GATE 2022 -AE 63

EE23BTECH11057 - Shakunaveti Sai Sri Ram Varun

Question: Which one of the following is the closed form for the generating function of the sequence $\{a\}_{n\geq 0}$ defined below?

$$a_n = \begin{cases} n+1 & ,\text{n is odd} \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

(A) $\frac{x(1+x)^2}{(1-x^2)^2} + \frac{1}{1-x}$
(B) $\frac{x(3-x^2)}{(1-x^2)^2} + \frac{1}{1-x}$
(C) $\frac{2x}{(1-x^2)^2} + \frac{1}{1-x}$
(D) $\frac{x}{(1-x^2)^2} + \frac{1}{1-x}$

(GATE CS 2022 QUESTION 36)

Solution:
For the given sequence:

Parameter	Description	Value
$X(z)$	Generating function for a sequence $\{a_n\}$	?
a_n	n^{th} term of the sequence	$(n+1)\,u(n)$ (when odd)
		$u(n)$ (when even)

TABLE I
INPUT VALUES

$$X(z) = \sum_{k=-\infty}^{\infty} u(2k)\, z^{-2k} + \sum_{k=-\infty}^{\infty} ((2k+2)\, u(2k+1))\, z^{-(2k+1)} \tag{2}$$

$$\implies X(z) = \left(1 + z^{-2} + z^{-4} + \dots\right) + \left(2z^{-1} + 4z^{-3} + 6z^{-5} + \dots\right) \tag{3}$$

$$\implies X(z) = \frac{1}{1-z^{-2}} + \left(2z^{-1} + 4z^{-3} + 6z^{-5} \dots\right) \quad |z| > 1 \tag{4}$$

$$\implies X(z) = \frac{1}{1-z^{-2}} + 2z^{-1}\left(\frac{1}{1-z^{-2}} + \frac{z^{-2}}{(1-z^{-2})^2}\right) \quad |z| > 1 \tag{5}$$

$$\therefore X(z) = \frac{1}{1-z^{-1}} + \frac{z^{-1}\left(1+z^{-2}\right)}{(1-z^{-2})^2} \quad |z| > 1 \tag{6}$$

(6) is the closed form of generating function required in the question.

Hence, option (A) is correct.

$$X(z) = X_1(z) + X_2(z) \tag{7}$$

$$X_1(z) = \frac{1}{1-z^{-1}} \quad |z| > 1 \tag{8}$$

$$\implies x_1(n) = u(n) \tag{9}$$

$$\implies a_n = x_1(n) + x_2(n) \tag{10}$$

To find inverse z-transform of $X_2(z)$ we use contour integration technique:

$$x_2(n) = \frac{1}{2\pi j} \oint_C X_2(z)\; z^{n-1}\, dz \tag{11}$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^n\left(z^2+1\right)}{(z^2-1)^2}\, dz \tag{12}$$

We can observe that we have two poles at $z = 1, -1$. And poles are repeated twice, thus by applying residue theorem two times for poles 1 and -1:

$$x_2(n) = \frac{1}{(1)!} \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 X_2(z) \right) + \frac{1}{(1)!} \lim_{z \to -1} \frac{d}{dz} \left((z+1)^2 X_2(z) \right) \tag{13}$$

$$\implies x_2(n) = \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 \frac{z^n \left(z^2+1\right)}{\left(z^2-1\right)^2} \right) + \lim_{z \to -1} \frac{d}{dz} \left((z+1)^2 \frac{z^n \left(z^2+1\right)}{\left(z^2-1\right)^2} \right) \tag{14}$$

$$\implies x_2(n) = \lim_{z \to 1} \frac{(z+1)^2 \left(nz^{n-1} + (n+2) z^{n+1}\right) - 2z^n \left(1+z^2\right)(z+1)}{(z+1)^4} + \lim_{z \to -1} \frac{(z-1)^2 \left(nz^{n-1} + (n+2) z^{n+1}\right) - 2z^n \left(1+z^2\right)(z-1)}{(z-1)^4} \tag{15}$$

on simplification, we get

$$x_2(n) = \frac{n + n(-1)^{n-1}}{2} \tag{16}$$

$$\therefore a_n = u(n) + \frac{n + n(-1)^{n-1}}{2} u(n) \tag{17}$$

Which is the sequence given in the Question.

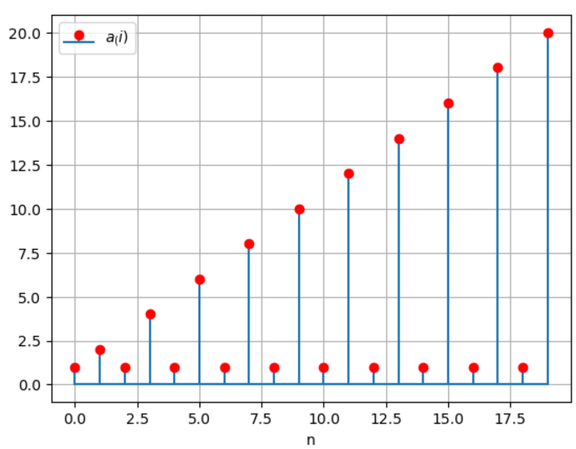

Fig. 1. Terms of the sequence given