SIGNAL PROCESSING

Through GATE

EE1205-TA Group

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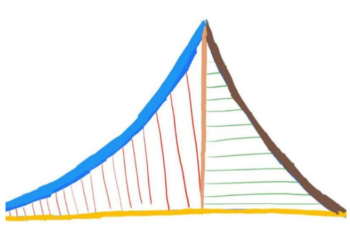


Contents

Introduction

This book provides solutions to signal processing problems in GATE.

Chapter 1

Harmonics

1.1 A damper with damping coefficient, c, is attached to a mass of 5 kg and spring of stiffness 5 kN/m as shown in figure. The system undergoes under-damped oscillations. If the ratio of the 3^{rd} amplitude to the 4^{th} amplitude of oscillations is 1.5, the value of c is ?

(GATE AE-62 (2022)) **Solution:**

Parameter	**Value**	**Description**
c	?	Damping Coefficient
k	5 kN/m	Stiffness
r	1.5	Ratio of 3^{rd} amplitude to 4^{th} amplitude of oscillations

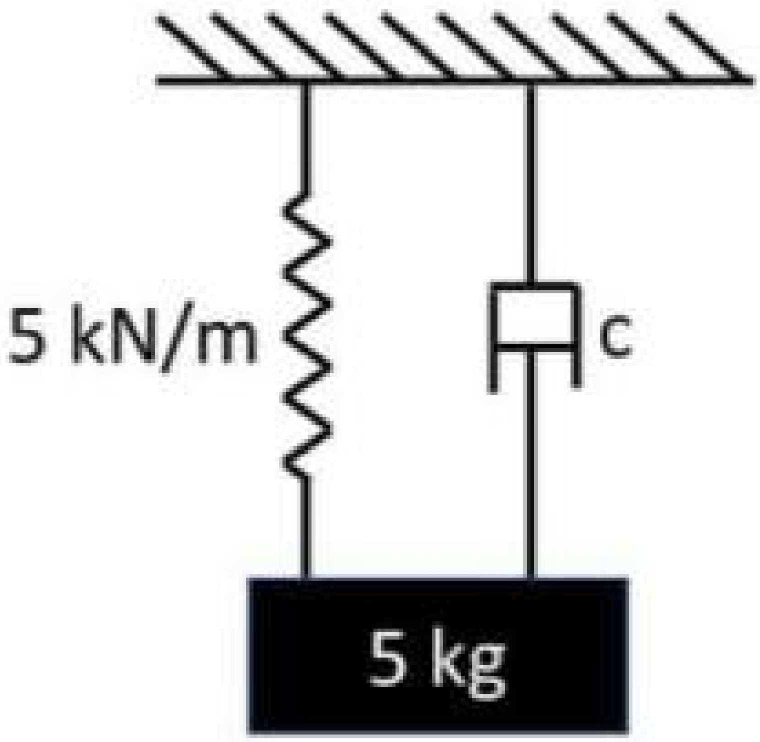
Table 1.1: Parameter Table (GATE AE-62)

Now, as the oscillation begins, from the Fig. 1.1 we write net force on the mass as,

$$F = F_1 + F_2 + mgu(t) \tag{1.1}$$

$$\implies m\frac{d^2x(t)}{dt^2} = -kx(t) - c\frac{dx(t)}{dt} + mgu(t) \tag{1.2}$$

$$\implies \frac{d^2x(t)}{dt^2} + \left(\frac{c}{m}\right)\frac{dx(t)}{dt} + \left(\frac{k}{m}\right)x(t) = gu(t) \tag{1.3}$$

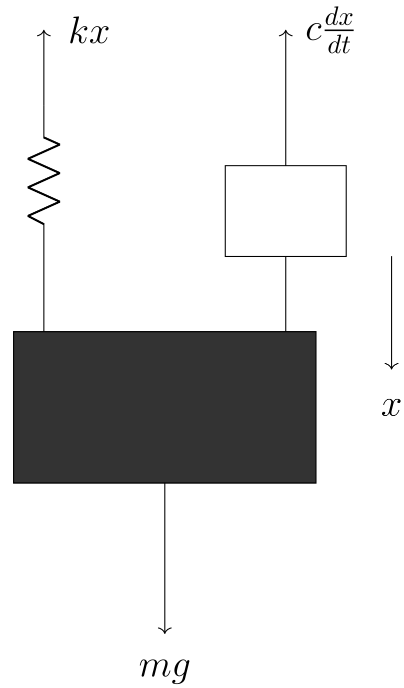


Now, taking the Laplace transform on both sides,

$$s^2X(s) + \frac{c}{m}sX(s) + \frac{k}{m}X(s) = \frac{g}{s} \tag{1.4}$$

$$\implies X(s) = \frac{g}{s\left(s^2 + \frac{c}{m}s + \frac{k}{m}\right)} \tag{1.5}$$

$$\implies X(s) = \frac{g}{s(s-s_1)(s-s_2)} \tag{1.6}$$



Where

$$s_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{1.7}$$

$$s_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{1.8}$$

From (1.6) we get,

$$\begin{aligned} \implies X(s) = \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_1(s - s_1)} - \frac{1}{s_2(s - s_2)}\right] \\ - \frac{g}{s_1 s_2}\left(\frac{1}{s}\right) \end{aligned} \tag{1.9}$$

Now again taking the inverse Laplace transform we have,

$$x(t) = -\frac{g}{s_1 s_2}u(t) + \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_1}e^{s_1 t} - \frac{1}{s_2}e^{s_2 t}\right]u(t) \tag{1.10}$$

$$\begin{aligned}\implies x(t) = &-\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gc}{2mk}\right)^2}\, e^{-ct/2m}u(t) \\ &\sin\left(\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t + \tan^{-1}\left(\frac{2mg\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}}{gc}\right)\right) \\ &-\frac{mg}{k}u(t)\end{aligned} \tag{1.11}$$

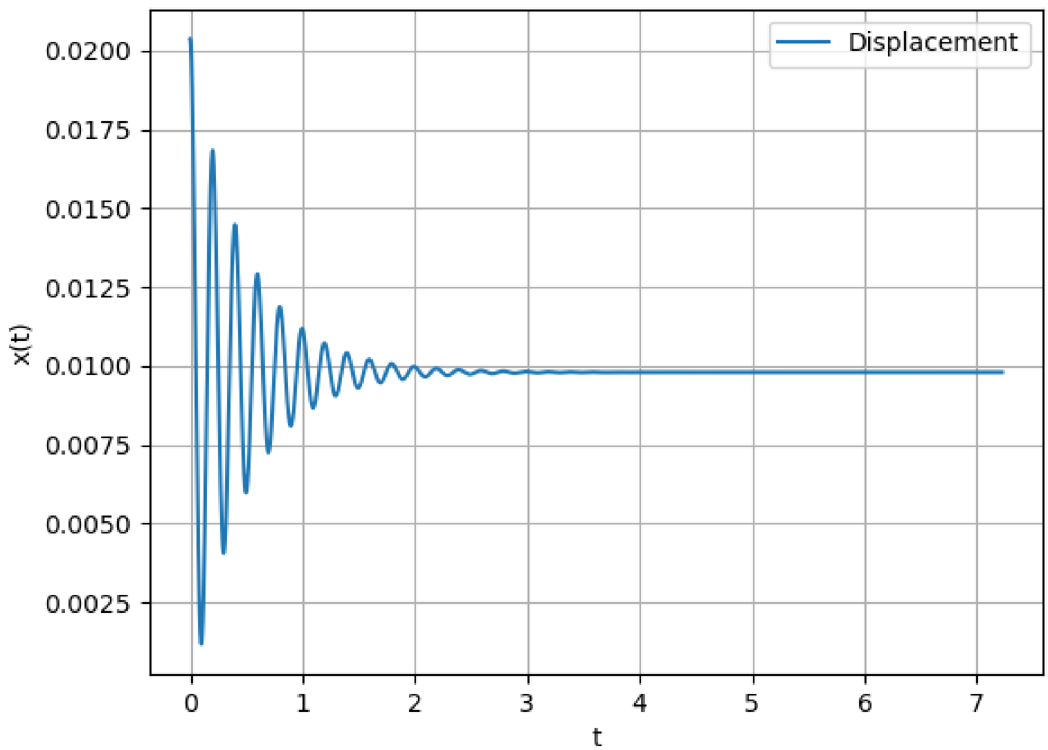
(Substituting the values of s_1 and s_2 from (1.7) and (1.8))

From (1.11), we have the ratio of 3^{rd} to 4^{th} amplitude,

$$\begin{aligned}&-\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gc}{2mk}\right)^2}\, e^{-3cT/2m} = \\ &-\frac{3}{2}\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gc}{2mk}\right)^2}\, e^{-4cT/2m}\end{aligned} \tag{1.12}$$

$$\implies e^{\pi c/\sqrt{mk}} = \frac{3}{2} \tag{1.13}$$

$$\implies c = \frac{\sqrt{mk}\ln\frac{3}{2}}{\pi} \tag{1.14}$$



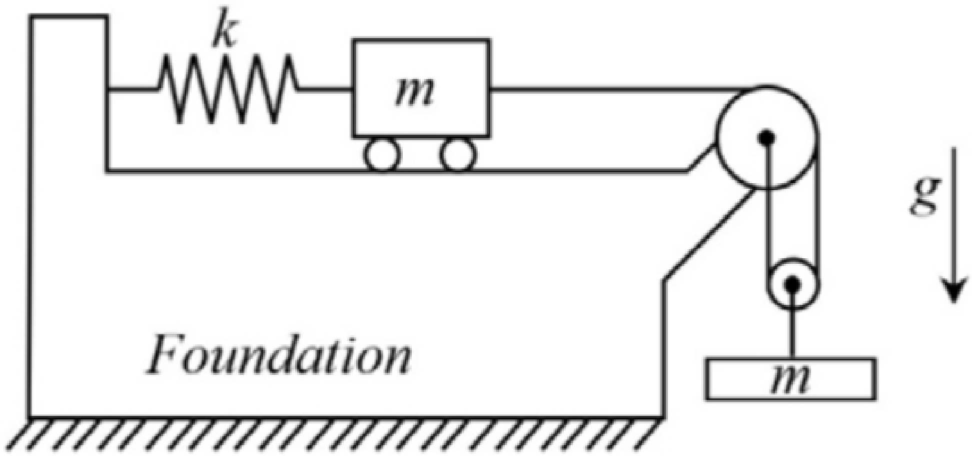
1.2 A spring-mass system having a mass m and spring constant k, placed horizontally on a foundation, is connected to a vertically hanging mass m with the help of an inextensible string. Ignore the friction in the pulleys and also the inertia of pulleys, string and spring. Gravity is acting vertically downward as shown. The natural frequency of the system in rad/s is

(A) $\sqrt{\frac{4k}{3m}}$

(B) $\sqrt{\frac{k}{2m}}$

(C) $\sqrt{\frac{k}{3m}}$

(D) $\sqrt{\frac{4k}{5m}}$



(GATE XE 2022)

Solution:

Parameters	**Description**	**Value**
$x(t)$	Displacement of mass m on foundation at time t	
$x(0)$	Displacement of mass m on foundation at time $t = 0$	0
$x'(0)$	Velocity of mass m on foundation at time $t = 0$	0

Table 1.2: Parameters

$$T - kx = m\frac{d^2x}{dt^2} \tag{1.15}$$

$$mg - 2T = m\frac{d^2\left(\frac{x}{2}\right)}{dt^2} \tag{1.16}$$

$$\implies mg - 2kx = \frac{5}{2}m\frac{d^2x}{dt^2} \tag{1.17}$$

$$\frac{d^2x}{dt^2} \xleftrightarrow{\mathcal{L}} s^2X(s) - sx(0) - x'(0) \tag{1.18}$$

$$t^n \xleftrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}} \tag{1.19}$$

From the Laplace transforms (1.18) and (1.19), we get

$$\frac{mg}{s} - 2kX(s) = \frac{5}{2}m\left(s^2X(s) - sx(0) - x'(0)\right) \tag{1.20}$$

$$\implies X(s) = \frac{\frac{2g}{5}}{s\left(s^2 + \frac{4k}{5m}\right)} \tag{1.21}$$

$$= \frac{mg}{2ks} - \frac{mgs}{2k\left(s^2 + \frac{4k}{5m}\right)} \tag{1.22}$$

$$\cos at \xleftrightarrow{\mathcal{L}} \frac{s}{s^2 + a^2} \tag{1.23}$$

From the Laplace transforms (1.19) and (1.23), we get

$$x(t) = \frac{mg}{2k}\left(1 - \cos\left(\sqrt{\frac{4k}{5m}}t\right)\right)u(t) \tag{1.24}$$

$$\implies \omega = \sqrt{\frac{4k}{5m}} \tag{1.25}$$

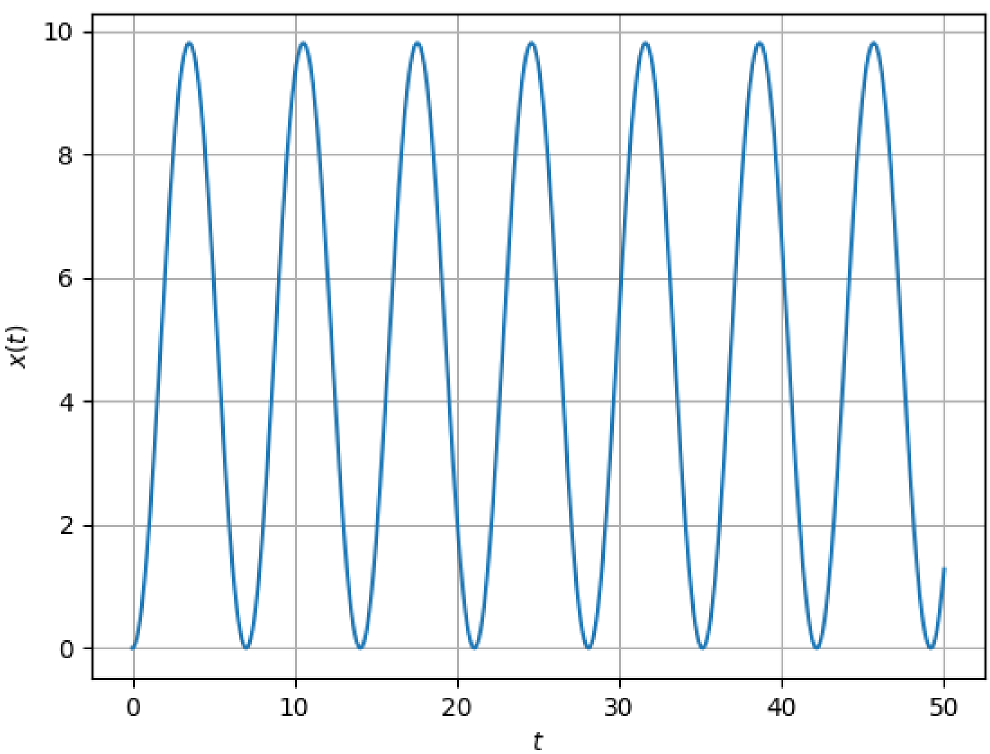


Figure 1.1: Plot of $x(t)$ for $m = 1kg$, $k = 1N/m^2$

1.3 The time delay between the peaks of the voltage signals $v_1(t) = \cos(6t + 60°)$ and $v_2(t) = -\sin(6t)$ is ____s

(A) $\frac{300\pi}{360}$

(B) $\frac{10\pi}{360}$

(C) $\frac{50\pi}{360}$

(D) $\frac{200\pi}{360}$

(GATE BM 2022 QUESTION 18)

Solution: From the values given in the Table 1.3:

Parameter	**Description**	**Value**
$v_1(t)$	Input voltage signal 1	$\cos(6t + 60°)$
$v_2(t)$	Input voltage signal 2	$-\sin(6t)$
$\Delta\phi$	Phase difference between two input signals	?
Δt	Time difference between maxima of two input signals	?
ω	angular frequency of input voltages	6

Table 1.3: input values

$$v_1(t) = \cos(6t + 60°) \tag{1.26}$$

$$v_2(t) = -\sin(6t) \tag{1.27}$$

$$\implies v_2(t) = \cos(6t + 90°) \tag{1.28}$$

From (1.27) and (1.28), phase difference between two voltage signals is 30°. From formula,

$$\Delta\phi = \frac{\Delta t}{\frac{2\pi}{\omega}} 360 \tag{1.29}$$

$$\therefore \Delta t = \frac{10\pi}{360} s \tag{1.30}$$

Hence, option B is correct.

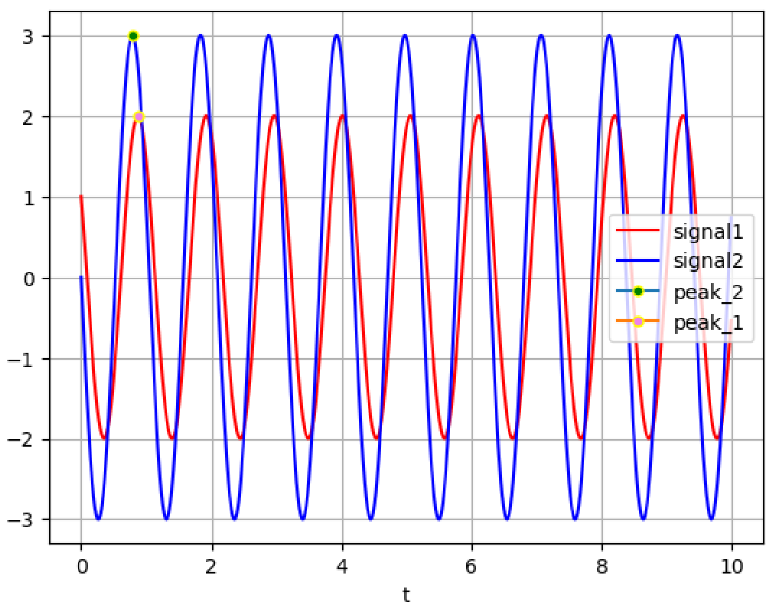


Figure 1.2: Figure of input voltage signals

Chapter 2

Filters

2.1 The network shown below has a resonant frequency of 150 kHz and bandwidth of 600 Hz. The Q-factor of the network is ______

(rounded off to one decimal place).

(GATE 2022 EC)

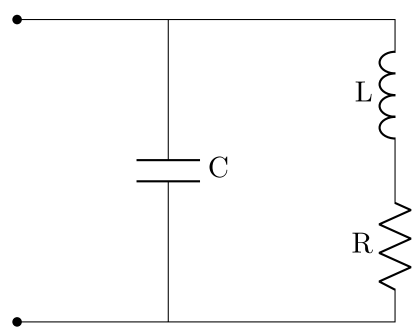


Figure 2.1: Circuit 1

Solution:

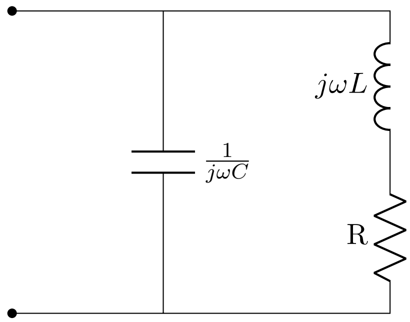


Figure 2.2: Circuit 2

Parameter	Description	Value
f_0	Resonant frequency	150 kHz
B	Bandwidth	600 Hz

Table 2.1: Parameters

At Resonance,

$$X_L = X_C \tag{2.1}$$

$$\omega_0 L = \frac{1}{\omega_0 C} \tag{2.2}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{2.3}$$

$$2\pi f_0 = \frac{1}{\sqrt{LC}} \tag{2.4}$$

$$\implies f_0 = \frac{1}{2\pi\sqrt{LC}} \tag{2.5}$$

Parameter	Description	Formula
Q	Quality factor	$\frac{X_L}{R}$
B	Bandwidth	$\frac{R}{2\pi L}$
ω_0	Radial resonant frequency	$2\pi f_0$
X_L	Inductive reactance	ωL
X_C	Capacitive reactance	$\frac{1}{\omega C}$

Table 2.2: Formulae

Using Table 2.2,

$$Q = \frac{X_L}{R} \tag{2.6}$$

$$= \frac{\omega_0 L}{R} \tag{2.7}$$

$$= \left(\frac{1}{\sqrt{LC}}\right)\frac{L}{R} \tag{2.8}$$

$$\implies Q = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{2.9}$$

From eq (2.5) and Table 2.2

$$\frac{f_0}{B} = \left(\frac{1}{2\pi\sqrt{LC}}\right)\frac{2\pi L}{R} \tag{2.10}$$

$$= \left(\frac{1}{\sqrt{LC}}\right)\frac{L}{R} \tag{2.11}$$

$$\implies \frac{f_0}{B} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{2.12}$$

From Table 2.1, eq (2.9) and eq (2.12),

$$Q = \frac{f_0}{B} \tag{2.13}$$

$$= \frac{150 \text{ x } 10^3}{600} \tag{2.14}$$

$$= 250 \tag{2.15}$$

$\therefore$ Q-factor is 250

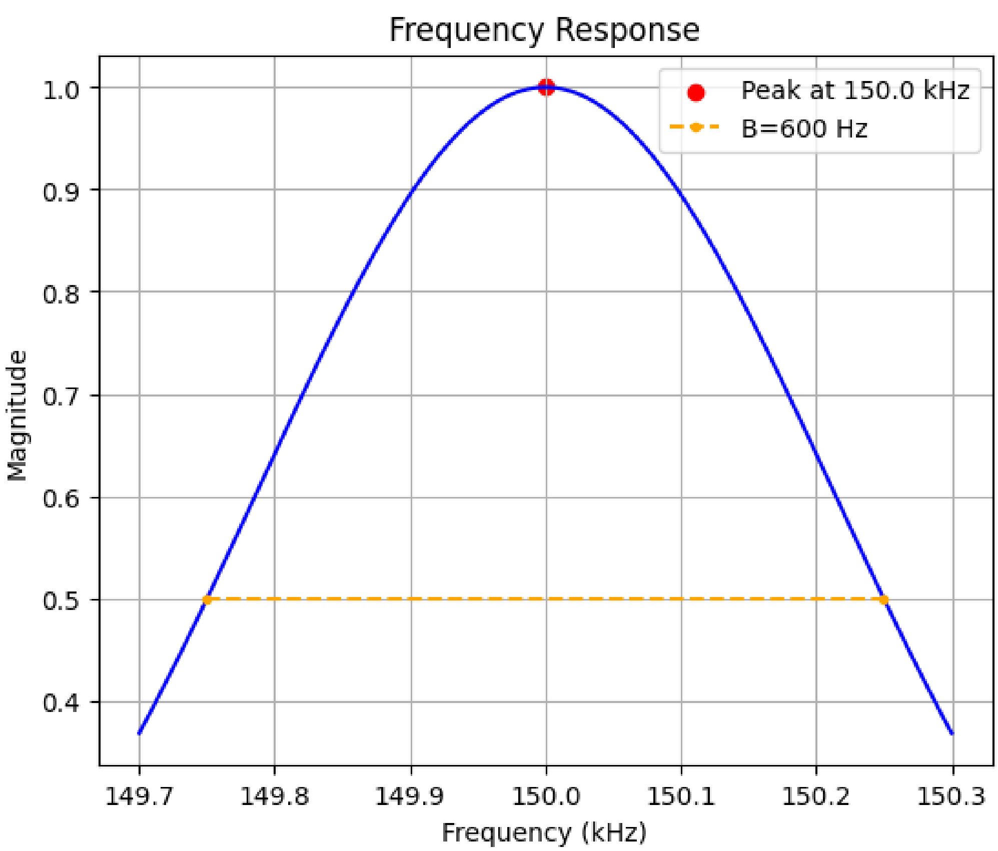


Figure 2.3: Plot of Q-factor

Chapter 3

Z-transform

3.1 Consider the following recursive iteration scheme for different values of variable P with the initial guess $x_1 = 1$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{P}{x_n}\right), \qquad n = 1, 2, 3, 4, 5$$

For $P = 2$, x_5 is obtained to be 1.414, rounded off to 3 decimal places. For $P = 3$, x_5 is obtained to be 1.732, rounded off to 3 decimal places.

If $P = 10$, the numerical value of x_5 is ______ . (*round off to three decimal places*)
(GATE CE 2022)

Solution:

Applying $A.M \geq G.M$ inequality,

$$\frac{x_n + \frac{P}{x_n}}{2} \geq \sqrt{P} \tag{3.1}$$

$$\implies x_{n+1} \geq \sqrt{P} \tag{3.2}$$

Solving the equation,

$$2x_{n+1}x_n - {x_n}^2 - P = 0 \tag{3.3}$$

Applying Z-transform we get,

$$\begin{align}
X\left(z\right) * X\left(z\right) &= \frac{PZ^{-1}}{\left(1-z^{-1}\right)\left(2-z^{-1}\right)} \tag{3.4}\\
&= P\left(\frac{z^{-1}}{1-z^{-1}} - \frac{z^{-1}}{2-z^{-1}}\right) \tag{3.5}
\end{align}$$

From the transformation pairs,

$$\begin{align}
x_{n-a} &\overset{\mathcal{Z}}{\longleftrightarrow} z^{-a}X\left(z\right) \tag{3.6}\\
x_{n_1} \times x_{n_2} &\overset{\mathcal{Z}}{\longleftrightarrow} X_1\left(z\right) * X_2\left(z\right) \tag{3.7}\\
\frac{u\left(n-1\right)}{a^n} &\overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{a-z^{-1}} \tag{3.8}
\end{align}$$

Now, applying inverse Z-tranform,

$$\begin{align}
x_n^2 &= P\left(u\left(n-1\right) - \frac{u\left(n-1\right)}{2^n}\right) \tag{3.9}\\
\implies x_n^2 &= P\left(1-\frac{1}{2^n}\right) \quad [\because n \geq 1] \tag{3.10}
\end{align}$$

Similarly,

$$\begin{align}
x_{n+1}^2 &= P\left(1-\frac{1}{2^{n+1}}\right) \tag{3.11}\\
\implies \lim_{n\to\infty}\frac{x_{n+1}}{x_n} &= \lim_{n\to\infty}\sqrt{\frac{P\left(1-\frac{1}{2^n}\right)}{P\left(1-\frac{1}{2^{n+1}}\right)}} \tag{3.12}\\
&= 1 \tag{3.13}
\end{align}$$

Hence, the system is convergent.

Now finding the limit of the sequence,

$$x^2 = \lim_{x \to \infty} P\left(1 - \frac{1}{2^n}\right) \tag{3.14}$$

$$\implies x = \pm\sqrt{P} \tag{3.15}$$

From (3.2) and (3.15),

$$x_{n+1} = \sqrt{P} \tag{3.16}$$

Therefore, for $P = 10$ the value of x_5 is,

$$x_5 = \sqrt{10} \tag{3.17}$$

$$\therefore x_5 = 3.162 \tag{3.18}$$

3.2 The block diagram of a two-tap high-pass FIR filter is shown below. The filter transfer function is given by $H(z) = Y(z)/X(z)$.

If the ratio of maximum to minimum value of $H(z)$ is 2 and $|H(z)|_{max} = 1$, the coefficients β_0 and β_1 are ________________ and ________________, respectively.

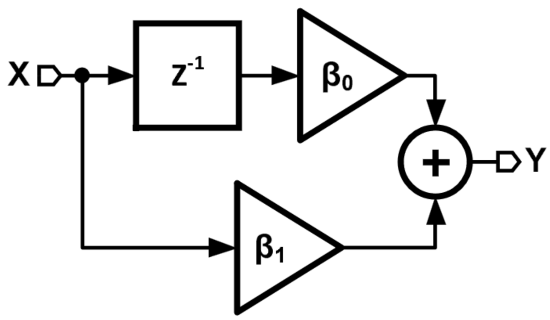


Figure 3.1: Block diagram

(A) 0.75, -0.25

(B) 0.67, 0.33

(C) 0.60, -0.40

(D) -0.64, 0.36

GATE BM 2022

Solution:

Results and Proofs:

Time Shift Property:

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{3.19}$$

$$x(n-n_0) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z) \tag{3.20}$$

Proof:

Let

$$y(n) = x(n - n_0) \tag{3.21}$$

Taking z-transform

$$\mathcal{Z}\left(y(n)\right) = \mathcal{Z}\left(x(n - n_0)\right) \tag{3.22}$$

$$\tag{3.23}$$

Simplifying LHS

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n} \tag{3.24}$$

From (3.21)

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n - n_0) z^{-n} \tag{3.25}$$

Let

$$n - n_0 = s \tag{3.26}$$

$$\implies n = s + n_0 \tag{3.27}$$

From (3.25) and (3.27)

$$Y(z) = \sum_{s=-\infty}^{\infty} x(s) z^{-(s+n_0)} \tag{3.28}$$

$$= z^{-n_0} \sum_{s=-\infty}^{\infty} x(s) z^{-s} \tag{3.29}$$

As variable in Z-transform is dummy, on replacing it, we get

$$Y(z) = z^{-n_0} \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{3.30}$$

$$= z^{-n_0} X(z) \tag{3.31}$$

From (3.22) and (3.31)

$$\mathcal{Z}\left(x(n-n_0)\right) = z^{-n_0} X(z) \tag{3.32}$$

Hence proved

Result:

$$z^{-n_0} X(z) \overset{\mathcal{Z}^-}{\longleftrightarrow} x(n-n_0) \tag{3.33}$$

Sol:

Variable	**Description**	**Value**
$H(z)$	Transfer Function	$\beta_0 z^{-1} + \beta_1$
$\lvert H(z)\rvert_{max}$	Maximum value of Transfer Function	1
$\lvert H(z)\rvert_{min}$	Minimum value of Transfer Function	$\frac{1}{2}$

Table 3.1: input parameters

In (3.33), put

$$n_0 = 1, \quad x(n) = \delta(n)$$

Since

$$1 \overset{\mathcal{Z}^-}{\longleftrightarrow} \delta(n)$$

$$z^{-1} \xleftrightarrow{\mathcal{Z}^-} \delta(n-1) \tag{3.34}$$

This is a unit delay in discrete time and represents unit amplitude sinosoidal signal. So,

$$z^{-1} = e^{-jw} \tag{3.35}$$

$$\implies \left|z^{-1}\right| = 1 \tag{3.36}$$

Since $H(z)$ is complex, on using Triangle Inequality, we get

$$|x+y| \leq |x| + |y| \tag{3.37}$$

And its corollary

$$||x| - |y|| \leq |x+y| \tag{3.38}$$

where x and y are complex numbers.

$$\left|\left|z^{-1}\beta_0\right| - |\beta_1|\right| \leq \left|z^{-1}\beta_0 + \beta_1\right| \leq \left|z^{-1}\beta_0\right| + |\beta_1| \tag{3.39}$$

From Table 3.1

$$\left|\left|z^{-1}\beta_0\right| - |\beta_1|\right| \leq |H(z)| \leq \left|z^{-1}\beta_0\right| + |\beta_1| \tag{3.40}$$

From (3.36)

$$||\beta_0| - |\beta_1|| \leq |H(z)| \leq |\beta_0| + |\beta_1| \tag{3.41}$$

So, we can conclude that

$$|H(z)|_{max} = |\beta_0| + |\beta_1| \tag{3.42}$$

Now from Table 3.1

$$1 = |\beta_0| + |\beta_1| \tag{3.43}$$

Similarly,

$$\frac{1}{2} = ||\beta_0| - |\beta_1|| \tag{3.44}$$

On solving (3.43) and (3.44), we get

$$|\beta_0| = 0.75, |\beta_1| = 0.25 \tag{3.45}$$

OR

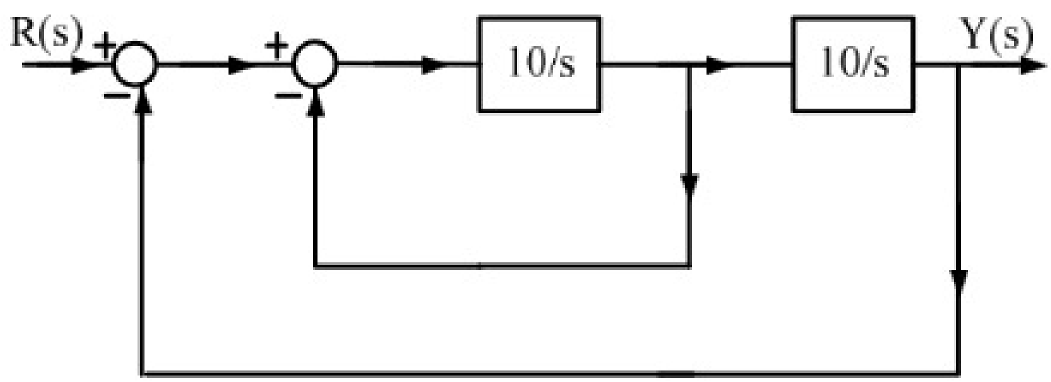
$$|\beta_0| = 0.25, |\beta_1| = 0.75 \tag{3.46}$$

Hence the correct answer is option (A)

Chapter 4

Systems

4.1 The damping ratio and undamped natural frequency of a closed loop system as shown in the figure, are denoted as ζ and ω_n, respectively. The values of ζ and ω_n are



(a) $\zeta = 0.5$ and $\omega_n = 10$ rad/s

(b) $\zeta = 0.1$ and $\omega_n = 10$ rad/s

(c) $\zeta = 0.707$ and $\omega_n = 10$ rad/s

(d) $\zeta = 0.707$ and $\omega_n = 100$ rad/s

(GATE EE 2022) **Solution:**

We will use Mason's Gain Formula to calculate the transfer function of this system.

Parameter	Description	Values
m	load of system	
k	stiffness of system	
ω_n	Natural frequency	$\sqrt{\frac{k}{m}}$
ζ	Damping ratio	$\frac{c}{2m\omega_n}$
y(t)	Output of system	
x(t)	Input to the system	
c	Damping coefficient	
T(s)	Transfer function of system	$\frac{Y(s)}{R(s)}$

Table 4.1: Parameter Table

First converting the given diagram to a signal flow graph :

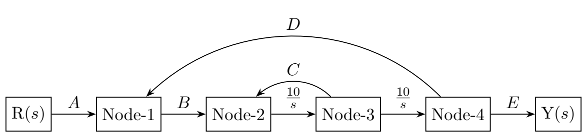


Figure 4.1: Signal Flow Diagram

Mason's Gain Formula is given by :

$$H\left(s\right) = \sum_{i=1}^{N}\left(\frac{P_i\Delta_i}{\Delta}\right) \tag{4.1}$$

This signal flow graph has only one forward path whose gain is given by :

$$P_1 = \frac{10}{s}\frac{10}{s} \tag{4.2}$$

$$= \frac{100}{s^2} \tag{4.3}$$

Parameter	**Description**
N	Number of forward paths
L	Number of loops
P_k	Forward path gain of k^{th} path
Δ_k	Associated path factor
Δ	Determinant of the graph

Table 4.2: Parameter Table - Mason's Gain Law

Parameter	**Formula**
Δ	$1 + \sum_{k=1}^{L} \left((-1)^k \text{ Product of gain of groups of k isolated loops} \right)$
Δ_k	Δ part of graph that is not touching k^{th} forward path

Table 4.3: Formula Table - Mason's Gain Law

The loop gain for loop between Node-2 and Node-3 is :

$$L_1 = \frac{10}{s}(-1) \tag{4.4}$$

$$= -\frac{10}{s} \tag{4.5}$$

The loop gain for loop between Node-1 and Node-4 is :

$$L_1 = \frac{10}{s}\frac{10}{s}(-1) \tag{4.6}$$

$$= -\frac{100}{s^2} \tag{4.7}$$

Using Table 4.3, Δ is :

$$\Delta = 1 - \left(-\frac{10}{s} - \frac{100}{s^2}\right) \tag{4.8}$$

$$= 1 + \frac{10}{s} + \frac{100}{s^2} \tag{4.9}$$

There are no two isolated loops available. Hence all further terms will b zero.

As both the loops are in contact with the only forward path,

$$\Delta_1 = 1 \tag{4.10}$$

Using equation (4.1) :

$$H\left(s\right) = \frac{\frac{100}{s^2}}{1 + \frac{10}{s} + \frac{100}{s^2}} \tag{4.11}$$

$$= \frac{100}{s^2 + 10s + 100} \tag{4.12}$$

Referring to Table 4.1, the general equation of the damping system is second order and can be written as :

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = x(t) \tag{4.13}$$

Take the Laplace transform and solve for $\frac{Y(s)}{X(s)}$:

$$\frac{Y\left(s\right)}{X\left(s\right)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.14}$$

$$\implies H\left(s\right) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.15}$$

Comparing equations (4.12) and (4.15) ,

$$\omega_n^2 = 100 \tag{4.16}$$

$$\implies \omega_n = 10 \text{ rad/s} \tag{4.17}$$

$$2\zeta\omega_n = 10 \tag{4.18}$$

$$\implies \zeta = 0.5 \tag{4.19}$$

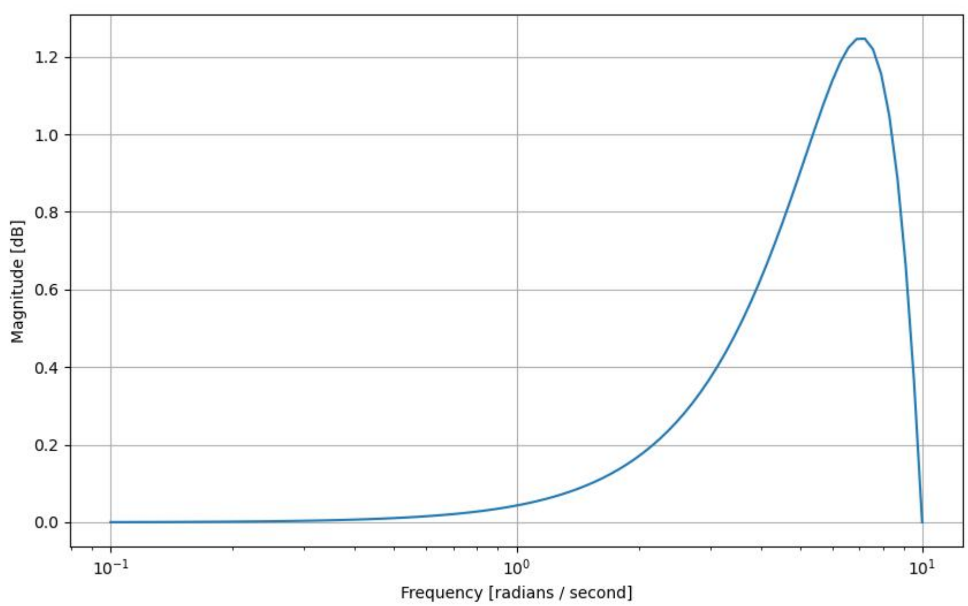


Figure 4.2: Magnitude plot

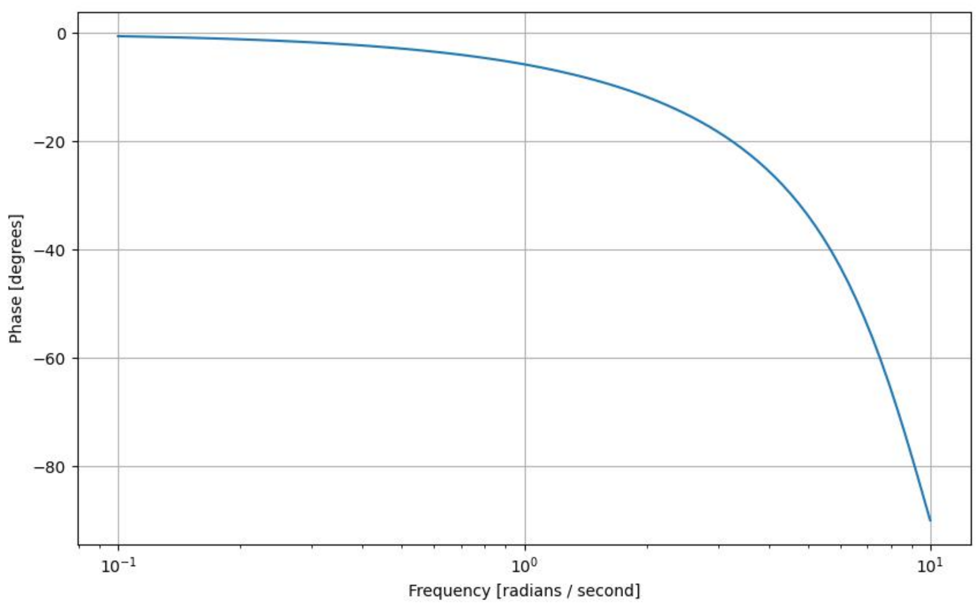
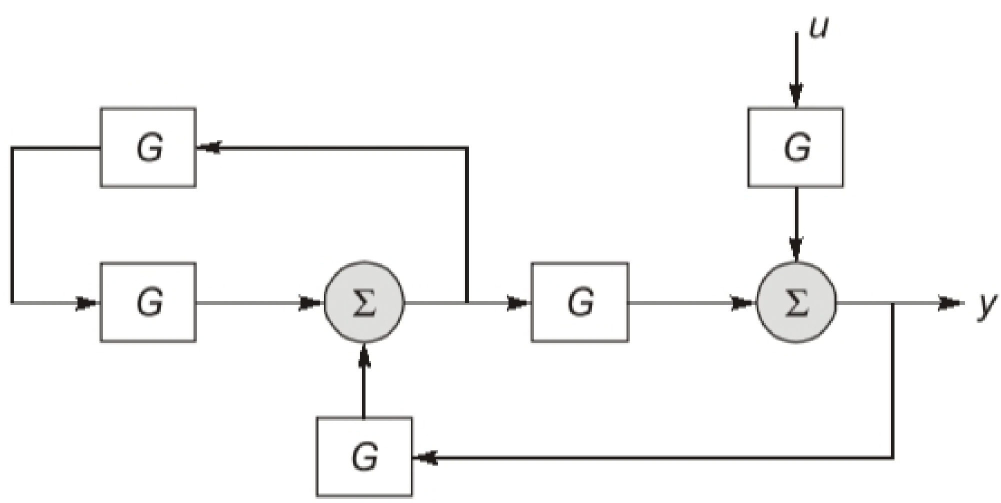


Figure 4.3: Phase plot

4.2 In the block diagram shown in the figure, the transfer function $G = \frac{K}{\tau s+1}$ with $K > 0$ and $\tau > 0$. The maximum value of K below which the system remains stable is _____(rounded off to two decimal places) (GATE CH 2022)

Solution:



Parameter	Value	Description
G	$\frac{K}{\tau s+1}$	Transfer function shown in blocks
Y		Laplace transform of y(output)
U		Laplace transform of u(input)
X,Z		Laplace transform of x and z
T	$\frac{Y}{U}$	Transfer function of complete system

Table 4.4: Parameters

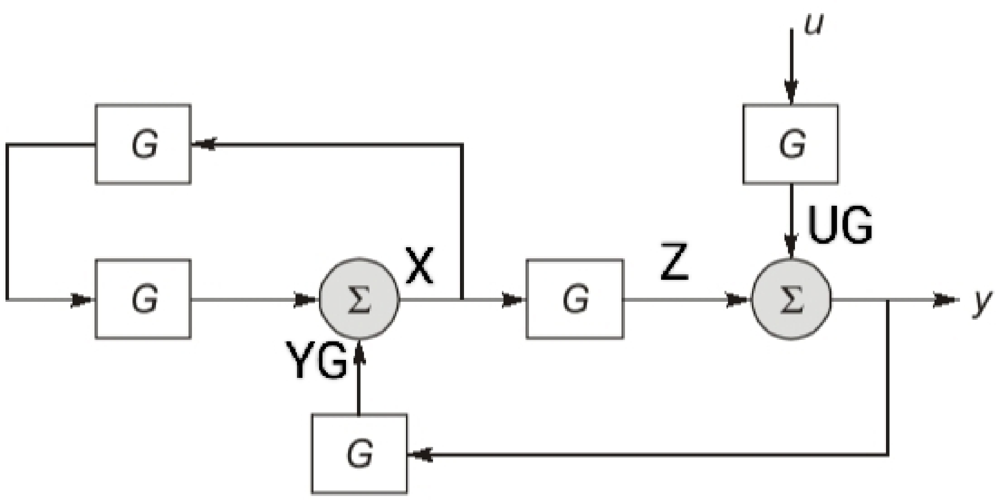


Figure 4.4: Block Diagram

$$X = XG^2 + YG \tag{4.20}$$

$$\implies X = \frac{YG}{1 - G^2} \tag{4.21}$$

$$Z = XG \tag{4.22}$$

$$Y = Z + UG \tag{4.23}$$

$$Y = XG + UG \tag{4.24}$$

$$Y = \frac{YG^2}{1 - G^2} + UG \tag{4.25}$$

$$\implies Y = \frac{UG(1 - G^2)}{1 - 2G^2} \tag{4.26}$$

From Table 4.4,

$$T = \frac{G(1-G^2)}{1-2G^2} \tag{4.27}$$

$$= \frac{K\left(1-\frac{K^2}{(\tau s+1)^2}\right)}{\left(1-\frac{2K^2}{(\tau s+1)^2}\right)(\tau s+1)} \tag{4.28}$$

$$= \frac{K(\tau^2 s^2 + 2\tau s + 1 - K^2)}{\tau^3 s^3 + 3\tau^2 s^2 + (3\tau - 2K^2\tau)s + 1 - 2K^2} \tag{4.29}$$

So, Characteristic equation :

$$\tau^3 s^3 + 3\tau^2 s^2 + (3\tau - 2K^2\tau)s + 1 - 2K^2 = 0 \tag{4.30}$$

For a characteristic equation $a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ...a_n = 0$,

s^n	a_0	a_2	a_4	...
s^{n-1}	a_1	a_3	a_5	...
s^{n-2}	$b_1 = \frac{a_1 a_2 - a_3 a_0}{a_1}$	$b_2 = \frac{a_1 a_4 - a_5 a_0}{a_1}$	...	..
s^{n-3}	$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$		
s^1	$\vdots$	$\vdots$		
s^0	a_n			

Table 4.5: Routh Array

From Table 4.5:

s^3	τ^3	$3\tau - 2K^2\tau$
s^2	$3\tau^2$	$1 - 2K^2$
s^1	$\frac{8}{3}\tau(1 - K^2)$	0
s^0	$1 - 2K^2$	

Table 4.6:

Given $\tau > 0$ and $K > 0$, for system to be stable,

$$1 - K^2 > 0 \tag{4.31}$$

$$1 - 2K^2 > 0 \tag{4.32}$$

$$\implies 0 < K < \frac{1}{\sqrt{2}} \tag{4.33}$$

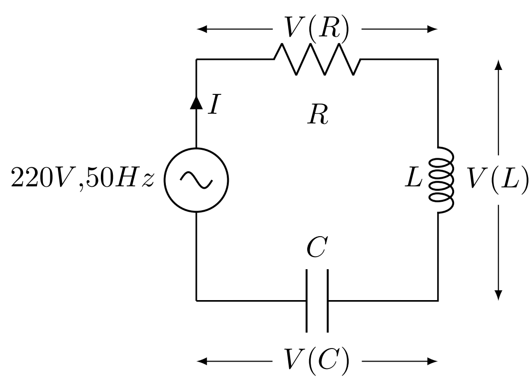
$$K_{max} \approx 0.71 \tag{4.34}$$

4.3 A series RLC circuit is connected to 220 V, 50 Hz supply. For a fixed a value of R and C, the inductor L is varied to deliver the maximum current. This value 0.4A and the corresponding potential drop across the capacitor is 330 V. The value of the inductor L is ? (Rounded off to two decimal places). (GATE BM 2022)

Solution:

Symbols	Description	Values
V_s	Input voltage	220 V and 50Hz
χ_L	Impedance across inductor	$j\omega L$
χ_C	Impedance across capacitor	$\frac{-j}{\omega C}$
Z	Impedance across the entire circuit	$\frac{1}{R+j\omega L+\frac{-j}{\omega C}}$

Table 4.7: Parameters, Descriptions, and Values



During maximum current $|Z|$ is minimum .

$$I = \frac{V_s}{Z} \tag{4.35}$$

$$= \frac{V_s}{R + \chi_L + \chi_C} \tag{4.36}$$

$$= \frac{V_s}{R + j\omega L + \frac{1}{j\omega C}} \tag{4.37}$$

$$|I| = \frac{|V_s|}{\sqrt{R + \left(\omega L - \frac{1}{\omega C}^2\right)}} \tag{4.38}$$

Varying L for maximum value of I :

$$\omega L = \frac{1}{\omega C} \tag{4.39}$$

Putting in (4.37):

$$I_{max} = \frac{V_s}{R} \tag{4.40}$$

I_{max} has same phase as V_s (Assume $\angle\phi$). For impedance across the capacitor :

$$V_C|_{I=I_{max}} = I_{max}\chi_C \tag{4.41}$$

$$-330\angle(90+\phi) = (0.4\angle\phi)\,\chi_C \tag{4.42}$$

$$-330\angle 90 = 0.4\chi_C \tag{4.43}$$

$$\implies \chi_C = -825j\Omega \tag{4.44}$$

For value of Capacitor and inductor, using (4.39) :

$$L = \frac{825}{100\pi}H \tag{4.45}$$

$$\approx 2.63\text{H} \tag{4.46}$$

$$C = 3.858 * 10^{-6}\text{F} \tag{4.47}$$

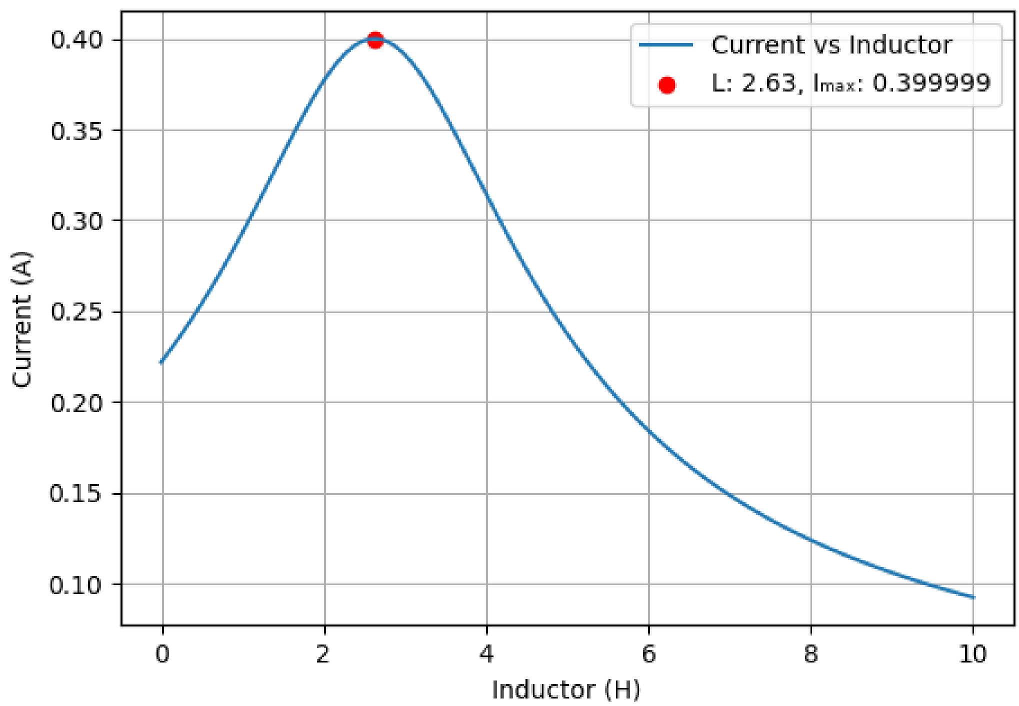


Figure 4.5: I vs L

4.4 The open loop transfer function of a unity gain negative feedback system is given by $G(s) = \frac{k}{s^2+4s-5}$. The range of k for which the system is stable,is (GATE EE 2022)

Solution:

Variable	**Description**	**value**
$G(s)$	Open loop transfer function	$\frac{k}{s^2+4s-5}$
1+G(s)	Characteristic equation	0

Table 4.8: A Table with input parameters

from Table4.8

Characteristic equation:

$$1 + G(s) = 0 \tag{4.48}$$

$$\implies 1 + \frac{k}{s^2 + 4s - 5} = 0 \tag{4.49}$$

$$\implies s^2 + 4s + (k - 5) = 0 \tag{4.50}$$

By routh table analysis, for a stable system:

$$\begin{array}{c|cc} s^2 & 1 & k-5 \\ s^1 & 4 & 0 \\ s^0 & \frac{4(k-5)-0}{4} & 0 \end{array}$$

$$\frac{4(k-5) - 0}{4} > 0 \tag{4.51}$$

$$k - 5 > 0 \tag{4.52}$$

$$\implies k > 5 \tag{4.53}$$

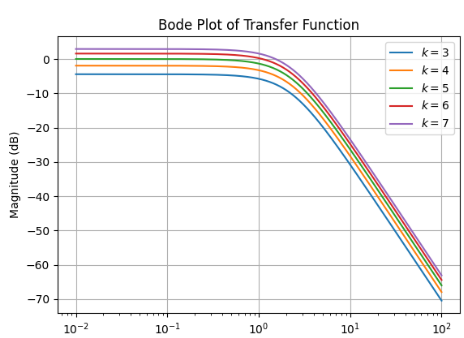


Figure 4.6: Graph showing $k < 5, k = 5, k > 5$

For an open transfer function to be stable, its magnitude in the bode plot should be positive for some positive frequency.

In the below graph we can observe that the above condition satisfies for k>5.

4.5 The signal flow graph of a system is shown. The expression for $\frac{Y(s)}{X(s)}$ is

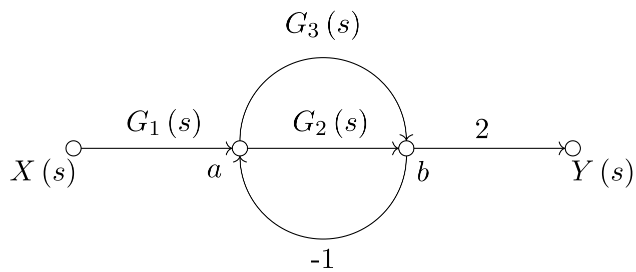


Figure 4.7: Signal Flow Graph of the System

(a) $\frac{2G_1(s)G_2(s)+2G_1(s)G_3(s)}{1+G_2(s)+G_3(s)}$

(b) $2+G_1(s)+G_3(s)+\frac{G_2(s)}{1+G_2(s)}$

(c) $G_1(s)+G_3(s)-\frac{G_2(s)}{2+G_2(s)}$

(d) $\frac{2G_1(s)G_2(s)+2G_1(s)G_3(s)-G_1(s)}{1+G_2(s)+G_3(s)}$

(GATE 2022 IN Question 37)

Solution:

Parameter	**Description**	**Value**
$Y(s)$	Output node variable	
$X(s)$	Input node variable	
$\frac{Y(s)}{R(s)}$	Transfer function	?
P_1	Forward Path Gain a-b through $G_2(s)$	$2G_1(s)G_2(s)$
P_2	Forward Path Gain a-b through $G_3(s)$	$2G_1(s)G_3(s)$
Δ_1	Determinant of Forward Path a-b through $G_2(s)$	1
Δ_2	Determinant of Forward Path a-b through $G_3(s)$	1
L_1	Gain of Loop a-b through $G_2(s)$ and back	$-G_2(s)$
L_2	Gain of Loop a-b through $G_3(s)$ and back	$-G_3(s)$
Δ	Determinant of System	$1+G_2(s)+G_3(s)$
n	Number of forward paths	2

Table 4.9: Variables Used

$$P_1 = (G_1(s))(G_2(s))(2) = 2G_1(s)G_2(s) \tag{4.54}$$

$$P_2 = (G_1(s))(G_3(s))(2) = 2G_1(s)G_3(s) \tag{4.55}$$

$$\Delta_1 = 1 - (0) = 1 \tag{4.56}$$

$$\Delta_2 = 1 - (0) = 1 \tag{4.57}$$

$$L_1 = -G_2(s) \tag{4.58}$$

$$L_2 = -G_3(s) \tag{4.59}$$

$$\Delta = 1 - (L_1 + L_2) = 1 + G_1(s) + G_2(s) \tag{4.60}$$

From 4.7, Using Mason's Gain Formula,

$$\frac{Y(s)}{X(s)} = \frac{\sum_{i=1}^{n} P_i\Delta_i}{\Delta} \tag{4.61}$$

$$= \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} \tag{4.62}$$

$$= \frac{2G_1(s)G_2(s)(1) + 2G_1(s)G_3(s)(1)}{1 + G_2(s) + G_3(s)} \tag{4.63}$$

$$\implies \frac{Y(s)}{X(s)} = \frac{2G_1(s)G_2(s) + 2G_1(s)G_3(s)}{1 + G_2(s) + G_3(s)} \tag{4.64}$$

4.6 The output of the system y(t) is related to its input x(t) according to the relation $y(t) = x(t)\, sin(2\pi t)$.This system is

(A) Linear and time-variant

(B) Non-Linear and time-invariant

(C) Linear and time-invariant

(D) Non-linear and time-variant

(GATE 2022 IN Question 14) **Solution:**

Symbol	**Value**	**Description**
$x(t)$		input signal
$y(t)$	$x(t)\, sin(2\pi t)$	output signal
τ		Time delay

Table 1: input parameters

From Table 1

$$y_1(t) \leftrightarrow x_1(t) \tag{4.65}$$

$$y_2(t) \leftrightarrow x_2(t) \tag{4.66}$$

$$ay_1(t) + by_2(t) \leftrightarrow ax_1(t) + bx_2(t) \tag{4.67}$$

$$ay_1(t) + by_2(t) = (ax_1(t) + bx_2(t))\, sin(2\pi t) \tag{4.68}$$

$\therefore$ satisfies principle of superposition

$$ky\left(t\right) \leftrightarrow kx\left(t\right) \tag{4.69}$$

$$ky\left(t\right) = k\left(x\left(t\right)sin\left(2\pi t\right)\right) \tag{4.70}$$

$\therefore$ satisfies principle of homogenity

$\therefore$ it is linear

Delay in input x(t):

$$y_1\left(t\right) = x\left(t-\tau\right)sin\left(2\pi t\right) \tag{4.71}$$

Delay in output y(t):

$$y\left(t-\tau\right) = x\left(t-\tau\right)sin\left(2\pi\left(t-\tau\right)\right) \tag{4.72}$$

$$y_2\left(t\right) = x\left(t-\tau\right)sin\left(2\pi\left(t-\tau\right)\right) \tag{4.73}$$

$$y_1\left(t\right) \neq y_2\left(t\right) \tag{4.74}$$

$\therefore$ it is time variant

$\therefore$ (A) linear and time variant

4.7 Two linear time-invariant systems with transfer functions

$$G_1(s) = \frac{10}{s^2 + s + 1}$$

and

$$G_2(s) = \frac{10}{s^2 + s\sqrt{10} + 10}$$

have unit step responses $y_1(t)$ and $y_2(t)$, respectively. Which of the following statements is/are true?

(a) $y_1(t)$ and $y_2(t)$ have the same percentage peak overshoot.

(b) $y_1(t)$ and $y_2(t)$ have the same steady state values.

(c) $y_1(t)$ and $y_2(t)$ have the same damped frequency of oscillation.

(d) $y_1(t)$ and $y_2(t)$ have the same 2% settling time.

(GATE 2022 EC Q50)

Solution: The general second-order transfer function is given by:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.75}$$

After comparing the coefficients of $G_1(s)$ and $G_2(s)$, as $\zeta = \frac{1}{2}$ is less than 1, the

Parameter	**Description**	**value**
$X_1(s)$	input	$\frac{1}{s}$
$X_2(s)$	input	$\frac{1}{s}$
$G_1(s)$	transfer function	$\frac{10}{s^2+s+1}$
$G_2(s)$	transfer function	$\frac{10}{s^2+s\sqrt{10}+10}$
$y_1(t)$	unit step response	−
$y_2(t)$	unit step response	−
ω_n	natural frequency	−
ζ	damping ratio	−

Table 4.11: Given Parameters

Tranfer function	ω_n	ζ
$G_1(s)$	1	$\frac{1}{2}$
$G_1(s)$	$\sqrt{10}$	$\frac{1}{2}$

Table 4.12: Given Parameters

system is underdamped.

$$Y(s) = X(s)\,G(s) \tag{4.76}$$

$$= \frac{1}{s}\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{4.77}$$

Applying inverse laplace transform,

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{1-\zeta^2}\sin(\omega_d t + \phi) \tag{4.78}$$

where ω_d is the damped frequency of oscillation.

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{4.79}$$

The percentage peak overshoot (PO):

$$PO = \left(\frac{y_{\text{max}} - y_{\text{ss}}}{y_{\text{ss}}}\right) \times 100\% \tag{4.80}$$

y_{max} is obtained by differentiating (4.78) with respect to time and equating it to zero, substituting the value in (4.78),

$$y_{\text{max}} = 1 + \frac{1}{\sqrt{1-\zeta^2}} \tag{4.81}$$

y_{ss} is obtained by final value theorem,

$$\begin{aligned} y_{\text{ss}} &= \lim_{s\to 0} sY(s) && (4.82)\\ &= \lim_{s\to 0} s\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s} && (4.83)\\ &= 1 && (4.84)\end{aligned}$$

Substituting the values of y_{max} and y_{ss} in (4.80),

$$PO = \frac{1}{\sqrt{1-\zeta^2}} \times 100\% \tag{4.85}$$

$y_1(t)$ and $y_2(t)$ have same ζ, they have same percentage peak overshoot.So, option (1) is correct.

The steady state value of $y(t)$ is given by final value theorem:

$$y_{1ss} = \lim_{s \to 0} sY_1(s) \tag{4.86}$$

$$= \lim_{s \to 0} s \frac{10}{s^2 + s + 1} \frac{1}{s} \tag{4.87}$$

$$= 10 \tag{4.88}$$

$$y_{2ss} = \lim_{s \to 0} sY_2(s) \tag{4.89}$$

$$= \lim_{s \to 0} s \frac{10}{s^2 + s\sqrt{10} + 10} \frac{1}{s} \tag{4.90}$$

$$= 1 \tag{4.91}$$

as both the unit step responses have different steady state values, option (2) is incorrect.

From (4.80), as ω_n is different for $y_1(t)$ and $y_2(t)$, they have different damped frequency of oscillation. Hence option (3) is incorrect.

Settling time T_s:

$$T_s = \frac{4}{\zeta \omega_n} \tag{4.92}$$

As, ω_n is different for $y_1(t)$ and $y_2(t)$, they have different 2% settling time, Hence option (4) is incorrect.

So, only option (1) is correct.

4.8 Consider a single-input-single-output (SISO) system with the transfer function

$$G_p(s) = \frac{2\left(s+1\right)}{\left(\frac{1}{2}s+1\right)\left(\frac{1}{4}s+1\right)}$$

where the time constants are in minutes. The system is forced by a unit step input at time $t = 0$. The time at which the output response reaches the maximum is ______ minutes (rounded off to two decimal places). (GATE CH 2022)

Solution:

Parameters	**Description**	**Value**
$y(t)$	Output response	
$G_p(s)$	Transfer function	$\frac{2(s+1)}{\left(\frac{1}{2}s+1\right)\left(\frac{1}{4}s+1\right)}$
$x(t)$	Input	$u(t)$
$X(s)$	Laplace transform of x(t)	$\frac{1}{s}$
$y'(t)$	$\frac{dy}{dt}$	

Table 4.13: Parameters

$$Y(s) = G_p(s)X(s) \tag{4.93}$$

$$= \frac{16\left(s+1\right)}{s\left(s+2\right)\left(s+4\right)} \tag{4.94}$$

$$= \frac{2}{s} + \frac{4}{s+2} - \frac{6}{s+4} \tag{4.95}$$

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} \tag{4.96}$$

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \tag{4.97}$$

From Laplace transforms (4.96) and (4.97), we get

$$y(t) = \left(2 + 4e^{-2t} - 6e^{-4t}\right) u(t) \tag{4.98}$$

For maximum value of $y(t)$,

$$y'(t) = 0 \tag{4.99}$$

$$\implies -8e^{-2t} + 24e^{-4t} = 0 \tag{4.100}$$

$$e^{2t} = 3 \tag{4.101}$$

$$\implies t = \frac{\ln 3}{2} \tag{4.102}$$

$$\approx 0.55 \tag{4.103}$$

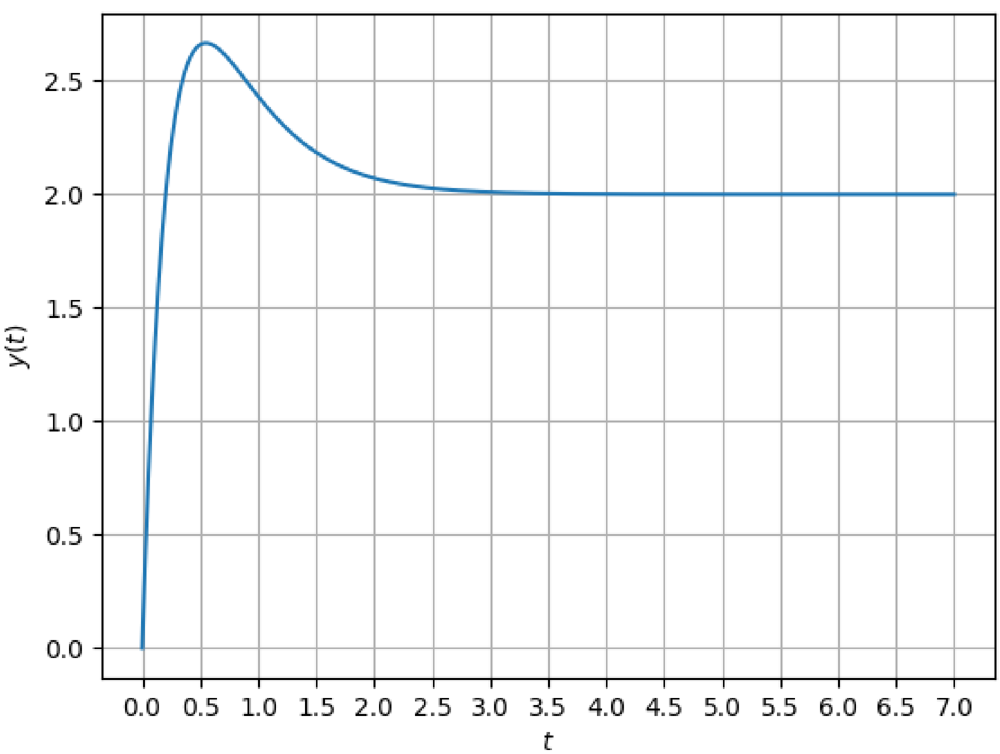
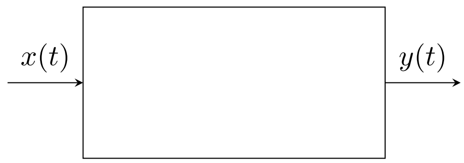


Figure 4.8: Plot of $y(t)$

4.9 Consider the system as shown below:



The system is described by the equation

$$y(t) = x(e^{-t}).$$

The system is:

(A) non-linear and causal.

(B) linear and non-causal.

(C) non-linear and non-causal.

(D) linear and causal.

(GATE EE 2022)

Solution:

Chapter 5

Sequences

5.1 Discrete signals $x(n)$ and $y(n)$ are shown below. The cross-correlation $r_{xy}(0)$ is:

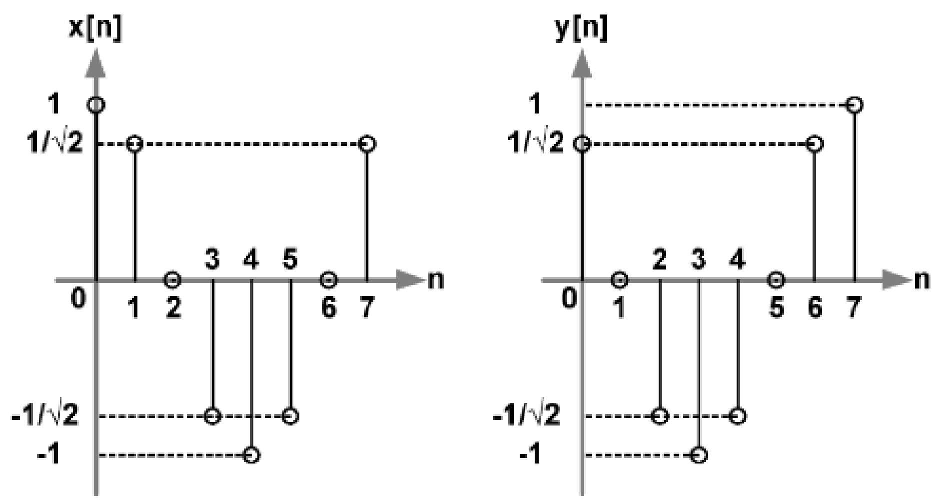


Figure 5.1: Question Figure

(GATE BM 2022)

Solution:

Parameter	Description	Value
$x(n)$	First Sequence	$x(n) = \begin{cases} 0 & ; n < 0 \\ \left(1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) & ; 0 \leq n \leq 7 \\ 0 & ; n > 7 \end{cases}$
$y(n)$	Second Sequence	$y(n) = \begin{cases} 0 & ; n < 0 \\ \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1\right) & ; 0 \leq n \leq 7 \\ 0 & ; n > 7 \end{cases}$
$r_{xy}(k)$	Cross-correlation	$\sum_{m=-\infty}^{\infty} x(m)\, y(m-k)$

Table 1: Parameter Table

It can be seen that :

$$y(n) = x(n+1) \tag{5.1}$$

From Table 1 :

$$r_{xy}(k) = \sum_{m=-\infty}^{\infty} x(m)\, y(m-k) \tag{5.2}$$

$$= x(k) * y(-k) \tag{5.3}$$

From (5.1):

$$r_{xy}(k) = x(k+1) * x(-k) \tag{5.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n+1)\, x(n+k) \tag{5.5}$$

By definition of x(n) from Table 1:

$$r_{xy}(k) = \sum_{n=0}^{6} x(n+1)\, x(n+k) \tag{5.6}$$

$$r_{xy}(0) = \sum_{n=0}^{6} x(n+1)\, x(n) \tag{5.7}$$

Using values from Fig. 5.1:

$$r_{xy}(0) = 2\sqrt{2} \tag{5.8}$$

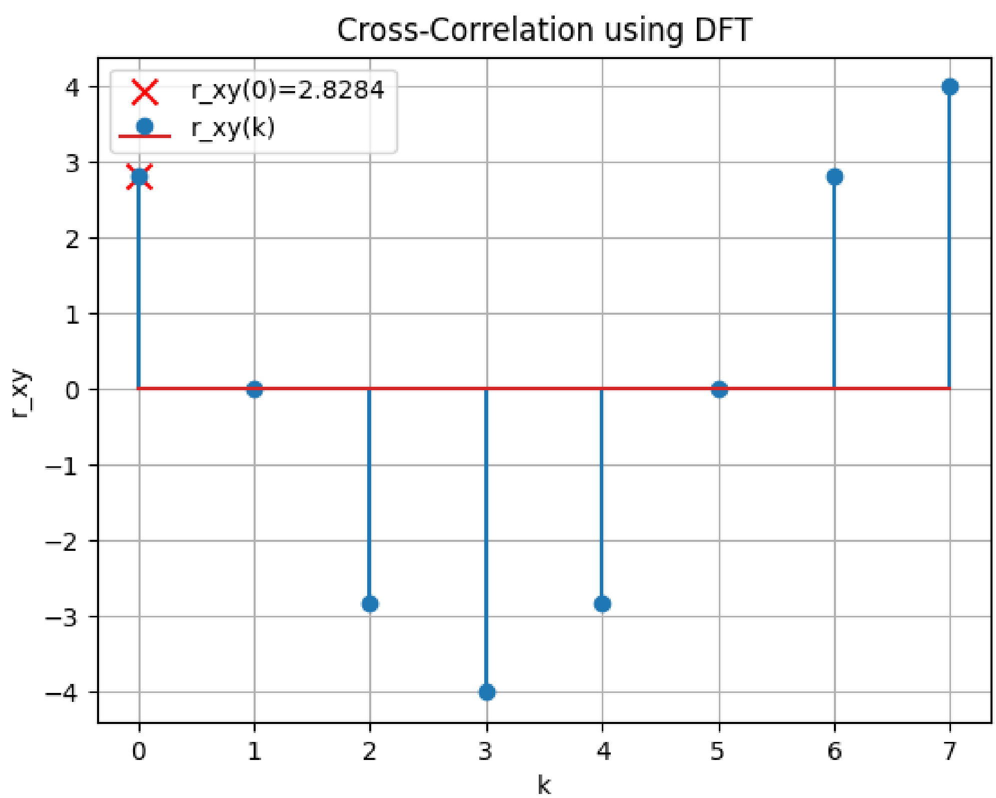


Figure 5.2: Verification of result by DFT

5.2 Which one of the following is the closed form for the generating function of the sequence $\{a\}_{n\geq 0}$ defined below?

$$a_n = \begin{cases} n+1 & \text{, n is odd} \\ 1 & \text{otherwise} \end{cases} \tag{5.9}$$

(A) $\frac{x(1+x)^2}{(1-x^2)^2} + \frac{1}{1-x}$

(B) $\frac{x\left(3-x^2\right)}{(1-x^2)^2} + \frac{1}{1-x}$

(C) $\frac{2x}{(1-x^2)^2} + \frac{1}{1-x}$

(D) $\frac{x}{(1-x^2)^2} + \frac{1}{1-x}$

(GATE CS 2022 QUESTION 36)

Solution: For the given sequence:

Parameter	Description	Value
$X(z)$	Generating function for a sequence $\{a_n\}$	?
a_n	n^{th} term of the sequence	$(n+1)\,u(n)$ (when odd)
		$u(n)$ (when even)

Table 5.2: input values

$$X(z) = \sum_{k=-\infty}^{\infty} u(2k)\, z^{-2k} + \sum_{k=-\infty}^{\infty} \left((2k+2)\, u(2k+1)\right) z^{-(2k+1)} \tag{5.10}$$

$$\implies X(z) = \left(1 + z^{-2} + z^{-4} + \dots\right) + \left(2z^{-1} + 4z^{-3} + 6z^{-5} + \dots\right) \tag{5.11}$$

$$\implies X(z) = \frac{1}{1-z^{-2}} + \left(2z^{-1} + 4z^{-3} + 6z^{-5} \dots\right) \quad |z| > 1 \tag{5.12}$$

$$\implies X(z) = \frac{1}{1-z^{-2}} + 2z^{-1}\left(\frac{1}{1-z^{-2}} + \frac{z^{-2}}{(1-z^{-2})^2}\right) \quad |z| > 1 \tag{5.13}$$

$$\therefore X(z) = \frac{1}{1-z^{-1}} + \frac{z^{-1}\left(1+z^{-2}\right)}{\left(1-z^{-2}\right)^2} \quad |z| > 1 \tag{5.14}$$

(5.14) is the closed form of generating function required in the question.

Hence, option (A) is correct.

$$X(z) = X_1(z) + X_2(z) \tag{5.15}$$

$$X_1(z) = \frac{1}{1-z^{-1}} \quad |z| > 1 \tag{5.16}$$

$$\implies x_1(n) = u(n) \tag{5.17}$$

$$\implies a_n = x_1(n) + x_2(n) \tag{5.18}$$

To find inverse z-transform of $X_2(z)$ we use contour integration technique:

$$x_2(n) = \frac{1}{2\pi j} \oint_C X_2(z)\ z^{n-1}\, dz \tag{5.19}$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^n\left(z^2+1\right)}{\left(z^2-1\right)^2}\, dz \tag{5.20}$$

We can observe that we have two poles at

$z = 1, -1$. And poles are repeated twice, thus by applying residue theorem two times for poles 1 and -1:

$$x_2(n) = \frac{1}{(1)!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2 X_2(z)\right) + \frac{1}{(1)!}\lim_{z\to -1}\frac{d}{dz}\left((z+1)^2 X_2(z)\right) \tag{5.21}$$

$$\implies x_2(n) = \lim_{z\to 1}\frac{d}{dz}\left((z-1)^2\frac{z^n\left(z^2+1\right)}{\left(z^2-1\right)^2}\right) + \lim_{z\to -1}\frac{d}{dz}\left((z+1)^2\frac{z^n\left(z^2+1\right)}{\left(z^2-1\right)^2}\right) \tag{5.22}$$

$$\implies x_2(n) = \lim_{z\to 1}\frac{(z+1)^2\left(nz^{n-1}+(n+2)\,z^{n+1}\right)-2z^n\left(1+z^2\right)(z+1)}{(z+1)^4}$$
$$+ \lim_{z\to -1}\frac{(z-1)^2\left(nz^{n-1}+(n+2)\,z^{n+1}\right)-2z^n\left(1+z^2\right)(z-1)}{(z-1)^4} \tag{5.23}$$

on simplification, we get

$$x_2(n) = \frac{n+n(-1)^{n-1}}{2} \tag{5.24}$$

$$\therefore a_n = u(n) + \frac{n+n(-1)^{n-1}}{2}u(n) \tag{5.25}$$

Which is the sequence given in the Question.

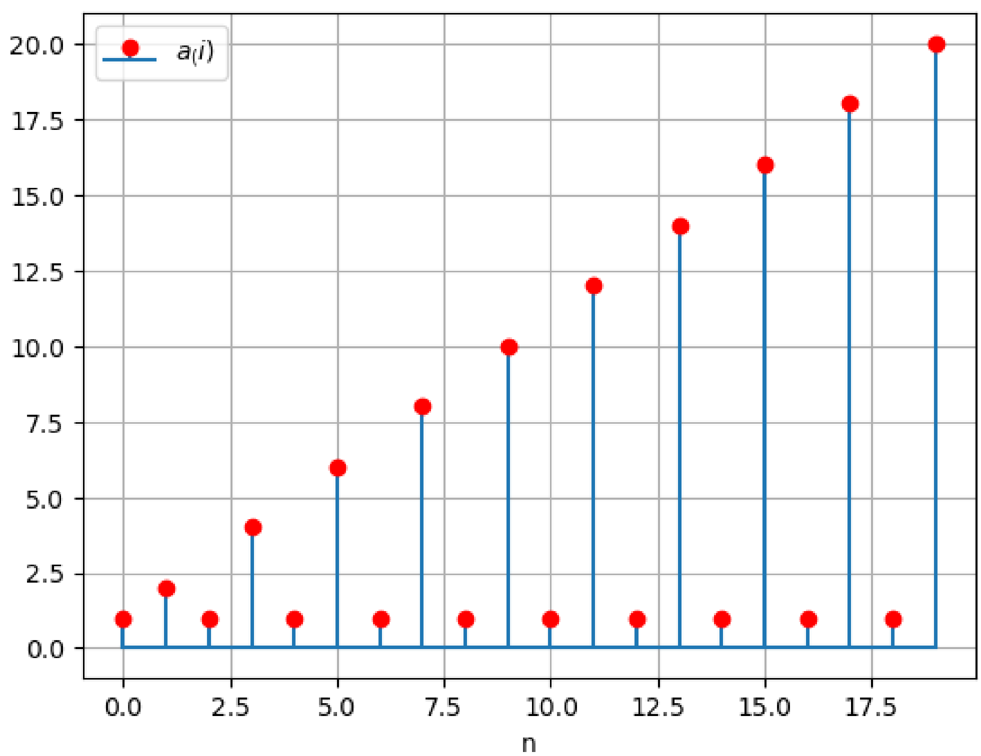


Figure 5.3: Terms of the sequence given

Chapter 6

Sampling

6.1

Chapter 7

Contour Integration

7.1 In the complex z-domain, the value of integral $\oint_C \frac{z^3-9}{3z-i}\,dz$ is

(a) $\frac{2\pi}{81} - 6i\pi$

(b) $\frac{2\pi}{81} + 6i\pi$

(c) $-\frac{2\pi}{81} + 6i\pi$

(d) $-\frac{2\pi}{81} - 6i\pi$

(GATE 2022 BM)

Solution:

Simplyfying the Contour Integral to the standard form we get,

$$\oint_C \frac{z^3-9}{3z-i}\,dz = \frac{1}{3}\oint_C \frac{z^3-9}{z-\frac{i}{3}}\,dz \tag{7.1}$$

From Cauchy's residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \sum R_j \tag{7.2}$$

We can observe a non-repeated pole at $z = \frac{i}{3}$ and thus $a = \frac{i}{3}$,

$$R = \lim_{z \to a} (z - a) f(z) \tag{7.3}$$

$$\implies R = \frac{1}{3} \lim_{z \to \frac{i}{3}} \left(z - \frac{i}{3} \right) \frac{z^3 - 9}{z - \frac{i}{3}} \tag{7.4}$$

$$= \frac{-i}{81} - 3 \tag{7.5}$$

Therefore, from (7.2) and (7.5)

$$\oint_C \frac{z^3 - 9}{3z - i} \, dz = \frac{2\pi}{81} - 6i\pi \tag{7.6}$$

7.2 Consider the function

$$f(z) = \frac{1}{(z+1)(z+2)(z+3)}$$

The residue of $f(z)$ at $z=-1$, is ______ (GATE 2022 IN)

Solution: Residue of a function $f(z)$ at a simple pole c is

$$\operatorname{Res}(f,c) = \lim_{z\to c}(z-c)\,f(z) \tag{7.7}$$

$$\implies \operatorname{Res}(f,-1) = \lim_{z\to -1}\frac{z+1}{(z+1)(z+2)(z+3)} \tag{7.8}$$

$$= \frac{1}{2} \tag{7.9}$$

$\therefore$ residue of $f(z)$ at $z=-1$ is $\frac{1}{2}$.

Chapter 8

Laplace Transform

8.1 Consider the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$. The boundary conditions are $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$. Then the value of y at $x = \frac{1}{2}$ (GATE AE 2022)

Solution:

Parameters	**Values**	**Description**
$y(0)$	0	y at $x = 0$
$y'(0)$	1	$\frac{dy}{dx}$ at $x = 0$

Table 8.1: Parameters

$$\frac{d^2y}{dx^2} \xleftrightarrow{\mathcal{L}} s^2Y(s) - sy(0) - y'(0) \tag{8.1}$$

$$\frac{dy}{dx} \xleftrightarrow{\mathcal{L}} sY(s) - y(0) \tag{8.2}$$

Applying Laplace Transform, using (8.1) and (8.2),

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + Y(s) = 0 \tag{8.3}$$

From Table 8.1,

$$(s^2 - 2s + 1)Y(s) - 1 = 0 \tag{8.4}$$

$$Y(s) = \frac{1}{(s-1)^2} \tag{8.5}$$

$$t^n \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{n!}{s^{n+1}} \tag{8.6}$$

$$e^{at}x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s-a) \tag{8.7}$$

Taking Inverse Laplace Transform for $Y(s)$, using (8.6) and (8.7),

$$y(x) = xe^x \tag{8.8}$$

$$\implies y\left(\frac{1}{2}\right) = \frac{\sqrt{e}}{2} \tag{8.9}$$

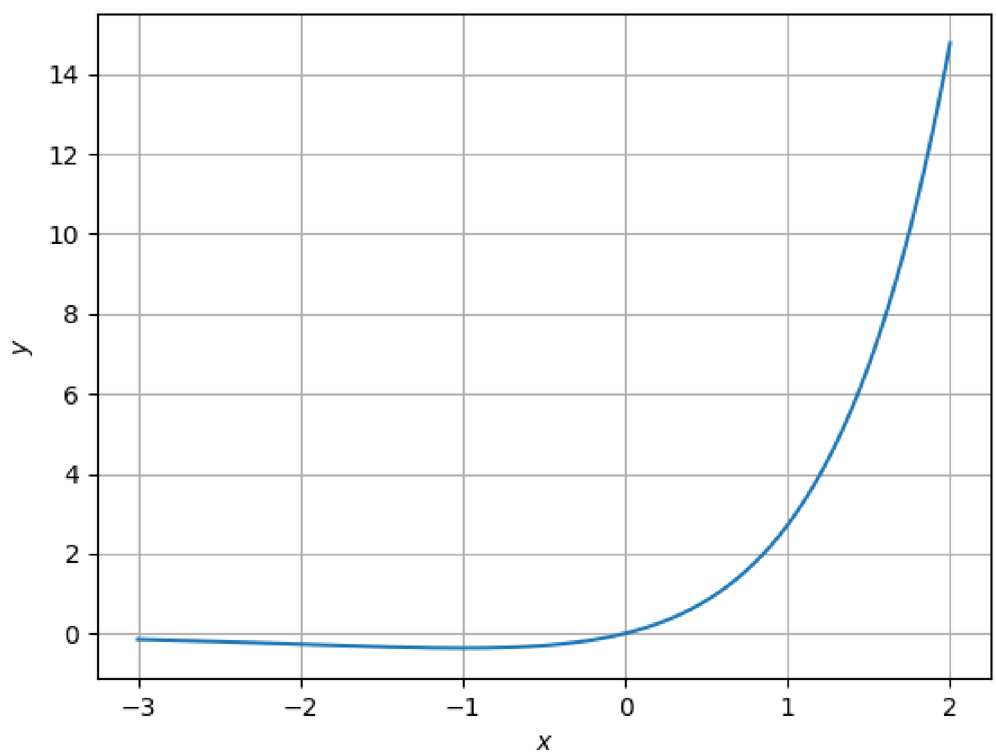


Figure 8.1: Plot of $y(x)$

8.2 A process described by the transfer function

$$G_p(s) = \frac{(10s+1)}{(5s+1)}$$

is forced by a unit step input at time $t = 0$. The output value immediately after the unit step input (at $t = 0^+$) is ? (Gate 2022 CH 34)

Solution:

Parameters	**Description**
$X(s)$	Laplace transform of $x(t)$
$Y(s)$	Laplace transform of $y(t)$
$G_p(s) = \frac{Y(s)}{X(s)}$	Transfer function
$x(t) = u(t)$	unit step function

Table 8.2: Given parameters

$$G_p(s) = \frac{Y(s)}{X(s)} = \frac{(10s+1)}{(5s+1)} \tag{8.10}$$

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} \tag{8.11}$$

From equation (8.11):

$$Y(s) = \frac{(10s+1)}{s\,(5s+1)} \tag{8.12}$$

$$= \frac{1}{s} + \frac{5}{5s+1} \tag{8.13}$$

Taking inverse laplace transformation,

$$\frac{1}{s} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} u(t) \tag{8.14}$$

$$\frac{1}{s-c} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} e^{ct}u(t) \tag{8.15}$$

$$y(t) = \left(1 + e^{\frac{-t}{5}}\right) u(t) \tag{8.16}$$

$$y(0^{+}) = 2 \tag{8.17}$$

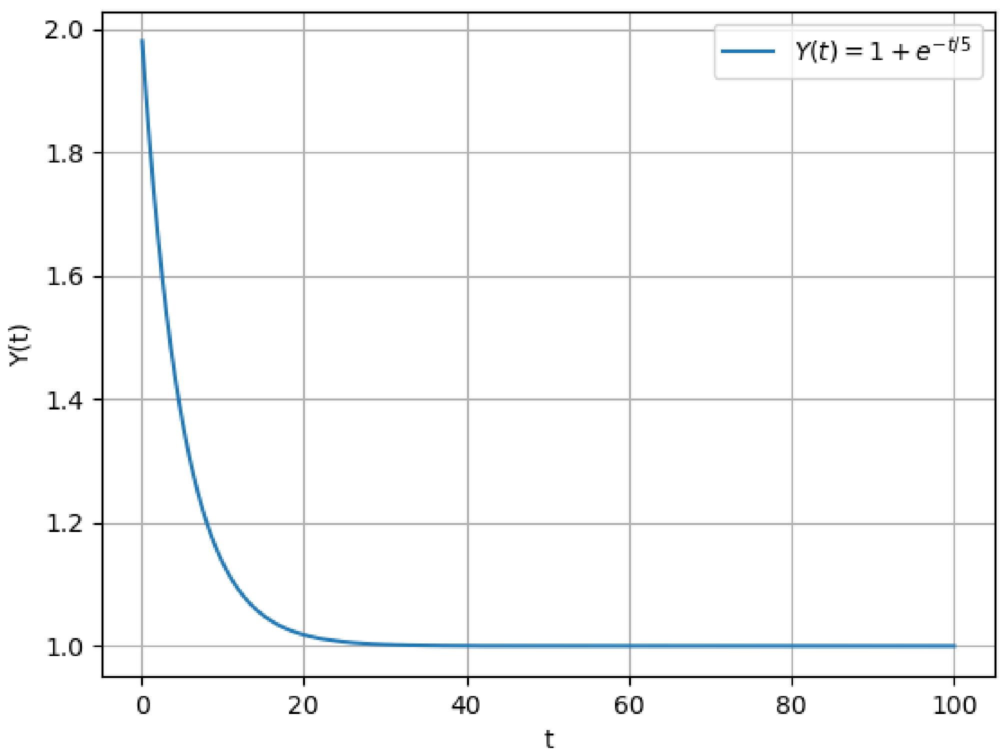


Figure 8.2: Graph of y(t)

8.3 The transfer function of a real system $H(S)$ is given as:

$$H(s) = \frac{As + B}{s^2 + Cs + D}$$

where A, B, C and D are positive constants. This system cannot operate as

(A) Low pass filter

(B) High pass filter

(C) Band pass filter

(D) An Integrator

(GATE EE 11 2022)

Solution: The transfer function $H(s)$ is given by:

$$H(s) = \frac{As + B}{s^2 + Cs + D} \tag{8.18}$$

Put $s = j\omega$ in (8.18):

$$H(j\omega) = \frac{A(j\omega) + B}{(j\omega)^2 + C(j\omega) + D} \tag{8.19}$$

$$|H(j\omega)| = \frac{\sqrt{(A\omega)^2 + B^2}}{\sqrt{(D - \omega^2)^2 + (\omega C)^2}} \tag{8.20}$$

a) Low Pass Filter:

At low frequency ($\omega = 0$):

$$|H(\omega = 0)| = \frac{B}{D} \tag{8.21}$$

$\therefore$ H(s) can operate as Low pass filter.

Parameter	Description
Low Pass Filter	The gain should be finite at low frequency
High Pass Filter	The gain should be finite at high frequency
Band Pass Filter	Finite gain over frequency band
Integrator	Transfer function should have at least one pole at origin

Table 8.3: Conditions

b) High Pass Filter:

At high frequency ($\omega = \infty$):

$$|H(\omega = \infty)| = 0 \tag{8.22}$$

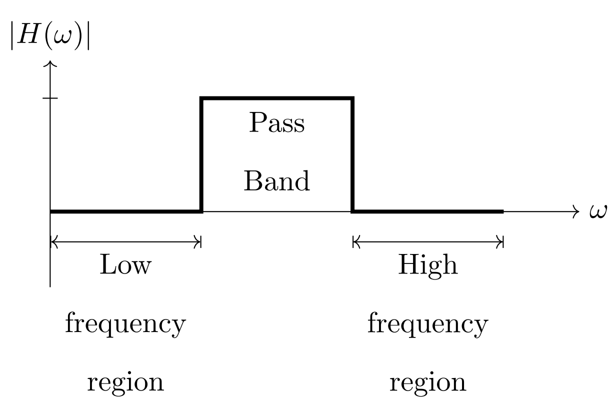
$\therefore$ $H(s)$ cannot operate as High pass filter.

c) Band Pass Filter:

Assuming B is a very less positive valued constant as compared to others:

$$|H(j\omega)| = \frac{(A\omega)}{\sqrt{(D-\omega^2)^2 + (\omega C)^2}} \tag{8.23}$$

$$\implies |H(\omega = 0)| = 0 \text{ and } |H(\omega = \infty)| = 0 \tag{8.24}$$



$\because$ $H(s)$ passes frequency between low and high frequencies.

$\therefore$ $H(s)$ can operate as a band pass filter.

d) Integrator:

At very high value of frequency($\omega \to \infty$):

$$H(s) \approx \frac{As}{s^2} \approx \frac{A}{s} \tag{8.25}$$

From Table 8.3:

$\therefore$ $H(s)$ can operate as an Integrator.

8.4 In a circuit, there is a series connection of an ideal resistor and an ideal capacitor. The conduction current (in Amperes) through the resistor is $2\sin\left(t+\frac{\pi}{2}\right)$. The displacement current (in Amperes) through the capacitor is ______.

(A) $2\sin(t)$

(B) $2\sin(t+\pi)$

(C) $2\sin\left(t+\frac{\pi}{2}\right)$

(D) 0

(GATE 2022 EC 24)

Solution:

Parameter	Description	Value
I_c	Conduction Current	$2\sin\left(t+\frac{\pi}{2}\right)$
A	Cross-sectional area	

Table 8.4: Parameters

Parameter	Description	Formula
Q	Charge	$\int I_c\, dt$
D	Electric Displacement	$\frac{Q}{A}$
J_D	Displacement current density	$\frac{\partial D}{\partial t}$
I_D	Displacement current	J_D x A

Table 8.5: Formulae

S Domain	Time Domain
$\frac{1}{s}$	$u(t)$
$\frac{-s}{a^2+s^2}$	$-\cos(at)$
$\frac{a}{a^2+s^2}$	$\sin(at)$
$\frac{1}{s+a}$	e^{-at}

Table 8.6: Laplace transforms

$$\mathcal{L}\left[\int f(t)\ dt\right] = \int_0^\infty \left[\int f(t)\ dt\right] e^{-st}\, dt \tag{8.26}$$

$$= \int_0^\infty u\, dv \quad \text{where} \begin{cases} u = \int f(t)\, dt \\ dv = e^{-st} dt \end{cases} \tag{8.27}$$

$$= uv - v \int du \tag{8.28}$$

$$= \frac{1}{s}\int f(t)\, dt|_0 + \frac{1}{s}\int_0^\infty f(t)\, e^{-st} dt \tag{8.29}$$

$$\implies \frac{1}{s}\int f(t)\, dt|_0 + \frac{1}{s}F(s) \tag{8.30}$$

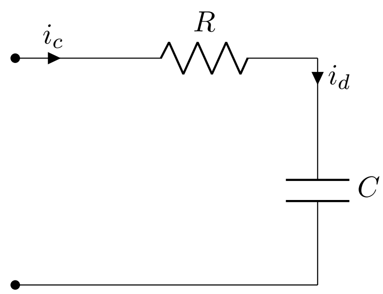


Figure 8.3: Circuit 1

From Table 8.5, Table 8.6 and eq (8.30)

$$I_c(s) = \frac{2s}{s^2+1} \tag{8.31}$$

$$Q_c(s) = \frac{2}{s(s^2+1)} \tag{8.32}$$

$$D(s) = \frac{1}{A}\left(\frac{2}{s(s^2+1)}\right) \tag{8.33}$$

$$J_D(s) = \frac{2}{A}\left(\frac{1}{s^2+1}\right) \tag{8.34}$$

$$I_D(s) = \frac{2}{s^2+1} \tag{8.35}$$

$$\implies I_D = 2\sin t \tag{8.36}$$

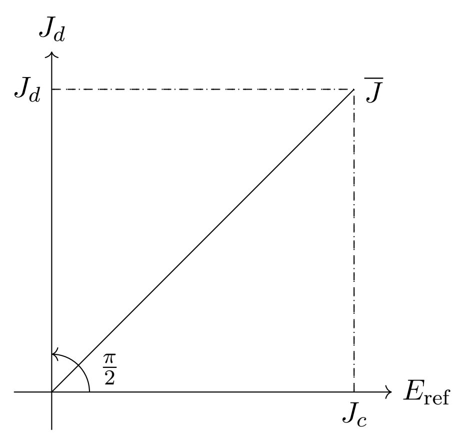


Figure 8.4: Phasor plot

From figure 8.4, phase of I_d is $\frac{\pi}{2}$

$$\therefore I_d = 2\sin\left(t + \frac{\pi}{2}\right) \tag{8.37}$$

$\therefore$ (C) is correct.

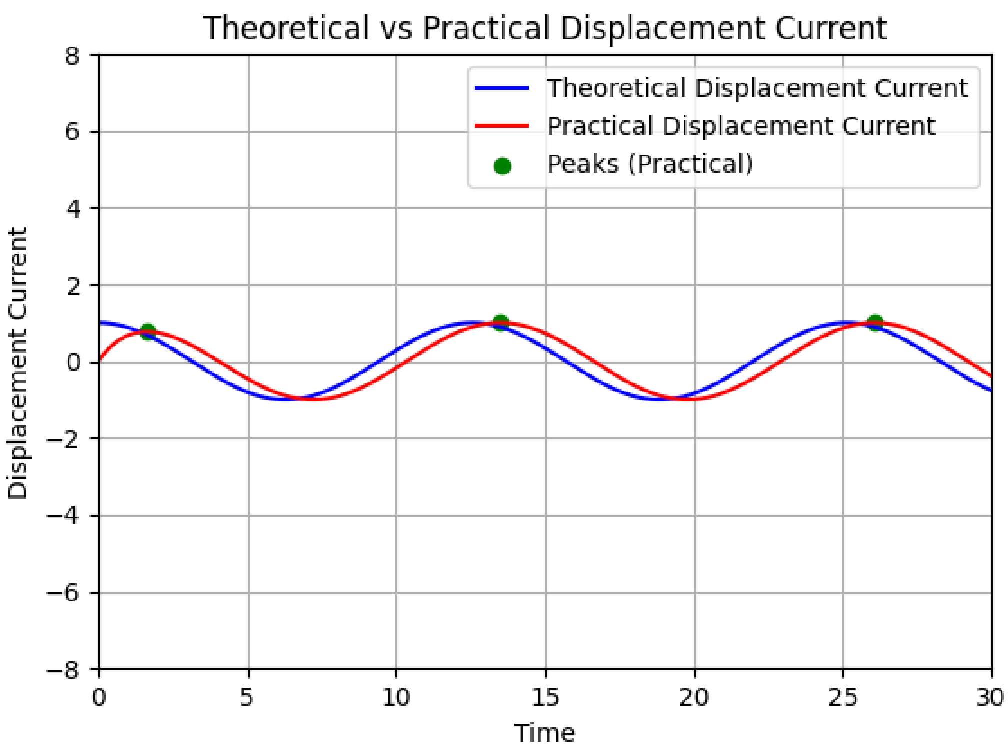


Figure 8.5: Thoritical vs Practical simulation

8.5 Given, $y = f(x)$; $\frac{d^2y}{dx2} + 4y = 0; y(0) = 0; \frac{dy}{dx}(0) = 1$. The problem is a/an

(a) initial value problem having soluition $y = x$

(b) boundary value problem having soluition $y = x$

(c) initial value problem having soluition $y = \frac{1}{2}\sin 2x$

(d) boundary value problem having soluition $y = \frac{1}{2}\sin 2x$

(GATE 2022 ES)

Solution:

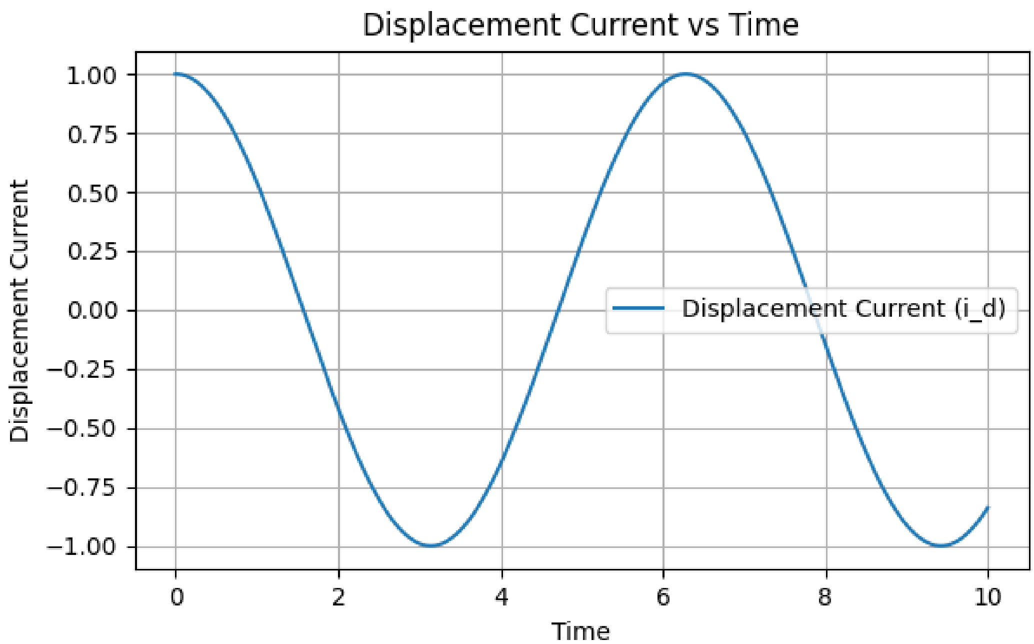


Figure 8.6: Displacement current

The above equation can be written as,

$$y''(t) + 4y(t) = 0 \tag{8.38}$$

Using the Laplace transformation pairs,

$$y''(t) \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - sy(0) - y'(0) \tag{8.39}$$

$$y(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) \tag{8.40}$$

$$\sin at \overset{\mathcal{L}}{\longleftrightarrow} \frac{a}{a^2 + s^2} \tag{8.41}$$

Applying Laplace transform for the equation we get,

$$s^2 Y(s) - 1 + 4Y(s) = 0 \tag{8.42}$$

$$\implies Y(s) = \frac{1}{4+s^2} \tag{8.43}$$

Now, applying inverse laplace transform we get,

$$y(t) = \frac{1}{2}\sin 2t \quad \text{(from (8.41))} \tag{8.44}$$

Since, the conditions at the same point(0) are mentioned, it is an initial valued problem having solution $y = \frac{1}{2}\sin 2x$.

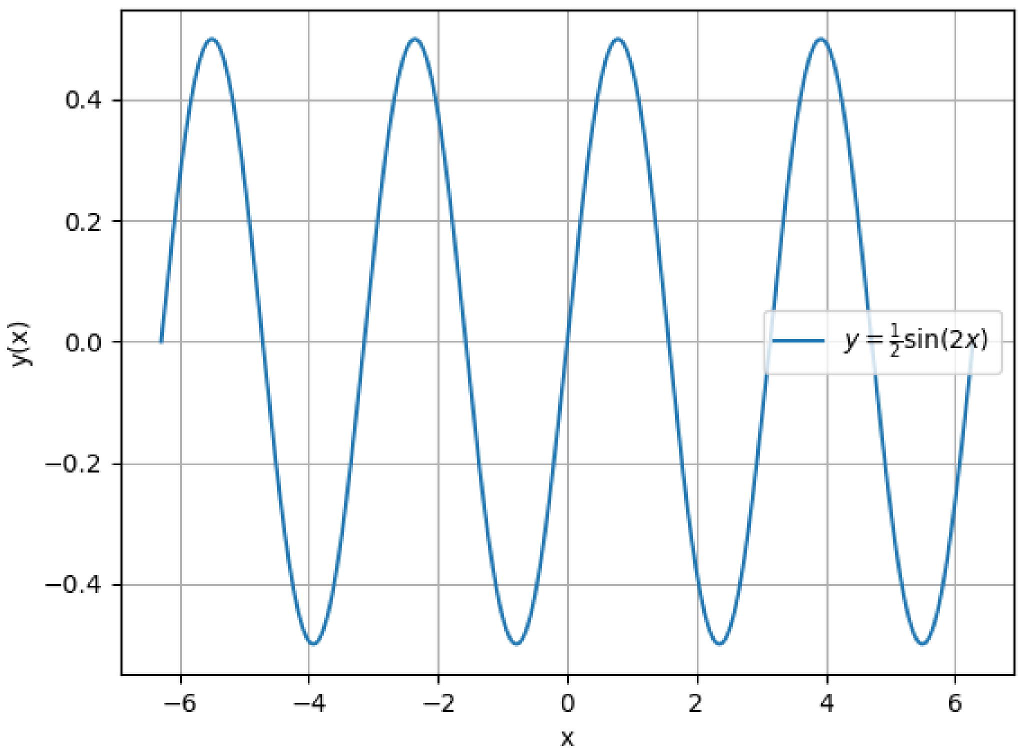


Figure 8.7: $y(x)$ vs x graph

8.6 Let a causal LTI system be governed by the following differential equation,

$$y(t) + \frac{1}{4}\frac{dy}{dt} = 2x(t) \tag{8.45}$$

where $x(t)$ and $y(t)$ are the input and output respectively. It's impulse response is (GATE EE-2022)

Solution: Solution:

From (8.45), corresponding Laplace transform,

$$Y(s) + \frac{1}{4}(sY(s) - y(0)) = 2X(s) \tag{8.46}$$

Since it is causal LTI system,

$$y(0) = 0 \tag{8.47}$$

$$\implies Y(s) + \frac{1}{4}sY(s) = 2X(s) \tag{8.48}$$

$$\implies Y(s) = X(s)\frac{8}{4+s} \tag{8.49}$$

$$\implies H(s) = \frac{8}{4+s} \quad ROC : Re(s) > -4 \tag{8.50}$$

Taking inverse laplace transform and applying causality conditions

$$h(t) = 8e^{-4t}u(t) \tag{8.51}$$

8.7 Assuming $s > 0$; Laplace transform for $f(x) = sin(ax)$ is

(A) $\frac{a}{s^2+a^2}$

(B) $\frac{s}{s^2+a^2}$

(C) $\frac{a}{s^2-a^2}$

(D) $\frac{s}{s^2-a^2}$

(GATE 2022 ES)

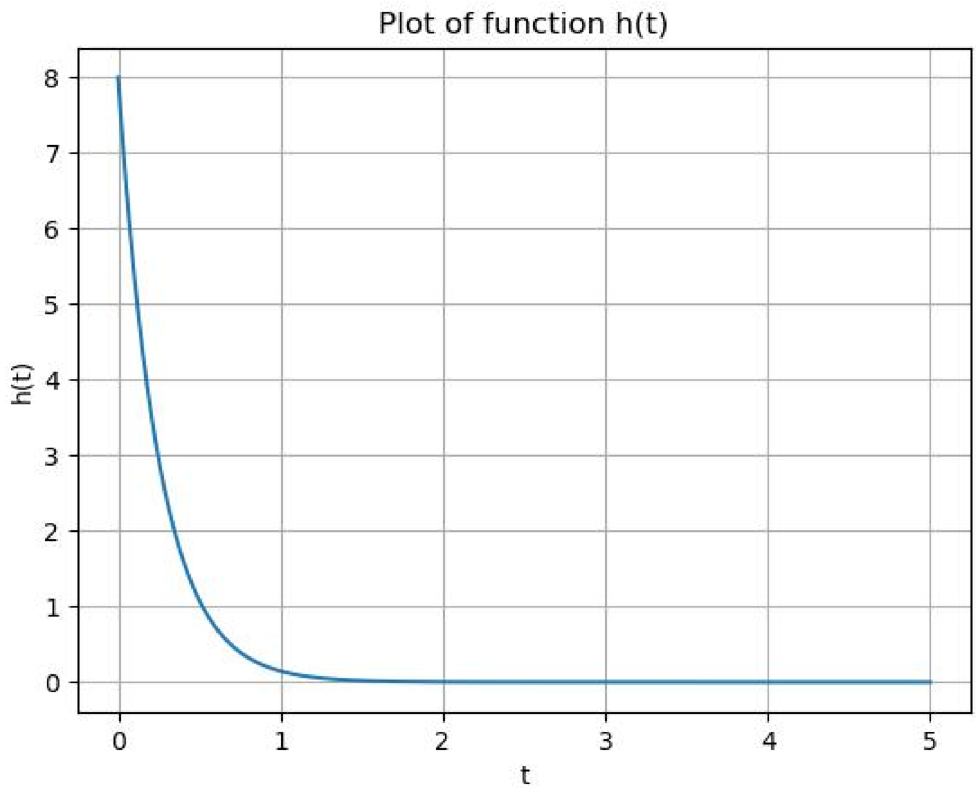


Figure 1: Plot of $h(n)$, taken from python3

Solution:

$$\mathcal{L}(f(x)) = \int_{-\infty}^{\infty} e^{-sx} f(x)\, dx \tag{8.52}$$

We can write $\quad \sin(ax) = \dfrac{e^{ax} - e^{-ax}}{2i}$ (8.53)

From (8.53)

$$\mathcal{L}\left(\sin\left(ax\right)\right) = \int_0^\infty e^{-sx}\left(\frac{e^{iax}-e^{-iax}}{2i}\right)dx \tag{8.54}$$

$$= \frac{1}{2i}\int_0^\infty e^{-x(s-ia)} - e^{-x(s+ia)}dx \tag{8.55}$$

$$= \frac{1}{2i}\left(\frac{e^{-x(s-ia)}}{-\left(s-ia\right)} + \frac{e^{-x(s+ia)}}{-\left(s+ia\right)}\right)_0^\infty \tag{8.56}$$

$$= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) \tag{8.57}$$

$$= \frac{a}{s^2+a^2} \tag{8.58}$$

So, option (A) is correct.

8.8 The input $x(t)$ to a system is related to its output $y(t)$ as

$$\frac{dy(t)}{dt} + y(t) = 3x(t-3)u(t-3)$$

Here $u(t)$ represents a unit-step function.

The transfer function of this system is

(A) $\frac{e^{-3s}}{s+3}$

(B) $\frac{3e^{-3s}}{s+1}$

(C) $\frac{3e^{-(s/3)}}{s+1}$

(D) $\frac{e^{-(s/3)}}{s+3}$

(GATE IN 2022)

Solution:

$$\frac{dy(t)}{dt} + y(t) = 3x(t-3)u(t-3) \tag{8.59}$$

By applying Laplace Transform on both sides

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \tag{8.60}$$

$$x(t-t_o) \overset{\mathcal{L}}{\longleftrightarrow} X(s)e^{-st_o} \tag{8.61}$$

$$sY(s) + Y(s) = 3X(s)e^{-3s} \tag{8.62}$$

$$Y(s)\,(s+1) = 3X(s)e^{-3s} \tag{8.63}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3e^{-3s}}{s+1} \quad (Re(s) > 0) \tag{8.64}$$

$$H(j\omega) = \frac{3e^{-3j\omega}}{1+j\omega} \tag{8.65}$$

$$= \frac{3\,(cos3\omega - jsin3\omega)}{1+j\omega} \tag{8.66}$$

$$|H(j\omega)| = \frac{3}{\sqrt{1+\omega^2}} \tag{8.67}$$

$$phase = \tan^{-1}\left(\frac{\omega\cos(3\omega) + \sin(3\omega)}{\omega\sin(3\omega) - \cos(3\omega)}\right) \tag{8.68}$$

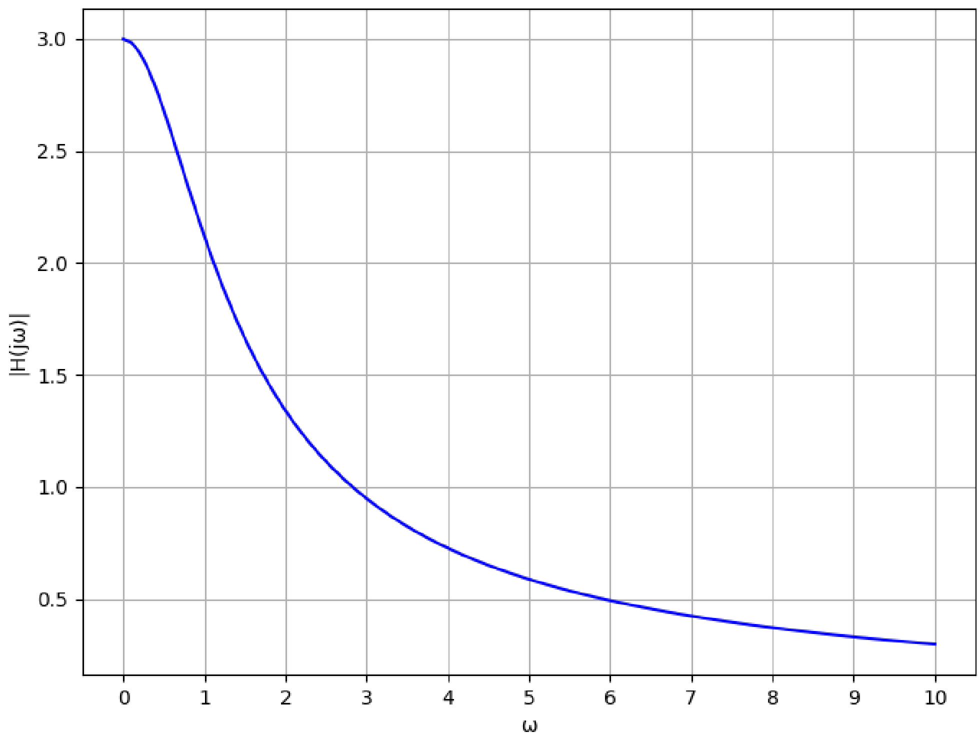


Figure 8.9: Plot for magnitude of transfer function

8.9 Let $x_1(t) = e^{-t}u(t)$ and $x_2(t) = u(t) - u(t-2)$, where $u(.)$ denotes the unit step function. If $y(t)$ denotes the convolution of $x_1(t)$ and $x_2(t)$,then $\lim_{t\to\infty} y(t) = $ ______.
(Rounded off to one decimal place)

(GATE EC 2022)

Solution:

$$y(t) = x_1(t) * x_2(t) \tag{8.69}$$

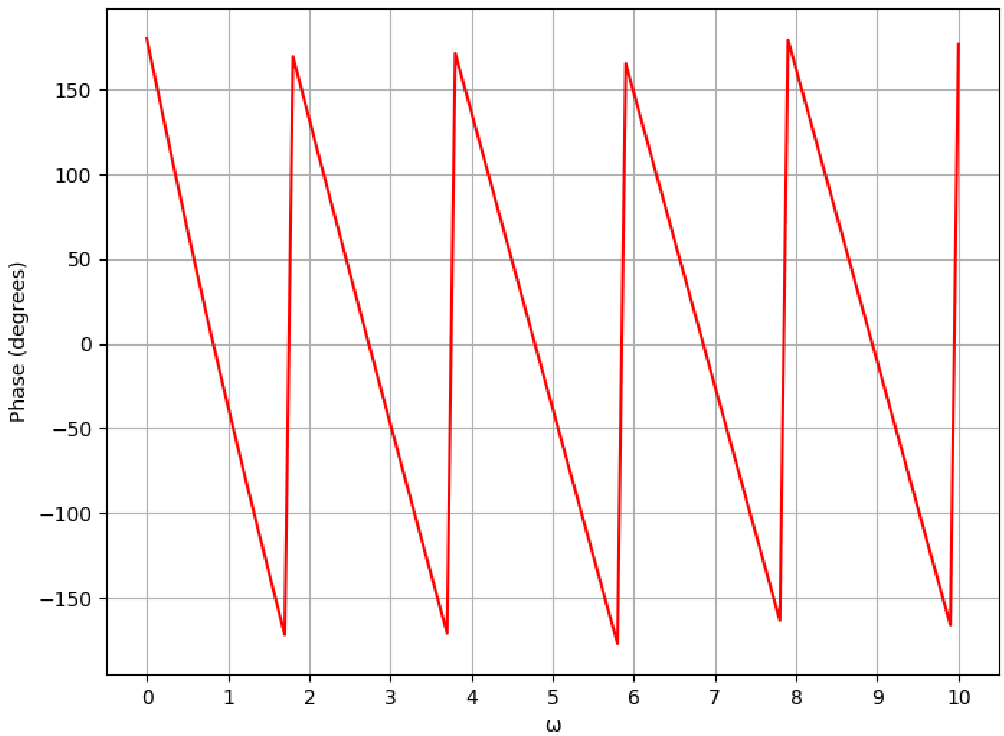


Figure 8.10: Plot for phase of transfer function

variable	value	description
$x_1(t)$	$e^{-t}u(t)$	given function 1
$x_2(t)$	$u(t) - u(t-2)$	given function 2
$y(t)$	-	convolution of $x_1(t)$ and $x_2(t)$

Table 8.7: Table: Input Parameters

from Table 8.7

$$y(t) = e^{-t}u(t) * (u(t) - u(t-2)) \tag{8.70}$$

By applying Laplace transform

$$Y(s) = X_1(s).X_2(s) \tag{8.71}$$

$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{1+s}, \quad \text{Re}(s) > -1 \tag{8.72}$$

$$u(t) - u(t-2) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1-e^{-2s}}{s}, \quad \text{Re}(s) > 0 \tag{8.73}$$

$$Y(s) = \left(\frac{1}{1+s}\right)\left(\frac{1-e^{-2s}}{s}\right), \quad \text{Re}(s) > 0 \tag{8.74}$$

$$= \frac{1-e^{-2s}}{s(s+1)} \tag{8.75}$$

Final value theorem

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{8.76}$$

$$\tag{8.77}$$

Proof:

$$\mathcal{L}\left[x\left(t\right)\right] = X\left(s\right) = \int_{0}^{\infty} x\left(t\right)e^{-st}\,dt \tag{8.78}$$

$$\mathcal{L}\left[\frac{dx\left(t\right)}{dt}\right] = \int_{0}^{\infty} \frac{d}{dt}\left(x\left(t\right)e^{-st}\right)\,dt \tag{8.79}$$

$$= sX\left(s\right) - x\left(0^{-}\right) \tag{8.80}$$

$$\lim_{s\to 0}\left[\int_{0}^{\infty} \frac{d}{dt}\left(x\left(t\right)e^{-st}\right)\,dt\right] = \lim_{s\to 0}\left[sX\left(s\right) - x\left(0^{-}\right)\right] \tag{8.81}$$

$$\int_{0}^{\infty} \frac{dx\left(t\right)}{dt}\,dt = \lim_{s\to 0}\left[sX\left(s\right) - x\left(0^{-}\right)\right] \tag{8.82}$$

$$\left[x\left(t\right)\right]_{0}^{\infty} = \lim_{s\to 0}\left[sX\left(s\right) - x\left(0^{-}\right)\right] \tag{8.83}$$

$$x\left(\infty\right) - x\left(0^{-}\right) = \lim_{s\to 0}\left[sX\left(s\right) - x\left(0^{-}\right)\right] \tag{8.84}$$

$$\implies x\left(\infty\right) = \lim_{s\to 0} sX\left(s\right) \tag{8.85}$$

$$\lim_{t\to\infty} x\left(t\right) = \lim_{s\to 0} sX\left(s\right) \tag{8.86}$$

By applying Final value theorem

$$\lim_{t\to\infty} y\left(t\right) = \lim_{s\to 0} sY\left(s\right) \tag{8.87}$$

$$= \lim_{s\to 0} s\left(\frac{1-e^{-2s}}{s\left(s+1\right)}\right) \tag{8.88}$$

$$= \lim_{s\to 0}\left(\frac{1-e^{-2s}}{\left(s+1\right)}\right) \tag{8.89}$$

$$= \left(\frac{1-e^{0}}{0+1}\right) \tag{8.90}$$

$$\lim_{t\to\infty} y\left(t\right) = 0 \tag{8.91}$$

8.10 A unity-gain negative-feedback control system has a loop-gain $L(s)$ given by

$$L(s) = \frac{6}{s(s-5)} \tag{8.92}$$

The closed loop system is ____

(a) Causal and stable

(b) Causal and unstable

(c) Non-causal and stable

(d) Non-causal and unstable

(GATE IN 2022)

Solution: From Table 8.8, the transfer function of the system is given by,

Parameter	**Description**	**Value**
$L(s)$	Forward loop transfer function	$\frac{6}{s(s-5)}$
$H(s)$	Feedback path transferfunction	1
$T(s)$	Transfer function	$\frac{L(s)}{1+L(s)H(s)}$

Table 8.8: Parameter Table

$$T(s) = \frac{\frac{6}{s(s-5)}}{1+1\frac{6}{s(s-5)}} \tag{8.93}$$

$$= \frac{6}{s^2-5s+6} \tag{8.94}$$

The poles of the system are given by the roots of the denominator of transfer function,

$$s^2-5s+6=0 \tag{8.95}$$

$\therefore$ The poles of the system are $s = 2$ and $s = 3$.

As the poles are positive, the output will increase without bound, causing the system to be unstable.

The transfer function of the system is ,

$$T(s) = \frac{6}{(s-2)(s-3)} \tag{8.96}$$

Clearly, it is dependent only on the past values. Hence, the system is causal.

Thus the correct option is B. The system is causal and unstable.

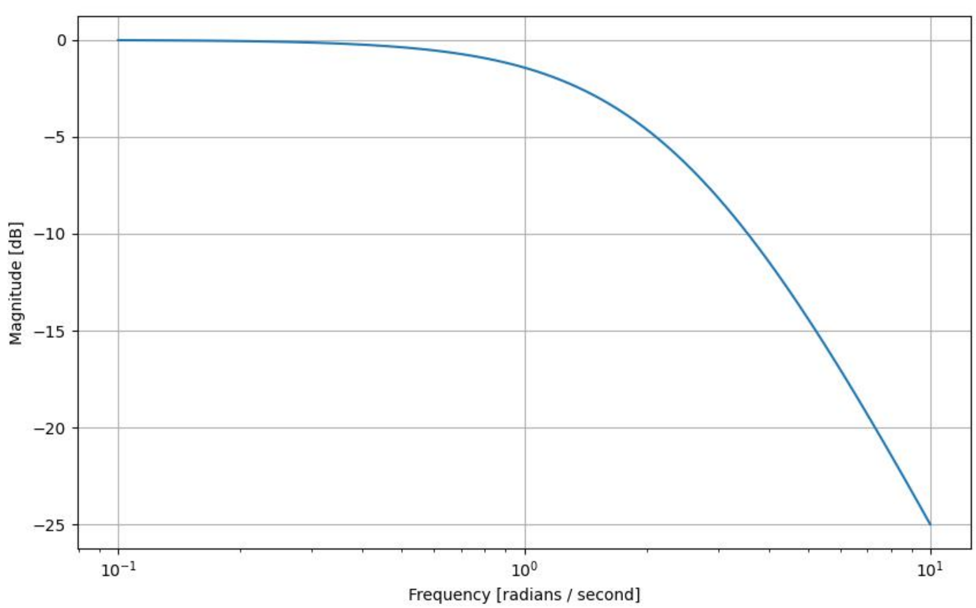


Figure 8.11: Magnitude plot for the transfer function

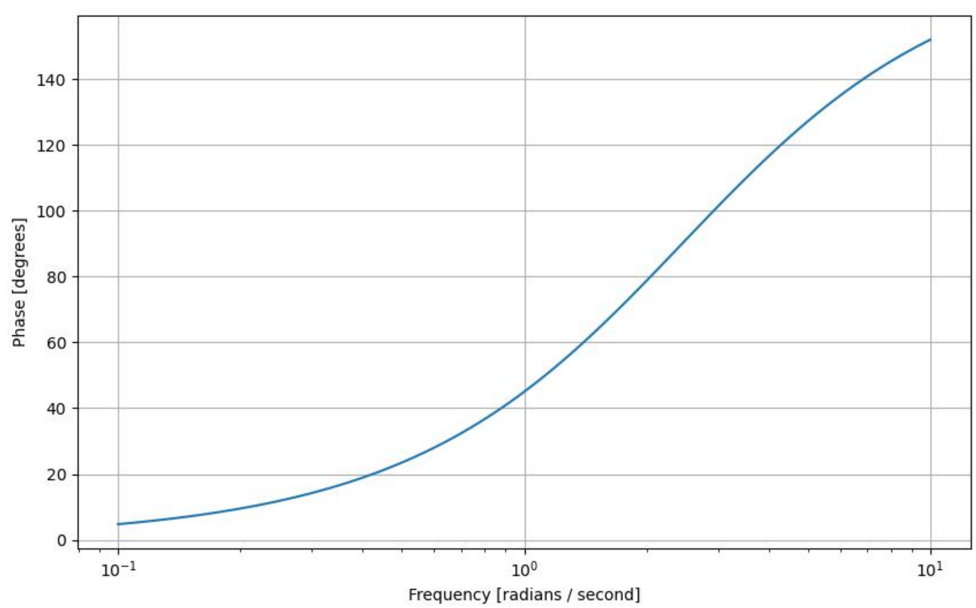


Figure 8.12: Phase plot for the transfer function

Chapter 9

Fourier transform

9.1 The Fourier transform X($j\omega$) of the signal

$x(t) = \frac{t}{(1+t^2)^2}$ is ________. GATE-2022-EC-15

(A) $\frac{\pi}{2j}\omega e^{-|\omega|}$

(B) $\frac{\pi}{2}\omega e^{-|\omega|}$

(C) $\frac{\pi}{2j}e^{-|\omega|}$

(D) $\frac{\pi}{2}e^{-|\omega|}$

Solution:

Symbol	Value	Description
$x(t)$	$\frac{t}{(1+t^2)^2}$	Signal
$X(\omega)$	$\int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$	Fourier transform of $x(t)$

Table 9.1: Variable description

The Fourier transform of the form x(t)=$e^{-a|t|}$ is

$$x(t) \xleftrightarrow{\text{F.T}} X(\omega) \tag{9.1}$$

$$X(\omega) = \frac{2a}{a^2+\omega^2} \tag{9.2}$$

Consider,

$$x(t) = e^{-|t|} \tag{9.3}$$

$$X(\omega) = \frac{2}{1+\omega^2} \tag{9.4}$$

By using differentiation property from (A.1.5),

$$tx(t) \xleftrightarrow{\text{F.T}} j\frac{d}{d\omega}X(\omega) \tag{9.5}$$

$$tx(t) \xleftrightarrow{\text{F.T}} j\left[\frac{d}{d\omega}\left(\frac{2}{1+\omega^2}\right)\right] \tag{9.6}$$

$$te^{-|t|} \xleftrightarrow{\text{F.T}} \frac{-4j\omega}{(1+\omega^2)^2} \tag{9.7}$$

Applying duality property from (A.2.3),

$$\frac{-4jt}{(1+t^2)^2} \xleftrightarrow{\text{F.T}} 2\pi(-\omega)e^{-|-\omega|} \tag{9.8}$$

$$\frac{t}{(1+t^2)^2} \xleftrightarrow{\text{F.T}} \frac{-2\pi\omega e^{-|\omega|}}{-4j} \tag{9.9}$$

$$\frac{t}{(1+t^2)^2} \xleftrightarrow{\text{F.T}} \frac{\pi}{2j}\omega e^{-|\omega|} \tag{9.10}$$

9.2 For a vector $\bar{x} = [x[0], x[1], \dots, x[7]]$, the 8-point discrete Fourier transform (DFT) is denoted by $\bar{X} = \text{DFT}(\bar{x}) = [X[0], X[1], \dots, X[7]]$, where

$$X[k] = \sum_{n=0}^{7} x[n] \exp\left(-j\frac{2\pi}{8}nk\right).$$

Here $j = \sqrt{-1}$. If $\bar{x} = [1, 0, 0, 0, 2, 0, 0, 0]$ and $\bar{y} = \text{DFT}(\text{DFT}(\bar{x}))$, then the value of $y[0]$ is. GATE-2022-EC-55

Solution:

Parameter	Description	Value
$\bar{X}$	$\text{DFT}(\bar{x})$	–
$\bar{x}$	vector	$[1, 0, 0, 0, 2, 0, 0, 0]$
$\bar{y}$	$\text{DFT}(\text{DFT}(\bar{x}))$	–

Table 9.2: Given Parameters

DFT of $\bar{x}$

$$X[k] = \sum_{n=0}^{7} x[n] \exp\left(-j\frac{2\pi}{8}nk\right) \tag{9.11}$$

As the only non-zero values in x are x[0] and x[4]:

$$X[k] = x[0] + x[4] \exp\left(-j\pi k\right) \tag{9.12}$$

After substituting the values of k ranging from 0 to 7,

$$\bar{X} = \text{DFT}(\bar{x}) = [X[0], X[1], \dots, X[7]] \tag{9.13}$$

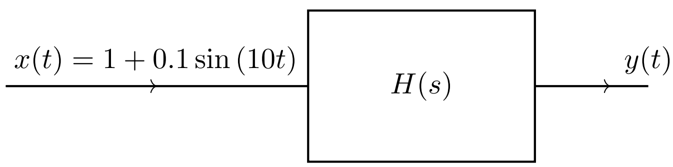
$$\bar{X} = [3, -1, 3, -1, 3, -1, 3, -1] \tag{9.14}$$

$$
\begin{aligned}
\bar{y} &= \mathrm{DFT}(\mathrm{DFT}(\bar{x})) && (9.15)\\
\bar{y} &= [3, -1, 3, -1, 3, -1, 3, -1] && (9.16)\\
y[0] &= \sum_{n=0}^{7} x[n] && (9.17)\\
&= x[0] + x[1] + \cdots + x[7] && (9.18)\\
&= 3 - 1 + 3 - 1 + 3 - 1 + 3 - 1 = 8 && (9.19)
\end{aligned}
$$

9.3 **Question:** An LTI system is shown in the figure where

$$H(s) = \frac{100}{s^2 + 0.1s + 10}$$

The steady state output of the system for an input $x(t)$ is given by $y(t) = a + b\sin(10t + \theta)$. The values of $'a'$ and $'b'$ are



Solution:

Symbol	Value	Description
$x(t)$	$1 + 0.1\sin(10t)$	Input Signal
$y(t)$	?	Output of the system
$H(s)$	$\frac{100}{s^2+0.1s+10}$	Impulse Response

Table 9.3: Given Information

(a) **Theory:** If a sinusoidal input is given to a system, whose transfer function is known, the output can be calculated as follows

$$y(t) = h(t) * x(t) \tag{9.20}$$

$$Y(s) = H(s)X(s) \tag{9.21}$$

Let $s = j\omega$

$$Y(j\omega) = H(j\omega)X(j\omega) \tag{9.22}$$

If Φ is the phase of $H(j\omega)$,

$$H(j\omega) = |H(j\omega)|\, e^{j\Phi(\omega)} \tag{9.23}$$

If $x(t) = \cos(\omega_0 t)$,

$$X(j\omega) = \pi\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right) \tag{9.24}$$

Now,

$$Y(j\omega) = \left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)|H(j\omega)|\, e^{j\Phi(\omega)} \tag{9.25}$$

$$\tag{9.26}$$

Since $|H(j\omega)|\,\delta(\omega - \omega_0)$ is zero everywhere except at ω_0

$$Y(j\omega) = |H(j\omega_0)|\, e^{j\Phi(\omega_0)}\delta(\omega - \omega_0) \tag{9.27}$$

$$+ |H(-j\omega_0)|\, e^{j\Phi(-j\omega_0)}\delta(\omega + \omega_0) \tag{9.28}$$

As $h(t)$ is real,

$$H(\omega) = H^*(-\omega)$$

$$\Phi(-\omega_0) = -\Phi(\omega_0)$$

Hence

$$Y(\omega) = |H(\omega_0)|\left(e^{j\Phi(\omega_0)}\delta(\omega - \omega_0) + e^{-j\Phi(\omega_0)}\delta(\omega + \omega_0)\right) \tag{9.29}$$

Taking Inverse Fourier Transform,

$$\delta(\omega - \omega_0) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2} e^{j\omega_0 t} \tag{9.30}$$

$$\implies y(t) = |H(\omega_0)| \frac{1}{2} \left(e^{j(\omega_0 t + \Phi(\omega_0))} + e^{-j(\omega_0 t + \Phi(\omega_0))} \right) \tag{9.31}$$

$$\implies y(t) = |H(\omega_0)| \cos(\omega_0 t + \Phi(\omega_0)) \tag{9.32}$$

(b) The given input can be assumed to be a superposition of $u(t)$ and $0.1 \sin(\omega_0 t) u(t)$.

$$\omega_0 = 0 \text{ and } \omega_0 = 10$$

for the constant input and the sinusoidal input respectively.

$$y(t) = |H(0)| + |H(10)| \sin(10t + \Phi(10)) \tag{9.33}$$

Here

$$H(\omega) = \frac{100}{(j\omega)^2 + 0.1(j\omega) + 10} \tag{9.34}$$

$$\implies H(\omega) = \frac{100}{10 - \omega^2 + j(0.1\omega)} \tag{9.35}$$

$$\implies |H(\omega)| = \frac{100}{\sqrt{(10 - \omega^2)^2 + (0.1\omega)^2}} \tag{9.36}$$

$$\therefore |H(0)| = 10 \text{ and } |H(10)| \approx 1 \tag{9.37}$$

The phase $\Phi(\omega)$ is given by

$$\Phi(\omega) = \tan^{-1} \frac{0.1\omega}{\omega^2 - 10} \tag{9.38}$$

$$\implies \Phi(10) = \tan^{-1} \frac{1}{90} \tag{9.39}$$

Hence the output of the system

$$y(t) = 10 + \sin\left(10t + \tan^{-1}\frac{1}{90}\right) \tag{9.40}$$

Hence $a = 10$ and $b = 1$

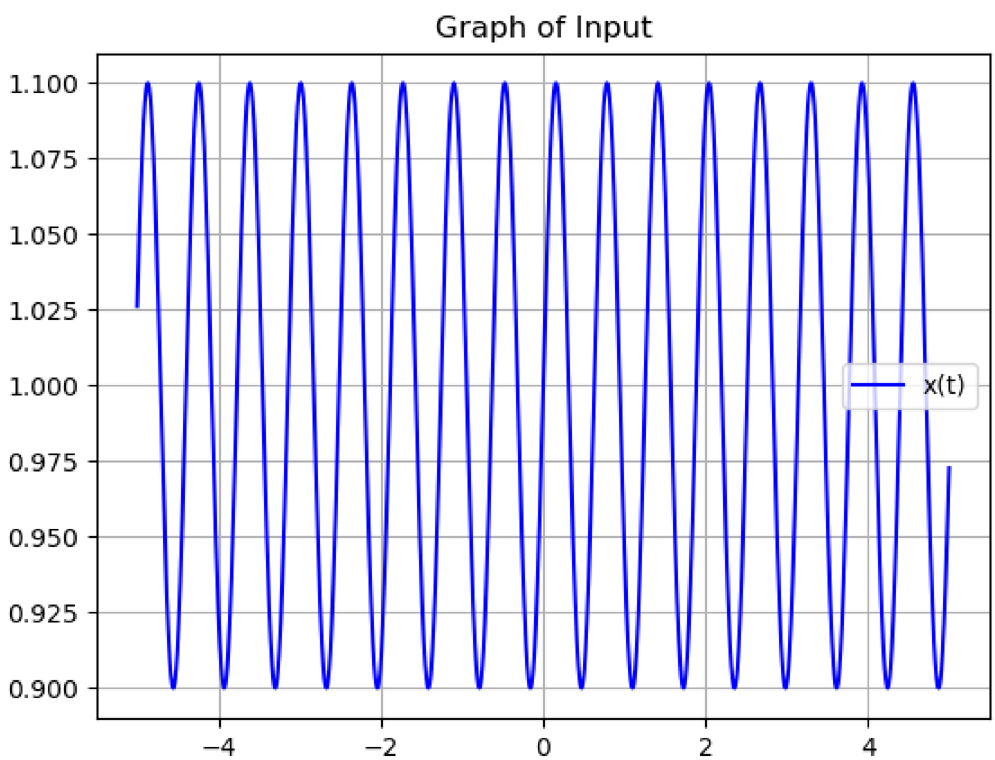


Figure 9.1: Input of the system, $x(t)$

9.4 A periodic function $f(x)$, with period 2, is defined as

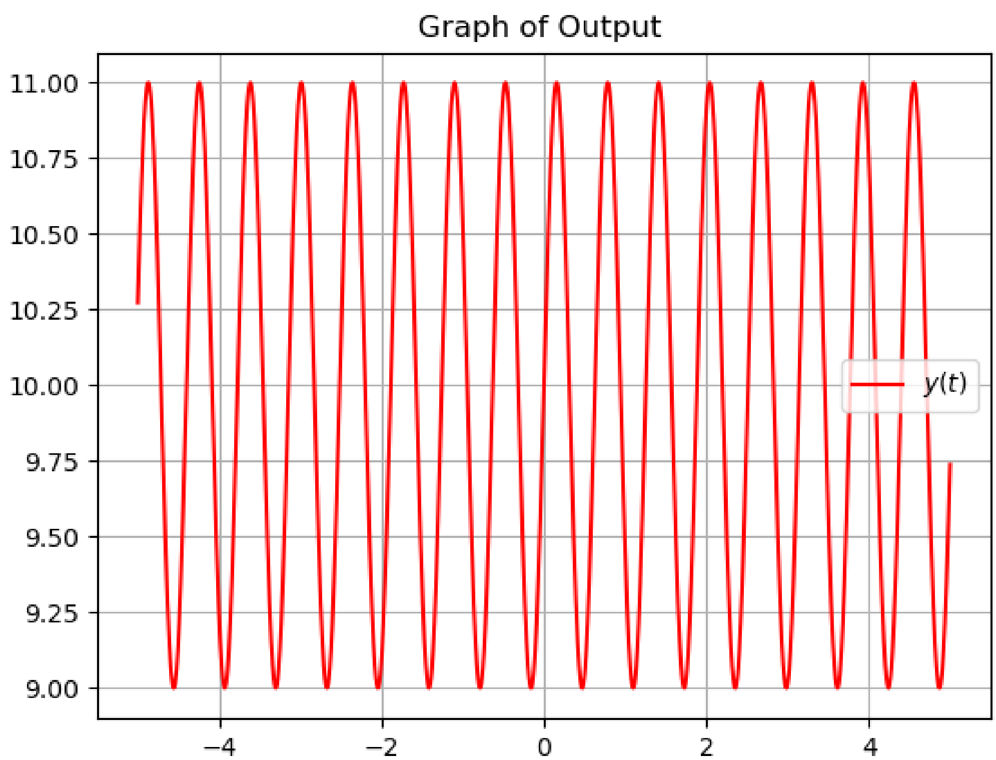


Figure 9.2: Output of the system, $y(t)$

$$f(x) = \begin{cases} -1-x & -1 \leq x < 0 \\ 1-x & 0 < x \leq 1 \end{cases} \tag{9.41}$$

The Fourier series of this function contains

A. Both $\cos(n\pi x)$ and $sin(n\pi x)$ where n=1,2,3...

B. Only $\sin(n\pi x)$ where n=1,2,3...

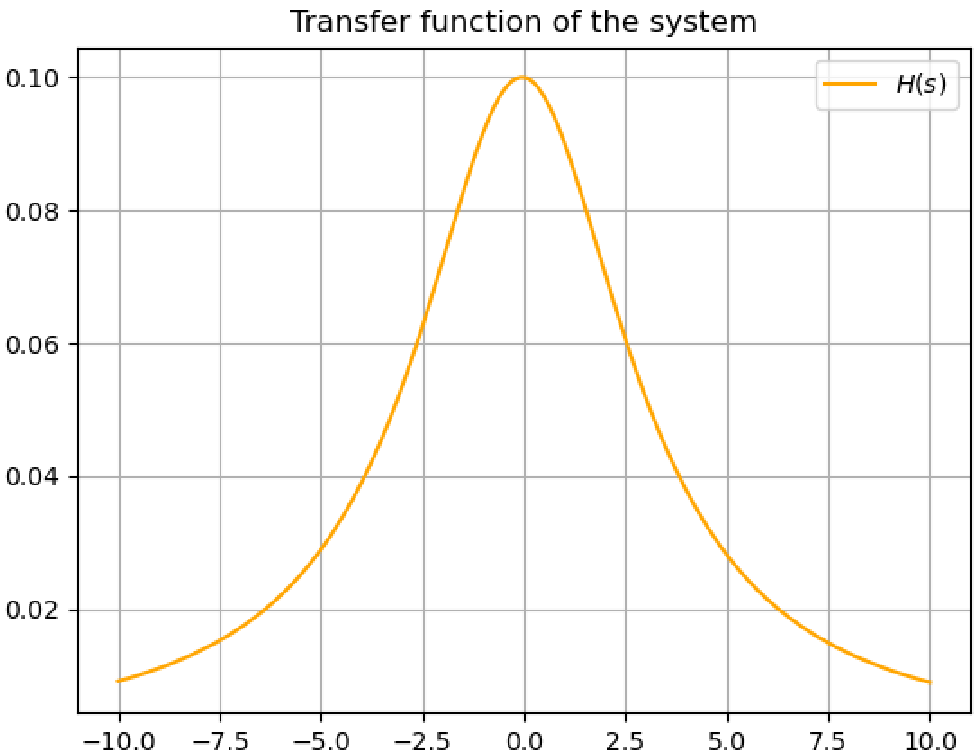


Figure 9.3: Transfer function of the system, $H(s)$

C. Only $\cos(n\pi x)$ where n=1,2,3...

D. Only $\cos(2n\pi x)$ where n=1,2,3... GATE IN 2022

Solution:

Parameter	**Description**
$f(x)$	Polynomial function
$2L$	Period of the Polynomial function
$c(n)$	Complex Fourier Coefficients
$a(0)$, $a(n)$, $b(n)$	Trigonometric Fourier Coefficients

Table 9.4: Input Parameters

The complex exponential Fourier Series of $f(x)$ is,

$$f(x) = \sum_{n=-\infty}^{\infty} c(n) e^{jn\omega x} \tag{9.42}$$

$$\implies c(n) = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{-jn\omega x}\, dx \tag{9.43}$$

For $n \neq 0$;

$$\begin{aligned}
c(n) &= \frac{1}{2} \int_{-1}^{1} f(x) e^{-jn\omega x}\, dx && (9.44)\\
&= \frac{1}{2} \left(\int_{-1}^{0} (-1 - x)\, e^{-jn\omega x}\, dx + \int_{0}^{1} (+1 - x)\, e^{-jn\omega x}\, dx \right) && (9.45)\\
&= \frac{1}{2} \left(- \int_{-1}^{0} e^{-jn\omega x}\, dx - \int_{-1}^{1} x e^{-jn\omega x}\, dx + \int_{0}^{1} e^{-jn\omega x}\, dx \right) && (9.46)\\
&= \frac{1}{2} \left[\frac{-1}{jn\omega} \left[- \left(1 - e^{+jn\omega}\right) + \left(e^{-jn\omega} - 1\right) \right] - \int_{-1}^{1} x e^{-jn\omega x}\, dx \right] && (9.47)\\
&= \frac{1}{2} \left[\frac{-1}{jn\omega} \left[-2 + e^{+jn\omega} + e^{-jn\omega} \right] + \left(\frac{e^{-jn\omega x}}{jn\omega} \left[x + \frac{1}{jn\omega} \right] \right)_{-1}^{1} \right] && (9.48)\\
&= \frac{-1}{jn\omega} \left[-1 + \cos(n\omega) \right] + \frac{1}{2(jn\omega)^2} \left[\left(e^{-jn\omega}\right)(1 + jn\omega) - \left(e^{jn\omega}\right)(-jn\omega + 1) \right] && (9.49)
\end{aligned}$$

$$\implies c(n) = \frac{-1}{(jn\omega)^2} \left[-jn\omega + j \sin(n\omega) \right] \tag{9.50}$$

For $n = 0$;

$$c(0) = \frac{1}{2} \int_{-1}^{1} f(x)\, dx \tag{9.51}$$

$$= \frac{1}{2} \left[\int_{-1}^{0} (-1 - x)\ dx + \int_{0}^{1} (1 - x)\ dx \right] \tag{9.52}$$

$$= \frac{1}{2} \left[\left(-x - \frac{x^2}{2} \right)_{-1}^{0} + \left(x - \frac{x^2}{2} \right)_{0}^{1} \right] \tag{9.53}$$

$$= \frac{1}{2} \left[0 - 1 + \frac{1}{2} + 1 - \frac{1}{2} - 0 \right] \tag{9.54}$$

$$\implies c(0) = 0 \tag{9.55}$$

The trigonometric Fourier Series of $f\,(x)$ is,

$$f\,(x) = a(0) + \sum_{n=1}^{\infty} \left\{ a(n) \cos\left(n\omega x\right) + b(n) \sin\left(n\omega x\right) \right\} \tag{9.56}$$

Finding the Fourier Coefficient a_0,

$$a(0) = c(0) \tag{9.57}$$

$$\implies a(0) = 0 \tag{9.58}$$

Finding the Fourier Coefficients $a(n)$,

$$a(n) = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(n\omega x\right)\ dx, n \geq 0 \tag{9.59}$$

$$= \frac{1}{L} \int_{-L}^{L} f(x) \left(e^{-jn\omega x} + e^{jn\omega x} \right)\ dx \tag{9.60}$$

$$\implies a(n) = c(n) + c(-n) \tag{9.61}$$

$$\implies a(n) = 0 \tag{9.62}$$

Finding the Fourier Coefficients $b(n)$,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\omega x)\ dx, n \geq 0 \tag{9.63}$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)j\left(e^{-jn\omega x} - e^{jn\omega x}\right)\ dx \tag{9.64}$$

$$\implies b(n) = j\left(c\left(n\right) - c\left(-n\right)\right) \tag{9.65}$$

$$\implies b(n) = \frac{-2}{(n\omega)^2}\left[-n\omega + \sin(n\omega)\right] \tag{9.66}$$

$\implies$ The trigonometric Fourier Series of $f\left(x\right)$ is,

$$f\left(x\right) = \sum_{n=1}^{\infty}\left\{0 + 0 + b(n)\sin\left(n\omega x\right)\right\} \tag{9.67}$$

$$f\left(x\right) = \sum_{n=1}^{\infty}\left\{\frac{-2}{(n\omega)^2}\left[-n\omega + \sin(n\omega)\right]\sin\left(n\omega x\right)\right\} \tag{9.68}$$

$$f\left(x\right) = \sum_{n=1}^{\infty}\left\{\frac{-2}{(n\pi)^2}\left[-n\pi + \sin(n\pi)\right]\sin\left(n\pi x\right)\right\} \tag{9.69}$$

$$f\left(x\right) = \sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}\sin\left(n\pi x\right)\right\} \tag{9.70}$$

$\therefore$ The Fourier series of this function contains only $\sin(n\pi x)$ where n=1,2,3...

9.5 A Simple closed path C in the Complex Plane is shown in the figure.

$$\oint_C \frac{2^z}{z^2-1}dz = -\jmath\pi A$$

Where $\jmath = \sqrt{-1}$, Then find the value of A is _____(Rounded of to two decimals)

(GATE 2022 EC)

Solution: Let

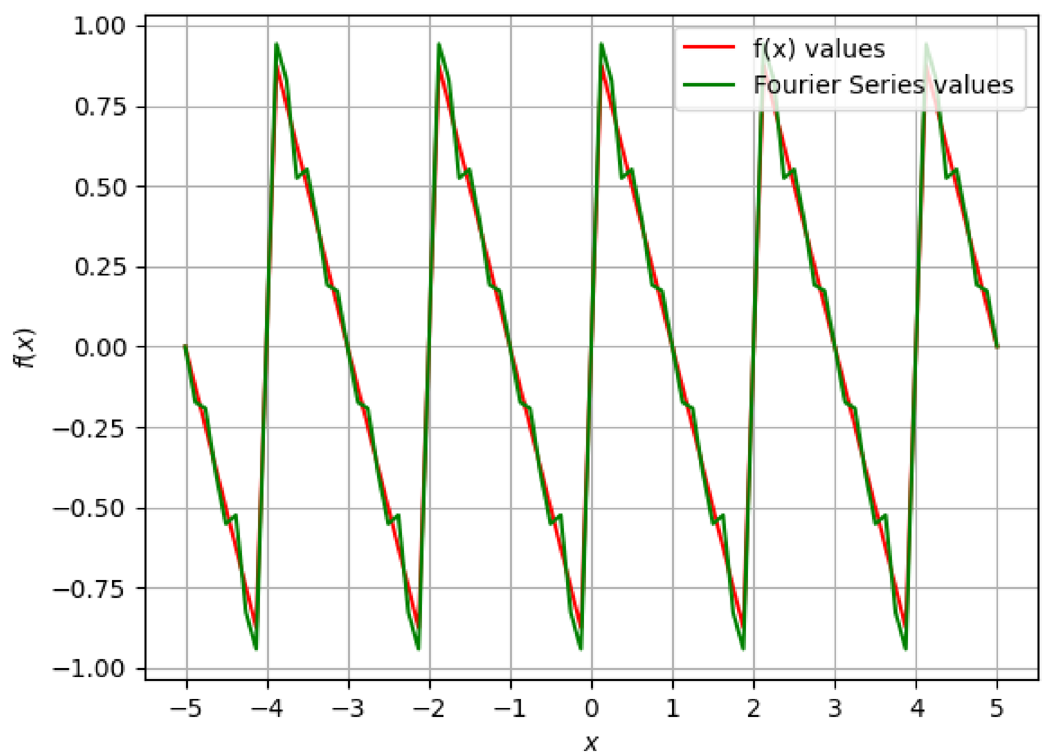
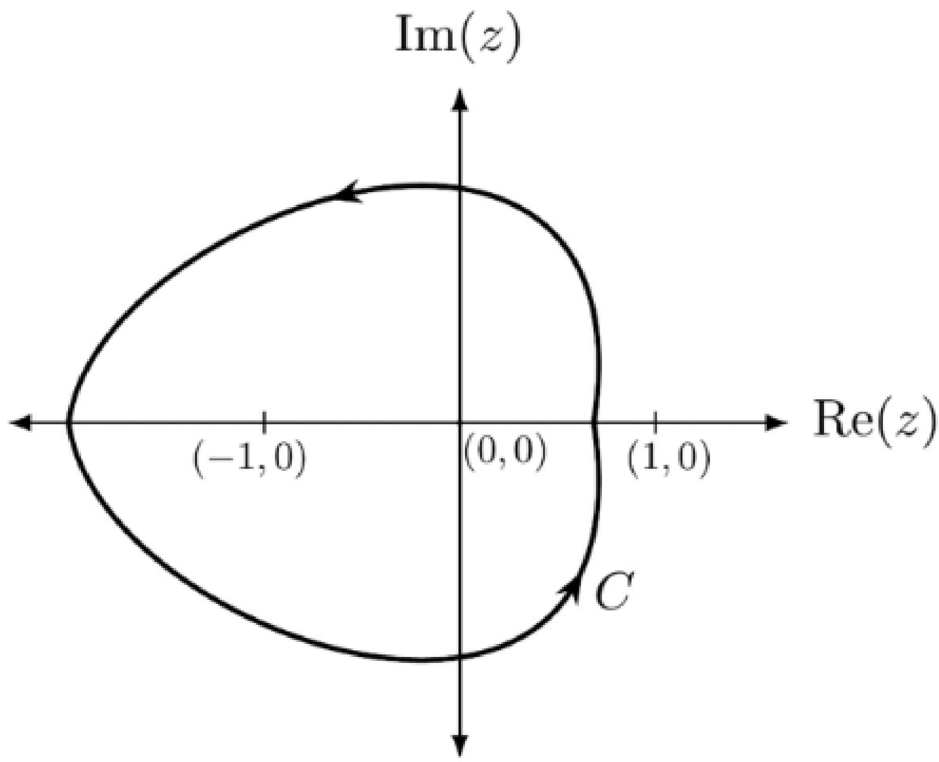


Figure 9.4:

$$f(z) = \frac{2^z}{z^2 - 1}$$



For poles

$$z^2 - 1 = 0 \tag{9.71}$$

$$\implies z = \pm 1 \tag{9.72}$$

As $Z = -1$ lies inside the C and $z = 1$ lies outside C

$$\oint_C f(z)dz = \oint_C \frac{\frac{2^z}{z-1}}{z+1}dz \tag{9.73}$$

$$= 2\pi \jmath \left(\frac{2^z}{z-1}\right) \text{ At } z = -1 \tag{9.74}$$

(By Cauchy's integral formula)

$$= 2\pi \jmath \left(\frac{-1}{4} \right) \tag{9.75}$$

$$= -\pi \jmath \left(\frac{1}{2} \right) \tag{9.76}$$

By comparing

$$A = \frac{1}{2} = 0.50 \tag{9.77}$$

Appendix A

Fourier transform

A.1 The Differentiation in frequency domain is as follows

Let x(t) be a signal such that,

$$x(t) \xleftrightarrow{\text{F.T}} X(\omega) \tag{A.1.1}$$

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{A.1.2}$$

$$\frac{d}{d\omega} X(\omega) = \int_{t=-\infty}^{\infty} x(t)(-jt) e^{-j\omega t} dt \tag{A.1.3}$$

$$j\frac{d}{d\omega} X(\omega) = \int_{t=-\infty}^{\infty} tx(t) e^{-j\omega t} dt \tag{A.1.4}$$

$$tx(t) \xleftrightarrow{\text{F.T}} j\frac{d}{d\omega} X(\omega) \tag{A.1.5}$$

A.2 The duality property is as follows

From inverse Fourier transform we get,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \tag{A.2.1}$$

Replacing t by -t and multiplying 2π on both sides we get,

$$2\pi x\left(-t\right)=\int_{-\infty}^{\infty}X\left(\omega\right)e^{-j\omega t}d\omega \tag{A.2.2}$$

$$X\left(t\right)\xleftrightarrow{\text{F.T}}2\pi x\left(-\omega\right) \tag{A.2.3}$$