NCERT Physics 12.7 Q21

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Question: Obtain the resonant frequency and Q-factor of a series LCR circuit with $L = 3.0\,H$, $C = 27\,\mu F$, and $R = 7.4\,\Omega$. It is desired to improve the sharpness of the resonance of the circuit by reducing its 'full width at half maximum' by a factor of 2. Suggest a suitable way.

Solution: Given parameters are:

Symbol	Value	Description
L	3.0 H	Inductance
C	27 μF	Capacitance
R	7.4 Ω	Resistance
Q		Quality Factor: ratio of voltage across inductor or capacitor to that across the resistor at resonance
ω_0	$\frac{1}{\sqrt{LC}}$	Angular Resonant Frequency

TABLE I: Given Parameters

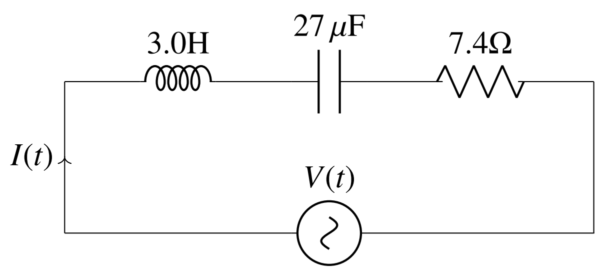

Fig. 1: LCR Circuit

1) Frequency Response of the Circuit
   From Kirchhoff's Voltage Law (KVL):
$$V(t) = V_R + V_L + V_C \tag{1}$$
   Using reactances from Fig. 2,
$$V(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s) \tag{2}$$
$$= I(s)\left(R + Ls + \frac{1}{sC}\right) \tag{3}$$
$$\implies I(s) = \frac{V(s)}{\left(R + Ls + \frac{1}{sC}\right)} \tag{4}$$

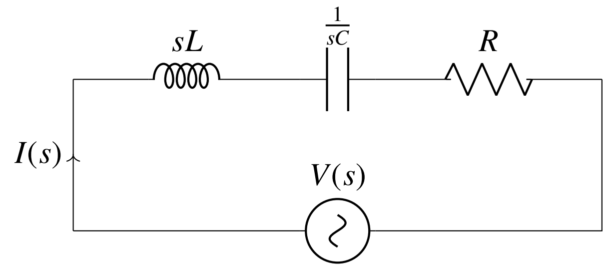

Fig. 2: LCR Circuit

At resonance, the circuit becomes purely resistive. The reactances of capacitor and inductor cancel out as follows:
$$Ls + \frac{1}{sC} = 0 \tag{5}$$
$$\implies s = j\frac{1}{\sqrt{LC}} \tag{6}$$
s can be expressed in terms of angular resonance frequency as
$$s = j\omega_0 \tag{7}$$
Comparing (6) and (7), we get
$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{8}$$

2) Quality Factor
   a) Using voltage across inductor,
$$Q = \left(\frac{V_L}{V_R}\right)_{\omega_0} = \frac{|sLI(s)|}{|RI(s)|} \tag{9}$$
$$= \frac{1}{\sqrt{LC}}\frac{L}{R} \tag{10}$$
$$= \frac{1}{R}\sqrt{\frac{L}{C}} \tag{11}$$
   b) Using voltage across capacitor,
$$Q = \left(\frac{V_C}{V_R}\right)_{\omega_0} = \frac{\left|\frac{I(s)}{sC}\right|}{|RI(s)|} \tag{12}$$
$$= \frac{\sqrt{LC}}{RC} \tag{13}$$
$$= \frac{1}{R}\sqrt{\frac{L}{C}} \tag{14}$$

3) Plot of Impedance vs Angular Frequency
   Impedance is defined as

$$H(s) = \frac{V(s)}{I(s)} \tag{15}$$

   Using (4),

$$H(s) = R + sL + \frac{1}{sC} \tag{16}$$

$$\implies H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \tag{17}$$

$$\implies |H(j\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{18}$$

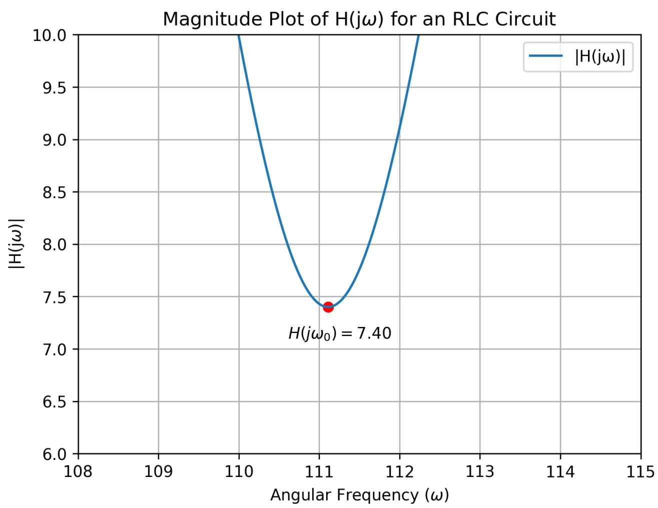

Fig. 3: Impedance vs ω (using values in Table I)