SIGNAL PROCESSING FUNDAMENTALS Through NCERT

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Contents

Introduction

This book introduces some concepts in signal processing through maths and physics problems in NCERT textbooks.

Chapter 1

Analog

1.1. Harmonics

1.1.1 Suppose that the electric field amplitude of an electromagnetic wave is $E_0 = 120\text{N/C}$ and that its frequency is $f = 50.0$ MHz.

(a) Determine, B_0, ω, k and λ

(b) Find expressions for $\mathbf{E}$ and $\mathbf{B}$

Solution:

Table 1.1: Input Parameters

Symbol	Description	value
f	frequency of source	50.0 MHz
E_0	Electric field amplitude	120 N/C
c	speed of light	3 x 10^8 m/s
$\mathbf{e_2}, \mathbf{e_3}$	Standard Basis vectors	N/A

Table 1.2: Formulae and Output

Symbol	Description	Formula	Value
$\mathbf{E}$	Electric field vector	$E_0 \sin(kx - 2\pi ft)\mathbf{e_2}$	$120 \sin[1.05x - 3.14\text{x}10^8 t]\mathbf{e_2}$
$\mathbf{B}$	Magnetic field vector	$B_0 \sin(kx - 2\pi ft)\mathbf{e_3}$	$(4x10^{-7}) \sin[1.05x - 3.14\text{x}10^8 t]\mathbf{e_3}$
B_0	Magnetic field strength	$\frac{E_0}{c}$	400nT
ω	Angular frequency	$2\pi f$	3.14 x 10^8m/s
k	Propagation constant	$\frac{2\pi f}{c}$	1.05rad/s
λ	Wavelength	$\frac{c}{f}$	6.0m

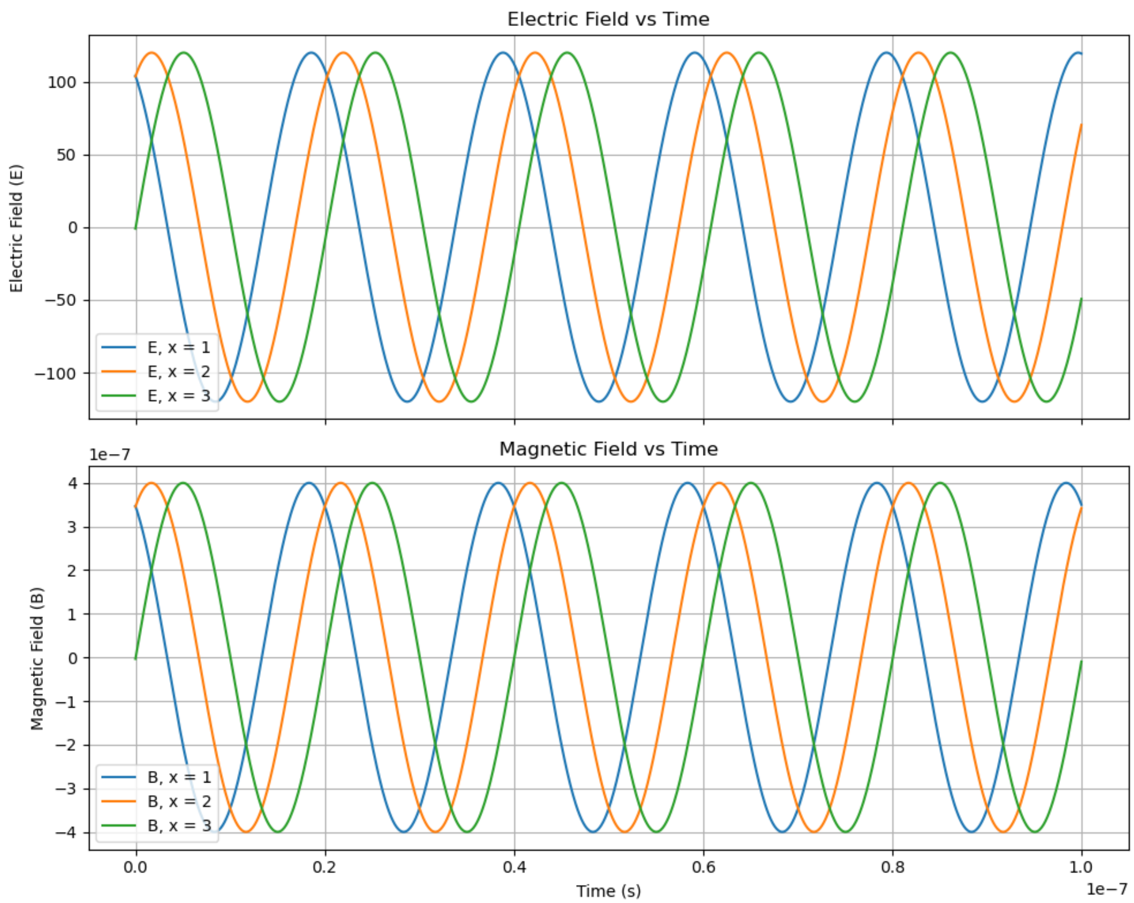


Figure 1.1.1: Graphs of **E** and **B**

1.1.2 A charged particle oscillates about its mean equilibrium position with a frequency of $10^9 Hz$. What is the frequency of the electromagnetic waves produced by the oscillator?

Solution:

Symbol	Value	Description
$y(t)$	$\cos(2\pi f_c t)$	Wave equation of electro-magnetic wave
f_c	10^9	Frequency of electromagnetic wave
t	seconds	Time

Table 1.1.2: Variable description

1.1.3 Given below are some functions of x and t to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represents (i) travelling wave, (ii) a stationary wave or (iii) none at all:

(a) $y = 2\cos(3x)\sin(10t)$

(b) $y = 2\sqrt{x - vt}$

(c) $y = 3\sin(5x - 0.5t) + 4\cos(5x - 0.5t)$

(d) $y = \cos x \sin t + \cos 2x \sin 2t$

Solution:

TRAVELLING WAVE	STATIONARY WAVE
$y(x,t) = A\sin(kx \pm \omega t)$	$y(x,t) = A\sin kx \cos \omega t$

PARAMETERS	DEFINITION
A	Amplitude
ω	Angular Velocity
x	Position
k	Wavenumber

Table 1.1.3: Travelling wave vs Stationary wave

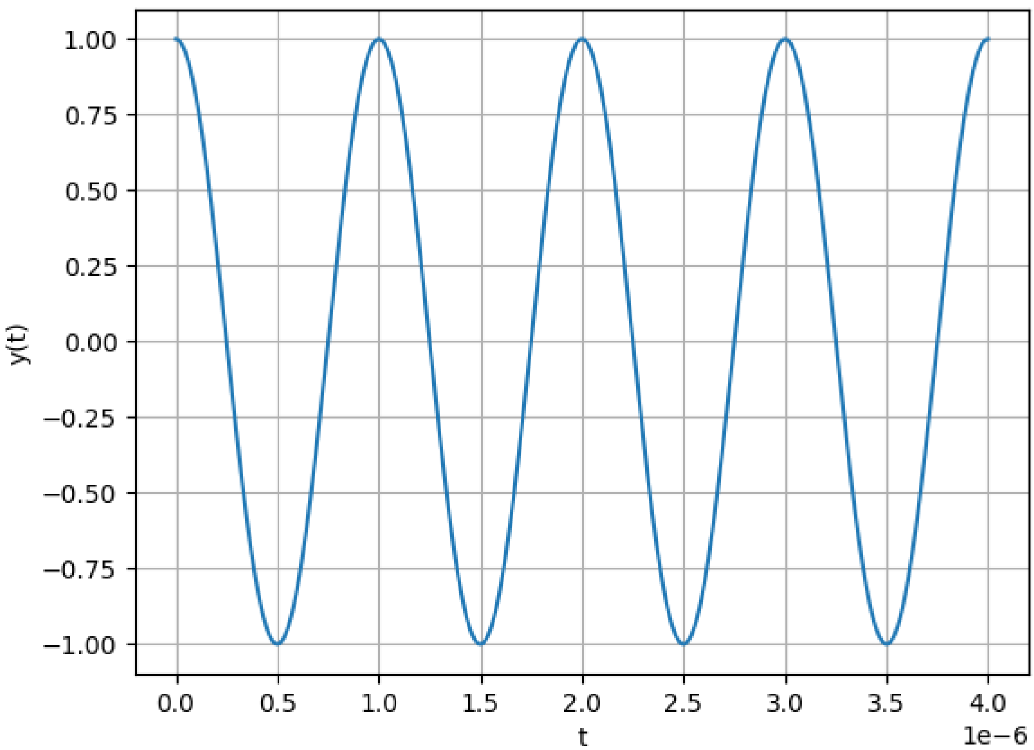


Figure 1.1.2: $y(t) = \cos\left(2\pi \times 10^9 t\right)$

Let us assume an equation:

$$y = A(x)\cos\left(\omega t + \phi\left(x\right)\right) \tag{1.1}$$

Fig. 1.1.3 and Fig. 1.1.3 are self explanatory for stationary and travelling waves.Fig. 1.1.3 and Fig. 1.1.3 are neither stationary nor travelling waves.

STATIONARY WAVE CONDITION	TRAVELLING WAVE CONDITION
(1) $A(x)$ should be a function of position x, and it can be expressed as $A(x) = A_0 cos(\omega t + \alpha)$ where A_0 is a constant, k is the wavenumber, x is the position and α is a phase constant.	(1) $A(x)$ should be a constant, and it can be expressed as $A(x) = A_0$ where A_0 is a constant number.
(2) $\phi(x)$ can be expressed as $\phi(x) = c$ where c is a constant.	(2) $\phi(x)$ represents a linear expression in x, and it can be expressed as $\phi(x) = kx + \theta$ where k is the wavenumber and θ is the phaseconstant.

Table 1.1.3: Travelling wave *vs* Stationary wave

1.1.4 For the travelling harmonic wave $y\,(x, t) = 2.0 \cos 2\pi\,(10t - 0.0080x + 0.35)$ where x and y are in cm and t in s. Calculate the phase difference between oscillatory motion of two points separated by a distance of

(a) $4m$

(b) $0.5m$

(c) $\lambda/2$

(d) $3\lambda/4$

Solution:

$$(\Delta\theta) = (2\pi f t - kx_1 + \phi) - (2\pi f t - kx_2 + \phi) \tag{1.2}$$

$$= k\,(x_2 - x_1) \tag{1.3}$$

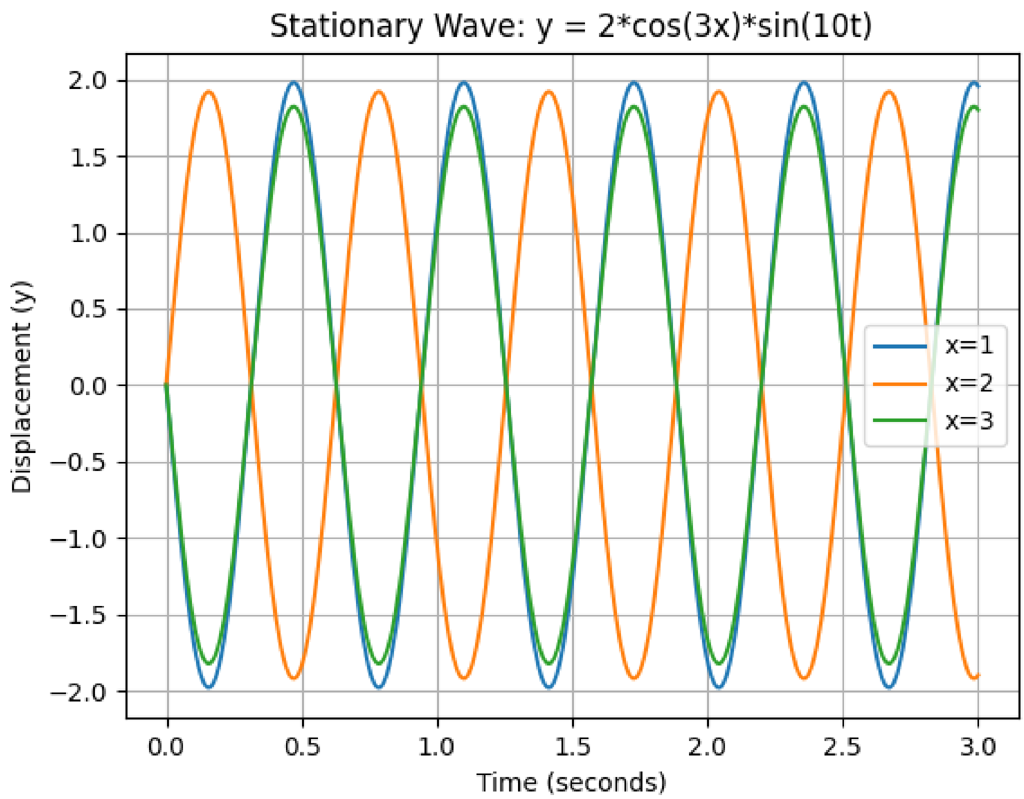


Figure 1.1.3: DIPLACEMENT *vs* TIME-graph1

1.1.5 (a) The peak voltage of an AC supply is 300 V. What is the rms voltage?

(b) The rms value of current in an AC circuit is 10 A. What is the peak current?

Solution:

Parameter	**Description**	**Value**
$y(x_i, t)$	equation of harmonic wave	$A\cos(2\pi ft - kx_i + \phi)$
k	angular wave number	$2\pi\,(0.008)$
$\lambda = \frac{2\pi}{k}$	wavelength	$125\,cm$
f	frequency	10
A	amplitude	2.0
ϕ	phase constant	$2\pi\,(0.35)$
θ_i	phase of i^{th} harmonic wave	$(2\pi ft - kx + \phi)$
x_i	position of i^{th} harmonic wave	
t	time	
$x_2 - x_1$	path difference	$400\,cm$
		$50\,cm$
		$\frac{\lambda}{2}$
		$\frac{3\lambda}{4}$

Table 1.1.4: Given parameters list

Parameter	**Description**	**subquestion**	**Value**
$\Delta\theta$	$\theta_1 - \theta_2$	(a)	6.4π radians
		(b)	0.8π radians
		(c)	π radians
		(d)	$\frac{3\pi}{2}$ radians

Table 1.1.4: Phase differences

parameter	value	description
$V(t)$	$V_0 \cdot \sin(2\pi ft + \phi)$	voltage in terms of time
$I(t)$	$I_0 \cdot \sin(2\pi ft + \phi)$	current in terms of time
V_0	$300\,\text{V}$	peak voltage
V_{rms}	$\sqrt{\frac{1}{T}\int_0^T [V(t)]^2\,dt}$	rms value of Voltage
I_{rms}	$10\,\text{A}$	rms value of current
I_0	$\sqrt{2} \times I_{\text{rms}}$	peak current
f	$50\,\text{Hz}$	frequency of the sinusoidal wave
T	$0.02\,\text{s}$	time period of sinusoidal wave

Table 1.1.5: Input Parameter Table

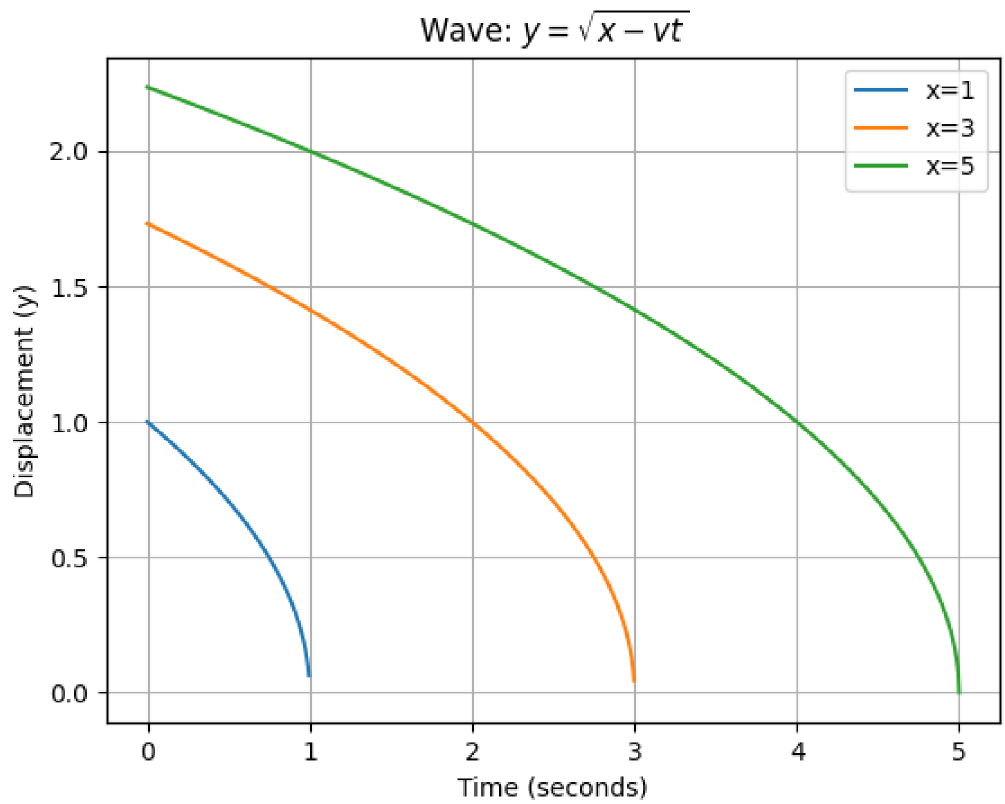


Figure 1.1.3: DIPLACEMENT vs TIME-graph2

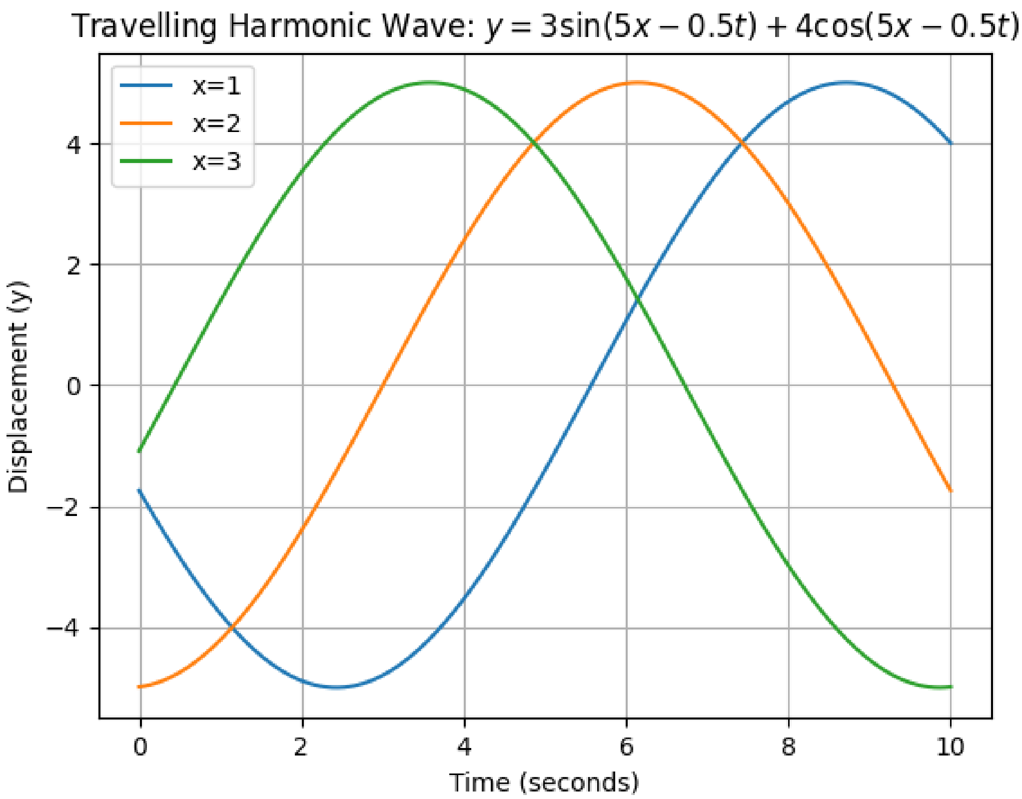


Figure 1.1.3: DIPLACEMENT vs TIME-graph3

(a)

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T [V(t)]^2\,dt \tag{1.4}$$

$$= f\int_0^{\frac{1}{f}} V_0^2 \cdot \sin^2(2\pi f t + \phi)\,dt \tag{1.5}$$

$$= \frac{1}{2}V_0^2\left(1 - \frac{1}{f}\int_0^{\frac{1}{f}} \cos(4\pi f t + 2\phi)\,dt\right) \tag{1.6}$$

$$= \frac{1}{2}V_0^2\left(1 - \frac{1}{f}\left[\frac{\sin(4\pi f t + 2\phi)}{4\pi f}\right]_0^{\frac{1}{f}}\right) \tag{1.7}$$

$$= \frac{1}{2}V_0^2\left(1 - \frac{1}{f}\cdot\frac{\sin\left(4\pi + 2\phi\right) - \sin(0 + 2\phi)}{4\pi f}\right) \tag{1.8}$$

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}} \tag{1.9}$$

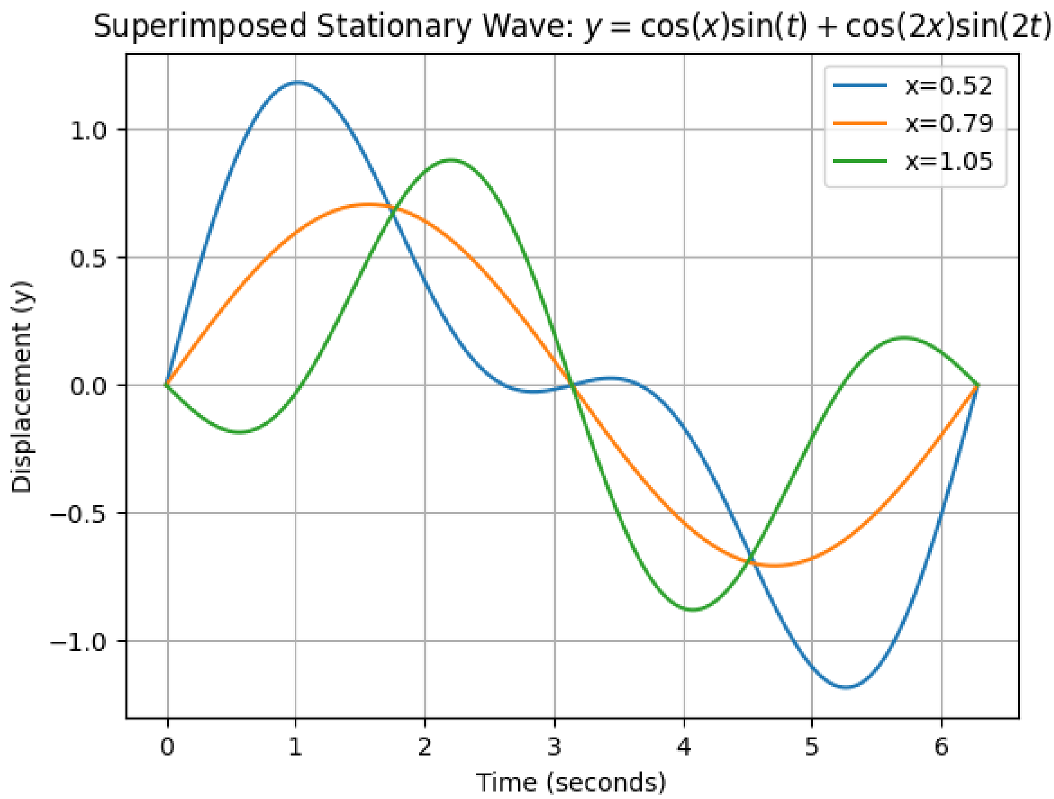


Figure 1.1.3: DIPLACEMENT *vs* TIME-graph4

To find the RMS voltage (V_{rms}) when the peak voltage (V_0) is 300V, you can use equation (1.9)

$$V_{\text{rms}} = \frac{300V}{\sqrt{2}} \approx 212.13V \tag{1.10}$$

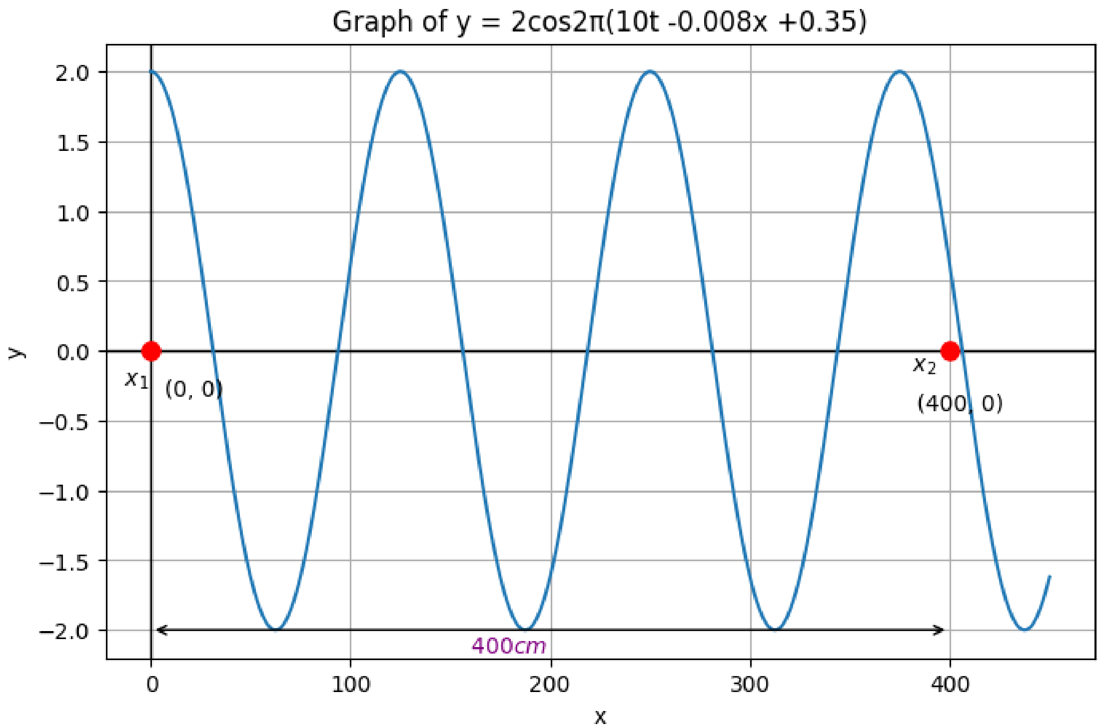


Figure 1.1.4:

(b)

$$I_{\text{rms}}^2 = \frac{1}{T}\int_0^T [I(t)]^2\, dt \tag{1.11}$$

$$= f\int_0^{\frac{1}{f}} I_0^2 \cdot \sin^2(2\pi f t + \phi)\, dt \tag{1.12}$$

$$= \frac{1}{2}I_0^2\left(1 - \frac{1}{f}\left[\frac{\sin(4\pi f t + 2\phi)}{4\pi f}\right]_0^{\frac{1}{f}}\right) \tag{1.13}$$

$$= \frac{1}{2}I_0^2\left(1 - \frac{1}{f}\cdot\frac{\sin(4\pi + 2\phi) - \sin(0 + 2\phi)}{4\pi f}\right) \tag{1.14}$$

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}} \tag{1.15}$$

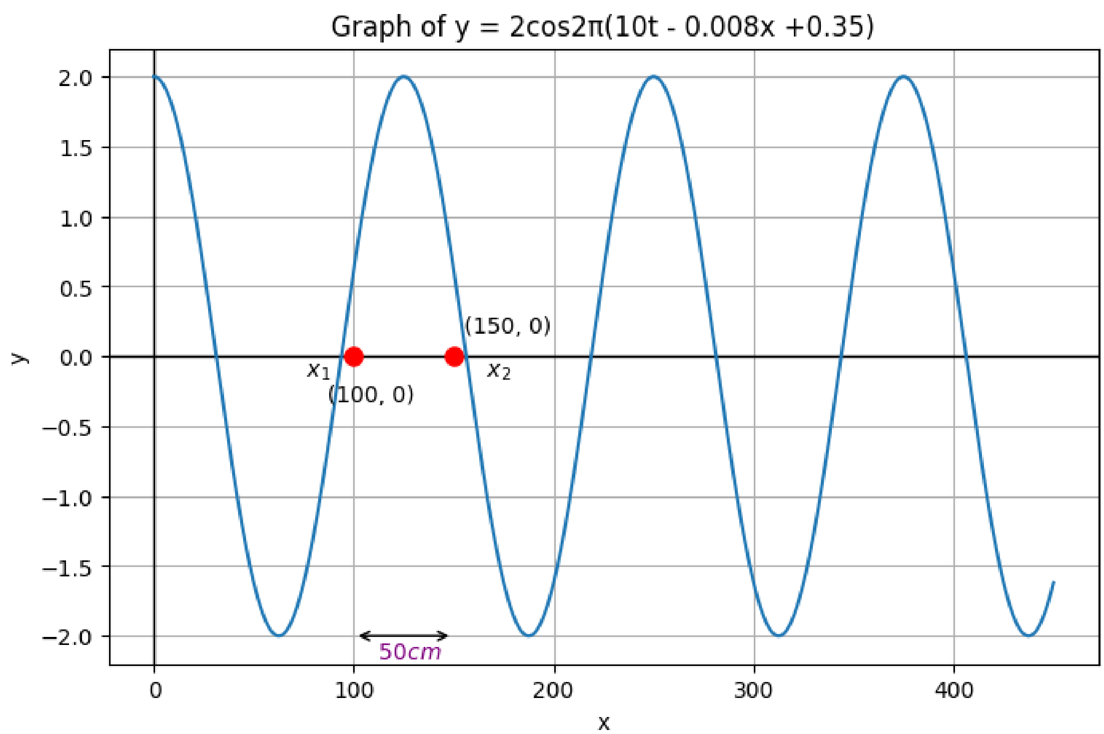


Figure 1.1.4:

To find the peak current (I_0) when the RMS current (I_{rms}) is given, you can use equation (1.15)

$$I_0 \approx 10\,\text{A} \times 1.414 \approx 14.14\,\text{A} \tag{1.16}$$

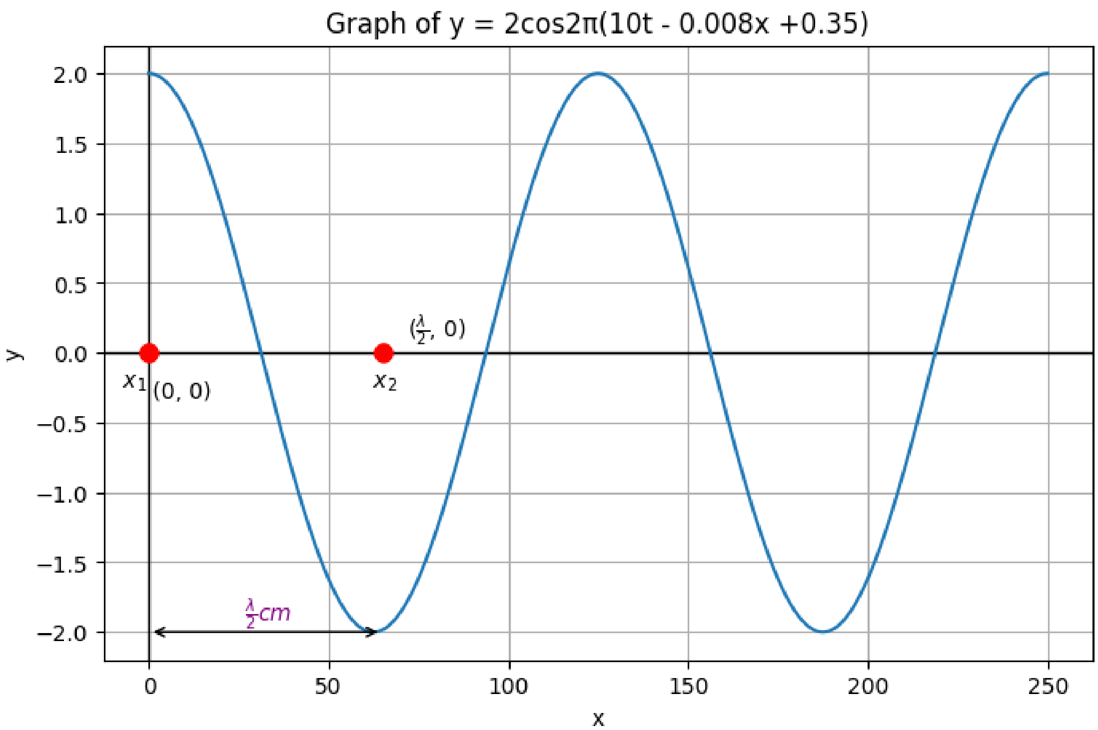
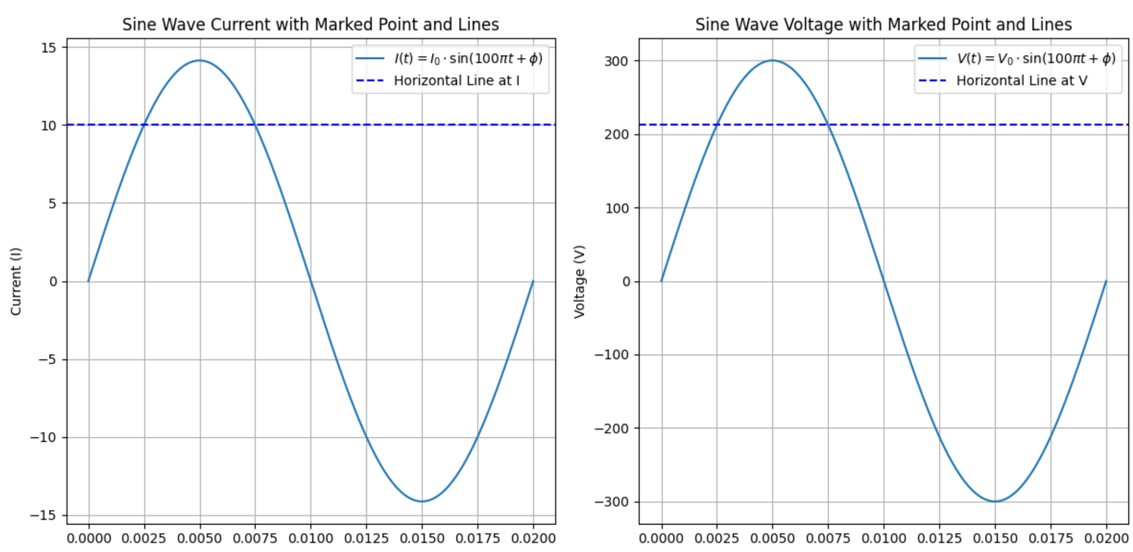

Graph of y = 2cos2π(10t - 0.008x +0.35)
y
x
$(\frac{\lambda}{2}, 0)$
x_1 (0, 0)
x_2
$\frac{\lambda}{2}cm$

Figure 1.1.4:


Sine Wave Current with Marked Point and Lines
$I(t) = I_0 \cdot \sin(100\pi t + \phi)$
Horizontal Line at I
Current (I)
Time (s)
Sine Wave Voltage with Marked Point and Lines
$V(t) = V_0 \cdot \sin(100\pi t + \phi)$
Horizontal Line at V
Voltage (V)
Time (s)

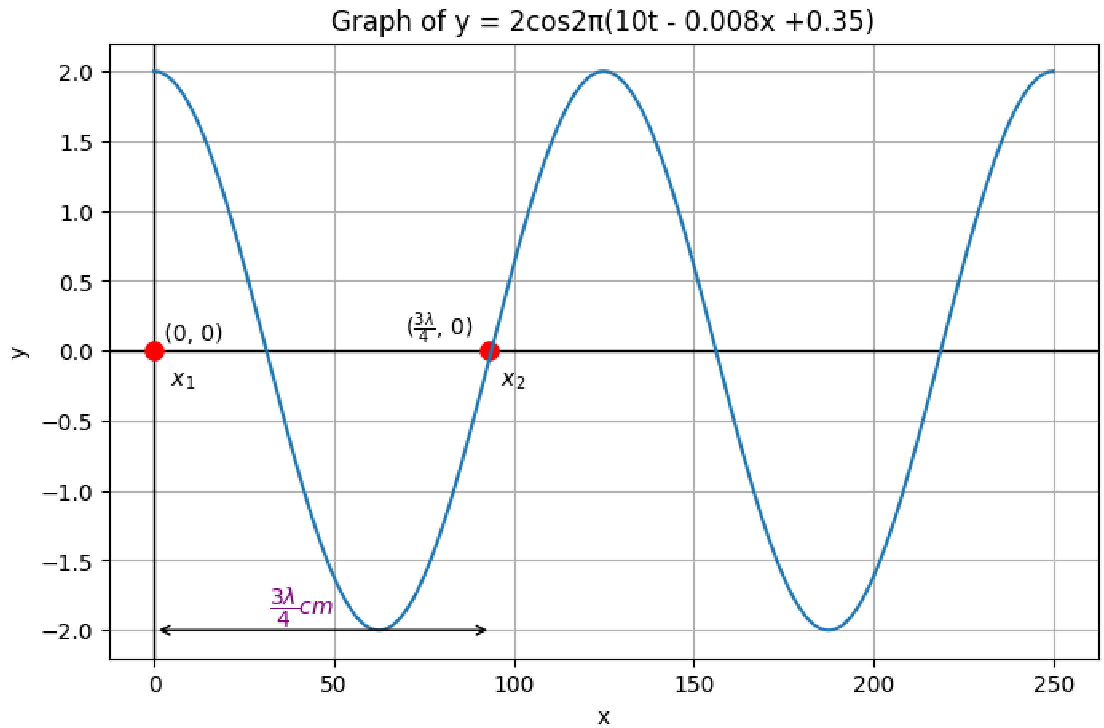


Figure 1.1.4:

1.1.6 In Young's double-slit experiment using monochromatic light of wavelength λ, the intensity of light at a point on the screen where path difference is λ, is K units. What is the intensity of light at a point where path difference is $\lambda/3$?

Solution:

From Table 1.1.6:

$$y\left(t\right) = A\sin(2\pi ft - kx_1) + A\sin(2\pi ft - kx_2) \tag{1.17}$$

$$y\left(t\right) = 2A\cos\left(\frac{k\Delta x}{2}\right)\sin\left(2\pi ft - \frac{k(x_1 + x_2)}{2}\right) \tag{1.18}$$

Parameter	**Description**	**Value**
$y_i\left(t\right)$	Equation of light from $S_{i^{\text{th}}}$	$A\sin(\omega t - kx_i)$
k	Wave number	$\frac{2\pi}{\lambda}$
I	Intensity of wave	$\propto A^2$
$\Delta x = x_1 - x_2$	Path difference	λ
		$\frac{\lambda}{3}$
K	Intensity of light at $\Delta x = \lambda$	
A	Amplitude of wave from source	
r	constant	$r \geq 0$

Table 1.1.6: Parameters

From Table 1.1.6 and equation (1.18):

$$\therefore I \propto 4A^2\cos^2\left(\frac{k\Delta x}{2}\right) \tag{1.19}$$

From Table 1.1.6 and equation (1.19):

$$\frac{K}{I_r} = \frac{4A^2\cos^2\left(\frac{2\pi}{2}\right)}{4A^2\cos^2\left(\frac{\pi}{3}\right)} \implies I_r = \frac{K}{4} \tag{1.20}$$

Hence, the Intensity of light at a point where path difference is $\frac{\lambda}{3}$ is $\frac{K}{4}$ units.

Parameter	**Description**	**Value**
I_r	Net Intensity of light at $\Delta x = \frac{\lambda}{3}$	$\frac{K}{4}$

Table 1.1.6:

Assuming $\Delta x = r\lambda$,

From equation (1.19):

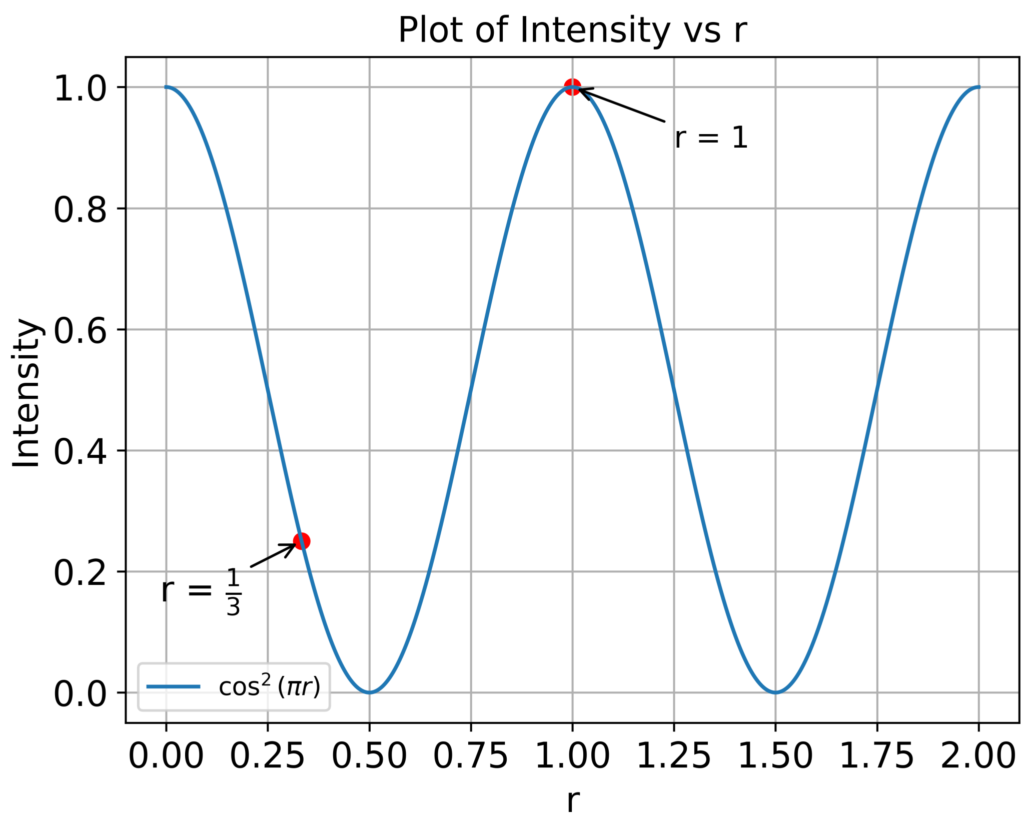


Figure 1.1.6:

1.2. Filters

1.2.1 Obtain the resonant frequency and Q-factor of a series LCR circuit with $L = 3.0\,H$, $C = 27\,\mu F$, and $R = 7.4\,\Omega$. It is desired to improve the sharpness of the resonance of the circuit by reducing its 'full width at half maximum' by a factor of 2. Suggest a suitable way.

Solution: Given parameters are:

Symbol	**Value**	**Description**
L	$3.0\,\text{H}$	Inductance
C	$27\,\mu\text{F}$	Capacitance
R	$7.4\,\Omega$	Resistance
Q		Quality Factor: ratio of voltage across inductor or capacitor to that across the resistor at resonance
ω_0	$\frac{1}{\sqrt{LC}}$	Angular Resonant Frequency

Table 1.11: Given Parameters

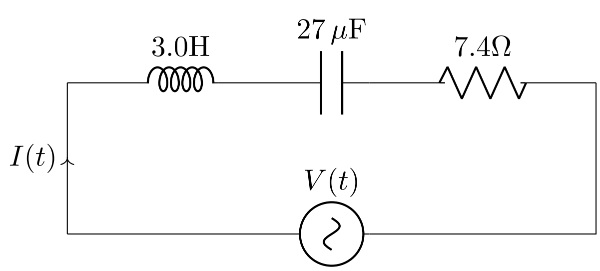


Figure 1.12: LCR Circuit

(a) Frequency Response of the Circuit

From Kirchhoff's Voltage Law (KVL):

$$V(t) = V_R + V_L + V_C \tag{1.21}$$

Using reactances from Fig. 1.13,

$$V(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s) \tag{1.22}$$

$$= I(s)\left(R + Ls + \frac{1}{sC}\right) \tag{1.23}$$

$$\implies I(s) = \frac{V(s)}{\left(R + Ls + \frac{1}{sC}\right)} \tag{1.24}$$

At resonance, the circuit becomes purely resistive. The reactances of capacitor

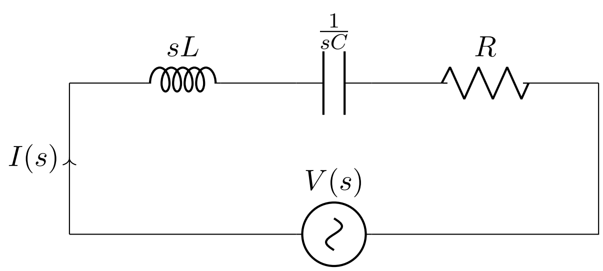


Figure 1.13: LCR Circuit

and inductor cancel out as follows:

$$Ls + \frac{1}{sC} = 0 \tag{1.25}$$

$$\implies s = j\frac{1}{\sqrt{LC}} \tag{1.26}$$

s can be expressed in terms of angular resonance frequency as

$$s = j\omega_0 \tag{1.27}$$

Comparing (1.26) and (1.27), we get

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{1.28}$$

(b) Quality Factor

i. Using voltage across inductor,

$$Q = \left(\frac{V_L}{V_R}\right)_{\omega_0} = \frac{|sLI(s)|}{|RI(s)|} \tag{1.29}$$

$$= \frac{1}{\sqrt{LC}}\frac{L}{R} \tag{1.30}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}} \tag{1.31}$$

ii. Using voltage across capacitor,

$$Q = \left(\frac{V_C}{V_R}\right)_{\omega_0} = \frac{\left|\frac{I(s)}{sC}\right|}{|RI(s)|} \tag{1.32}$$

$$= \frac{\sqrt{LC}}{RC} \tag{1.33}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}} \tag{1.34}$$

(c) Plot of Impedance vs Angular Frequency

Impedance is defined as

$$H(s) = \frac{V(s)}{I(s)} \tag{1.35}$$

Using (1.24),

$$H(s) = R + sL + \frac{1}{sC} \tag{1.36}$$

$$\implies H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \tag{1.37}$$

$$\implies |H(j\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{1.38}$$

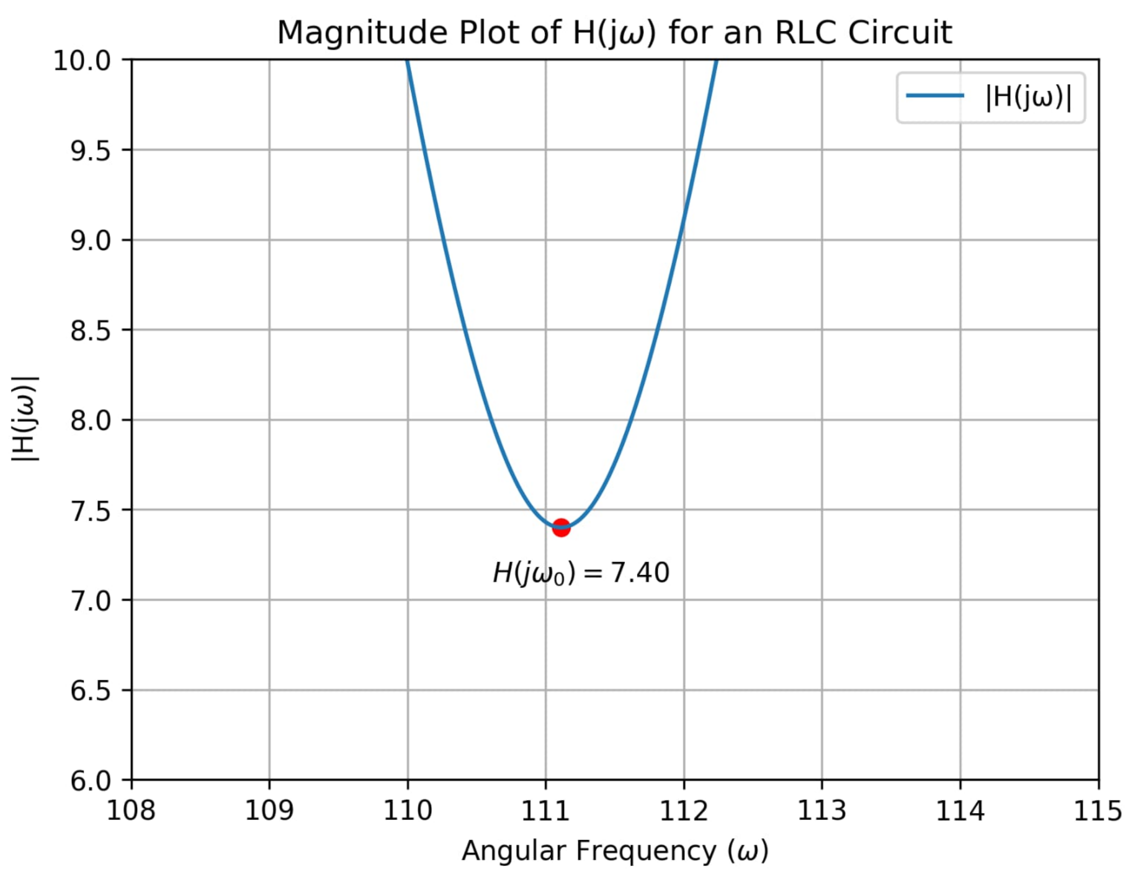


Figure 1.14: Impedance vs ω (using values in Table 1.11)

Chapter 2

Discrete

2.1. Z-transform

2.1.1 Show that

$$\frac{1\times 2^2+2\times 3^2+\cdots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\cdots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$$

Solution:

Parameter	Description	Value
n	Integer	.... -2,-1,0,1, 2, ...
$x_1(n)$	General term of Numerator	$\left(n^3+5n^2+8n+4\right)\cdot u\left(n\right)$
$x_2(n)$	General Term of Denominator	$\left(n^3+4n^2+5n+2\right)\cdot u\left(n\right)$
$y_1\left(n\right)$	Sum of terms of numerator	?
$y_2\left(n\right)$	Sum of terms of denominator	?
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}}, \{z\in\mathbb{C}:\lvert z\rvert>1\}$
ROC	Region of convergence	$\left\{z:\left\lvert\sum_{n=-\infty}^{\infty}x(n)z^{-n}\right\rvert<\infty\right\}$

Table 1: Parameter Table

1. Analysis of Numerator:

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n)\, z^{-n} \tag{2.1}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 5n^2 + 8n + 4\right) u(n)\, z^{-n} \tag{2.2}$$

Using results of equations (B.3.2) to (B.3.5) we get:

$$\therefore X_1(z) = \frac{4 + 2z^{-1}}{\left(1 - z^{-1}\right)^4}, |z| > 1 \tag{2.3}$$

From (A.3.2)

$$y_1(n) = x_1(n) * u(n) \tag{2.4}$$

$$Y_1(z) = X_1(z)\, U(z) \tag{2.5}$$

$$= \frac{4 + 2z^{-1}}{\left(1 - z^{-1}\right)^5}, |z| > 1 \tag{2.6}$$

Using partial fractions:

$$\begin{aligned} Y_1(z) = \frac{22z^{-1}}{(1 - z^{-1})} + \frac{48z^{-2}}{(1 - z^{-1})^2} + \frac{52z^{-3}}{(1 - z^{-3})^3}, \\ + \frac{28z^{-4}}{(1 - z^{-1})^4} + \frac{6z^{-5}}{(1 - z^{-1})^5} + 4, |z| > 1 \end{aligned} \tag{2.7}$$

Substituting results of equation (B.4.6) to (B.4.9) in equation (2.7):

$$y_1(n) = \frac{3n^4 + 26n^3 + 81n^2 + 106n + 48}{12} u(n) \tag{2.8}$$

$$= \frac{(3n + 8)(n + 1)(n + 2)(n + 3)}{12} u(n) \tag{2.9}$$

2. Analysis of Denominator:

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \tag{2.10}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 4n^2 + 5n + 2\right) u(n) z^{-n} \tag{2.11}$$

Using results of equation (B.3.2) to (B.3.5) we get:

$$\therefore X_2(z) = \frac{2 + 4z^{-1}}{(1 - z^{-1})^4}, |z| > 1 \tag{2.12}$$

From (A.3.2)

$$y_2(n) = x_2(n) * u(n) \tag{2.13}$$

$$Y_2(z) = X_2(z) U(z) \tag{2.14}$$

$$= \frac{2 + 4z^{-1}}{(1 - z^{-1})^5}, |z| > 1 \tag{2.15}$$

Using partial fractions:

$$\begin{aligned} Y_2(z) = &\frac{14z^{-1}}{(1 - z^{-1})} + \frac{36z^{-2}}{(1 - z^{-1})^2} + \frac{44z^{-3}}{(1 - z^{-3})^3} \\ &+ \frac{26z^{-4}}{(1 - z^{-1})^4} + \frac{6z^{-5}}{(1 - z^{-1})^5} + 2, |z| > 1 \end{aligned} \tag{2.16}$$

Substituting results of equation (B.4.6) to (B.4.9) in equation (2.16):

$$y_2(n) = \frac{3n^4 + 22n^3 + 57n^2 + 62n + 24}{12} u(n) \tag{2.17}$$

$$= \frac{(3n + 4)(n + 1)(n + 2)(n + 3)}{12} u(n) \tag{2.18}$$

As the sequence start from $n = 0$, in RHS of question n should be replaced by $n + 1$:

$$\frac{y_1(n)}{y_2(n)} = \frac{3n + 8}{3n + 4} \tag{2.19}$$

Hence Prooved.

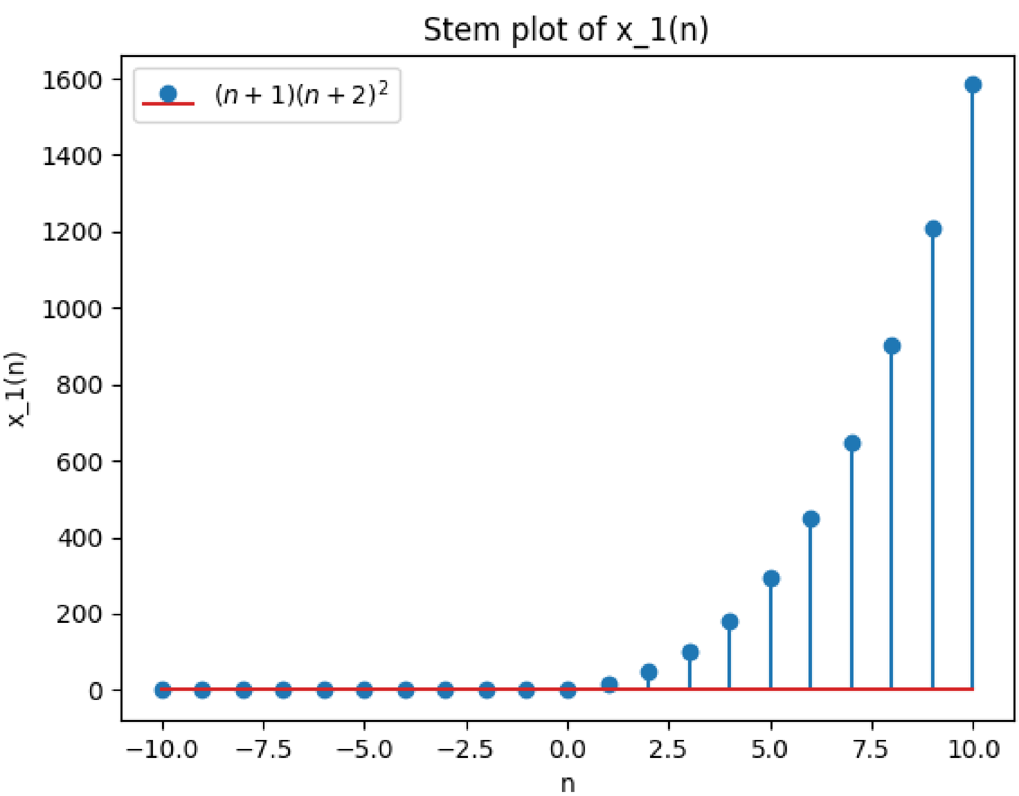


Figure 2.1: Stem Plot of $x_1(n)$

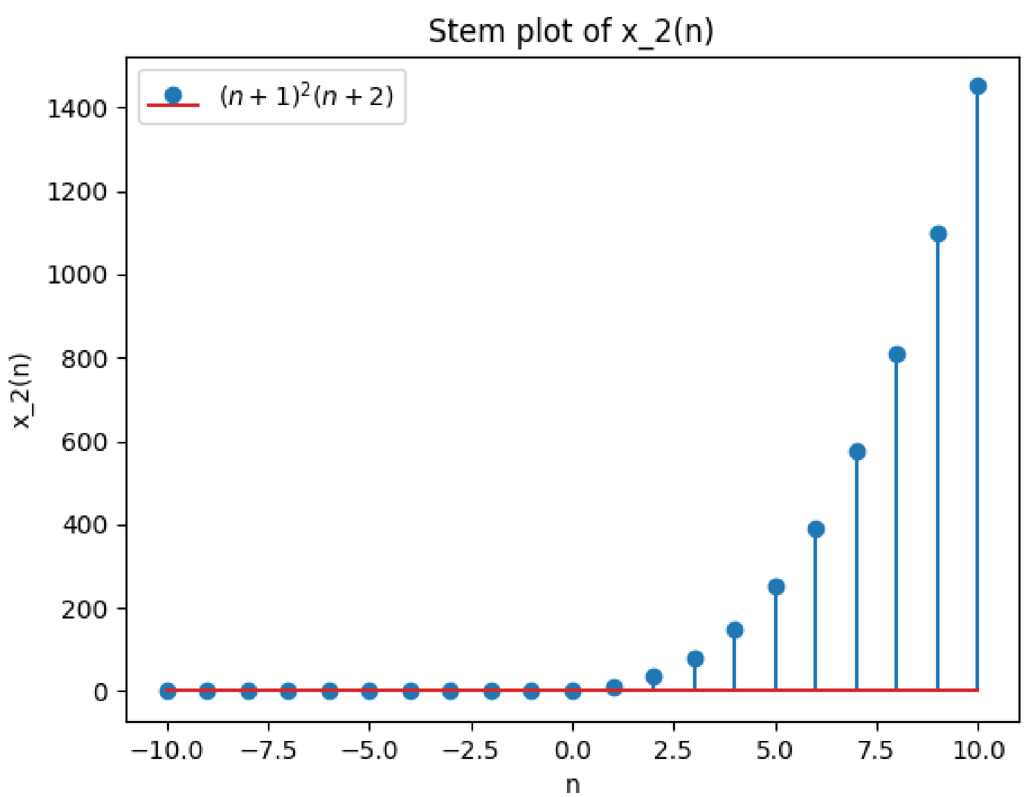


Figure 2.2: Stem Plot of $x_2(n)$

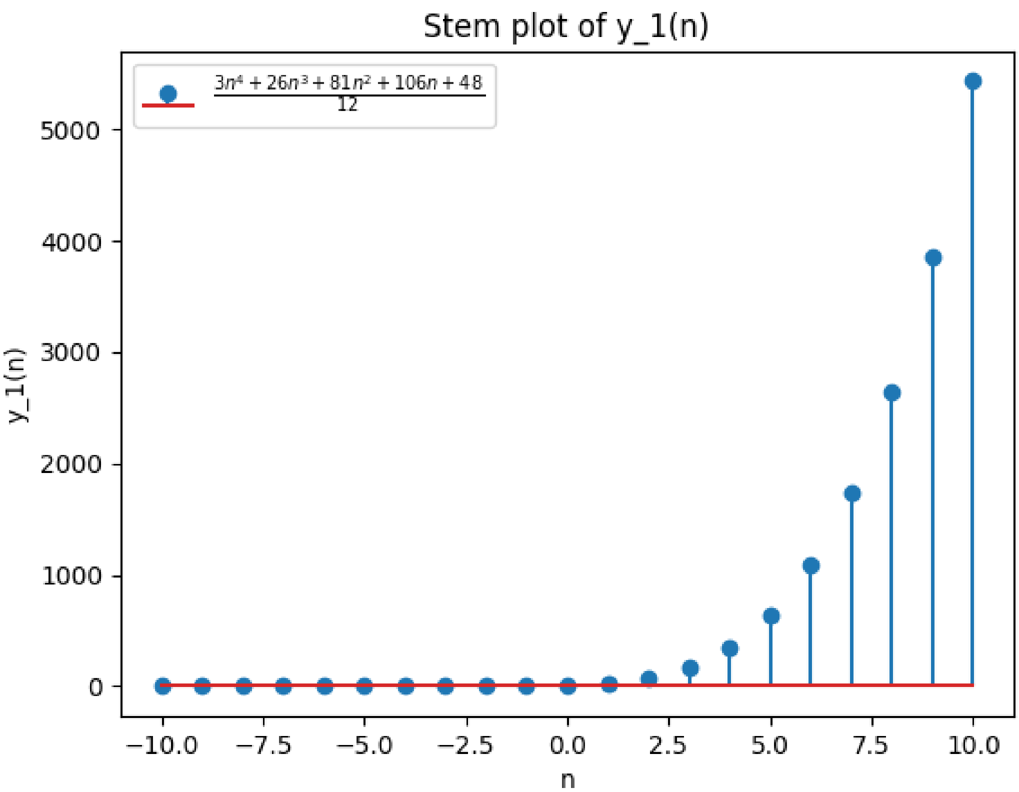


Figure 2.3: Stem Plot of $y_1(n)$

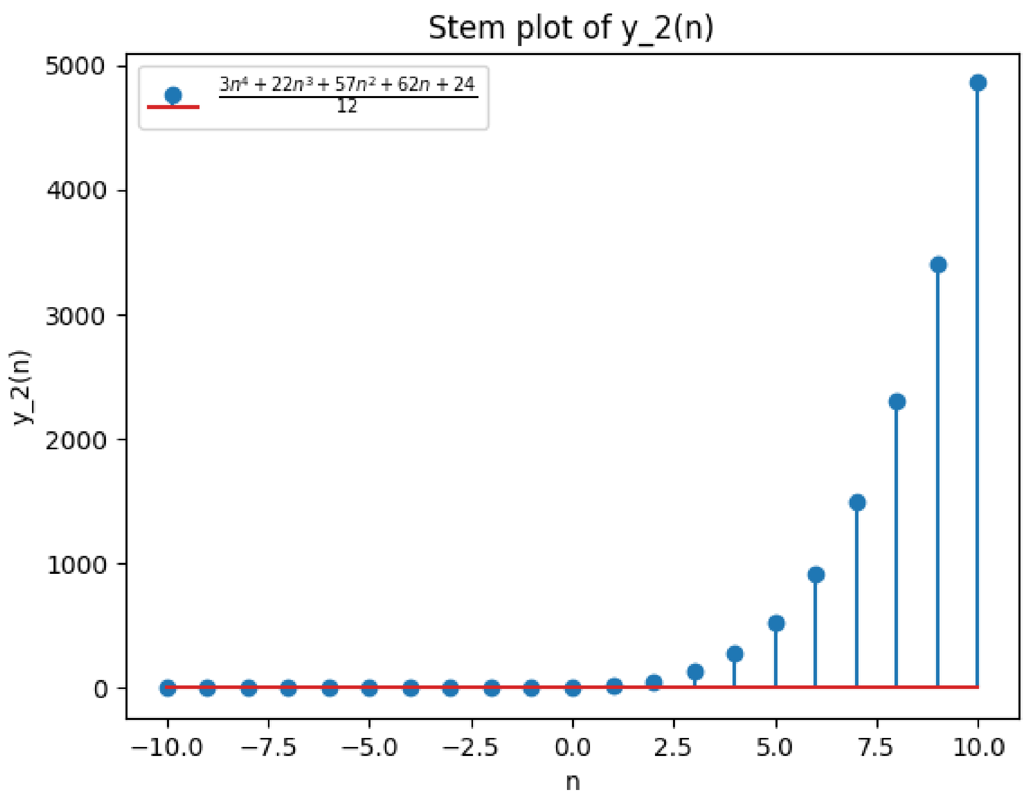


Figure 2.4: Stem Plot of $y_2(n)$

2.1.2 Write the five terms at n = 1, 2, 3, 4, 5 of the sequence and obtain the Z-transform of the series

$$x(n) = -1, \qquad n = 0 \tag{2.20}$$

$$= \frac{x(n-1)}{n}, \qquad n > 0 \tag{2.21}$$

$$= 0, \qquad n < 0 \tag{2.22}$$

Solution:

$$x(1) = \frac{x(0)}{1} = -1 \tag{2.23}$$
$$x(2) = \frac{x(1)}{2} = -\frac{1}{2} \tag{2.24}$$
$$x(3) = \frac{x(2)}{3} = -\frac{1}{(2)(3)} = -\frac{1}{6} \tag{2.25}$$
$$x(4) = \frac{x(3)}{4} = -\frac{1}{(2)(3)(4)} = -\frac{1}{24} \tag{2.26}$$
$$x(5) = \frac{x(4)}{5} = -\frac{1}{(2)(3)(4)(5)} = -\frac{1}{120} \tag{2.27}$$
$$x(n) = \frac{-1}{n!}(u(n)) \tag{2.28}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{2.29}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \tag{2.30}$$

using (2.28),

$$= \sum_{n=-\infty}^{\infty} \frac{-1}{n!} u(n)\, z^{-n} \tag{2.31}$$
$$= \sum_{n=0}^{\infty} \frac{-1}{n!} z^{-n} \tag{2.32}$$
$$= -e^{z^{-1}} \qquad \{z \in \mathbb{C} : z \neq 0\} \tag{2.33}$$

Symbol	**Value**	**Description**
$x(n)$	$\frac{-1}{n!}$	general term of the series
$X(z)$	$-e^{z^{-1}}$	Z-transform of x(n)
$u(n)$		unit step function

Table 2.2: Parameters

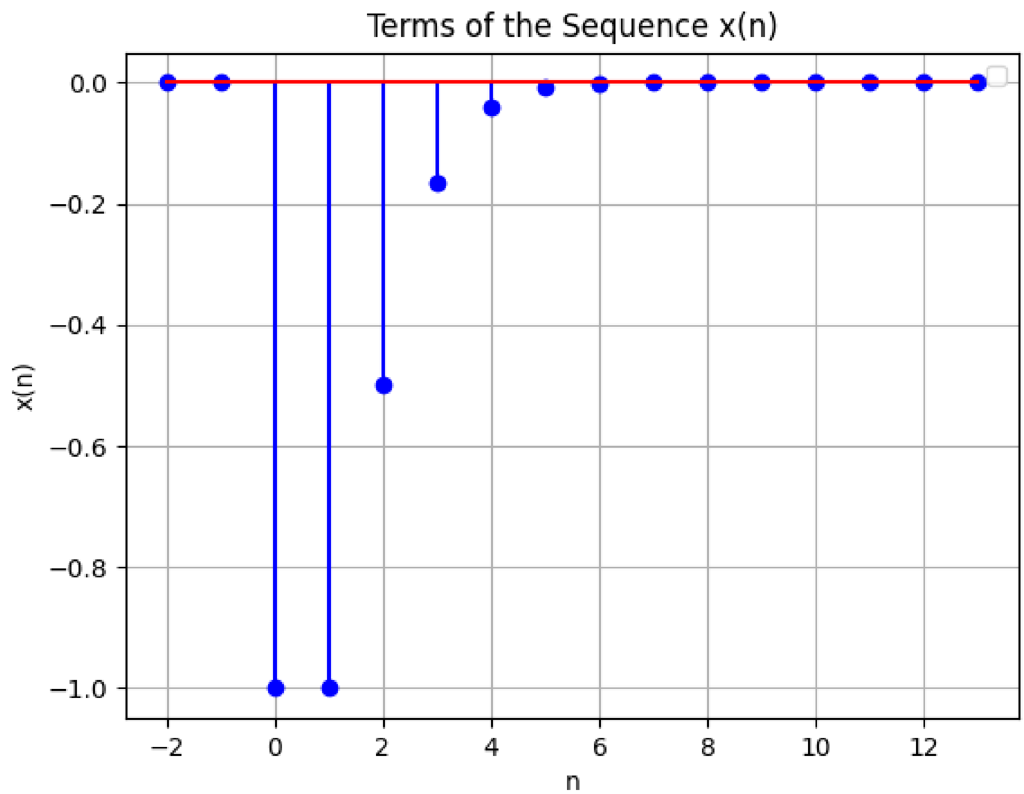


Figure 2.5: Plot of x(n) vs n

2.1.3 Subba Rao started work in 1995 at an annual salary of Rs. 5000 and received an increment of Rs. 200 each year. In which year did his income reach Rs. 7000?

Solution:

Parameter	Value	Description
$x(0)$	5000	Initial Income
d	200	Annual Increment (Common Difference)
$x(n)$	$(x(0) + nd)u(n)$	n^{th} term of the AP

Table 2.3: Input Parameters

From the values given in Table 2.3:

$$7000 = 5000 + 200n \tag{2.34}$$

$$\implies 2000 = 200n \tag{2.35}$$

$$\therefore n = 10 \tag{2.36}$$

Let Z-transform of $x(n)$ be $X(z)$.

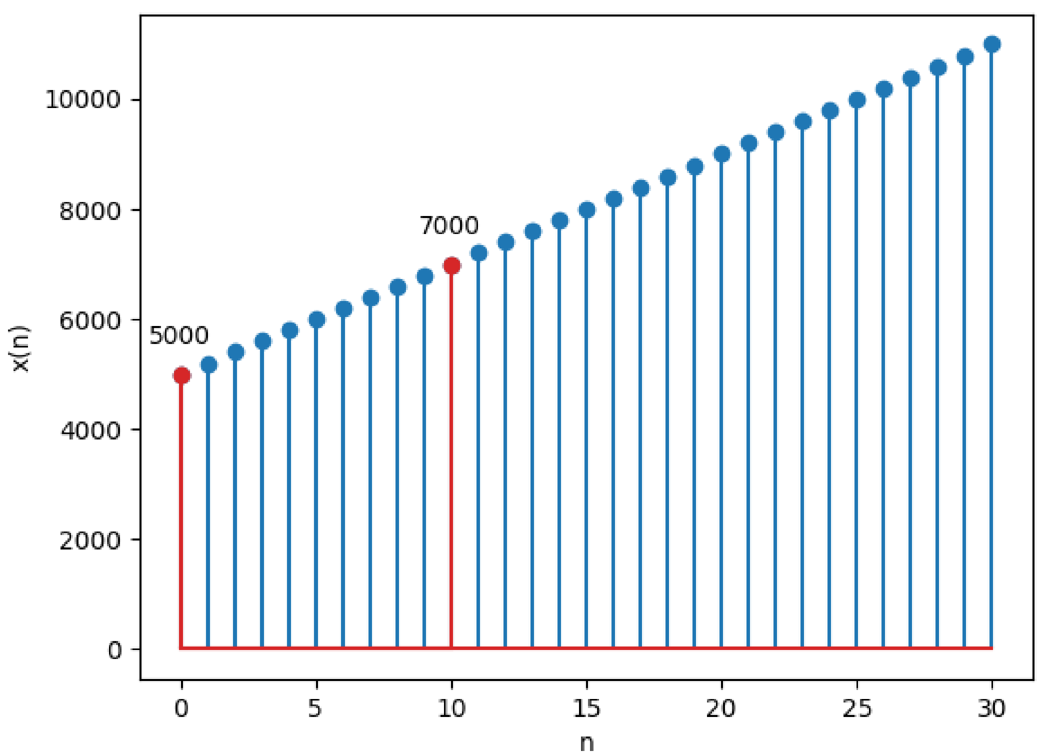


Figure 2.6: Plot of $x(n)$ vs n. See Table 2.3 for details.

$$X(z) = \frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{2.37}$$

Using the values from Table 2.3:

$$X(z) = \frac{5000}{1-z^{-1}} + \frac{200z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{2.38}$$

2.1.4 Consider the sequence whose n^{th} term is given by 2^n. Find the first 6 terms of this sequence.

Solution:

Variable	**Description**	**Value**
$x(n)$	general term of sequence	$2^n u(n)$

Table 2.4: input parameters

$$X(Z) = \frac{1}{1-2z^{-1}} \quad |z| > |2| \tag{2.39}$$

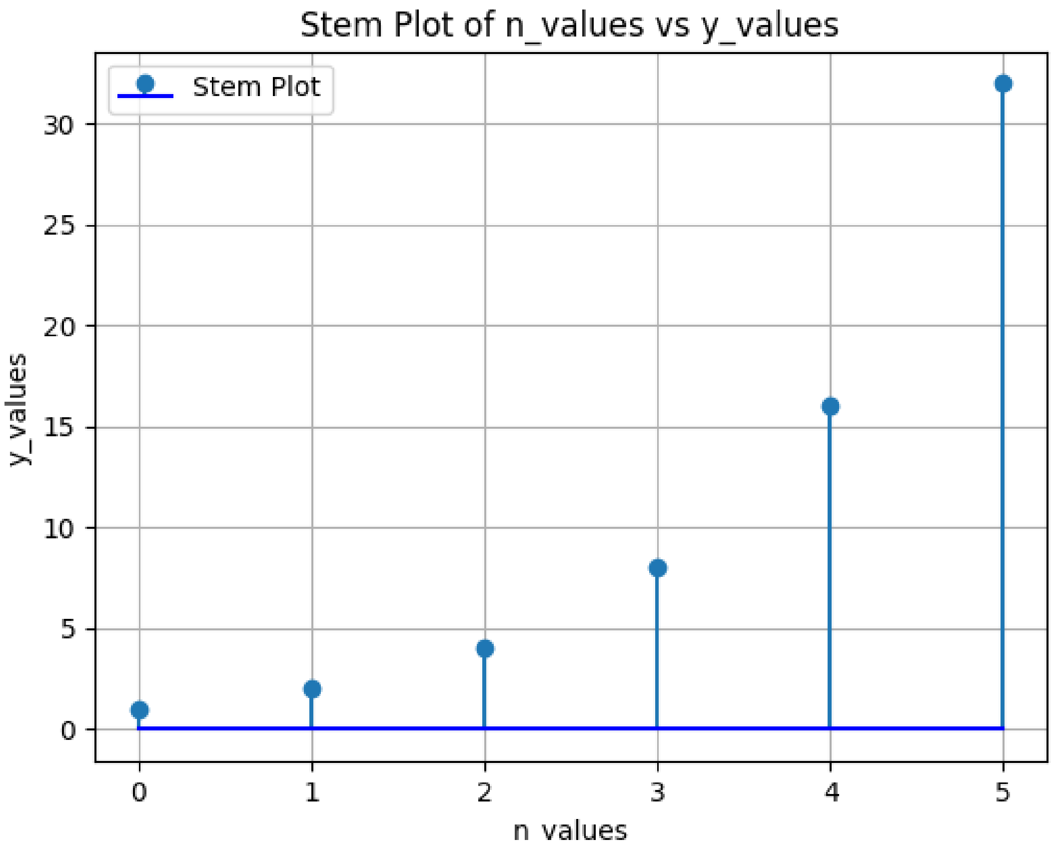


Figure 2.7: Six terms of given sequence

2.1.5 If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289, find the sum of first n terms.

Solution:

Variable	**Description**
$x(0)$	First term of the AP
d	Common difference of the AP
$y(n)$	Sum of $n+1$ terms of the AP
$x(n)$	General term

Table 2.5: Variables Used

$$y(n) = \frac{n+1}{2}\left(2x(0) + nd\right) u(n) \tag{2.40}$$

$$y(6) = 49 \tag{2.41}$$

$$y(16) = 289 \tag{2.42}$$

Then,

$$x(0) + 3d = 7 \tag{2.43}$$

$$x(0) + 8d = 17 \tag{2.44}$$

From equations 2.43 and 2.44, the augmented matrix is:

$$\begin{pmatrix} 1 & 3 & 7 \\ 1 & 8 & 17 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 3 & 7 \\ 0 & 5 & 10 \end{pmatrix} \tag{2.45}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{5} R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & 10 \end{pmatrix} \tag{2.46}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{5}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \tag{2.47}$$

$$\implies \begin{pmatrix} x\left(0\right) \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.48}$$

$$x\left(n\right) = \left(1 + 2n\right) u\left(n\right) \tag{2.49}$$

$$X\left(z\right) = \frac{1}{1 - z^{-1}} + \frac{2z^{-1}}{\left(1 - z^{-1}\right)^2} \quad \left\{z \in \mathbb{C} : \left|z\right| > 1\right\} \tag{2.50}$$

$$y\left(n\right) = x\left(n\right) * u\left(n\right) \tag{2.51}$$

$$Y\left(z\right) = X\left(z\right)\, U\left(z\right) \tag{2.52}$$

$$\implies Y\left(z\right) = \left(\frac{1}{1 - z^{-1}} + \frac{2z^{-1}}{\left(1 - z^{-1}\right)^2}\right)\left(\frac{1}{1 - z^{-1}}\right) \tag{2.53}$$

$$= \frac{1}{\left(1 - z^{-1}\right)^2} + \frac{2z^{-1}}{\left(1 - z^{-1}\right)^3} \tag{2.54}$$

$$\left(n + 1\right) u\left(n\right) \xleftrightarrow{\mathcal{Z}} \frac{1}{\left(1 - z^{-1}\right)^2} \left\{z \in \mathbb{C} : \left|z\right| > 1\right\} \tag{2.55}$$

$$n\left(\left(n + 1\right) u\left(n\right)\right) \xleftrightarrow{\mathcal{Z}} \frac{2z^{-1}}{\left(1 - z^{-1}\right)^3} \left\{z \in \mathbb{C} : \left|z\right| > 1\right\} \tag{2.56}$$

From equations (B.11.1) and (B.11.2), taking the inverse Z Transform,

$$y(n) = (n+1)\,u(n) + n\,((n+1)\,u(n)) \tag{2.57}$$

$$\implies y(n) = (n+1)^2\,u(n) \tag{2.58}$$

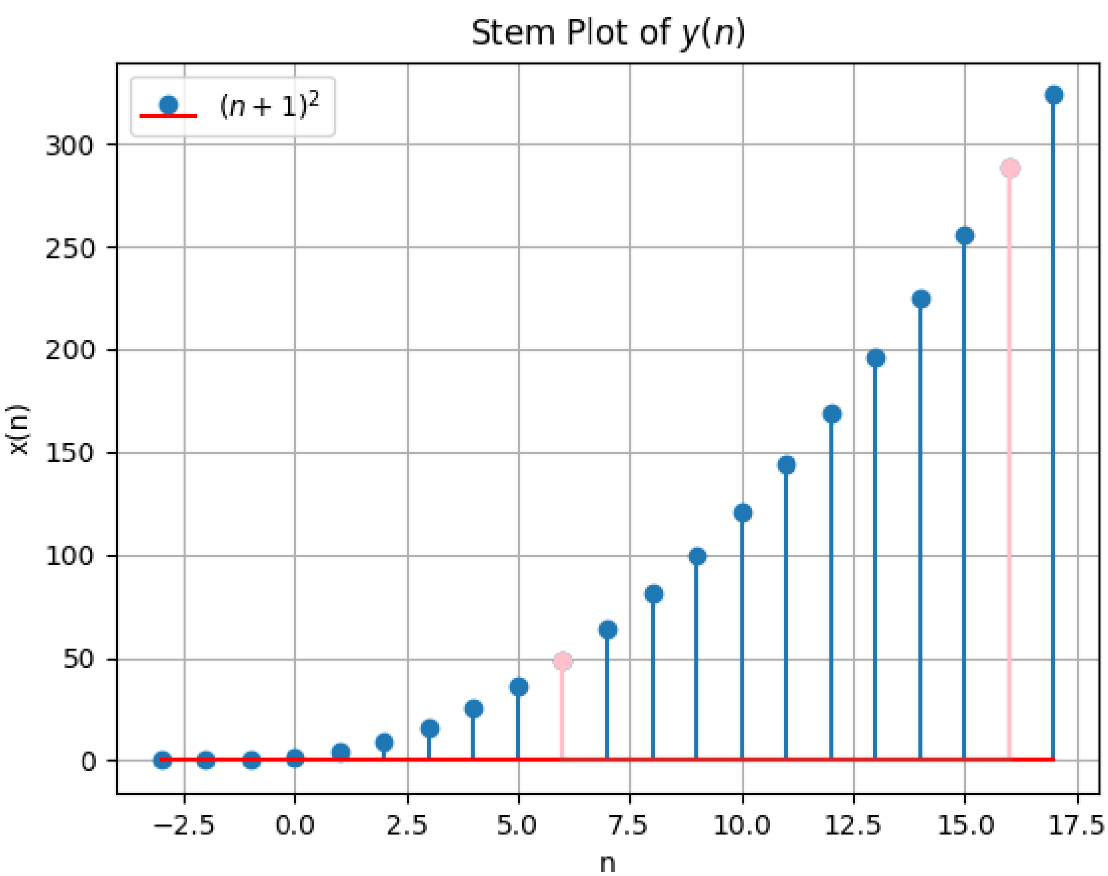


Figure 2.8: Stem Plot of y(n)

2.1.6 Write the first five terms of the sequence and obtain the corresponding series:

$a_1 = a_2 = 2,\, a_n = a_{n-1} - 1,\, n > 2$

Solution:

Parameter	Description	Value
$x(0)$	First term	2
$x(1)$	Second term	2
ROC	Region of convergence	$\{z : \lvert\sum_{n=-\infty}^{\infty} x(n)z^{-n}\rvert < \infty\}$
$x(n)$	General term	$x(n) = \begin{cases} ? & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$

Table 1: Parameter Table

$$x(n) - x(n-1) = 2u(n) - 2u(n-1) - u(n-2) \tag{2.59}$$

$$X(z) - z^{-1}X(z) = \frac{2}{(1-z^{-1})} - \frac{z^{-2}}{(1-z^{-1})} - \frac{2z^{-1}}{(1-z^{-1})} \tag{2.60}$$

$$X(z) = \frac{2 - 2z^{-1} - z^{-2}}{(1-z^{-1})^2}, |z| > 1 \tag{2.61}$$

Using partial fractions

$$X(z) = \frac{2z^{-1}}{(1-z^{-1})} - \frac{z^{-2}}{(1-z^{-1})^2} + 2 \tag{2.62}$$

Taking inverse Z-transform by result of equation (B.4.6) in equation (2.62):

$$x(n) = 2u(n) + (1-n)u(n-1) \tag{2.63}$$

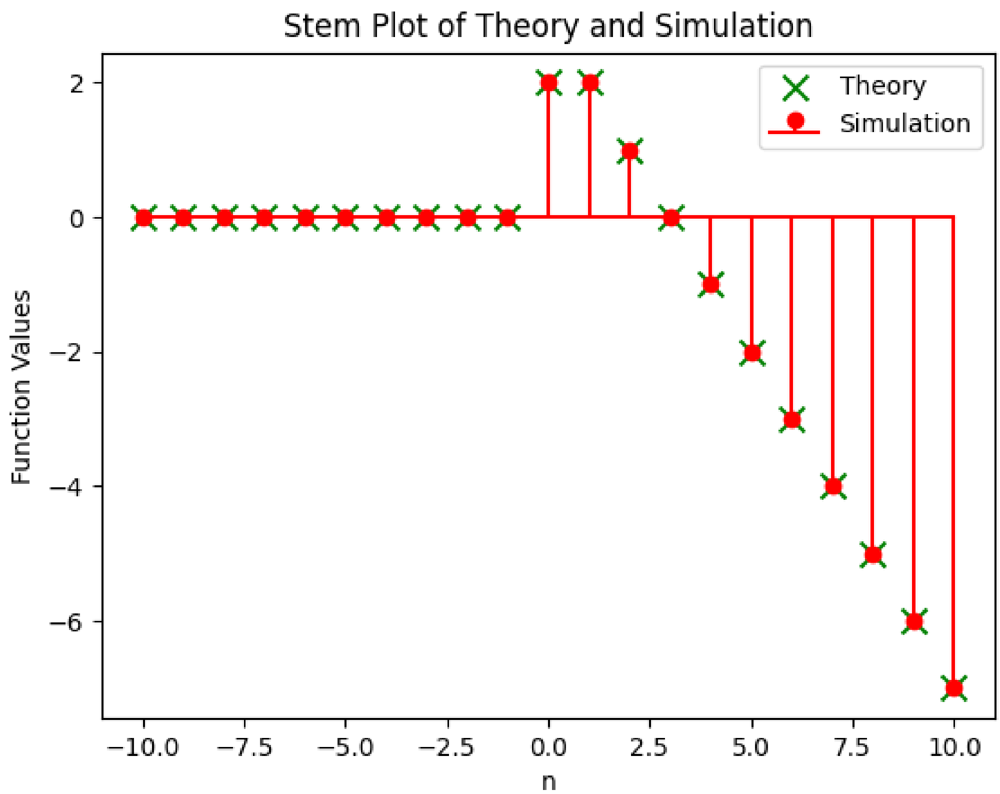


Figure 2.9: Comparison of Theory and Simulated Values

From the figureFig. 2.9 we can see that the theoretical and simulated values overlap.

2.2. Sequences

2.2.1 Find the number of terms in each of the following APs.

(a) 7, 13, 19, ... 205

(b) 18, $15\frac{1}{2}$, 13, ... -47

Solution: The number of terms in the AP x(n) is given by:

Parameter	**Used to denote**	**Values**
$x_i(n)$	n^{th} term of i^{th} series $(i = (1, 2))$	$(x_i(0) + nd_i)\, u(n)$
$x_i(0)$	First term of i^{th} AP	$x_1(0) = 7$ $x_2(0) = 18$
d_i	Commmon difference of i^{th} AP	$d_1 = 6$ $d_2 = -2.5$

Table 2.7: Parameter Table

$$\frac{x(n) - x(0)}{d} + 1 \tag{2.64}$$

$$X_i(z) = \frac{x_i(0)}{1 - z^{-1}} + d_i \frac{z^{-1}}{(1 - z^{-1})^2} \text{ , for i=1,2} \tag{2.65}$$

$$\text{ROC} : |z| > 1 \text{ as it is an AP} \tag{2.66}$$

(a)

$$x_1(n) = (7 + (n)\, 6)\, u(n) \tag{2.67}$$

Using the values in Table 2.7 and equation (2.64),

$$k_1 = \frac{205 - 7}{6} + 1 = 34 \tag{2.68}$$

Using the values in Table 2.7 and equation (2.65) :

$$X_1(z) = \frac{7 - z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{2.69}$$

ROC is $|z| > 1$

(b)

$$x_2(n) = \left(18 + n(-2.5)\,u(n)\right) \tag{2.70}$$

Using the values in Table 2.7 and equation (2.64),

$$k_2 = \frac{-47 - 18}{-2.5} + 1 = 27 \tag{2.71}$$

Using the values in Table 2.7 and equation (2.65) :

$$X_2(z) = \frac{18 - (20.5)\,z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{2.72}$$

ROC is $|z| > 1$.

2.2.2 For what value of n, are the nth terms of two A.Ps: 63, 65, 67,... and 3, 10, 17,... equal?

Solution:

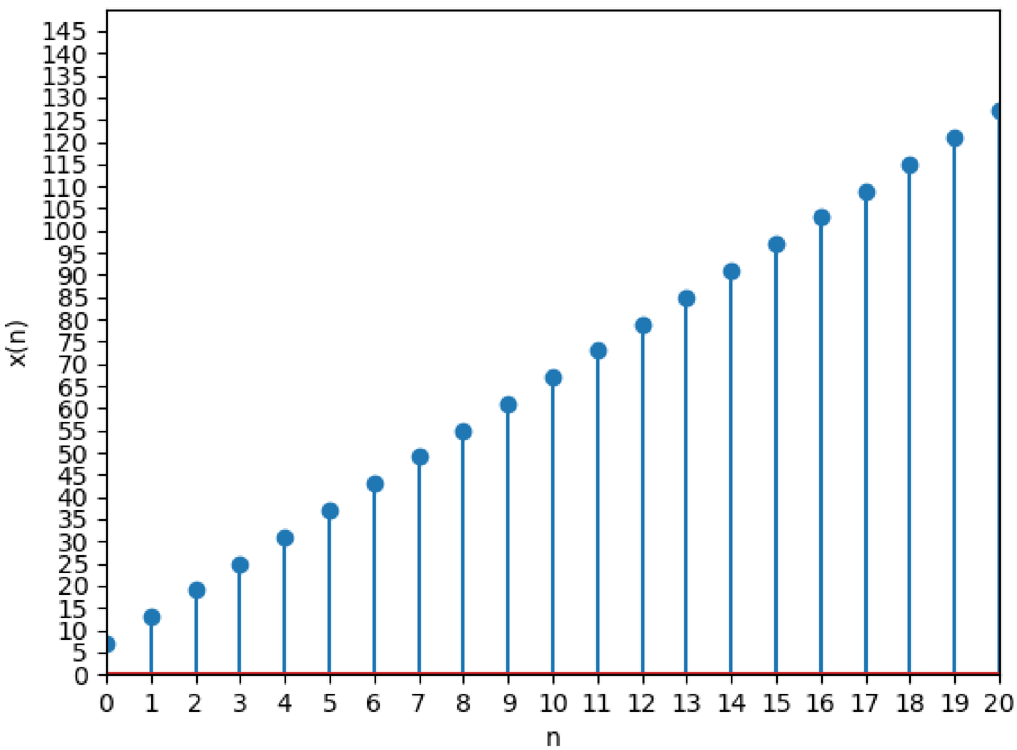


Figure 2.10: Plot of $x_1\left(n\right)$

$$x_i\left(n\right) = x\left(0\right)u\left(n\right) + dnu\left(n\right) \tag{2.73}$$

$$X\left(z\right) = \frac{x\left(0\right)}{1-z^{-1}} + \frac{dz^{-1}}{\left(1-z^{-1}\right)^2} \quad |z| > 1 \tag{2.74}$$

(a)

$$x_1\left(n\right) = 63u\left(n\right) + 2nu\left(n\right) \tag{2.75}$$

$$X_1\left(z\right) = \frac{63}{1-z^{-1}} + \frac{2z^{-1}}{\left(1-z^{-1}\right)^2} \quad |z| > 1 \tag{2.76}$$

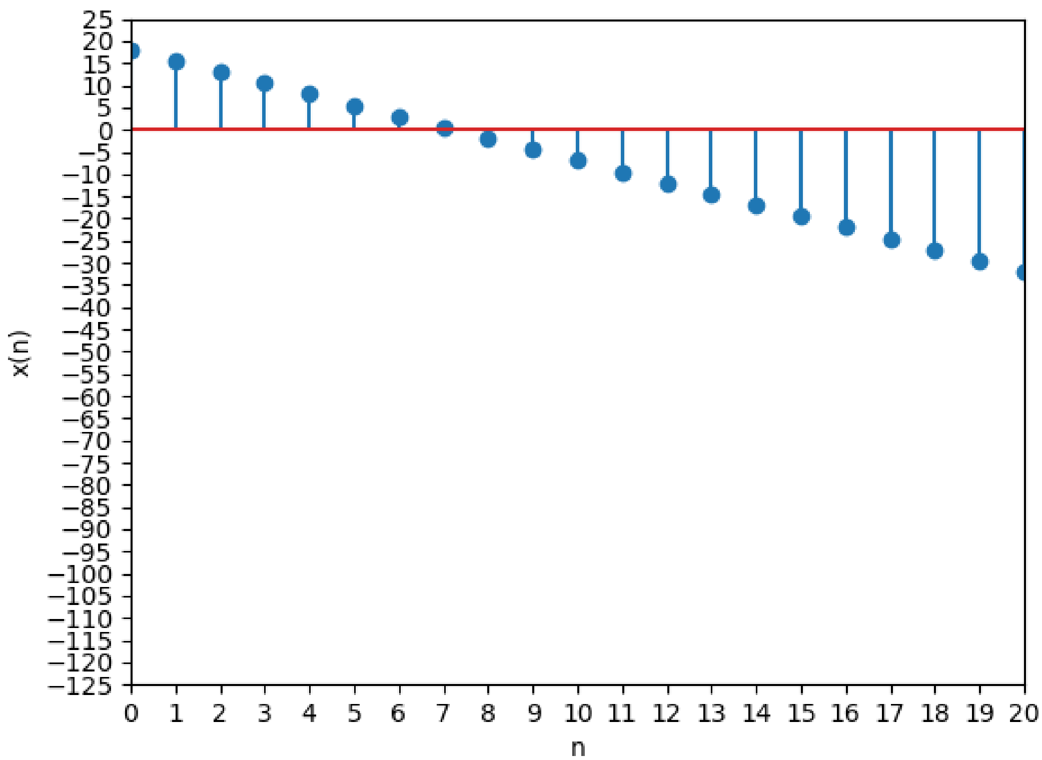


Figure 2.11: Plot of $x_2(n)$

(b)

$$x_2(n) = 3u(n) + 7nu(n) \tag{2.77}$$

$$X_2(z) = \frac{3}{1-z^{-1}} + \frac{7z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{2.78}$$

Parameter	Sub-question	Description	Value
$x_i(0)$	$x_1(0)$	1^{st} term of 1^{st} A.P.	63
	$x_2(0)$	1^{st} term of 2^{nd} A.P.	3
d_i	d_1	Common difference of 1^{st} A.P.	2
	d_2	Common difference of 2^{nd} A.P.	7

Table 2.8: input values

(c) given,

$$x_1(n) = x_2(n) \tag{2.79}$$

$$\therefore 63 + 2n = 7n + 3 \tag{2.80}$$

$$\implies n = 12 \tag{2.81}$$

2.2.3 Two APs have the same common difference.The difference between their 100th terms is 100,what is the difference between their 1000th terms?

Solution:

$$x(n) = \{x(0) + nd\}u(n) \tag{2.82}$$

$$x(99) - y(99) = 100 \tag{2.83}$$

$$\implies (x(0) + 99d) - (y(0) + 99d) = 100 \tag{2.84}$$

$$\implies x(0) - y(0) = 100 \tag{2.85}$$

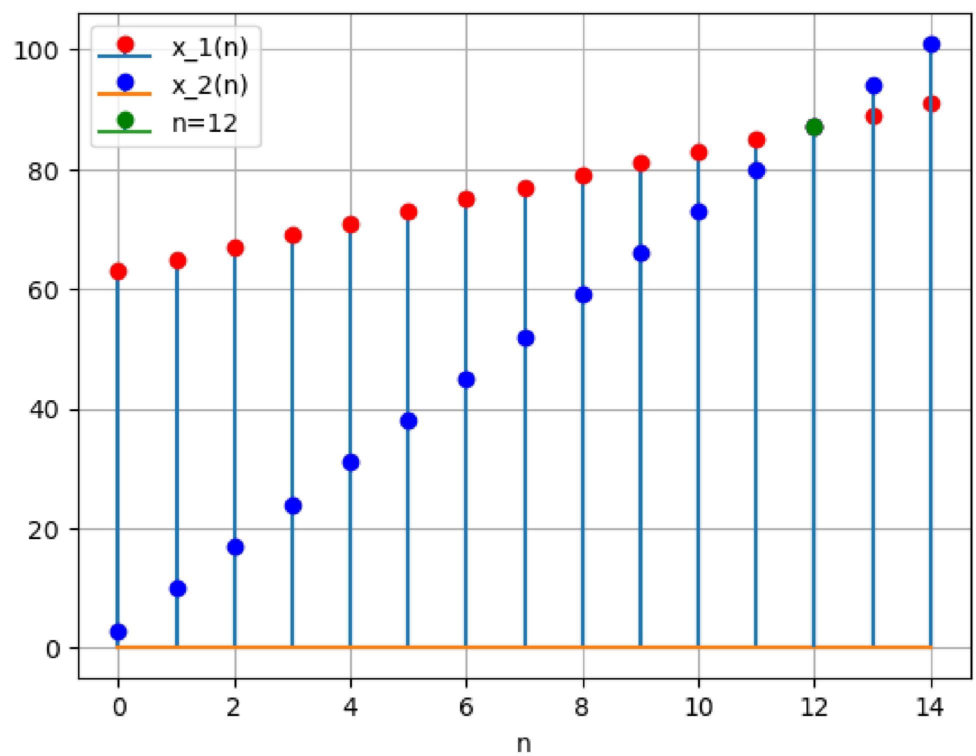


Figure 2.12: Graphs of $x_1\left(n\right)$ and $x_2\left(n\right)$ and both are equal at $n = 12$

$$x(n) - y(n) = (x(0) + nd) - (y(0) + nd) \tag{2.86}$$

$$= x(0) - y(0) \tag{2.87}$$

$$= 100 \tag{2.88}$$

$$\implies x(999) - y(999) = 100 \tag{2.89}$$

Variable	**Description**	**Value**
$x(n)$	n^{th} term of X	none
$y(n)$	n^{th} term of Y	none
d	common difference between the terms of AP	none
$x(99) - y(99)$	difference of 99^{th} terms of X and Y	100

Table 2.9: input parameters

Let

$$x(n) = \{101, 106, 111, ...\} \tag{2.90}$$

$$y(n) = \{1, 6, 11, ...\} \tag{2.91}$$

2.2.4 Check whether -150 is a term of the AP: 11,8,5,2,....

Solution:

$$x(n) = x(0) + nd \tag{2.92}$$

$$n = \frac{x(n) - x(0)}{d} \tag{2.93}$$

$$x(n) - x(0) \equiv 0 \pmod{d} \tag{2.94}$$

On substitutings values

$$-161 \equiv 2 \pmod{-3} \tag{2.95}$$

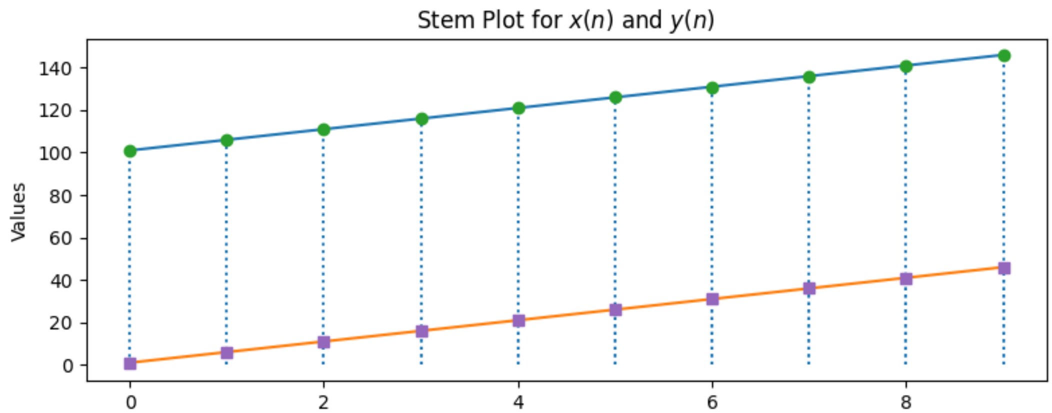


Figure 2.13:

Thus -150 is not a term of the given AP.

$$\boxed{x(n) = (11 - 3n) \times u(n)} \tag{2.96}$$

$$X(z) = \frac{11}{1 - z^{-1}} - \frac{3z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{2.97}$$

Variable	Description	Value
$x(0)$	First term of AP	11
d	Common difference	-3
$x(n)$	General term of given AP	None

Table 2.10: Input parameters

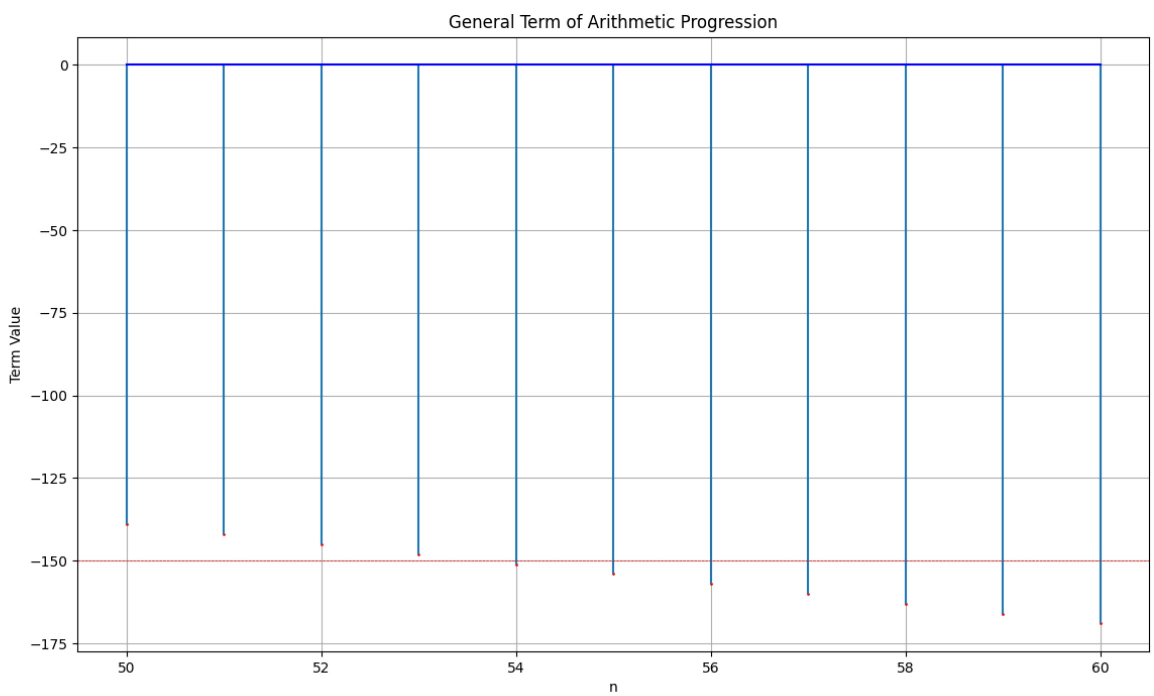


Figure 2.14: Representation of x(n)

2.2.5 Write the first five terms of the sequence $a_n = \frac{n(n^2+5)}{4}$.

Solution:

$$x(n) = \left(\frac{n^3 + 3n^2 + 8n + 6}{4}\right) u(n) \tag{2.98}$$

$$n^k u(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-1)^k z^k \frac{d^k}{dz^k} U(z) \tag{2.99}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{2.100}$$

$$n^2 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1+z^{-1})}{(1-z^{-1})^3} \quad |z| > 1 \tag{2.101}$$

$$n^3 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4} \quad |z| > 1 \tag{2.102}$$

Referencing the equations from (2.100), (2.101), and (2.102).

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1+4z^{-1}+z^{-2})}{4(1-z^{-1})^4} + \frac{3(z^{-1})(1+z^{-1})}{4(1-z^{-1})^3} + \frac{2z^{-1}}{(1-z^{-1})^2} + \frac{3}{2(1-z^{-1})} \quad |z| > 1 \tag{2.103}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{3}{2(1-z^{-1})^3} + \frac{3z^{-2}}{2(1-z^{-1})^4} \quad |z| > 1 \tag{2.104}$$

2.2.6 (a) 30th term of the AP: 10, 7, 4, ... is

(b) 11th term of the AP: $-3, -\frac{1}{2}, 2, \ldots$ is

Solution:

Parameter	**value**	**Description**
$x_i(0)$	10	First term
	-3	
d_i	-3	Common difference
	$\frac{5}{2}$	
$x_1(29)$	?	30th term
$x_2(10)$	?	11th term

Table 2.11: Input Parameters

$$x_i(n) = [x_i(0) + nd_i]\, u(n) \tag{2.105}$$

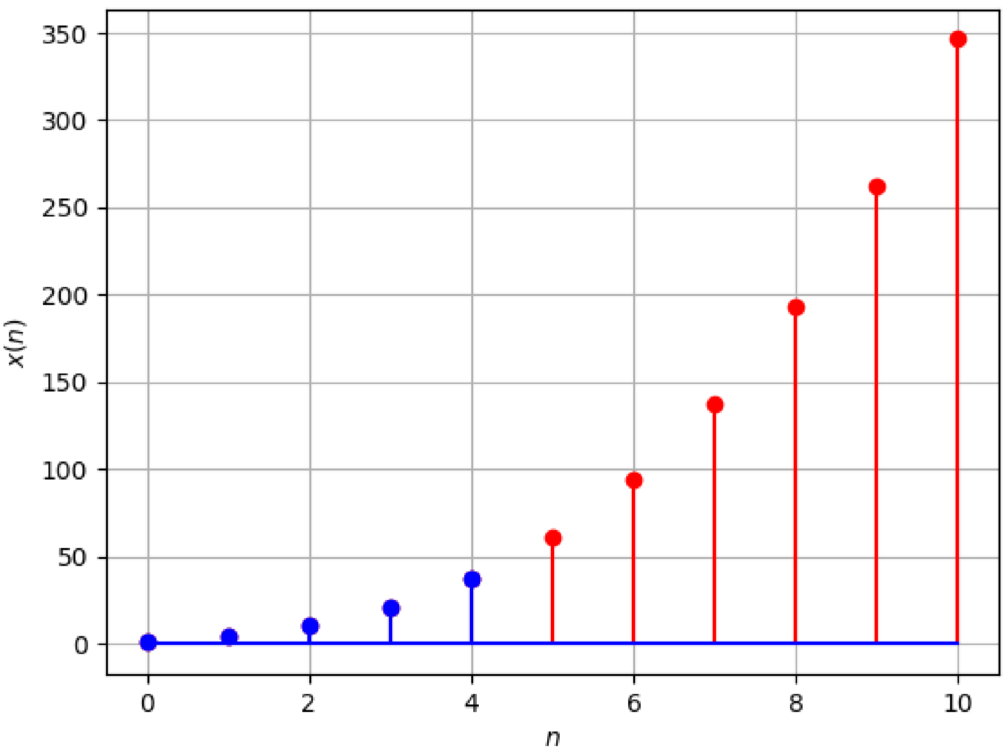


Figure 2.15: Plot of equation(2.98)

(a) From (2.105) Table 2.11 :

$$x_1(n) = [10 - 3n]\, u(n) \tag{2.106}$$

$$x_1(29) = -77 \tag{2.107}$$

$$X_1(z) = \frac{10 - 13z^{-1}}{(1 - z^{-1})^2} \qquad |z| > 1 \tag{2.108}$$

(b) From (2.105) and Table 2.11 :

$$x_2(n) = \left[-3 + \frac{5}{2}n\right]u(n) \tag{2.109}$$

$$x_2(10) = 22 \tag{2.110}$$

$$X_2(z) = \frac{5.5z^{-1} - 3}{(1 - z^{-1})^2} \qquad |z| > 1 \tag{2.111}$$

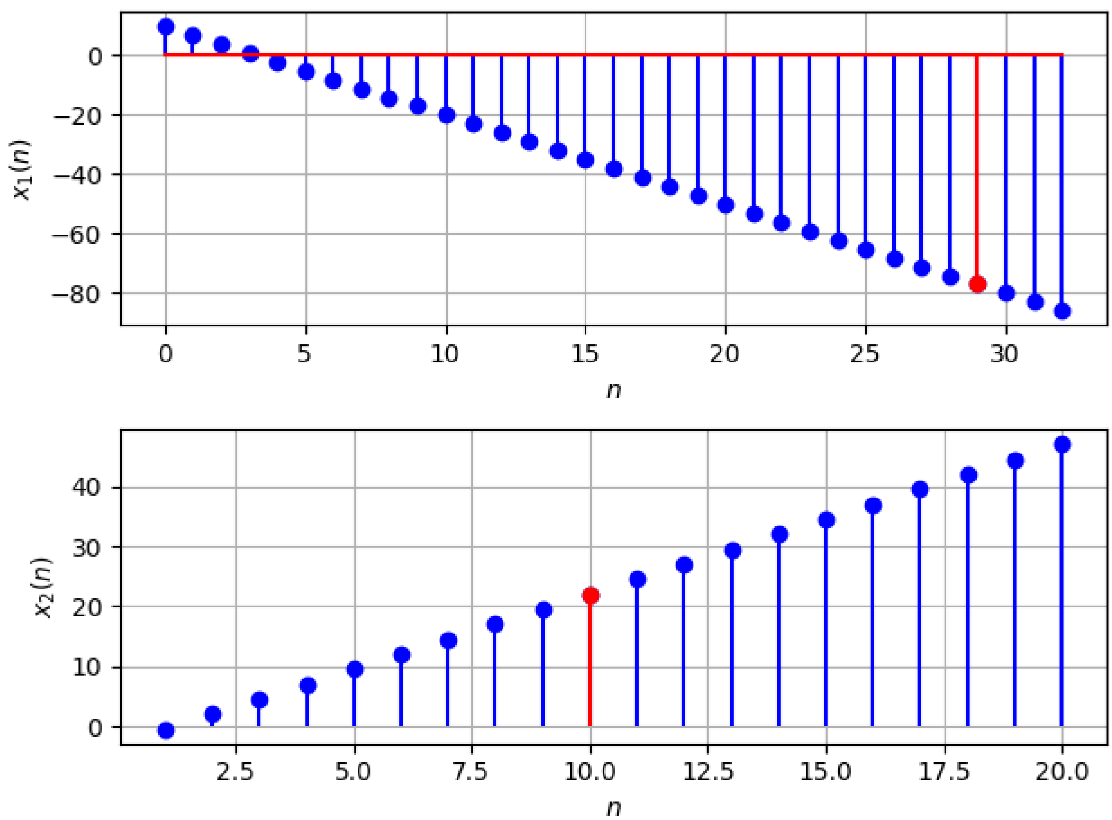


Figure 2.16: stem plots of $x_1(n)$ and $x_2(n)$

2.2.7 Write the first five terms of the sequence whose nth term is $\frac{2n-3}{6}$ and obtain the Z transform of the series **Solution:**

$$x(n) = \frac{2n-1}{6}(u(n)) \tag{2.112}$$

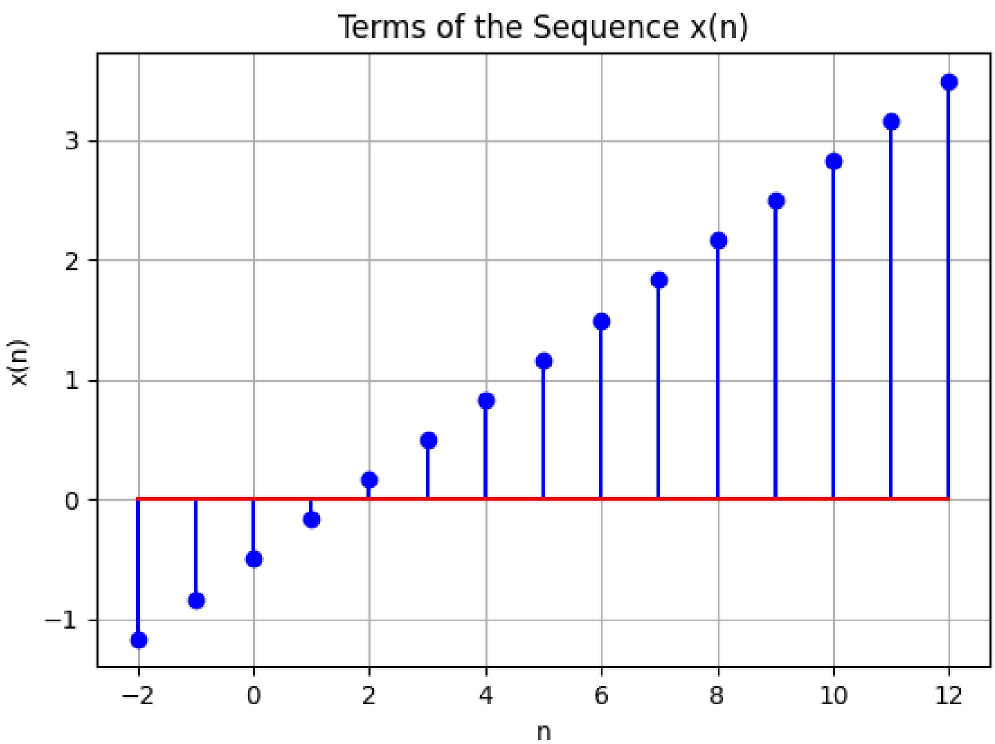


Figure 2.17: Plot of x(n) vs n

$$X(z) = \frac{3z^{-1}-1}{6(1-z^{-1})^2} \quad |z| > 1 \tag{2.113}$$

2.2.8 For what values of x, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P ?

Solution: Let r be the common ratio

Variable	Description	Value
$x(0)$	First term of the GP	$-\left(\frac{2}{7}\right)$
$x(1)$	Second term of the GP	x
$x(2)$	Third term of the GP	$-\left(\frac{7}{2}\right)$
r	Common ratio of the GP	
$x(n)$	General term	$x(0)\, r^n\, u(n)$

Table 2.12: Variables Used

From Table 2.12:

$$\implies \frac{x}{\left(-\frac{2}{7}\right)} = \frac{\left(-\frac{7}{2}\right)}{x} = r \tag{2.114}$$

$$x^2 = \left(-\frac{2}{7}\right) \cdot \left(-\frac{7}{2}\right) \tag{2.115}$$

$$x = \pm 1 \tag{2.116}$$

$$\implies r = \pm\frac{7}{2} \tag{2.117}$$

The signal corresponding to this is

$$x(n) = \left(-\frac{2}{7}\right)\left(\pm\frac{7}{2}\right)^n u(n) \tag{2.118}$$

Applying z-Transform :

$$\implies X_1(z) = \left(\frac{1}{7}\right)\left(\frac{4}{7z^{-1}+2}\right) \quad |z| > \frac{7}{2} \tag{2.119}$$

$$\implies X_2(z) = \left(\frac{1}{7}\right)\left(\frac{4}{7z^{-1}-2}\right) \quad |z| > \frac{7}{2} \tag{2.120}$$

2.2.9 Find the 20^{th} and n^{th} terms of the G.P $\frac{5}{2}$, $\frac{5}{4}$, $\frac{5}{8}$,.....

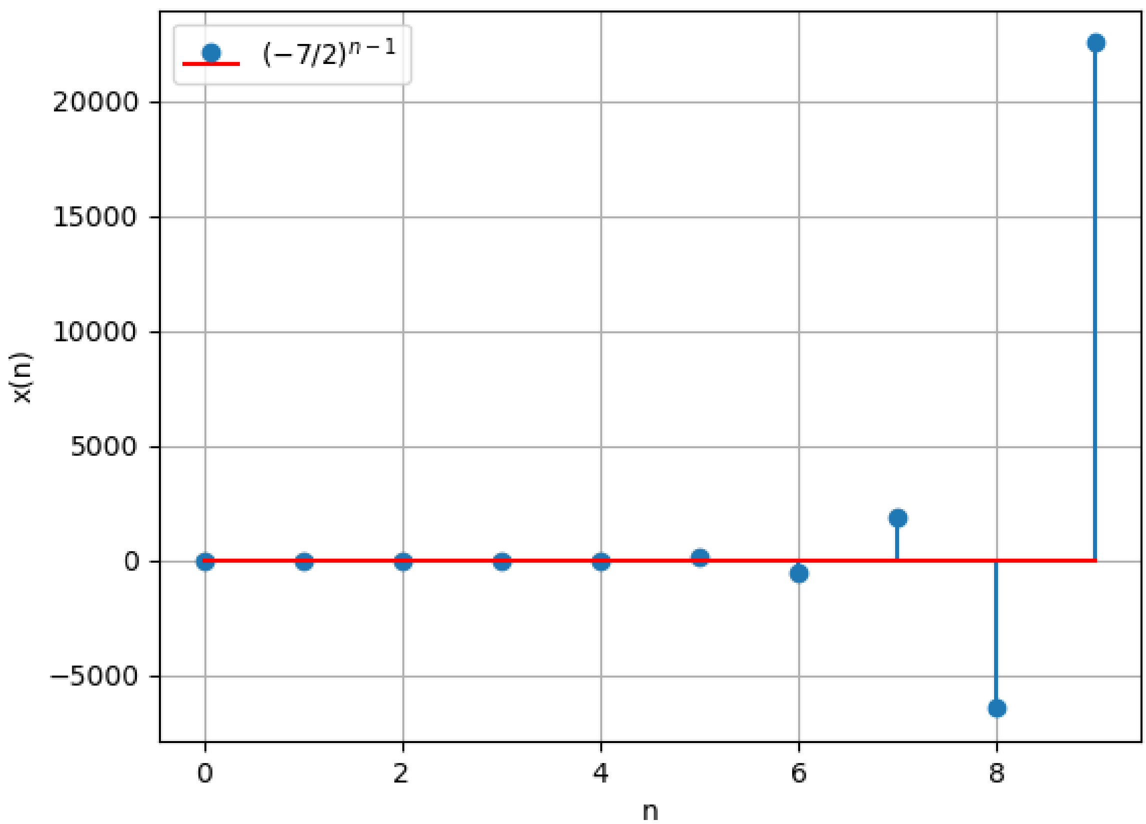


Figure 2.18: Stem Plot of x_1(n)

Solution:

From Table 2.13: Z-Transform of $x(n)$:

$$\implies X(z) = \frac{5}{2}\left(\frac{1}{1-\frac{z^{-1}}{2}}\right); \left\{z \in \mathbb{C} : |z| > \frac{1}{2}\right\} \tag{2.121}$$

2.2.10 Which term of the following sequences:

(a) $2, 2\sqrt{2}, 4 \ldots$ is 128 (b) $\sqrt{3}, 3, 3\sqrt{3} \ldots$ is 729

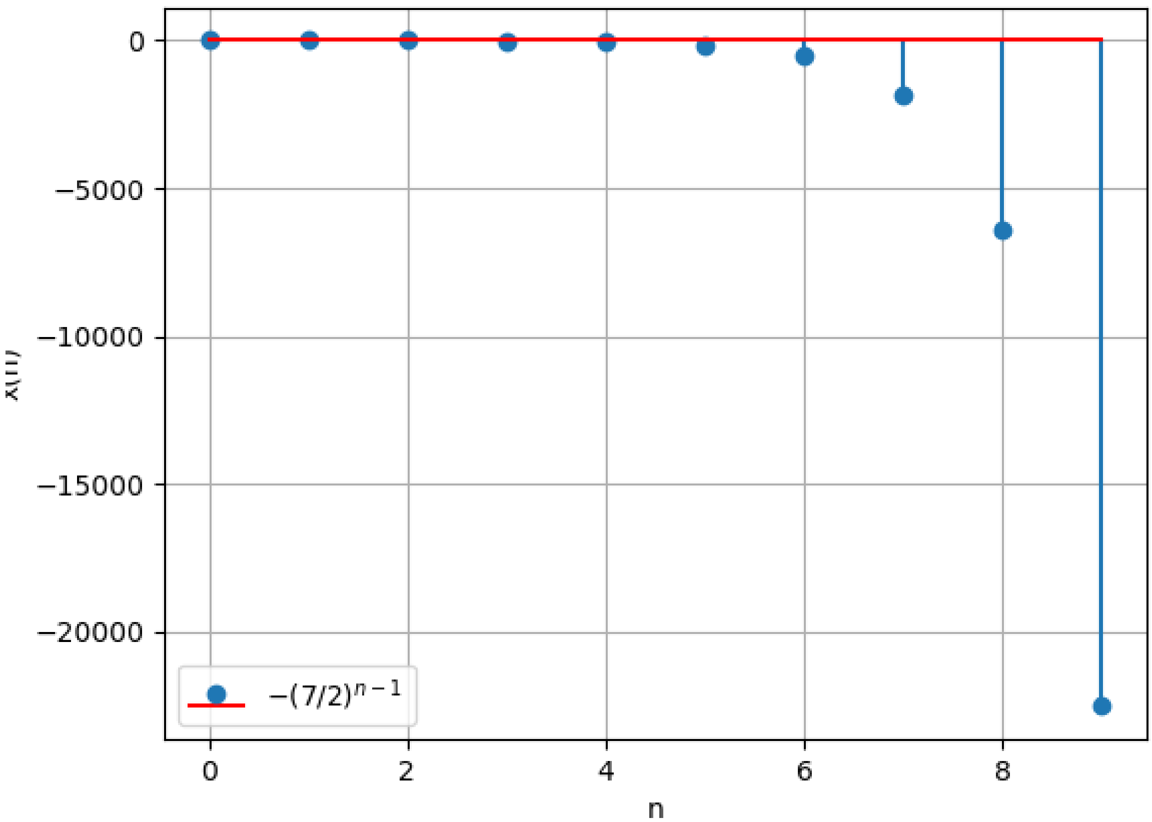


Figure 2.19: Stem Plot of $x_2(n)$

(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27} \ldots$ is $\frac{1}{19683}$

Solution: For a general GP series and $k > 0$,

$$x(k) = x(0)\, r^k \tag{2.122}$$

$$\therefore k = \log_r \frac{x(k)}{x(0)} \tag{2.123}$$

Parameter	**Description**	**Value**
$x(0)$	First Term	$\frac{5}{2}$
$r = \frac{x(1)}{x(0)}$	Common Ratio	$\frac{1}{2}$
$x(n)$	n^{th} Term	$\frac{5}{2}\left(\frac{1}{2}\right)^{n} \cdot u(n)$
$x(19)$	20^{th} Term	$\frac{5}{2}\left(\frac{1}{2}\right)^{19}$
$u(n)$	Unit step function	

Table 2.13: Parameters

And the Z-transform $X(z)$:

$$X(z) = \frac{x(0)}{1 - rz^{-1}} \quad |z| > |r| \tag{2.124}$$

(a) By Table 2.14, (2.123) and Table 2.14:

$$x_1(n) = x_1(0)\, r_1^n u(n) \tag{2.125}$$

$$k_1 = \log_{r_1} \frac{128}{x_1(0)} \tag{2.126}$$

$$\therefore k_1 = 12 \tag{2.127}$$

$$X_1(z) = \frac{2}{1 - \sqrt{2}z^{-1}} \quad |z| > \sqrt{2} \tag{2.128}$$

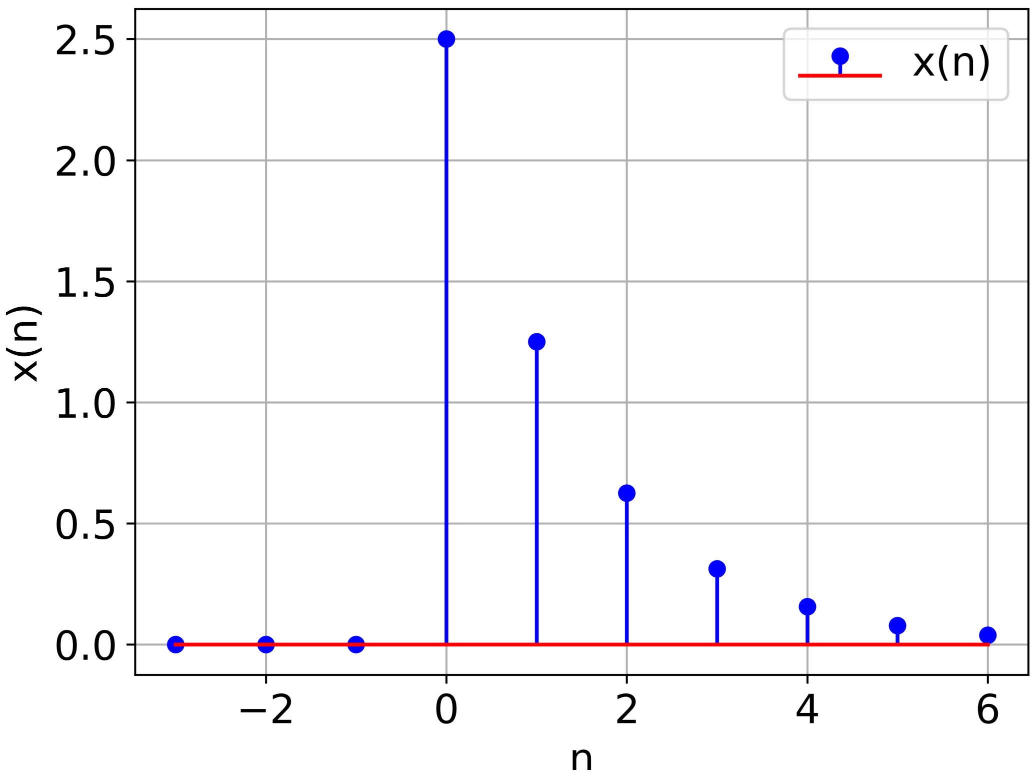


Figure 2.20:

(b) By (2.123), (2.124) and Table 2.14:

$$x_2(n) = x_2(0)\, r_2^n u(n) \tag{2.129}$$

$$k_2 = \log_{r_2} \frac{729}{x_2(0)} \tag{2.130}$$

$$\therefore k_2 = 11 \tag{2.131}$$

$$X_2(z) = \frac{\sqrt{3}}{1-\sqrt{3}z^{-1}} \quad |z| > \sqrt{3} \tag{2.132}$$

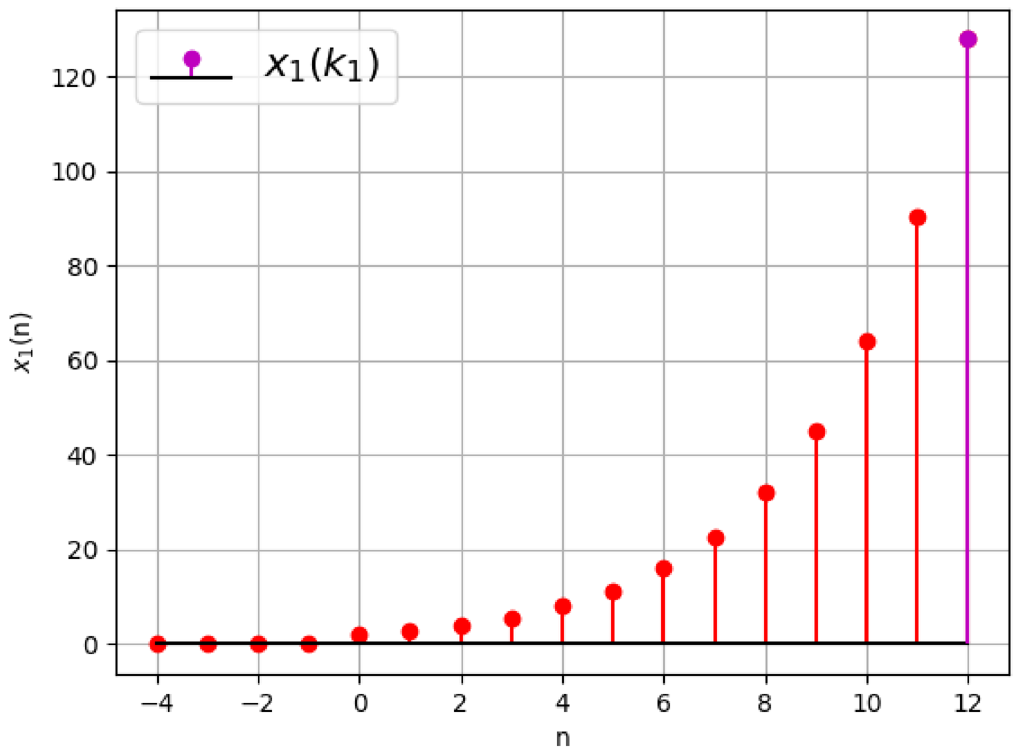


Figure 1: Plot of $x_1(n)$ vs n. See Table 2.14

(c) By (2.123), (2.124) and Table 2.14:

$$x_3(n) = x_3(0)\, r_3^n u(n) \tag{2.133}$$

$$k_3 = \log_{r_3} \frac{1}{19683 x_3(0)} \tag{2.134}$$

$$\therefore k_3 = 8 \tag{2.135}$$

$$X_3(z) = \frac{1}{3 - z^{-1}} \quad |z| > \frac{1}{3} \tag{2.136}$$

Find the 20^{th} and n^{th} terms of the G.P $\frac{5}{2}$, $\frac{5}{4}$, $\frac{5}{8}$,.....

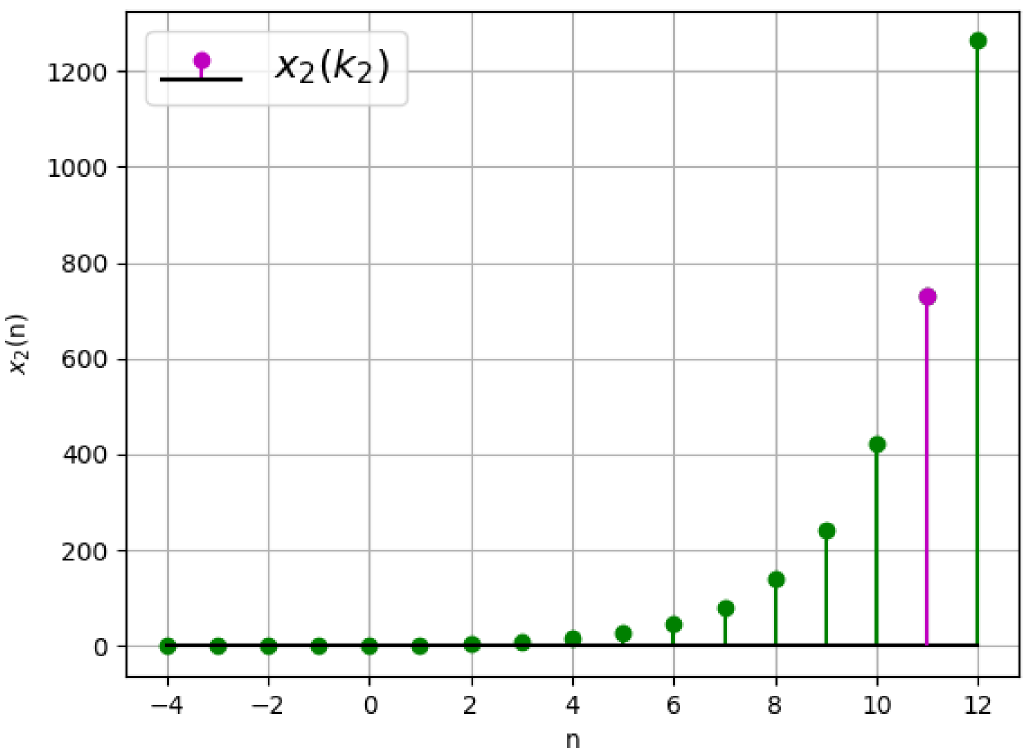


Figure 2: Plot of $x_2(n)$ vs n. See Table 2.14

Parameter	**Description**	**Value**
r_i	Common ratio of G.P (a),(b),(c)	$\sqrt{2}, \sqrt{3}, \frac{1}{3}$
$x_i(0)$	Initial Values	$2, \sqrt{3}, \frac{1}{3}$
$x_i(k_i)$	Given Values	$128, 729, \frac{1}{19683}$
k_i	Desired index	$12, 11, 8$
$x_i\left(n\right)$	Series	$x_i\left(0\right) r_i^n u\left(n\right)$
$X_i\left(z\right)$	Z-Transform of $x_i\left(n\right)$	$\frac{x(0)}{1-rz^{-1}}$

Table 2.14: Table of parameters

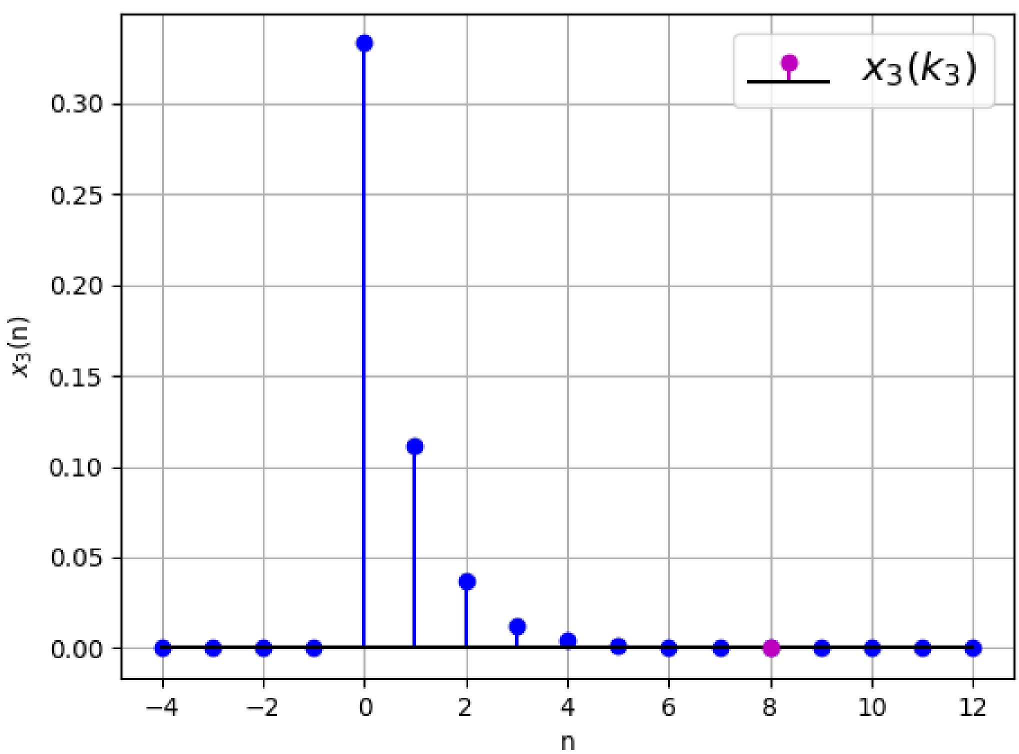


Figure 3: Plot of $x_3(n)$ vs n. See Table 2.14

2.2.11 The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2^{nd} hour, 4^{th} hour and n^{th} hour?

Solution: From Table 2.15:

Parameter	Value	Description
$x(0)$	30	Initial no. of bacteria
r	2	Ratio of no. of bacteria at end of hour to start of hour (Common Ratio)
$x(n)$	$r^n x(0)u(n)$	n^{th} term of the GP

Table 2.15: Input Parameters

$$x(2) = 120 \tag{2.137}$$

$$x(4) = 480 \tag{2.138}$$

$$x(n) = 30(2^n)u(n) \tag{2.139}$$

$$X(z) = \frac{30z^{-1}}{1 - 2z^{-1}} \quad |z| > 2 \tag{2.140}$$

2.2.12 Ramkali saved Rs 5 in the first week of a year and then increased her weekly savings by Rs 1.75. If in the nth week, her weekly savings become Rs 20.75, find n.

Solution:

Parameter	Value	Description
$x(0)$	5	First term of AP
d	1.75	Common difference of AP
$x(n)$	20.75	n^{th} term of AP

Table 2.16: Parameter List

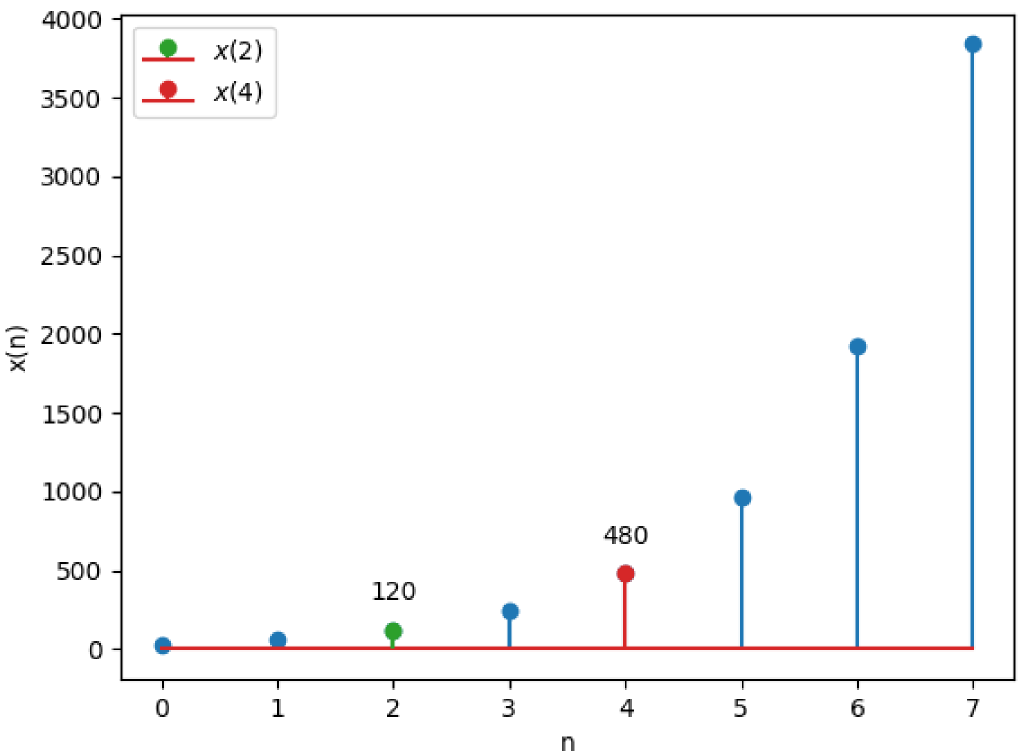


Figure 2.24: Plot of $x(n)$ vs n. See Table 2.15 for details.

$$x(n) = x(0) + (n)(d) \tag{2.141}$$

$$20.75 = 5 + (n)(1.75) \tag{2.142}$$

$$\implies 15.75 = (n)(1.75) \tag{2.143}$$

$$\implies n = \frac{15.75}{1.75} \tag{2.144}$$

$$\implies n = 9 \tag{2.145}$$

$$x(n) = 5u(n) + 1.75nu(n) \tag{2.146}$$

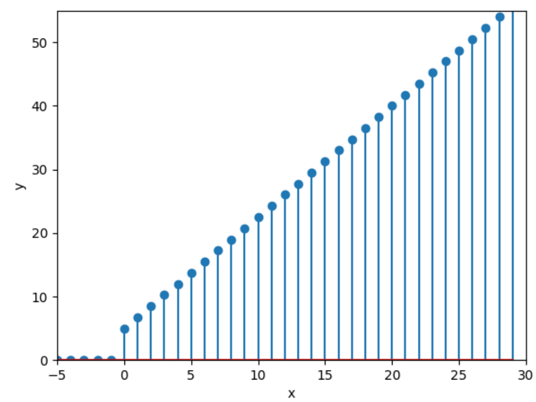


Figure 2.25: Plot of $x(n) = 5 + 1.75n$

The Z-transform of a sequence $x(n)$ is given by:

$$X(z) = \frac{5z^{-1}}{1 - z^{-1}} + \frac{1.75z^{-1}}{(1 - z^{-1})^2}; |z| > 1 \tag{2.147}$$

2.2.13 Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an $A.P.$, is equal to twice the m^{th} terms.

Solution:

For an AP,

$$x\left(n\right) = [x\left(0\right) + nd]u\left(n\right) \tag{2.148}$$

$$\implies x\left(m+n\right) + x\left(m-n\right) = [x\left(0\right) + \left(m+n\right)d] + [x\left(0\right) + \left(m-n\right)d] \tag{2.149}$$

$$= 2[x\left(0\right) + md] \tag{2.150}$$

$$\therefore x\left(m+n\right) + x\left(m-n\right) = 2x\left(m\right) \tag{2.151}$$

PARAMETER	VALUE	DESCRIPTION
$x(0)$	$x(0)$	First term
d	d	common difference
$x(n)$	$[x(0)+nd]u(n)$	General term of the series

Table 2.17: Parameter Table1

2.2.14 The sum of the first three terms of a G.P is 39/10 and their product is 1. Find the common ratio and the terms.

Solution:

$$y(n) = x(0)\left(\frac{r^{n+1}-1}{r-1}\right)u(n) \tag{2.152}$$

From Table 2.19 and (2.152) :

$$y(2) = x(0)\left(\frac{r^3-1}{r-1}\right) \tag{2.153}$$

$$\frac{39}{10} = x(0)\left(r^2+r+1\right) \tag{2.154}$$

$$\implies \frac{39r}{10} = r^2+r+1 \quad (\because x(0)r = 1) \tag{2.155}$$

$$\implies (2r-5)(5r-2) = 0 \tag{2.156}$$

$$\implies r = \frac{2}{5} \quad or \quad \frac{5}{2} \tag{2.157}$$

(a) If $r = \frac{2}{5}$, then terms are $\frac{5}{2}$, 1, $\frac{2}{5}$.

(b) If $r = \frac{5}{2}$, then terms are $\frac{2}{5}$, 1, $\frac{5}{2}$.

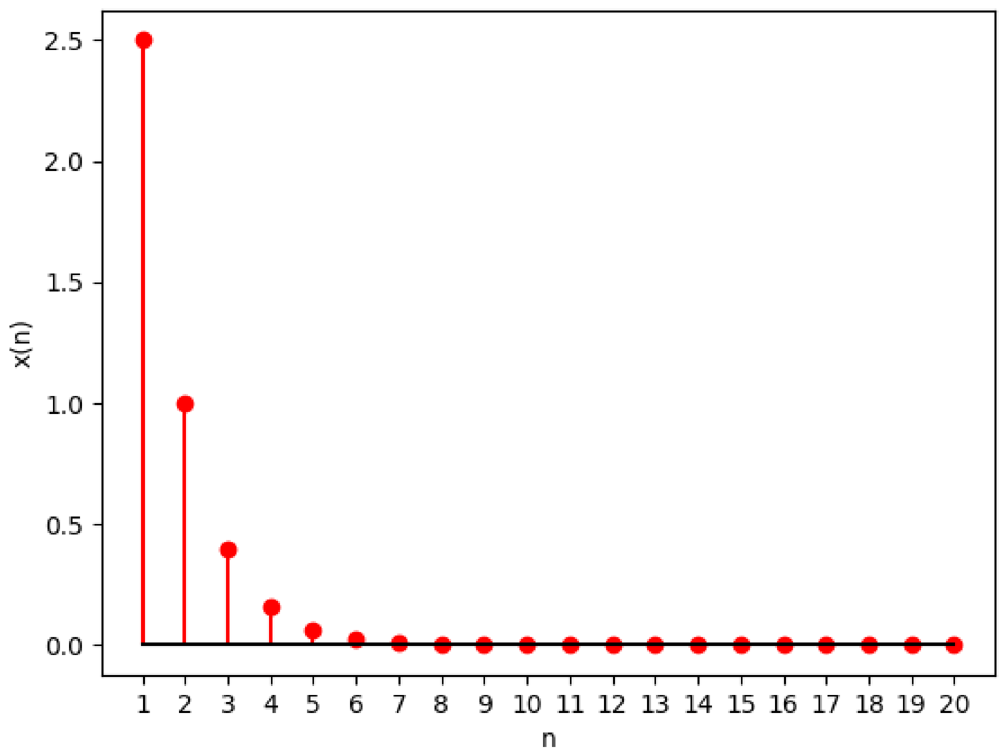


Figure 2.26: stem plots of GP if $r = \frac{2}{5}$

2.2.15 The ratio of the A.M and G.M of two positive numbers a and b is $m : n$. Show that $a : b = \left(m + \sqrt{m^2 - n^2}\right) : \left(m - \sqrt{m^2 - n^2}\right)$.

Solution: Expressing A.M and G.M in terms of a and b:

$$\frac{a+b}{2\sqrt{ab}} = \frac{m}{n} \tag{2.158}$$

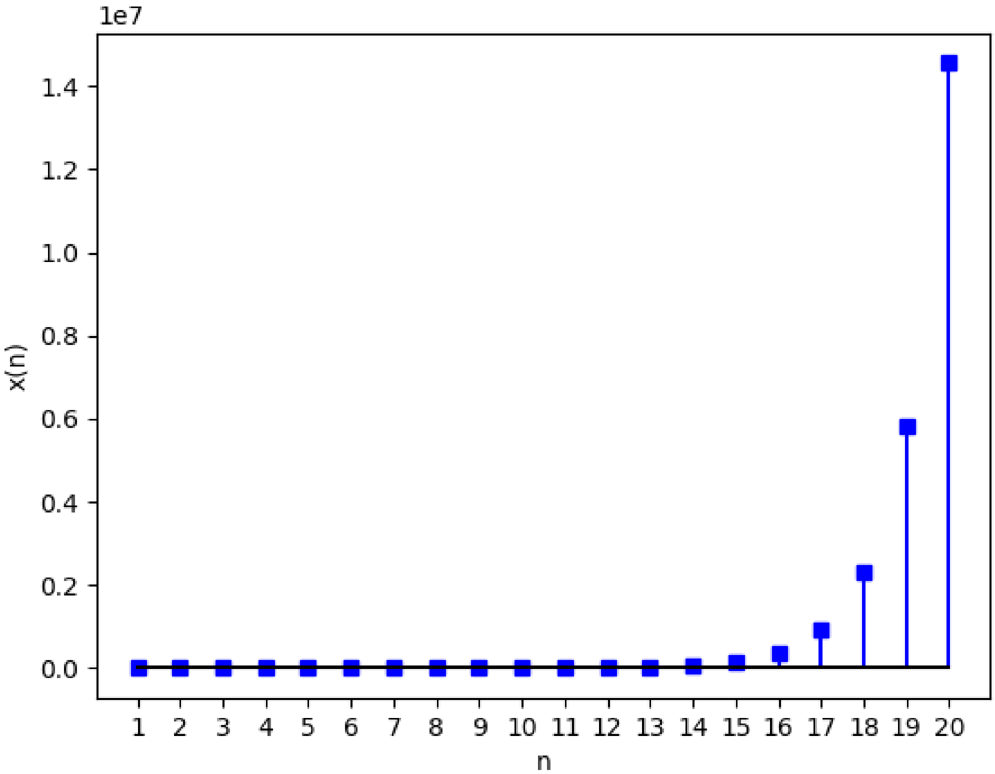


Figure 2.27: stem plots of GP if $r = \frac{5}{2}$

Let's assume that $x = \sqrt{\frac{a}{b}}$. Then, we have:

$$\frac{a}{b} = x^2 \tag{2.159}$$

Substituting the value of x in equation (2.158):

$$\frac{1+x^2}{2x} = \frac{m}{n} \tag{2.160}$$

$$\frac{1}{x} + x = \frac{2m}{n} \tag{2.161}$$

$$x^2 - \frac{2m}{n}x + 1 = 0 \tag{2.162}$$

$$\implies x = \frac{m}{n} \pm \frac{\sqrt{m^2 - n^2}}{n} \tag{2.163}$$

Since $x = \sqrt{\frac{a}{b}}$, x must be positive.

$$x = \frac{m + \sqrt{m^2 - n^2}}{n} \tag{2.164}$$

Referencing the value of x from equation(2.159).

$$\frac{a}{b} = \left(\frac{m + \sqrt{m^2 - n^2}}{n}\right)^2 \tag{2.165}$$

Multiplying both the numerator and denominator with $\left(m - \sqrt{m^2 - n^2}\right)$:

$$\frac{a}{b} = \frac{1}{n^2} \frac{\left(m + \sqrt{m^2 - n^2}\right)^2 \left(m - \sqrt{m^2 - n^2}\right)}{\left(m - \sqrt{m^2 - n^2}\right)} \tag{2.166}$$

$$\implies a : b = \left(m + \sqrt{m^2 - n^2}\right) : \left(m - \sqrt{m^2 - n^2}\right) \tag{2.167}$$

nth term of the AP :

$$y(n) = \left[a + n\left(b - a\right)\right] u(n) \tag{2.168}$$

$$n^k u(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-1)^k z^k \frac{d^k}{dz^k} U(z) \tag{2.169}$$

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{\left(1 - z^{-1}\right)} \quad |z| > |1| \tag{2.170}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \quad |z| > |1| \tag{2.171}$$

Referencing the equations from (2.170),(2.171).

$$y(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{a}{\left(1 - z^{-1}\right)} + \frac{\left(b - a\right) z^{-1}}{\left(1 - z^{-1}\right)^2} \quad |z| > |1| \tag{2.172}$$

nth term of the GP :

$$y(n) = a\left(\frac{b}{a}\right)^n u(n) \tag{2.173}$$

$$r^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{\left(1 - rz^{-1}\right)} \quad |z| > |r| \tag{2.174}$$

Referencing the equation from (2.174).

$$y(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{a^2 z^{-1}}{\left(a - bz^{-1}\right)} \quad |z| > \left|\frac{b}{a}\right| \tag{2.175}$$

$x\left(0\right)$	3
d	2
m	6
n	2
$x\left(m+n\right)$	19
$x\left(m-n\right)$	11
$x\left(m\right)$	15

Table 2.18: Verified Values

Parameter	**Value**	**Description**
$x(0)$		First term
r		Common ratio
$x(0)^3r^3$	1	Product of terms
$x(0) + x(0)r + x(0)r^2$	$\frac{39}{10}$	Sum of terms

Table 2.19: Input Parameters

Chapter 3

Contour Integration

3.1 Find the sum of the first 15 multiples of 8.

Solution:

PARAMETER	VALUE	DESCRIPTION
$x(0)$	8	First term
d	8	common difference
$x(n)$	$[8+8n]u(n)$	General term of the series

Table 3.1: Parameter Table1

For an AP,

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{\left(1 - z^{-1}\right)^2} \tag{3.1}$$

$$\implies X(z) = \frac{8}{1 - z^{-1}} + \frac{8z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{3.2}$$

$$= \frac{8}{\left(1 - z^{-1}\right)^2}, \quad |z| > 1 \tag{3.3}$$

$$y(n) = x(n) * u(n) \tag{3.4}$$

$$\implies Y(z) = X(z)\, U(z) \tag{3.5}$$

$$Y(z) = \left(\frac{8}{\left(1 - z^{-1}\right)^2}\right)\left(\frac{1}{1 - z^{-1}}\right) \tag{3.6}$$

$$= \frac{8}{\left(1 - z^{-1}\right)^3}, \quad |z| > 1 \tag{3.7}$$

Using Contour Integration to find the inverse Z-transform,

$$y(14) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{13}\, dz \tag{3.8}$$

$$= \frac{1}{2\pi j} \oint_C \frac{8z^{13}}{\left(1 - z^{-1}\right)^3}\, dz \tag{3.9}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{3.10}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{8z^{16}}{(z-1)^3}\right) \tag{3.11}$$

$$= 4 \lim_{z \to 1} \frac{d^2}{dz^2} (z^{16}) \tag{3.12}$$

$$= 960 \tag{3.13}$$

$$\therefore \boxed{y(14) = 960} \tag{3.14}$$

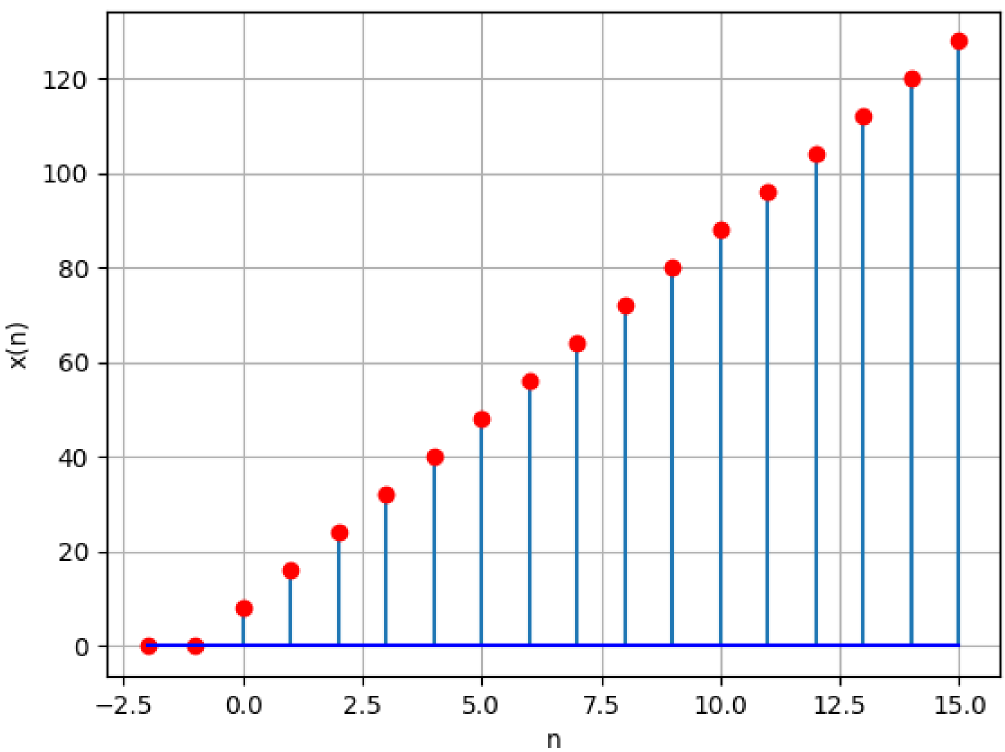


Figure 3.1: Plot of x(n) vs n

3.2 If the sum of n terms of an A.P. is $3n^2 + 5n$ and its m^{th} term is 164, find the value of m.

Solution:

$$\begin{aligned}
Y(z) &= \sum_{n=0}^{\infty} y(n)\, z^{-n} && (3.15)\\
&= \frac{2\left(4 - z^{-1}\right)}{\left(1 - z^{-1}\right)^3}, \qquad |z| > 1 && (3.16)\\
U(z) &= \frac{1}{1 - z^{-1}}, \qquad |z| > 1 && (3.17)\\
X(z) &= \frac{Y(z)}{U(z)} && (3.18)\\
&= 2\left(\frac{1}{1 - z^{-1}}\right) + 6\left(\frac{1}{\left(1 - z^{-1}\right)^2}\right) && (3.19)\\
&= \frac{8z^2 - 2z}{(z-1)^2} && (3.20)
\end{aligned}$$

Using Contour Integration to find the inverse Z-transform,

$$\begin{aligned}
x(n) &= \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1} dz && (3.21)\\
&= \frac{1}{2\pi j} \oint_C \frac{\left(8z^{n+1} - 2z^n\right) dz}{(z-1)^2} && (3.22)\\
&= \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) && (3.23)\\
&= \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 \frac{8z^{n+1} - 2z^n}{(z-1)^2}\right) && (3.24)\\
&= \lim_{z \to 1} \left(8(n+1)\, z^n - 2nz^{n-1}\right) && (3.25)\\
\implies x(n) &= (6n+8)\,(u(n)) && (3.26)\\
164 &= (6m+8)\,(u(m)) && (3.27)\\
\implies m &= 26 && (3.28)
\end{aligned}$$

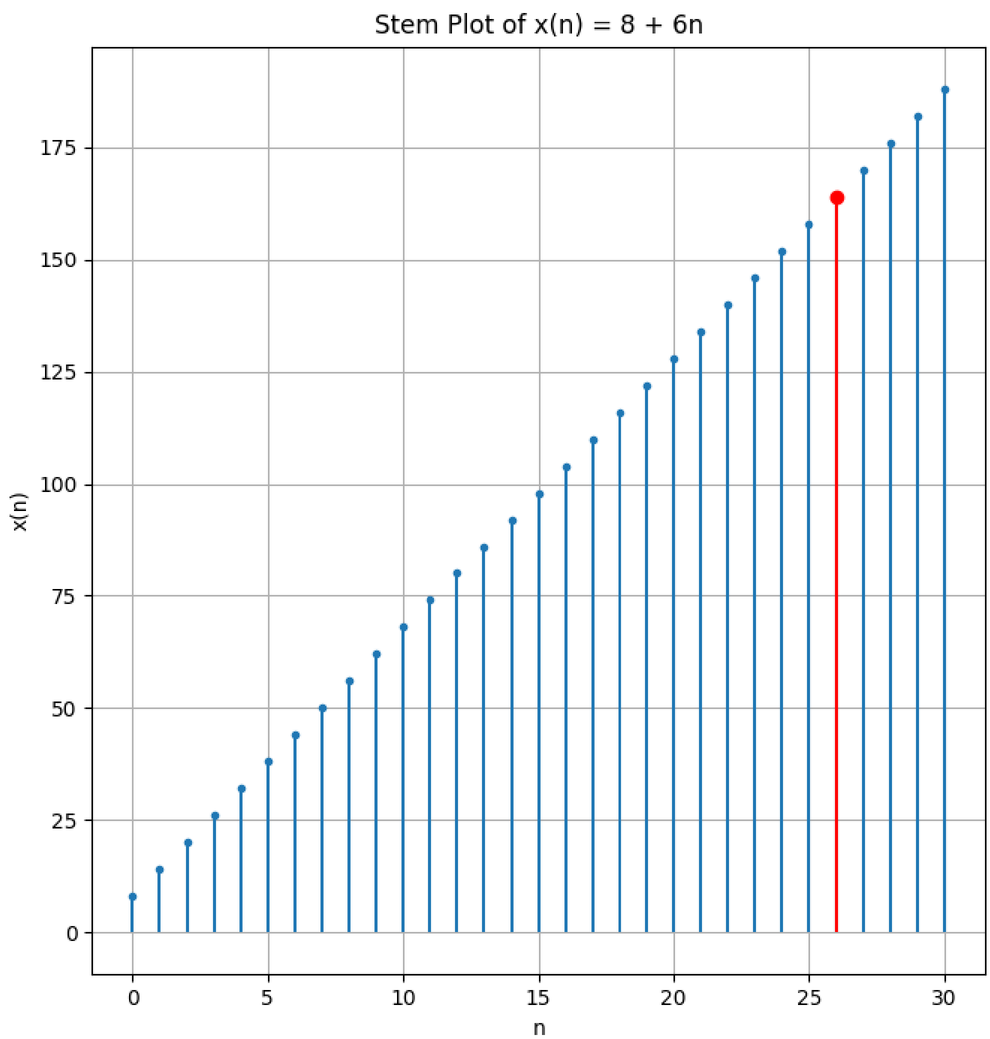


Figure 3.2: Plot of x(n) vs n

Symbol	Remarks
$y(n) = \left(3n^2 + 11n + 8\right)(u(n))$	Sum of n terms
$x(m-1)$	164
$y(n)$	$x(n) * u(n)$

Table 3.2: Parameters

3.3 Show that a_0 , a_1 , a_2 , . . ., a_n , . . . form an AP where an is defined as below :

(a) $a_n = (3 + 4n)$

(b) $a_n = (9 - 5n)$

Also find the sum of the first 15 terms in each case. **Solution:**

Parameter	**Description**	**Value**
$x_i(n)$	i^{th}Discrete signal	$(3+4n)u(n)$
		$(9-5n)u(n)$
$x_i(0)$	First term of i^{th}AP	3
		9
d_i	common difference of i^{th}AP	4
		-5

Table 3.3: Given parameters

(a) From equation (B.10.6)

$$X(z) = \frac{3}{1 - z^{-1}} + \frac{4.z^{-1}}{(1 - z^{-1})^2}; |z| > 1 \tag{3.29}$$

$$\because y(n) = x(n) * u(n) \tag{3.30}$$

$$Y(z) = X(z)U(z) \tag{3.31}$$

$$= \left[\frac{3}{(1 - z^{-1})^2} + \frac{4z^{-1}}{(1 - z^{-1})^3}\right] \tag{3.32}$$

Using contour integration for inverse Z transformation,

$$y(14) = \frac{1}{2\pi j}\int Y(z)z^{13}dz \tag{3.33}$$

$$= \frac{1}{2\pi j}\int \frac{3.z^{15}}{(z - 1)^2}dz + \frac{1}{2\pi j}\int \frac{4.z^{15}}{(z - 1)^3}dz \tag{3.34}$$

$$\because R = \frac{1}{(m - 1)!}\lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}}\left((z - a)^m f(z)\right) \tag{3.35}$$

$$R_1 = \frac{1}{1!}\lim_{z \to 1} \frac{d}{dz}\left((z - 1)^2 . \frac{3.z^{15}}{(z - 1)^2}\right) \tag{3.36}$$

$$= 45 \tag{3.37}$$

$$R_2 = \frac{1}{2!}\lim_{z \to 1} \frac{d^2}{dz^2}\left((z - 1)^3 . \frac{4.z^{15}}{(z - 1)^3}\right) \tag{3.38}$$

$$= 420 \tag{3.39}$$

$$\implies y(14) = R_1 + R_2 \tag{3.40}$$

$$= 465 \tag{3.41}$$

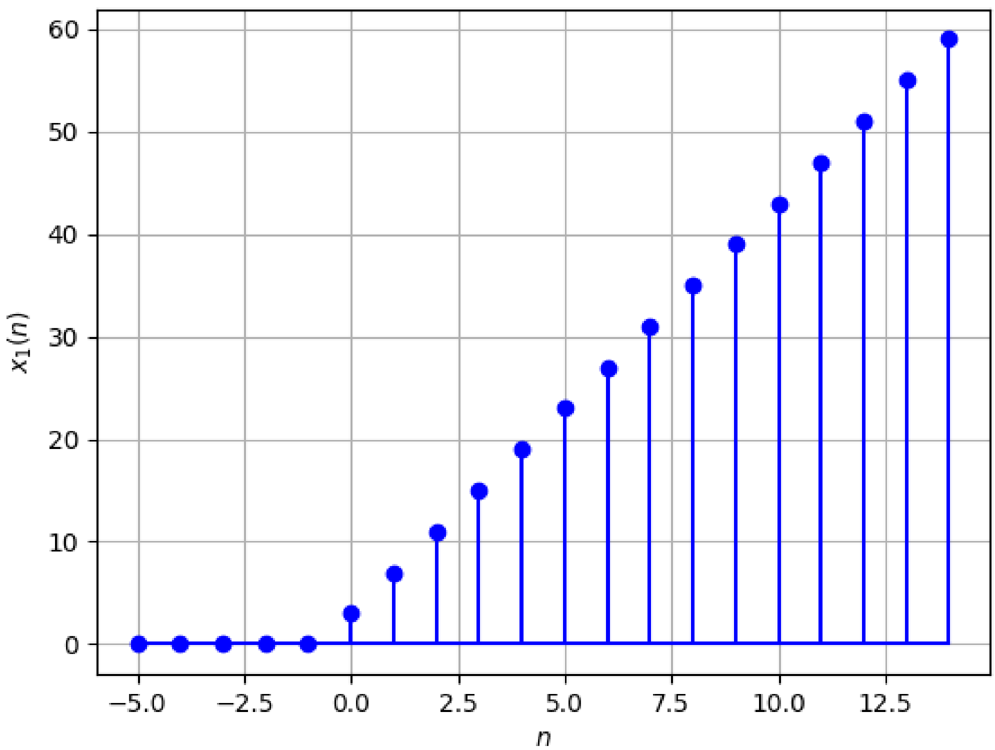


Figure 3.3: $x_1(n) = (3 + 4n)u(n)$

(b) From equation (B.10.6)

$$X(z) = \frac{9}{1 - z^{-1}} - \frac{5.z^{-1}}{(1 - z^{-1})^2}; |z| > 1 \tag{3.42}$$

$$\because y(n) = x(n) * u(n) \tag{3.43}$$

$$Y(z) = X(z)U(z) \tag{3.44}$$

$$= \left[\frac{9}{(1 - z^{-1})^2} - \frac{5z^{-1}}{(1 - z^{-1})^3}\right] \tag{3.45}$$

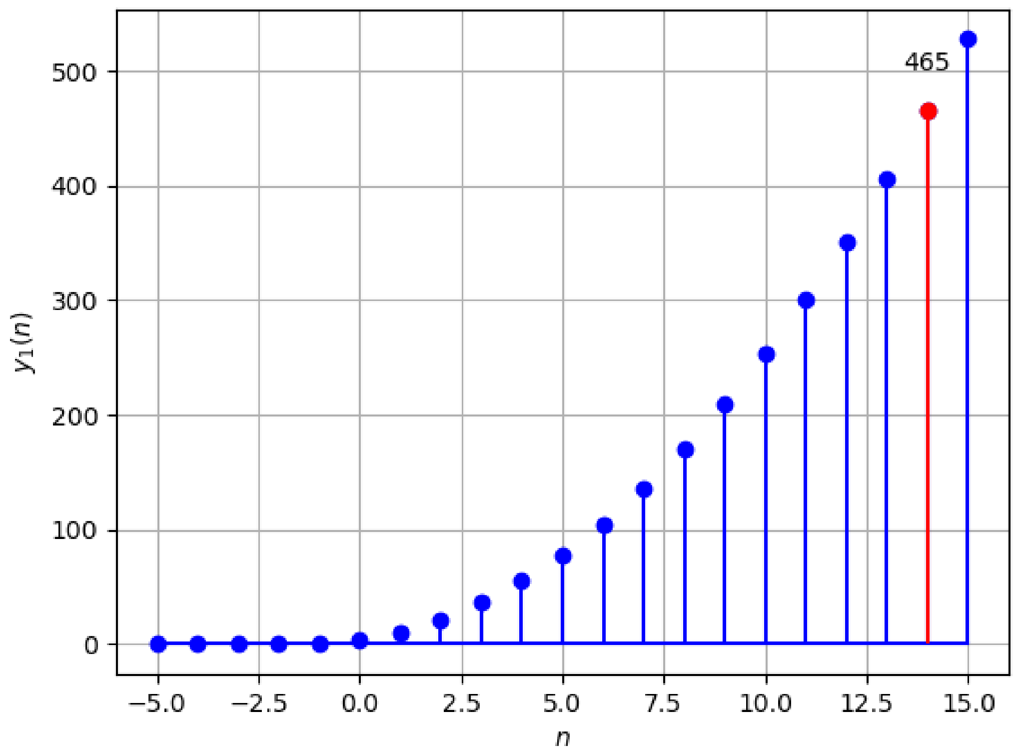


Figure 3.4: $x_1(n) = (2n^2 + 5n + 3)u(n)$

Using contour integration for inverse Z transformation,

$$y(14) = \frac{1}{2\pi j}\int Y(z)z^{13}dz \tag{3.46}$$

$$= \frac{1}{2\pi j}\int \frac{9.z^{15}}{(z-1)^2}dz - \frac{1}{2\pi j}\int \frac{5.z^{15}}{(z-1)^3}dz \tag{3.47}$$

$$\because R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{3.48}$$

$$R_1 = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2.\frac{9.z^{15}}{(z-1)^2}\right) \tag{3.49}$$

$$= 135 \tag{3.50}$$

$$R_2 = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3.\frac{5.z^{15}}{(z-1)^3}\right) \tag{3.51}$$

$$= 525 \tag{3.52}$$

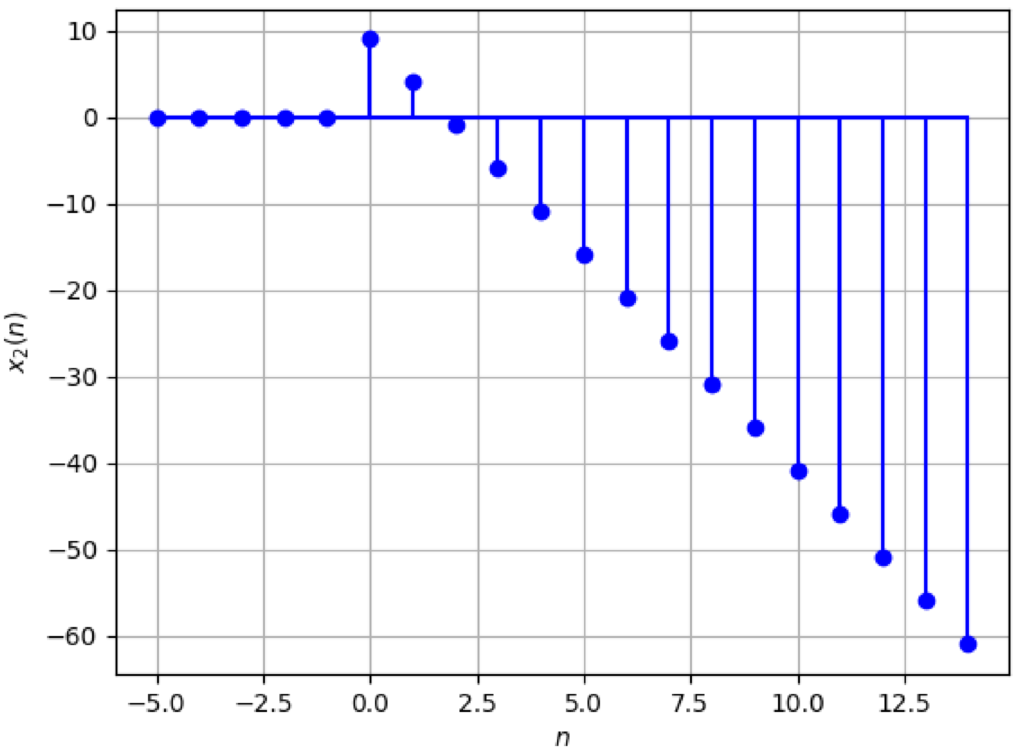


Figure 3.5: $x_2(n) = (9 - 5n)u(n)$

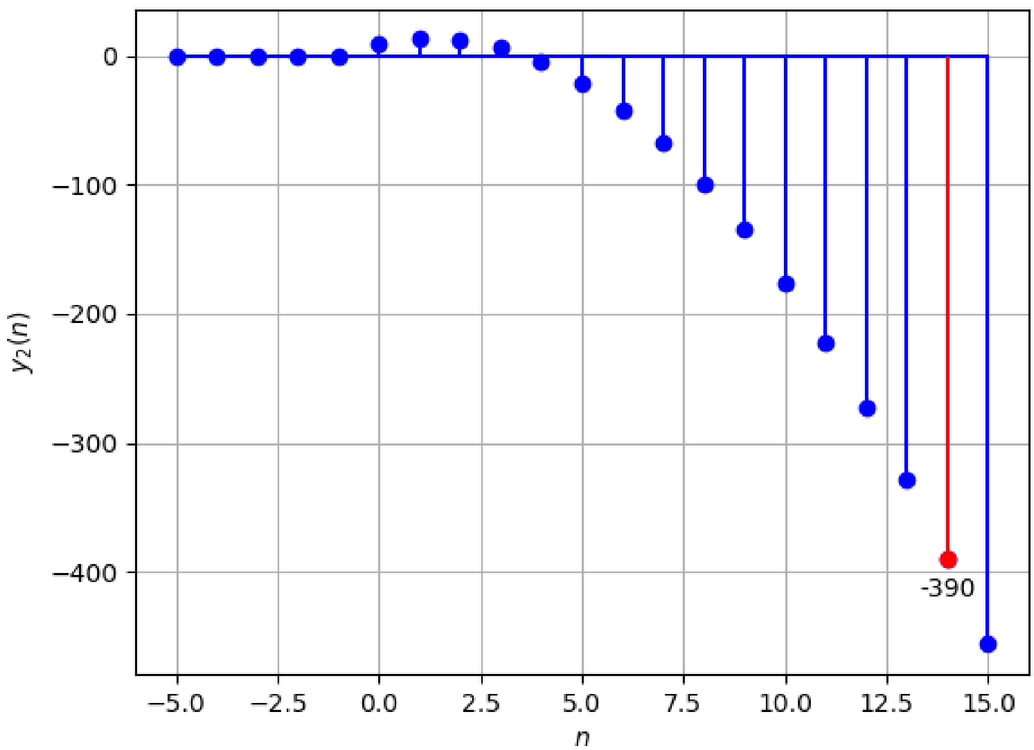


Figure 3.6: $x_2(n) = (-5n^2 + 13n + 18)u(n)$

3.4 If the sum of n terms of an AP is $(pn + qn^2)$, where p and q are constants, find the common difference. **Solution:**

Symbol	**Value**	**Description**
$y(n)$	$(pn + qn^2)$	Sum of n terms
$x(n)$		n^{th} term of AP
d	$x(n+1) - x(n)$	Common Difference

Table 3.4: Given Parameters

Sum of n terms, as a discrete signal:

$$y(n) = (pn + qn^2)u(n) \tag{3.55}$$

Taking the Z-Transform,

(a) $\mathcal{Z}\{u(n)\}$

$$u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - z^{-1}}\{|z| > 1\} \tag{3.56}$$

(b) $\mathcal{Z}\{nu(n)\}$

$$nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1 - z^{-1})^2}\ \{|z| > 1\} \tag{3.57}$$

(c) $\mathcal{Z}\{n^2u(n)\}$

$$n^2u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}\ \{|z| > 1\} \tag{3.58}$$

Taking the Z-Transform of (3.55) using (3.57) and (3.58)

$$Y(z) = p\left(\frac{z^{-1}}{(1 - z^{-1})^2}\right) + q\left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}\right) \tag{3.59}$$

Now,

$$y(n) = x(n) * u(n) \tag{3.60}$$

$$\implies Y(z) = X(z)U(z) \tag{3.61}$$

$$\implies X(z) = \frac{Y(z)}{U(z)} \tag{3.62}$$

Using (3.56) in (3.62),

$$X(z) = p\left(\frac{z^{-1}}{(1-z^{-1})}\right) + q\left(\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^2}\right) \tag{3.63}$$

Using contour integration for inverse Z-Transform:

$$\begin{aligned} x(n) &= \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz & (3.64)\\ &= \frac{1}{2\pi j}\oint_C \left[p\left(\frac{z^{-1}}{(1-z^{-1})}\right) + q\left(\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^2}\right)\right] z^{n-1}dz & (3.65) \end{aligned}$$

Calculating the residues R_1 and R_2 at pole $z = 1$:

$$\begin{aligned} R_1 &= \frac{1}{0!}\lim_{z\to 1}(z-1)\left(p\left(\frac{z^{-1}}{1-z^{-1}}\right)\right) z^{n-1} & (3.66)\\ &= p & (3.67)\\ R_2 &= \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2 q\left(\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^2}\right)\right) z^{n-1} & (3.68)\\ &= q\lim_{z\to 1}\frac{d}{dz}\left(z^n + z^{n-1}\right) & (3.69)\\ &= q(2n-1) & (3.70)\\ \implies x(n) &= R_1 + R_2 & (3.71)\\ &= p + q(2n-1) & (3.72) \end{aligned}$$

Writing x(n) as a discrete signal we get:

$$x(n) = (p-q)u(n) + 2qnu(n) \tag{3.73}$$

To simplify, use $n = 0$:

$$y(0) = x(0) \tag{3.74}$$

$$\implies 0 = (p - q)u(0) + 2q(0)u(0) \tag{3.75}$$

$$\implies p = q \tag{3.76}$$

$\therefore$ (3.73) an be written as:

$$x(n) = 2qnu(n) \tag{3.77}$$

Common difference is given by:

$$d = x(n+1) - x(n) \tag{3.78}$$

$$= 2q \tag{3.79}$$

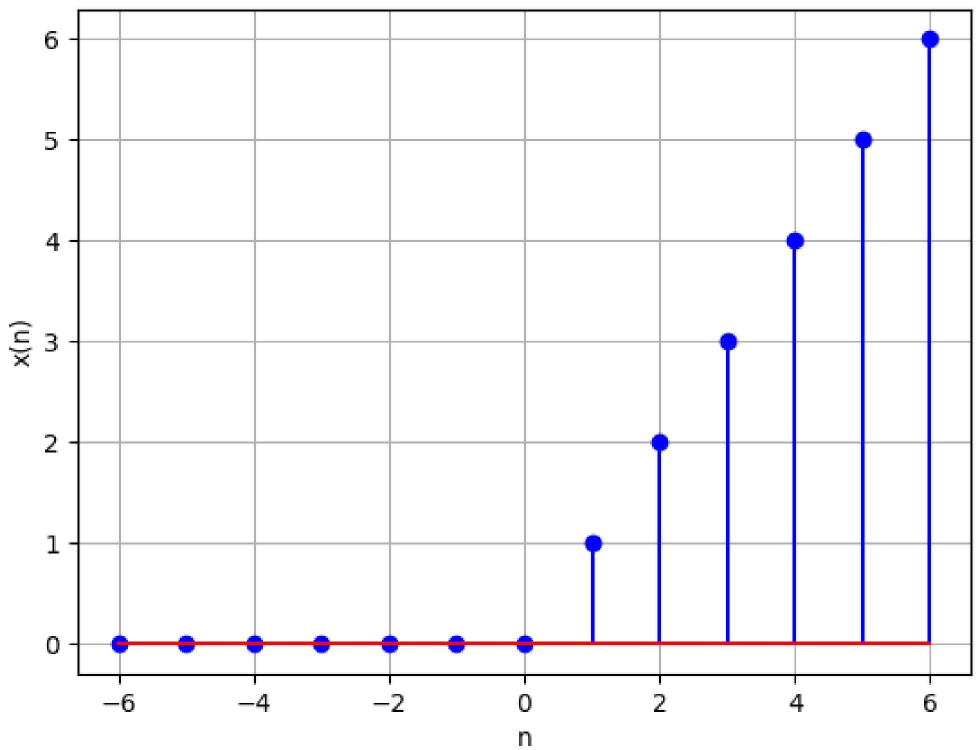


Figure 3.7: Plot of x(n) vs n for $p = q = 0.5$

3.5 Find the sum of the first 40 positive integers divisible by 6

Solution:

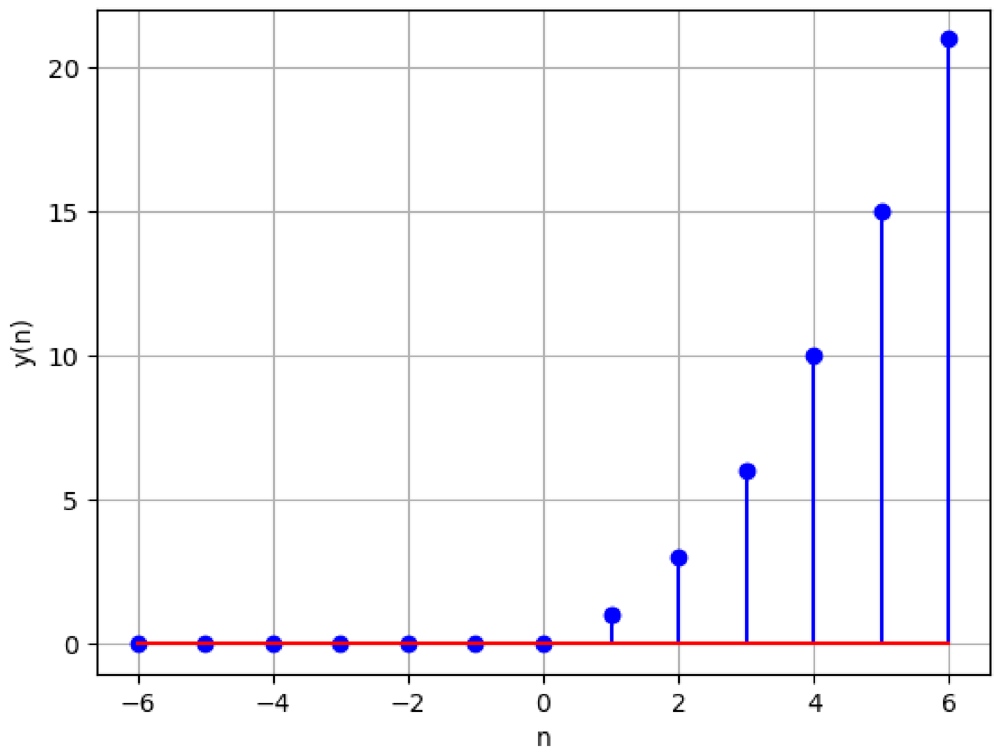


Figure 3.8: Plot of y(n) vs n for $p = q = 0.5$

$$x(n) = (6 + 6n)(u(n)) \tag{3.80}$$

$$\implies X(z) = \frac{6}{1 - z^{-1}} + \frac{6z^{-1}}{(1 - z^{-1})^2} \quad \text{(B.10.6)} \tag{3.81}$$

$$\implies X(z) = \frac{6}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{3.82}$$

$$y(n) = x(n) * u(n) \tag{3.83}$$

$$\implies Y(z) = X(z)\,U(z) \tag{3.84}$$

$$= \frac{6}{(1 - z^{-1})^3}, \quad |z| > 1 \tag{3.85}$$

Parameter	Description	Value
x(0)	First Term	6
d	Common Difference	6

Table 3.5: Parameter Table 10.5.3.12

Using contour integration to find the inverse Z-transform:

$$\implies y(39) = \frac{1}{2\pi j} \oint_C Y(z)\; z^{38}\, dz \tag{3.86}$$

$$= \frac{1}{2\pi j} \oint_C \frac{6z^{41}}{(z-1)^3}\, dz \tag{3.87}$$

We can observe that there is only a three times repeated pole at z=1,

$$\implies R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{3.88}$$

$$= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{6z^{41}}{(z-1)^3}\right) \tag{3.89}$$

$$= 3 \lim_{z \to 1} \frac{d^2}{dz^2} \left(z^{41}\right) \tag{3.90}$$

$$= 4920 \tag{3.91}$$

$$\therefore \boxed{y(39) = 4920} \tag{3.92}$$

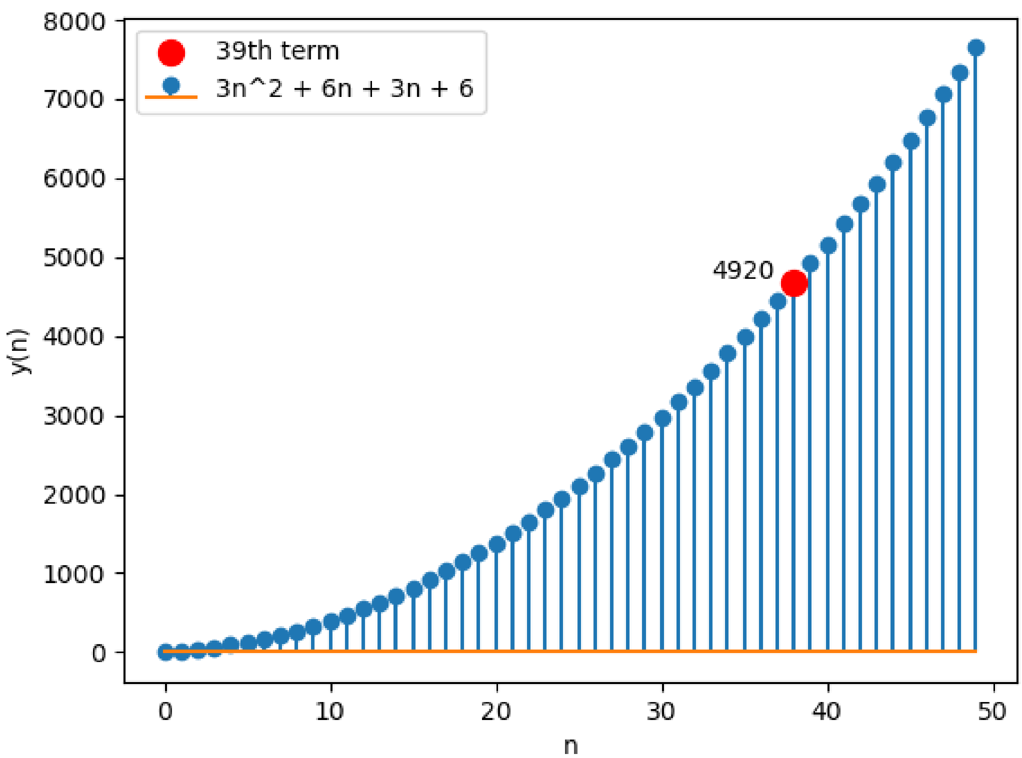


Figure 3.9: Plot of y(n) vs n

Chapter 4

Laplace Transform

4.0.1 You are riding in an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of

(a) The spring constant K

(b) The damping constant b for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg.

Solution: The parameters are :

Parameter	**Value(SI)**	**Description**
x_0	0.15	Initial spring compression
m	750	Mass
g	9.8	Gravitational acc
k	mg/x_0	Spring constant
b		Damping constant

Table 4.1: Input Parameters

Parameter	Value(SI)	Description
x		Spring Extension
F_1	kx	Spring Force
F_2	$b\frac{dx}{dt}$	Damping Force
s		Complex Frequency
s_1, s_2		Values of s

Table 4.2: Intermediate Parameters

Initially the automobile is in rest, so we can use,

$$mg = kx_0 \tag{4.1}$$

$$\implies k = \frac{mg}{x_0} \tag{4.2}$$

Now, as the oscillation begins, from the FBD, we have net force on the mass as,

$$F = F_1 + F_2 + mgu(t) \tag{4.3}$$

$$\implies -m\frac{d^2x(t)}{dt^2} = kx(t) + b\frac{dx(t)}{dt} + mgu(t) \tag{4.4}$$

$$\implies \frac{d^2x(t)}{dt^2} + \left(\frac{b}{m}\right)\frac{dx(t)}{dt} + \left(\frac{k}{m}\right)x(t) = -gu(t) \tag{4.5}$$

Now, taking the Laplace transform on both sides,

$$s^2X(s) + \frac{b}{m}sX(s) + \frac{k}{m}X(s) = -\frac{g}{s} \tag{4.6}$$

$$\implies X(s) = -\frac{g}{s\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} \tag{4.7}$$

$$\implies X(s) = -\frac{g}{s(s-s_1)(s-s_2)} \tag{4.8}$$

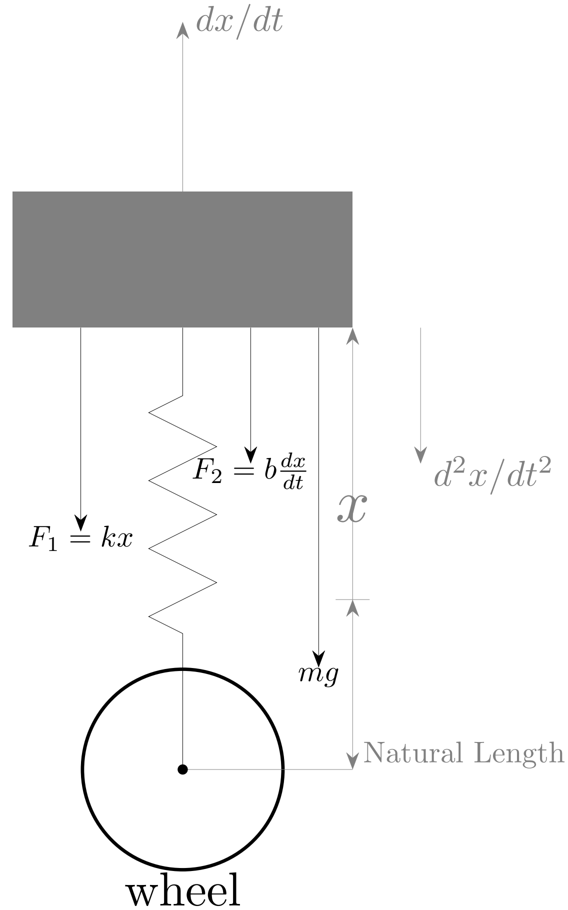


Figure 4.1: FBD of the damped oscillation system

Where

$$s_1 = -\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \tag{4.9}$$

$$s_2 = -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \tag{4.10}$$

From (4.8) we get,

$$\Longrightarrow X(s) = \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_2(s - s_2)} - \frac{1}{s_1(s - s_1)}\right] + \frac{g}{s_1 s_2}\left(\frac{1}{s}\right) \tag{4.11}$$

Now again taking the inverse Laplace transform we have,

$$x(t) = \frac{g}{s_1 s_2}u(t) + \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_2}e^{s_2 t} - \frac{1}{s_1}e^{s_1 t}\right]u(t) \tag{4.12}$$

$$\Longrightarrow x(t) = \sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2}\, e^{-bt/2m}u(t) \sin\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\, t + \tan^{-1}\left(\frac{2mg\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}}{gb}\right)\right) + \frac{mg}{k}u(t) \tag{4.13}$$

(Substituting the values of s_1 and s_2 from (4.9) and (4.10))

From (4.13) we have the amplitude after one time period T,

$$\frac{1}{2}\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2} = \sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gb}{2mk}\right)^2}\, e^{-bT/2m} \tag{4.14}$$

$$\Longrightarrow e^{\pi b/\sqrt{mk}} = 2 \tag{4.15}$$

$$\Longrightarrow b = \frac{\sqrt{mk}\ln 2}{\pi} \tag{4.16}$$

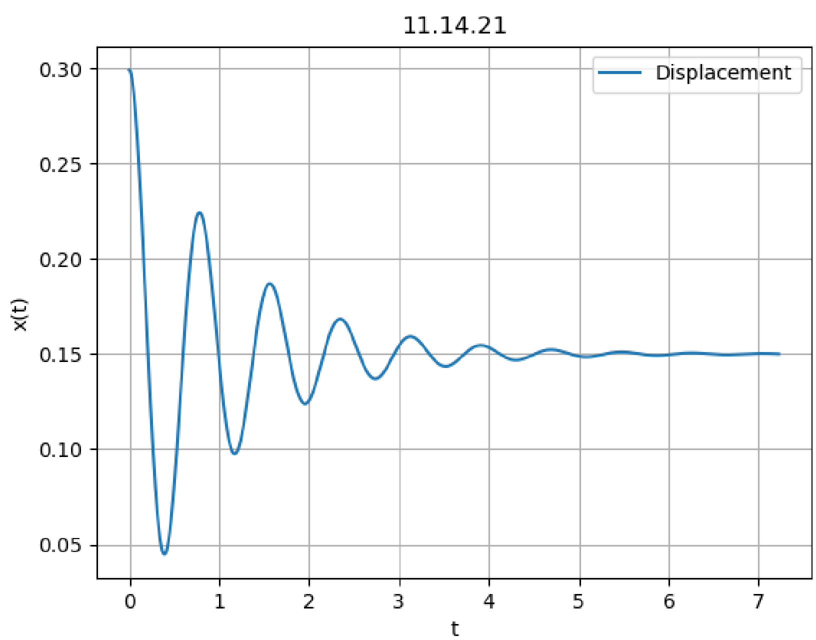


Figure 4.2: Displacement Vs. Time Graph

Appendix A

Convolution

A.1 The convolution sum is defined as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{A.1.1}$$

A.2 The unit step function is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{A.2.1}$$

A.3 If

$$x(n) = 0, \quad n < 0, \tag{A.3.1}$$

from (A.1.1),

$$x(n) * u(n) = \sum_{k=0}^{n} x(k) \tag{A.3.2}$$

Appendix B

Z-transform

B.1 The Z-transform of $p(n)$ is defined as

$$P(z) = \sum_{n=-\infty}^{\infty} p(n) z^{-n} \tag{B.1.1}$$

B.2 If

$$p(n) = p_1(n) * p_2(n), \tag{B.2.1}$$

$$P(z) = P_1(z) P_2(z) \tag{B.2.2}$$

B.3

$$n x(n) \overset{\mathcal{Z}}{\longleftrightarrow} -z X'(z) \tag{B.3.1}$$

From (B.3.1)

$$\Longrightarrow nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{\left(1-z^{-1}\right)^2}, |z|>1 \tag{B.3.2}$$

$$\Longrightarrow n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^3}, |z|>1 \tag{B.3.3}$$

$$\Longrightarrow n^3u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4}, |z|>1 \tag{B.3.4}$$

$$\Longrightarrow n^4u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{\left(1-z^{-1}\right)^5} \tag{B.3.5}$$

$$\text{where } |z|>1$$

B.4

$$x(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k}X(z) \tag{B.4.1}$$

Using (B.4.1):

$$nu(n-1) \overset{\mathcal{Z}}{\longleftrightarrow} z\frac{2z^{-2}}{\left(1-z^{-1}\right)^2} \tag{B.4.2}$$

Now ,

$$\frac{(n-1)}{2}u(n-2) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-2}}{\left(1-z^{-1}\right)^2} \tag{B.4.3}$$

$$\frac{(n-1)(n-2)}{6}u(n-3) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-3}}{\left(1-z^{-1}\right)^3} \tag{B.4.4}$$

$$\vdots$$

$$\frac{(n-1)(n-2)\ldots(n-k+1)}{(k-1)!}u(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-k}}{\left(1-z^{-1}\right)^k} \tag{B.4.5}$$

$$\Longrightarrow Z^{-1}\left[\frac{z^{-2}}{(1-z^{-1})^2}\right] = (n-1)\,u\,(n-1) \tag{B.4.6}$$

$$\Longrightarrow Z^{-1}\left[\frac{z^{-3}}{(1-z^{-1})^3}\right] = \frac{(n-1)\,(n-2)}{2}u\,(n-1) \tag{B.4.7}$$

$$\Longrightarrow Z^{-1}\left[\frac{z^{-4}}{(1-z^{-1})^4}\right] = \frac{(n-1)\,(n-2)\,(n-3)}{6}u\,(n-1) \tag{B.4.8}$$

$$\Longrightarrow Z^{-1}\left[\frac{z^{-5}}{(1-z^{-1})^5}\right] = \frac{(n-1)\,(n-2)\,(n-3)\,(n-4)}{24} \tag{B.4.9}$$

$$u\,(n-1)$$

B.5 For a Geometric progression

$$x\,(n) = x\,(0)\,r^n u\,(n)\,, \tag{B.5.1}$$

$$\Longrightarrow X\,(z) = \sum_{n=-\infty}^{\infty} x\,(n)\,z^{-n} = \sum_{n=0}^{\infty} x\,(0)\,r^n z^{-n} \tag{B.5.2}$$

$$= \sum_{n=0}^{\infty} x\,(0)\left(rz^{-1}\right)^n \tag{B.5.3}$$

$$= \frac{x\,(0)}{1-rz^{-1}}, \quad |z| > |r| \tag{B.5.4}$$

B.6 Substituting $r = 1$ in (B.5.4),

$$u(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{B.6.1}$$

B.7 From (B.3.1) and (B.6.1),

$$nu(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \tag{B.7.1}$$

B.8 For an AP,

$$x(n) = [x(0) + nd]\, u(n) = x(0)u(n) + dnu(n) \tag{B.8.1}$$

$$\implies X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{B.8.2}$$

upon substituting from (B.6.1) and (B.7.1).

B.9 From (A.3.2), the sum to n terms of a GP can be expressed as

$$y(n) = x(n) * u(n) \tag{B.9.1}$$

where $x(n)$ is defined in (B.5.1). From (B.2.2), (B.5.4) and (B.6.1),

$$Y(z) = X(z)\, U(z) \tag{B.9.2}$$

$$= \left(\frac{x(0)}{1 - rz^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right) \quad |z| > |r| \cap |z| > |1| \tag{B.9.3}$$

$$= \frac{x(0)}{(1 - rz^{-1})(1 - z^{-1})} \quad |z| > |r| \tag{B.9.4}$$

which can be expressed as

$$Y(z) = \frac{x(0)}{r - 1}\left(\frac{r}{1 - rz^{-1}} - \frac{1}{1 - z^{-1}}\right) \tag{B.9.5}$$

using partial fractions. Again, from (B.5.4) and (B.6.1), the inverse of the above can be expressed as

$$y(n) = x(0)\left(\frac{r^{n+1} - 1}{r - 1}\right) u(n) \tag{B.9.6}$$

B.10 For the AP $x(n)$, the sum of first $n+1$ terms can be expressed as

$$y(n) = \sum_{k=0}^{n} x(k) \tag{B.10.1}$$

$$\implies y(n) = \sum_{k=-\infty}^{\infty} x(k)u(n-k) \tag{B.10.2}$$

$$= x(n) * u(n) \tag{B.10.3}$$

Taking the Z-transform on both sides, and substituting (B.8.2) and (B.6.1),

$$Y(z) = X(z)U(z) \tag{B.10.4}$$

$$\implies Y(z) = \left(\frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2}\right)\frac{1}{1-z^{-1}} \quad |z| > 1 \tag{B.10.5}$$

$$= \frac{x(0)}{(1-z^{-1})^2} + \frac{dz^{-1}}{(1-z^{-1})^3}, \quad |z| > 1 \tag{B.10.6}$$

B.11 From (B.4.1) and (B.7.1),

$$(n+1)u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-z^{-1})^2}, \quad |z| > 1, \tag{B.11.1}$$

From (B.11.1) and (B.3.1),

$$n(n+1)u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^3}, \quad |z| > 1, \tag{B.11.2}$$

B.12 Taking the inverse Z-transform of (B.10.6),

$$y(n) = x(0)\left[(n+1)u(n)\right] + \frac{d}{2}\left[n(n+1)u(n)\right] \tag{B.12.1}$$

$$= \frac{n+1}{2}\left\{2x(0) + nd\right\}u(n) \tag{B.12.2}$$

Appendix C

Contour Integration

C.1

$$x(n) \xrightarrow{Z} X(z) \tag{C.1.1}$$

$$\implies X(z) = \sum_{k=-\infty}^{\infty} x(k)\, z^{-k} \tag{C.1.2}$$

Multiplying both side with z^{k-1} and integrating on a contour integral enclosing the region of convergence. Where C is a counter-clockwise closed contour in region of convergence.

$$\frac{1}{2\pi j}\oint_C X(z)\, z^{k-1} dz = \frac{1}{2\pi j}\oint_C \sum_{k=-\infty}^{\infty} x(k)\, z^{-n+k-1} dz \tag{C.1.3}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, \frac{1}{2\pi j}\oint_C z^{-n+k-1}.dz \tag{C.1.4}$$

From cauchy's integral theorem

$$\frac{1}{2\pi j}\oint_C z^{-k} dz = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases} \tag{C.1.5}$$

$$= \delta(1-k) \tag{C.1.6}$$

So eq (C.1.4) becomes

$$\frac{1}{2\pi j}\oint_C X(z)\, z^{k-1} dz = \sum_{k=-\infty}^{\infty} x(k)\,\delta(k-n) \tag{C.1.7}$$

$$\implies x(n) = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1} dz \tag{C.1.8}$$

Contour integrals like (C.1.8) can be evaluated using Cauchy's residue theorem.

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1} dz \tag{C.1.9}$$

$$= \sum \left[\text{Residue of } X(z)\, z^{n-1} \text{ at poles inside } C\right] \tag{C.1.10}$$