
SIGNAL PROCESSING FUNDAMENTALS Through NCERT

G. V. V. Sharma



Copyright ©2024 by G. V. V. Sharma.

<https://creativecommons.org/licenses/by-sa/3.0/>

and

<https://www.gnu.org/licenses/fdl-1.3.en.html>

Contents

Introduction	iii
1 Analog	1
1.1 Harmonics	1
1.2 Filters	17
2 Discrete	23
2.1 Z-transform	23
2.2 Sequences	33
A Convolution	39
B Z-transform	41

Introduction

This book introduces some concepts in signal processing through maths and physics problems in NCERT textbooks.

Chapter 1

Analog

1.1. Harmonics

1.1.1 Suppose that the electric field amplitude of an electromagnetic wave is $E_0 = 120\text{N/C}$ and that its frequency is $f = 50.0\text{ MHz}$.

- (a) Determine, B_0, ω, k and λ
- (b) Find expressions for \mathbf{E} and \mathbf{B}

Solution:

Table 1.1: Input Parameters

Symbol	Description	value
f	frequency of source	50.0 MHz
E_0	Electric field amplitude	120 N/C
c	speed of light	3×10^8 m/s
$\mathbf{e}_2, \mathbf{e}_3$	Standard Basis vectors	N/A

Table 1.2: Formulae and Output

Symbol	Description	Formula	Value
E	Electric field vector	$E_0 \sin(kx - 2\pi ft)\mathbf{e}_2$	$120 \sin[1.05x - 3.14 \times 10^8 t]\mathbf{e}_2$
B	Magnetic field vector	$B_0 \sin(kx - 2\pi ft)\mathbf{e}_3$	$(4 \times 10^{-7}) \sin[1.05x - 3.14 \times 10^8 t]\mathbf{e}_3$
B_0	Magnetic field strength	$\frac{E_0}{c}$	400nT
ω	Angular frequency	$2\pi f$	$3.14 \times 10^8 \text{m/s}$
k	Propagation constant	$\frac{2\pi f}{c}$	1.05rad/s
λ	Wavelength	$\frac{c}{f}$	6.0m

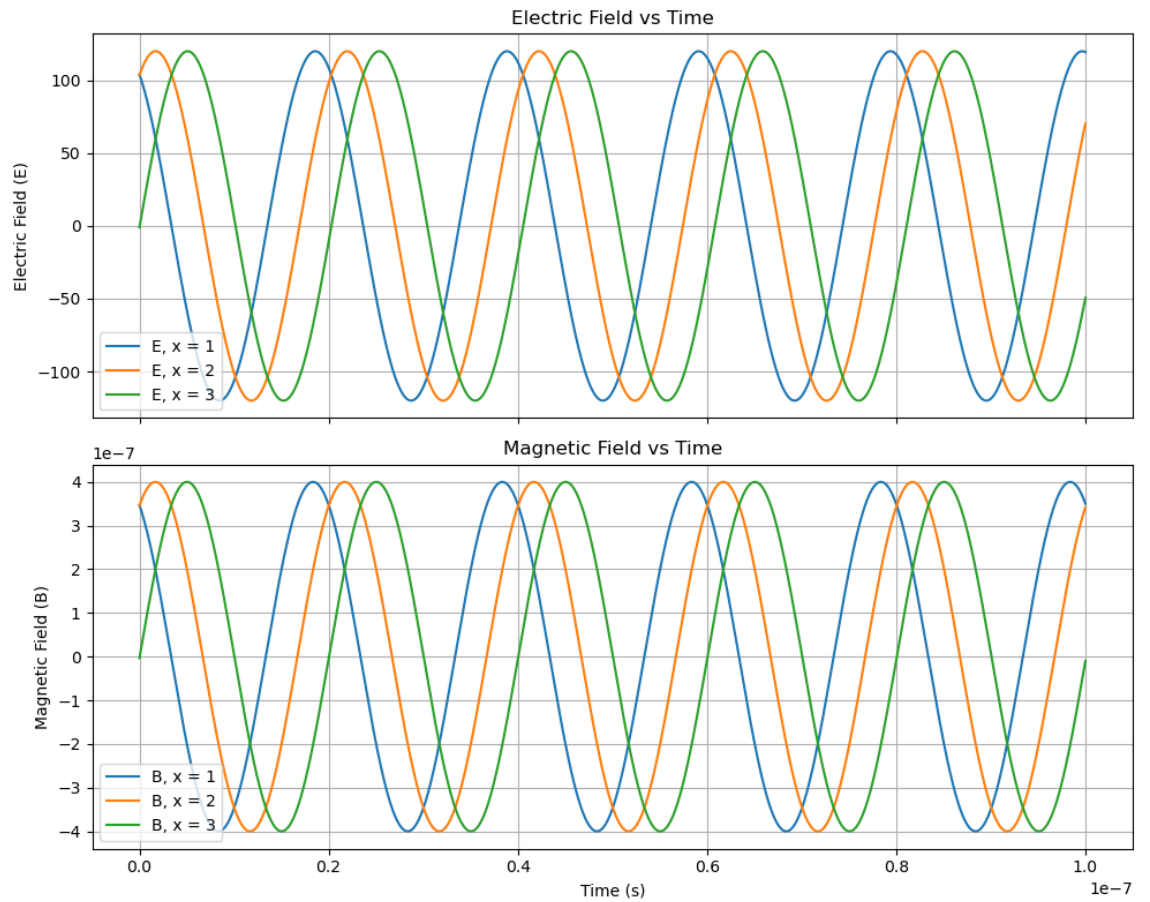


Figure 1.1.1: Graphs of \mathbf{E} and \mathbf{B}

1.1.2 A charged particle oscillates about its mean equilibrium position with a frequency of 10^9 Hz . What is the frequency of the electromagnetic waves produced by the oscillator?

Solution:

Symbol	Value	Description
$y(t)$	$\cos(2\pi f_c t)$	Wave equation of electro-magnetic wave
f_c	10^9	Frequency of electromagnetic wave
t	seconds	Time

Table 1.1.2: Variable description

1.1.3 Given below are some functions of x and t to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represents (i) travelling wave, (ii) a stationary wave or (iii) none at all:

(a) $y = 2 \cos(3x) \sin(10t)$

(b) $y = 2\sqrt{x - vt}$

(c) $y = 3 \sin(5x - 0.5t) + 4 \cos(5x - 0.5t)$

(d) $y = \cos x \sin t + \cos 2x \sin 2t$

Solution:

TRAVELLING WAVE	STATIONARY WAVE
$y(x, t) = A \sin(kx \pm \omega t)$	$y(x, t) = A \sin kx \cos \omega t$
PARAMETERS	DEFINITION
A	Amplitude
ω	Angular Velocity
x	Position
k	Wavenumber

Table 1.1.3: Travelling wave *vs* Stationary wave



Figure 1.1.2: $y(t) = \cos(2\pi \times 10^9 t)$

Let us assume an equation:

$$y = A(x) \cos(\omega t + \phi(x)) \quad (1.1)$$

Fig. 1.1.3 and Fig. 1.1.3 are self explanatory for stationary and travelling waves. Fig. 1.1.3 and Fig. 1.1.3 are neither stationary nor travelling waves.

STATIONARY WAVE CONDITION	TRAVELLING WAVE CONDITION
(1) $A(x)$ should be a function of position x , and it can be expressed as $A(x) = A_0 \cos(\omega t + \alpha)$ where A_0 is a constant, k is the wavenumber, x is the position and α is a phase constant.	(1) $A(x)$ should be a constant, and it can be expressed as $A(x) = A_0$ where A_0 is a constant number.
(2) $\phi(x)$ can be expressed as $\phi(x) = c$ where c is a constant.	(2) $\phi(x)$ represents a linear expression in x , and it can be expressed as $\phi(x) = kx + \theta$ where k is the wavenumber and θ is the phase constant.

Table 1.1.3: Travelling wave *vs* Stationary wave

1.1.4 For the travelling harmonic wave $y(x, t) = 2.0 \cos 2\pi(10t - 0.0080x + 0.35)$ where x and y are in cm and t in s . Calculate the phase difference between oscillatory motion of two points separated by a distance of

- (a) $4m$
- (b) $0.5m$
- (c) $\lambda/2$
- (d) $3\lambda/4$

Solution:

$$(\Delta\theta) = (2\pi ft - kx_1 + \phi) - (2\pi ft - kx_2 + \phi) \quad (1.2)$$

$$= k(x_2 - x_1) \quad (1.3)$$



Figure 1.1.3: DIPLACEMENT *vs* TIME-graph1

1.1.5 (a) The peak voltage of an AC supply is 300 V. What is the rms voltage?

(b) The rms value of current in an AC circuit is 10 A. What is the peak current?

Solution:

Parameter	Description	Value
$y(x_i, t)$	equation of harmonic wave	$A \cos(2\pi ft - kx_i + \phi)$
k	angular wave number	2π (0.008)
$\lambda = \frac{2\pi}{k}$	wavelength	125 cm
f	frequency	10
A	amplitude	2.0
ϕ	phase constant	2π (0.35)
θ_i	phase of i^{th} harmonic wave	$(2\pi ft - kx + \phi)$
x_i	position of i^{th} harmonic wave	
t	time	
$x_2 - x_1$	path difference	400 cm
		50 cm
		$\frac{\lambda}{2}$
		$\frac{3\lambda}{4}$

Table 1.1.4: Given parameters list

Parameter	Description	subquestion	Value
$\Delta\theta$	$\theta_1 - \theta_2$	(a)	6.4π radians
		(b)	0.8π radians
		(c)	π radians
		(d)	$\frac{3\pi}{2}$ radians

Table 1.1.4: Phase differences

parameter	value	description
$V(t)$	$V_0 \cdot \sin(2\pi ft + \phi)$	voltage in terms of time
$I(t)$	$I_0 \cdot \sin(2\pi ft + \phi)$	current in terms of time
V_0	300 V	peak voltage
V_{rms}	$\sqrt{\frac{1}{T} \int_0^T [V(t)]^2 dt}$	rms value of Voltage
I_{rms}	10 A	rms value of current
I_0	$\sqrt{2} \times I_{\text{rms}}$	peak current
f	50 Hz	frequency of the sinusoidal wave
T	0.02 s	time period of sinusoidal wave

Table 1.1.5: Input Parameter Table



Figure 1.1.3: DIPLACEMENT *vs* TIME-graph2



Figure 1.1.3: DIPLACEMENT *vs* TIME-graph3

(a)

$$V_{\text{rms}}^2 = \frac{1}{T} \int_0^T [V(t)]^2 dt \quad (1.4)$$

$$= f \int_0^{\frac{1}{f}} V_0^2 \cdot \sin^2(2\pi ft + \phi) dt \quad (1.5)$$

$$= \frac{1}{2} V_0^2 \left(1 - \frac{1}{f} \int_0^{\frac{1}{f}} \cos(4\pi ft + 2\phi) dt \right) \quad (1.6)$$

$$= \frac{1}{2} V_0^2 \left(1 - \frac{1}{f} \left[\frac{\sin(4\pi ft + 2\phi)}{4\pi f} \right]_0^{\frac{1}{f}} \right) \quad (1.7)$$

$$= \frac{1}{2} V_0^2 \left(1 - \frac{1}{f} \cdot \frac{\sin(4\pi + 2\phi) - \sin(0 + 2\phi)}{4\pi f} \right) \quad (1.8)$$

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}} \quad 10 \quad (1.9)$$



Figure 1.1.3: DIPLACEMENT *vs* TIME-graph4

To find the RMS voltage (V_{rms}) when the peak voltage (V_0) is 300V, you can use equation (1.9)

$$V_{\text{rms}} = \frac{300V}{\sqrt{2}} \approx 212.13V \quad (1.10)$$



Figure 1.1.4:

(b)

$$I_{\text{rms}}^2 = \frac{1}{T} \int_0^T [I(t)]^2 dt \quad (1.11)$$

$$= f \int_0^{\frac{1}{f}} I_0^2 \cdot \sin^2(2\pi ft + \phi) dt \quad (1.12)$$

$$= \frac{1}{2} I_0^2 \left(1 - \frac{1}{f} \left[\frac{\sin(4\pi ft + 2\phi)}{4\pi f} \right]_0^{\frac{1}{f}} \right) \quad (1.13)$$

$$= \frac{1}{2} I_0^2 \left(1 - \frac{1}{f} \cdot \frac{\sin(4\pi + 2\phi) - \sin(0 + 2\phi)}{4\pi f} \right) \quad (1.14)$$

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}} \quad (1.15)$$



Figure 1.1.4:

To find the peak current (I_0) when the RMS current (I_{rms}) is given, you can use equation (1.15)

$$I_0 \approx 10 \text{ A} \times 1.414 \approx 14.14 \text{ A} \quad (1.16)$$



Figure 1.1.4:





Figure 1.1.4:

1.1.6 In Young's double-slit experiment using monochromatic light of wavelength λ , the intensity of light at a point on the screen where path difference is λ , is K units. What is the intensity of light at a point where path difference is $\lambda/3$?

Solution:

From Table 1.1.6:

$$y(t) = A \sin(2\pi ft - kx_1) + A \sin(2\pi ft - kx_2) \quad (1.17)$$

$$y(t) = 2A \cos\left(\frac{k\Delta x}{2}\right) \sin\left(2\pi ft - \frac{k(x_1 + x_2)}{2}\right) \quad (1.18)$$

Parameter	Description	Value
$y_i(t)$	Equation of light from $S_{i^{\text{th}}}$	$A \sin(\omega t - kx_i)$
k	Wave number	$\frac{2\pi}{\lambda}$
I	Intensity of wave	$\propto A^2$
$\Delta x = x_1 - x_2$	Path difference	λ
		$\frac{\lambda}{3}$
K	Intensity of light at $\Delta x = \lambda$	
A	Amplitude of wave from source	
r	constant	$r \geq 0$

Table 1.1.6: Parameters

From Table 1.1.6 and equation (1.18):

$$\therefore I \propto 4A^2 \cos^2 \left(\frac{k\Delta x}{2} \right) \quad (1.19)$$

From Table 1.1.6 and equation (1.19):

$$\frac{K}{I_r} = \frac{4A^2 \cos^2 \left(\frac{2\pi}{2} \right)}{4A^2 \cos^2 \left(\frac{\pi}{3} \right)} \implies I_r = \frac{K}{4} \quad (1.20)$$

Hence, the Intensity of light at a point where path difference is $\frac{\lambda}{3}$ is $\frac{K}{4}$ units.

Parameter	Description	Value
I_r	Net Intensity of light at $\Delta x = \frac{\lambda}{3}$	$\frac{K}{4}$

Table 1.1.6:

Assuming $\Delta x = r\lambda$,

From equation (1.19):

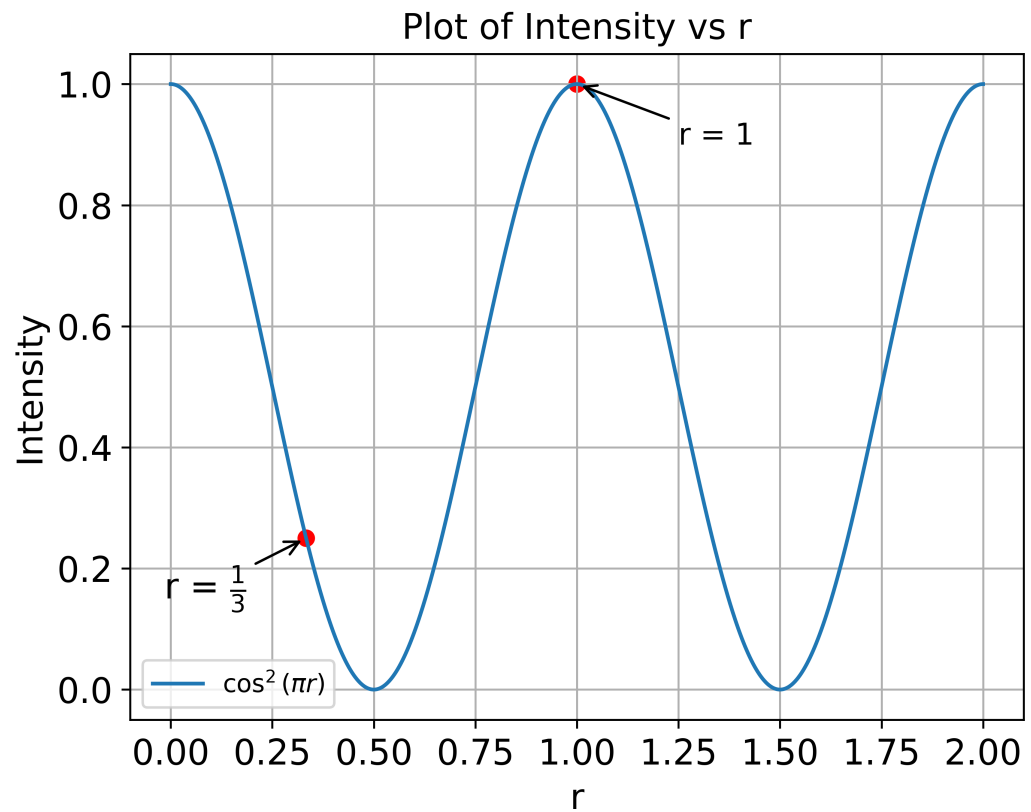


Figure 1.1.6:

1.2. Filters

1.2.1 Obtain the resonant frequency and Q-factor of a series LCR circuit with $L = 3.0 H$, $C = 27 \mu F$, and $R = 7.4 \Omega$. It is desired to improve the sharpness of the resonance of the circuit by reducing its ‘full width at half maximum’ by a factor of 2. Suggest a

suitable way.

Solution: Given parameters are:

Symbol	Value	Description
L	3.0 H	Inductance
C	27 μF	Capacitance
R	7.4 Ω	Resistance
Q		Quality Factor: ratio of voltage across inductor or capacitor to that across the resistor at resonance
ω_0	$\frac{1}{\sqrt{LC}}$	Angular Resonant Frequency

Table 1.11: Given Parameters

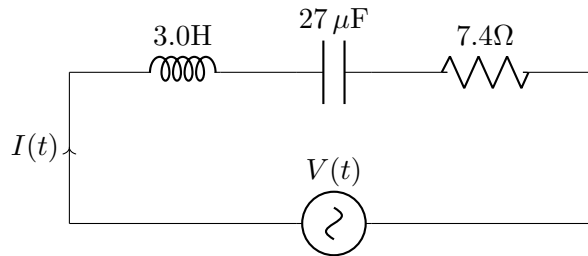


Figure 1.12: LCR Circuit

(a) Frequency Response of the Circuit

From Kirchhoff's Voltage Law (KVL):

$$V(t) = V_R + V_L + V_C \quad (1.21)$$

Using reactances from Fig. 1.13,

$$V(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s) \quad (1.22)$$

$$= I(s) \left(R + Ls + \frac{1}{sC} \right) \quad (1.23)$$

$$\Rightarrow I(s) = \frac{V(s)}{\left(R + Ls + \frac{1}{sC} \right)} \quad (1.24)$$

At resonance, the circuit becomes purely resistive. The reactances of capacitor



Figure 1.13: LCR Circuit

and inductor cancel out as follows:

$$Ls + \frac{1}{sC} = 0 \quad (1.25)$$

$$\Rightarrow s = j \frac{1}{\sqrt{LC}} \quad (1.26)$$

s can be expressed in terms of angular resonance frequency as

$$s = j\omega_0 \quad (1.27)$$

Comparing (1.26) and (1.27), we get

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad (1.28)$$

(b) Quality Factor

i. Using voltage across inductor,

$$Q = \left(\frac{V_L}{V_R} \right)_{\omega_0} = \frac{|sLI(s)|}{|RI(s)|} \quad (1.29)$$

$$= \frac{1}{\sqrt{LC}} \frac{L}{R} \quad (1.30)$$

$$= \frac{1}{R} \sqrt{\frac{L}{C}} \quad (1.31)$$

ii. Using voltage across capacitor,

$$Q = \left(\frac{V_C}{V_R} \right)_{\omega_0} = \frac{\left| \frac{I(s)}{sC} \right|}{|RI(s)|} \quad (1.32)$$

$$= \frac{\sqrt{LC}}{RC} \quad (1.33)$$

$$= \frac{1}{R} \sqrt{\frac{L}{C}} \quad (1.34)$$

(c) Plot of Impedance vs Angular Frequency

Impedance is defined as

$$H(s) = \frac{V(s)}{I(s)} \quad (1.35)$$

Using (1.24),

$$H(s) = R + sL + \frac{1}{sC} \quad (1.36)$$

$$\Rightarrow H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \quad (1.37)$$

$$\Rightarrow |H(j\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2} \quad (1.38)$$



Figure 1.14: Impedance vs ω (using values in Table 1.11)

Chapter 2

Discrete

2.1. Z-transform

2.1.1 Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

Solution:

Parameter	Description	Value
n	Integer -2,-1,0,1, 2, ...
$x_1(n)$	General term of Numerator	$(n^3 + 5n^2 + 8n + 4) \cdot u(n)$
$x_2(n)$	General Term of Denominator	$(n^3 + 4n^2 + 5n + 2) \cdot u(n)$
$y_1(n)$	Sum of terms of numerator	?
$y_2(n)$	Sum of terms of denominator	?
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}}, \{z \in \mathbb{C} : z > 1\}$
ROC	Region of convergence	$\{z : \sum_{n=-\infty}^{\infty} x(n)z^{-n} < \infty\}$

Table 1: Parameter Table

1. Analysis of Numerator:

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} \quad (2.1)$$

$$= \sum_{n=-\infty}^{\infty} (n^3 + 5n^2 + 8n + 4) u(n) z^{-n} \quad (2.2)$$

Using results of equations (B.3.2) to (B.3.5) we get:

$$\therefore X_1(z) = \frac{4 + 2z^{-1}}{(1 - z^{-1})^4}, |z| > 1 \quad (2.3)$$

From (A.3.2)

$$y_1(n) = x_1(n) * u(n) \quad (2.4)$$

$$Y_1(z) = X_1(z) U(z) \quad (2.5)$$

$$= \frac{4 + 2z^{-1}}{(1 - z^{-1})^5}, |z| > 1 \quad (2.6)$$

Using partial fractions:

$$\begin{aligned} Y_1(z) &= \frac{22z^{-1}}{(1 - z^{-1})} + \frac{48z^{-2}}{(1 - z^{-1})^2} + \frac{52z^{-3}}{(1 - z^{-1})^3}, \\ &+ \frac{28z^{-4}}{(1 - z^{-1})^4} + \frac{6z^{-5}}{(1 - z^{-1})^5} + 4, |z| > 1 \end{aligned} \quad (2.7)$$

Substituting results of equation (B.4.6) to (B.4.9) in equation (2.7):

$$y_1(n) = \frac{3n^4 + 26n^3 + 81n^2 + 106n + 48}{12} u(n) \quad (2.8)$$

$$= \frac{(3n + 8)(n + 1)(n + 2)(n + 3)}{12} u(n) \quad (2.9)$$

2. Analysis of Denominator:

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \quad (2.10)$$

$$= \sum_{n=-\infty}^{\infty} (n^3 + 4n^2 + 5n + 2) u(n) z^{-n} \quad (2.11)$$

Using results of equation (B.3.2) to (B.3.5) we get:

$$\therefore X_2(z) = \frac{2 + 4z^{-1}}{(1 - z^{-1})^4}, |z| > 1 \quad (2.12)$$

From (A.3.2)

$$y_2(n) = x_2(n) * u(n) \quad (2.13)$$

$$Y_2(z) = X_2(z) U(z) \quad (2.14)$$

$$= \frac{2 + 4z^{-1}}{(1 - z^{-1})^5}, |z| > 1 \quad (2.15)$$

Using partial fractions:

$$\begin{aligned} Y_2(z) &= \frac{14z^{-1}}{(1 - z^{-1})} + \frac{36z^{-2}}{(1 - z^{-1})^2} + \frac{44z^{-3}}{(1 - z^{-1})^3} \\ &\quad + \frac{26z^{-4}}{(1 - z^{-1})^4} + \frac{6z^{-5}}{(1 - z^{-1})^5} + 2, |z| > 1 \end{aligned} \quad (2.16)$$

Substituting results of equation (B.4.6) to (B.4.9) in equation (2.16):

$$y_2(n) = \frac{3n^4 + 22n^3 + 57n^2 + 62n + 24}{12} u(n) \quad (2.17)$$

$$= \frac{(3n + 4)(n + 1)(n + 2)(n + 3)}{12} u(n) \quad (2.18)$$

As the sequence start from $n = 0$, in RHS of question n should be replaced by $n + 1$:

$$\frac{y_1(n)}{y_2(n)} = \frac{3n + 8}{3n + 4} \quad (2.19)$$

Hence Prooved.



Figure 2.1: Stem Plot of $x_1(n)$



Figure 2.2: Stem Plot of $x_2(n)$



Figure 2.3: Stem Plot of $y_1(n)$



Figure 2.4: Stem Plot of $y_2(n)$

2.1.2 Write the five terms at $n = 1, 2, 3, 4, 5$ of the sequence and obtain the Z-transform of the series

$$x(n) = -1, \quad n = 0 \quad (2.20)$$

$$= \frac{x(n-1)}{n}, \quad n > 0 \quad (2.21)$$

$$= 0, \quad n < 0 \quad (2.22)$$

Solution:

$$x(1) = \frac{x(0)}{1} = -1 \quad (2.23)$$

$$x(2) = \frac{x(1)}{2} = -\frac{1}{2} \quad (2.24)$$

$$x(3) = \frac{x(2)}{3} = -\frac{1}{(2)(3)} = -\frac{1}{6} \quad (2.25)$$

$$x(4) = \frac{x(3)}{4} = -\frac{1}{(2)(3)(4)} = -\frac{1}{24} \quad (2.26)$$

$$x(5) = \frac{x(4)}{5} = -\frac{1}{(2)(3)(4)(5)} = -\frac{1}{120} \quad (2.27)$$

$$x(n) = \frac{-1}{n!} (u(n)) \quad (2.28)$$

$$x(n) \xleftrightarrow{\mathcal{Z}} X(z) \quad (2.29)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad (2.30)$$

using (2.28),

$$= \sum_{n=-\infty}^{\infty} \frac{-1}{n!} u(n) z^{-n} \quad (2.31)$$

$$= \sum_{n=0}^{\infty} \frac{-1}{n!} z^{-n} \quad (2.32)$$

$$= -e^{z^{-1}} \quad \{z \in \mathbb{C} : z \neq 0\} \quad (2.33)$$

Symbol	Value	Description
$x(n)$	$\frac{-1}{n!}$	general term of the series
$X(z)$	$-e^{z^{-1}}$	Z-transform of x(n)
$u(n)$		unit step function

Table 2.2: Parameters



Figure 2.5: Plot of $x(n)$ vs n

2.1.3 Subba Rao started work in 1995 at an annual salary of Rs. 5000 and received an increment of Rs. 200 each year. In which year did his income reach Rs. 7000?

Solution:

Parameter	Value	Description
$x(0)$	5000	Initial Income
d	200	Annual Increment (Common Difference)
$x(n)$	$(x(0) + nd)u(n)$	n^{th} term of the AP

Table 2.3: Input Parameters

From the values given in Table 2.3:

$$7000 = 5000 + 200n \quad (2.34)$$

$$\Rightarrow 2000 = 200n \quad (2.35)$$

$$\therefore n = 10 \quad (2.36)$$

Let Z-transform of $x(n)$ be $X(z)$.

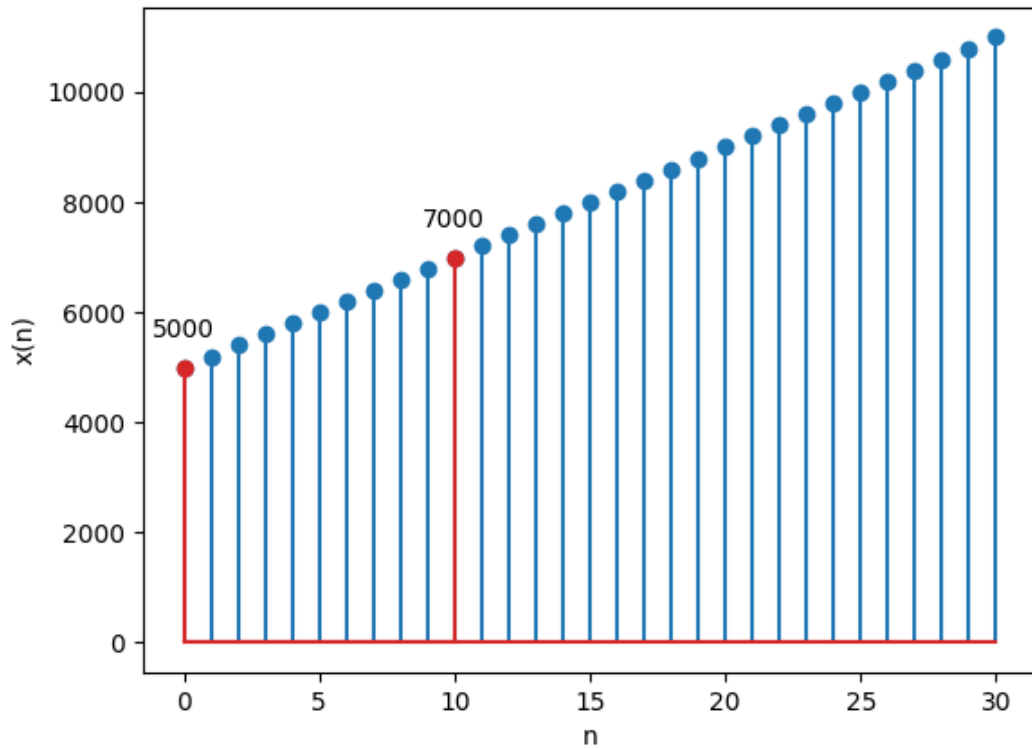


Figure 2.6: Plot of $x(n)$ vs n . See Table 2.3 for details.

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (2.37)$$

Using the values from Table 2.3:

$$X(z) = \frac{5000}{1 - z^{-1}} + \frac{200z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (2.38)$$

2.2. Sequences

2.2.1 Two APs have the same common difference. The difference between their 100th terms is 100, what is the difference between their 1000th terms?

Solution:

$$x(n) = \{x(0) + nd\}u(n) \quad (2.39)$$

$$x(99) - y(99) = 100 \quad (2.40)$$

$$\implies (x(0) + 99d) - (y(0) + 99d) = 100 \quad (2.41)$$

$$\implies x(0) - y(0) = 100 \quad (2.42)$$

$$x(n) - y(n) = (x(0) + nd) - (y(0) + nd) \quad (2.43)$$

$$= x(0) - y(0) \quad (2.44)$$

$$= 100 \quad (2.45)$$

$$\implies x(999) - y(999) = 100 \quad (2.46)$$

Variable	Description	Value
$x(n)$	n^{th} term of X	none
$y(n)$	n^{th} term of Y	none
d	common difference between the terms of AP	none
$x(99) - y(99)$	difference of 99^{th} terms of X and Y	100

Table 2.4: input parameters

Let

$$x(n) = \{101, 106, 111, \dots\} \quad (2.47)$$

$$y(n) = \{1, 6, 11, \dots\} \quad (2.48)$$

2.2.2 Check whether -150 is a term of the AP: 11,8,5,2,....

Solution:

$$x(n) = x(0) + nd \quad (2.49)$$

$$n = \frac{x(n) - x(0)}{d} \quad (2.50)$$

$$x(n) - x(0) \equiv 0 \pmod{d} \quad (2.51)$$

On substitutings values

$$-161 \equiv 2 \pmod{-3} \quad (2.52)$$

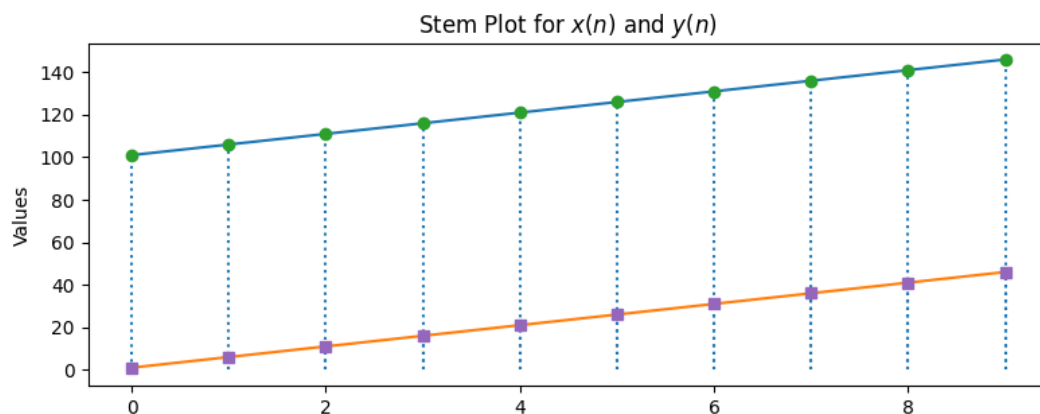


Figure 2.7:

Thus -150 is not a term of the given AP.

$$\boxed{x(n) = (11 - 3n) \times u(n)} \quad (2.53)$$

$$X(z) = \frac{11}{1 - z^{-1}} - \frac{3z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (2.54)$$

Variable	Description	Value
$x(0)$	First term of AP	11
d	Common difference	-3
$x(n)$	General term of given AP	None

Table 2.5: Input parameters

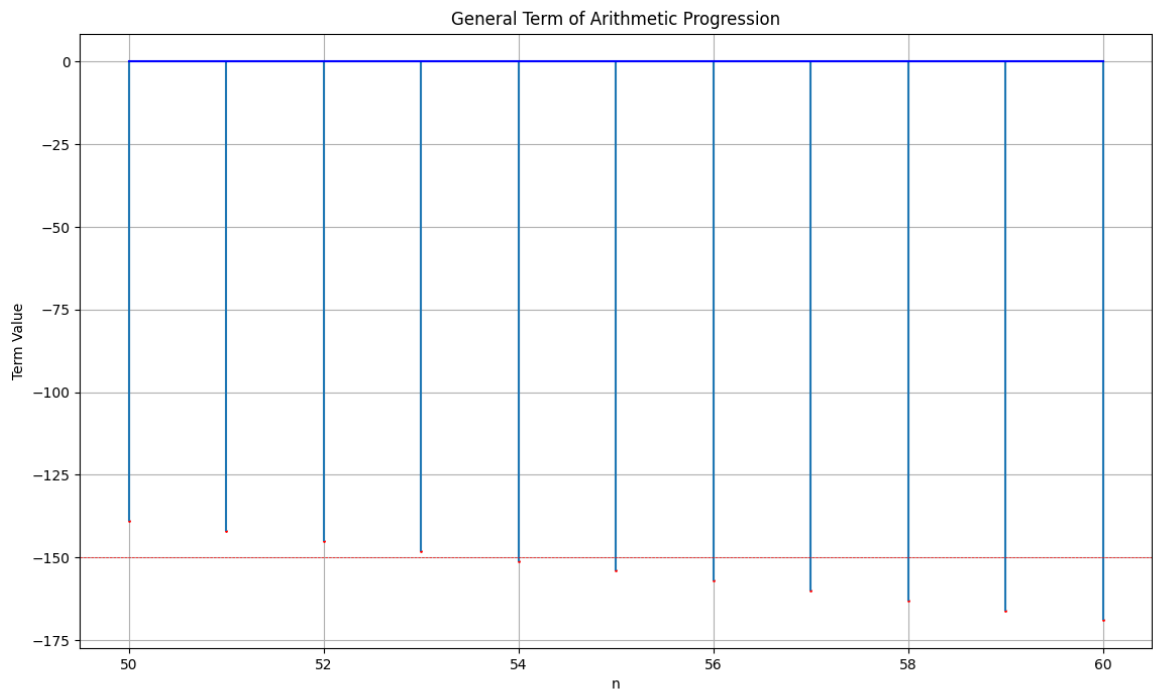


Figure 2.8: Representation of $x(n)$

2.2.3 Write the first five terms of the sequence $a_n = \frac{n(n^2+5)}{4}$.

Solution:

$$x(n) = \left(\frac{n^3 + 3n^2 + 8n + 6}{4} \right) u(n) \quad (2.55)$$

$$n^k u(n) \xleftrightarrow{\mathcal{Z}} (-1)^k z^k \frac{d^k}{dz^k} U(z) \quad (2.56)$$

$$nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \quad (2.57)$$

$$n^2 u(n) \xleftrightarrow{\mathcal{Z}} \frac{(z^{-1})(1 + z^{-1})}{(1 - z^{-1})^3} \quad |z| > 1 \quad (2.58)$$

$$n^3 u(n) \xleftrightarrow{\mathcal{Z}} \frac{(z^{-1})(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4} \quad |z| > 1 \quad (2.59)$$

Referencing the equations from (2.57), (2.58), and (2.59).

$$x(n) \xleftrightarrow{\mathcal{Z}} \frac{(z^{-1})(1 + 4z^{-1} + z^{-2})}{4(1 - z^{-1})^4} + \frac{3(z^{-1})(1 + z^{-1})}{4(1 - z^{-1})^3} + \frac{2z^{-1}}{(1 - z^{-1})^2} + \frac{3}{2(1 - z^{-1})} \quad |z| > 1 \quad (2.60)$$

$$x(n) \xleftrightarrow{\mathcal{Z}} \frac{3}{2(1 - z^{-1})^3} + \frac{3z^{-2}}{2(1 - z^{-1})^4} \quad |z| > 1 \quad (2.61)$$

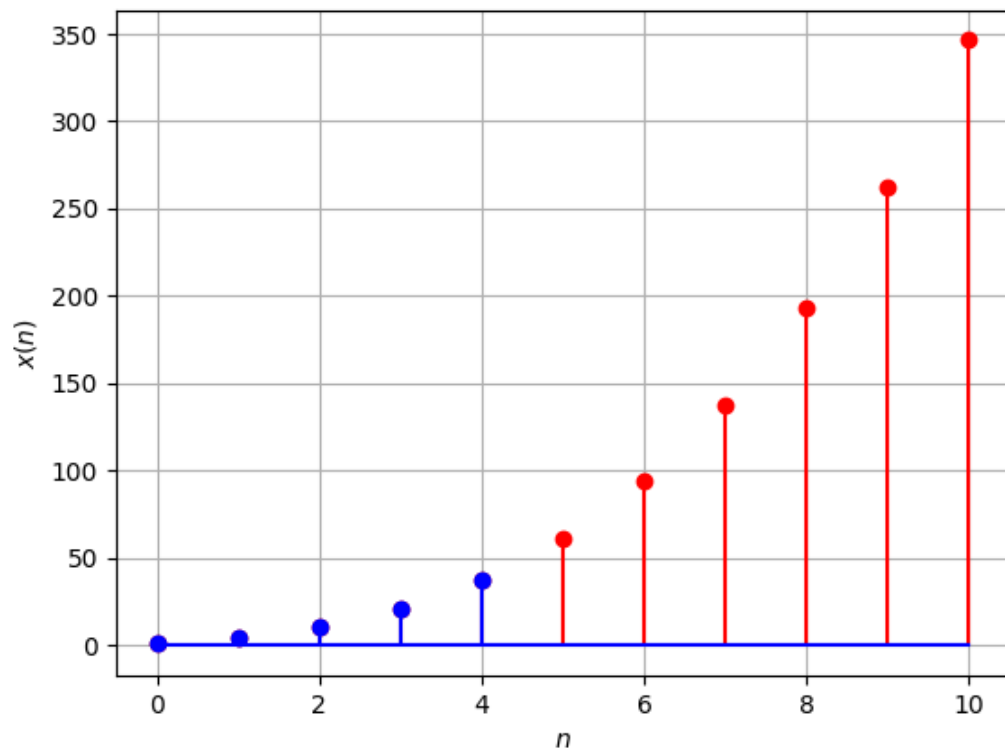


Figure 2.9: Plot of equation(2.55)

Appendix A

Convolution

A.1 The convolution sum is defined as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) \quad (\text{A.1.1})$$

A.2 The unit step function is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (\text{A.2.1})$$

A.3 If

$$x(n) = 0, \quad n < 0, \quad (\text{A.3.1})$$

from (A.1.1),

$$x(n) * u(n) = \sum_{k=0}^n x(k) \quad (\text{A.3.2})$$

Appendix B

Z-transform

B.1 The Z-transform of $p(n)$ is defined as

$$P(z) = \sum_{n=-\infty}^{\infty} p(n)z^{-n} \quad (\text{B.1.1})$$

B.2 If

$$p(n) = p_1(n) * p_2(n), \quad (\text{B.2.1})$$

$$P(z) = P_1(z)P_2(z) \quad (\text{B.2.2})$$

B.3

$$nx(n) \xleftrightarrow{\mathcal{Z}} -zX'(z) \quad (\text{B.3.1})$$

From (B.3.1)

$$\implies nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1-z^{-1})^2}, |z| > 1 \quad (\text{B.3.2})$$

$$\implies n^2u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3}, |z| > 1 \quad (\text{B.3.3})$$

$$\implies n^3u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}, |z| > 1 \quad (\text{B.3.4})$$

$$\implies n^4u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5} \quad (\text{B.3.5})$$

where $|z| > 1$

B.4

$$x(n-k) \xleftrightarrow{\mathcal{Z}} z^{-k}X(z) \quad (\text{B.4.1})$$

Using (B.4.1):

$$nu(n-1) \xleftrightarrow{\mathcal{Z}} z \frac{2z^{-2}}{(1-z^{-1})^2} \quad (\text{B.4.2})$$

Now ,

$$\frac{(n-1)}{2}u(n-2) \xleftrightarrow{\mathcal{Z}} \frac{z^{-2}}{(1-z^{-1})^2} \quad (\text{B.4.3})$$

$$\frac{(n-1)(n-2)}{6}u(n-3) \xleftrightarrow{\mathcal{Z}} \frac{z^{-3}}{(1-z^{-1})^3} \quad (\text{B.4.4})$$

\vdots

$$\frac{(n-1)(n-2)\dots(n-k+1)}{(k-1)!}u(n-k) \xleftrightarrow{\mathcal{Z}} \frac{z^{-k}}{(1-z^{-1})^k} \quad (\text{B.4.5})$$

$$\Rightarrow Z^{-1} \left[\frac{z^{-2}}{(1 - z^{-1})^2} \right] = (n - 1) u(n - 1) \quad (\text{B.4.6})$$

$$\Rightarrow Z^{-1} \left[\frac{z^{-3}}{(1 - z^{-1})^3} \right] = \frac{(n - 1)(n - 2)}{2} u(n - 1) \quad (\text{B.4.7})$$

$$\Rightarrow Z^{-1} \left[\frac{z^{-4}}{(1 - z^{-1})^4} \right] = \frac{(n - 1)(n - 2)(n - 3)}{6} u(n - 1) \quad (\text{B.4.8})$$

$$\Rightarrow Z^{-1} \left[\frac{z^{-5}}{(1 - z^{-1})^5} \right] = \frac{(n - 1)(n - 2)(n - 3)(n - 4)}{24} u(n - 1) \quad (\text{B.4.9})$$

B.5 For a Geometric progression

$$x(n) = x(0) r^n u(n), \quad (\text{B.5.1})$$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} x(0) r^n z^{-n} \quad (\text{B.5.2})$$

$$= \sum_{n=0}^{\infty} x(0) (rz^{-1})^n \quad (\text{B.5.3})$$

$$= \frac{x(0)}{1 - rz^{-1}}, \quad |z| > |r| \quad (\text{B.5.4})$$

B.6 Substituting $r = 1$ in (B.5.4),

$$u(n) \xleftrightarrow{Z} U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (\text{B.6.1})$$

B.7 From (B.3.1) and (B.6.1),

$$nu(n) \xleftrightarrow{Z} \frac{z^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \quad (\text{B.7.1})$$

B.8 For an AP,

$$x(n) = [x(0) + nd] u(n) = x(0)u(n) + dnu(n) \quad (\text{B.8.1})$$

$$\implies X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \quad (\text{B.8.2})$$

upon substituting from (B.6.1) and (B.7.1).

B.9 From (A.3.2), the sum to n terms of a GP can be expressed as

$$y(n) = x(n) * u(n) \quad (\text{B.9.1})$$

where $x(n)$ is defined in (B.5.1). From (B.2.2), (B.5.4) and (B.6.1),

$$Y(z) = X(z) U(z) \quad (\text{B.9.2})$$

$$= \left(\frac{x(0)}{1 - rz^{-1}} \right) \left(\frac{1}{1 - z^{-1}} \right) \quad |z| > |r| \cap |z| > |1| \quad (\text{B.9.3})$$

$$= \frac{x(0)}{(1 - rz^{-1})(1 - z^{-1})} \quad |z| > |r| \quad (\text{B.9.4})$$

which can be expressed as

$$Y(z) = \frac{x(0)}{r - 1} \left(\frac{r}{1 - rz^{-1}} - \frac{1}{1 - z^{-1}} \right) \quad (\text{B.9.5})$$

using partial fractions. Again, from (B.5.4) and (B.6.1), the inverse of the above can be expressed as

$$y(n) = x(0) \left(\frac{r^{n+1} - 1}{r - 1} \right) u(n) \quad (\text{B.9.6})$$

B.10 For the AP $x(n)$, the sum of first $n + 1$ terms can be expressed as

$$y(n) = \sum_{k=0}^n x(k) \quad (\text{B.10.1})$$

$$\Rightarrow y(n) = \sum_{k=-\infty}^{\infty} x(k)u(n-k) \quad (\text{B.10.2})$$

$$= x(n) * u(n) \quad (\text{B.10.3})$$

Taking the Z-transform on both sides, and substituting (B.8.2) and (B.6.1),

$$Y(z) = X(z)U(z) \quad (\text{B.10.4})$$

$$\Rightarrow Y(z) = \left(\frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2} \right) \frac{1}{1-z^{-1}} \quad |z| > 1 \quad (\text{B.10.5})$$

$$= \frac{x(0)}{(1-z^{-1})^2} + \frac{dz^{-1}}{(1-z^{-1})^3}, \quad |z| > 1 \quad (\text{B.10.6})$$

B.11 From (B.4.1) and (B.7.1),

$$(n+1)u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{(1-z^{-1})^2}, \quad |z| > 1, \quad (\text{B.11.1})$$

From (B.11.1) and (B.3.1),

$$n(n+1)u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1-z^{-1})^3}, \quad |z| > 1, \quad (\text{B.11.2})$$

B.12 Taking the inverse Z-transform of (B.10.6),

$$y(n) = x(0) [(n+1)u(n)] + \frac{d}{2} [n(n+1)u(n)] \quad (\text{B.12.1})$$

$$= \frac{n+1}{2} \{2x(0) + nd\} u(n) \quad (\text{B.12.2})$$

