1.1.4.

Assuming R = Asymptotic risk for 1-NN and R^* is the Bayes risk. As shown in 1.1.3, the correctly classified point will have probability $(1+\alpha) r^*(x)(1-r^*(x))$. Now, the NN risk R is

$$R = \lim_n E[r(x, x_n')].$$

Now, by the dominated convergence theorem

$$R = E[\lim_n r(x, x_n')]$$

and we know

$$R = E[r(x)]$$

$$\Rightarrow R = E[(1+\alpha)\hat{\eta}_1(x)\hat{\eta}_2(x)]$$

$$= E[(1+\alpha) r^*(x)(1-r^*(x))]$$

Since, Bayes risk R^* is the expectation of r^*:

$$R = (1+\alpha) R^*(1-R^*).$$

or $R \leq (1+\alpha) R^*(1-R^*)$.

~~1.(2)~~ 1.1.3

We know that the conditional risk $r(x, x_n')$ is given by the conditional independence of θ and θ_n'.

$r(x, x_n') = E(L(\theta, \theta_n'))$

$= Pr\{\theta \neq \theta_n' | x, x_n'\}$

$= Pr\{\theta = 1 | x\} Pr\{\theta_n' = 2 | x_n'\}$

$+ Pr\{\theta = 2 | x\} Pr\{\theta_n' = 1 | x_n'\}$

So, we get

$r(x, x_n') = \hat{\eta}_1(x) \hat{\eta}_2(x_n') + \hat{\eta}_2(x) \hat{\eta}_1(x_n')$

We see that f_1, f_2 are not continuous at x as they are trivially taken and hence they converge with probability one.

$r(x, x_n') = 2\hat{\eta}_1(x_0) \hat{\eta}_2(x_0)$

Conditional Bayes risk as show in 1.1.1 is

$r^*(x) = \min\{\eta(x), \alpha(1-\eta(x))\}$

We get by symmetry

$r(x) \leq (1+\alpha) r^*(x) (1 - r^*(x))$.

1.1.1

Let us assume that $L(i,j)$ be the loss incurred by assigning individual categories.

Probability densities at x are $f_1(x), f_2(x) \cdots f_M(x)$.

Now the prior probability for M categories is

$$\hat{\eta}_i = \frac{\eta_i f_i}{\sum \eta_i f_i}, \quad i = 1, 2 \cdots M.$$

Thus, for random variable x posterior probability will be, $\hat{\eta}(x)$. the corresponding loss is

$$r_i(x) = \sum_{i=1}^{M} \eta_i(x) L(i,j)$$

for a given x, the conditional loss is minimum when assigned to category j for which $r_i(x)$ is lowest. The minimizing decision rule δ^* will be know. Now the risk $r^*(x)$ will be different for different values as the difference of α is given.

Hence,

$$r^*(x) = \min_i \left\{ \sum_{i=1}^{M} \hat{\eta}_i(x) \, L(i,j) \right\}$$

or $r^*(x) = \min \{ \eta(x), \alpha(1-\eta(x)) \}$

3.

1.2.1

Prior probability $= \hat{\eta}_i = \frac{\eta_i p_i}{\sum \eta_i p_i}$

~~form~~ this, we get

$r(x, x'_n) = \hat{\eta}_1(x)\hat{\eta}_2(x'_n) + \hat{\eta}_2(x)\hat{\eta}_1(x'_n)$

or

$r(x) = 2\hat{\eta}_1(x)\hat{\eta}_2(x)$

By symmetry of r^* in $\hat{\eta}_1$

$r(x) = 2\hat{\eta}_1(x)\hat{\eta}_2(x) = 2\hat{\eta}_1(x)(1-\hat{\eta}_1(x))$

$= 2r^*(x)(1-r^*(x))$

or $r(x) = r^*(x) + (1-2r^*(x))\, g(r^*(x))$