Gate Question

EE:1205 Signals and Systems
Indian Institute of Technology, Hyderabad

Abhey Garg
EE23BTECH11202

I. Question GATE PH 56

Consider the complex function

$$f(z) = \frac{z^2 \sin z}{(z-\pi)^4}$$

At $z = \pi$, which of the following options is (are) correct?

(A) The order of the pole is 4
(B) The order of the pole is 3
(C) The residue at the pole is $\frac{\pi}{6}$
(D) The residue at the pole is $\frac{2\pi}{3}$

(GATE PH 2023)

II. Solution

TABLE 1
Input Parameters

Parameter	Used to denote	Values
m	order of pole at z = π	?
$Res(f,\pi)$	Residue of pole	?

$$\text{m = minimum k such that } lim_{z\to\pi}(z-\pi)^3 f(z) \text{ is finite} \tag{1}$$

$$= \lim_{z\to\pi}(z-\pi)^3 \left(\frac{z^2 \sin z}{(z-\pi)^4}\right) \tag{2}$$

$$= \lim_{z\to\pi}\left(\frac{z^2 \sin z}{(z-\pi)}\right) \tag{3}$$

$$= \lim_{z\to\pi}\left(\frac{2z\sin(z) - z^2\cos(z)}{1}\right) \tag{4}$$

Which is a finite value , so $m = 3$

$$\text{Res}(f,\pi) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-\pi)^m f(z)\right]\Big|_{z=\pi} \tag{5}$$

$$\text{Res}(f,\pi) = \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-\pi)^3\frac{z^2\sin z}{(z-\pi)^4}\right]\Bigg|_{z=\pi} \tag{6}$$

$$\text{Res}(f,\pi) = \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{z^2\sin z}{(z-\pi)}\right]\Bigg|_{z=\pi} \tag{7}$$

$$= \frac{1}{2!}\left(-\frac{z^2-2z(z-\pi)}{(z-\pi)^2}\right)\cos(z) + \left(\frac{z^2-2}{z-\pi} - \frac{2z^2}{(z-\pi)^3} - \frac{4z}{(z-\pi)^2}\right)\sin(z) \tag{8}$$

Applying L'Hopital rule Since $\sin(\pi) = 0$ and $\cos(\pi) = -1$, this simplifies to:

$$\text{Res}(f,\pi) = \frac{1}{2!}\frac{\pi^2-6}{3} = \frac{\pi^2-6}{6} \tag{9}$$