SIGNAL PROCESSING

Through GATE

EE1205-TA Group

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Contents

Introduction

This book provides solutions to signal processing problems in GATE.

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Chapter 3

Z-transform

3.1 Consider the following recursive iteration scheme for different values of variable P with the initial guess $x_1 = 1$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{P}{x_n}\right), \qquad n = 1, 2, 3, 4, 5$$

For $P = 2$, x_5 is obtained to be 1.414, rounded off to 3 decimal places. For $P = 3$, x_5 is obtained to be 1.732, rounded off to 3 decimal places.

If $P = 10$, the numerical value of x_5 is ______ . (*round off to three decimal places*) (GATE CE 2022)

Solution:

Applying $A.M \geq G.M$ inequality,

$$\frac{x_n + \frac{P}{x_n}}{2} \geq \sqrt{P} \tag{3.1}$$

$$\implies x_{n+1} \geq \sqrt{P} \tag{3.2}$$

Solving the equation,

$$2x_{n+1}x_n - {x_n}^2 - P = 0 \tag{3.3}$$

Applying Z-transform we get,

$$X(z) * X(z) = \frac{PZ^{-1}}{(1 - z^{-1})(2 - z^{-1})} \tag{3.4}$$

$$= P\left(\frac{z^{-1}}{1 - z^{-1}} - \frac{z^{-1}}{2 - z^{-1}}\right) \tag{3.5}$$

From the transformation pairs,

$$x_{n-a} \overset{\mathcal{Z}}{\longleftrightarrow} z^{-a}X(z) \tag{3.6}$$

$$x_{n_1} \times x_{n_2} \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) * X_2(z) \tag{3.7}$$

$$\frac{u(n-1)}{a^n} \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{a - z^{-1}} \tag{3.8}$$

Now, applying inverse Z-tranform,

$$x_n^2 = P\left(u(n-1) - \frac{u(n-1)}{2^n}\right) \tag{3.9}$$

$$\implies x_n^2 = P\left(1 - \frac{1}{2^n}\right) \quad [\because n \geq 1] \tag{3.10}$$

Similarly,

$$x_{n+1}^2 = P\left(1 - \frac{1}{2^{n+1}}\right) \tag{3.11}$$

$$\implies \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \sqrt{\frac{P\left(1 - \frac{1}{2^n}\right)}{P\left(1 - \frac{1}{2^{n+1}}\right)}} \tag{3.12}$$

$$= 1 \tag{3.13}$$

Hence, the system is convergent.

Now finding the limit of the sequence,

$$x^2 = \lim_{x \to \infty} P\left(1 - \frac{1}{2^n}\right) \tag{3.14}$$

$$\implies x = \pm\sqrt{P} \tag{3.15}$$

From (3.2) and (3.15),

$$x_{n+1} = \sqrt{P} \tag{3.16}$$

Therefore, for $P = 10$ the value of x_5 is,

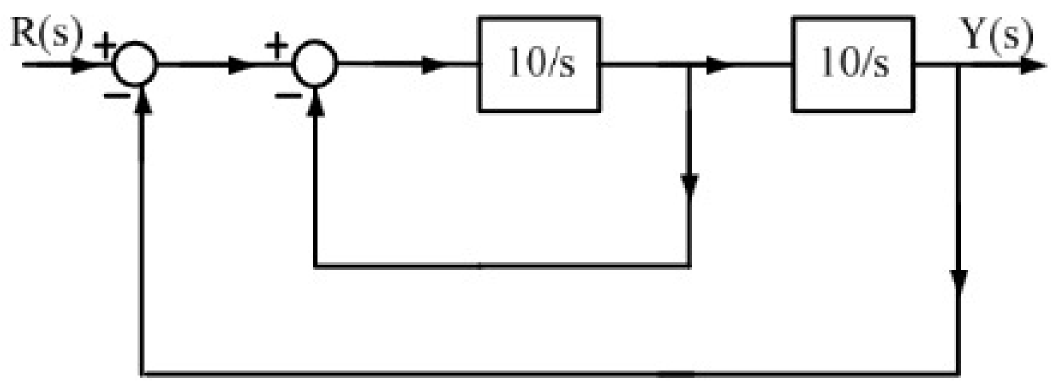
$$x_5 = \sqrt{10} \tag{3.17}$$

$$\therefore x_5 = 3.162 \tag{3.18}$$

Chapter 4

Systems

4.1 The damping ratio and undamped natural frequency of a closed loop system as shown in the figure, are denoted as ζ and ω_n, respectively. The values of ζ and ω_n are



(a) $\zeta = 0.5$ and $\omega_n = 10$ rad/s

(b) $\zeta = 0.1$ and $\omega_n = 10$ rad/s

(c) $\zeta = 0.707$ and $\omega_n = 10$ rad/s

(d) $\zeta = 0.707$ and $\omega_n = 100$ rad/s

(GATE EE 2022) **Solution:**

We will use Mason's Gain Formula to calculate the transfer function of this system.

Parameter	**Description**	**Values**
m	load of system	
k	stiffness of system	
ω_n	Natural frequency	$\sqrt{\frac{k}{m}}$
ζ	Damping ratio	$\frac{c}{2m\omega_n}$
y(t)	Output of system	
x(t)	Input to the system	
c	Damping coefficient	
T(s)	Transfer function of system	$\frac{Y(s)}{R(s)}$

Table 4.1: Parameter Table

First converting the given diagram to a signal flow graph :

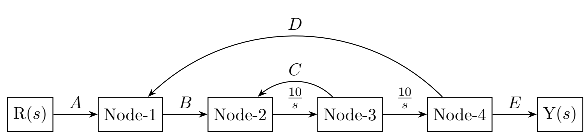


Figure 4.1: Signal Flow Diagram

Mason's Gain Formula is given by :

$$H\left(s\right) = \sum_{i=1}^{N} \left(\frac{P_i \Delta_i}{\Delta}\right) \tag{4.1}$$

This signal flow graph has only one forward path whose gain is given by :

$$P_1 = \frac{10}{s}\frac{10}{s} \tag{4.2}$$
$$= \frac{100}{s^2} \tag{4.3}$$

Parameter	Description
N	Number of forward paths
L	Number of loops
P_k	Forward path gain of k^{th} path
Δ_k	Associated path factor
Δ	Determinant of the graph

Table 4.2: Parameter Table - Mason's Gain Law

Parameter	Formula
Δ	$1 + \sum_{k=1}^{L} \left((-1)^k \text{ Product of gain of groups of k isolated loops} \right)$
Δ_k	Δ part of graph that is not touching k^{th} forward path

Table 4.3: Formula Table - Mason's Gain Law

The loop gain for loop between Node-2 and Node-3 is :

$$L_1 = \frac{10}{s}(-1) \tag{4.4}$$

$$= -\frac{10}{s} \tag{4.5}$$

The loop gain for loop between Node-1 and Node-4 is :

$$L_1 = \frac{10}{s}\frac{10}{s}(-1) \tag{4.6}$$

$$= -\frac{100}{s^2} \tag{4.7}$$

Using Table 4.3, Δ is :

$$\Delta = 1 - \left(-\frac{10}{s} - \frac{100}{s^2}\right) \tag{4.8}$$

$$= 1 + \frac{10}{s} + \frac{100}{s^2} \tag{4.9}$$

There are no two isolated loops available. Hence all further terms will b zero.

As both the loops are in contact with the only forward path,

$$\Delta_1 = 1 \tag{4.10}$$

Using equation (4.1) :

$$H\left(s\right) = \frac{\frac{100}{s^2}}{1 + \frac{10}{s} + \frac{100}{s^2}} \tag{4.11}$$

$$= \frac{100}{s^2 + 10s + 100} \tag{4.12}$$

Referring to Table 4.1, the general equation of the damping system is second order and can be written as :

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = x(t) \tag{4.13}$$

Take the Laplace transform and solve for $\frac{Y(s)}{X(s)}$:

$$\frac{Y\left(s\right)}{X\left(s\right)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.14}$$

$$\implies H\left(s\right) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.15}$$

Comparing equations (4.12) and (4.15) ,

$$\omega_n^2 = 100 \tag{4.16}$$

$$\implies \omega_n = 10 \text{ rad/s} \tag{4.17}$$

$$2\zeta\omega_n = 10 \tag{4.18}$$

$$\implies \zeta = 0.5 \tag{4.19}$$

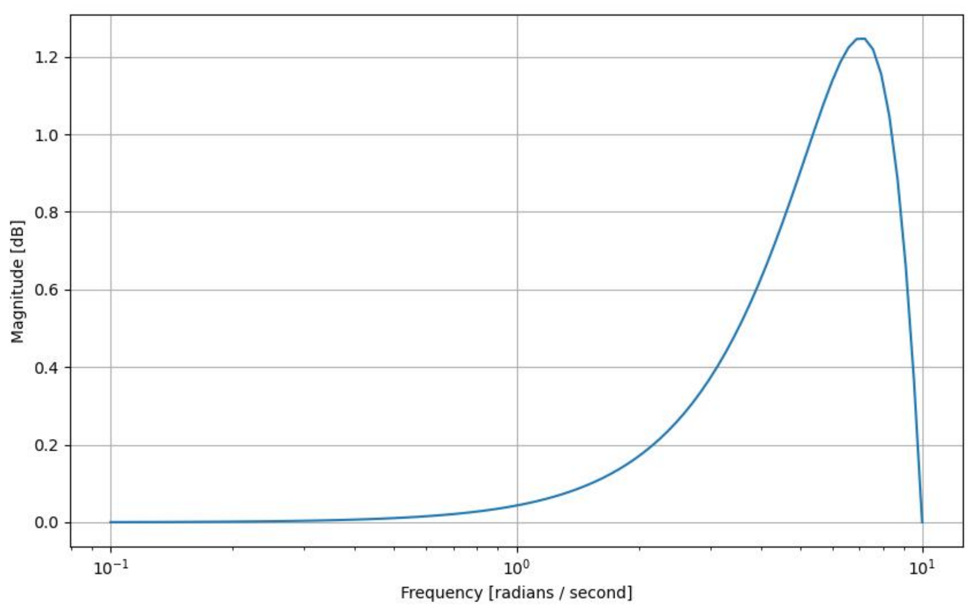


Figure 4.2: Magnitude plot

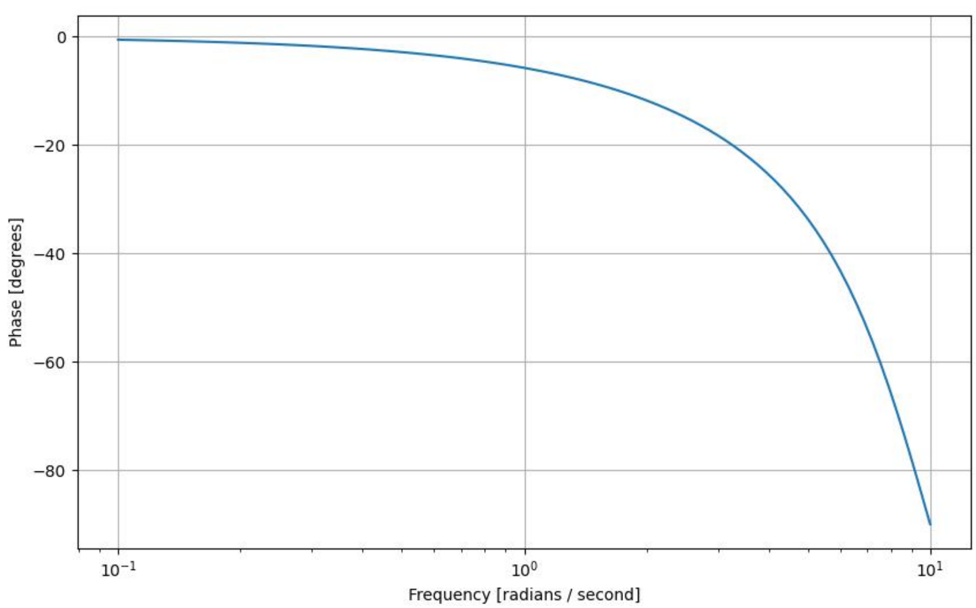


Figure 4.3: Phase plot

Chapter 5

5.1

Chapter 6

Sampling

6.1

Chapter 7

Contour Integration

7.1

Chapter 8

Laplace Transform

8.1 Consider the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$. The boundary conditions are $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$. Then the value of y at $x = \frac{1}{2}$ (GATE AE 2022)

Solution:

Parameters	**Values**	**Description**
$y(0)$	0	y at $x = 0$
$y'(0)$	1	$\frac{dy}{dx}$ at $x = 0$

Table 8.1: Parameters

$$\frac{d^2y}{dx^2} \xleftrightarrow{\mathcal{L}} s^2Y(s) - sy(0) - y'(0) \tag{8.1}$$

$$\frac{dy}{dx} \xleftrightarrow{\mathcal{L}} sY(s) - y(0) \tag{8.2}$$

Applying Laplace Transform, using (8.1) and (8.2),

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + Y(s) = 0 \tag{8.3}$$

From Table 8.1,

$$(s^2 - 2s + 1)Y(s) - 1 = 0 \tag{8.4}$$

$$Y(s) = \frac{1}{(s-1)^2} \tag{8.5}$$

$$t^n \overset{\mathcal{L}}{\longleftrightarrow} \frac{n!}{s^{n+1}} \tag{8.6}$$

$$e^{at}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s-a) \tag{8.7}$$

Taking Inverse Laplace Transform for $Y(s)$, using (8.6) and (8.7),

$$y(x) = xe^x \tag{8.8}$$

$$\implies y\left(\frac{1}{2}\right) = \frac{\sqrt{e}}{2} \tag{8.9}$$

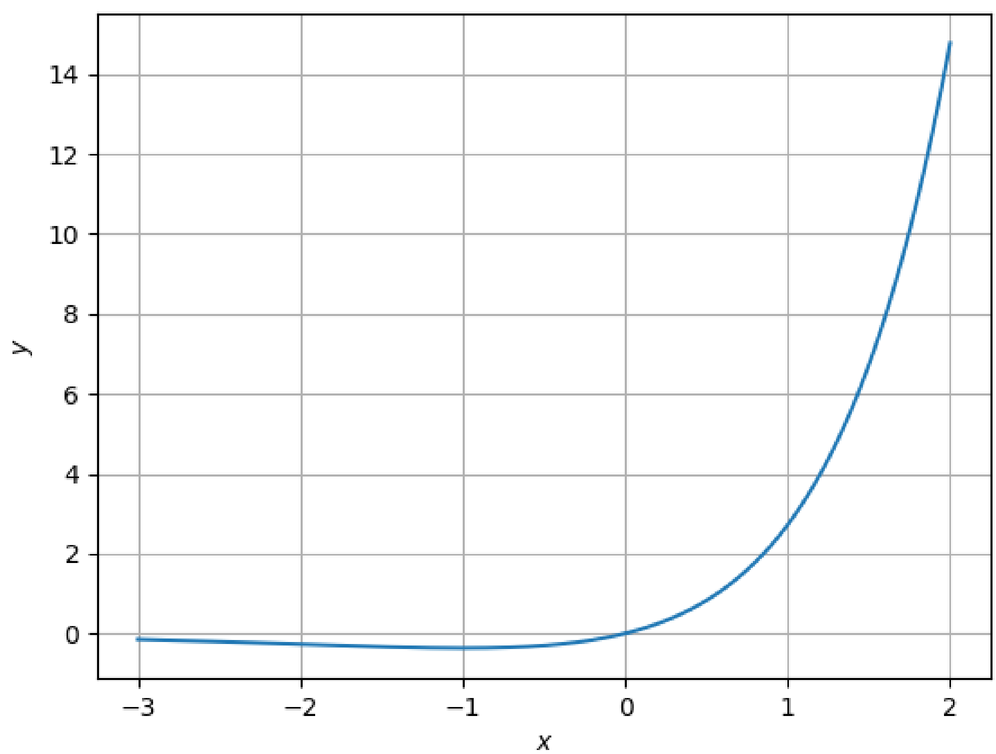


Figure 8.1: Plot of $y(x)$

8.2 A process described by the transfer function

$$G_p(s) = \frac{(10s+1)}{(5s+1)}$$

is forced by a unit step input at time $t = 0$. The output value immediately after the unit step input (at $t = 0^+$) is ? (Gate 2022 CH 34)

Solution:

Parameters	**Description**
$X(s)$	Laplace transform of $x(t)$
$Y(s)$	Laplace transform of $y(t)$
$G_p(s) = \frac{Y(s)}{X(s)}$	Transfer function
$x(t) = u(t)$	unit step function

Table 8.2: Given parameters

$$G_p(s) = \frac{Y(s)}{X(s)} = \frac{(10s+1)}{(5s+1)} \tag{8.10}$$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \tag{8.11}$$

From equation (8.11):

$$Y(s) = \frac{(10s+1)}{s(5s+1)} \tag{8.12}$$

$$= \frac{1}{s} + \frac{5}{5s+1} \tag{8.13}$$

Taking inverse laplace transformation,

$$\frac{1}{s} \xleftrightarrow{\mathcal{L}^{-1}} u(t) \tag{8.14}$$

$$\frac{1}{s-c} \xleftrightarrow{\mathcal{L}^{-1}} e^{ct}u(t) \tag{8.15}$$

$$y(t) = \left(1 + e^{\frac{-t}{5}}\right) u(t) \tag{8.16}$$

$$y(0^+) = 2 \tag{8.17}$$

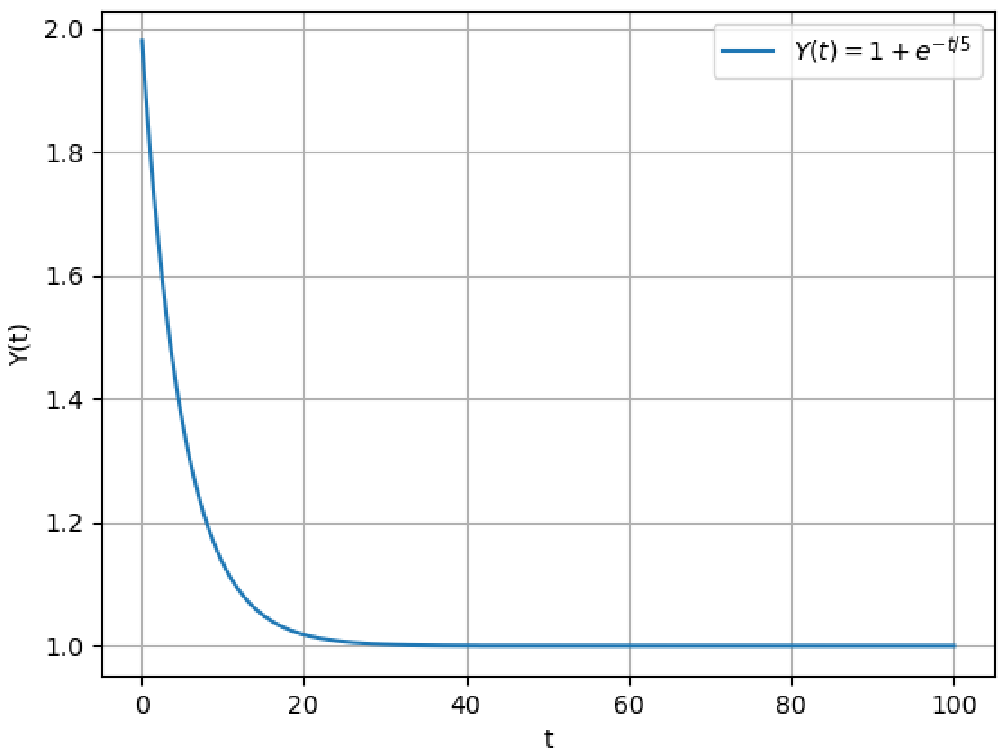


Figure 8.2: Graph of y(t)

Chapter 9

Fourier transform

9.1