NCERT 11.9.5 26Q

EE23BTECH11015 - DHANUSH V NAYAK*

RESULTS AND DERIVATIONS

1) By the differentiation property :

$$nx(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-z)\frac{dX(z)}{dz} \tag{1}$$

$$\implies nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2}, |z| > 1 \tag{2}$$

$$\implies n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(z^{-1}+1\right)}{(1-z^{-1})^3}, |z| > 1 \tag{3}$$

$$\implies n^3u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{(1-z^{-1})^4}, |z| > 1 \tag{4}$$

$$\implies n^4u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{(1-z^{-1})^5} \tag{5}$$

where $|z| > 1$

2) Time shifting property:

$$x(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k}X(z) \tag{6}$$

3) By (6)

$$u(n-1) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})} \tag{7}$$

By (1)

$$nu(n-1) \overset{\mathcal{Z}}{\longleftrightarrow} z\frac{2z^{-2}}{(1-z^{-1})^2} \tag{8}$$

Now ,

$$\frac{(n-1)}{2}u(n-2) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-2}}{(1-z^{-1})^2} \tag{9}$$

$$\frac{(n-1)(n-2)}{6}u(n-3) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-3}}{(1-z^{-1})^3} \tag{10}$$

$$\vdots$$

$$\frac{(n-1)(n-2)\ldots(n-k+1)}{(k-1)!}u(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-k}}{(1-z^{-1})^k} \tag{11}$$

$$\implies Z^{-1}\left[\frac{z^{-2}}{(1-z^{-1})^2}\right] = (n-1)u(n-1) \tag{12}$$

$$\implies Z^{-1}\left[\frac{z^{-3}}{(1-z^{-1})^3}\right] = \frac{(n-1)(n-2)}{2}u(n-1) \tag{13}$$

$$\implies Z^{-1}\left[\frac{z^{-4}}{(1-z^{-1})^4}\right] = \frac{(n-1)(n-2)(n-3)}{6}u(n-1) \tag{14}$$

$$\implies Z^{-1}\left[\frac{z^{-5}}{(1-z^{-1})^5}\right] = \frac{(n-1)(n-2)(n-3)(n-4)}{24} \tag{15}$$

$$u(n-1)$$

4) The convolution sum is defined as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{16}$$

from (16),

$$x(n) * u(n) = \sum_{k=0}^{n} x(k) \tag{17}$$

Question: Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

Solution:

Parameter	Description	Value
n	Integer	.... -2,-1,0,1, 2, ...
$x_1(n)$	General term of Numerator	$\left(n^3 + 5n^2 + 8n + 4\right) \cdot u(n)$
$x_2(n)$	General Term of Denominator	$\left(n^3 + 4n^2 + 5n + 2\right) \cdot u(n)$
$y_1(n)$	Sum of terms of numerator	?
$y_2(n)$	Sum of terms of denominator	?
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}}, \{z \in \mathbb{C} : \|z\| > 1\}$
ROC	Region of convergence	$\left\{z : \left\|\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right\| < \infty\right\}$

TABLE 1: Parameter Table

1. Analysis of Numerator:

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} \tag{18}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 5n^2 + 8n + 4\right) u(n) z^{-n} \tag{19}$$

Using results of equations (2) to (5) we get:

$$\therefore X_1(z) = \frac{4 + 2z^{-1}}{\left(1 - z^{-1}\right)^4}, |z| > 1 \tag{20}$$

From (17)

$$y_1(n) = x_1(n) * u(n) \tag{21}$$

$$Y_1(z) = X_1(z) U(z) \tag{22}$$

$$= \frac{4 + 2z^{-1}}{\left(1 - z^{-1}\right)^5}, |z| > 1 \tag{23}$$

Using partial fractions:

$$Y_1(z) = \frac{22z^{-1}}{\left(1 - z^{-1}\right)} + \frac{48z^{-2}}{\left(1 - z^{-1}\right)^2} + \frac{52z^{-3}}{\left(1 - z^{-3}\right)^3}, \tag{24}$$

$$+ \frac{28z^{-4}}{\left(1 - z^{-1}\right)^4} + \frac{6z^{-5}}{\left(1 - z^{-1}\right)^5} + 4, |z| > 1$$

Substituting results of equation (12) to (15) in equation (24):

$$y_1(n) = \frac{3n^4 + 26n^3 + 81n^2 + 106n + 48}{12} u(n) \tag{25}$$

$$= \frac{(3n+8)(n+1)(n+2)(n+3)}{12} u(n) \tag{26}$$

2. Analysis of Denominator:

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \tag{27}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 4n^2 + 5n + 2\right) u(n) z^{-n} \tag{28}$$

Using results of equation (2) to (5) we get:

$$\therefore X_2(z) = \frac{2 + 4z^{-1}}{\left(1 - z^{-1}\right)^4}, |z| > 1 \tag{29}$$

From (17)

$$y_2(n) = x_2(n) * u(n) \tag{30}$$

$$Y_2(z) = X_2(z) U(z) \tag{31}$$

$$= \frac{2 + 4z^{-1}}{\left(1 - z^{-1}\right)^5}, |z| > 1 \tag{32}$$

Using partial fractions:

$$Y_2(z) = \frac{14z^{-1}}{\left(1 - z^{-1}\right)} + \frac{36z^{-2}}{\left(1 - z^{-1}\right)^2} + \frac{44z^{-3}}{\left(1 - z^{-3}\right)^3} \tag{33}$$

$$+ \frac{26z^{-4}}{\left(1 - z^{-1}\right)^4} + \frac{6z^{-5}}{\left(1 - z^{-1}\right)^5} + 2, |z| > 1$$

Substituting results of equation (12) to (15) in equation (33):

$$y_2(n) = \frac{3n^4 + 22n^3 + 57n^2 + 62n + 24}{12} u(n) \tag{34}$$

$$= \frac{(3n+4)(n+1)(n+2)(n+3)}{12} u(n) \tag{35}$$

As the sequence start from $n = 0$, in RHS of question n should be replaced by $n + 1$:

$$\frac{y_1(n)}{y_2(n)} = \frac{3n+8}{3n+4} \tag{36}$$

Hence Prooved.

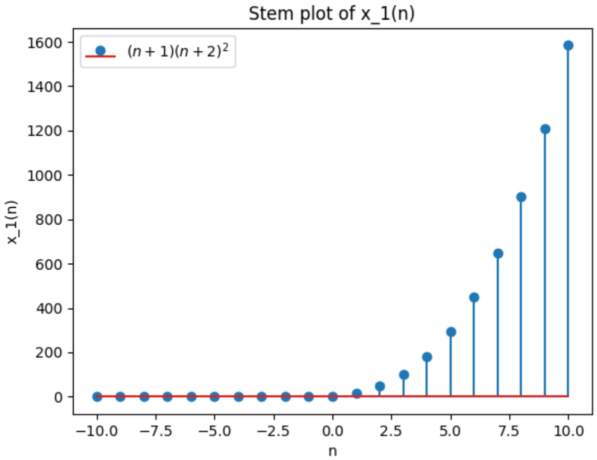

Fig. 1: Stem Plot of $x_1(n)$

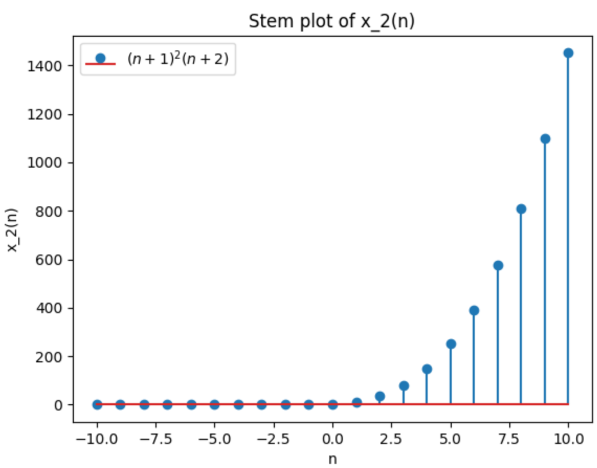

Fig. 2: Stem Plot of $x_2(n)$

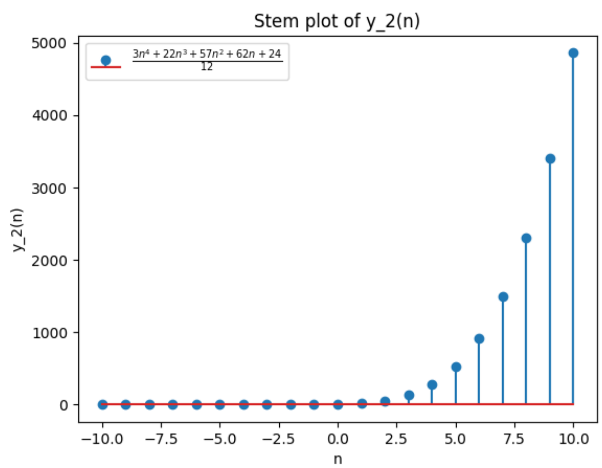

Fig. 4: Stem Plot of $y_2(n)$

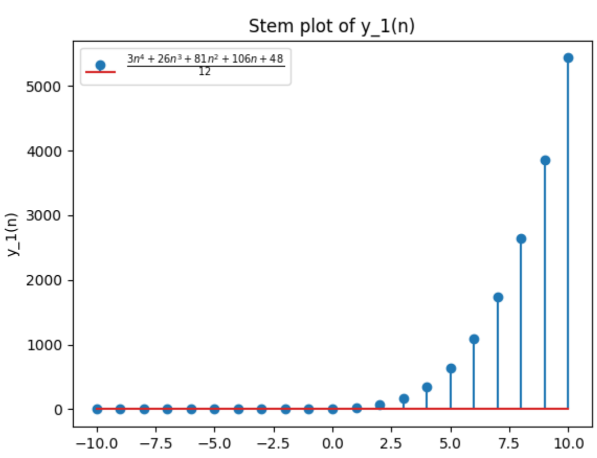

Fig. 3: Stem Plot of $y_1(n)$