GATE: BM - 36.2022

EE22BTECH11219 - Rada Sai Sujan

Question

In the complex z-domain, the value of integral $\oint_C \frac{z^3-9}{3z-i}\, dz$ is

(a) $\frac{2\pi}{81} - 6i\pi$
(b) $\frac{2\pi}{81} + 6i\pi$
(c) $-\frac{2\pi}{81} + 6i\pi$
(d) $-\frac{2\pi}{81} - 6i\pi$

(GATE 2022 BM)

Solution:
Simplyfying the Contour Integral to the standard form we get,

$$\oint_C \frac{z^3-9}{3z-i}\, dz = \frac{1}{3}\oint_C \frac{z^3-9}{z-\frac{i}{3}}\, dz \tag{1}$$

From Cauchy's residue theorem,

$$\oint_C f(z)\, dz = 2\pi i \sum R_j \tag{2}$$

We can observe a non-repeated pole at $z = \frac{i}{3}$ and thus $a = \frac{i}{3}$,

$$R = \lim_{z\to a}(z-a)\, f(z) \tag{3}$$

$$\implies R = \frac{1}{3}\lim_{z\to\frac{i}{3}}\left(z-\frac{i}{3}\right)\frac{z^3-9}{z-\frac{i}{3}} \tag{4}$$

$$= \frac{-i}{81} - 3 \tag{5}$$

Therefore, from (2) and (5)

$$\oint_C \frac{z^3-9}{3z-i}\, dz = \frac{2\pi}{81} - 6i\pi \tag{6}$$