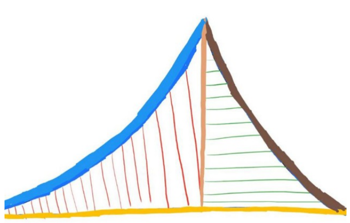
SIGNAL PROCESSING

Through GATE

EE1205-TA Group

Author: Sayyam Palrecha



Contents

Introduction

This book provides solutions to signal processing problems in GATE.

Chapter 1

Harmonics

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Chapter 3

Z-transform

3.1 Consider the following recursive iteration scheme for different values of variable P with the initial guess $x_1 = 1$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{P}{x_n}\right), \qquad n = 1, 2, 3, 4, 5$$

For $P = 2$, x_5 is obtained to be 1.414, rounded off to 3 decimal places. For $P = 3$, x_5 is obtained to be 1.732, rounded off to 3 decimal places.

If $P = 10$, the numerical value of x_5 is ______ . (*round off to three decimal places*)

(GATE CE 2022)

Solution:

Applying $A.M \geq G.M$ inequality,

$$\frac{x_n + \frac{P}{x_n}}{2} \geq \sqrt{P} \tag{3.1}$$

$$\implies x_{n+1} \geq \sqrt{P} \tag{3.2}$$

Solving the equation,

$$2x_{n+1}x_n - {x_n}^2 - P = 0 \tag{3.3}$$

Applying Z-transform we get,

$$X(z) * X(z) = \frac{PZ^{-1}}{(1 - z^{-1})(2 - z^{-1})} \tag{3.4}$$

$$= P\left(\frac{z^{-1}}{1 - z^{-1}} - \frac{z^{-1}}{2 - z^{-1}}\right) \tag{3.5}$$

From the transformation pairs,

$$x_{n-a} \overset{\mathcal{Z}}{\longleftrightarrow} z^{-a}X(z) \tag{3.6}$$

$$x_{n_1} \times x_{n_2} \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) * X_2(z) \tag{3.7}$$

$$\frac{u(n-1)}{a^n} \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{a - z^{-1}} \tag{3.8}$$

Now, applying inverse Z-tranform,

$$x_n^2 = P\left(u(n-1) - \frac{u(n-1)}{2^n}\right) \tag{3.9}$$

$$\implies x_n^2 = P\left(1 - \frac{1}{2^n}\right) \quad [\because n \geq 1] \tag{3.10}$$

Similarly,

$$x_{n+1}^2 = P\left(1 - \frac{1}{2^{n+1}}\right) \tag{3.11}$$

$$\implies \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \sqrt{\frac{P\left(1 - \frac{1}{2^n}\right)}{P\left(1 - \frac{1}{2^{n+1}}\right)}} \tag{3.12}$$

$$= 1 \tag{3.13}$$

Hence, the system is convergent.

Now finding the limit of the sequence,

$$x^2 = \lim_{x \to \infty} P\left(1 - \frac{1}{2^n}\right) \tag{3.14}$$

$$\implies x = \pm\sqrt{P} \tag{3.15}$$

From (3.2) and (3.15),

$$x_{n+1} = \sqrt{P} \tag{3.16}$$

Therefore, for $P = 10$ the value of x_5 is,

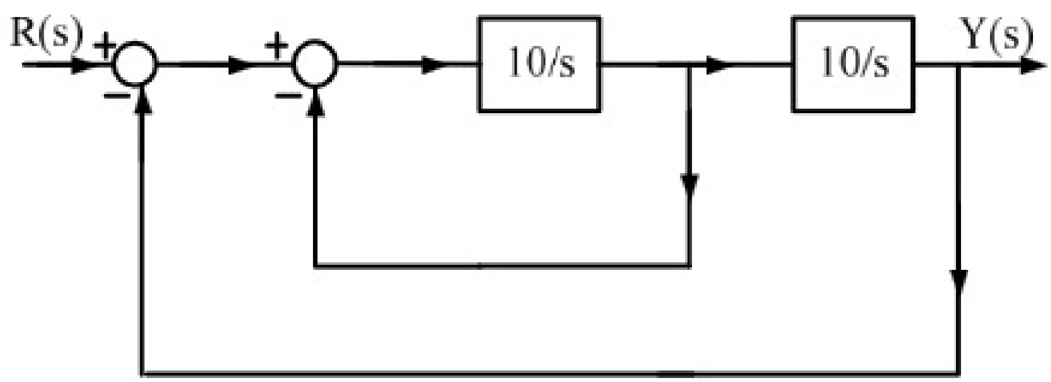
$$x_5 = \sqrt{10} \tag{3.17}$$

$$\therefore x_5 = 3.162 \tag{3.18}$$

Chapter 4

Systems

4.1 The damping ratio and undamped natural frequency of a closed loop system as shown in the figure, are denoted as ζ and ω_n, respectively. The values of ζ and ω_n are



(a) $\zeta = 0.5$ and $\omega_n = 10$ rad/s

(b) $\zeta = 0.1$ and $\omega_n = 10$ rad/s

(c) $\zeta = 0.707$ and $\omega_n = 10$ rad/s

(d) $\zeta = 0.707$ and $\omega_n = 100$ rad/s

(GATE EE 2022) **Solution:**

We will use Mason's Gain Formula to calculate the transfer function of this system.

Parameter	**Description**	**Values**
m	load of system	
k	stiffness of system	
ω_n	Natural frequency	$\sqrt{\frac{k}{m}}$
ζ	Damping ratio	$\frac{c}{2m\omega_n}$
y(t)	Output of system	
x(t)	Input to the system	
c	Damping coefficient	
T(s)	Transfer function of system	$\frac{Y(s)}{R(s)}$

Table 4.1: Parameter Table

First converting the given diagram to a signal flow graph :

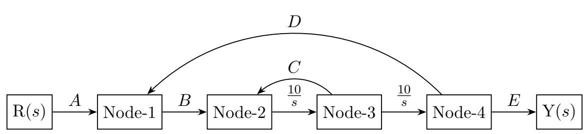


Figure 4.1: Signal Flow Diagram

Mason's Gain Formula is given by :

$$H(s) = \sum_{i=1}^{N} \left(\frac{P_i \Delta_i}{\Delta} \right) \tag{4.1}$$

This signal flow graph has only one forward path whose gain is given by :

$$P_1 = \frac{10}{s}\frac{10}{s} \tag{4.2}$$

$$= \frac{100}{s^2} \tag{4.3}$$

Parameter	**Description**
N	Number of forward paths
L	Number of loops
P_k	Forward path gain of k^{th} path
Δ_k	Associated path factor
Δ	Determinant of the graph

Table 4.2: Parameter Table - Mason's Gain Law

Parameter	**Formula**
Δ	$1 + \sum_{k=1}^{L} \left((-1)^k \text{ Product of gain of groups of k isolated loops} \right)$
Δ_k	Δ part of graph that is not touching k^{th} forward path

Table 4.3: Formula Table - Mason's Gain Law

The loop gain for loop between Node-2 and Node-3 is :

$$L_1 = \frac{10}{s}(-1) \tag{4.4}$$
$$= -\frac{10}{s} \tag{4.5}$$

The loop gain for loop between Node-1 and Node-4 is :

$$L_1 = \frac{10}{s}\frac{10}{s}(-1) \tag{4.6}$$
$$= -\frac{100}{s^2} \tag{4.7}$$

Using Table 4.3, Δ is :

$$\Delta = 1 - \left(-\frac{10}{s} - \frac{100}{s^2}\right) \tag{4.8}$$
$$= 1 + \frac{10}{s} + \frac{100}{s^2} \tag{4.9}$$

There are no two isolated loops available. Hence all further terms will b zero.

As both the loops are in contact with the only forward path,

$$\Delta_1 = 1 \tag{4.10}$$

Using equation (4.1) :

$$H(s) = \frac{\frac{100}{s^2}}{1 + \frac{10}{s} + \frac{100}{s^2}} \tag{4.11}$$
$$= \frac{100}{s^2 + 10s + 100} \tag{4.12}$$

Referring to Table 4.1, the general equation of the damping system is second order and can be written as :

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = x(t) \tag{4.13}$$

Take the Laplace transform and solve for $\frac{Y(s)}{X(s)}$:

$$\frac{Y(s)}{X(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.14}$$
$$\implies H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.15}$$

Comparing equations (4.12) and (4.15) ,

$$\omega_n^2 = 100 \tag{4.16}$$

$$\implies \omega_n = 10 \text{ rad/s} \tag{4.17}$$

$$2\zeta\omega_n = 10 \tag{4.18}$$

$$\implies \zeta = 0.5 \tag{4.19}$$

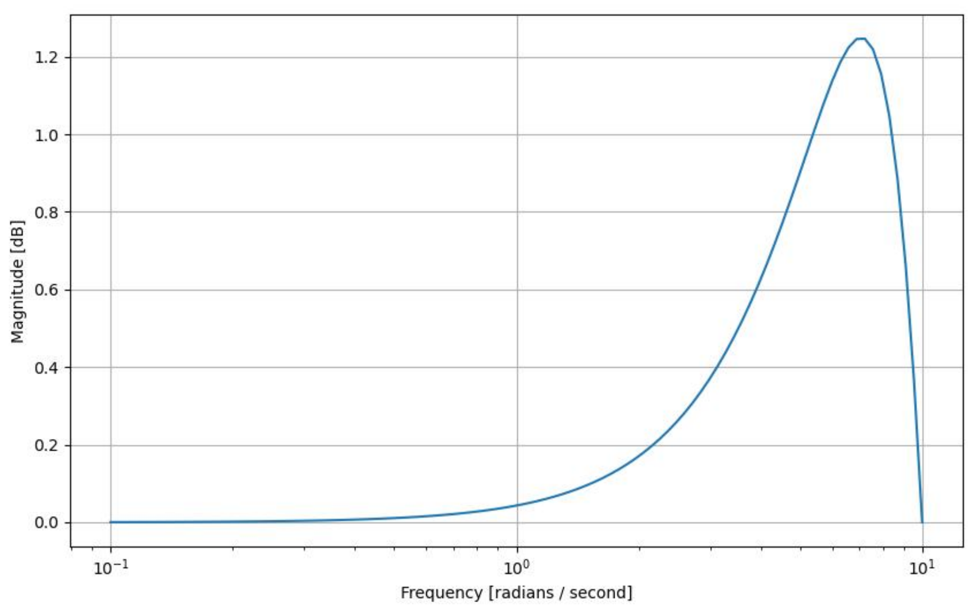


Figure 4.2: Magnitude plot

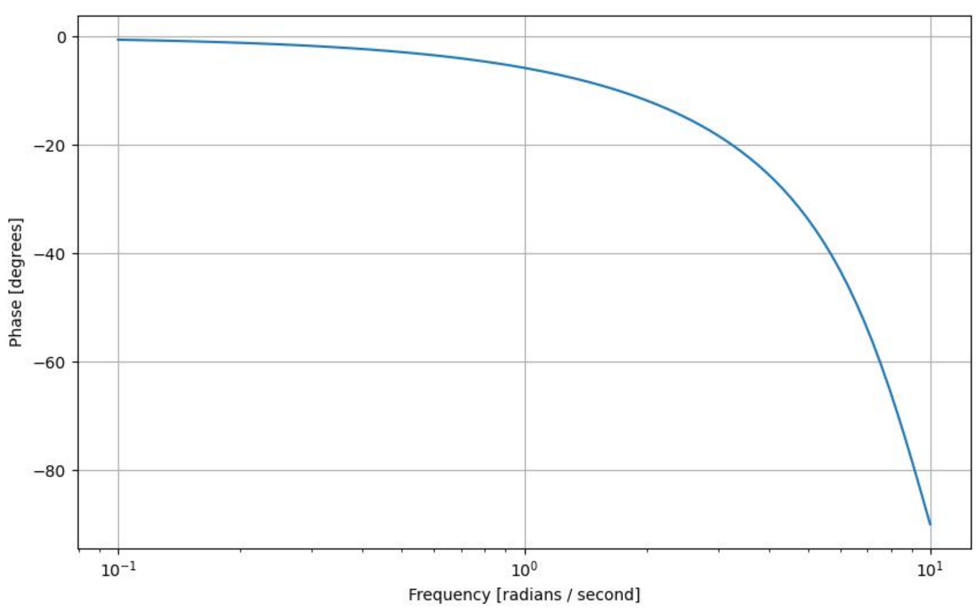


Figure 4.3: Phase plot

Chapter 5

5.1

Chapter 6

Sampling

6.1

Chapter 7

Contour Integration

7.1

Chapter 8

Laplace Transform

8.1 Consider the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$. The boundary conditions are $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$. Then the value of y at $x = \frac{1}{2}$ (GATE AE 2022)

Solution:

Parameters	**Values**	**Description**
$y(0)$	0	y at $x = 0$
$y'(0)$	1	$\frac{dy}{dx}$ at $x = 0$

Table 8.1: Parameters

$$\frac{d^2y}{dx^2} \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - sy(0) - y'(0) \tag{8.1}$$
$$\frac{dy}{dx} \overset{\mathcal{L}}{\longleftrightarrow} sY(s) - y(0) \tag{8.2}$$

Applying Laplace Transform, using (8.1) and (8.2),

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + Y(s) = 0 \tag{8.3}$$

From Table 8.1,

$$(s^2 - 2s + 1)Y(s) - 1 = 0 \tag{8.4}$$

$$Y(s) = \frac{1}{(s-1)^2} \tag{8.5}$$

$$t^n \xleftrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}} \tag{8.6}$$

$$e^{at}x(t) \xleftrightarrow{\mathcal{L}} X(s-a) \tag{8.7}$$

Taking Inverse Laplace Transform for $Y(s)$, using (8.6) and (8.7),

$$y(x) = xe^x \tag{8.8}$$

$$\implies y\left(\frac{1}{2}\right) = \frac{\sqrt{e}}{2} \tag{8.9}$$

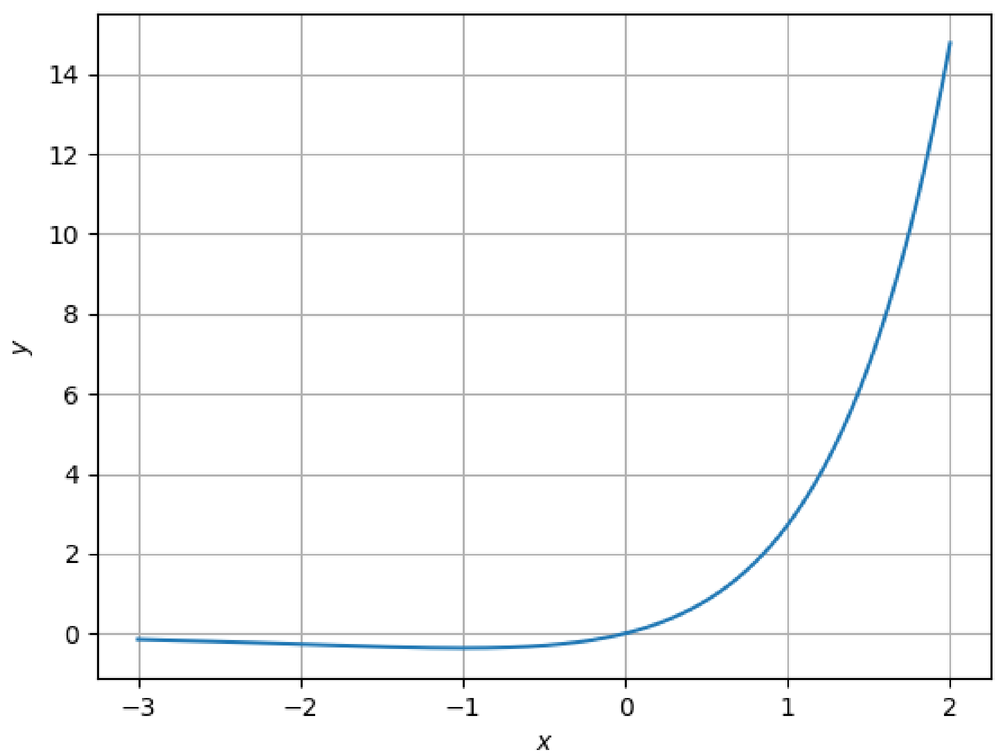


Figure 8.1: Plot of $y(x)$

8.2 A process described by the transfer function

$$G_p(s) = \frac{(10s+1)}{(5s+1)}$$

is forced by a unit step input at time $t = 0$. The output value immediately after the unit step input (at $t = 0^+$) is ? (Gate 2022 CH 34)

Solution:

Parameters	**Description**
$X(s)$	Laplace transform of $x(t)$
$Y(s)$	Laplace transform of $y(t)$
$G_p(s) = \frac{Y(s)}{X(s)}$	Transfer function
$x(t) = u(t)$	unit step function

Table 8.2: Given parameters

$$G_p(s) = \frac{Y(s)}{X(s)} = \frac{(10s+1)}{(5s+1)} \tag{8.10}$$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \tag{8.11}$$

From equation (8.11):

$$Y(s) = \frac{(10s+1)}{s\,(5s+1)} \tag{8.12}$$

$$= \frac{1}{s} + \frac{5}{5s+1} \tag{8.13}$$

Taking inverse laplace transformation,

$$\frac{1}{s} \xleftrightarrow{\mathcal{L}^{-1}} u(t) \tag{8.14}$$

$$\frac{1}{s-c} \xleftrightarrow{\mathcal{L}^{-1}} e^{ct}u(t) \tag{8.15}$$

$$y(t) = \left(1 + e^{\frac{-t}{5}}\right) u(t) \tag{8.16}$$

$$y(0^+) = 2 \tag{8.17}$$

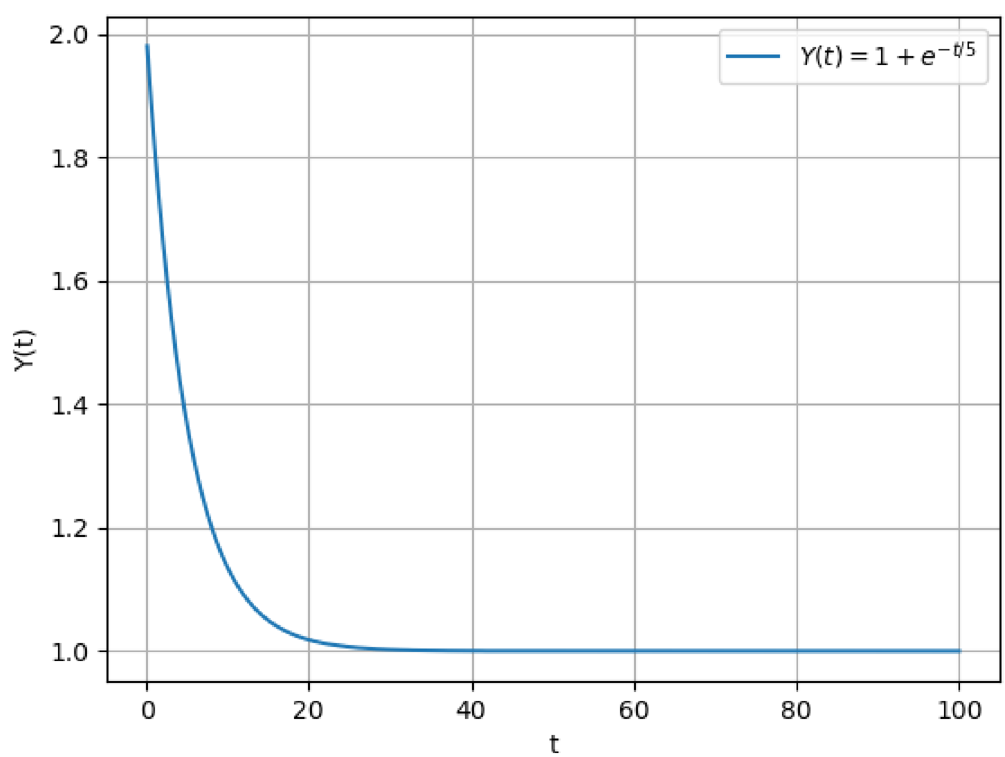


Figure 8.2: Graph of y(t)

8.3 The transfer function of a real system $H(S)$ is given as:

$$H(s) = \frac{As + B}{s^2 + Cs + D}$$

where A, B, C and D are positive constants. This system cannot operate as

(A) Low pass filter

(B) High pass filter

(C) Band pass filter

(D) An Integrator

(GATE EE 11 2022)

Solution: The transfer function $H(s)$ is given by:

$$H(s) = \frac{As + B}{s^2 + Cs + D} \tag{8.18}$$

Put $s = j\omega$ in (8.18):

$$H(j\omega) = \frac{A(j\omega) + B}{(j\omega)^2 + C(j\omega) + D} \tag{8.19}$$

$$|H(j\omega)| = \frac{\sqrt{(A\omega)^2 + B^2}}{\sqrt{(D - \omega^2)^2 + (\omega C)^2}} \tag{8.20}$$

a) Low Pass Filter:

At low frequency ($\omega = 0$):

$$|H(\omega = 0)| = \frac{B}{D} \tag{8.21}$$

$\therefore$ H(s) can operate as Low pass filter.

Parameter	**Description**
Low Pass Filter	The gain should be finite at low frequency
High Pass Filter	The gain should be finite at high frequency
Band Pass Filter	Finite gain over frequency band
Integrator	Transfer function should have at least one pole at origin

Table 8.3: Conditions

b) High Pass Filter:

At high frequency ($\omega = \infty$):

$$|H(\omega = \infty)| = 0 \tag{8.22}$$

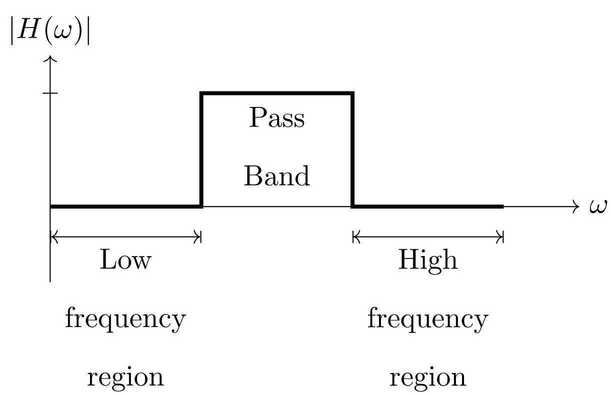
$\therefore$ $H(s)$ cannot operate as High pass filter.

c) Band Pass Filter:

Assuming B is a very less positive valued constant as compared to others:

$$|H(j\omega)| = \frac{(A\omega)}{\sqrt{(D-\omega^2)^2 + (\omega C)^2}} \tag{8.23}$$

$$\implies |H(\omega = 0)| = 0 \text{ and } |H(\omega = \infty)| = 0 \tag{8.24}$$



$\because$ $H(s)$ passes frequency between low and high frequencies.

$\therefore H(s)$ can operate as a band pass filter.

d) Integrator:

At very high value of frequency($\omega \to \infty$):

$$H(s) \approx \frac{As}{s^2} \approx \frac{A}{s} \tag{8.25}$$

From Table 8.3:

$\therefore H(s)$ can operate as an Integrator.

8.4 In a circuit, there is a series connection of an ideal resistor and an ideal capacitor. The conduction current (in Amperes) through the resistor is $2\sin\left(t+\frac{\pi}{2}\right)$. The displacement current (in Amperes) through the capacitor is ____.

(A) $2\sin(t)$

(B) $2\sin(t+\pi)$

(C) $2\sin\left(t+\frac{\pi}{2}\right)$

(D) 0

(GATE 2022 EC 24)

Solution:

Parameter	Description	Value
I_c	Conduction Current	$2\sin\left(t+\frac{\pi}{2}\right)$
A	Cross-sectional area	

Table 8.4: Parameters

Parameter	Description	Formula
Q	Charge	$\int I_c\,dt$
D	Electric Displacement	$\frac{Q}{A}$
J_D	Displacement current density	$\frac{\partial D}{\partial t}$
I_D	Displacement current	J_D x A

Table 8.5: Formulae

S Domain	Time Domain
$\frac{1}{s}$	$u(t)$
$\frac{-s}{a^2+s^2}$	$-\cos(at)$
$\frac{a}{a^2+s^2}$	$\sin(at)$
$\frac{1}{s+a}$	e^{-at}

Table 8.6: Laplace transforms

$$\begin{aligned}
\mathcal{L}\left[\int f(t)\ dt\right] &= \int_0^\infty \left[\int f(t)\ dt\right] e^{-st}\, dt && (8.26)\\
&= \int_0^\infty u\, dv \quad \text{where} \begin{cases} u = \int f(t)\, dt \\ dv = e^{-st} dt \end{cases} && (8.27)\\
&= uv - v\int du && (8.28)\\
&= \frac{1}{s}\int f(t)\, dt|_0 + \frac{1}{s}\int_0^\infty f(t)\, e^{-st} dt && (8.29)\\
&\implies \frac{1}{s}\int f(t)\, dt|_0 + \frac{1}{s} F(s) && (8.30)
\end{aligned}$$

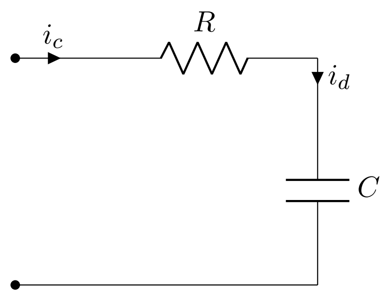


Figure 8.3: Circuit 1

From Table 8.5, Table 8.6 and eq (8.30)

$$I_c(s) = \frac{2s}{s^2+1} \tag{8.31}$$

$$Q_c(s) = \frac{2}{s(s^2+1)} \tag{8.32}$$

$$D(s) = \frac{1}{A}\left(\frac{2}{s(s^2+1)}\right) \tag{8.33}$$

$$J_D(s) = \frac{2}{A}\left(\frac{1}{s^2+1}\right) \tag{8.34}$$

$$I_D(s) = \frac{2}{s^2+1} \tag{8.35}$$

$$\implies I_D = 2\sin t \tag{8.36}$$

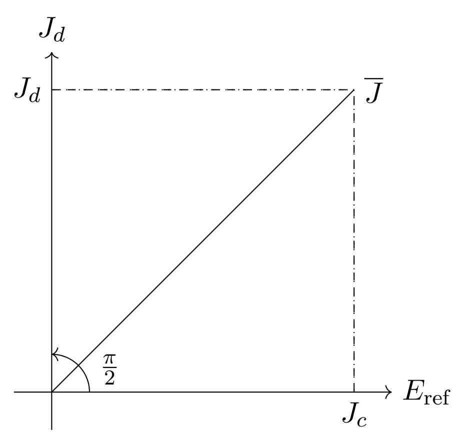


Figure 8.4: Phasor plot

From figure 8.4, phase of I_d is $\frac{\pi}{2}$

$$\therefore I_d = 2\sin\left(t + \frac{\pi}{2}\right) \tag{8.37}$$

$\therefore$ (C) is correct.

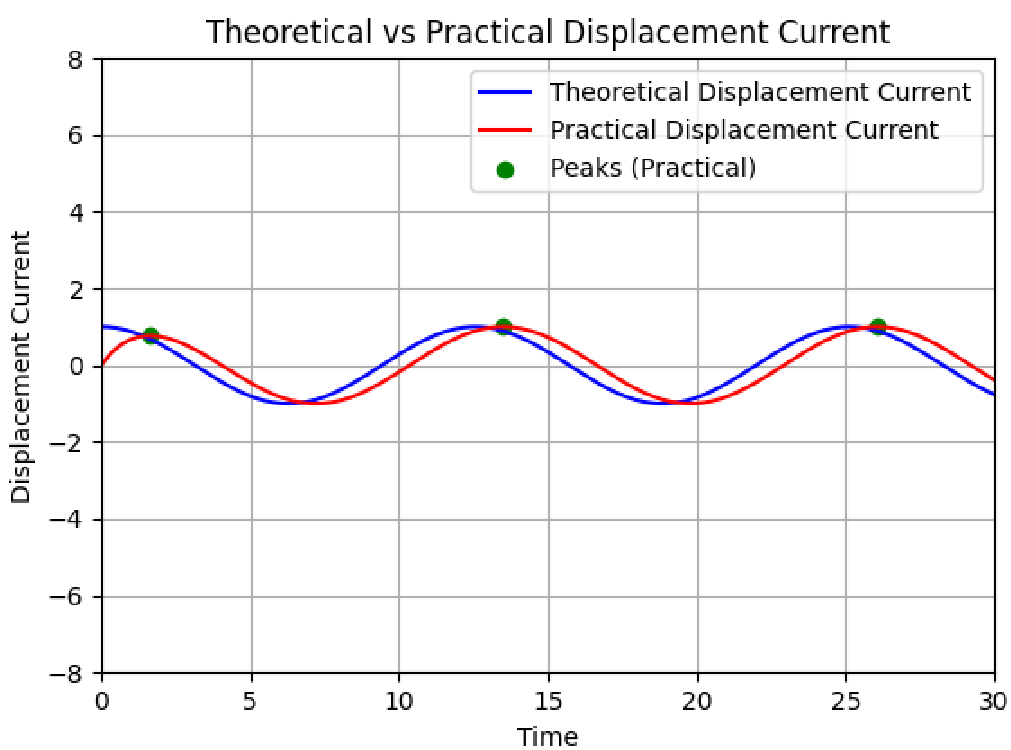


Figure 8.5: Thoritical vs Practical simulation

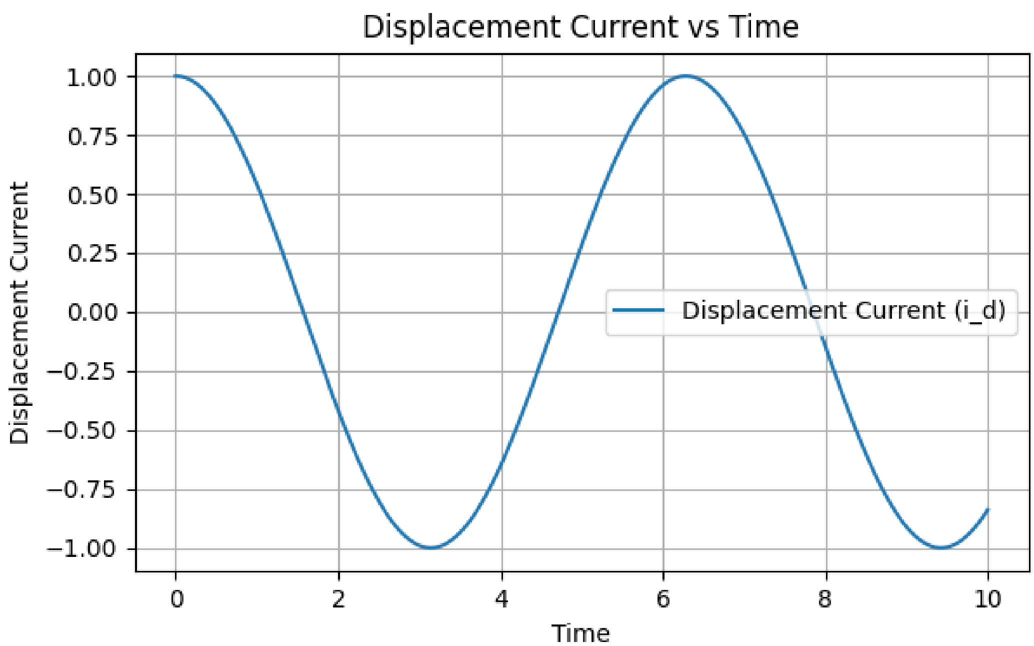


Figure 8.6: Displacement current

Chapter 9

Fourier transform

9.1 The Fourier transform $X(j\omega)$ of the signal

$x(t) = \frac{t}{(1+t^2)^2}$ is ________. GATE-2022-EC-15

(A) $\frac{\pi}{2j}\omega e^{-|\omega|}$

(B) $\frac{\pi}{2}\omega e^{-|\omega|}$

(C) $\frac{\pi}{2j}e^{-|\omega|}$

(D) $\frac{\pi}{2}e^{-|\omega|}$

Solution:

Symbol	Value	Description
$x(t)$	$\frac{t}{(1+t^2)^2}$	Signal
$X(\omega)$	$\int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$	Fourier transform of $x(t)$

Table 9.1: Variable description

The Fourier transform of the form x(t)=$e^{-a|t|}$ is

$$x\left(t\right) \overset{\text{F.T}}{\longleftrightarrow} X\left(\omega\right) \tag{9.1}$$

$$X\left(\omega\right) = \frac{2a}{a^2+\omega^2} \tag{9.2}$$

Consider,

$$x\left(t\right) = e^{-|t|} \tag{9.3}$$

$$X\left(\omega\right) = \frac{2}{1+\omega^2} \tag{9.4}$$

By using differentiation property from (A.1.5),

$$tx\left(t\right) \overset{\text{F.T}}{\longleftrightarrow} j\frac{d}{d\omega}X\left(\omega\right) \tag{9.5}$$

$$tx\left(t\right) \overset{\text{F.T}}{\longleftrightarrow} j\left[\frac{d}{d\omega}\left(\frac{2}{1+\omega^2}\right)\right] \tag{9.6}$$

$$te^{-|t|} \overset{\text{F.T}}{\longleftrightarrow} \frac{-4j\omega}{\left(1+\omega^2\right)^2} \tag{9.7}$$

Applying duality property from (A.2.3),

$$\frac{-4jt}{\left(1+t^2\right)^2} \overset{\text{F.T}}{\longleftrightarrow} 2\pi\left(-\omega\right)e^{-|-\omega|} \tag{9.8}$$

$$\frac{t}{\left(1+t^2\right)^2} \overset{\text{F.T}}{\longleftrightarrow} \frac{-2\pi\omega e^{-|\omega|}}{-4j} \tag{9.9}$$

$$\frac{t}{\left(1+t^2\right)^2} \overset{\text{F.T}}{\longleftrightarrow} \frac{\pi}{2j}\omega e^{-|\omega|} \tag{9.10}$$

Appendix A

Fourier transform

A.1 The Differentiation in frequency domain is as follows

Let x(t) be a signal such that,

$$x(t) \xleftrightarrow{\text{F.T}} X(\omega) \tag{A.1.1}$$

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \tag{A.1.2}$$

$$\frac{d}{d\omega} X(\omega) = \int_{t=-\infty}^{\infty} x(t)\,(-jt)\, e^{-j\omega t} dt \tag{A.1.3}$$

$$j\frac{d}{d\omega} X(\omega) = \int_{t=-\infty}^{\infty} tx(t)\, e^{-j\omega t} dt \tag{A.1.4}$$

$$tx(t) \xleftrightarrow{\text{F.T}} j\frac{d}{d\omega} X(\omega) \tag{A.1.5}$$

A.2 The duality property is as follows

From inverse Fourier transform we get,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega \tag{A.2.1}$$

Replacing t by -t and multiplying 2π on both sides we get,

$$2\pi x\left(-t\right) = \int_{-\infty}^{\infty} X\left(\omega\right) e^{-j\omega t} d\omega \tag{A.2.2}$$

$$X\left(t\right) \xleftrightarrow{\text{F.T}} 2\pi x\left(-\omega\right) \tag{A.2.3}$$