SIGNAL PROCESSING
Through GATE

G. V. V. Sharma

Contents

Introduction

This book provides solutions to signal processing problems in GATE.

Chapter 1

Harmonics

1.1 Let y(t)=x($4t$),where x(t) is a continous-time periodic signal of 100s.the fundamental period of y(t) is (**rounded off to the nearest integer**)   (GATE IN 2023)

Solution:

1.2 In the circuit shown below, it is observed that the amplitude of voltage across the resistor is the same as the amplitude of the sorce voltage. What is the angular frequency ω_0(in rad/s)?

$$100\cos(\omega_0 t)$$
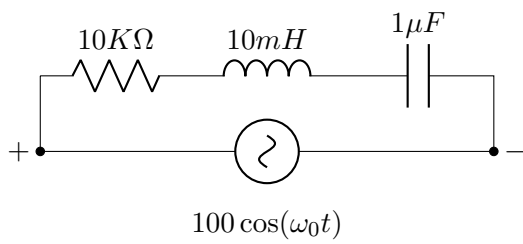
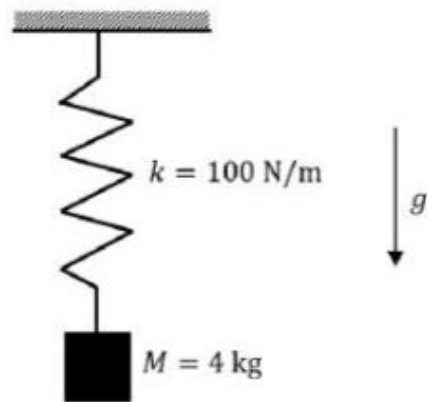(GATE BM 2023) **Solution:**

1.3 For a regular sinusoidal wave propagating in deep water having wave height of 3.5 m and wave period of 9 s, the wave steepness is _____ (round off to three decimal places).   Gate 2023 NM 33 **Solution:**

1.4 A spring mass system is shown in the figure . Take the value of acceleration due to gravity as $g = 9.81m/s^2$.The static deflection due to weight and the time period of the oscillations,respectively,are

Solution:

1.5 In the circuit shown below, the amplitudes of the voltage across the resistor and the capacitor are equal. What is the value of the angular frequency ω_o (in rad/s)? (Round off the answer to one decimal place.)   (GATE BM 32 2023)

1 kΩ

$100\cos(\omega_o t)$ 100μF
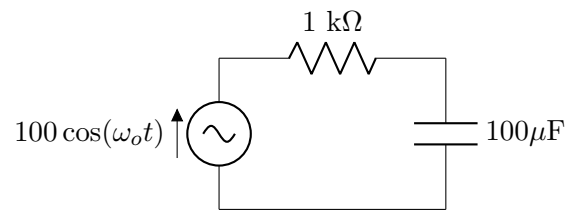
Solution:

Chapter 2

Filters

2.1 For the circuit given below, choose the angular frequency ω_0 at which voltage across capacitor has maximum amplitude?

Figure 2.1: circuit

(A) 1000

(B) 100

(C) 1

(D) 0

(GATE BM 2023 Question 16)

Solution:

Parameter	Description	Value
$V_i(j\omega)$	Input voltage	100
$v_c(t)$	Potential difference across Capacitor	?
$V_c(s)$	Potential difference across Capacitor	$V_c(s)$
$H(s)$	Transfer function	$\frac{V_c(s)}{V_i(s)}$
V_o	Amplitude of input voltage	$100\,\mathrm{V}$
R	Resistance in circuit	$1\,\mathrm{k\Omega}$
C	Capacitance in circuit	$100\,\mu\mathrm{F}$
ω_o	Angular frequency of input voltage	ω_o

Table 2.1: input values

$$V_c(s) = \frac{V_1(s)\frac{1}{sC}}{R + \frac{1}{sC}} \tag{2.1}$$

$$\implies H(s) = \frac{1}{1 + sRC} \tag{2.2}$$

$$\implies H(j\omega) = \frac{1}{1 + j\omega RC} \tag{2.3}$$

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \tag{2.4}$$

$$v_c(t) = \frac{100}{\sqrt{1 + (\omega_o RC)^2}}\left(\cos\omega_o t + \arctan\left(\frac{1}{\omega_o RC}\right)\right) \tag{2.5}$$

Maximum amplitude of $v_c(t)$ occurs at $\omega_o = 0$

$$\therefore \omega_o = 0 \tag{2.6}$$

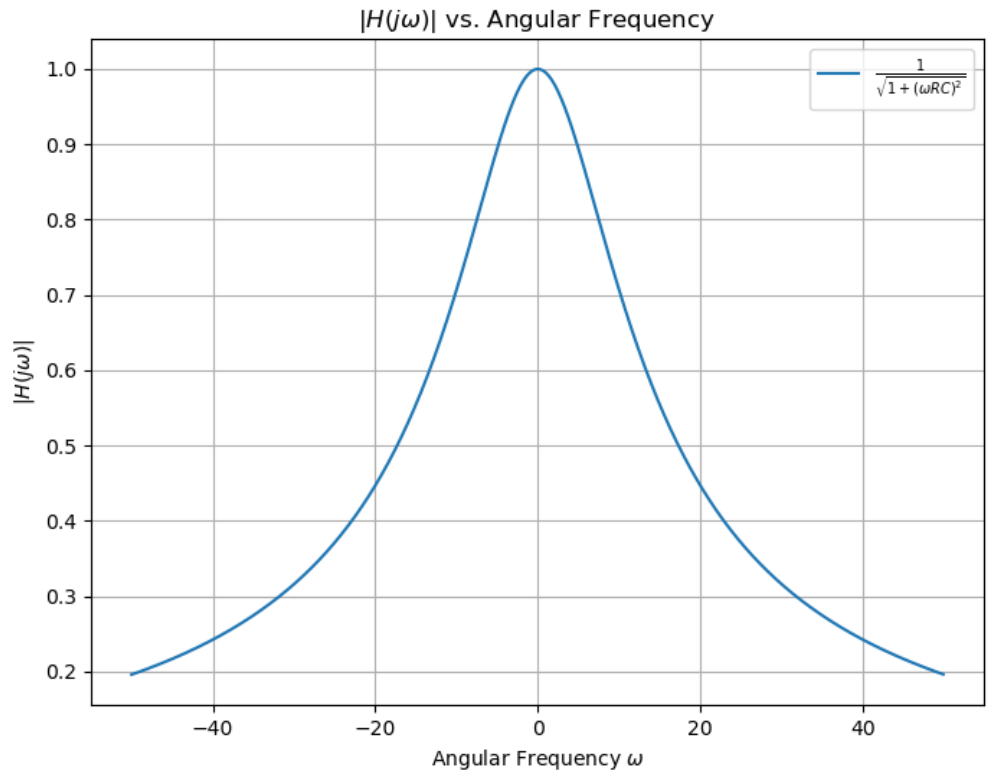$\therefore$ maximum value of $v_c(t)$ at steady state is 100 Volts.

Figure 2.2: $|H(j\omega)|$

2.2 In the following circuit, the switch S is open for $t < 0$ and closed for $t \geq 0$. What is the steady state voltage (in Volts) across the capacitor when the switch is closed?

(GATE BM 2023 Question 30)

2.3 A finite impulse response (FIR) filter has only two non-zero samples in its impulse response $h[n]$, namely $h[0] = h[1] = 1$. The Discrete Time Fourier Transform (DTFT) of $h[n]$ equals $H(e^{j\omega})$, as a function of the normalized angular frequency ω. For the range $|\omega| \leq \pi$, $\left|H(e^{j\omega})\right|$ is equal to
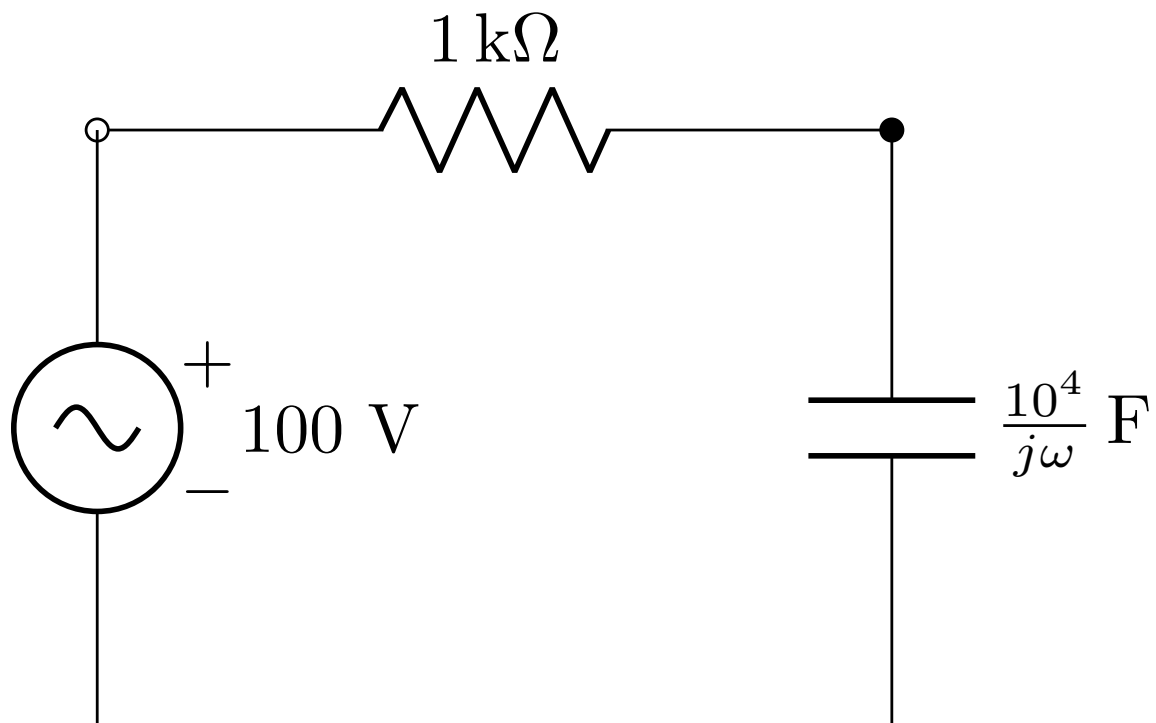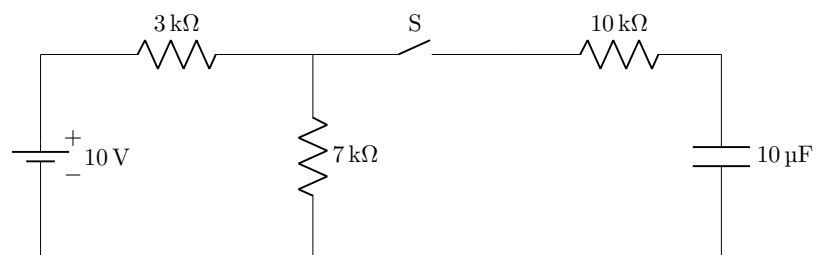
Figure 2.3: circuit in ω-domain

Figure 2.4: circuit

(A) $2\left|\cos(\omega)\right|$
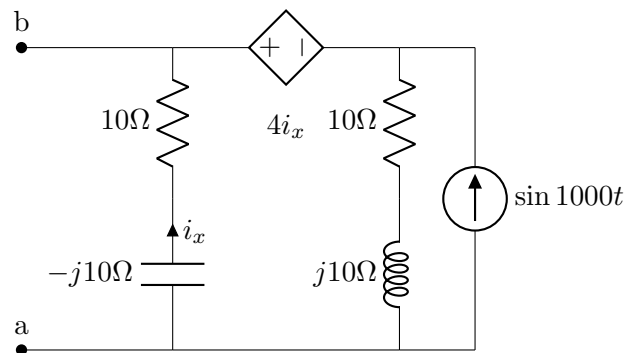
(B) $2\left|\sin(\omega)\right|$

(C) $2\left|\cos\left(\frac{\omega}{2}\right)\right|$

(D) $2\left|\sin\left(\frac{\omega}{2}\right)\right|$

2.4 For the circuit shown,if $i = \sin 1000t$, the instantaneous value of the Thevenin's voltage(in volts) across the terminals a anb b at time t=5ms is

b

10Ω $4i_x$ 10Ω

i_x

$\sin 1000t$

$-j10\Omega$ $j10\Omega$

a

2.5 In the circuit shown ,$\omega = 100\pi$rads/s, R1=R2=2.2Ω and L=7mH. the capacitance C for which Y_{in} is purely real is mF

$V_0\cos(\omega t)$ R_1 R_2 L   C
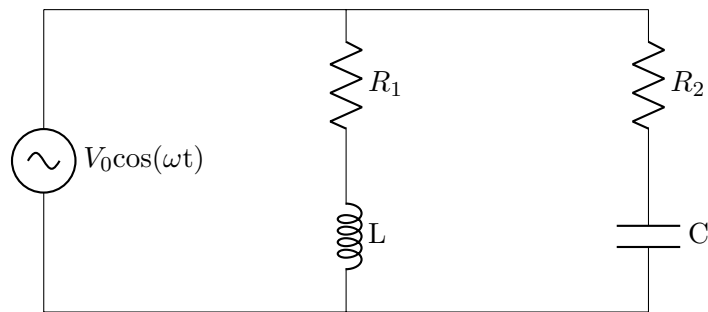
Solution:

2.6 An input voltage in the form of a square wave of frequency $1\,kHz$ is given to a circuit, which results in the output shown schematically below. Which one of the following options is the CORRECT representation of the circuit?   (GATE PH 2023 Q37)
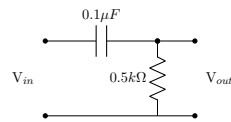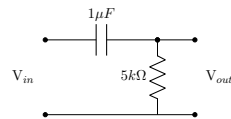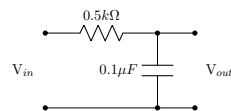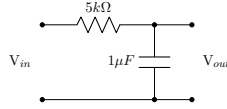

Figure 2.5:

(a)

(b)

(c)

13

(d)

Solution:

Symbol	Value	Description
$V_{in}(t)$		Input Voltage
$\mathcal{V}_{in}(f)$		Fourier Transform of $V_{in}(t)$
$V_{out}(t)$		Output Voltage
$\mathcal{V}_{out}(f)$		Fourier Transform of $V_{out}(t)$
f	$1000Hz$	Input Wave Frequency
T	$\frac{1}{f} = 10^{-3}s$	Input Wave Time Period
R	(a) $0.5k\Omega$ (b) $5k\Omega$	Resistance
C	(a) $0.1\mu F$ (b) $1\mu F$	Capacitance
τ	RC	Time Constant
Z	$R + \frac{1}{sC}$	Impedance
$H(f)$	$\frac{V_{out}}{V_{in}}$	General Transfer Function
$H_R(f)$	$\frac{V_{R,out}}{V_{in}}$	Transfer Function for Resistor
$H_C(f)$	$\frac{V_{C,out}}{V_{in}}$	Transfer Function for Capacitor

Table 2.2: Given Parameters

Input waveform is a square wave (Fig. 2.7), so we take its Fourier Transform as shown in Fig. 2.8

$$V_{in}(t) = 2\left(2\left[\frac{\left(t - \frac{T}{4}\right)}{T}\right] - \left[\frac{2\left(t - \frac{T}{4}\right)}{T}\right]\right) + 1 \tag{2.7}$$

$$V_{in}(t) \xleftrightarrow{\mathcal{F}} \mathcal{V}_{in}(f) \tag{2.8}$$
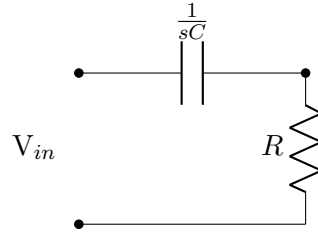
Figure 2.6: Series RC Circuit in s-domain

$$s = j2\pi f \tag{2.9}$$

$$\implies Z = R + \frac{1}{sC} \tag{2.10}$$

$$= R + \frac{1}{j2\pi fC} \tag{2.11}$$

$$H(f) = \frac{V_{out}}{V_{in}} \tag{2.12}$$

Across R,

$$H_R(f) = \frac{R}{R + \frac{1}{j2\pi fC}} \tag{2.13}$$

$$\implies \mathcal{V}_{out}(f) = H_R(f)\mathcal{V}_{in}(f) \tag{2.14}$$

Across C,

$$H_C(f) = \frac{\frac{1}{j2\pi fC}}{R + \frac{1}{j2\pi fC}} \tag{2.15}$$

$$\implies \mathcal{V}_{out}(f) = H_C(f)\mathcal{V}_{in}(f) \tag{2.16}$$

$$\mathcal{V}_{out}(f) \xleftrightarrow{\mathcal{F}} V_{out}(t) \tag{2.17}$$
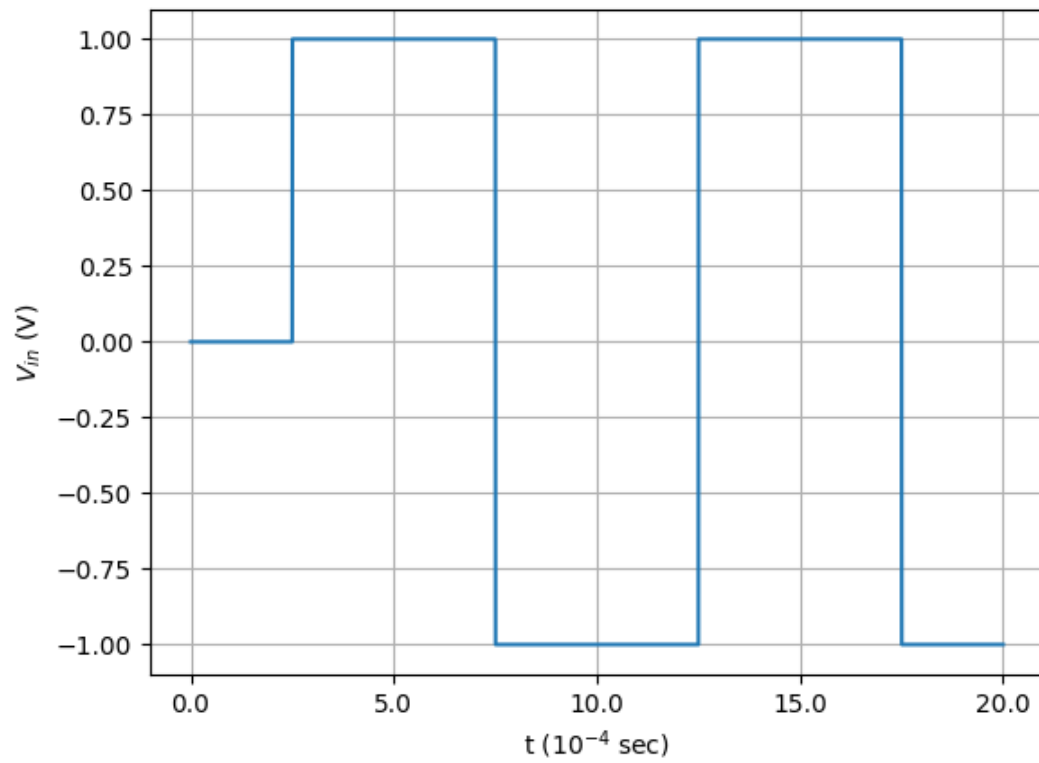
Figure 2.7: Input Square Waveform $(V_{in}(t))$

$\mathcal{V}_{in}(f)$ was input into all four circuits and Inverse Fourier Transform was taken of the response. All responss are plotted below:

As Fig. 2.10 resembles question Fig. 2.5, option (a) is the correct answer.

16
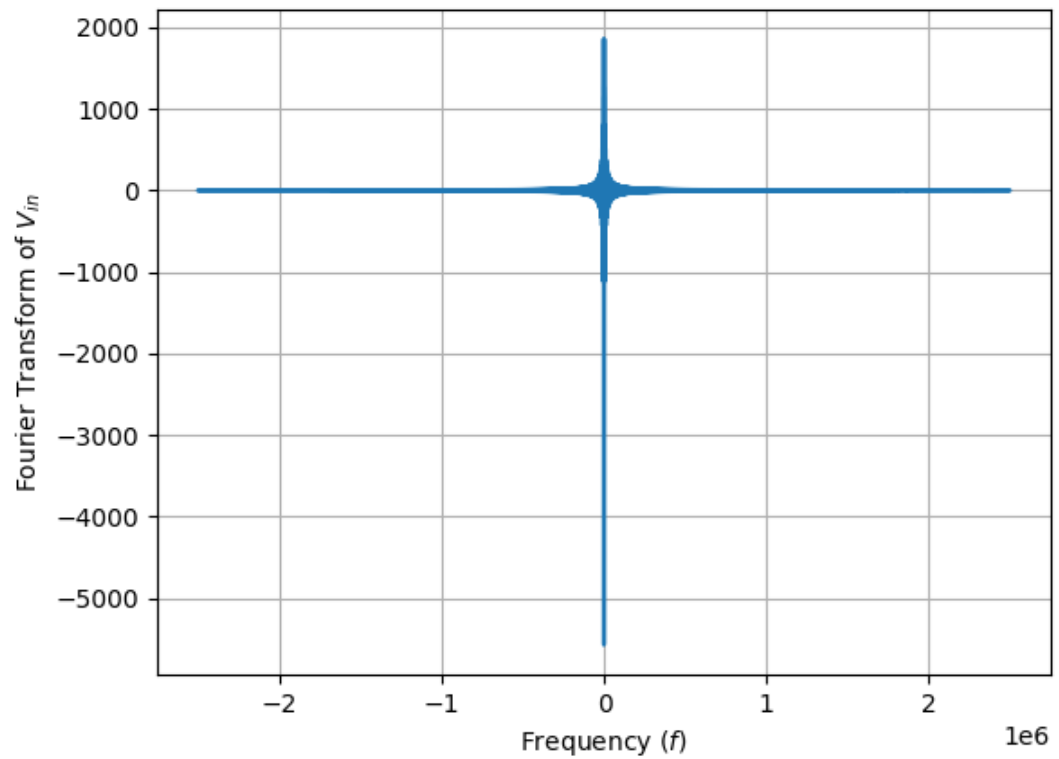
Figure 2.8: $\mathcal{V}_{in}(f)$ (Fourier Transform of $V_{in}(t)$)

2.7 In the circuit shown below, switch S was closed for long time. If the switch is opened at $t = 0$, the maximum magnitude of the voltage V_R , in volts is (rounded off to the nearest integer) (GATE 2023 EC 35)

Solution:
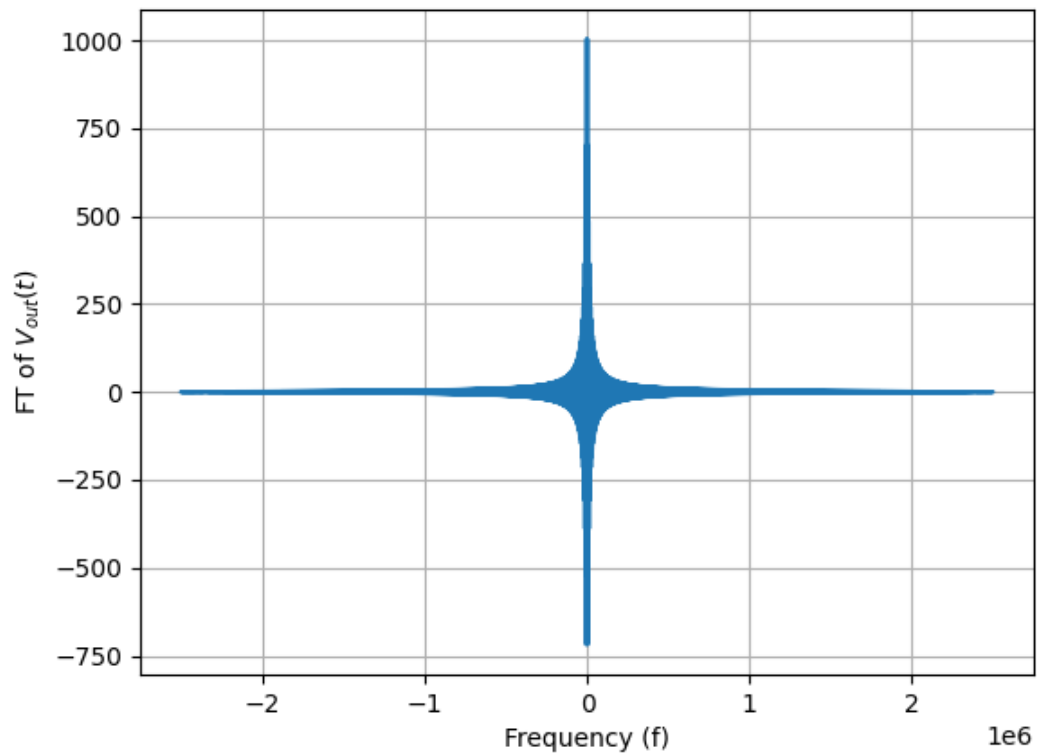
Figure 2.9: Opt A: Fourier Transform of $V_{out}(t)$

2.8 A signal $x(t) = 2\cos(180\pi t)\cos(60\pi t)$ is sampled at 200 Hz and then passed through an ideal low pass filter having cut-off frequency of 100 Hz.

The maximum Frequency present in the filtered signal in Hz is ———— (Round off to the nearest integer.)   (GATE 2023 EE) **Solution:**
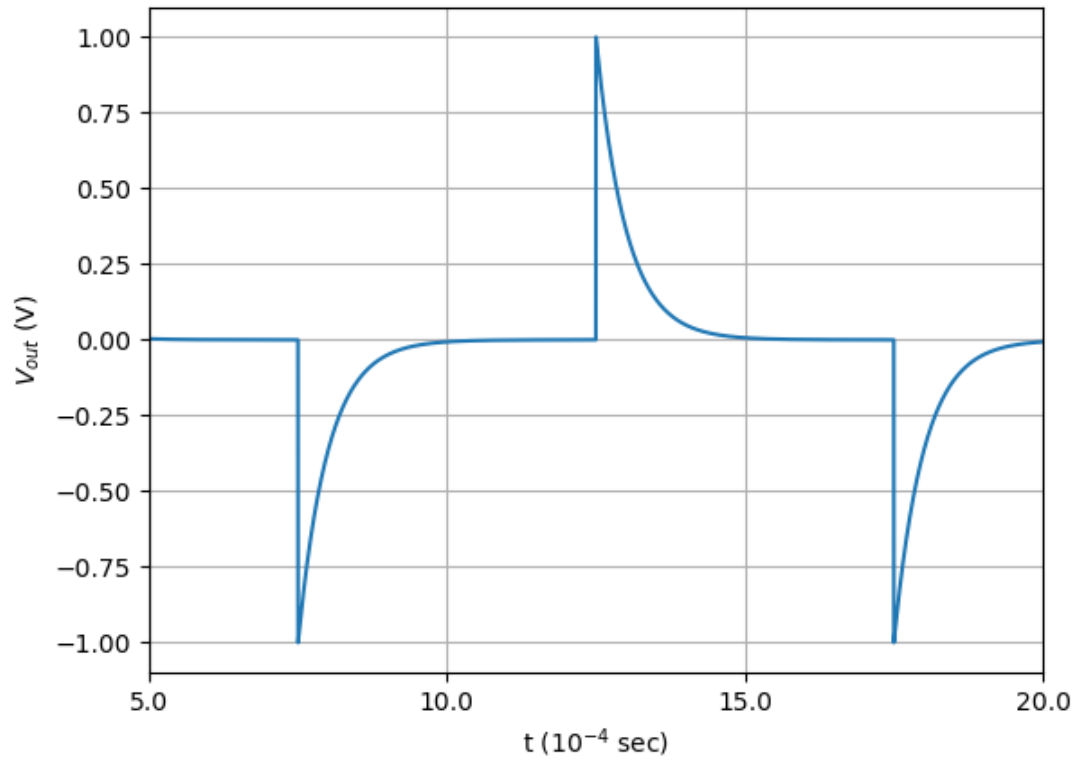
Figure 2.10: Opt A: Response

2.9 In the circuit shown, the input voltage $V_{in} = 100mV$. The switch and the opamp are ideal. At time $t = 0$, the intial charge stored in the $10nF$ capacitor is $1nC$, with the polarity as indicated in the figure. The switch S is controlled using a $1KHz$ square-wave voltage signal V_s as shown. Whenever V_s is 'High', S is in position '1' and when V_s is 'Low', S is in position '2'.

At $t = 20ms$, the magnitude of the voltage V_o will be

(GATE IN 2023)

Solution:
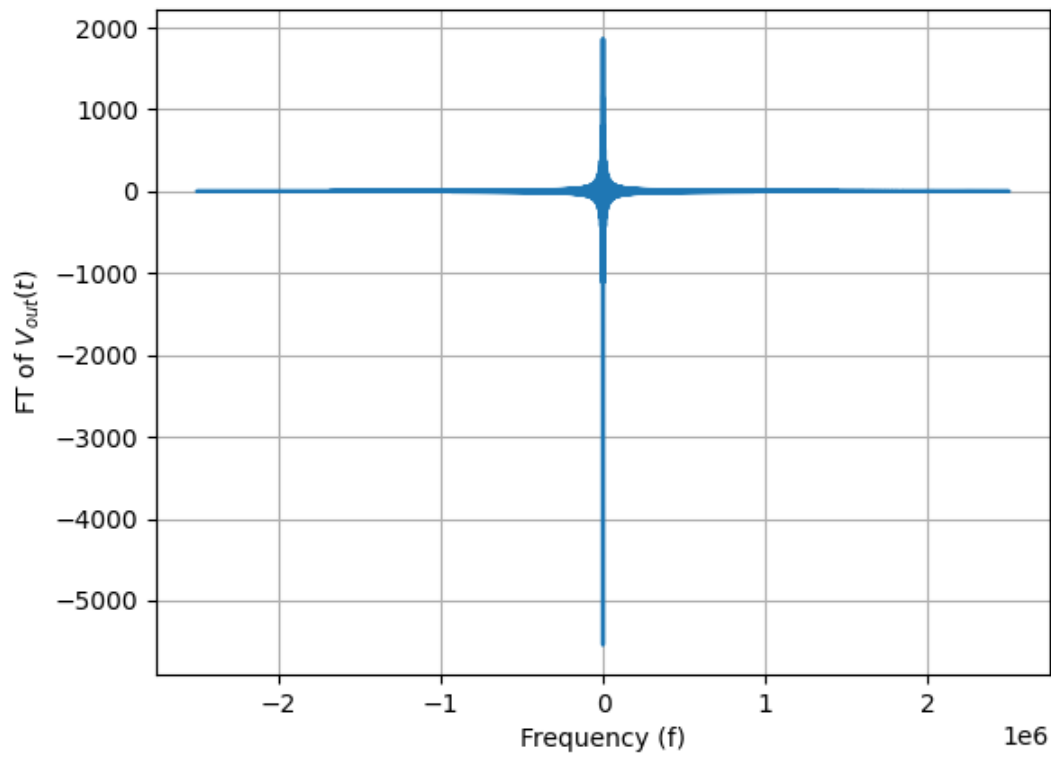
Figure 2.11: Opt B: Fourier Transform of $V_{out}(t)$

2.10 The value of parameters of the circuit shown in the figure are $R_1 = 2\Omega$, $R_2 = 2\Omega$, $R_3 = 3\Omega$, $L = 10mH$, $C = 100\mu F$. For time $t < 0$, the circuit is at steady state with the switch $'K'$ in closed condition. If the switch is opened at $t = 0$, the value of the voltage across the inductor (V_L) at $t = 0^+$ in Volts is _____ (Round off to 1 decimal
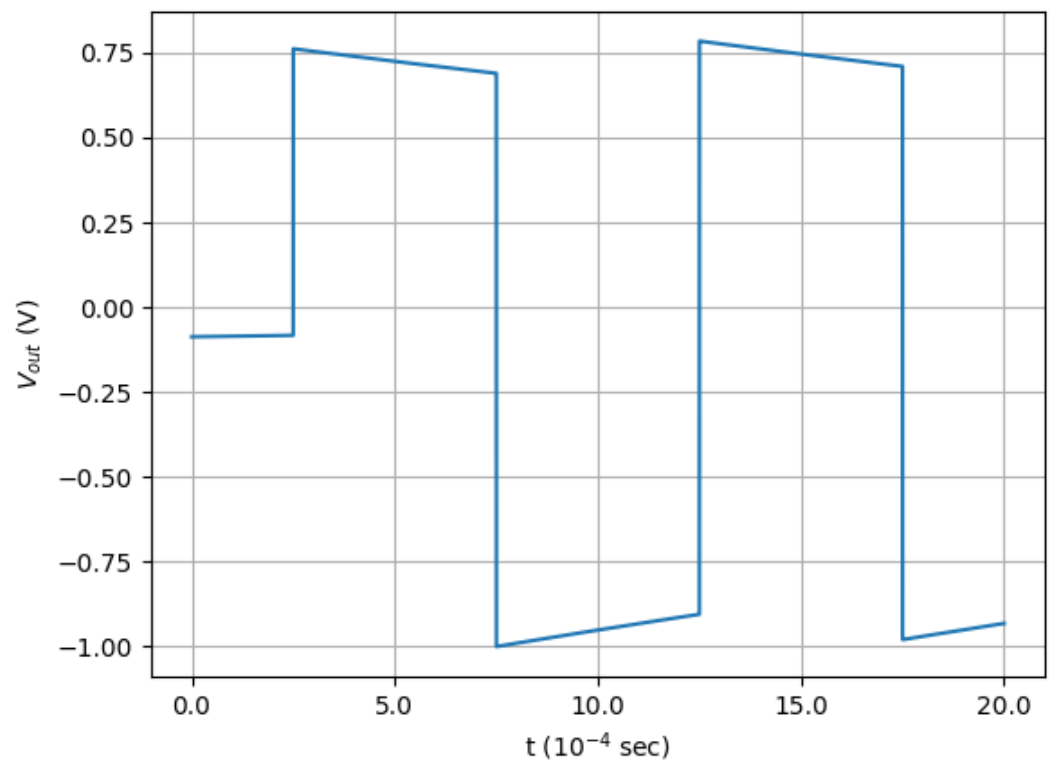
Figure 2.12: Opt B: Response

place).

(GATE

21
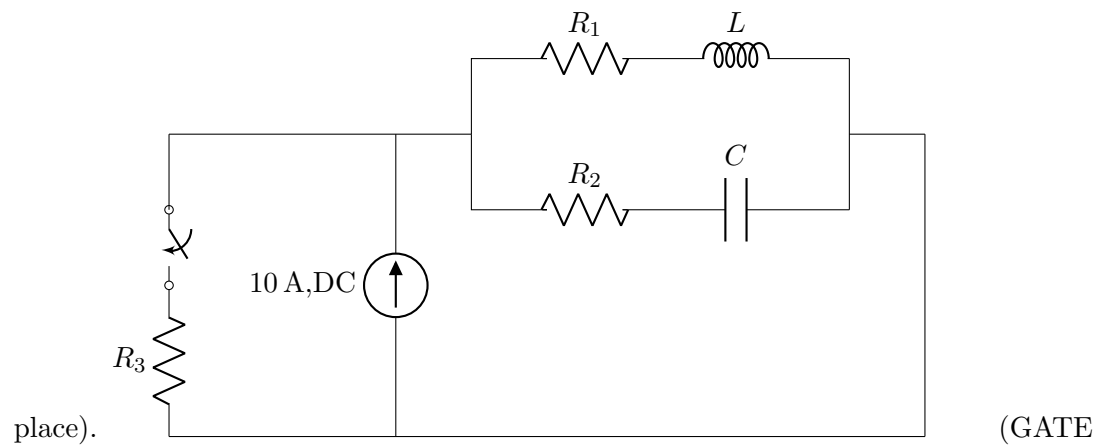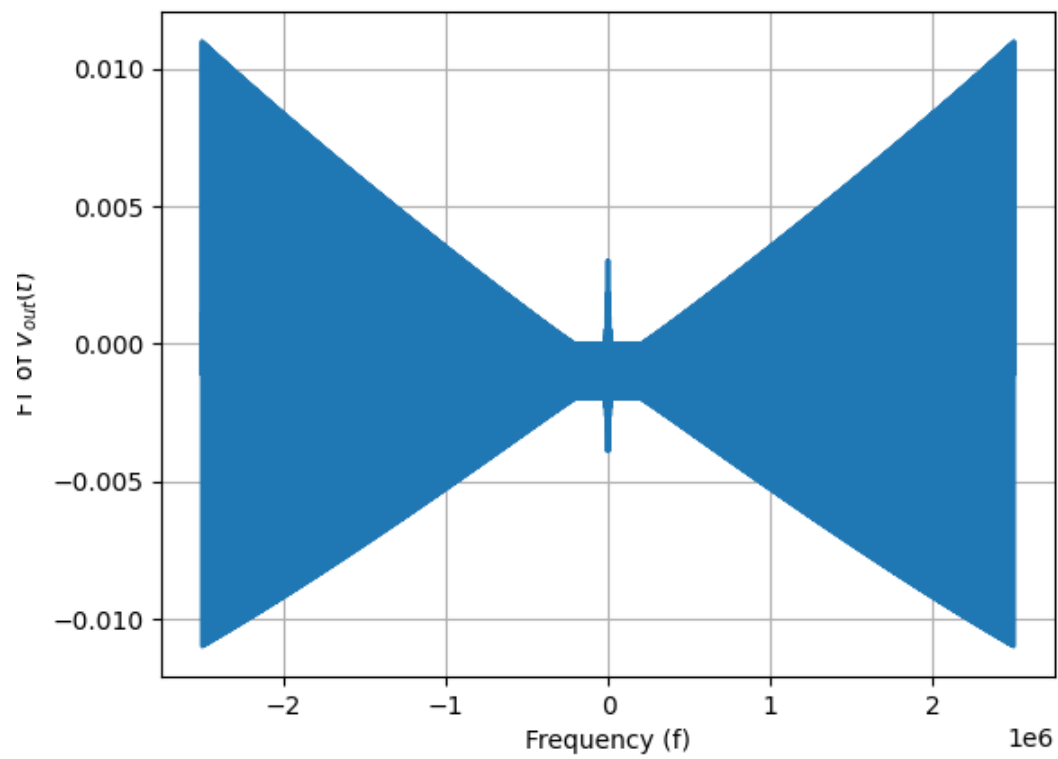
Figure 2.13: Opt C: Fourier Transform of $V_{out}(t)$

2023 EE 29Q) **Solution:**
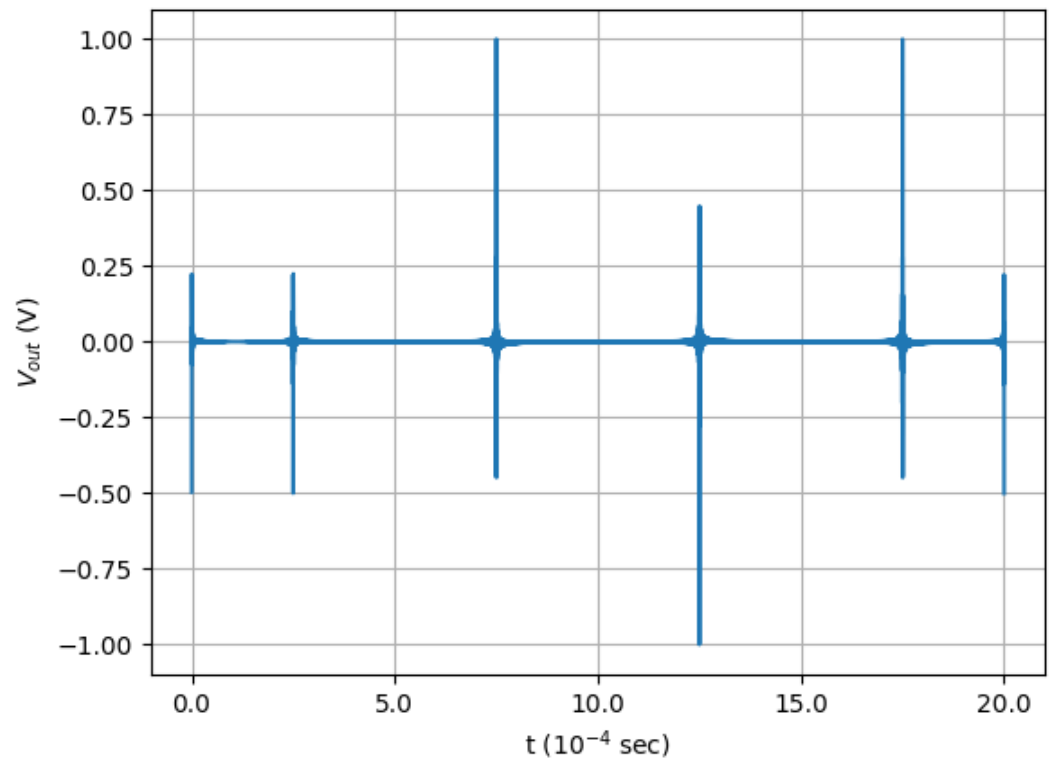
Figure 2.14: Opt C: Response

2.11 The op amps in the circuit are ideal. The input signals are $V_{S1} = 3 + 0.10 \sin(300t), \mathrm{V}$ and $V_{S2} = -2 + 0.11 \sin(300t) \, \mathrm{V}$. The average value of the voltage V_0 is _____ volts (rounded off to two decimal places).   (GATE IN 2023)
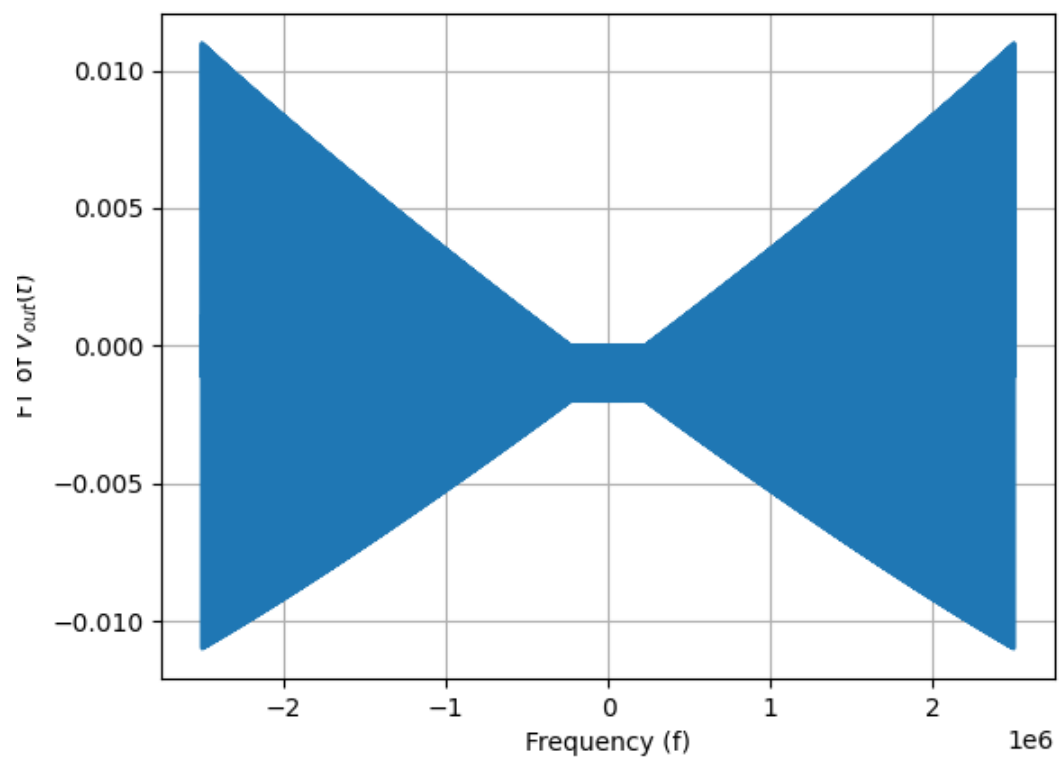
Figure 2.15: Opt D: Fourier Transform of $V_{out}(t)$
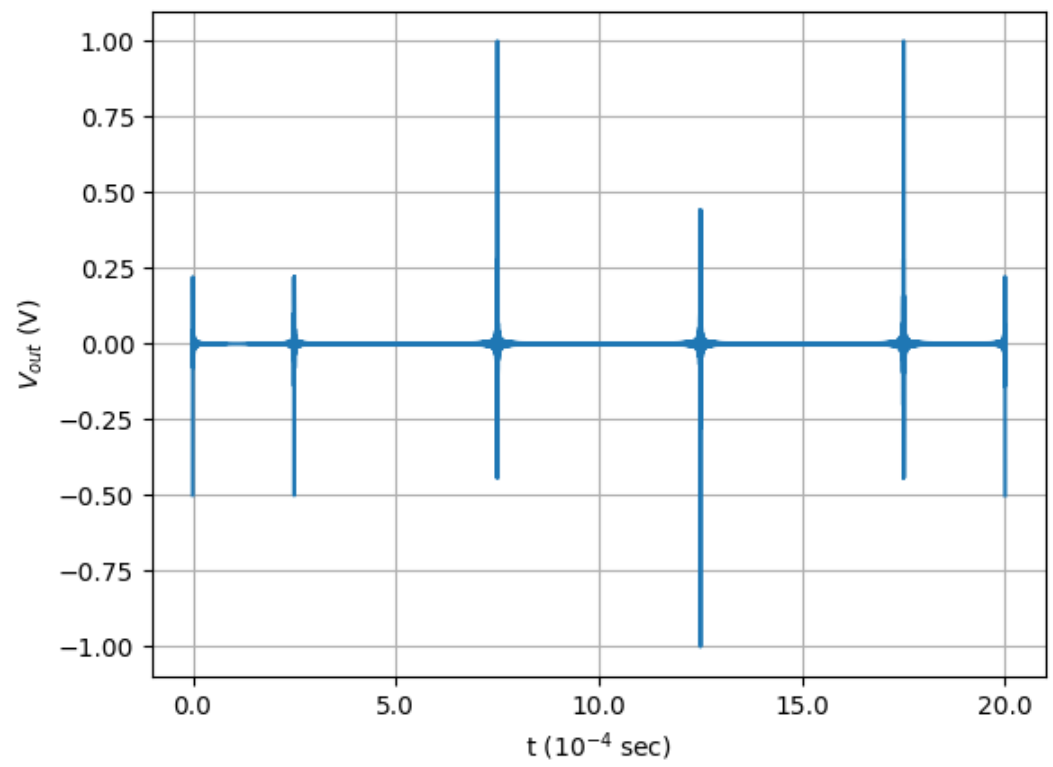
Figure 2.16: Opt D: Response
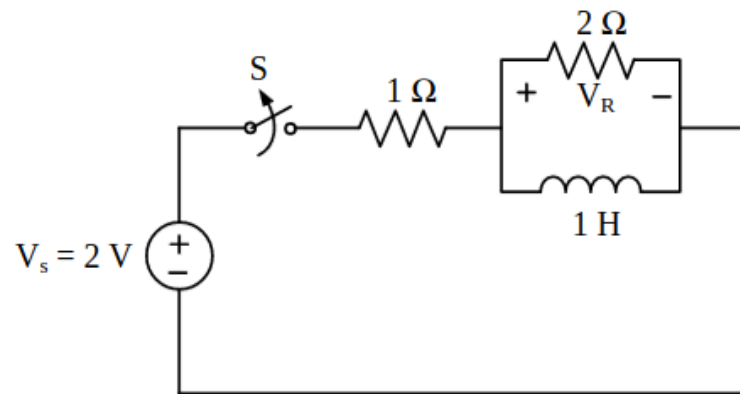
Figure 2.17:
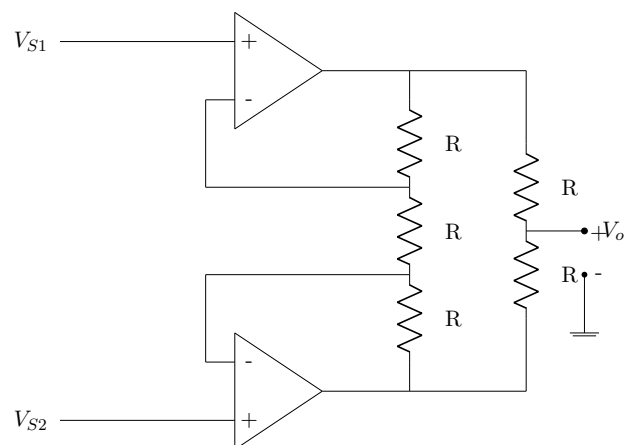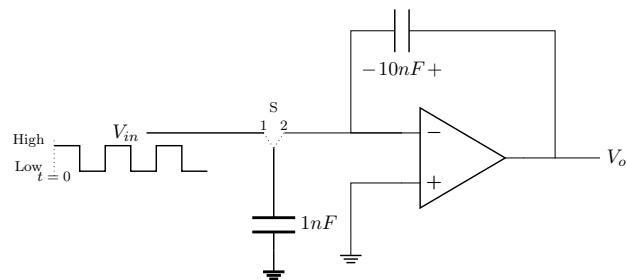
Chapter 3

Z-transform

Chapter 4

Systems

4.1 Consider a unity-gain negative feedback system consisting of the plant $G(s)$ and a proportional-integral controller. Let the proportional gain and integral gain be 3 and 1, respectively. For a unit step reference input, the final values of the controller output and the plant output, respectively, are

$$G(s) = \frac{1}{(s-1)}$$

(GATE EE 2023)

Solution:

Parameter	Description	Value
K_p	Proportional Gain	3
K_i	Integral Gain	1
$r(t)$	Reference Input	$u(t)$
$w(t)$	Controller Output	?
$y(t)$	Plant Output	?
$e(t)$	Error Input	$r(t) - y(t)$

Table 1: Parameter Table

From theFig. 4.1:

$$E\left(s\right) = U\left(s\right) - Y\left(s\right) \tag{4.1}$$

$$W\left(s\right) = 3E\left(s\right) + \frac{1}{s}E\left(s\right) \tag{4.2}$$

$$Y\left(s\right) = G\left(s\right)W\left(s\right) \tag{4.3}$$

Some results:

$$tx\left(t\right) \xleftrightarrow{\mathcal{L}} -\frac{dX\left(s\right)}{ds} \tag{4.4}$$

$$e^{-at}x\left(t\right) \xleftrightarrow{\mathcal{L}} X\left(s+a\right) \tag{4.5}$$

By using (4.4) and (4.5):

$$e^{-t}u\left(t\right) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}, Re\left(s\right) > -1 \tag{4.6}$$

$$te^{-t}u\left(t\right) \xleftrightarrow{\mathcal{L}} \frac{1}{\left(s+1\right)^2}, Re\left(s\right) > -1 \tag{4.7}$$
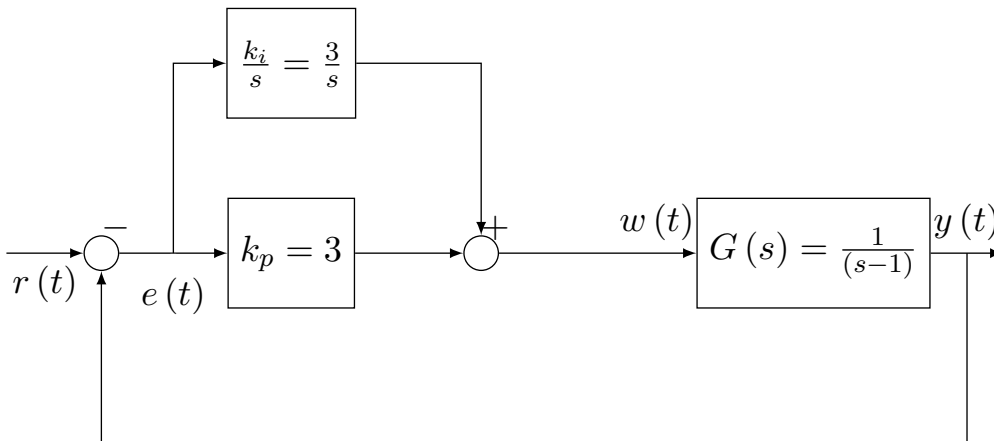
Figure 4.1: Block Diagram of System

(a) **Plant Output:**

From (4.1) , (4.2) and (4.3):

$$Y(s) = \frac{3s+1}{s(s+1)^2}, Re(s) > -1 \tag{4.8}$$

Final Value Theorem:

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) \tag{4.9}$$

Using (4.9) on Y(s):

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) \tag{4.10}$$

$$= 1 \tag{4.11}$$

Taking partial fraction of (4.8) :

$$Y(s) = \frac{1}{s} + \frac{2}{(s+1)^2} - \frac{1}{s+1} \tag{4.12}$$

Using (4.6) and (4.7):

$$\therefore y(t) = u(t) + 2te^{-t}u(t) - e^{-t}u(t) \tag{4.13}$$

(b) **Controller Output:**

From (4.2)

$$W(s) = \frac{3}{s} + \frac{1}{s^2} - Y(s)\left(3 + \frac{1}{s}\right) \tag{4.14}$$

Substituting (4.8)

$$W\left(s\right) = \frac{\left(s-1\right)\left(3s+1\right)}{s\left(s+1\right)^2}, Re\left(s\right) > -1 \qquad (4.15)$$

Using (4.9) on W(s)

$$\lim_{t\to\infty} w\left(t\right) = \lim_{s\to 0} sW\left(s\right) \qquad (4.16)$$

$$= -1 \qquad (4.17)$$

Taking partial fraction of equation(4.15) :

$$W\left(s\right) = -\frac{1}{s} - \frac{4}{\left(s+1\right)^2} + \frac{4}{s+1} \qquad (4.18)$$

Using equations (4.6) and (4.7) and taking inverse lapalace transform:

$$w\left(t\right) = -u\left(t\right) - 4te^{-t}u\left(t\right) + 4e^{-t}u\left(t\right) \qquad (4.19)$$
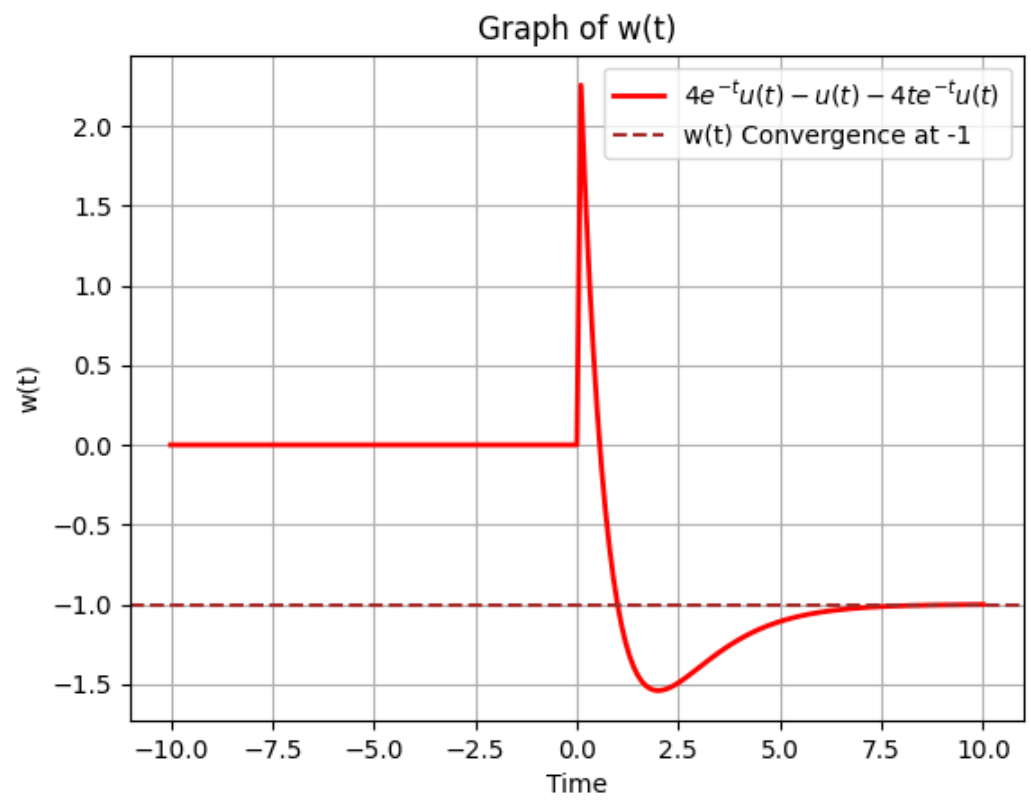
Figure 4.2: $w\left(t\right)$ converges at -1.
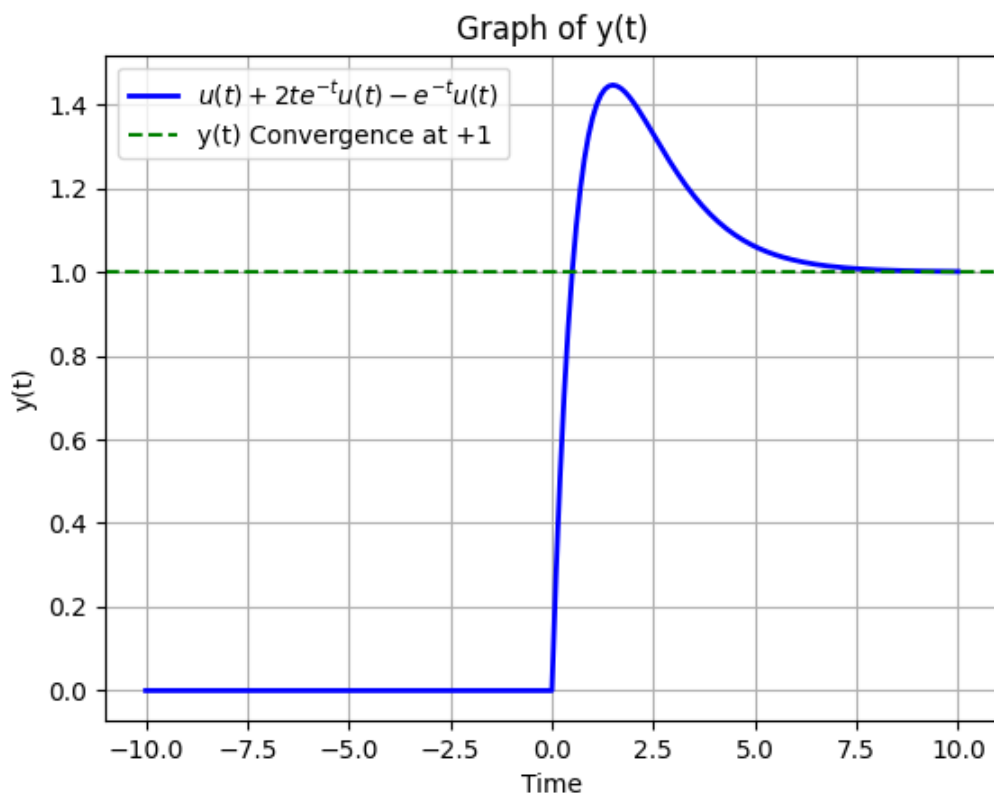
Figure 4.3: $y(t)$ converges at $+1$

4.2 Level (h) in a steam boiler is controlled by manipulating the flow rate (F) of the break-up(fresh) water using a proportional (P) controller. The transfer function between the output and the manipulated input is

$$\frac{h\,(s)}{F\,(s)} = \frac{0.25\,(1 - s)}{s\,(2s + 1)}$$

The measurement and the valve transfer functions are both equal to 1. A process engineer wants to tune the controller so that the closed loop response gives the decaying oscillations under the servo mode. Which one of the following is the CORRECT value of the controller gain to be used by the engineer?

(a) 0.25

(b) 2

(c) 4

(d) 6

Solution:

Closed loop signal transfer function of the above block diagram can be given by,

$$T\,(s) = \frac{G\,(s)}{1 + G\,(s)\,H\,(s)} \qquad (4.20)$$

From Fig. 4.4 and Table 4.2 for a unit impulse, $X\,(s) = 1$

35
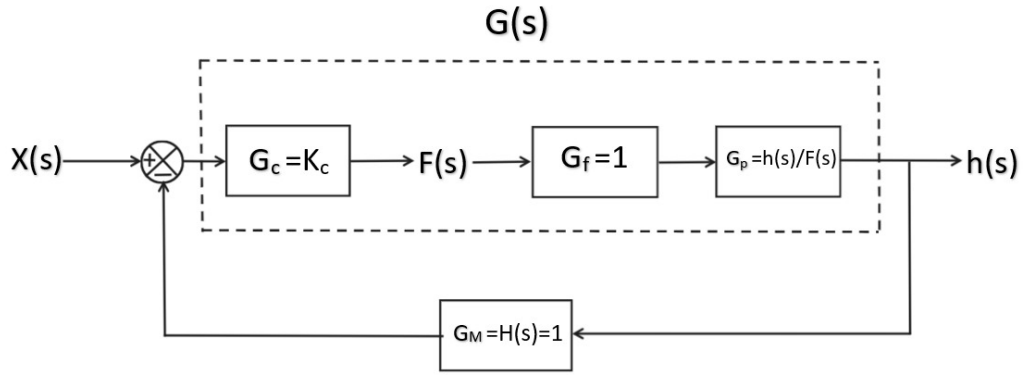
Figure 4.4: Closed loop Block diagram

$$h\left(s\right) = T\left(s\right) \times X\left(s\right) \tag{4.21}$$

$$h\left(s\right) = \frac{\left(1 - s\right) K_c}{8s^2 + \left(4 - K_c\right) s + K_c} \tag{4.22}$$

$$\implies h\left(s\right) = \frac{\left(1 - s\right) K_c}{8\left(s - s_1\right)\left(s - s_2\right)} \tag{4.23}$$

Where,

$$s_1 = \frac{\left(K_c - 4\right)}{16} + \sqrt{\left(\frac{K_c - 4}{16}\right)^2 - \frac{K_c}{8}} \tag{4.24}$$

$$s_2 = \frac{\left(K_c - 4\right)}{16} - \sqrt{\left(\frac{K_c - 4}{16}\right)^2 - \frac{K_c}{8}} \tag{4.25}$$

From (4.23) we get,

$$h\left(s\right) = \frac{K_c}{8\left(s_1 - s_2\right)} \left(\frac{1 - s_1}{s - s_1} - \frac{1 - s_2}{s - s_2}\right) \tag{4.26}$$

Now taking the inverse laplace transform we have,

$$h\left(t\right) = \frac{K_c}{8\left(s_1 - s_2\right)} \left[\left(1 - s_1\right)e^{s_1 t} - \left(1 - s_2\right)e^{s_2 t}\right] u\left(t\right) \tag{4.27}$$

$$\implies h\left(t\right) = e^{\frac{K_c - 4}{16}} \left(A_1 e^{\sqrt{\left(\frac{K_c - 4}{16}\right)^2 - \frac{K_c}{8}}} - A_2 e^{-\sqrt{\left(\frac{K_c - 4}{16}\right)^2 - \frac{K_c}{8}}}\right) u\left(t\right) \tag{4.28}$$

Where,

$$A_1 = \frac{K_c}{8} \left(\frac{1 - s_1}{s_1 - s_2}\right) \tag{4.29}$$

$$A_2 = \frac{K_c}{8} \left(\frac{1 - s_2}{s_1 - s_2}\right) \tag{4.30}$$

Now applying the condition for underdamped oscillations,

$$\left(\frac{K_c - 4}{16}\right)^2 - \frac{K_c}{8} < 0 \tag{4.31}$$

$$\implies K_c \in \left(20 - \sqrt{384}, 20 + \sqrt{384}\right) \tag{4.32}$$

For the system to be stable,

$$\frac{K_c - 4}{8} < 0 \tag{4.33}$$

$$\implies K_c < 4 \tag{4.34}$$

From (4.32) and (4.34)

$$K_c \in (0.4, 4) \tag{4.35}$$

(4.35) represents the ROC,($R_e\{s\} < 0$)

$$\implies K_c = 2 \tag{4.36}$$
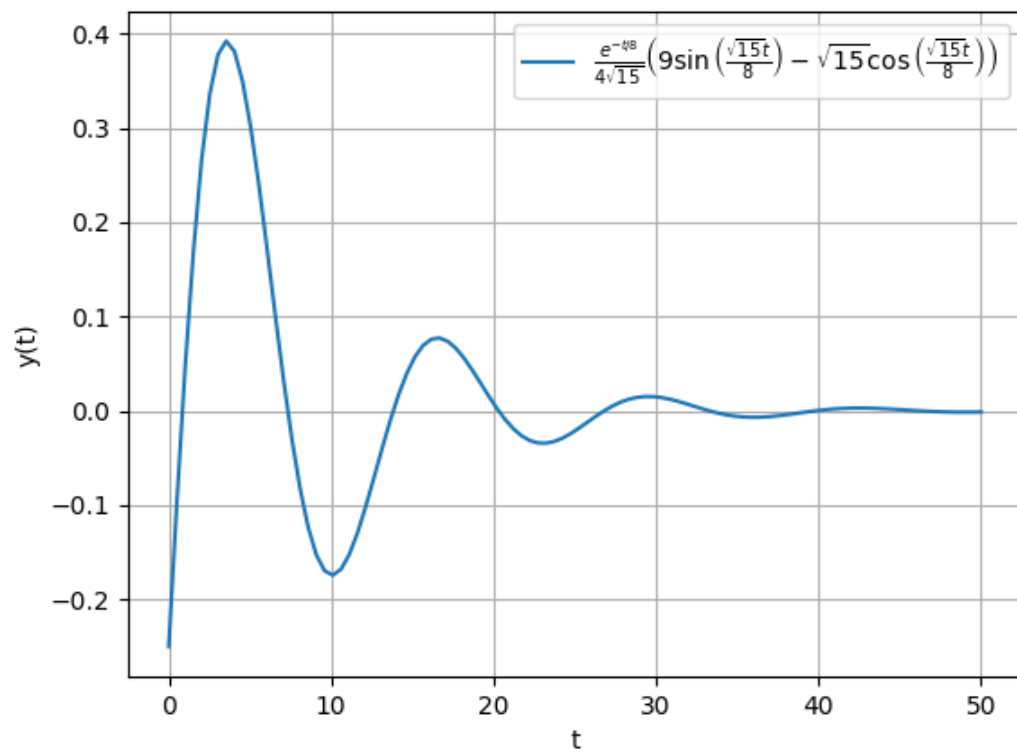
Figure 4.5: y(t) vs t graph

4.3 The figure shows a block of mass m $=$ 20 kg attached to a pair of identical linear springs, each having a spring constant k $=$ 1000 N/m. The block oscillates on a frictionless horizontal surface. Assuming free vibrations, the time taken by the block to complete ten oscillations is _____ seconds . (Rounded off to two decimal places) Take $\pi = 3.14$.

(GATE ME 2023)
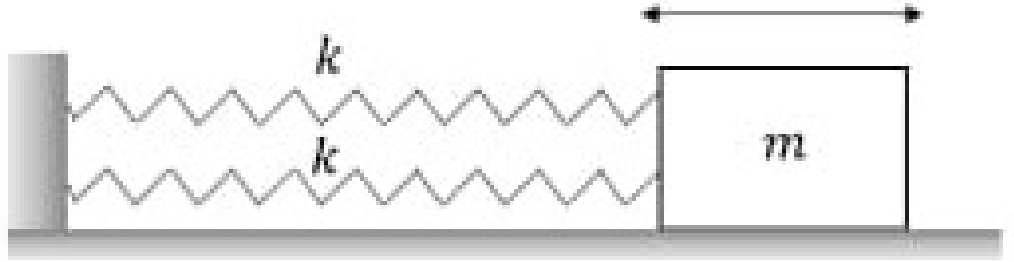
Solution:

$$F = ma \tag{4.37}$$

$$F = -kx \tag{4.38}$$

$$\implies ma + kx = 0 \tag{4.39}$$

$$\therefore m\frac{d^2x}{dt^2} + kx = 0 \tag{4.40}$$

The Laplace transform of the terms is ,

$$x \xleftrightarrow{\mathcal{L}} X(s) \tag{4.41}$$

$$\frac{d^2x}{dt^2} \xleftrightarrow{\mathcal{L}} s^2X(s) - sx(0) - \dot{x}(0) \tag{4.42}$$

39

Using equation (4.41) and (4.42) in equation (4.40),

$$m\left(s^2 X\left(s\right) - sx\left(0\right) - \dot{x}\left(0\right)\right) + kX\left(s\right) = 0 \qquad (4.43)$$

$$ms^2 X\left(s\right) - msA + kX\left(s\right) = 0 \qquad (4.44)$$

$$X\left(s\right) = \frac{msA}{ms^2 + k} \qquad (4.45)$$

$$= \frac{sA}{s^2 + \frac{k}{m}} \qquad (4.46)$$

The inverse Laplace transform of such terms is given by,

$$\frac{s}{s^2 + a^2} \xleftrightarrow{\mathcal{L}^-} cos\left(at\right) u\left(t\right) \qquad (4.47)$$

$\therefore$ the inverse Laplace of (4.46) is,

$$x\left(t\right) = Acos\left(\sqrt{\frac{k}{m}}t\right) \qquad (4.48)$$

From equation (4.48) and Table 4.3 ,the time to complete one oscillation is,

$$T_n = \frac{2\pi}{\sqrt{\frac{k}{m}}} \qquad (4.49)$$

$$= \frac{\pi}{5} \qquad (4.50)$$

$\therefore$ the time required for 10 oscillations is ,

$$10T_n = 2\pi \qquad (4.51)$$

$$= 6.28s \qquad (4.52)$$
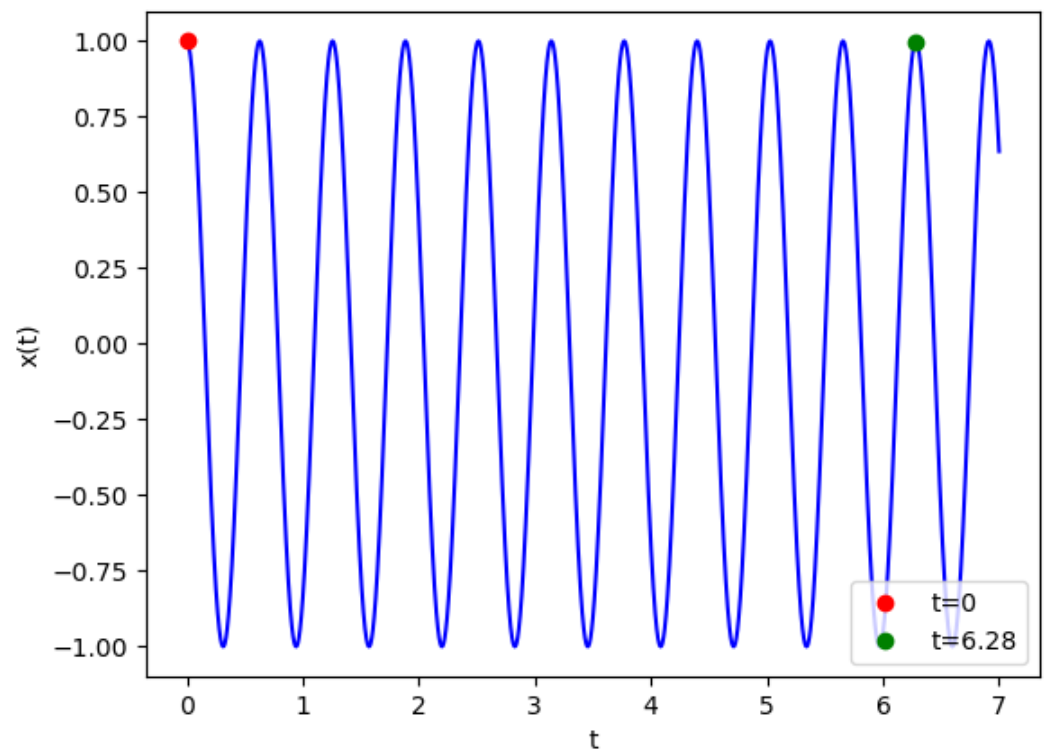
Figure 4.6: Plot of $x\left(t\right)$

4.4 A system has transfer function

$$\frac{Y(s)}{X(s)} = \frac{s - \pi}{s + \pi}$$

let $u(t)$ be the unit step function. The input $x(t)$ that results in a steady-state output

$y(t) = sin(\pi t)$ is __.   (GATE IN 2023)

Solution:
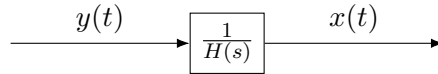
Figure 4.7: Block diagram of the inverse system
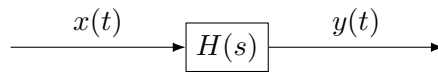
Figure 4.8: Block diagram of the system

$$H(s) = \frac{s - \pi}{s + \pi} \tag{4.53}$$

from Fig. 4.7

$$\frac{1}{H(s)} = \frac{s + \pi}{s - \pi} \tag{4.54}$$

Converting transfer function to frequency response, we get

$$\frac{1}{H(j\omega)} = \frac{j\omega + \pi}{j\omega - \pi} \tag{4.55}$$

from Table 4.4 $\omega = \pi$

$$\frac{1}{H(j\pi)} = \frac{j + 1}{j - 1} = -j = e^{-j\frac{\pi}{2}} \tag{4.56}$$

42

from (4.56)

$$\left| \frac{1}{H(j\pi)} \right| = 1 \quad \text{and} \quad \arg\left(\frac{1}{H(j\pi)} \right) = -90° \tag{4.57}$$

$$y(t) = \sin(\pi t) \tag{4.58}$$

$$\sin(\pi t) \overset{\infty}{\underset{\overline{\mathcal{H}(|\omega)}}{\longleftrightarrow}} \left| \frac{1}{H(j\omega)} \right| \sin\left(\pi t + \arg\left(\frac{1}{H(j\omega)} \right) \right) \tag{4.59}$$

Therefore, by (4.58) and (4.59), we get

$$x(t) = \sin\left(\pi t - \frac{\pi}{2} \right) \tag{4.60}$$
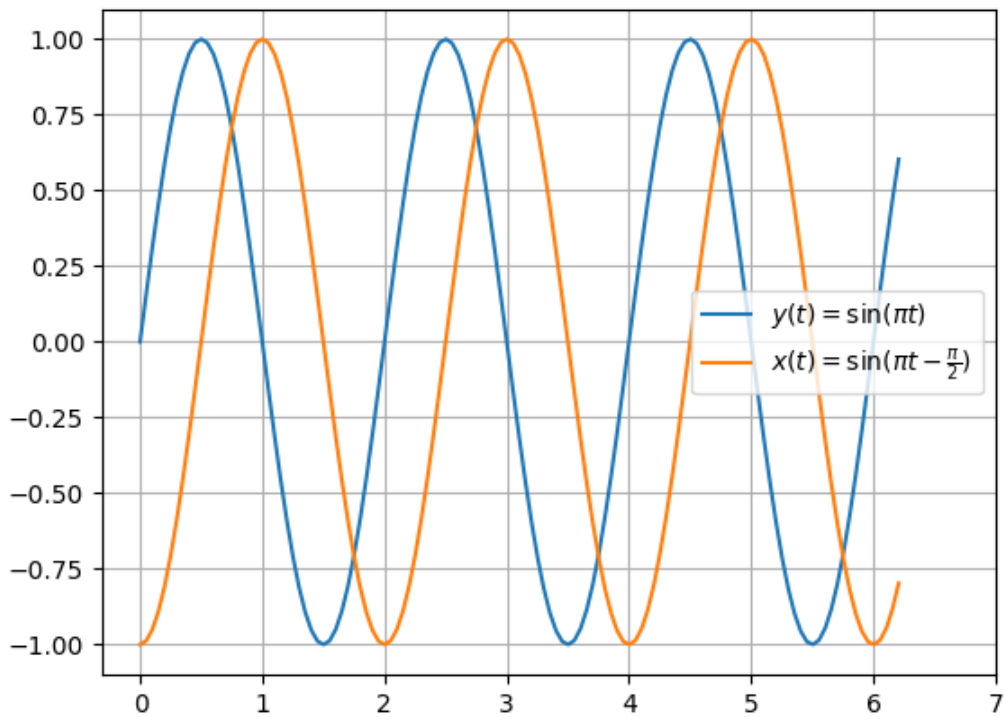
Figure 4.9: Plot of $x(t)$ and $y(t)$ taken from Python

4.5 The state equation of a second order system is

$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t), \quad \boldsymbol{x}(0)$ is the initial condition.

Suppose λ_1 and λ_2 are two distinct eigenvalues of A, and ν_1 and ν_2 are the corresponding eigenvectors. For constants α_1 and α_2, the solution, $\boldsymbol{x}(t)$, of the state equation is

(A) $\sum_{i=1}^{2} \alpha_i e^{\lambda_i t} \boldsymbol{\nu_i}$

(B) $\sum_{i=1}^{2} \alpha_i e^{2\lambda_i t} \boldsymbol{\nu_i}$

(C) $\sum_{i=1}^{2} \alpha_i e^{3\lambda_i t} \boldsymbol{\nu_i}$

(D) $\sum_{i=1}^{2} \alpha_i e^{4\lambda_i t} \nu_{\mathbf{i}}$

4.6 Consider the complex function

$$f(z) = \frac{z^2 \sin z}{(z - \pi)^4}$$

At $z = \pi$, which of the following options is (are) correct?

(A) The order of the pole is 4

(B) The order of the pole is 3

(C) The residue at the pole is $\frac{\pi}{6}$

(D) The residue at the pole is $\frac{2\pi}{3}$

4.7 A buoy of virtual mass 30 kg oscillates in a fluid medium as a single degree of freedom system. If the total damping in the system is set as 188.5 N-s/m, such that the oscillation just ceases to occur, then the natural period of the system is _____ s (round off to one decimal place) (GATE MN 2023 question 63)

4.8 Which of the following statement(s) is/are true?

(a) If an LTI system is causal, it is stable.

(b) A discrete time LTI system is causal if and only if its response to a step input $u[n]$ is 0 for $n < 0$.

(c) If a discrete time LTI system has an impulse response $h[n]$ of finite duration the system is stable.

(d) If the impulse response $0 < |h[n]| < 1$ for all n, then the LTI system is stable.

(GATE EE 2023 question 27)

Solution:

4.9 The outlet concentration C_A of a plug flow reactor (PFR) is controlled by manipulating the inlet concentration C_{A0}. The following transfer function describes the dynamics of this PFR.

$$\frac{C_A(s)}{C_{A0}(s)} = e^{-(\frac{V}{F})(k+s)}$$

In the above question, V=$1m^3$, F=$0.1m^3 min^{-1}$ and k=$0.5 min^{-1}$. The measurement and valve transfer functions are both equal to 1. The ultimate gain, defined as the proportional controller gain that produces sustained oscillations, for this system is (GATE 2023 CH 61)

Solution:

4.10 For the block diagram shown in the figure, the transfer function $\frac{Y(s)}{R(s)}$ is

$R(s)$

2

3

$\frac{1}{s}$

$Y(s)$

1
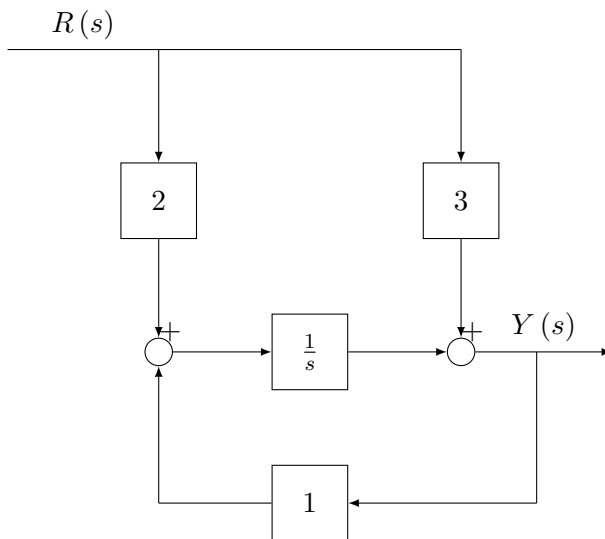
Figure 4.11: Block diagram

Solution:

4.11 In the following block diagram, $R(s)$ and $D(s)$ are two inputs. The output Y(s) is expressed as $Y(s) = G_1(s)R(s) + G_2(s)D(s)$.

$G_1(s)$ and $G_2(s)$ are given by   GATE 2023 EC Question 42

Solution:

4.12 In the table shown below, match the signal type with its spectral characteristics.GATE 2023 EC

(a) $(i) \rightarrow (a)$, $(ii) \rightarrow (b)$, $(iii) \rightarrow (c)$, $(iv) \rightarrow (d)$

(b) $(i) \rightarrow (a)$, $(ii) \rightarrow (c)$, $(iii) \rightarrow (b)$, $(iv) \rightarrow (d)$

(c) $(i) \rightarrow (d)$, $(ii) \rightarrow (b)$, $(iii) \rightarrow (c)$, $(iv) \rightarrow (a)$

(d) $(i) \rightarrow (a)$, $(ii) \rightarrow (c)$, $(iii) \rightarrow (d)$, $(iv) \rightarrow (b)$

Solution:

4.13 The impulse response of an LTI system is $h(t) = \delta(t) + 0.5\delta(t-4)$, where $\delta(t)$ is continuous-time unit impulse signal. If the input signal $x(t) = \cos\left(\frac{7\pi t}{4}\right)$, the output is (GATE IN 2023)

Solution:

4.14 Consider the state-space description of an LTI system with matrices

$$
A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -2 \end{bmatrix}, \quad D = 1
$$

For the input, $\sin(\omega t)$, $\omega > 0$, the value of ω for which the steady-state output of the system will be zero, is _____ (Round off to the nearest integer). (GATE 2023 EE Q46)

Solution:

4.15 A causal, discrete time system is described by the difference equation $y[n] = 0.5y[n-1] + x[n]$, for all n, where $y[n]$ denotes the output sequence and $x[n]$ denotes the input sequence. Which of the following statements is/are TRUE?

(a) he system has an impulse response described by $0.5^n u[-n]$ where $u[n]$ is the unit step sequence.

(b) The system is stable in the bounded input, bounded output sense.

(c) The system has an infinite number of non-zero samples in its impulse response

(d) The system has a finite number of non-zero samples in its impulse response.

(GATE 2023 BM-26)

Solution:

PARAMETER	DESCRIPTION
G_c	Proportional controller's transfer function
G_f	Valve transfer function
G_p	Process transfer function
G_M	Measurement transfer function
$G(s)$	Open loop transfer function
$T(s)$	Transfer function of system

Table 4.2: PARAMETER TABLE 1

Parameter	Description	Value
k_i	spring constant	1000 N/m
m	mass of block	20Kg
k	Equivalent spring constant	$k_1 + k_2$ (parallel)
ω_n	Natural frequency	$\sqrt{\frac{k}{m}}$
T	Time period of an oscillation	$\frac{2\pi}{\omega_n}$
x	Displacement of block	
a	Acceleration of block	$\frac{d^2x}{dt^2}$
F	Force on block	
A	Amplitude of oscillation	x(0)

Table 4.3: Parameter Table

Variable	Description	Value
$x(t)$	input function	none
$y(t)$	output function	$\sin(\pi t)$
$H(s)$	Transfer-function	$\frac{s-\pi}{s+\pi}$

Table 4.4: Input parameters

Signal Type	Spectral Characterstics
(i) Continuous, aperiodic	(a) Continuous, aperiodic
(ii) Continuous, periodic	(b) Continuous, periodic
(iii) Discrete, aperiodic	(c) Discrete, aperiodic
(iii) Discrete, periodic	(c) Discrete, periodic

Table 4.5:
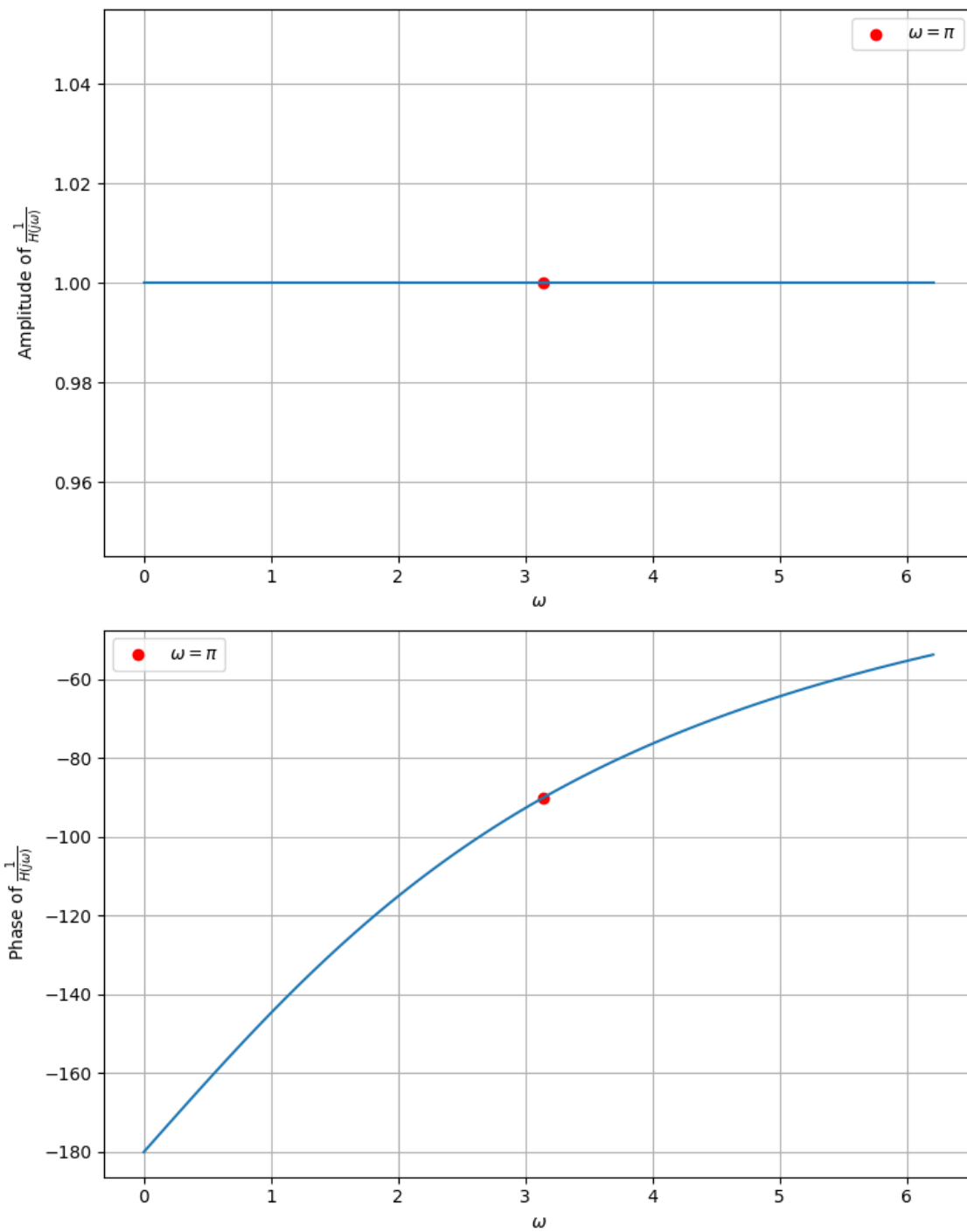
Figure 4.10: Amplitude and phase of $\frac{1}{H(j\omega)}$
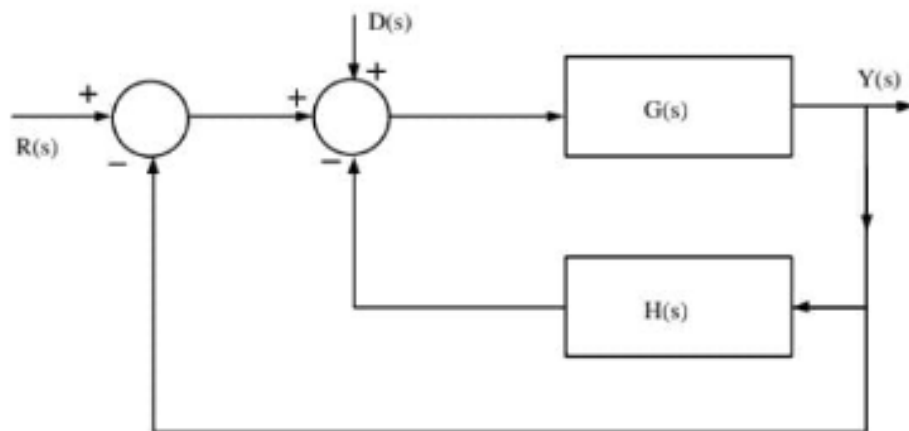
Chapter 5

Sequences

5.1 Consider the discrete time signal $x[n] = u[-n+5] - u[n+3]$, where

$$u[n] = \begin{cases} 1; n \geq 0 \\ 0; n < 0 \end{cases}$$

The smallest n for which $x[n] = 0$ is? (GATE IN 2023)

Solution: From Fig. 1, the minimum value of n is given as

$$n = -3 \tag{5.1}$$
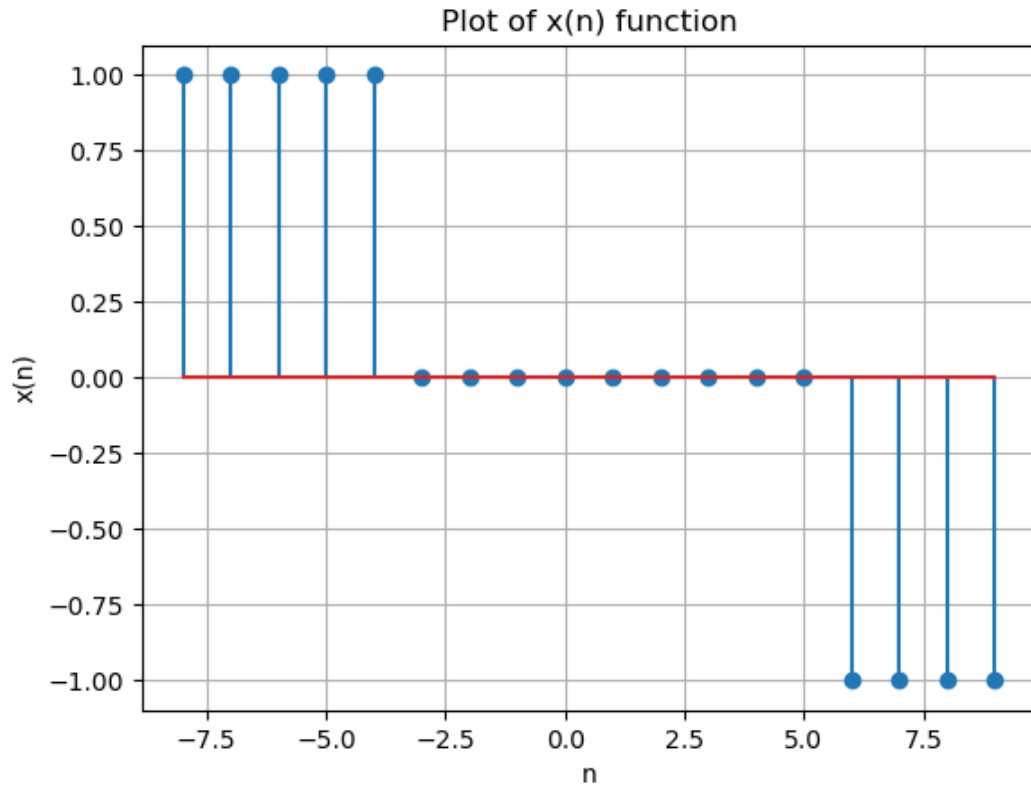
Figure 1: Plot of function $x(n)$ taken from python3

5.2 Two sequences $x_1[n]$ and $x_2[n]$ are described as follows:

$$x_1[0] = x_2[0] = 1 \tag{5.2}$$

$$x_1[1] = x_2[2] = 2 \tag{5.3}$$

$$x_1[2] = x_2[1] = 1 \tag{5.4}$$

$x_1[n] = x_2[n] = 0$ for all $n < 0$ and $n > 2$

If $x[n]$ is obtained by convoluting $x_1[n]$ with $x_2[n]$, which of the following equa-

60

tions is/are TRUE?

(A) $x[2] = x[3]$

(B) $x[1] = 2$

(C) $x[4] = 3$

(D) $x[2] = 5$

Solution:

5.3 A series (S) is given as S=1+3+5+7+9+..... The sum of the first 50 terms of S is

_____   (GATE 2023 BT 32) **Solution:**

5.4 For the signals x(t) and y(t) shown in the figure, $z\left(t\right) = x\left(t\right) * y\left(t\right)$ is maximum at $t = T_1$. Then T_1 in seconds is .... (Round off to the nearest integer) **Solution:**

5.5 A series of natural numbers F_1, F_2, F_3, F_4, F_5, F_6, F_7,....obeys $F_{n+1} = F_n + F_{n-1}$ for all integers $n \geq 2$. If $F_6 = 37$, and $F_7 = 60$, then what is F_1?

[GATE CS 2023]

Solution:

5.6 The Lucas sequence L_n is defined by the recurrence relation:

$$L_n = L_{n-1} + L_{n-2}, for n \geq 3$$

with $L_1{=}1$ and $L_2{=}3$

Which one of the option given is TRUE?

(a) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

(b) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{3}\right)^n$

(c) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{3}\right)^n$

(d) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$

(GATE 2023 CS 15)

Solution:

Chapter 6

Sampling

6.1 An 8 bit ADC converts analog voltage in the range of 0 to $+5\,V$ to the corresponding digital code as per the conversion characteristics shown in figure. For $V_{in} = 1.9922\,V$, which of the following digital output, given in hex, is true?

(a) $64H$

(b) $65H$

(c) $66H$

(d) $67H$

(GATE EE 40)

Solution:
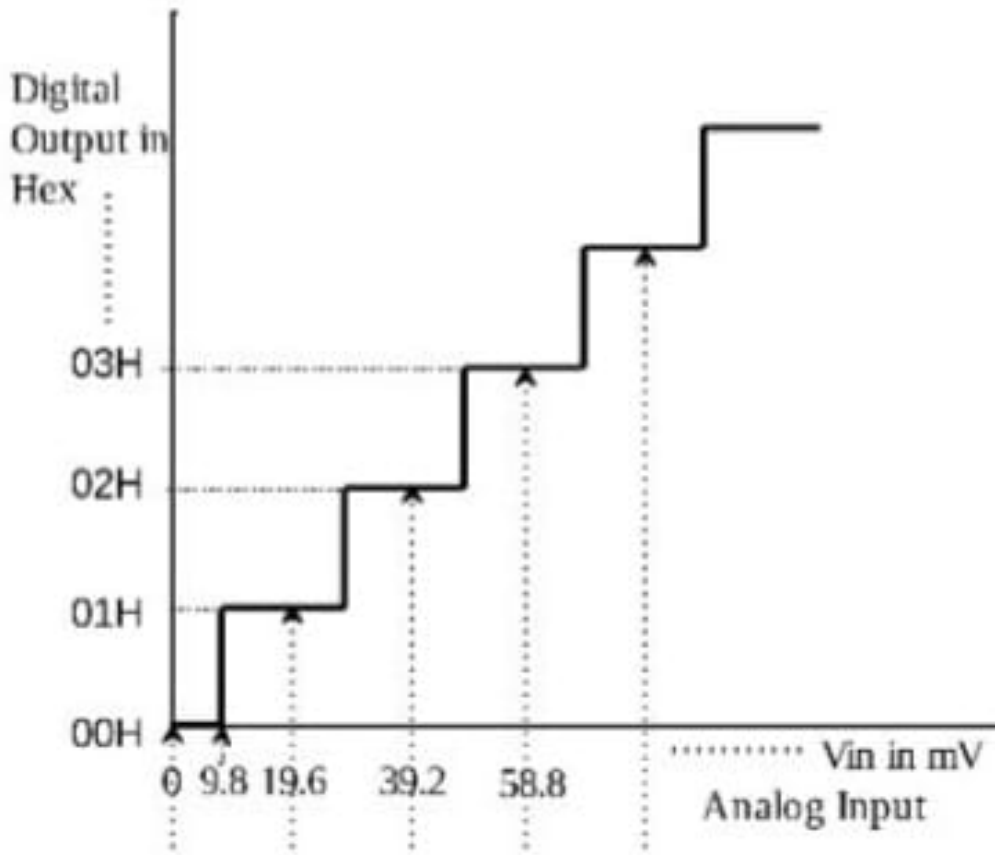
Figure 6.1:

Calculating the step-size:

$$\Delta V_{in} = \frac{V_{max} - V_{min}}{2^n - 1} \tag{6.1}$$

$$= \frac{5 - 0}{2^8 - 1} \tag{6.2}$$

$$= \frac{5}{255} \tag{6.3}$$

$$\implies V_{out} = \frac{V_{in}}{\Delta V_{in}} \tag{6.4}$$

$$= \frac{1.9922 \times 255}{5} \tag{6.5}$$

$$= 101.59 \tag{6.6}$$

$$\approx 102_{10} \tag{6.7}$$

Symbol	Value	Description
n	8	Number of bits of ADC
V_{min}	$0V$	Minimum Analog Voltage
V_{max}	$5V$	Maximum Analog Voltage
V_{in}	$1.9922V$	Input Voltage
V_{out}		Output Voltage

Table 6.1: Given Parameters

$\therefore$ correct answer is option (c).
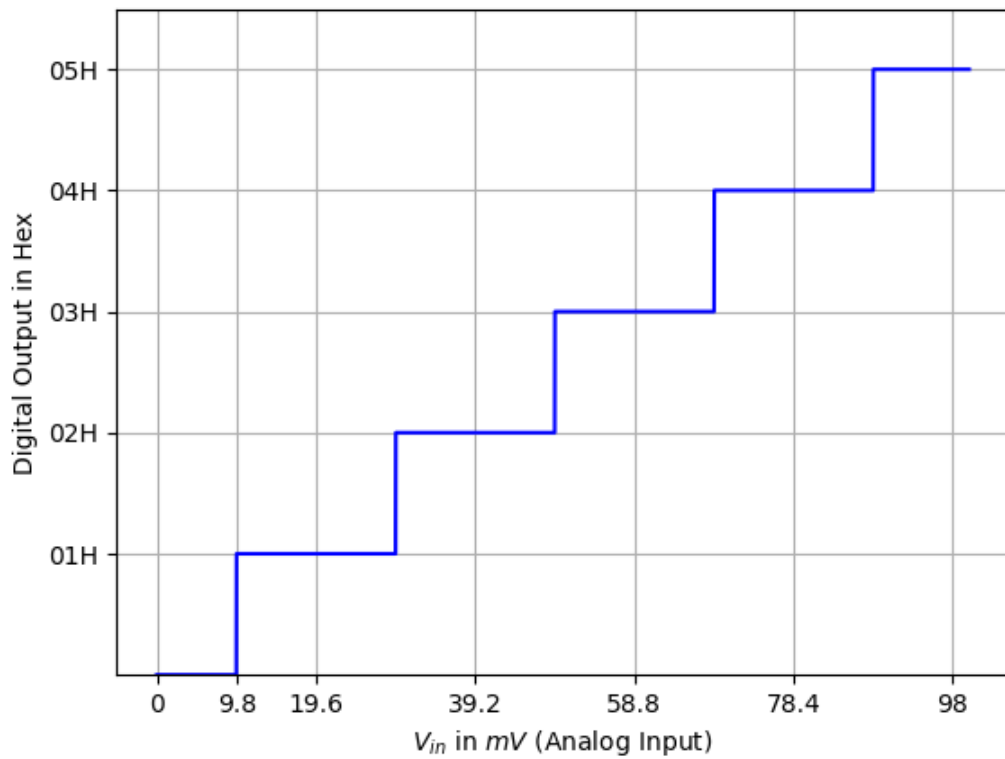
Figure 6.2:

6.2 Let x1(t) and x2(t) be two band-limited signals having bandwidth B $= 4\pi \times 10^3$ rad/s each. In the figure below, the Nyquist sampling frequency, in rad/s, required to sample y(t), is

$x_1(t)$ ── ⊗ ──
         cos($4\pi \times 10^3 t$)
                    ⊕ ─ y(t)
$x_2(t)$ ── ⊗ ──
         cos($12\pi \times 10^3 t$)
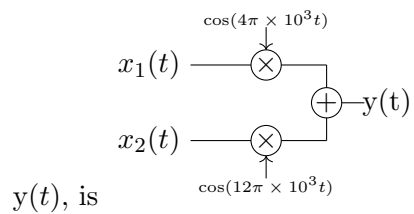
(a) $20\pi \times 10^3$

(b) $40\pi \times 10^3$

(c) $8\pi \times 10^3$

(d) $32\pi \times 10^3$

Solution:

6.3 An 8 bit successive approximation Analog to Digital Converter (ADC) has a clock frequency of 1 MHz. Assume that the start conversion and end conversion signals occupy one clock cycle each. Among the following options, what is the maximum frequency that this ADC can sample without aliasing?

    a) 0.9 kHz

    b) 9.9 kHz

    c) 49.9 kHz

    d) 99.9 kHz

(GATE BM 2023)

Solution:

6.4 The period of the discrete-time signal x[n] described by the equation below is N = (Round off to the nearest integer).

$$x[n] = 1 + 3\sin\left(\frac{15\pi}{8}n + \frac{3\pi}{4}\right) - 5\sin\left(\frac{\pi}{3}n - \frac{\pi}{4}\right)$$

(GATE 2023 EE) **Solution:**

Chapter 7

Contour Integration

7.1 The value of the contour integral, $\oint_C \frac{z+2}{z^2+2z+2} \, dz$, where the contour C is $\{z : |z + 1 - \frac{3}{2}i| = 1\}$, taken in the counter clockwise direction, is

(A) $-\pi(1 + j)$

(B) $\pi(1 + j)$

(C) $\pi(1 - j)$

(D) $-\pi(1 - j)$

(GATE EC 2023)

Solution:

7.2 The function $f(Z) = \frac{1}{Z-1}$ of a complex variable Z is integrated on a closed contour in an anti-clockwise direction. For which of the following contours, does this integral have a non-zero value?

$(A)|z - 2| = 0.01$

$(B)|z - 1| = 0.1$

$(C)|z - 3| = 5$

$(D)|z| = 2$

(GATE 2023 BM)

Solution:

7.3 Consider the contour integral $\oint \frac{dz}{z^4+z^3-2z^2}$, along the curve $|z| = 3$ oriented in the counterclockwise direction. If $\mathrm{Res}[f, z_0]$ denotes the residue of $f(z)$ at the point z_0, then which of the following are TRUE?

- (A) $\mathrm{Res}[f, 0] = -\frac{1}{4}$
- (B) $\mathrm{Res}[f, 1] = \frac{1}{3}$
- (C) $\mathrm{Res}[f, -2] = -\frac{1}{12}$
- (D) $\mathrm{Res}[f, 2] = -1$

Solution:

Chapter 8

Laplace Transform

8.1 The number of zeroes of the polynomial $P(s) = s^3 + 2s^2 + 5s + 80$ in the right side of the plane? (GATE IN 2023)

Solution: The table below shows the Routh array of the nth- order characteristic polynomial :

$$a_0 s^n + a_1 s^{n-1}...... + a_{n-1} s^1 + a_n s^0 \tag{8.1}$$

s^n	a_0	a_2	a_4	...
s^{n-1}	a_1	a_3	a_5	...
s^{n-2}	$b_1 = \dfrac{a_1 a_2 - a_3 a_0}{a_1}$	$b_2 = \dfrac{a_1 a_4 - a_5 a_0}{a_1}$	...	..
s^{n-3}	$c_1 = \dfrac{b_1 a_3 - b_2 a_1}{b_1}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$		
s^1	$\vdots$	$\vdots$		
s^0	a_n			

Table 8.1: Routh Array

Characteristic Equation:

$$s^3 + 2s^2 + 5s + 80 = 0 \qquad (8.2)$$

From Table 8.1:

s^3	1	5
s^2	2	80
s^1	$\dfrac{2 \times 5 - 80 \times 1}{2} = -35$	
s^0	$\dfrac{-35 \times 80}{-35} = 80$	

Table 8.2:

From Table 8.2:

Since there are 2 sign changes in the first column of the Routh tabulation. So, the number of zeros in the right half of the s-plane will be 2.

Figure 8.1:

8.2 The circuit shown in the figure is initially in the steady state with the switch K in open condition and $\overline{K}$ in closed condition. The switch K is closed and $\overline{K}$ is opened simultaneously at the instant $t = t_1$, where $t_1 > 0$. The minimum value of t_1 in milliseconds such that there is no transient in the voltage across the $100\ \mu F$ capacitor, is _____ (Round off to 2 decimal places) (GATE EE 2023)
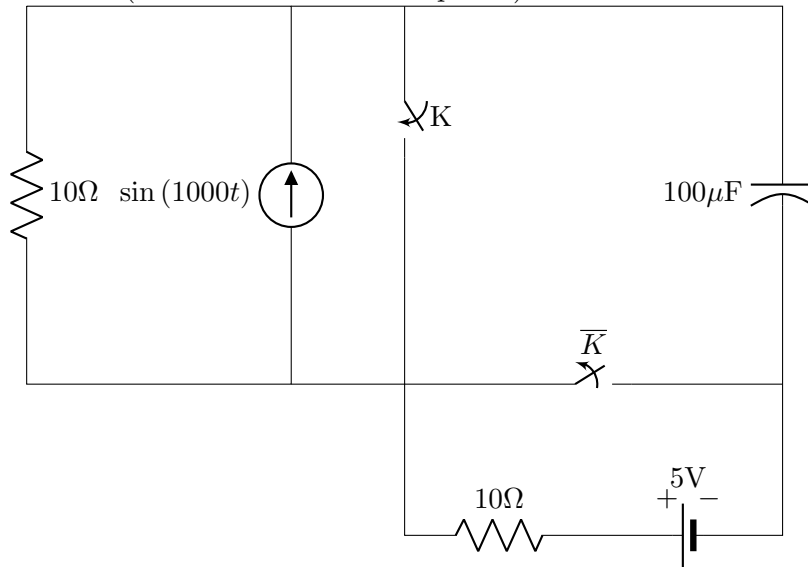
Solution:

(a) Switch K is open and $\overline{K}$ is closed.

Using Current divider rule,

$$I_1\,(j\omega) = \frac{10}{10 + \frac{1}{j\omega C}} \tag{8.3}$$

$$V_1\,(j\omega) = \frac{10}{1 + 10j\omega C} \tag{8.4}$$

$$|V_1\,(j\omega)| = 5\sqrt{2} \tag{8.5}$$
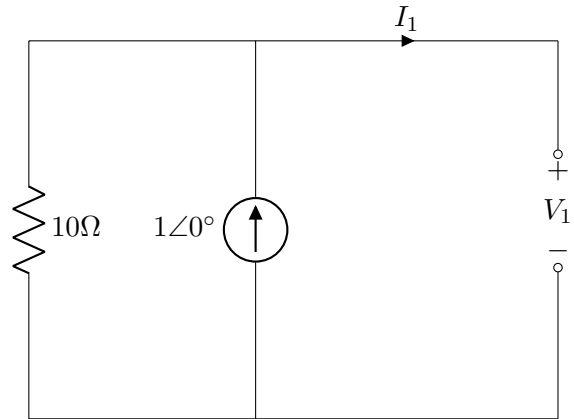
Figure 8.2: K is open and $\overline{K}$ is closed

From Table 8.3

$$V_1(t) = 5\sqrt{2}\sin\left(\omega t - \frac{\pi}{4}\right) \tag{8.6}$$

(b) Switch K is closed and $\overline{K}$ is open.
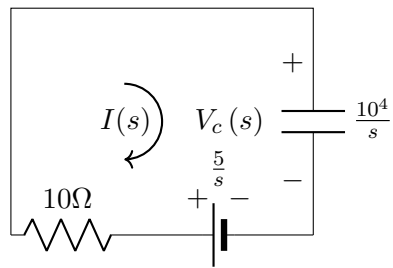
Figure 8.3: K is closed and $\overline{K}$ is open

The capacitor is charged. Thus, acts as a voltage source.

From eq(8.6) and Table 8.4

$$V_1(s) = \frac{5000 - 5s}{s^2 + 10^6} \tag{8.7}$$

$$I(s) = \frac{\frac{5}{s} - V_1(s)}{10 + \frac{10^4}{s}} \tag{8.8}$$

$$V_c(s) = \frac{5}{s} - 10\left(\frac{5 - V_1(s)}{1 + 10^{-3}s}\right) \tag{8.9}$$

For transient analysis,

$$\frac{5 - V_1(s)}{1 + 10^{-3}s} = 0 \tag{8.10}$$

$$\implies V_1(s) = 5 \tag{8.11}$$

$$\frac{10^7}{(s^2 + 10^6)(s + 10^3)} = \frac{5}{s} \tag{8.12}$$

$$\frac{5}{s + 10^3} + \frac{10^3 - s}{s^2 + 10^6} = \frac{5}{s} \tag{8.13}$$

$$\frac{-s}{s^2 + 10^6} + \frac{10^3}{s^2 + 10^6} + \frac{1}{s + 10^3} = \frac{1}{s} \tag{8.14}$$

From Table 8.4

$$-\cos(1000t_1) + \sin(1000t_1) + e^{-10^3 t_1} = 1 \tag{8.15}$$

$$\implies t_1 \approx 1.57\text{msec} \tag{8.16}$$
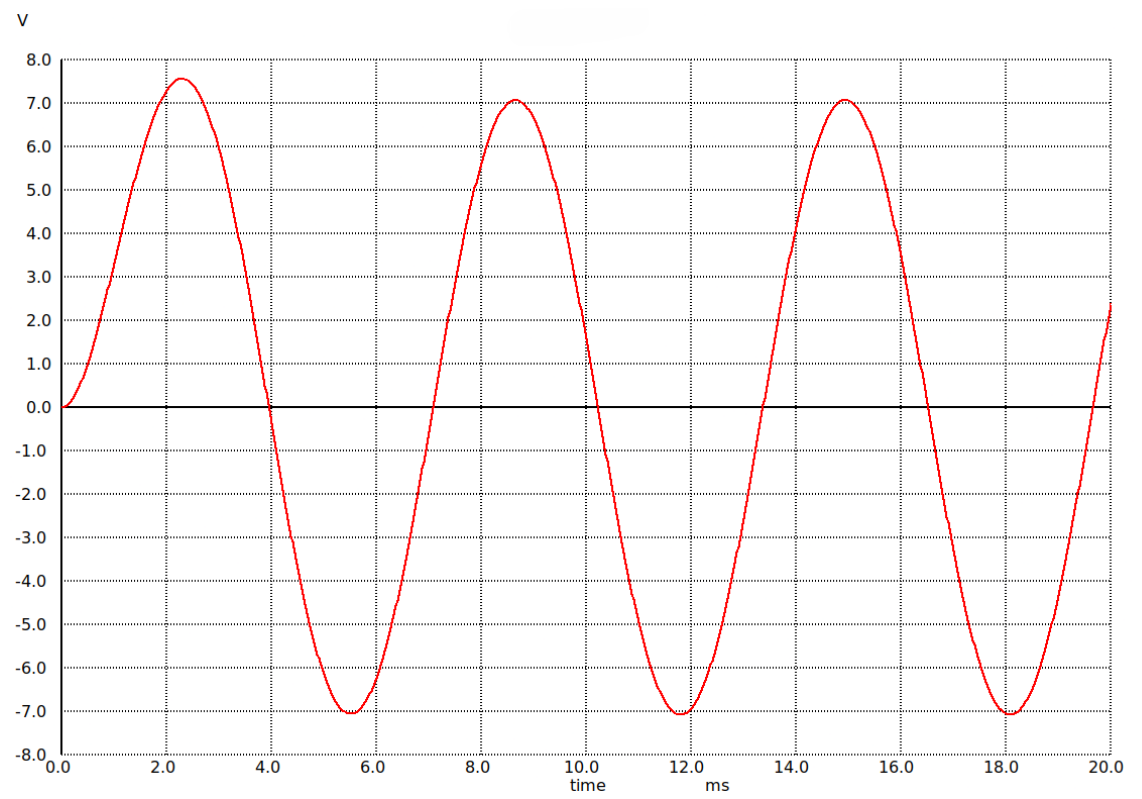
Figure 8.4: plot of V_1 vs time
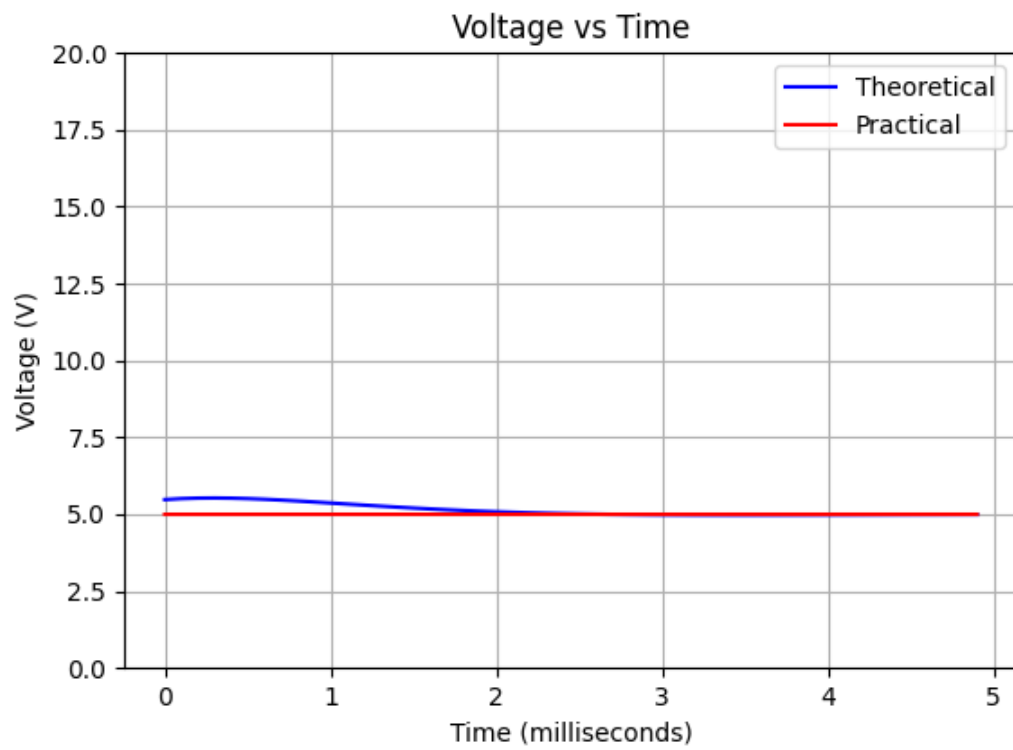
Figure 8.5: plot of V_c vs time

Parameter	Description	Remarks
ω	frequency of sine-wave	1000 rad s^{-1}
$V_1(t)$	Voltage across capacitor	$|V_1(j\omega)|\sin(\omega t - \angle V_1(j\omega))$
$\angle V_1(j\omega)$	phase of $V_1(j\omega)$	$\frac{-\pi}{4}$
C	Capacitance	100μF

Table 8.3: Parameters

S Domain	Time Domain
$\frac{1}{s}$	$u(t)$
$\frac{-s}{a^2+s^2}$	$-\cos(at)$
$\frac{a}{a^2+s^2}$	$\sin(at)$
$\frac{1}{s+a}$	e^{-at}

Table 8.4: Laplace transforms

8.3 $y = e^{mx} + e^{-mx}$ is the solution of which differential equation?

    1. $\frac{dy}{dx} - my = 0$

    2. $\frac{dy}{dx} + my = 0$

    3. $\frac{d^2y}{dx^2} + m^2 y = 0$

    4. $\frac{d^2y}{dx^2} - m^2 y = 0$

(GATE AG 2023) **Solution:**

8.4 A cascade control strategy is shown in the figure below. The transfer function between the output (y) and the secondary disturbance (d_2) is defined as

$$G_{d2}(s) = \frac{y(s)}{d_2(s)}$$

. Which one of the following is the CORRECT expression for the transfer function $G_{d2}(s)$?

Figure 8.6:

A. $\frac{1}{(11s+21)(0.1s+1)}$

B. $\frac{1}{(s+1)(0.1s+1)}$

C. $\frac{(s+1)}{(s+2)(0.1s+1)}$

D. $\frac{(s+1)}{(s+1)(0.1s+1)}$

(GATE CH 2023) **Solution:**

8.5 In the differential equation $\frac{dy}{dx} + \alpha xy = 0$, α is a positive constant. If $y = 1.0$ at $x = 0.0$, and $y = 0.8$ at $x = 1.0$, the value of α is (rounded off to three decimal places). (GATE CE 2023) **Solution:**

8.6 The switch S_1 was closed and S_2 was open for a long time. At t=0,switch S_1 is opened and S_2 is closed,simultaneously. The value of $i_c(0^+)$, in amperes, is .   (GATE EC 2023)
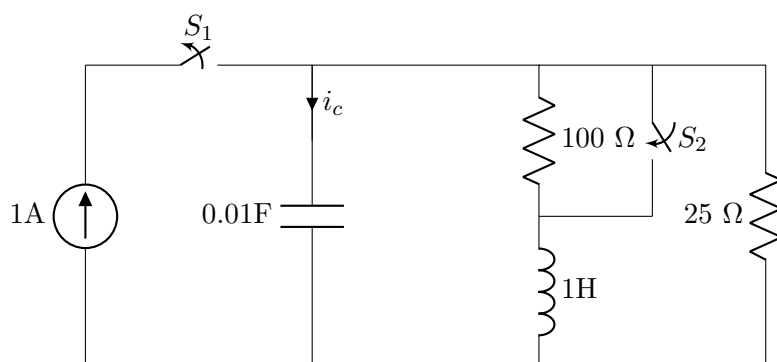
8.7 The continuous time signal $x(t)$ is described by:

$$x(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{elsewhere} \end{cases} \tag{8.17}$$

If $y(t)$ represents $x(t)$ convolved with itself, which of the following options is/are TRUE?

A $y(t) = 0$ for all $t < 0$

B $y(t) = 0$ for all $t > 1$

C $y(t) = 0$ for all $t > 3$

D $\int_{0.1}^{0.75} \frac{dy(t)}{dt} \, dt \neq 0$

Solution:

8.8 The Z-transform of a discrete signal $x(n)$ is

$$X(z) = \frac{4z}{\left(z - \frac{1}{5}\right)\left(z - \frac{2}{3}\right)(z - 3)} \quad \text{with ROC}= R \qquad (8.18)$$

Which one of the following statements is TRUE?

(a) Discrete time Fourier transform of $x[n]$ converges if R is $|z| > 3$

(b) Discrete time Fourier transform of $x[n]$ converges if R is $\frac{2}{3} < |z| < 3$

(c) Discrete time Fourier transform of $x[n]$ converges if R is such that $x[n]$ is a left-sided sequence.

(d) Discrete time Fourier transform of $x[n]$ converges if R is such that $x[n]$ is a right-sided sequence.

GATE EE 2023 **Solution:**

8.9 The phase margin of the transfer function $G(s) = \frac{2(1-s)}{(1+s)^2}$ is _____ degrees. (rounded off to the nearest integer). (GATE IN 2023)

Solution:

Parameters	Description
ω_c	crossover frequency
$\angle G(j\omega)$	phase angle of the transfer function
PM	$\angle G(j\omega_c) + 180°$; Phase Margin

Table 8.5: Parameters

Considering $s = j\omega$,

$$G(j\omega) = \frac{2(1 - j\omega)}{(1 + j\omega)^2} \tag{8.19}$$

$$= \frac{2(1 - j\omega)^3}{|1 + j\omega|^4} \tag{8.20}$$

$$= \frac{2}{(1 + \omega^2)^2}(1 - j\omega)^3 \tag{8.21}$$

$$= \frac{2}{\sqrt{1 + \omega^2}}(e^{-j\omega})^3 \tag{8.22}$$

$$\implies |G(j\omega)| = \frac{2}{\sqrt{1 + \omega^2}} \tag{8.23}$$

$$\implies \angle G(j\omega) = 3\tan^{-1}(-\omega) \tag{8.24}$$

At $\omega = \omega_c$, $Gain = 0$

$$\implies |G(j\omega_c)| = 1 \tag{8.25}$$

$$\frac{2}{\sqrt{1 + \omega_c^2}} = 1 \tag{8.26}$$

$$\implies \omega_c = \sqrt{3} \tag{8.27}$$

$$\angle G(j\omega_c) = 3\tan^{-1}(-\sqrt{3}) \tag{8.28}$$

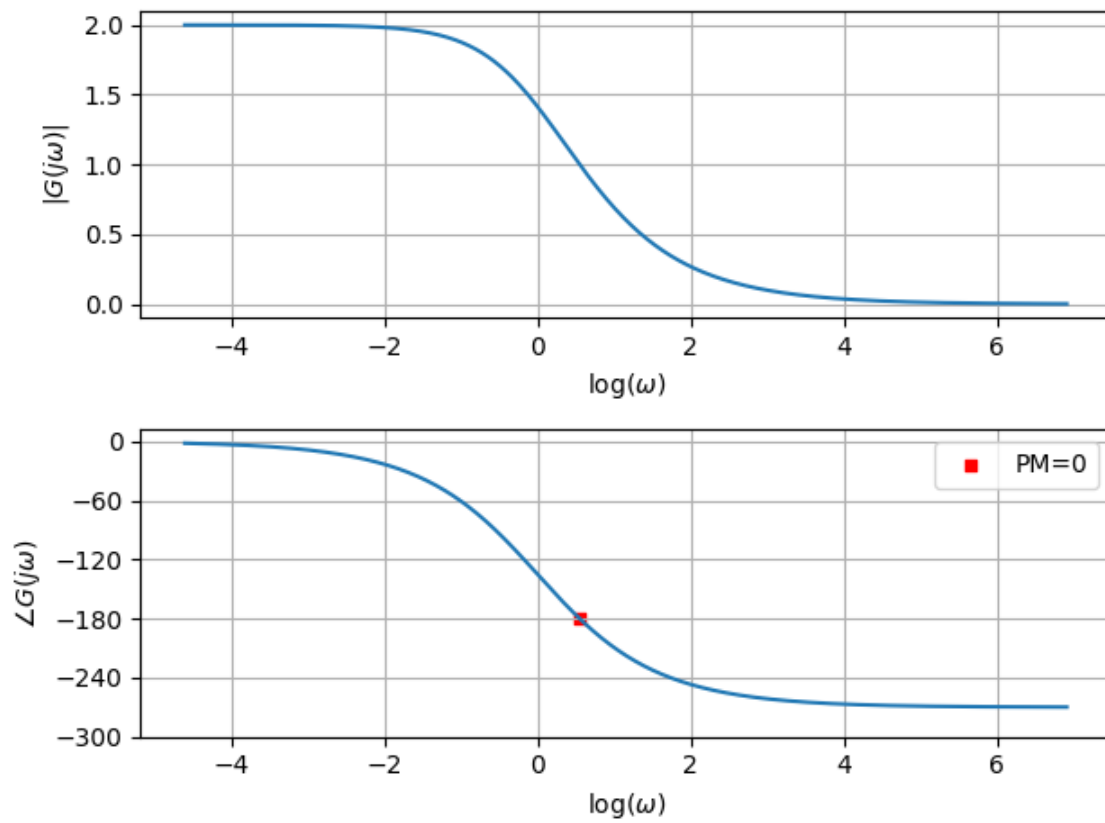$$= -180° \tag{8.29}$$

From Table 8.5,

$$PM = 0° \qquad (8.30)$$

Figure 8.7: Bode Plot of Transfer Function $G(s)$

8.10 Consider the second-order linear differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0, \ x \geq 1$$

with the initial conditions

$$y(x = 1) = 6, \quad \frac{dy}{dx}\Big|_{x=1} = 2.$$

Then the value of y at $x = 2$ is _____.

GATE ME 2023

Solution:

8.11 The transfer function of a measuring instrument is

$$G_m(s) = \frac{1.05}{2s + 1} exp(-s)$$

At time $t = 0$, a step change of $+1$ unit is introduced in the input of this instrument.The time taken by the instrument to show an increase of 1 unit in its output is(rounded off to two decimal places).

(GATE CH 2023) **Solution:**

8.12 The laplace transform of $x_1(t) = e^{-t}u(t)$ is $X_1(s)$, where $u(t)$ is the unit step function. The laplace transform of $x_2(t) = e^t u(-t)$ is $X_2(s)$. Which one of the following statements is TRUE?

   (a) The region of convergence of $X_1(s)$ is $Re(s) \geq 0$

   (b) The region of convergence of $X_2(s)$ is confined to the left half-plane of s.

   (c) The region of convergence of $X_1(s)$ is confined to the right half-plane of s.

   (d) the imaginary axis in the s-plane is included in both the region of convergence of $X_1(s)$ and the region of convergence of $X_2(s)$.

(GATE BM 2023)

Solution:

Symbols	Description
$X_1(s)$	Laplace transform of $x_1(t)$
$X_2(s)$	Laplace transform of $x_2(t)$
$u(t)$	Unit step function

Table 8.6: Parameters, Descriptions

(a) Laplace transform of $x_1(t)$ is given by :

$$X_1(s) = \int_{-\infty}^{\infty} e^{-t} e^{-st} u(t) \, dt \qquad (8.31)$$

Let $s = \sigma + j\omega$:

$$X_1(s) = \int_{0}^{\infty} e^{-t(\sigma+1)} e^{-tj\beta} \, dt \qquad (8.32)$$

$$= \left[\frac{-e^{-t(\sigma+1)} e^{-tj\beta}}{(\sigma+1) + j\beta} \right]_{0}^{\infty} \qquad (8.33)$$

For $X_1(s)$ to be convergent, $\left| -e^{-t(\sigma+1)} e^{-tj\beta} \right|$ must converge $\forall t \epsilon\, (0, \infty)$, so:

$$\left| e^{-tj\beta} \right| = |1| \,, \forall \beta \epsilon \mathbb{R} \implies Im\,(s)\, \epsilon \mathbb{R} \qquad (8.34)$$

$$\sigma + 1 > 0 \implies \Re\,(s) > -1 \qquad (8.35)$$

Putting the limits :

$$X_1(s) = \frac{1}{s+1}, \Re\,(s) > -1 \qquad (8.36)$$

(b) Laplace transform of $x_2(t)$ is given by :

$$X_2(s) = \int_{-\infty}^{\infty} e^{t} e^{-st} u(-t) \, dt \qquad (8.37)$$

Let $s = \sigma + j\omega$:

$$= \int_{-\infty}^{0} e^{t(1-\sigma)} e^{-tj\beta} \, dt \tag{8.38}$$

$$= \left[\frac{e^{t(1-\sigma)} e^{-tj\beta}}{(1-\sigma) - j\beta} \right]_{-\infty}^{0} \tag{8.39}$$

For $X_2(s)$ to be convergent, $\left| e^{t(1-\sigma)} e^{-tj\beta} \right|$ must converge $\forall t \epsilon (-\infty, 0)$, so:

$$\left| e^{-tj\beta} \right| = |1| , \forall \beta \epsilon \mathbb{R} \implies Im(s) \, \epsilon \mathbb{R} \tag{8.40}$$

$$1 - \sigma > 0 \implies \Re(s) < 1 \tag{8.41}$$

Putting the limits

$$X_2(s) = \frac{1}{1-s}, \Re(s) < 1 \tag{8.42}$$
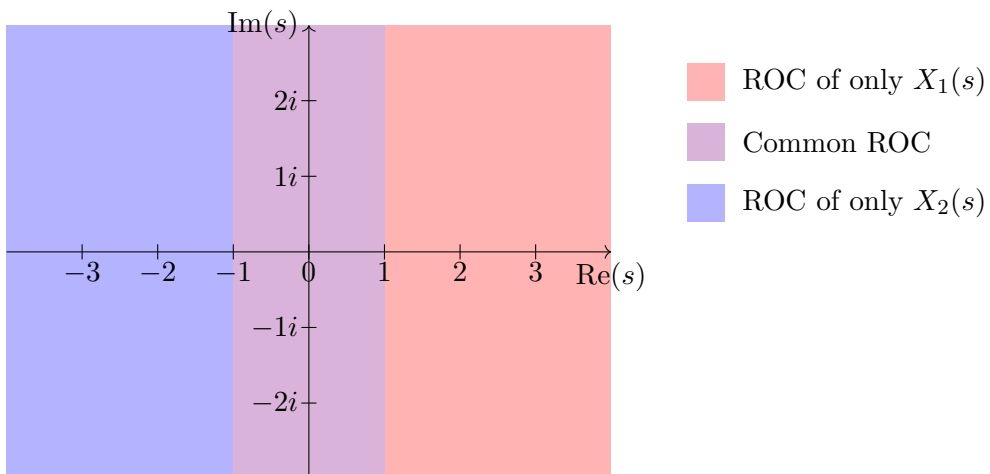
Figure 8.8: Representation of ROCs of $X_1(s)$ and $X_2(s)$

Based on the overlap of regions of convergence of $X_1(s)$ and $X_2(s)$ from 8.8 , we can conclude that option 4) is correct .

8.13 Given that $\frac{dy}{dx} = 2x + y$ and $y = 1$,when $x = 0$ Using Runge-Kutta fourth order method,the value of y at $x = 0.2$ is   (GATE 2023 AG 50)

Solution:

8.14 The magnitude and phase plots shown in the figure match with the transfer- function

Bode Diagram
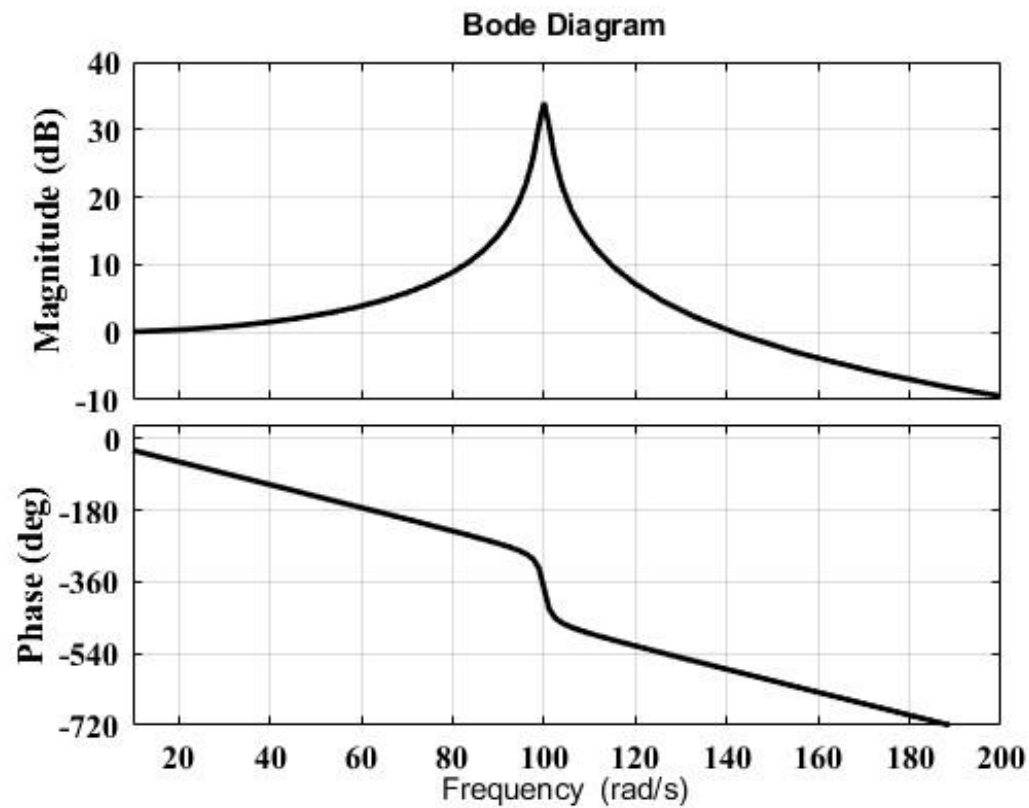
(a) $\frac{10000}{s^2+2s+10000}$

(b) $\frac{10000}{s^2+2s+10000}e^{-0.05s}$

(c) $\frac{10000}{s^2+2s+10000}e^{-0.5\times10^{-12}s}$

(d) $\frac{100}{s^2+2s+100}$

(GATE IN 2023) **Solution:**

8.15 The Laplace transform of the continuous-time signal $x(t) = e^{-3t}u(t-5)$ is _____, where $u(t)$ denotes the continuous-time unit step signal.

A) $\frac{e^{-5s}}{s+3}$, Real$\{s\} > -3$

B) $\frac{e^{-5(s-3)}}{s-3}$, Real$\{s\} > 3$

C) $\frac{e^{-5(s+3)}}{s+3}$, Real$\{s\} > -3$

D) $\frac{e^{-5(s-3)}}{s+3}$, Real$\{s\} > -3$

Solution:

Parameter	Description	Value
$x(t)$	Given Function	$x(t) = e^{-3t}u(t)$
$X(s)$	Laplace Transform of $x(t)$	$\frac{e^{-5(s+3)}}{s+3}$

Table 1: Table of parameters

$$e^{-3t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+3} \quad \Re(s) > -3 \qquad (8.43)$$

Using time shifting,

$$e^{-3(t-5)}u(t-5) \xleftrightarrow{\mathcal{L}} \frac{e^{-5s}}{s+3} \tag{8.44}$$

$$e^{-15}e^{-3(t-5)}u(t-5) \xleftrightarrow{\mathcal{L}} e^{-15}\frac{e^{-5s}}{s+3} \tag{8.45}$$

$$e^{-3t}u(t-5) \xleftrightarrow{\mathcal{L}} \frac{e^{-5(s+3)}}{s+3} \tag{8.46}$$

$$\therefore x(t) \xleftrightarrow{\mathcal{L}} \frac{e^{-5(s+3)}}{s+3} \quad \Re(s) > -3 \tag{8.47}$$
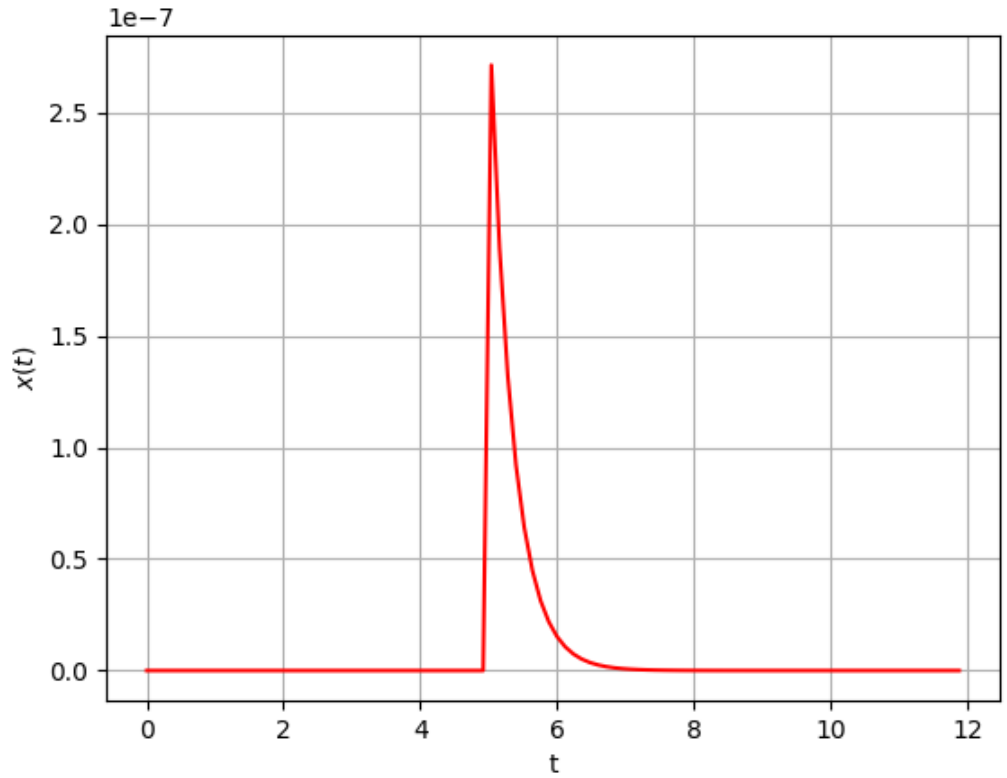
Figure 3: Plot of $x(t)$ vs t. See Table 1

103

8.16 The solution $x(t), t \geq 0$, to the differential equation $\ddot{x} = -k\dot{x}, k > 0$ with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$ is

Solution:

8.17 Consider the differential equation

$$x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0 \quad \text{for } x \geq 1$$

with initial conditions $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 1$. The value of y at $x = 2$ is ?

8.18 The time-dependent growth of a bacterial population is governed by the equation

$$\frac{dx}{dt} = x \left(1 - \frac{x}{200} \right) \tag{8.48}$$

where x is the population size at time t. The initial population size is $x_0 = 100$ at $x = 0$. As $t \to \infty$, the population size of bacteria asymptotically approaches

(a) 150

(b) 200

(c) 300

(d) 500

(GATE BM 2023) **Solution:**

8.19 Consider a lead compensator of the form

$$K(s) = \frac{1+\frac{s}{a}}{1+\frac{s}{\beta a}}, \quad \beta > 1, \quad a > 0$$

The frequency at which this compensator produces maximum phase lead is $4\,\mathrm{rad/s}$. At this frequency, the gain amplification provided by the controller, assuming an asymptotic Bode-magnitude plot of $K(s)$, is $6\,\mathrm{dB}$. The values of a and β, respectively, are

(a) $1, 16$

(b) $2, 4$

(c) $3, 5$

(d) $2.66, 2.25$

(GATE EC 2023) **Solution:**

8.20 Second order ordinary differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$ has values $y = 2$ and $\frac{dy}{dx} = 1$ at $x = 0$. The value of y at $x = 1$ is? $(round\ off\ to\ three\ decimal\ places)$

[GATE-ES 2023]

Solution:

8.21 A continuous-time system that is initially at rest is described by,

$$\frac{dy(t)}{dt} + 3y(t) = 2x(t)$$

where $x(t)$ is the input voltage and $y(t)$ is the output voltage.

The impulse response of the system is? (GATE 2023 EE) **Solution:**

8.22 Consider the equation $\frac{dy}{dx} + ay = \sin \omega x$, where a and ω are constants. Given $y = 1$ at $x = 0$, correct all the correct statement(s) from the following as $x \to \infty$.

(A) $y \to 0$ if $a \neq 0$

(B) $y \to 1$ if $a = 0$

(C) $y \to A exp(|a|x)$ if $a < 0$; A is constant

(D) $y \to B \sin(\omega x + C)$ if $a > 0$; B and C are constants

(GATE AE 2023) **Solution:**

8.23 **Question** : The position $x(t)$ of a particle, at constant ω, is described by the equation

$$\frac{d^2x}{dt^2} = -\omega^2 x. \tag{8.49}$$

The initial conditions are $x(t = 0) = 1$ and $\frac{dx}{dt}\Big|_{t=0} = 0$. The position of the particle at $t = \frac{3\pi}{\omega}$ is _____ (in integer). (GATE CH 2023) **Solution:**

8.24 **Question**:The initial value problem $\frac{dy}{dt} + 2y = 0, y(0) = 1$ is solved numerically using the forward Euler's method with a constant and positive time step of δ.

Let y_n represent the numerical solution obtained after n steps. The condition $|y_{n+1}| \leq |y_n|$ is satisfied if and only if δ does not exceed

(GATE ME 2023) **Solution:**

8.25 In the context of signals and systems, determine the phase cross-over frequency of the open-loop transfer function

$$G(s) = \frac{k \cdot s \cdot (1 + sT_1) \cdot (1 + sT_2)}{s}$$

with positive constants k, T_1, T_2 are positive constants.The phase crossover fequency,in rad/s,is

(a) $\frac{1}{\sqrt{T_1 T_2}}$

(b) $\frac{1}{T_1 T_2}$

(c) $\frac{1}{T_1 \sqrt{T_2}}$

(d) $\frac{1}{\sqrt{T_2 T_1}}$

(GATE EC 2023) **Solution:**

Chapter 9

Fourier transform

9.1 Let a frequency modulated (FM) signal : $x(t) = A\cos(\omega_c t + k_f \int_{-\infty}^{t} m(\lambda)d\lambda)$, where $m(t)$is a message signal of bandwidth W. It is passed through a non-linear system with output $y(t) = 2x(t) + 5(x(t))^2$. Let $B_T$denote the FM bandwidth. The minimum value of ω_c required to recover $x(t)$ from $y(t)$ is:

(A) $B_T + W$

(B) $\dfrac{3}{2}B_T$

(C) $2B_T + W$

(D) $\dfrac{5}{2}B_T$

Solution:

9.2 Let an input $x[n]$ having discrete-time Fourier transform $X(e^{j\Omega}) = 1 - e^{-j\Omega} + 2e^{-3j\Omega}$ be passed through an LTI system. The frequency response of the LTI system is $H(e^{j\Omega}) = 1 - \frac{1}{2}e^{-2j\Omega}$ The output $y[n]$ of the system is

(GATE EC 2023) **Solution:**

9.3 The Fourier transform $X(\omega)$ of $x(t) = e^{-t^2}$ is

Note: $\int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}$

A) $\sqrt{\pi} e^{\frac{\omega^2}{2}}$

B) $\frac{e^{\frac{-\omega^2}{4}}}{2\sqrt{\pi}}$

C) $\sqrt{\pi} e^{\frac{-\omega^2}{4}}$

D) $\sqrt{\pi} e^{\frac{-\omega^2}{2}}$

Gate 2023 EC Question 28

9.4 Let $x(t) = 10 \cos(10.5\omega t)$ be passed through an LTI system with impulse response $h(t) = \pi \left(\frac{\sin(\omega t)}{\pi t} \right)^2 \cos(10\omega t)$. The output of the system is:

(GATE EC 2023) **Solution:**

9.5 Q27) Let m(t) be a strictly band-limited signal with bandwidth B and energy E. Assuming $\omega_0 = 10B$, the energy in the signal $m(t)\cos(\omega_0 t)$

(A) $\frac{E}{4}$

(B) $\frac{E}{2}$

(C) E

(D) $2E$ (GATE EC 2023)

Solution:

9.6 The following function is defined over the integral $[-L, L]$:

$$f(x) = px^4 + qx^5$$

It is expressed as a Fourier series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \sin\left(\frac{\pi x}{L}\right) + b_n \cos\left(\frac{\pi x}{L}\right) \right\}$$

which options amongst the following are true?

(a) a_n, $n = 1, 2, .., \infty$ depend on p

(b) a_n, $n = 1, 2, .., \infty$ depend on q

(c) b_n, $n = 1, 2, .., \infty$ depend on p

(d) b_n, $n = 1, 2, .., \infty$ depend on q

(GATE 2023 CE Question 25)

Solution:

9.7 A continuous real-valued signal $x(t)$ has finite positive energy and $x(t) = 0$, $\forall\, t < 0$. From the list given below, select ALL the signals whose continuous-time Fourier transform is purely imaginary.

(a) $x(t) + x(-t)$

(b) $x(t) - x(-t)$

(c) $j(x(t) + x(-t))$

(d) $j(x(t) - x(-t))$

(GATE IN 2023)

Solution:

9.8 Let $x_1(t) = u(t + 1.5) - u(t - 1.5)$ and $x_2(t)$ is shown in the figure below. For $y(t) = x_1(t) * x_2(t)$, the $\int_{-\infty}^{\infty} y(t)\, dt$ is _____.
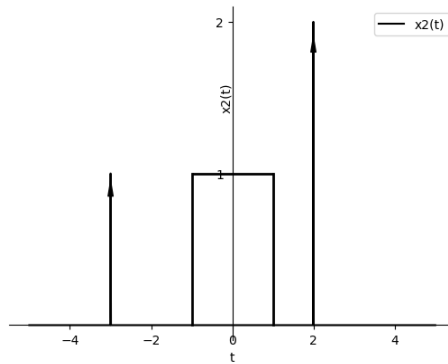

Figure 9.1: Figure

(GATE IN 2023)

Solution:

9.9 Consider a discrete-time signal with period $N = 5$. Let the discrete-time Fourier series (DTFS) representation be $x[n] = \sum\limits_{k=0}^{4} a_k e^{\frac{jk2\pi n}{5}}$ where $a_0 = 1$, $a_1 = 3j$, $a_2 = 2j$, $a_3 = -2j$, $a_4 = -3j$. The value of the sum $\sum\limits_{n=0}^{4} x[n] \sin\left(\frac{4\pi n}{5}\right)$ is

(A) -10

(B) 10

(C) -2

(D) 2

Gate 2023 EC 47 **Solution:**

9.10 A continuous time, band-limited signal $x(t)$ has its Fourier transform described by:

$$X(f) = \begin{cases} 1 - \frac{|f|}{200} & \text{if } |f| \leq 200 \\ 0 & \text{if } |f| > 200 \end{cases}$$

The signal is uniformly sampled at a sampling rate of 600 Hz. The Fourier transform of the signal is $X_s(f)$. What is the value of $\frac{X_s(600)}{X_s(500)}$?

(GATE 2023 BM) **Solution:**

9.11 The magnitude and phase plots of an LTI systems are shown in figure. Find the transfer function.
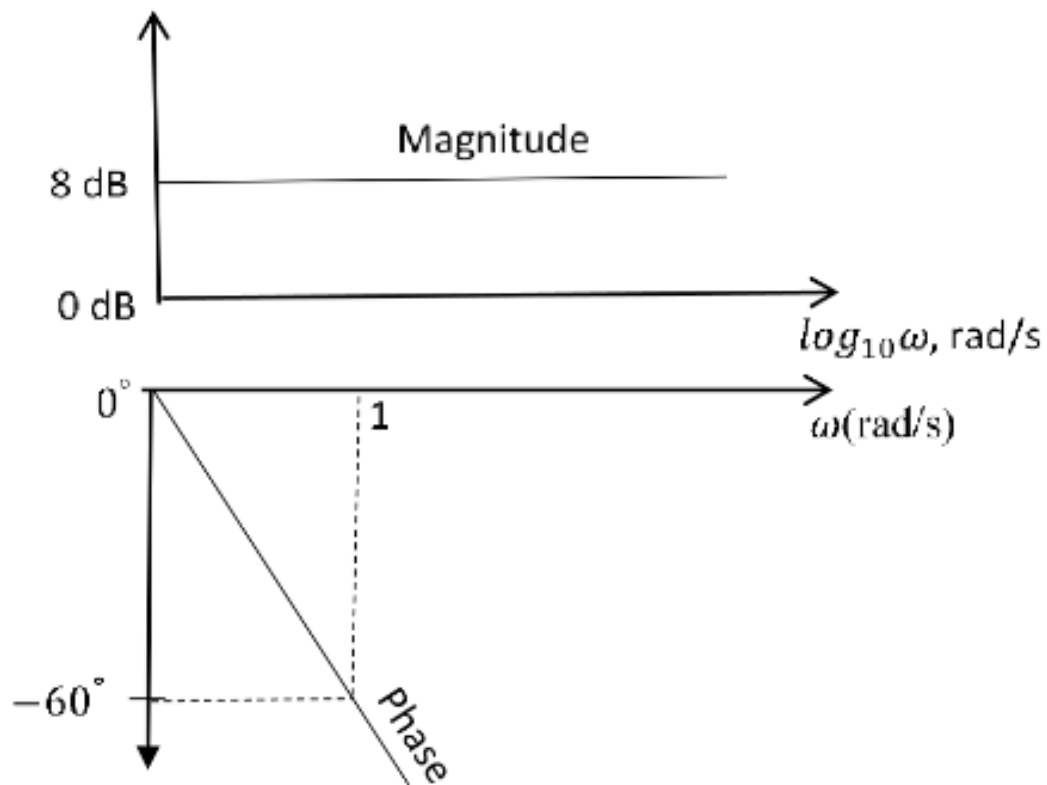

Figure 9.2:

(a) $2.511e^{-0.0032s}$

(b) $\frac{e^{-2.514s}}{s+1}$

(c) $1.04e^{-2.514s}$

(d) $2.511e^{-1.047s}$

Solution: From Fig. 9.2

$$|H(j\omega)| = 8 \tag{9.1}$$

$$\angle H(j\omega) = \frac{-\pi}{3}\omega \tag{9.2}$$

Substituting the values from Fig. 9.2, magnitude of transfer function is:

$$8 = 20\log_{10}(|H(j\omega)|) \tag{9.3}$$

$$|H(j\omega)| = 10^{0.4} = 2.511 \tag{9.4}$$

Substituting the values from Fig. 9.2, The direction of the transfer function is:

$$\frac{H(j\omega)}{|H(j\omega)|} = e^{-j\frac{\pi}{3}\omega} \tag{9.5}$$

$$H(j\omega) = 2.511e^{-j\frac{\pi}{3}\omega} \tag{9.6}$$

$$= 2.511e^{-1.047s} \tag{9.7}$$

9.12 The value of the convolution of $f(x) = 3\cos(2x)$ and $g(x) = \frac{1}{3}\sin(2x)$ where $x \in [0, 2\pi)$, at $x = \frac{\pi}{3}$, is (Rounded off to 2 decimal places)

(GATE 2023 GE)

Solution:

9.13 A system is described by the following differential equation

$$0.01\frac{d^2y(t)}{dt^2} + 0.2\frac{dy(t)}{dt} + y(t) = 6x(t)$$

where time t is in seconds. If $x(t)$ is the unit step input applied at $t = 0$ s to this system, the magnitude of the output at $t = 1$ s is _____. (Round off the answer to two decimal places.) (GATE-2023.BM)

Solution:

9.14 In the differential equation $\frac{dy}{dx} + \alpha xy = 0$, α is a positive constant. If $y = 1.0$ at $x = 0.0$, and $y = 0.8$ at $x = 1.0$, the value of α is (rounded off to three decimal places). (GATE CE 30 2023)

Solution:

Parameter	Value
x	0.0
	1.0
y	1.0
	0.8

Table 9.1: Given parameters

Let, $t = x$

$$\frac{dy}{dt} + \alpha ty = 0 \tag{9.8}$$

$$\int \frac{dy}{y} = -\int \alpha t \, dt \tag{9.9}$$

$$\ln(|y|) = -\frac{\alpha t^2}{2} + c \tag{9.10}$$

$$y(t) = e^c \cdot e^{-\frac{\alpha t^2}{2}} \tag{9.11}$$

Taking Fourier Transform:

where,

$$\frac{dy}{dt} \xleftrightarrow{\mathcal{F}} j2\pi f Y(f) \tag{9.12}$$

$$\alpha t y(t) \xleftrightarrow{\mathcal{F}} a \frac{j}{2\pi} \frac{d}{df} Y(f) \tag{9.13}$$

From equation (9.12) and (9.13):

$$\frac{4\pi^2 f}{\alpha} Y(f) + \frac{d}{df} Y(f) = 0 \tag{9.14}$$

$$Y(f) = K e^{-\frac{4\pi^2 f^2}{2\alpha}} \tag{9.15}$$

Substituting x and y values:

$$c = \ln(1) = 0 \tag{9.16}$$

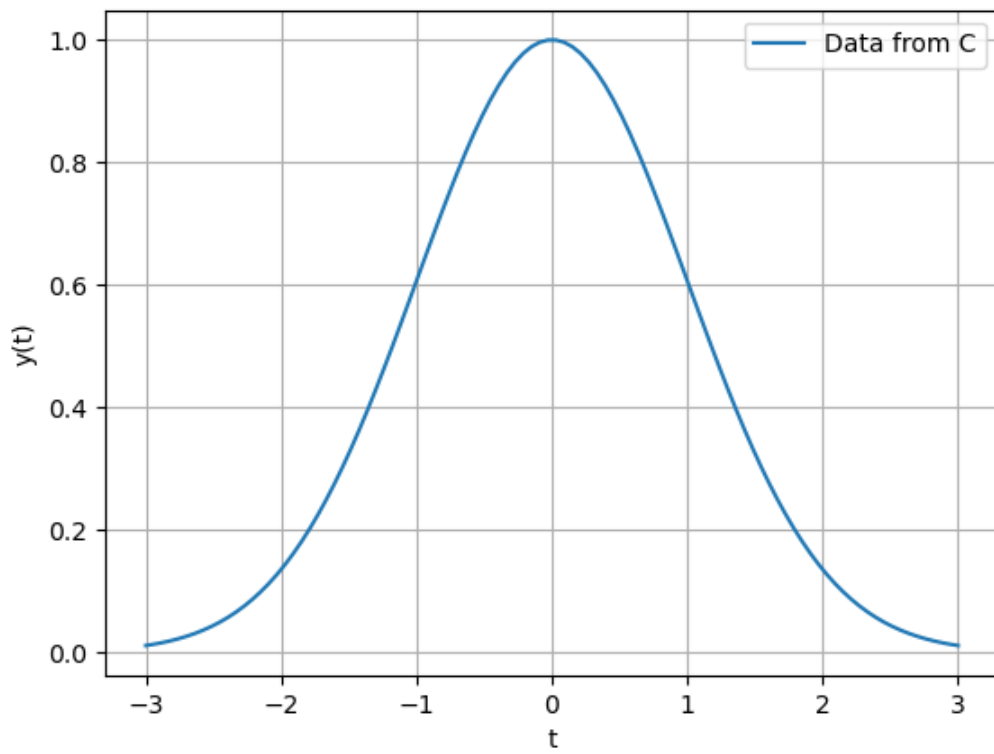$$\alpha = -2\ln(0.8) = 0.446 \tag{9.17}$$

Figure 9.2: Graph of y(t)

Chapter 10

FFT

10.1 The discrete-time Fourier transform of a signal $\mathrm{x}\,[n]$ is $X\,(\Omega) = (1 + \cos\Omega)\,e^{-j\Omega}$.

Consider that $x_p\,[n]$ is a periodic signal of period $N = 5$ such that

$$x_p\,[n] = x[n], \text{for n= 0, 1, 2} \tag{10.1}$$

$$= 0, \text{for n= 3, 4} \tag{10.2}$$

Note that $x_p\,[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn}$. The magnitude of the Fourier series coefficient a_3 is _____ (Round off to 3 decimal places). (GATE EE 2023) **Solution:** Using Euler's form of representation of complex numbers,

$$e^{j\Omega} = \cos\Omega + j\sin\Omega \tag{10.3}$$

$X\,(\Omega)$ can be expressed as,

$$X\,(\Omega) = \frac{1}{2} + e^{-j\Omega} + \frac{e^{-j2\Omega}}{2} \tag{10.4}$$

Symbol	Value	Description
$X\left(\Omega\right)$	$\left(1+\cos\Omega\right)e^{-j\Omega}$	Frequency function
Ω	ωF_s	angular frequency
ω	$\omega \in \left(-\pi, \pi\right)$	radian frequency
F_s	$1Hz$	Sampling frequency
$X\left(\omega\right)$	$\displaystyle\sum_{n=-\infty}^{\infty} x\left(n\right)e^{-j\omega n}$	D.T.F.T
$x\left(n\right)$	$x\left(n\right)$	Signal
$X\left(k\right)$	$\displaystyle\frac{1}{N}\sum_{n=0}^{N-1} x\left(n\right)e^{-\frac{j2\pi}{N}kn}$	Fourier coefficient
N	5	Period of the signal

Table 10.1: variable description

As sampling frequency is $1Hz$ $(\omega = \Omega)$ from DTFT(discrete time fourier transform) we get,

$$X(\Omega) = \sum_{n=0}^{n=2} x(n) e^{-j\Omega n}, \Omega \in (-\pi, \pi) \tag{10.5}$$

$$\implies \sum_{n=0}^{n=2} x(n) e^{-j\Omega n} = \frac{1}{2} + e^{-j\Omega} + \frac{e^{-j2\Omega}}{2} \tag{10.6}$$

On comparing coefficients we get,

$x(n) = \left\{ \frac{1}{2}, 1, \frac{1}{2} \right\}$

$$x_p(n) = \left\{ \frac{1}{2}, 1, \frac{1}{2}, 0, 0 \right\} \text{ with period, N=5} \tag{10.7}$$

$$X(3) = \frac{1}{5} \sum_{n=0}^{4} x(n) e^{-\frac{j6\pi}{5} n} \tag{10.8}$$

$$|X(3)| = 0.038 \tag{10.9}$$

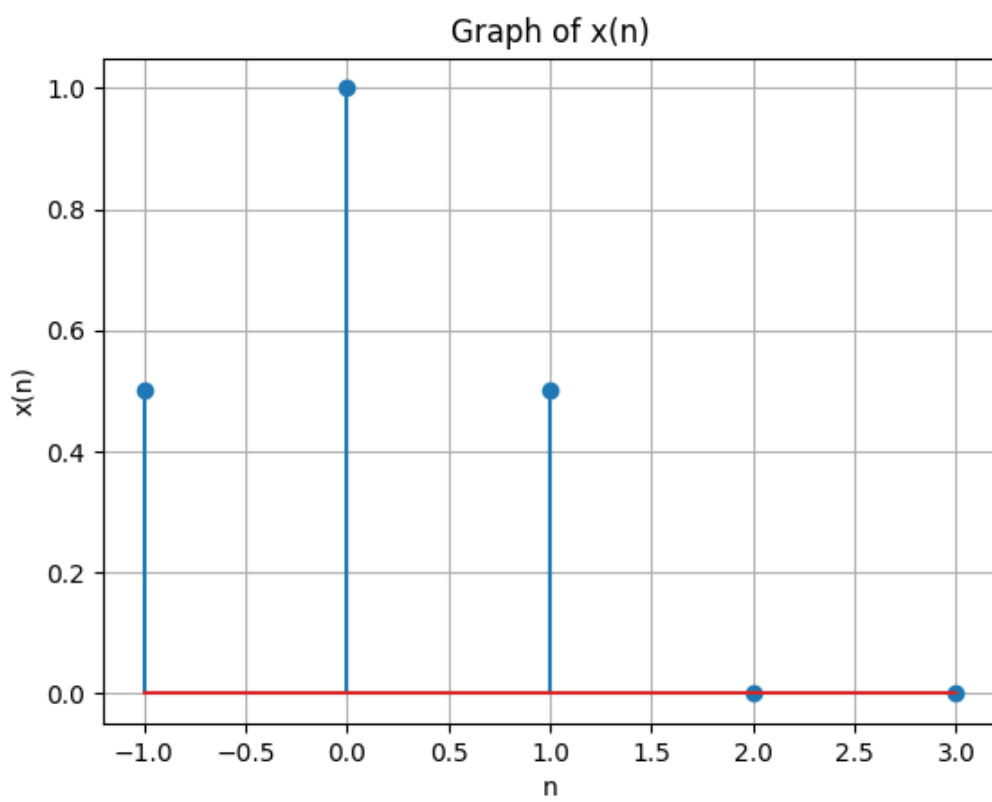| Fourier series coefficient	Real part	Imaginary part
X(0)	0.4000	0
X(1)	0.081	-0.249
X(2)	-0.031	-0.225
X(3)	-0.031	0.225
X(4)	0.081	0.249

Table 10.2: variable description
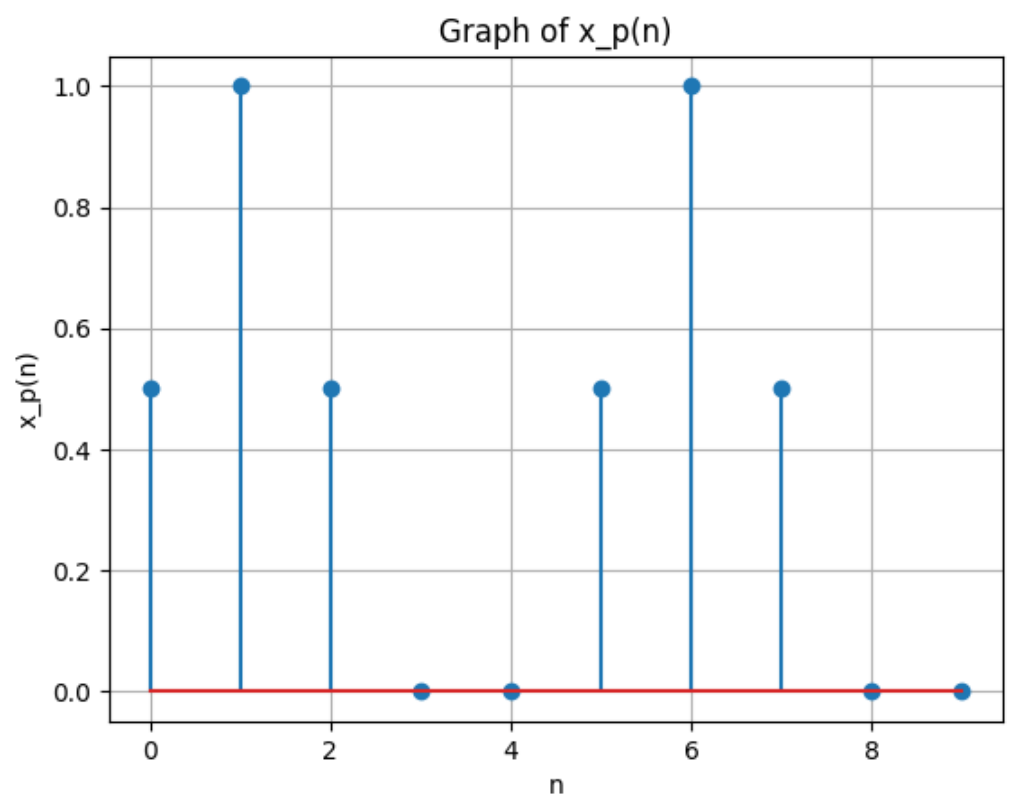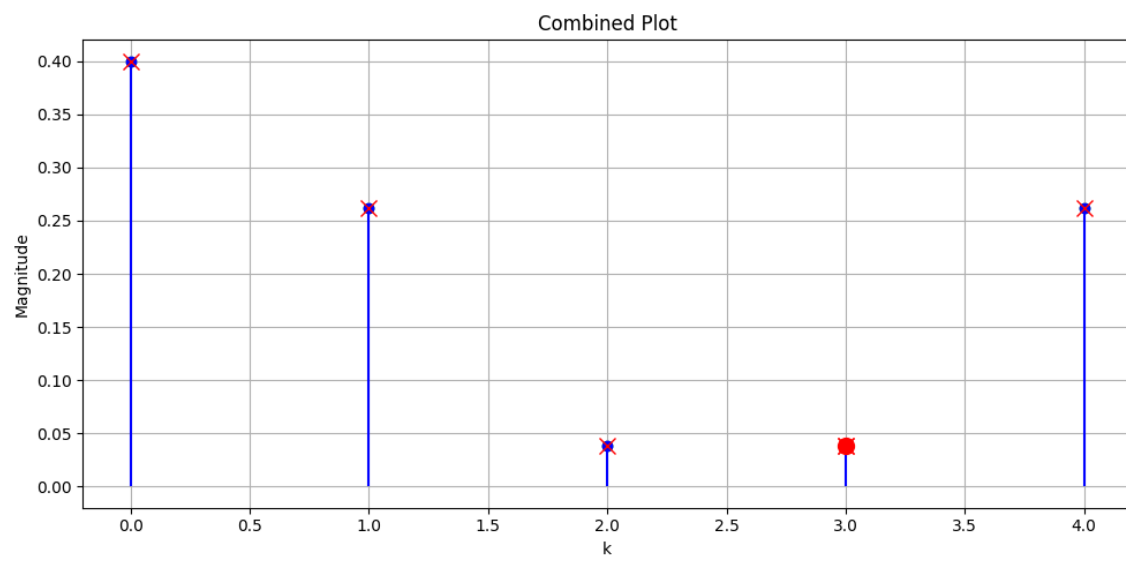
Figure 10.1: Plot of x(n) *vs* n

Figure 10.2: Plot of $x_p(n)$ vs n

Figure 10.3: Plot of $x_p(n)$ vs n