21AIE201-INTRODUCTION TO ROBOTICS

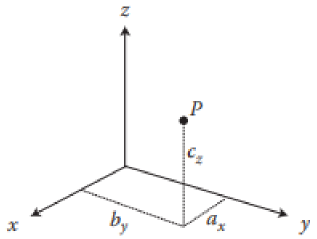
Lecture 8


Kinematics of Serial Robots: Position Analysis

- Kinematics is a branch of physics and a subdivision of classical mechanics concerned with the geometrically possible motion of a body or system of bodies without consideration of the forces involved.
- Within kinematics, one studies position, velocity, acceleration (and even higher-order derivatives of position) w.r.t. time

Representation of a Point in Space

A point P in space can be represented relative to a reference frame as:



$$P = a_x\mathbf{i} + b_y\mathbf{j} + c_z\mathbf{k}$$

$$\mathbf{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \\ w \end{bmatrix} \qquad \text{where } a_x = \frac{P_x}{w},\ b_y = \frac{P_y}{w},\ c_z = \frac{P_z}{w}$$

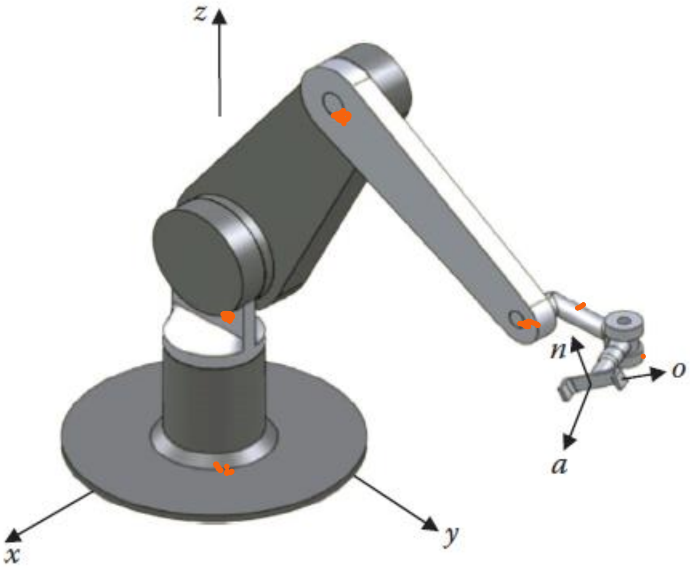
A vector is described as **P=3i+5j+2k**. Express the vector in matrix form. If it were to describe a direction as a unit vector

Example: A vector p is 5 units long and is in the direction of a unit vector q described as follows. Express the vector in matrix form.

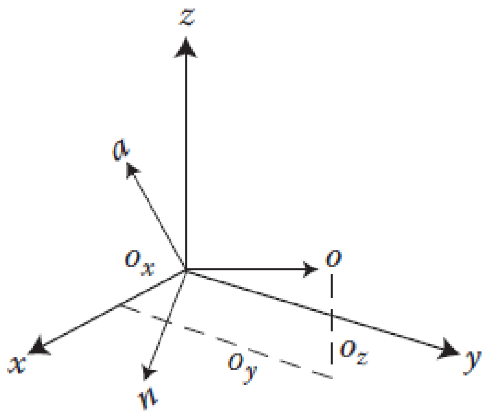
$$\mathbf{q}_{unit} = \begin{bmatrix} 0.371 \\ 0.557 \\ q_z \\ 0 \end{bmatrix}$$

Representation of a Frame at the Origin of a Fixed-Reference Frame

- A frame is generally represented by three mutually orthogonal axes (such as x, y, and z).
- Since we may have more than one frame at any given time, we use axes x, y, and z to represent the fixed Universe reference frame $\mathbf{F_{x,y,z}}$
- And a set of axes n, o, and a to represent another (moving) frame $\mathbf{F_{n,o,a}}$ relative to the Universe frame.

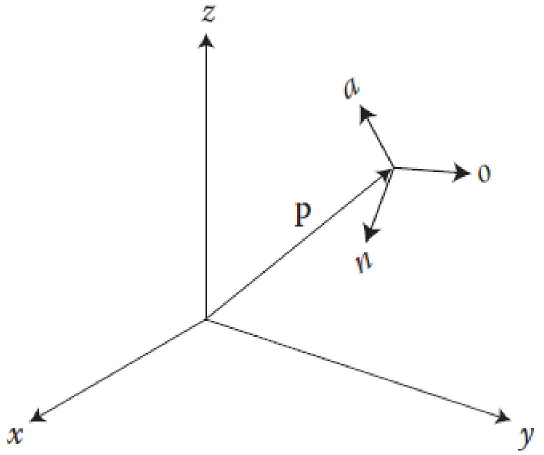


Representation of a Frame at the Origin of a Fixed-Reference Frame



$$F = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}$$

Representation of a Frame Relative to a Fixed Reference Frame

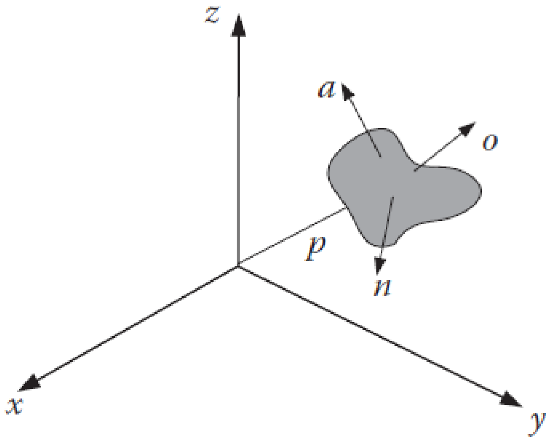


$$F = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An example of representation of a frame.

$$F = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0.707 & -0.707 & 5 \\ 0 & 0.707 & 0.707 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Representation of a Rigid Body



$$F_{object} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The three unit vectors **n**, **o**, **a** are mutually perpendicular, and
- Each unit vector representing directional cosines must be equal to 1

These constraints translate into the following six constraint equations:

1) $\mathbf{n} \cdot \mathbf{o} = 0$ (the dot product of $\mathbf{n}$ and $\mathbf{o}$ vectors must be zero)
2) $\mathbf{n} \cdot \mathbf{a} = 0$
3) $\mathbf{a} \cdot \mathbf{o} = 0$
4) $|\mathbf{n}| = 1$ (the magnitude of the lenght of the vector must be 1)
5) $|\mathbf{o}| = 1$
6) $|\mathbf{a}| = 1$

Rigid Body Transformations

- To start with basics of robotics we should first know what is a frame in 2D/3D world.
- A frame is nothing but a coordinate axis attached to a body as shown in below figures.

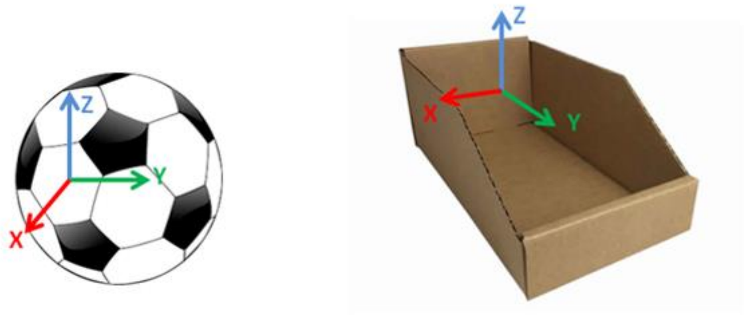
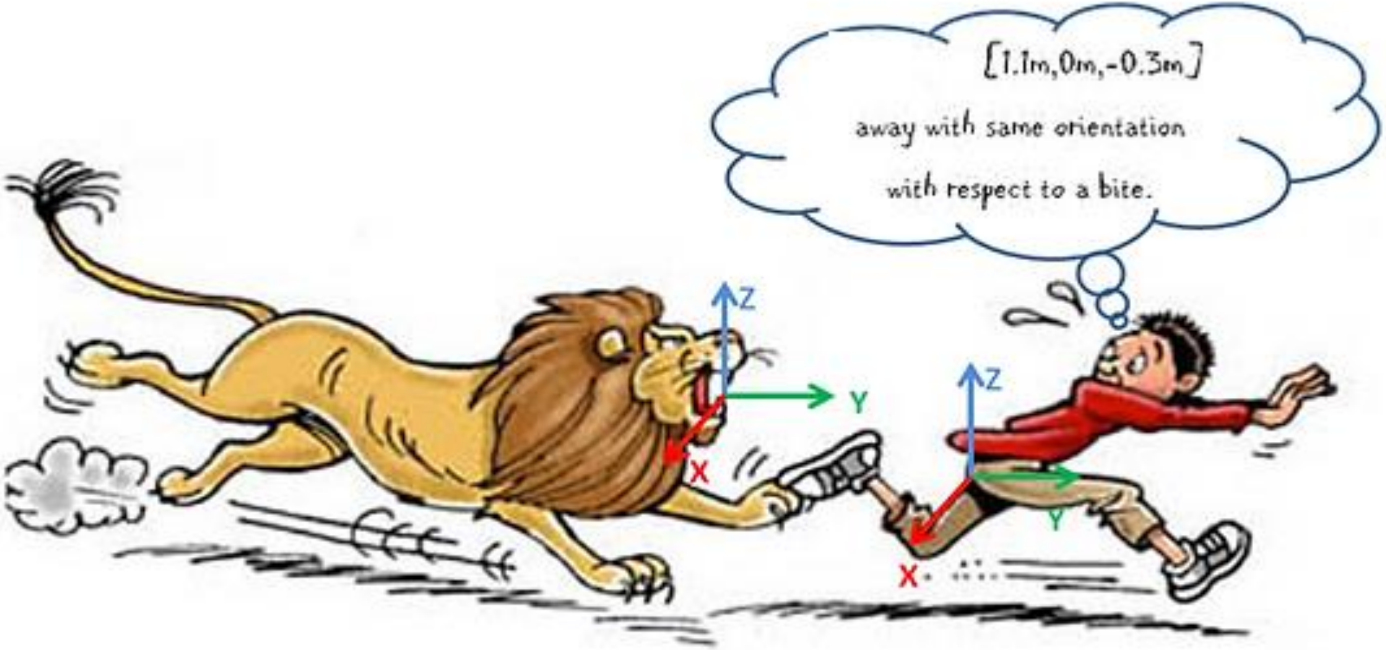
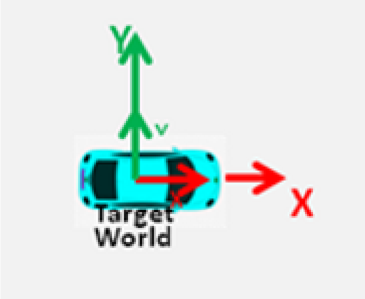


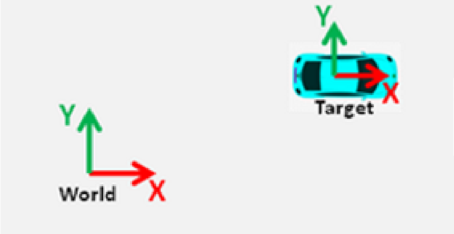
Fig. 1 3D Frames attached to objects

Fig. 2 2D Frames attached to objects


[1.1m,0m,-0.3m]
away with same orientation
with respect to a bite.
Z
Y
X
Z
Y
X



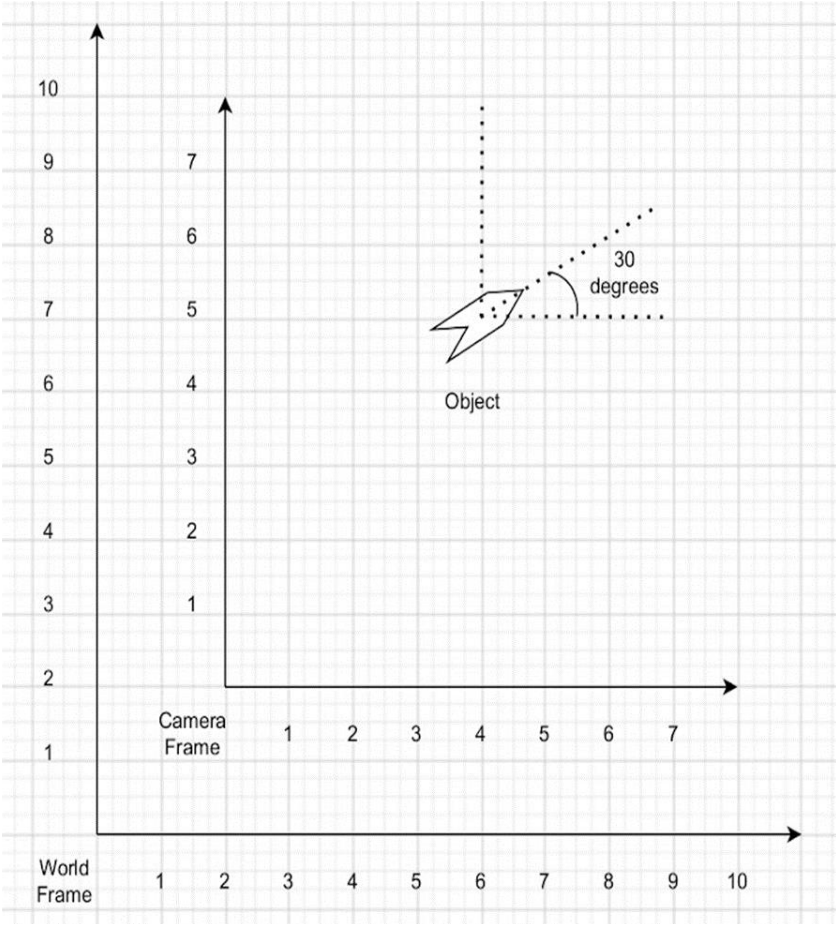
$T_{world\text{-}target}$ = (x,y,theta)= (0m,0m,0 degree)



$T_{world\text{-}target}$ = (x,y,theta)= (1m,1m,0 degree)

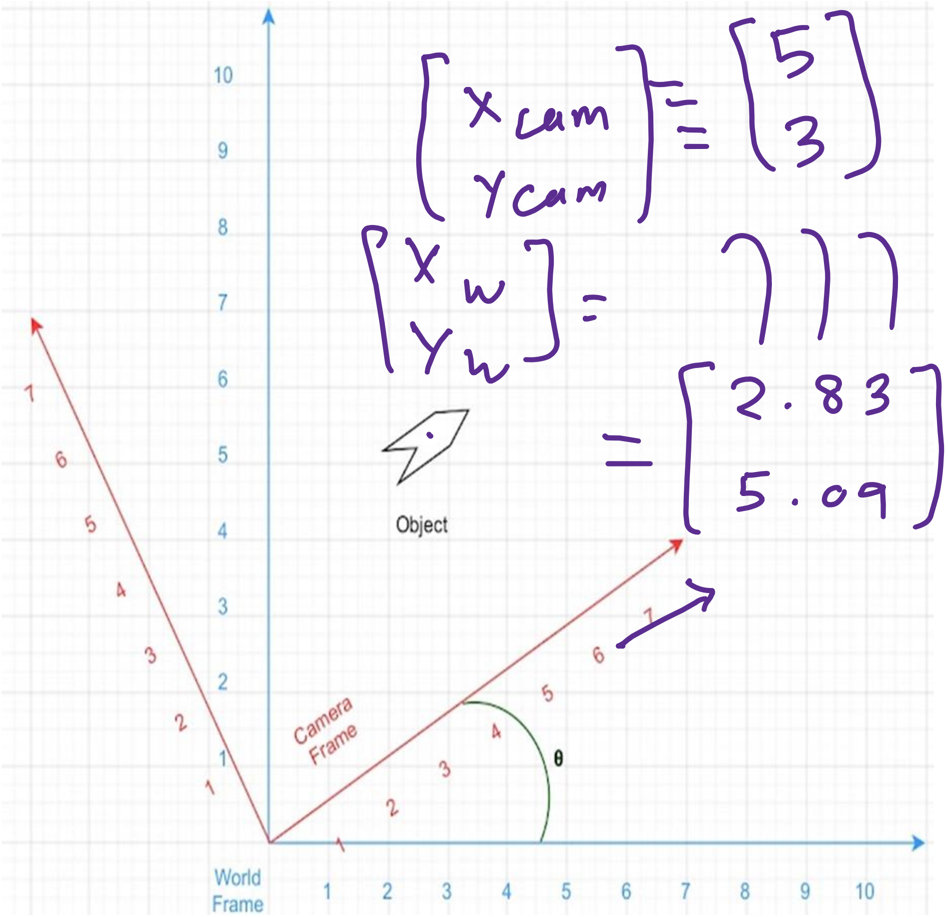
$T_{world\text{-}target}$ = (x,y,theta)= (1m,0m,45 degrees)

Define the (x,y) position of the object w.r.t Camera Frame. Also, how would you convert position of the object from Camera Frame to World Frame ? Suppose you have an arbitrary point(x,y) in the Camera Frame, define the point w.r.t World Frame using translation matrix



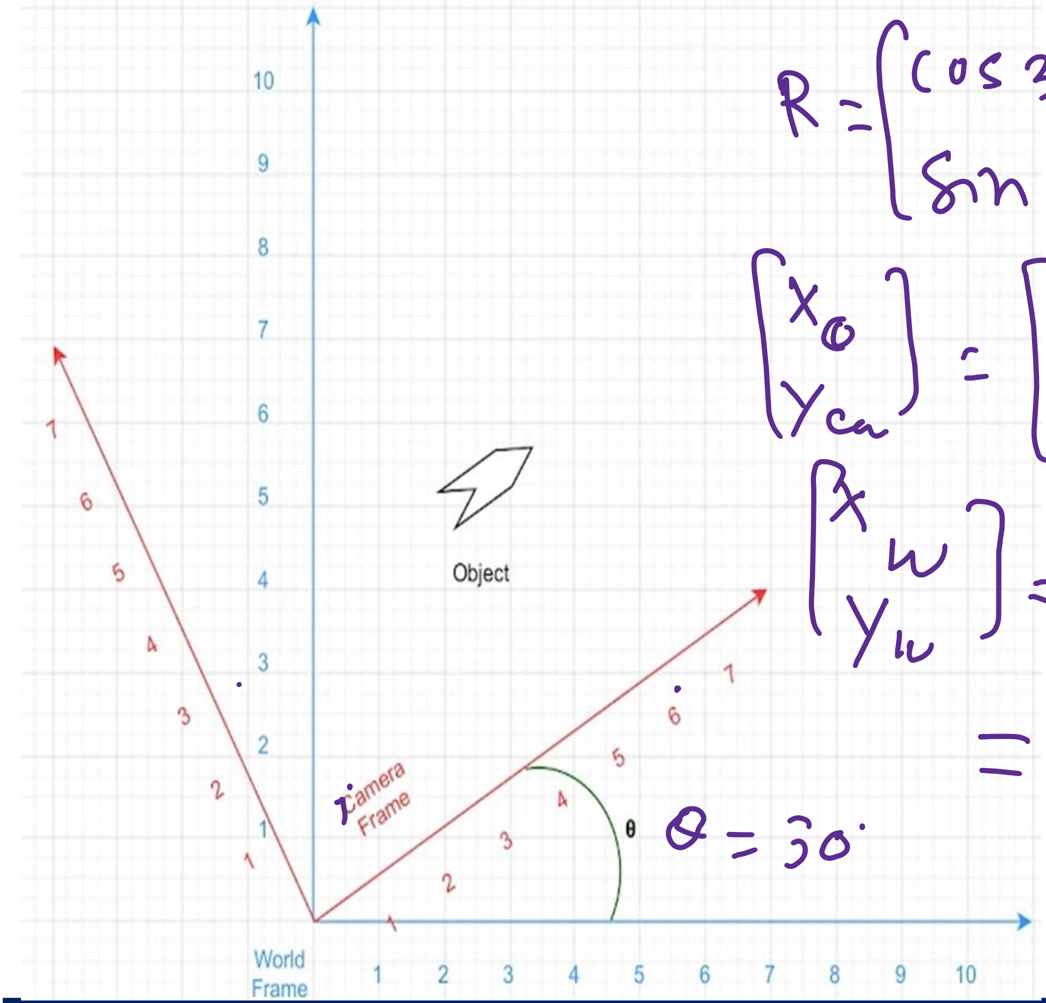
Define the (x,y) position of the object w.r.t Camera Frame. Also, how would you convert position of the object from Camera Frame to World Frame ? Suppose you have an arbitrary point(x,y) in the Camera Frame, define the point w.r.t World Frame using rotation matrix (assume (x,y) = (5,3) and θ as 30 degrees)

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$\begin{bmatrix} x_{cam} \\ y_{cam} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \qquad = \begin{bmatrix} 2.83 \\ 5.09 \end{bmatrix}$$



$\theta = 30^\circ$

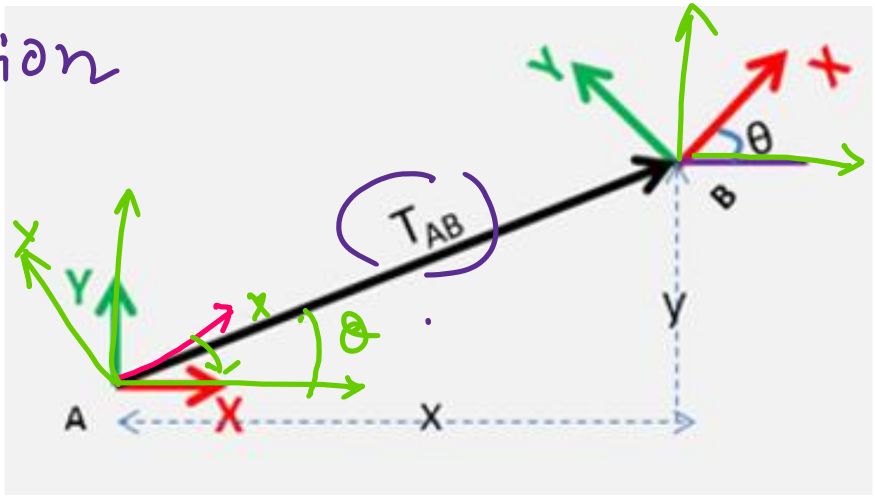
$$R = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}_{2\times 2}$$

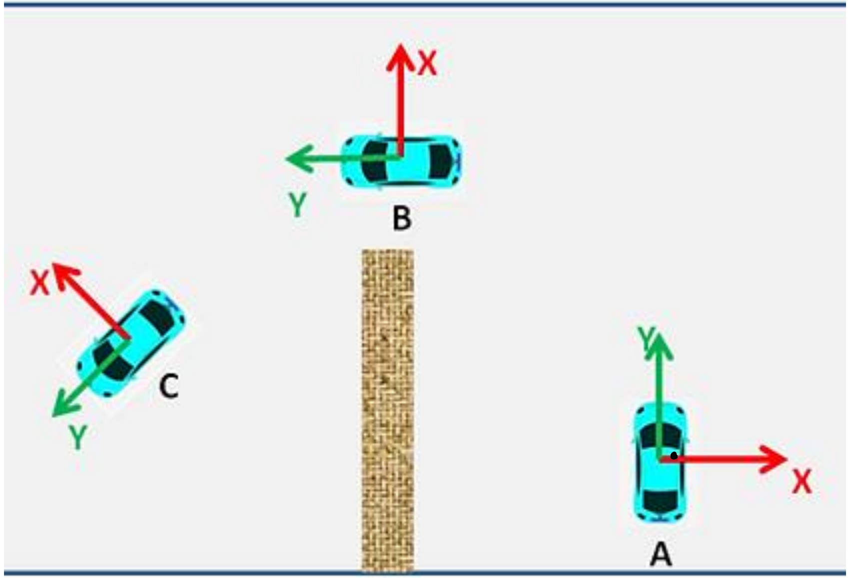
$$\begin{bmatrix} X_{o} \\ Y_{ca} \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}_{2\times 1}$$

$$\begin{bmatrix} X_{w} \\ Y_{w} \end{bmatrix} = [R] \times \begin{bmatrix} X_{c} \\ Y_{c} \end{bmatrix}$$

$$= \begin{bmatrix} 2.83 \\ 5.04 \end{bmatrix}_{2\times 1}$$

$T \rightarrow$ Transformation





Assume there are three cars in a parking area A, B and C as shown

TRANSFORMATION FROM A TO B

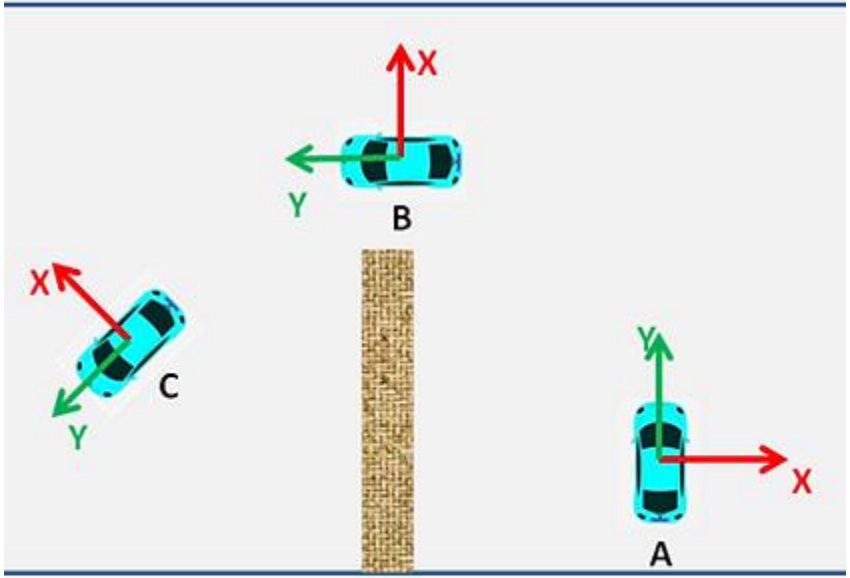
Car B is at [x1,y1] units and rotated at an angle θ1 w.r.t my car. Hence I can write the transformation matrix T_{AB} as below,

$$T_{AB} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & x_1 \\ \sin\theta_1 & \cos\theta_1 & y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(Y,\theta) = \begin{array}{c} \begin{matrix} & Y & \end{matrix} \\ \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$R(X,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 1 & \sin\theta & \cos\theta \end{bmatrix}$$



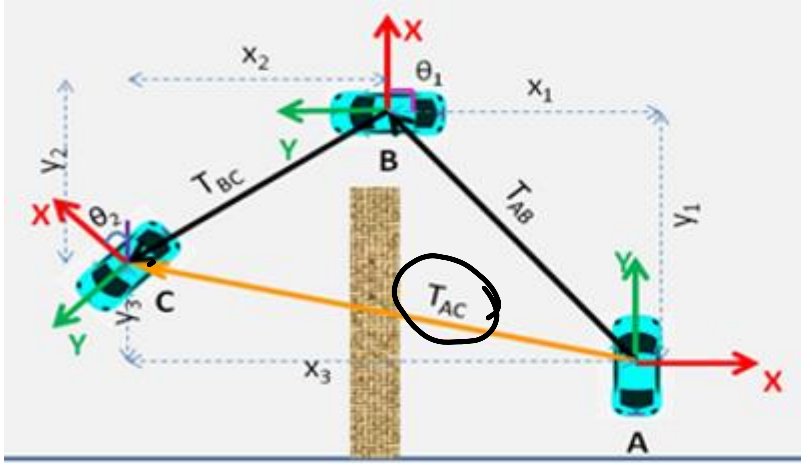
Assume there are three cars in a parking area A, B and C as shown

TRANSFORMATION BETWEEN B TO C

Similarly person in car B has figured out that car C is at $[x_2, y_2]$ units and rotated at an angle θ_2 with respect to his car B.

$$T_{BC} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & x_2 \\ \sin\theta_2 & \cos\theta_2 & y_2 \\ 0 & 0 & 1 \end{bmatrix}$$

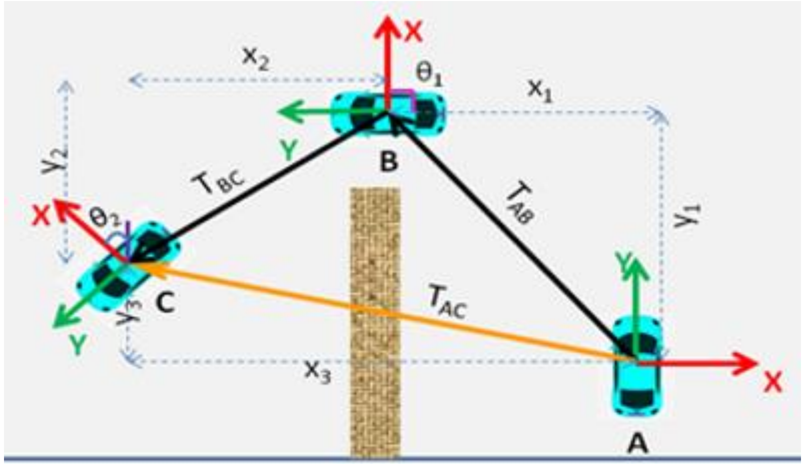
Now our task is to find out the position and orientation of car C without seeing it directly. Let's say car C is at $[x_3,y_3]$ units and rotated at an angle θ_3 with respect to my car, car A.



To find our x_3,y_3 and θ_3, we will take the help of person in car B to get T_{BC} matrix. Therefore I know T_{AB} and TBC. That's it I have all required information to calculate x_3,y_3 and θ_3.

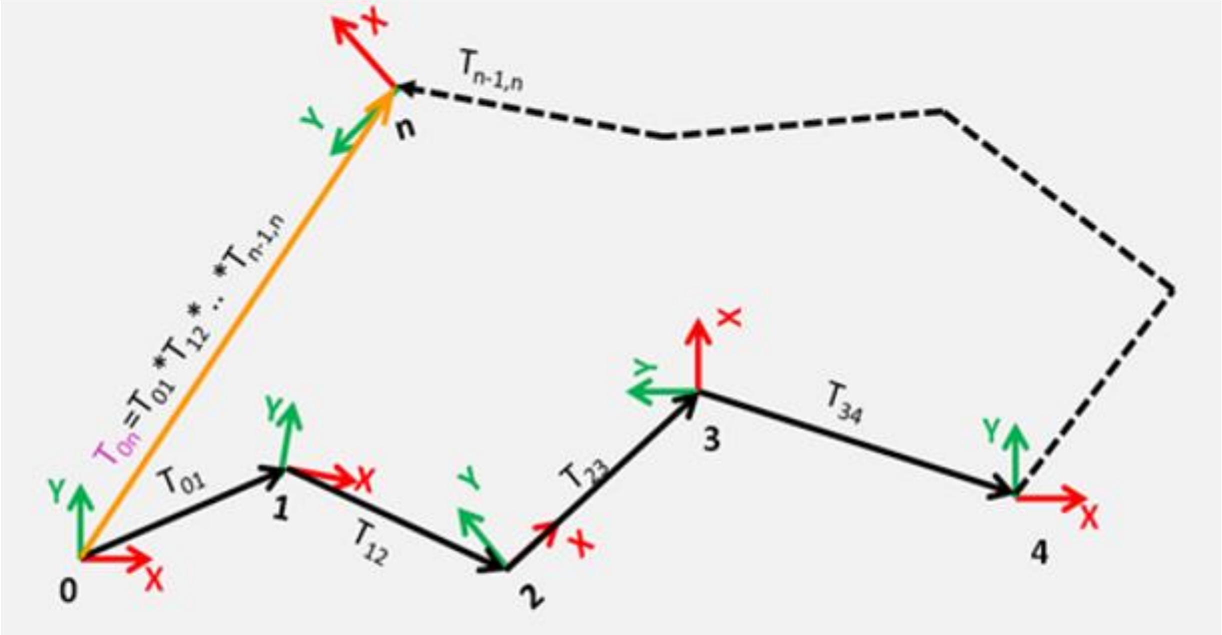
$T_{AC} =$

Transformation of Car C w.rt Car A with the help of Car B



$$T_{AC} = \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & x_1 \\ \sin\theta_1 & \cos\theta_1 & y_1 \\ 0 & 0 & 1 \end{bmatrix}}_{T_{AB}} * \underbrace{\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & x_2 \\ \sin\theta_2 & \cos\theta_2 & y_2 \\ 0 & 0 & 1 \end{bmatrix}}_{T_{BC}}$$

$$\begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & x_3 \\ \sin\theta_3 & \cos\theta_3 & y_3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 * \cos\theta_2 - \sin\theta_1 * \sin\theta_2 & -\cos\theta_1 * \sin\theta_2 - \sin\theta_1 * \cos\theta_2 & \cos\theta_1 * x_2 - \sin\theta_1 * y_2 + x_1 \\ \sin\theta 1 * \cos\theta_2 + \cos\theta_1 * \sin\theta_2 & -\sin\theta_1 * \sin\theta_2 + \cos\theta_1 * \cos\theta_2 & \sin\theta_1 * x_2 + \cos\theta_1 * y_2 + y_1 \\ 0 & 0 & 1 \end{bmatrix}$$



$$T_{0n}=T_{01}*T_{12}*T_{23}*\ldots\ldots..T_{n-2,n-1}*T_{n-1,n}$$

NOTE

2D transformations are mostly used in planar manipulators and autonomous mobile robots, self driving cars.

Time for Discussions

Thank You!