GATE 2022 Assignment
EE1205 Signals and Systems

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Question:
Let $x_1(t) = e^{-t}u(t)$ and $x_2(t) = u(t) - u(t-2)$, where $u(.)$ denotes the unit step function. If $y(t)$ denotes the convolution of $x_1(t)$ and $x_2(t)$,then $\lim_{t\to\infty} y(t) =$ ____. (Rounded off to one decimal place)
(GATE EC 2022)
Solution:

variable	value	description
$x_1(t)$	$e^{-t}u(t)$	given function 1
$x_2(t)$	$u(t) - u(t-2)$	given function 2
$y(t)$	-	convolution of $x_1(t)$ and $x_2(t)$

TABLE I
TABLE: INPUT PARAMETERS

$$y(t) = x_1(t) * x_2(t) \tag{1}$$

from Table I

$$y(t) = e^{-t}u(t) * (u(t) - u(t-2)) \tag{2}$$

By applying Laplace transform

$$Y(s) = X_1(s) .X_2(s) \tag{3}$$

$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{1+s}, \quad \text{Re}(s) > -1 \tag{4}$$

$$u(t) - u(t-2) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1-e^{-2s}}{s}, \quad \text{Re}(s) > 0 \tag{5}$$

$$Y(s) = \left(\frac{1}{1+s}\right)\left(\frac{1-e^{-2s}}{s}\right), \quad \text{Re}(s) > 0 \tag{6}$$

$$= \frac{1-e^{-2s}}{s(s+1)} \tag{7}$$

Final value theorem

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{8}$$

$$\tag{9}$$

Proof:

$$\mathcal{L}\left[x(t)\right] = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt \tag{10}$$

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] = \int_0^\infty \frac{d}{dt}\left(x(t)\, e^{-st}\right) dt \tag{11}$$

$$= sX(s) - x(0^-) \tag{12}$$

$$\lim_{s\to 0}\left[\int_0^\infty \frac{d}{dt}\left(x(t)\, e^{-st}\right) dt\right] = \lim_{s\to 0}\left[sX(s) - x(0^-)\right] \tag{13}$$

$$\int_0^\infty \frac{dx(t)}{dt}\, dt = \lim_{s\to 0}\left[sX(s) - x(0^-)\right] \tag{14}$$

$$\left[x(t)\right]_0^\infty = \lim_{s\to 0}\left[sX(s) - x(0^-)\right] \tag{15}$$

$$x(\infty) - x(0^-) = \lim_{s\to 0}\left[sX(s) - x(0^-)\right] \tag{16}$$

$$\implies x(\infty) = \lim_{s\to 0} sX(s) \tag{17}$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{18}$$

By applying Final value theorem

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \tag{19}$$

$$= \lim_{s\to 0} s\left(\frac{1 - e^{-2s}}{s(s+1)}\right) \tag{20}$$

$$= \lim_{s\to 0}\left(\frac{1 - e^{-2s}}{(s+1)}\right) \tag{21}$$

$$= \left(\frac{1 - e^{0}}{0+1}\right) \tag{22}$$

$$\lim_{t\to\infty} y(t) = 0 \tag{23}$$