Gate Question

EE:1205 Signals and Systems
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I. Question GATE PH 56

Consider the complex function

$$f(z) = \frac{z^2 \sin z}{(z-\pi)^4}$$

At $z = \pi$, which of the following options is (are) correct?

(A) The order of the pole is 4
(B) The order of the pole is 3
(C) The residue at the pole is $\frac{\pi}{6}$
(D) The residue at the pole is $\frac{2\pi}{3}$

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II. Solution

The order of the pole is determined by the highest power of $(z-\pi)$ in the denominator. In this case, the highest power is 4, so the order of the pole is 4.

Now, let's find the residue. The residue a_{-1} is the coefficient of $(z-\pi)^{-1}$ in the Laurent series. To find a_{-1}, we can rewrite the function as:

$$f(z) = \frac{z^2 \sin z}{(z-\pi)^4} = \frac{g(z)}{(z-\pi)^3} \tag{1}$$

$$\tag{2}$$

where $g(z) = z^2 \sin z$.

Now, we can use the formula for the n-th coefficient in the Laurent series:

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-c)^{n+1}}\, dz. \tag{3}$$

For n = -1, this becomes:

$$a_{-1} = \frac{1}{2\pi i} \oint_C \frac{g(z)}{(z-\pi)^3}\, dz. \tag{4}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C \frac{z^2 \sin z}{(z-\pi)^3}\, dz. \tag{5}$$

This is a standard residue calculation, and the result will be the coefficient of $(z-\pi)^{-1}$,
which is the residue at $z = \pi$.

$$a_{-1} = \frac{2\pi}{2!} \left(\frac{d^2}{dz^2} \sin z \right)_{z=\pi}. \tag{6}$$

$$a_{-1} = \frac{\pi}{3}. \tag{7}$$