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# SIGNAL PROCESSING

## Through GATE

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G. V. V. Sharma



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# Introduction

This book provides solutions to signal processing problems in GATE.



## Chapter 1

# Harmonics

- 1.1 Let  $y(t)=x(4t)$ , where  $x(t)$  is a continuous-time periodic signal of 100s. the fundamental period of  $y(t)$  is (**rounded off to the nearest integer**) (GATE IN 2023)

**Solution:**





## Chapter 2

## Filters

2.1 For the circuit given below, choose the angular frequency  $\omega_0$  at which voltage across capacitor has maximum amplitude?

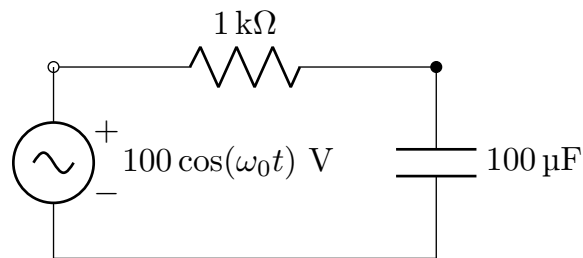


Figure 2.1: circuit

- (A) 1000
- (B) 100
- (C) 1
- (D) 0

(GATE BM 2023 Question 16)

2.2 A finite impulse response (FIR) filter has only two non-zero samples in its impulse response  $h[n]$ , namely  $h[0] = h[1] = 1$ . The Discrete Time Fourier Transform (DTFT)

of  $h[n]$  equals  $H(e^{j\omega})$ , as a function of the normalized angular frequency  $\omega$ . For the range  $|\omega| \leq \pi$ ,  $|H(e^{j\omega})|$  is equal to

(A)  $2|\cos(\omega)|$

(B)  $2|\sin(\omega)|$

(C)  $2|\cos(\frac{\omega}{2})|$

(D)  $2|\sin(\frac{\omega}{2})|$

(GATE BM 2023 Question 17)

## Chapter 3

# Z-transform



## Chapter 4

# Systems

- 4.1 Consider a unity-gain negative feedback system consisting of the plant  $G(s)$  and a proportional-integral controller. Let the proportional gain and integral gain be 3 and 1, respectively. For a unit step reference input, the final values of the controller output and the plant output, respectively, are

$$G(s) = \frac{1}{(s-1)}$$

(GATE EE 2023)

**Solution:**

Parameter	Description	Value
$K_p$	Proportional Gain	3
$K_i$	Integral Gain	1
$r(t)$	Reference Input	$u(t)$
$w(t)$	Controller Output	?
$y(t)$	Plant Output	?
$e(t)$	Error Input	$r(t) - y(t)$

Table 1: Parameter Table

From the Fig. 4.1:

$$E(s) = U(s) - Y(s) \quad (4.1)$$

$$W(s) = 3E(s) + \frac{1}{s}E(s) \quad (4.2)$$

$$Y(s) = G(s)W(s) \quad (4.3)$$

Some results:

$$tx(t) \xleftrightarrow{\mathcal{L}} -\frac{dX(s)}{ds} \quad (4.4)$$

$$e^{-at}x(t) \xleftrightarrow{\mathcal{L}} X(s+a) \quad (4.5)$$

By using (4.4) and (4.5):

$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}, \text{Re}(s) > -1 \quad (4.6)$$

$$te^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)^2}, \text{Re}(s) > -1 \quad (4.7)$$



Figure 4.1: Block Diagram of System

(a) **Plant Output:**

From (4.1) , (4.2) and (4.3):

$$Y(s) = \frac{3s+1}{s(s+1)^2}, \text{Re}(s) > -1 \quad (4.8)$$

Final Value Theorem:

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s) \quad (4.9)$$

Using (4.9) on Y(s):

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) \quad (4.10)$$

$$= 1 \quad (4.11)$$

Taking partial fraction of (4.8) :

$$Y(s) = \frac{1}{s} + \frac{2}{(s+1)^2} - \frac{1}{s+1} \quad (4.12)$$

Using (4.6) and (4.7):

$$\therefore y(t) = u(t) + 2te^{-t}u(t) - e^{-t}u(t) \quad (4.13)$$

(b) **Controller Output:**

From (4.2)

$$W(s) = \frac{3}{s} + \frac{1}{s^2} - Y(s) \left( 3 + \frac{1}{s} \right) \quad (4.14)$$

Substituting (4.8)

$$W(s) = \frac{(s-1)(3s+1)}{s(s+1)^2}, \operatorname{Re}(s) > -1 \quad (4.15)$$

Using (4.9) on  $W(s)$

$$\lim_{t \rightarrow \infty} w(t) = \lim_{s \rightarrow 0} sW(s) \quad (4.16)$$

$$= -1 \quad (4.17)$$

Taking partial fraction of equation(4.15) :

$$W(s) = -\frac{1}{s} - \frac{4}{(s+1)^2} + \frac{4}{s+1} \quad (4.18)$$

Using equations (4.6) and (4.7) and taking inverse lapalace transform:

$$w(t) = -u(t) - 4te^{-t}u(t) + 4e^{-t}u(t) \quad (4.19)$$





Figure 4.2:  $w(t)$  converges at -1.

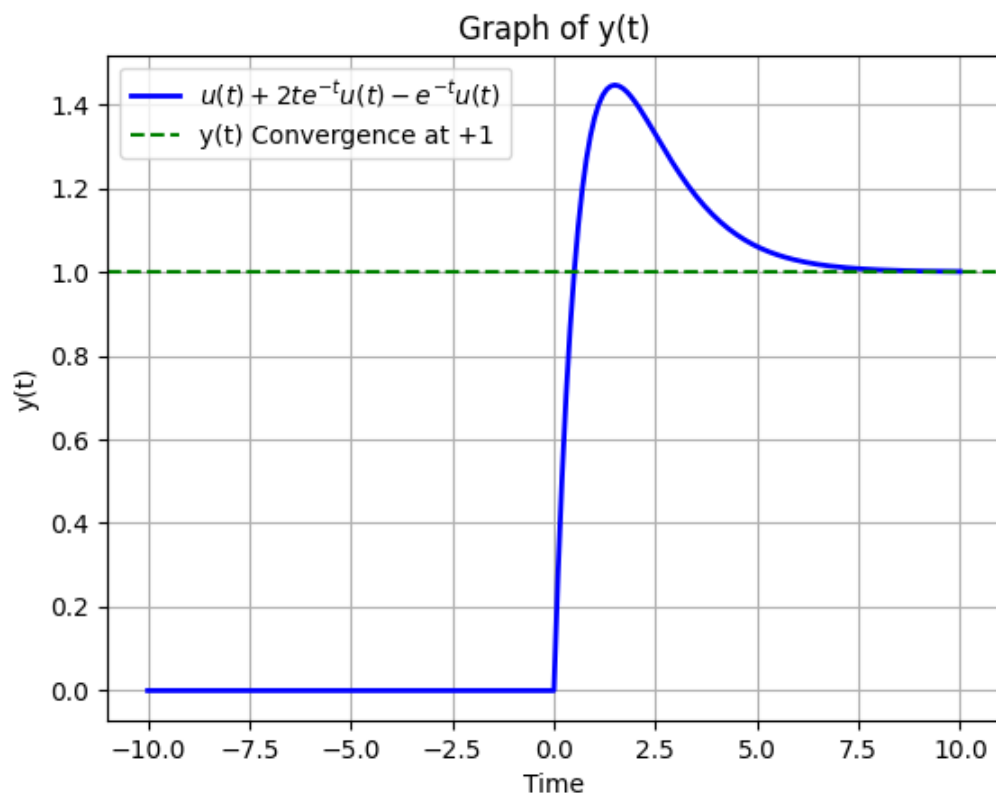


Figure 4.3:  $y(t)$  converges at +1

4.2 Level ( $h$ ) in a steam boiler is controlled by manipulating the flow rate ( $F$ ) of the break-up(fresh) water using a proportional ( $P$ ) controller. The transfer function between the output and the manipulated input is

$$\frac{h(s)}{F(s)} = \frac{0.25(1-s)}{s(2s+1)}$$

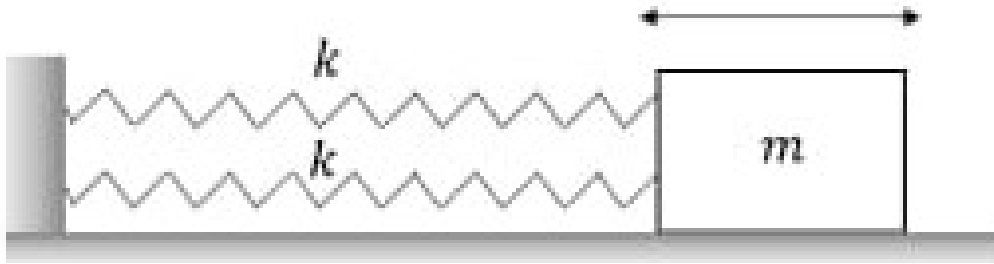
The measurement and the valve transfer functions are both equal to 1. A process engineer wants to tune the controller so that the closed loop response gives the decaying oscillations under the servo mode. Which one of the following is the CORRECT value of the controller gain to be used by the engineer?

- (a) 0.25
- (b) 2
- (c) 4
- (d) 6

GATE CH 2023

**Solution:**

- 4.3 The figure shows a block of mass  $m = 20$  kg attached to a pair of identical linear springs, each having a spring constant  $k = 1000$  N/m. The block oscillates on a frictionless horizontal surface. Assuming free vibrations, the time taken by the block to complete ten oscillations is \_\_\_\_\_ seconds . (Rounded off to two decimal places)  
Take  $\pi = 3.14$ . (GATE ME 2023)



- 4.4 A system has transfer function

$$\frac{Y(s)}{X(s)} = \frac{s - \pi}{s + \pi}$$

let  $u(t)$  be the unit step function. The input  $x(t)$  that results in a steady-state output  $y(t) = \sin(\pi t)$  is \_\_\_\_\_. (GATE IN 2023)

**Solution:**

4.5 The state equation of a second order system is

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) \text{ is the initial condition.}$$

Suppose  $\lambda_1$  and  $\lambda_2$  are two distinct eigenvalues of  $A$ , and  $\nu_1$  and  $\nu_2$  are the corresponding eigenvectors. For constants  $\alpha_1$  and  $\alpha_2$ , the solution,  $\mathbf{x}(t)$ , of the state equation is

(A)  $\sum_{i=1}^2 \alpha_i e^{\lambda_i t} \nu_i$

(B)  $\sum_{i=1}^2 \alpha_i e^{2\lambda_i t} \nu_i$

(C)  $\sum_{i=1}^2 \alpha_i e^{3\lambda_i t} \nu_i$

(D)  $\sum_{i=1}^2 \alpha_i e^{4\lambda_i t} \nu_i$

GATE 2023 EC Question 43



## Chapter 5

# Sequences

5.1 Consider the discrete time signal  $x[n] = u[-n + 5] - u[n + 3]$ , where

$$u[n] = \begin{cases} 1; n \geq 0 \\ 0; n < 0 \end{cases}$$

The smallest  $n$  for which  $x[n] = 0$  is?

**Solution:** From Fig. 1, the minimum value of  $n$  is given as

$$n = -3 \tag{5.1}$$



Figure 1: Plot of function  $x(n)$  taken from python3

5.2 Two sequences  $x_1[n]$  and  $x_2[n]$  are described as follows:

$$x_1[0] = x_2[0] = 1 \quad (5.2)$$

$$x_1[1] = x_2[2] = 2 \quad (5.3)$$

$$x_1[2] = x_2[1] = 1 \quad (5.4)$$

$$x_1[n] = x_2[n] = 0 \text{ for all } n < 0 \text{ and } n > 2$$

If  $x[n]$  is obtained by convoluting  $x_1[n]$  with  $x_2[n]$ , which of the following equa-



tions is/are TRUE?

(A)  $x[2] = x[3]$

(B)  $x[1] = 2$

(C)  $x[4] = 3$

(D)  $x[2] = 5$

**Solution:**

5.3 A series ( $S$ ) is given as  $S=1+3+5+7+9+\dots$ . The sum of the first 50 terms of  $S$  is

\_\_\_\_\_ **Solution:**

## Chapter 6

# Sampling

6.1 An 8 bit ADC converts analog voltage in the range of 0 to  $+5\text{ V}$  to the corresponding digital code as per the conversion characteristics shown in figure. For  $V_{in} = 1.9922\text{ V}$ , which of the following digital output, given in hex, is true?

(a)  $64H$

(b)  $65H$

(c)  $66H$

(d)  $67H$

(GATE EE 40)

**Solution:**

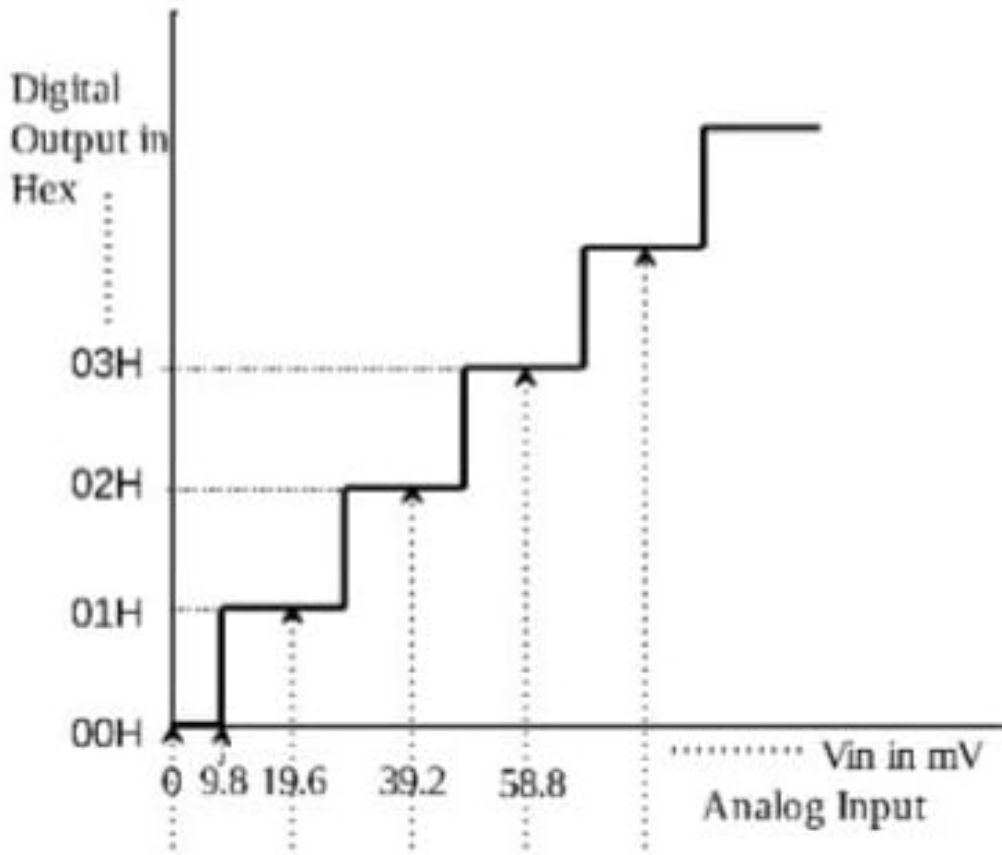


Figure 6.1:

Calculating the step-size:

$$\Delta V_{in} = \frac{V_{max} - V_{min}}{2^n - 1} \quad (6.1)$$

$$= \frac{5 - 0}{2^8 - 1} \quad (6.2)$$

$$= \frac{5}{255} \quad (6.3)$$

$$\Rightarrow V_{out} = \frac{V_{in}}{\Delta V_{in}} \quad (6.4)$$

$$= \frac{1.9922 \times 255}{5} \quad (6.5)$$

$$= 101.59 \quad (6.6)$$

$$\approx 102_{10} \quad (6.7)$$

Symbol	Value	Description
$n$	8	Number of bits of ADC
$V_{min}$	0V	Minimum Analog Voltage
$V_{max}$	5V	Maximum Analog Voltage
$V_{in}$	1.9922V	Input Voltage
$V_{out}$		Output Voltage

Table 6.1: Given Parameters

$\therefore$  correct answer is option (c).

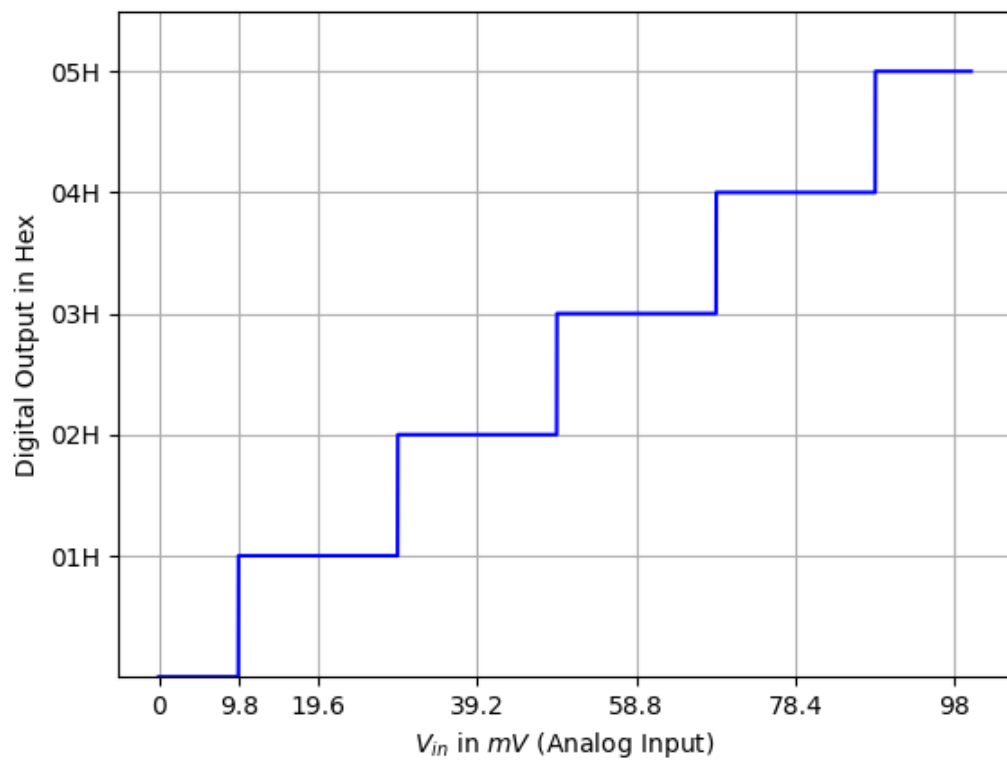


Figure 6.2:

## Chapter 7

# Contour Integration





## Chapter 8

# Laplace Transform

8.1 The number of zeroes of the polynomial  $P(s) = s^3 + 2s^2 + 5s + 80$  in the right side of the plane? (GATE IN 2023)

**Solution:** The table below shows the Routh array of the  $n^{th}$ - order characteristic polynomial :

$$a_0s^n + a_1s^{n-1} + \dots + a_{n-1}s^1 + a_ns^0 \quad (8.1)$$

$s^n$	$a_0$	$a_2$	$a_4$	...
$s^{n-1}$	$a_1$	$a_3$	$a_5$	...
$s^{n-2}$	$b_1 = \frac{a_1a_2 - a_3a_0}{a_1}$	$b_2 = \frac{a_1a_4 - a_5a_0}{a_1}$	...	..
$s^{n-3}$	$c_1 = \frac{b_1a_3 - b_2a_1}{b_1}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$		
$s^1$	$\vdots$	$\vdots$		
$s^0$	$a_n$			

Table 8.1: Routh Array

Characteristic Equation:

$$s^3 + 2s^2 + 5s + 80 = 0 \quad (8.2)$$

From Table 8.1:

$s^3$	1	5
$s^2$	2	80
$s^1$	$\frac{2 \times 5 - 80 \times 1}{2} = -35$	
$s^0$	$\frac{-35 \times 80}{-35} = 80$	

Table 8.2:

From Table 8.2:

Since there are 2 sign changes in the first column of the Routh tabulation. So, the number of zeros in the right half of the s-plane will be 2.

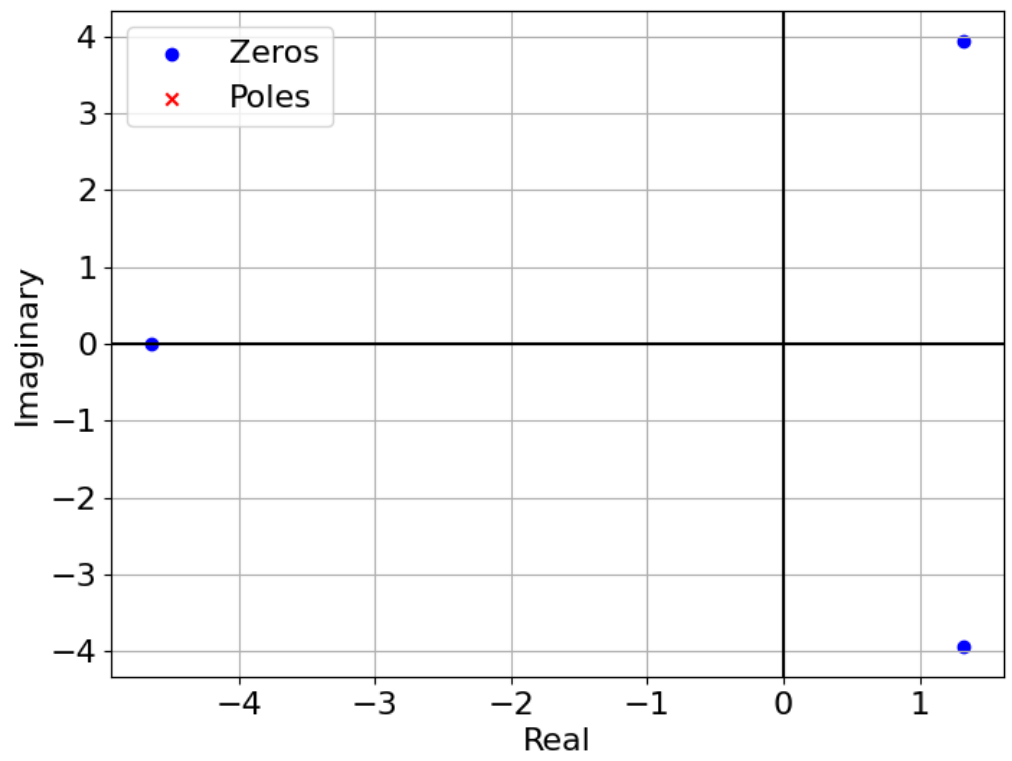
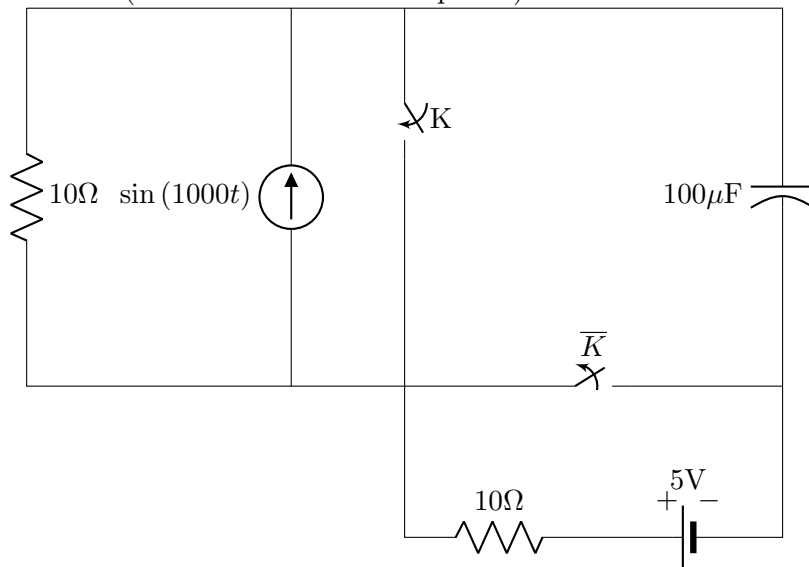


Figure 8.1:

8.2 The circuit shown in the figure is initially in the steady state with the switch K in open condition and  $\overline{K}$  in closed condition. The switch K is closed and  $\overline{K}$  is opened simultaneously at the instant  $t = t_1$ , where  $t_1 > 0$ . The minimum value of  $t_1$  in milliseconds such that there is no transient in the voltage across the  $100\ \mu\text{F}$  capacitor, is \_\_\_\_ (Round off to 2 decimal places) (GATE EE 2023)



8.3  $y = e^{mx} + e^{-mx}$  is the solution of which differential equation?

1.  $\frac{dy}{dx} - my = 0$
2.  $\frac{dy}{dx} + my = 0$
3.  $\frac{d^2y}{dx^2} + m^2y = 0$
4.  $\frac{d^2y}{dx^2} - m^2y = 0$

(GATE AG 2023) **Solution:**

8.4 A cascade control strategy is shown in the figure below. The transfer function between the output ( $y$ ) and the secondary disturbance ( $d_2$ ) is defined as

$$G_{d2}(s) = \frac{y(s)}{d_2(s)}$$

. Which one of the following is the CORRECT expression for the transfer function  $G_{d2}(s)$ ?



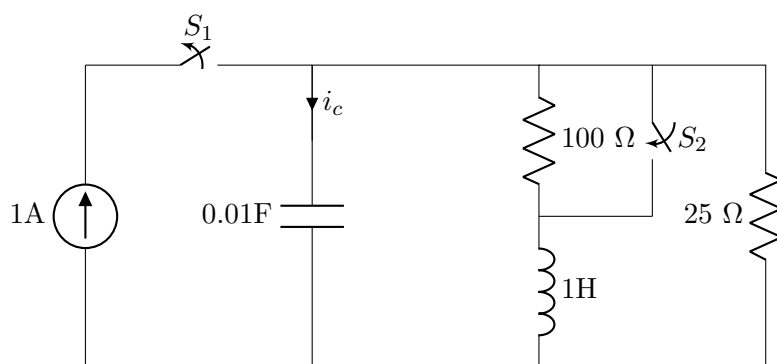
Figure 8.2:

- A.  $\frac{1}{(11s+21)(0.1s+1)}$
- B.  $\frac{1}{(s+1)(0.1s+1)}$
- C.  $\frac{(s+1)}{(s+2)(0.1s+1)}$
- D.  $\frac{(s+1)}{(s+1)(0.1s+1)}$

(GATE CH 2023) **Solution:**

8.5 In the differential equation  $\frac{dy}{dx} + \alpha xy = 0$ ,  $\alpha$  is a positive constant. If  $y = 1.0$  at  $x = 0.0$ , and  $y = 0.8$  at  $x = 1.0$ , the value of  $\alpha$  is (rounded off to three decimal places). (GATE CE 2023) **Solution:**

8.6 The switch  $S_1$  was closed and  $S_2$  was open for a long time. At  $t=0$ , switch  $S_1$  is opened and  $S_2$  is closed, simultaneously. The value of  $i_c(0^+)$ , in amperes, is . (GATE EC 2023)





8.7 The continuous time signal  $x(t)$  is described by:

$$x(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{elsewhere} \end{cases} \quad (8.3)$$

If  $y(t)$  represents  $x(t)$  convolved with itself, which of the following options is/are TRUE?

A  $y(t) = 0$  for all  $t < 0$

B  $y(t) = 0$  for all  $t > 1$

C  $y(t) = 0$  for all  $t > 3$

D  $\int_{0.1}^{0.75} \frac{dy(t)}{dt} dt \neq 0$

**Solution:**

8.8 The Z-transform of a discrete signal  $x(n)$  is

$$X(z) = \frac{4z}{\left(z - \frac{1}{5}\right)\left(z - \frac{2}{3}\right)(z - 3)} \text{ with ROC} = R \quad (8.4)$$

Which one of the following statements is TRUE?

- (a) Discrete time Fourier transform of  $x[n]$  converges if  $R$  is  $|z| > 3$
- (b) Discrete time Fourier transform of  $x[n]$  converges if  $R$  is  $\frac{2}{3} < |z| < 3$
- (c) Discrete time Fourier transform of  $x[n]$  converges if  $R$  is such that  $x[n]$  is a left-sided sequence.
- (d) Discrete time Fourier transform of  $x[n]$  converges if  $R$  is such that  $x[n]$  is a right-sided sequence.

**GATE EE 2023 Solution:**

8.9 The phase margin of the transfer function  $G(s) = \frac{2(1-s)}{(1+s)^2}$  is \_\_\_\_\_ degrees. (rounded off to the nearest integer). (GATE IN 2023)

**Solution:**

8.10 Consider the second-order linear differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0, \quad x \geq 1$$

with the initial conditions

$$y(x=1) = 6, \quad \left. \frac{dy}{dx} \right|_{x=1} = 2.$$

Then the value of  $y$  at  $x = 2$  is \_\_\_\_\_.

GATE ME 2023

**Solution:**

## Chapter 9

# Fourier transform

9.1 The discrete-time Fourier transform of a signal  $x[n]$  is  $X(\Omega) = (1 + \cos \Omega) e^{-j\Omega}$ .

Consider that  $x_p[n]$  is a periodic signal of period  $N = 5$  such that

$$x_p[n] = x[n], \text{ for } n = 0, 1, 2 \quad (9.1)$$

$$= 0, \text{ for } n = 3, 4 \quad (9.2)$$

Note that  $x_p[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn}$ . The magnitude of the Fourier series coefficient  $a_3$  is \_\_\_\_\_ (Round off to 3 decimal places). (GATE EE 2023) **Solution:**

9.2 Let a frequency modulated (FM) signal :  $x(t) = A \cos(\omega_c t + k_f \int_{-\infty}^t m(\lambda) d\lambda)$  , where  $m(t)$  is a message signal of bandwidth  $W$ . It is passed through a non-linear system with output  $y(t) = 2x(t) + 5(x(t))^2$ . Let  $B_T$  denote the FM bandwidth. The minimum value of  $\omega_c$  required to recover  $x(t)$  from  $y(t)$  is:

(A)  $B_T + W$

(B)  $\frac{3}{2} B_T$

(C)  $2B_T + W$

(D)  $\frac{5}{2} B_T$

**Solution:**

9.3 Let an input  $x[n]$  having discrete-time Fourier transform  $X(e^{j\Omega}) = 1 - e^{-j\Omega} + 2e^{-3j\Omega}$  be passed through an LTI system. The frequency response of the LTI system is  $H(e^{j\Omega}) = 1 - \frac{1}{2}e^{-2j\Omega}$ . The output  $y[n]$  of the system is

(GATE EC 2023) **Solution:**

9.4 The Fourier transform  $X(\omega)$  of  $x(t) = e^{-t^2}$  is

Note:  $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$

A)  $\sqrt{\pi} e^{\frac{\omega^2}{2}}$

B)  $\frac{e^{-\frac{\omega^2}{4}}}{2\sqrt{\pi}}$

C)  $\sqrt{\pi} e^{-\frac{\omega^2}{4}}$

D)  $\sqrt{\pi} e^{-\frac{\omega^2}{2}}$

Gate 2023 EC Question 28



9.5 Q27) Let  $m(t)$  be a strictly band-limited signal with bandwidth  $B$  and energy  $E$ .

Assuming  $\omega_0 = 10B$ , the energy in the signal  $m(t) \cos(\omega_0 t)$

(A)  $\frac{E}{4}$

(B)  $\frac{E}{2}$

(C)  $E$

(D)  $2E$

(GATE EC 2023)

**Solution:**

