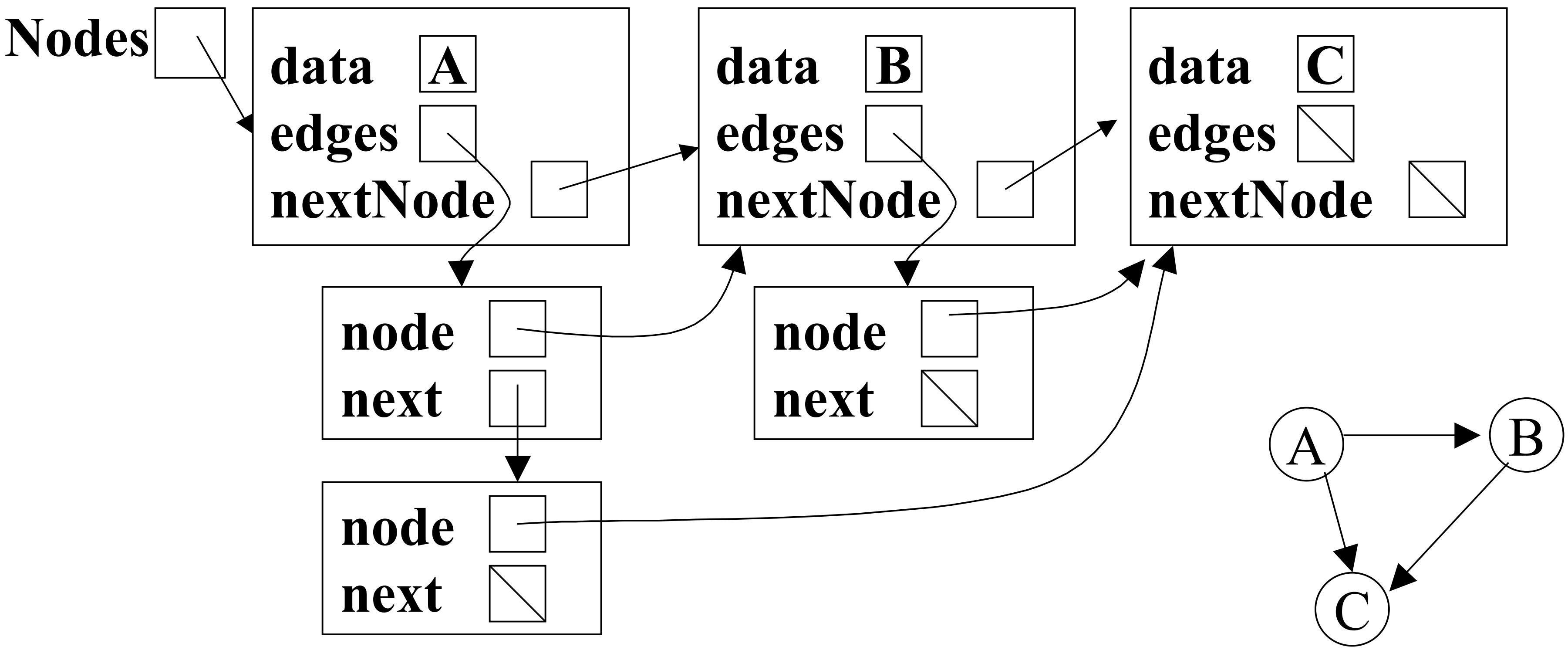
- **Adjacency list**
  - **for each node, linked list of edges**

```
public Gnode{
  String data;
  Edge edges;
  Gnode nextNode;
  …
}
```

```
public Edge{
  Gnode node;
  Edge next;
  …
}
```

Representing Graphs

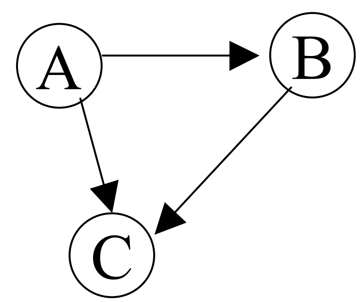
- **Adjacency list**
  - for each node, linked list of edges



Representing Graphs

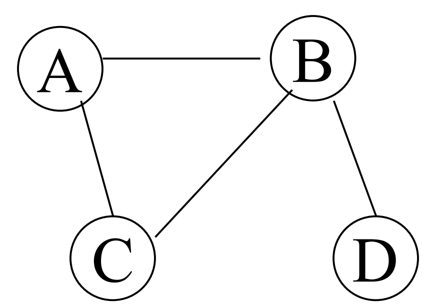
- **Adjacency matrix**
  - **n x n boolean matrix: is there an arc?**

From \To	A	B	C
A	F	T	T
B	F	F	T
C	F	F	F



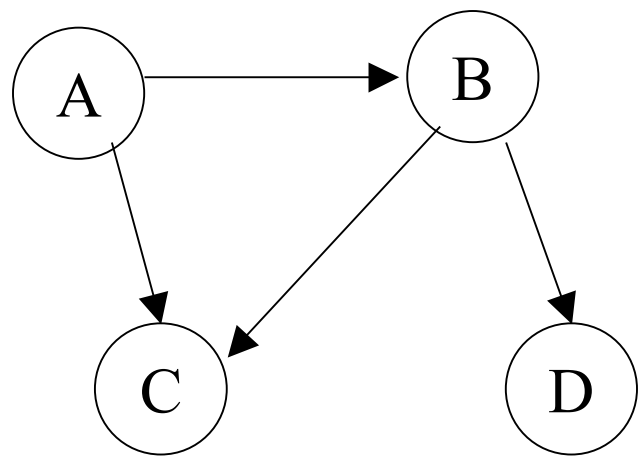
Graph Concepts

- **Neighbors of a vertex: vertices that it shares an arc with**
  - **Neighbors of A are B and C**
- **Degree of a vertex: number of neighbors**
  - **Degree of A is 2, degree of B is 3**



Graph Concepts

- **In degree (in a digraph): number of vertices that have arcs to this vertex**
  - **In degree of B is 1**
- **Out degree (in a digraph): number of vertices that have arcs from this vertex**
  - **Out degree of B is 2**



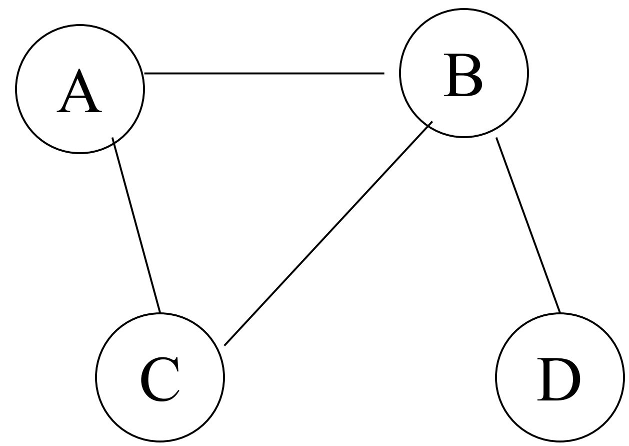
Graph Concepts

- **(Simple) Path**
  - **Sequence of arcs**

    (A,B),(B,C)
  - **May not revisit a vertex**

    ~~**(B,A),(A,C),(C,B),(B,D)**~~
  - **Except last vertex may = first**

    (B,A),(A,C),(C,B)
- **Vertex A is reachable from B if there is a path from B to A**

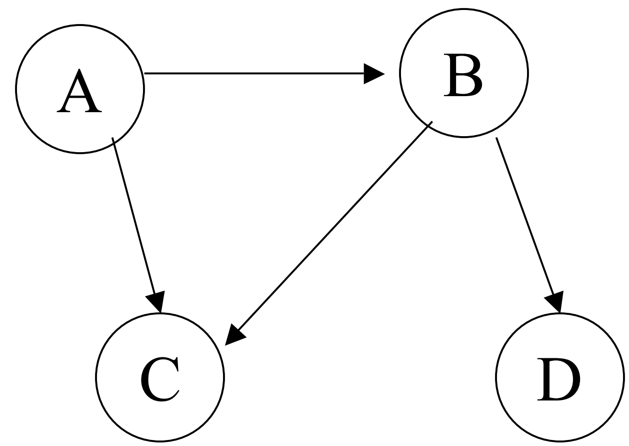


Graph Concepts

- (Simple) Path
  - On digraph must follow arc directions

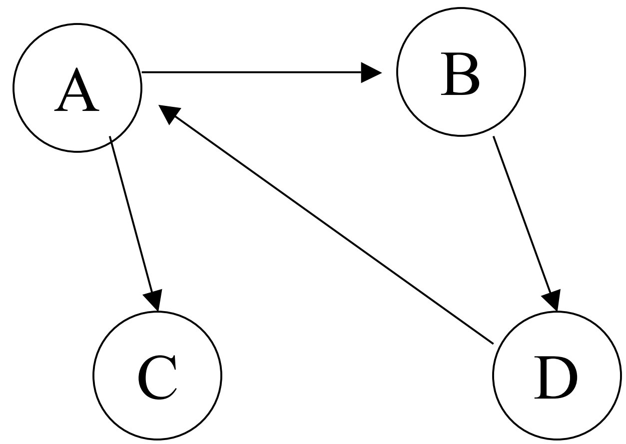
    (A,B),(B,D)

    ~~(A,C),(C,B)~~



Graph Concepts

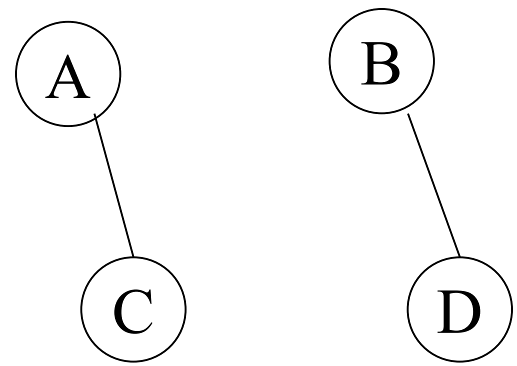
- A cycle is a path from a node back to itself
  - (A, B)(B, D)(D, A)
- A graph with no cycles is called acyclic



Graph Concepts

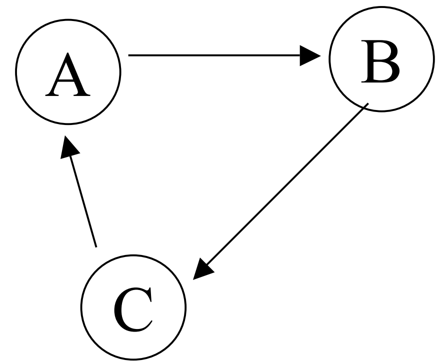
- **Connected Graph**

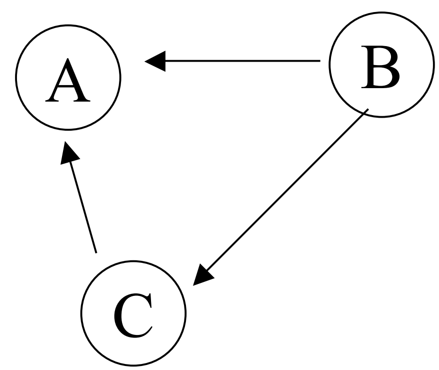
  For any two vertices X and Y there is a path from X to Y.



Graph Concepts

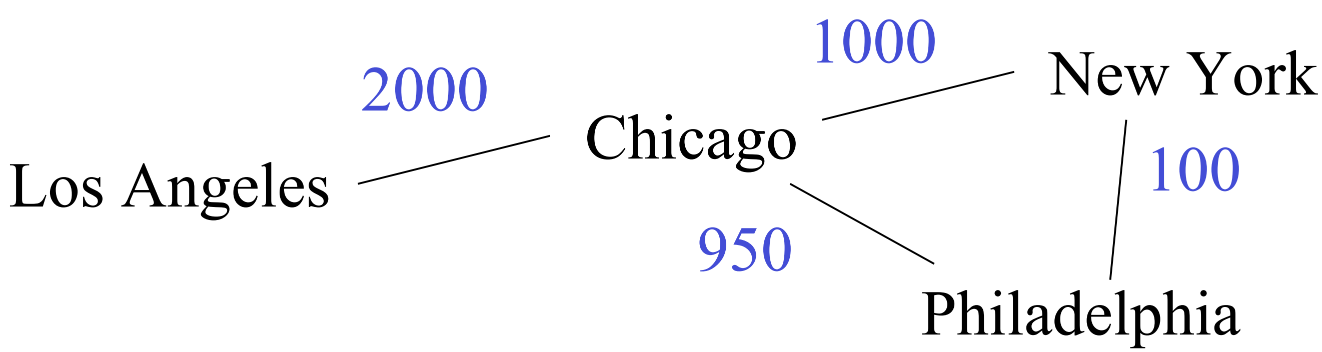
- **Strongly Connected Digraph**
  For any two vertices X and Y there is a path from X to Y. (Paths must follow arc directions)
- **Weakly Connected Digraph**
  Corresponding graph is connected (i.e., ignoring arc direction)





Graph Concepts

- **Weighted graph: each arc has a numerical weight**



Graph Traversals

- **Need to mark vertices to prevent infinite loops**
- **Need driver in case not connected**
- **Otherwise like tree traversals**
- **Depth first**

```
dfsG(v)
    if (marked(v)) return;
    process v;
    mark v;
    for each vn in neighbors(v)
      dfsG(vn)
```

Graph Traversals

- **Need driver in case not connected**

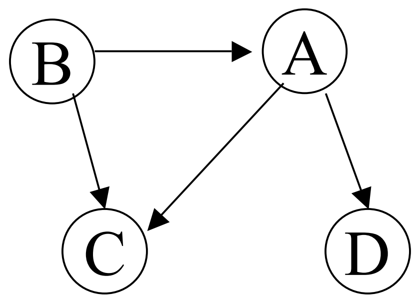
  For v in vertices

      dfsG(v)

DFS Graph Traversal

- Enters a vertex v
- Visits all vertices reachable from v (that have not yet been visited
- Leaves v

Graph Traversals



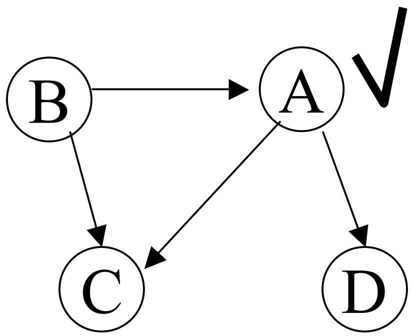
Driver

v = <A>

dfsG

v = <A>

Graph Traversals

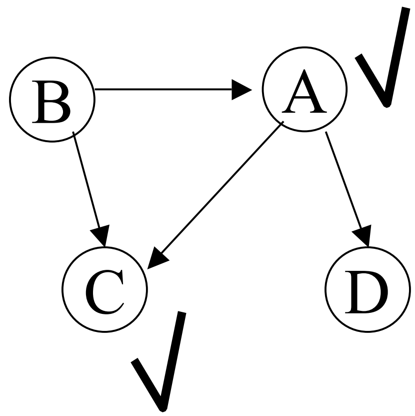


Driver
v = <A>

dfsG
v = <A>
vn = <C>

dfsG
v = <C>

Graph Traversals



Driver

v = <A>

dfsG

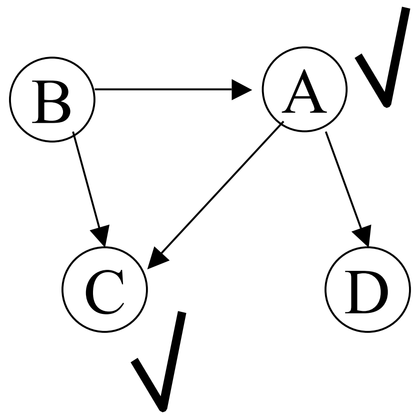
v = <A>

vn = <C>

dfsG

v = <C>

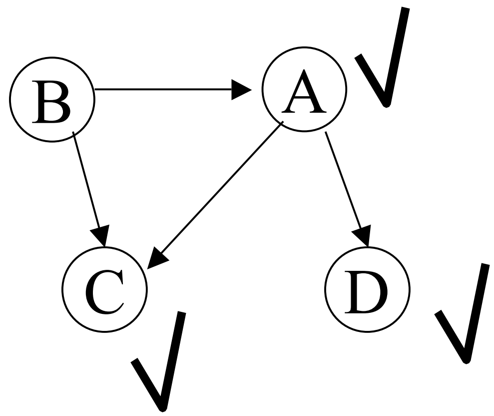
Graph Traversals



Driver
v = <A>

dfsG
v = <A>
vn = <D>

Graph Traversals

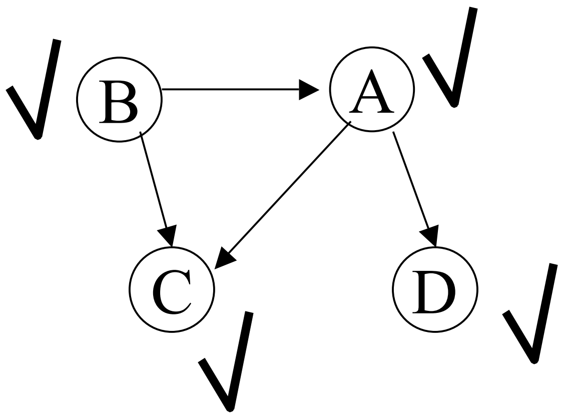


Driver
v = <A>

dfsG
v = <A>
vn = <D>

dfsG
v = <D>

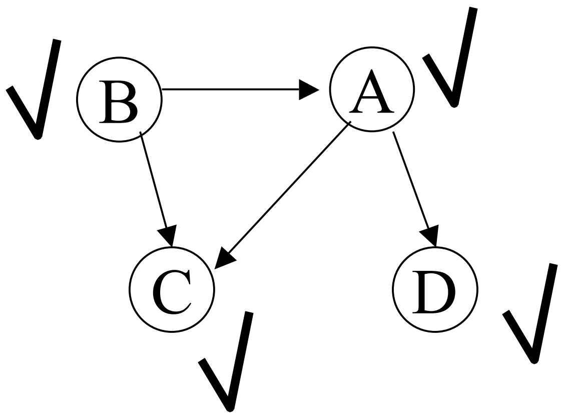
Graph Traversals



Driver
v = <B>

dfsG
v = <B>

Graph Traversals



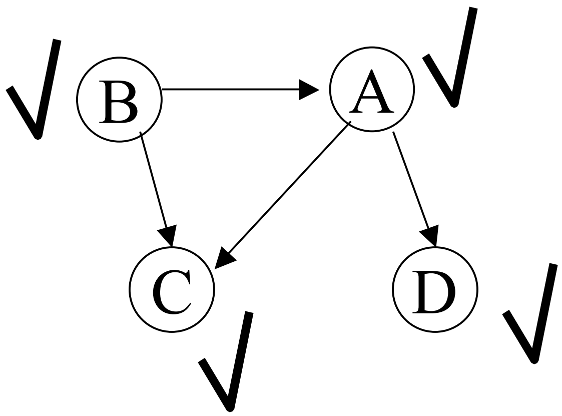
Driver

v = <C>

dfsG

v = <C>

Graph Traversals



Driver
v = <D>

dfsG
v = <D>

Graph Traversals

- **Time:**
  - **Visit each vertex**
  - **inspect each arc**
  - **driver**

  O(n + e) n vertices, e edges

Topological Sort

- **Acyclic Digraph <=> partial order**
- **Topsort: find total order consistent with partial order**

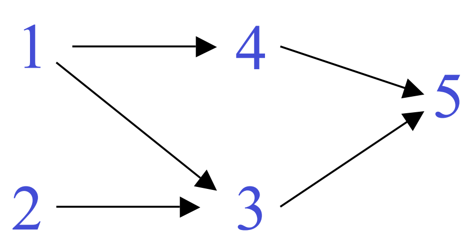
1 **a=1;**

2 **b=2;**

3 **c=a*b;**

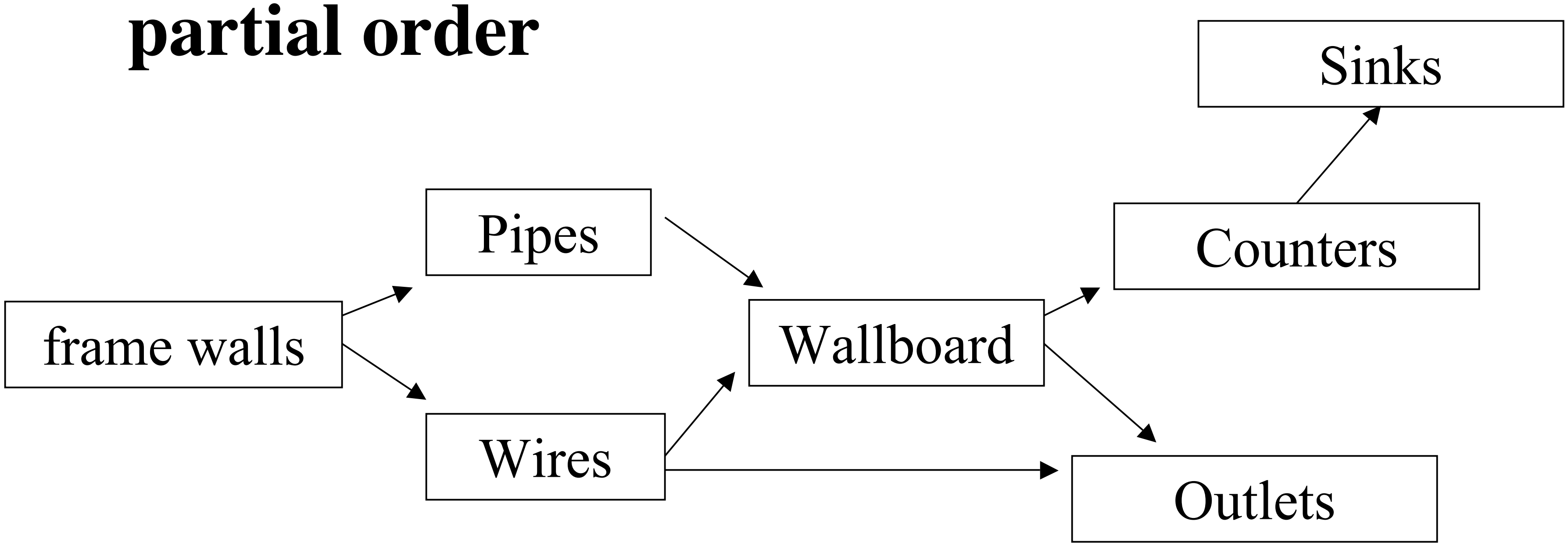
4 **d=a+4;**

5 **c=c+d**



Topological Sort

- **Acyclic Digraph <=> partial order**
- **Topsort: find total order consistent with partial order**



Topsort Algorithms

- Most work by assigning numbers to vertices
  - vertex order = numerical order
- Depth first
- Breadth First

DFS Topsort Algorithm

- **Algorithm:**
  - Do DFS
  - Number vertices as you leave them
- **Problem: leave vertex *after* leave reachable vertices, but needs number *smaller* than reachable vertices**
  - Solution: number with decreasing numbers

BFS Topsort Algorithm

- **Keep a "predecessor count" for each vertex**
  - **Initially: in degree**
  - **When a predecessor is numbered, decrement count of its successors**

BFS Topsort Algorithm

- **enqueue all sources**
- **while not queue.isEmpty**

  v = queue.dequeue( )

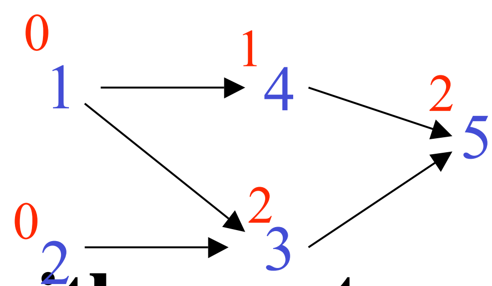
  number v (increasing numbers)

  decrement predecessor counts of v’s neighbors

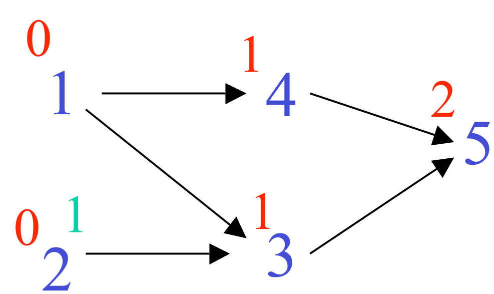
    if count becomes 0, enqueue neighbor

BFS Topsort Algorithm

- Keep predecessor count for each vertex



- Find vertex with count == 0
  - number it
  - decrement counts of neighbors



New: Shortest Path

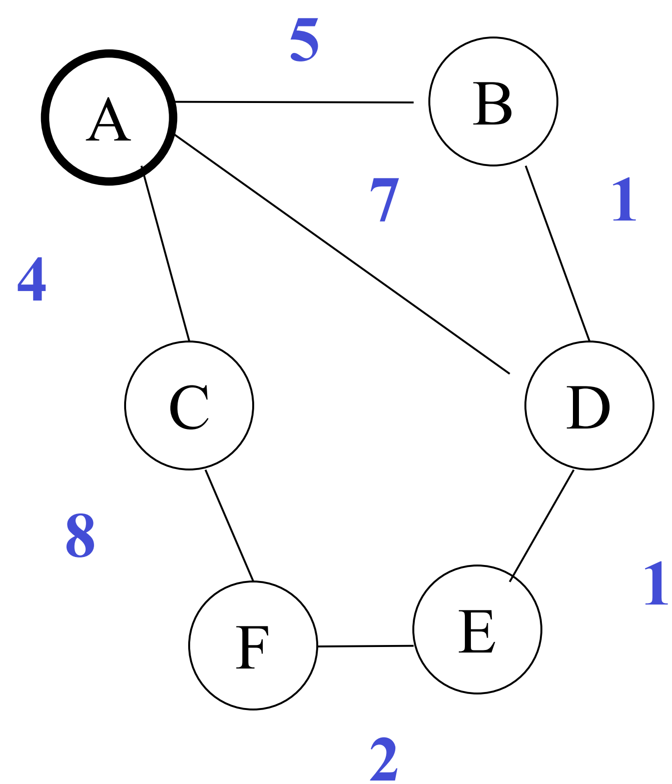
- weighted digraph
  - weights are all > 0
- "length" of a path = sum of weights of arcs on path
- given start vertex, end vertex, find shortest path from start to end

Dijkstra's Algorithm

- Grow a tree of paths from start
  - tree is subgraph of original digraph
  - grow it one vertex at a time
  - only add a vertex when we know where to put it so that path to vertex from root in tree is shortest in digraph
  - when we add end vertex to tree the shortest path from start to end is given by path in tree

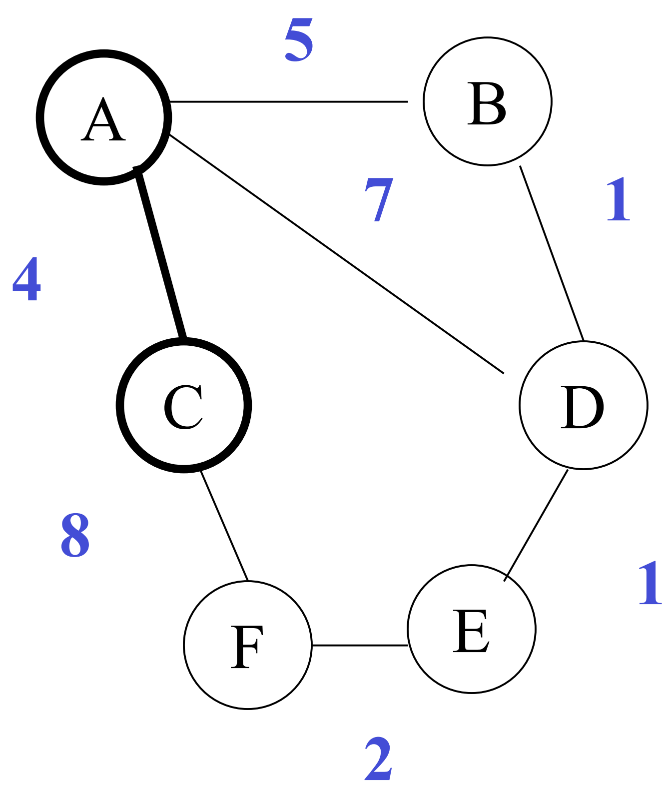
Example

Node	Status	LinK	Distance
A	Tree	--	0
B	Fringe	A	5
C	Fringe	A	4
D	Fringe	A	7
E			
F			



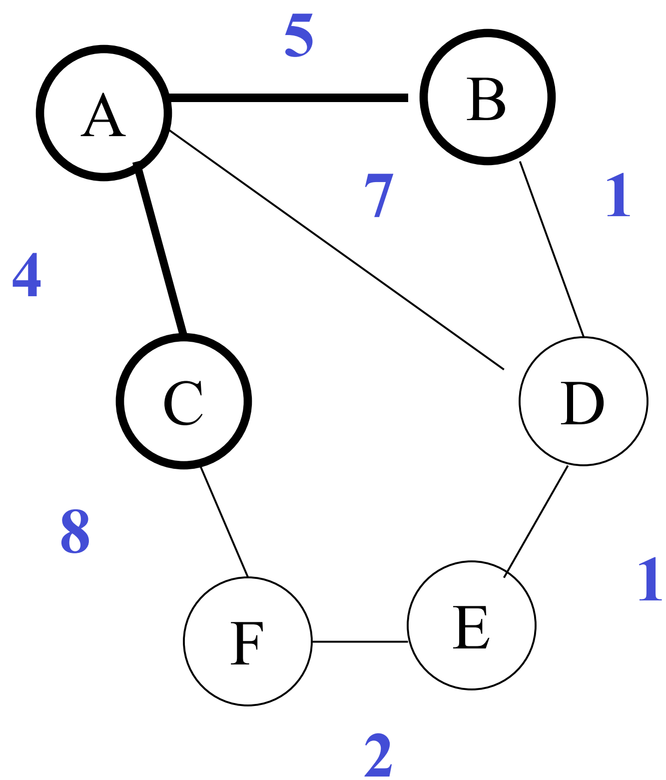
Example

Node	Status	LinK	Distance
A	Tree	--	0
B	Fringe	A	5
C	Tree	A	4
D	Fringe	A	7
E			
F	Fringe	C	12



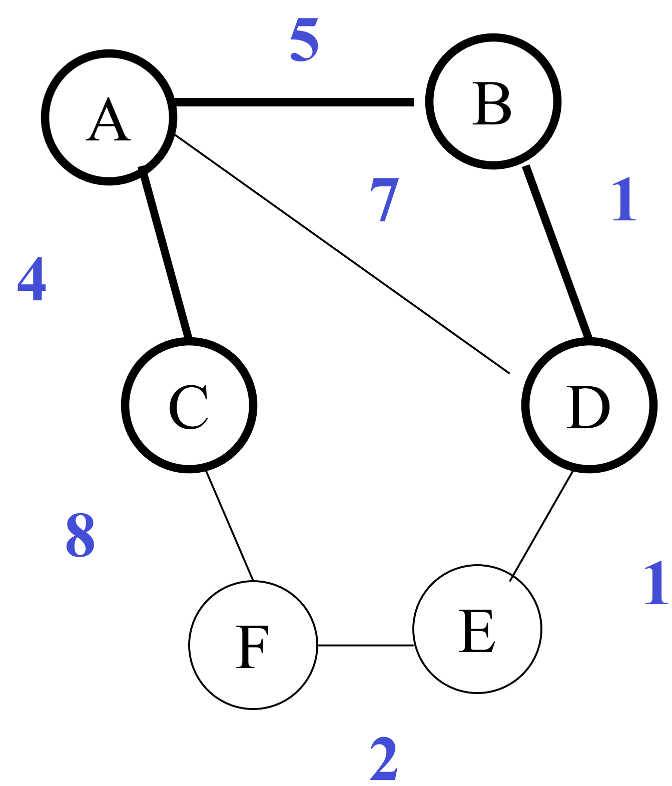
Example

Node	Status	LinK	Distance
A	Tree	--	0
B	Tree	A	5
C	Tree	A	4
D	Fringe	B	6
E			
F	Fringe	C	12



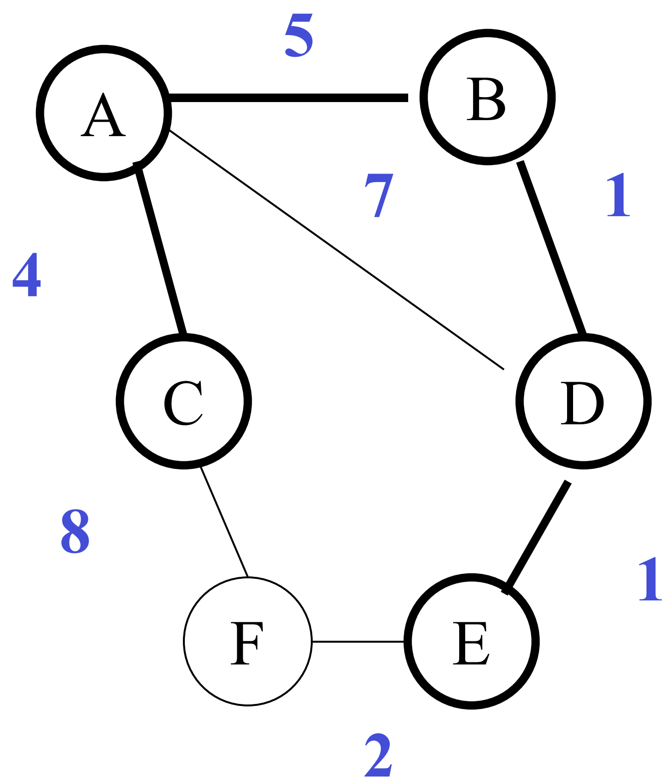
Example

Node	Status	LinK	Distance
A	Tree	--	0
B	Tree	A	5
C	Tree	A	4
D	Tree	B	6
E	Fringe	D	7
F	Fringe	C	12



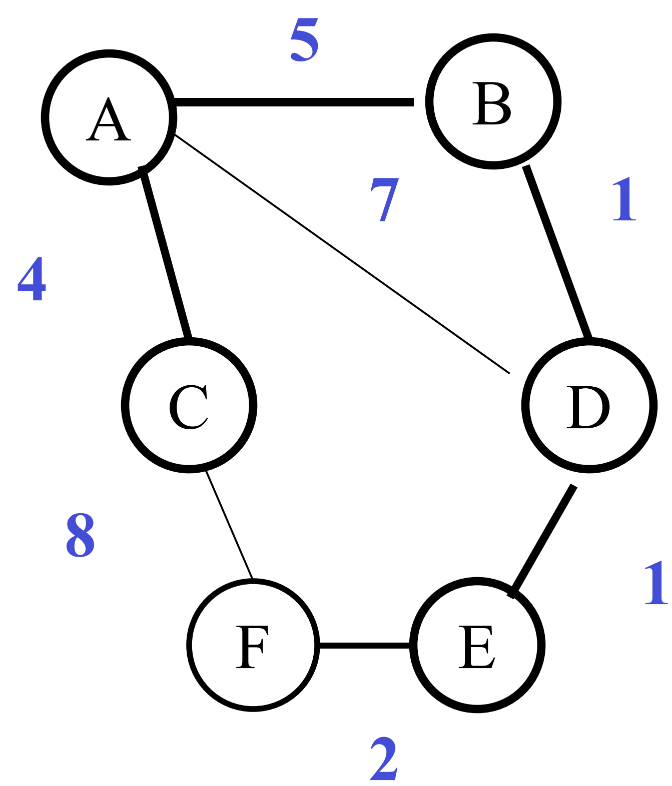
Example

Node	Status	LinK	Distance
A	Tree	--	0
B	Tree	A	5
C	Tree	A	4
D	Tree	B	6
E	Tree	D	7
F	Fringe	E	9



Example

Node	Status	Link	Distance
A	Tree	--	0
B	Tree	A	5
C	Tree	A	4
D	Tree	B	6
E	Tree	D	7
F	Tree	E	9



Dijkstra's Algorithm

- **How can we be sure we are attaching vertex at right point?**
  - **assume tree so far is shortest paths**
  - **choose vertex X and arc (Y, X), Y in tree and X not:**
    - **choose X and Y such that path start, … , Y,X has minimum weight of all possible X and Y**

Dijkstra’s Algorithm

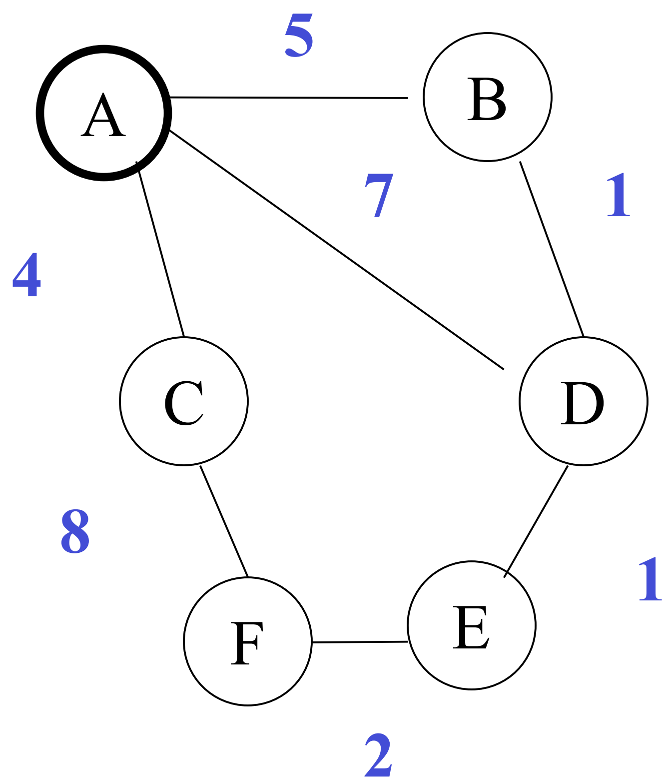
- **But what if some other path is shorter?**
  - **Other path must include some vertices in tree, some not in tree**
  - **Let (A,B) be arc in this shorter path such that A is in tree and B is not**
  - **Path start, …,A,B is longer than path we have found start, … , Y, X so path start,…,A,B,…,X must be longer than path start, … , Y, X**

New: Minimum Spanning Tree

- Spanning Tree: a subgraph with
  - All the nodes
  - Some of the edges
  - A tree, I.E., one path between any pair of nodes
- Minimum spanning tree
  - A spanning tree
  - With minimum total edge weight

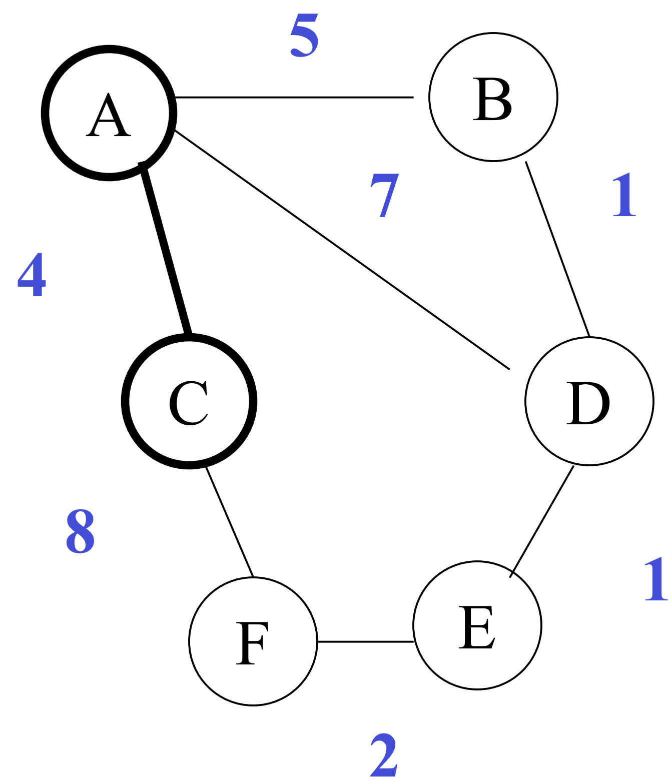
Example

Node	Status	Link	Weight
A	Tree	--	0
B	Fringe	A	5
C	Fringe	A	4
D	Fringe	A	7
E			
F			



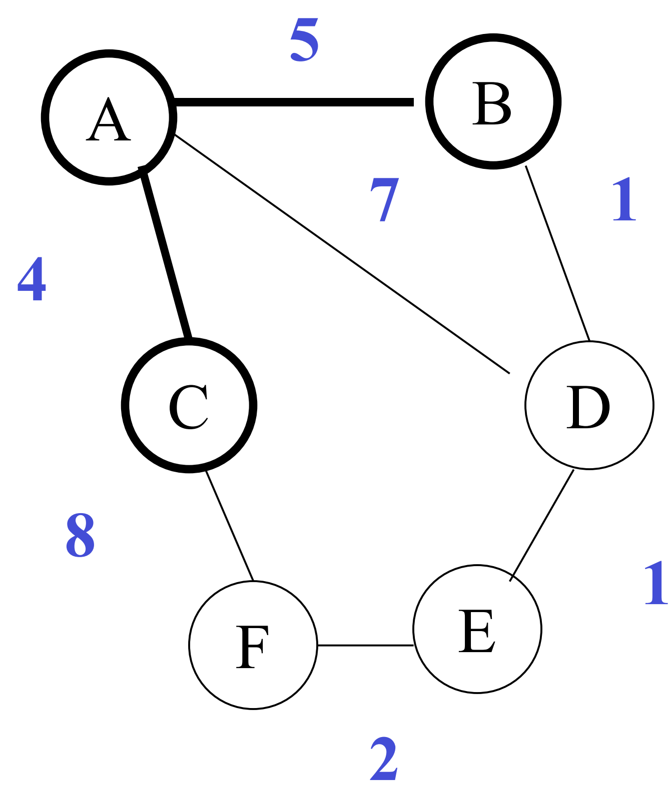
Example

Node	Status	Link	Weight
A	Tree	--	0
B	Fringe	A	5
C	Tree	A	4
D	Fringe	A	7
E			
F	Fringe	C	8



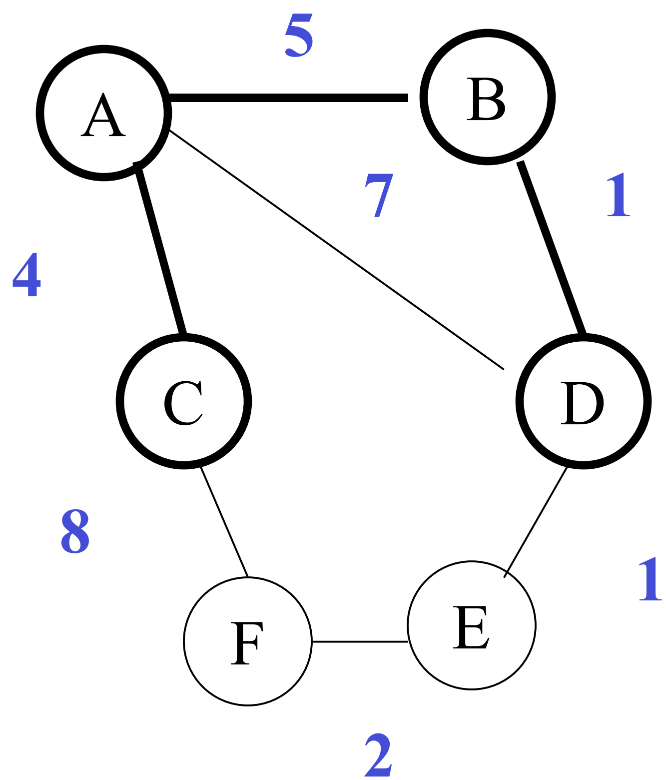
Example

Node	Status	Link	Weight
A	Tree	--	0
B	Tree	A	5
C	Tree	A	4
D	Fringe	B	1
E			
F	Fringe	C	8



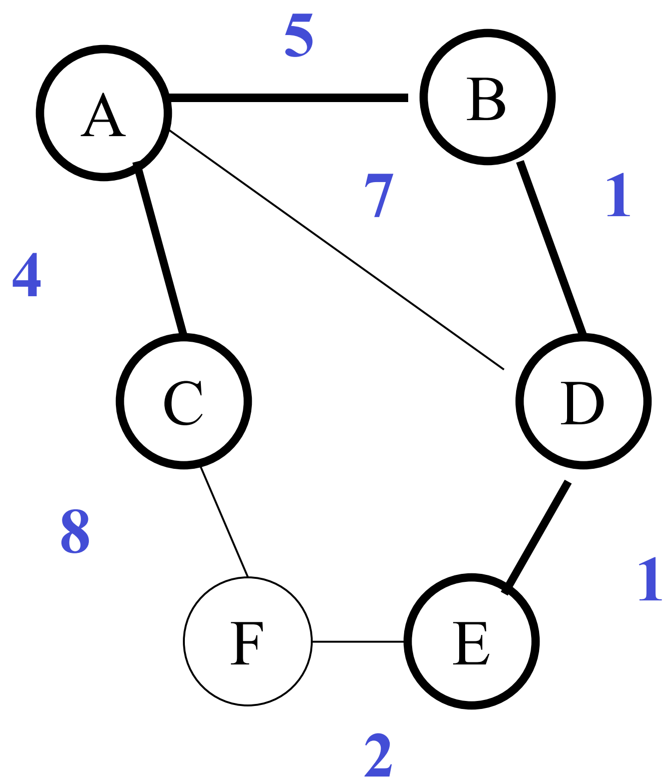
Example

Node	Status	Link	Weight
A	Tree	--	0
B	Tree	A	5
C	Tree	A	4
D	Tree	A	6
E	Fringe	D	1
F	Fringe	C	8



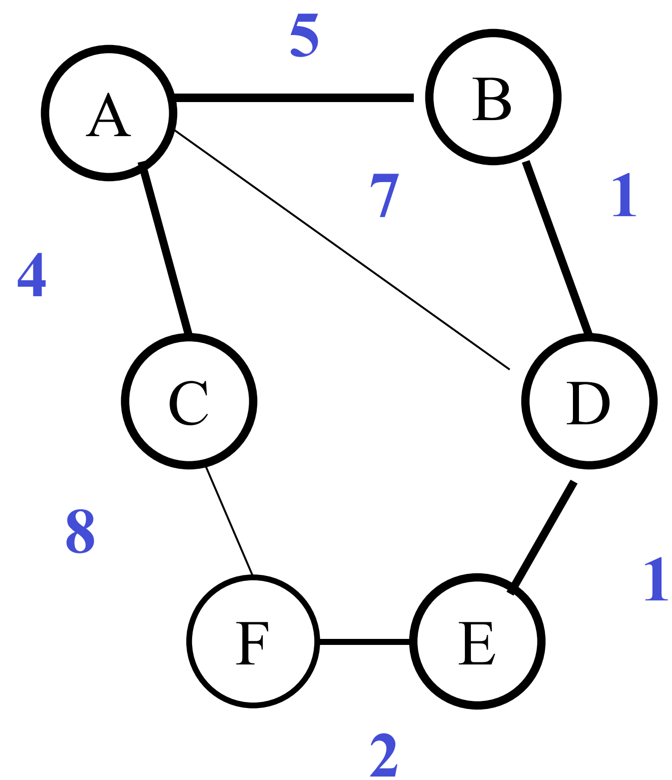
Example

Node	Status	Link	Weight
A	Tree	--	0
B	Tree	A	5
C	Tree	A	4
D	Tree	A	6
E	Tree	D	1
F	Fringe	E	2



Example

Node	Status	Link	Distance
A	Tree	--	0
B	Tree	A	5
C	Tree	A	4
D	Tree	B	1
E	Tree	D	1
F	Tree	E	2

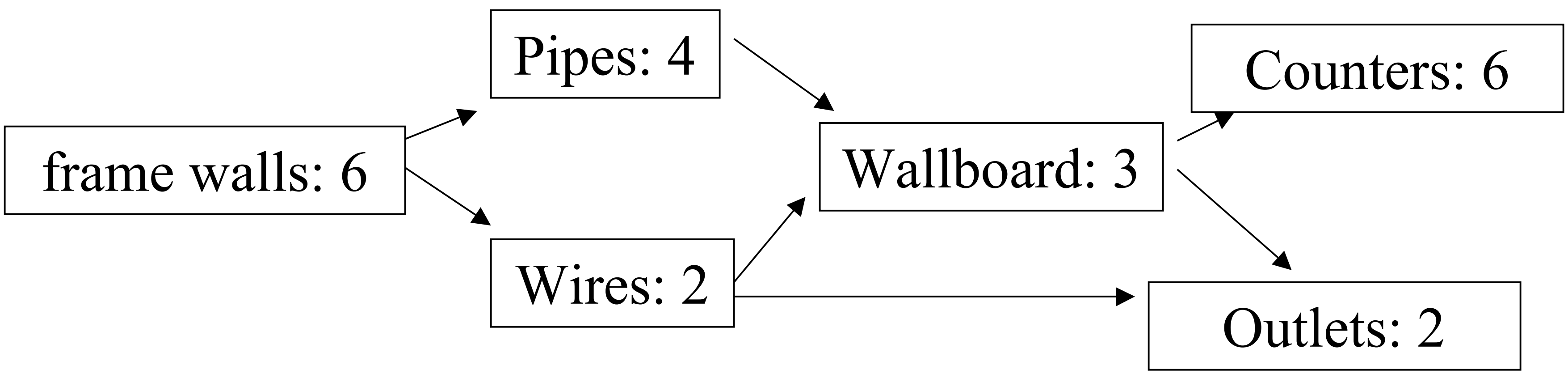


Proof

- Suppose we have just added node Y and edge XY to the tree.
- Suppose there is some spanning tree T that does not include XY with lower weight than any that does include XY. Let AB be an edge on that tree with A in current partial tree and B not. AB must have greater weight than XY.
- Let T' be T but with XY instead of AB. T' is still a spanning tree, with less weight than T
- Contradiction

PERT Algorithm

- **PERT (Project Evaluation and Review Technique)**
  - **Graph representing steps of a job**
    - **Edges: predecesors - "must be done before"**
    - **Vertices: tasks - labeled with time taken**

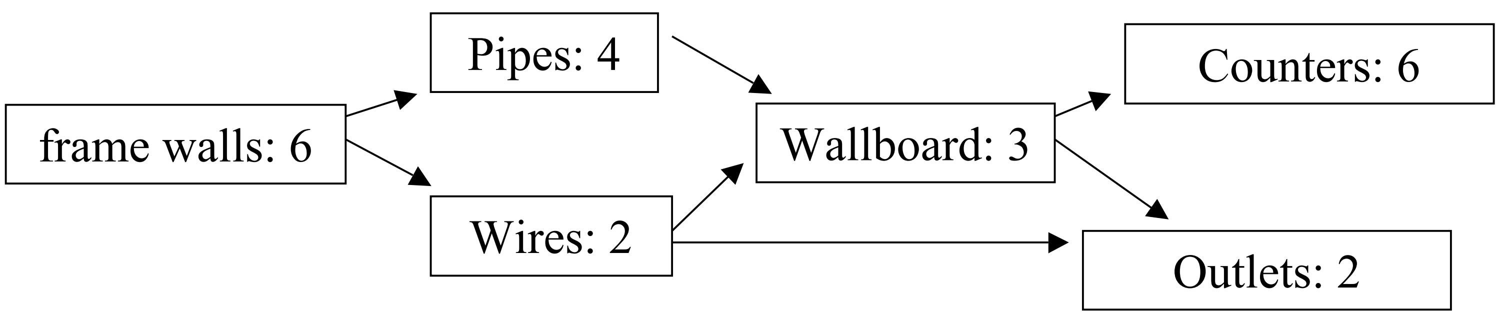


Critical Path

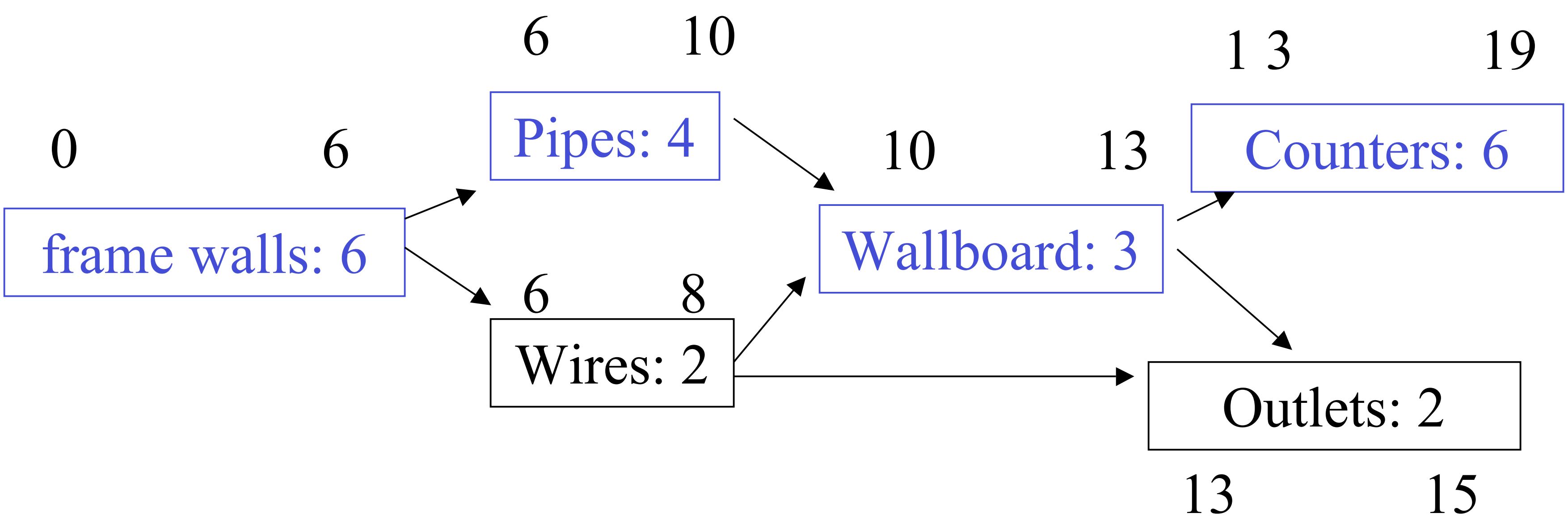
- **Suppose each task is started as soon as all predecessor finish**
  - **Suppose a task takes longer than specified: will entire job take longer?**

    Yes: task on critical path

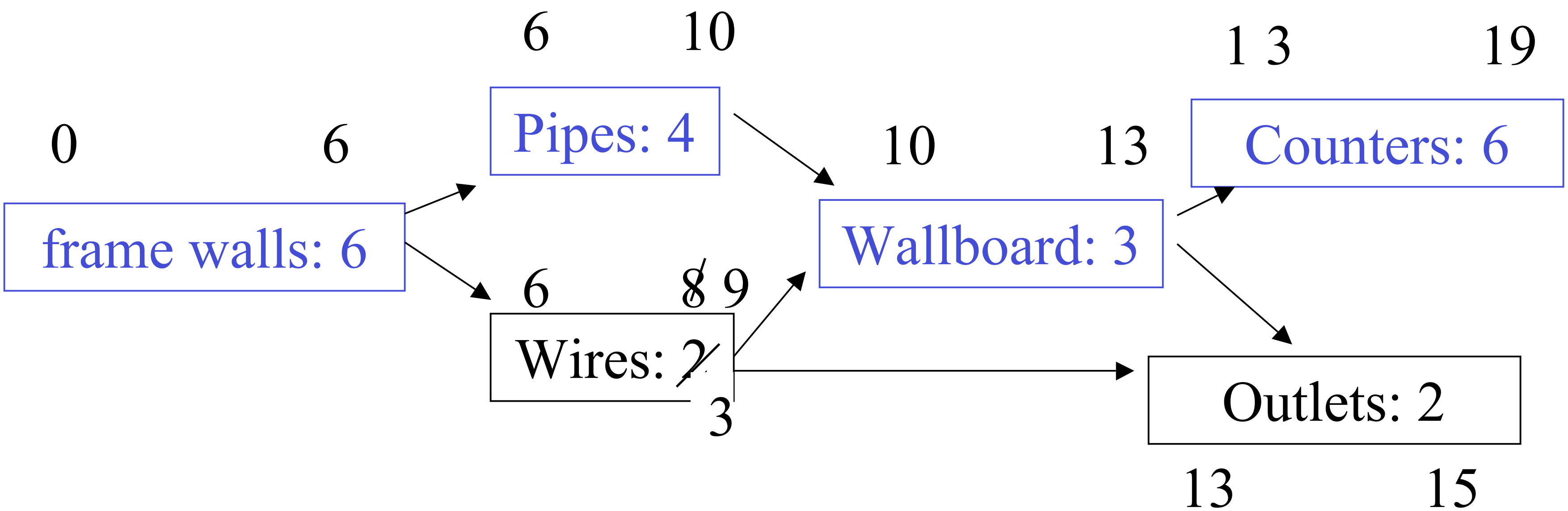
    No: not on critical path



Critical Path



Slack


0
6
frame walls: 6
6
10
Pipes: 4
6
8 9
Wires: 2
3
10
13
Wallboard: 3
1 3
19
Counters: 6
Outlets: 2
13
15

Algorithm

- **Define:**
  - **EFT Earliest finish time**
  - **LFT Latest finish time**
  - **EST Earliest start time**
  - **LST Latest start time**
- **Process tasks in topsort order**
  - **EST(n)=max(EFT(p), p in predecessors of n)**
  - **EFT(n) = EST(n) + Time(n)**

Algorithm

- **Process tasks in reverse topsort order**
  - **LFT(n)=min(LST(p), p in Successors of n)**
  - **LST(n) = EFT(n) - Time(n)**
- **On critical path if LFT == EFT and LST == ESTs**