SIGNAL PROCESSING FUNDAMENTALS Through NCERT

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Contents

Introduction

This book introduces some concepts in signal processing through maths and physics problems in NCERT textbooks.

Chapter 1

Analog

1.1. Harmonics

1.1.1 A charged particle oscillates about its mean equilibrium position with a frequency of $10^9 Hz$. What is the frequency of the electromagnetic waves produced by the oscillator?

Solution:

Symbol	Value	Description
$y(t)$	$\cos(2\pi f_c t)$	Wave equation of electro-magnetic wave
f_c	10^9	Frequency of electromagnetic wave
t	seconds	Time

Table 1.1: Variable description

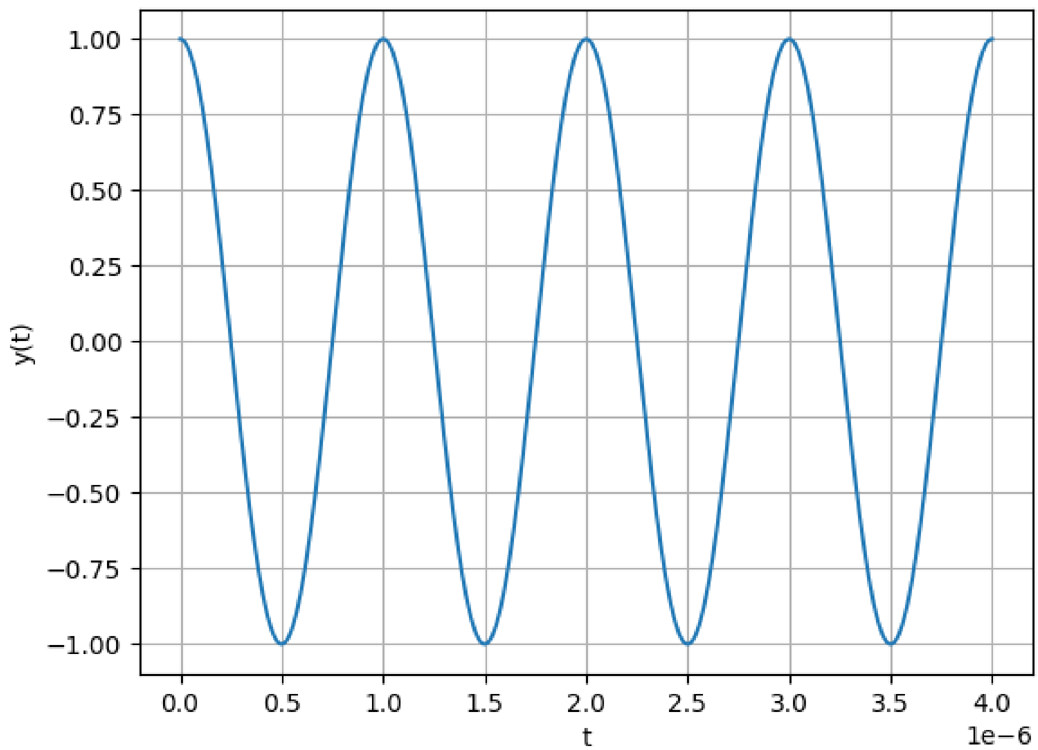

Figure 1.1: $y(t) = \cos\left(2\pi \times 10^9 t\right)$

Given below are some functions of x and t to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represents (i) travelling wave, (ii) a stationary wave or (iii) none at all:

(a) $y = 2\cos(3x)\sin(10t)$

(b) $y = 2\sqrt{x - vt}$

(c) $y = 3\sin(5x - 0.5t) + 4\cos(5x - 0.5t)$

(d) $y = \cos x \sin t + \cos 2x \sin 2t$

Solution:

TRAVELLING WAVE	STATIONARY WAVE
$y(x,t) = A\sin(kx \pm \omega t)$	$y(x,t) = A\sin kx \cos\omega t$
PARAMETERS	DEFINITION
A	Amplitude
ω	Angular Velocity
x	Position
k	Wavenumber

Table 1.2: Travelling wave *vs* Stationary wave

Let us assume an equation:

$$y = A(x)\cos(\omega t + \phi(x)) \tag{1.1}$$

Fig. 1.2 and Fig. 1.4 are self explanatory for stationary and travelling waves.Fig. 1.3 and Fig. 1.5 are neither stationary nor travelling waves.

STATIONARY WAVE CONDITION	TRAVELLING WAVE CONDITION
(1) $A(x)$ should be a function of position x, and it can be expressed as $A(x) = A_0 cos(\omega t + \alpha)$ where A_0 is a constant, k is the wavenumber, x is the position and α is a phase constant.	(1) $A(x)$ should be a constant, and it can be expressed as $A(x) = A_0$ where A_0 is a constant number.
(2) $\phi(x)$ can be expressed as $\phi(x) = c$ where c is a constant.	(2) $\phi(x)$ represents a linear expression in x, and it can be expressed as $\phi(x) = kx + \theta$ where k is the wavenumber and θ is the phaseconstant.

Table 1.3: Travelling wave *vs* Stationary wave

1.1.2 For the travelling harmonic wave $y\,(x,t) = 2.0\cos 2\pi\,(10t - 0.0080x + 0.35)$ where x and y are in cm and t in s. Calculate the phase difference between oscillatory motion of two points separated by a distance of

(a) $4m$

(b) $0.5m$

(c) $\lambda/2$

(d) $3\lambda/4$

Solution:

$$(\Delta\theta) = (2\pi ft - kx_1 + \phi) - (2\pi ft - kx_2 + \phi) \tag{1.2}$$

$$= k\,(x_2 - x_1) \tag{1.3}$$

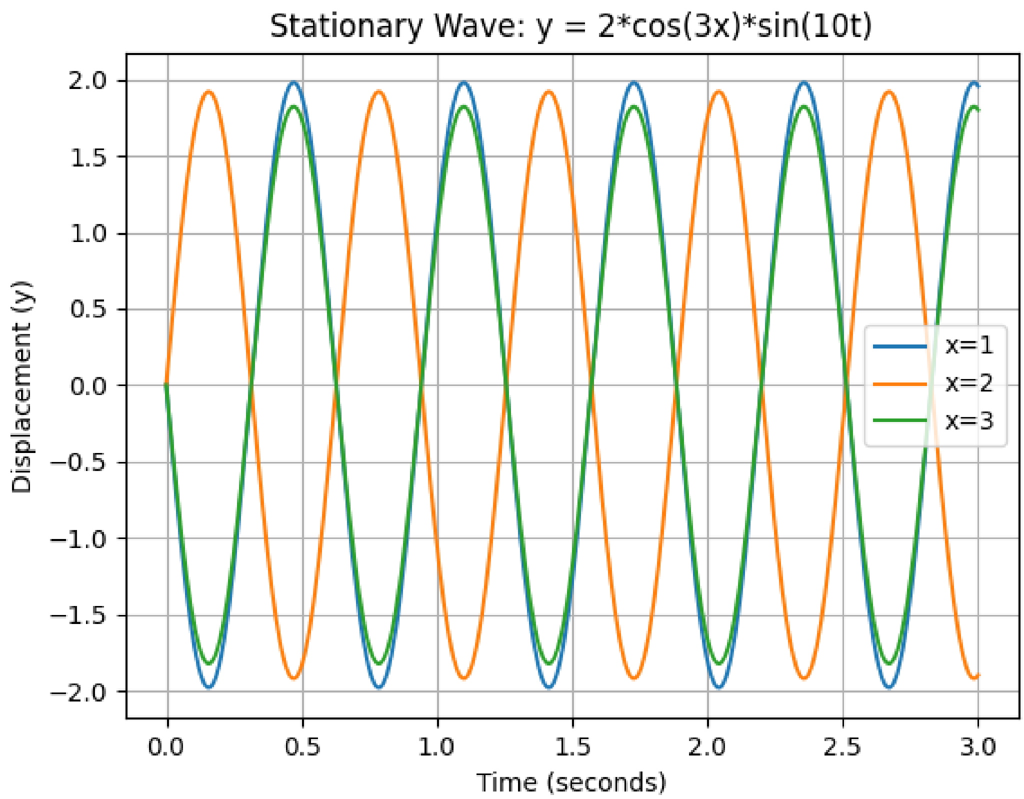

Figure 1.2: DIPLACEMENT *vs* TIME-graph1

1.2. Filters

1.2.1 Obtain the resonant frequency and Q-factor of a series LCR circuit with $L = 3.0\,H$, $C = 27\,\mu F$, and $R = 7.4\,\Omega$. It is desired to improve the sharpness of the resonance of the circuit by reducing its 'full width at half maximum' by a factor of 2. Suggest a suitable way.

Solution: Given parameters are:

(a) Frequency Response of the Circuit

Parameter	**Description**	**Value**
$y\left(x_i, t\right)$	equation of harmonic wave	$A\cos\left(2\pi ft - kx_i + \phi\right)$
k	angular wave number	$2\pi\left(0.008\right)$
$\lambda = \dfrac{2\pi}{k}$	wavelength	$125\,cm$
f	frequency	10
A	amplitude	2.0
ϕ	phase constant	$2\pi\left(0.35\right)$
θ_i	phase of i^{th} harmonic wave	$\left(2\pi ft - kx + \phi\right)$
x_i	position of i^{th} harmonic wave	
t	time	
$x_2 - x_1$	path difference	$400\,cm$
		$50\,cm$
		$\dfrac{\lambda}{2}$
		$\dfrac{3\lambda}{4}$

Table 1.4: Given parameters list

Parameter	**Description**	**subquestion**	**Value**
$\Delta\theta$	$\theta_1 - \theta_2$	(a)	6.4π radians
		(b)	0.8π radians
		(c)	π radians
		(d)	$\dfrac{3\pi}{2}$ radians

Table 1.5: Phase differences

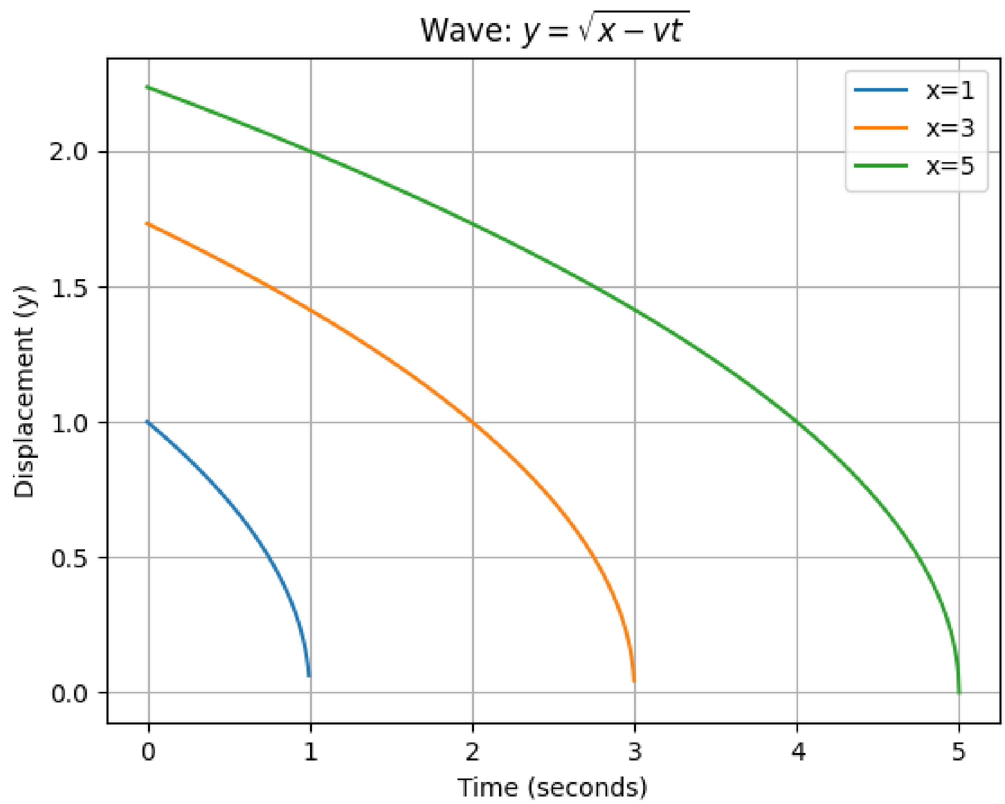

Figure 1.3: DIPLACEMENT *vs* TIME-graph2

From Kirchhoff's Voltage Law (KVL):

$$V(t) = V_R + V_L + V_C \tag{1.4}$$

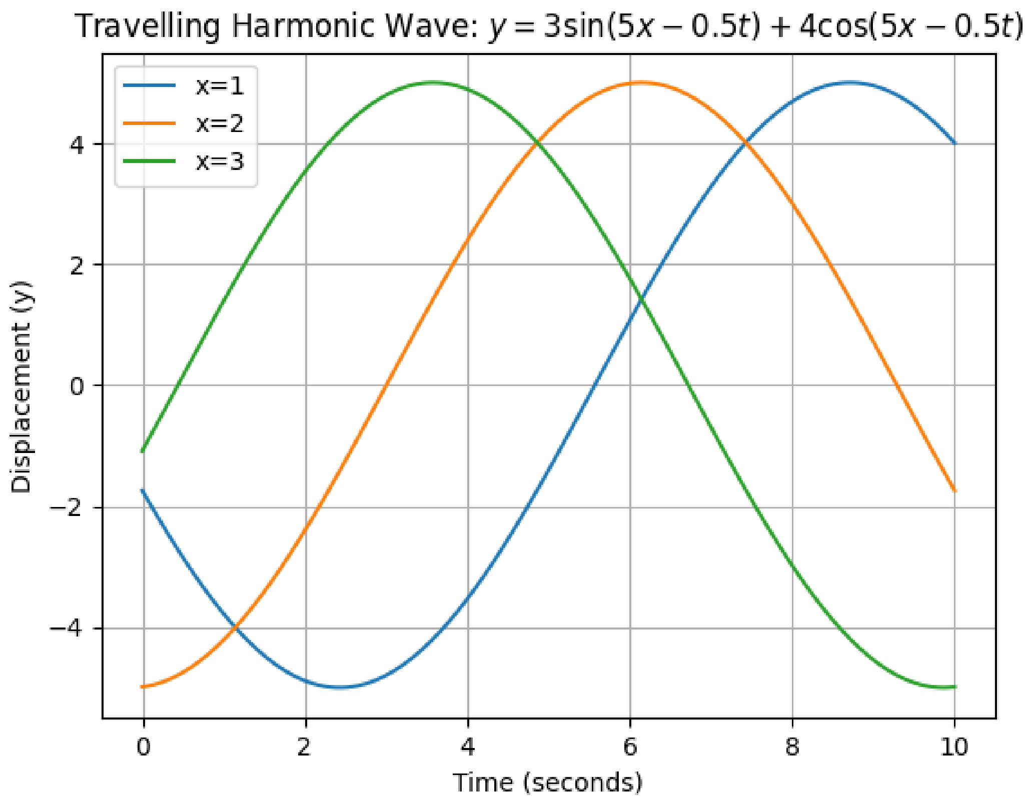

Figure 1.4: DIPLACEMENT vs TIME-graph3

Using reactances from Fig. 1.11,

$$V(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s) \tag{1.5}$$

$$= I(s)\left(R + Ls + \frac{1}{sC}\right) \tag{1.6}$$

$$\implies I(s) = \frac{V(s)}{\left(R + Ls + \frac{1}{sC}\right)} \tag{1.7}$$

At resonance, the circuit becomes purely resistive. The reactances of capacitor

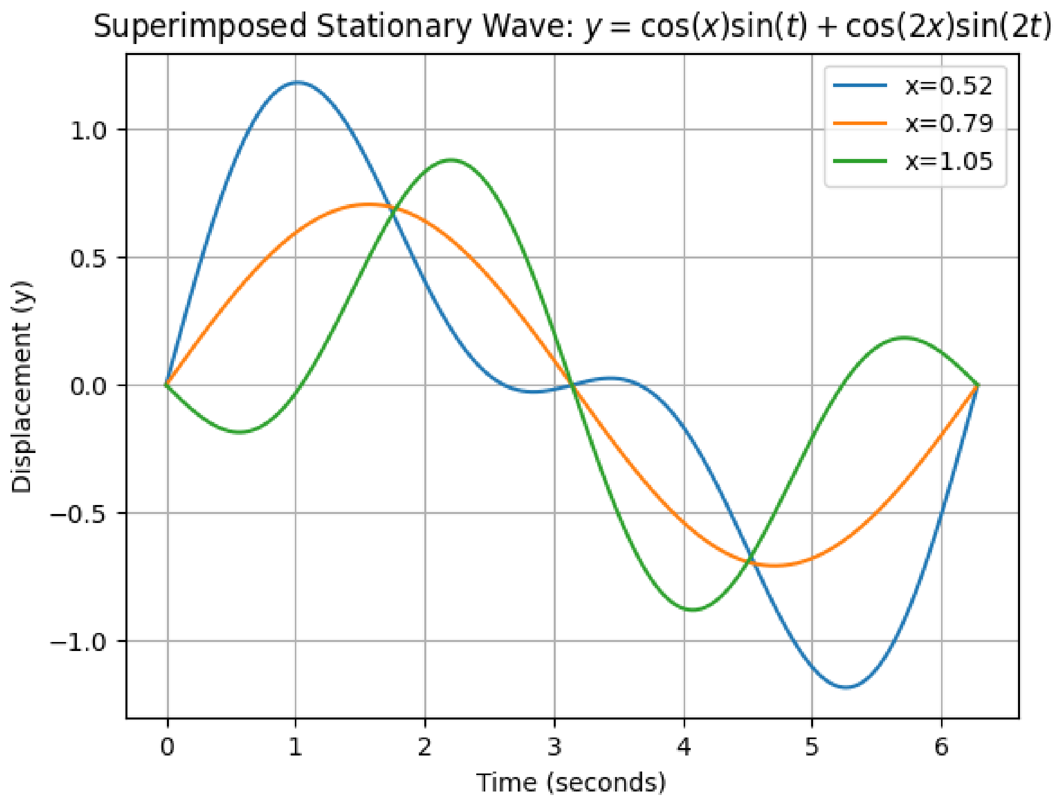

Figure 1.5: DIPLACEMENT vs TIME-graph4

and inductor cancel out as follows:

$$Ls + \frac{1}{sC} = 0 \tag{1.8}$$

$$\implies s = j\frac{1}{\sqrt{LC}} \tag{1.9}$$

s can be expressed in terms of angular resonance frequency as

$$s = j\omega_0 \tag{1.10}$$

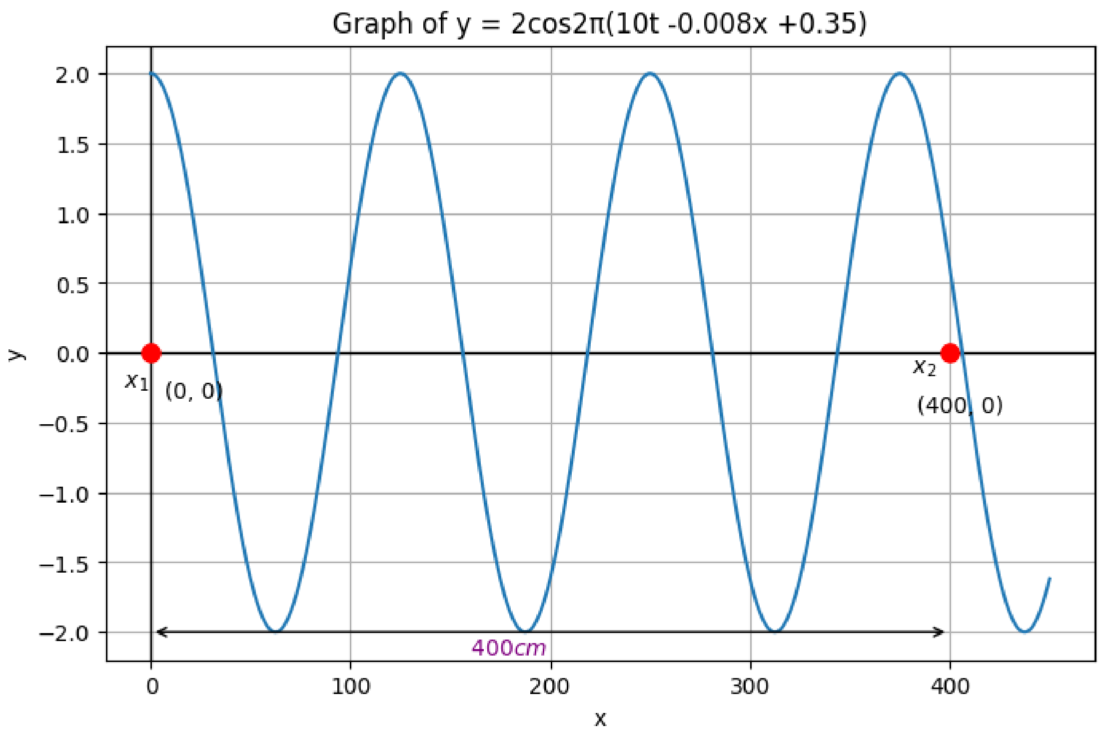

Figure 1.6:

Comparing (1.9) and (1.10), we get

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{1.11}$$

(b) Quality Factor

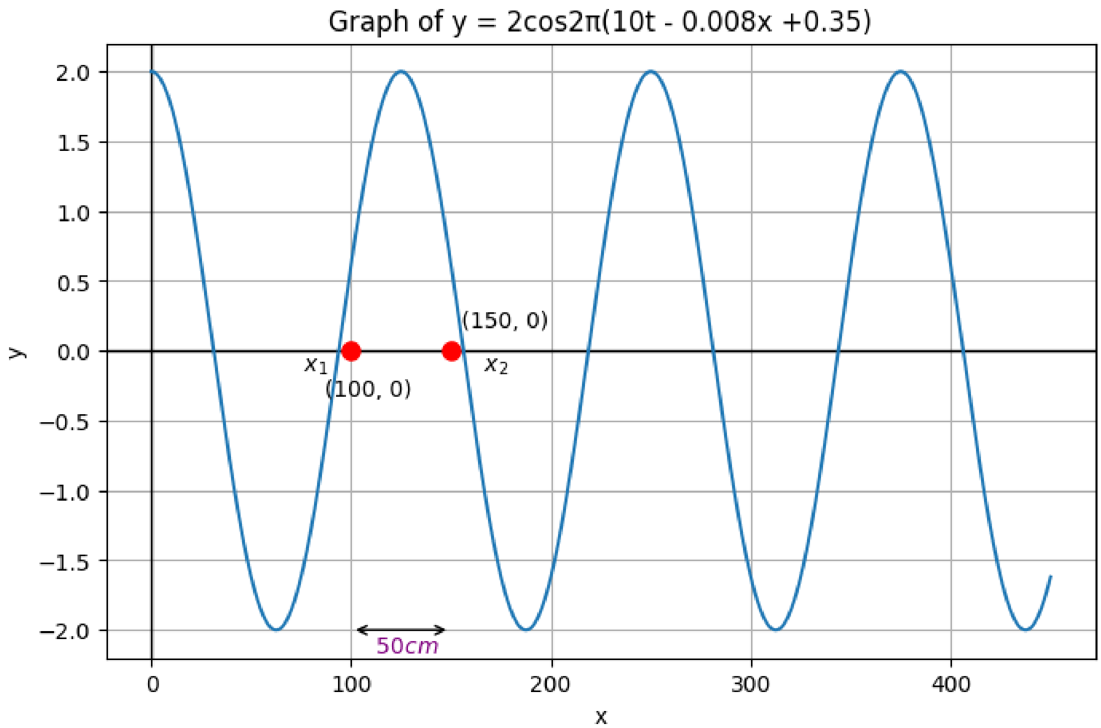

Figure 1.7:

i. Using voltage across inductor,

$$Q = \left(\frac{V_L}{V_R}\right)_{\omega_0} = \frac{|sLI(s)|}{|RI(s)|} \tag{1.12}$$

$$= \frac{1}{\sqrt{LC}}\frac{L}{R} \tag{1.13}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}} \tag{1.14}$$

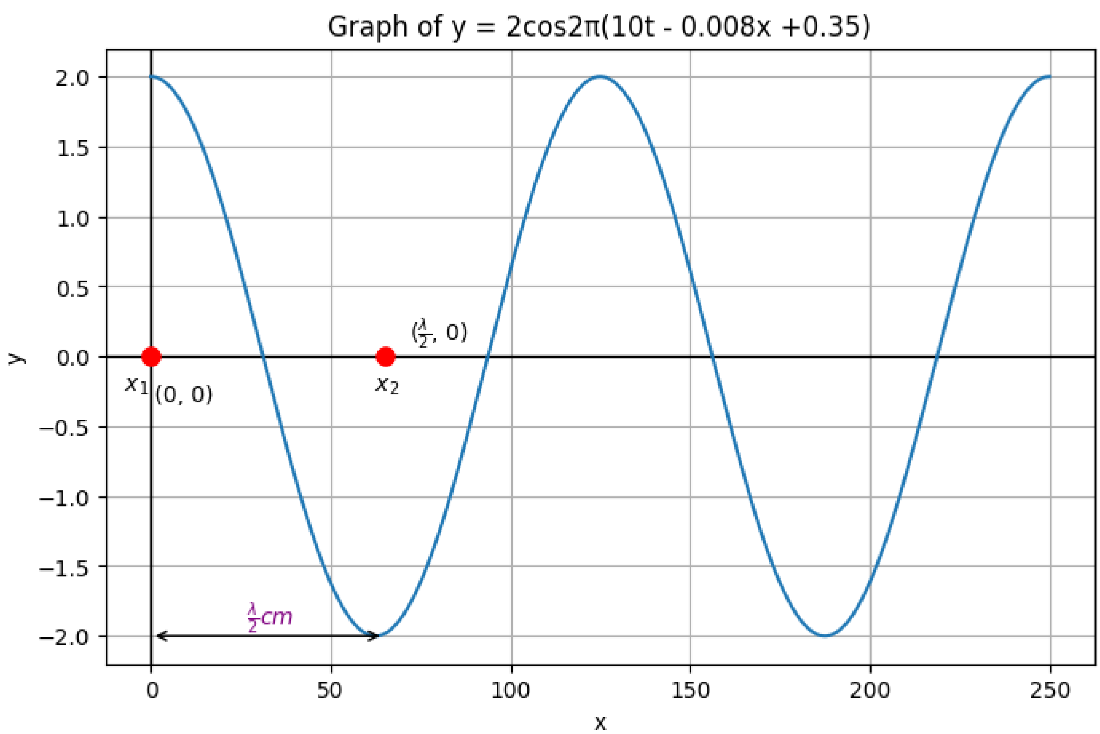

Figure 1.8:

ii. Using voltage across capacitor,

$$Q = \left(\frac{V_C}{V_R}\right)_{\omega_0} = \frac{\left|\frac{I(s)}{sC}\right|}{|RI(s)|} \tag{1.15}$$

$$= \frac{\sqrt{LC}}{RC} \tag{1.16}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}} \tag{1.17}$$

(c) Plot of Impedance vs Angular Frequency

Impedance is defined as

$$H(s) = \frac{V(s)}{I(s)} \tag{1.18}$$

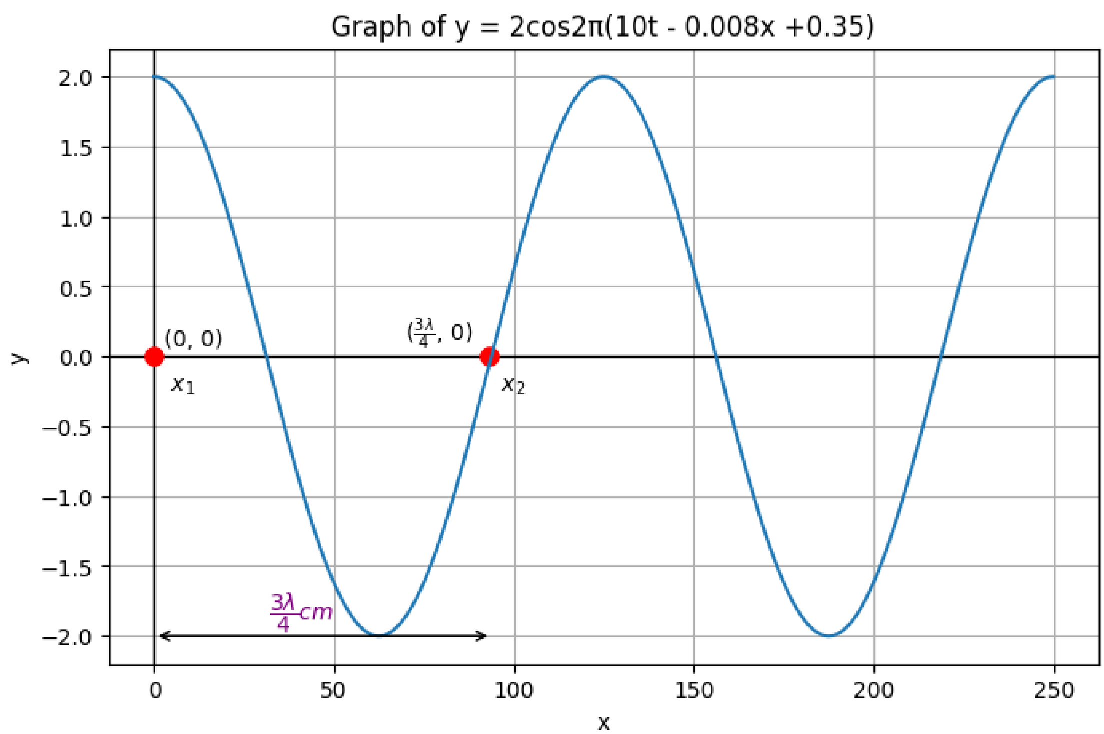

Figure 1.9:

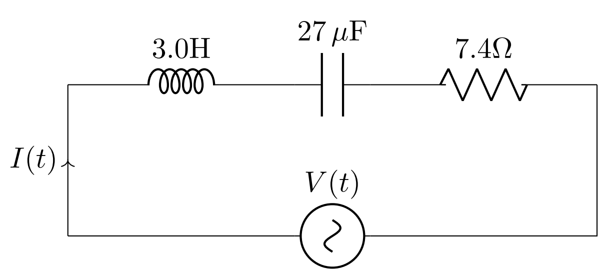

Figure 1.10: LCR Circuit

Using (1.7),

$$H(s) = R + sL + \frac{1}{sC} \tag{1.19}$$

$$\implies H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \tag{1.20}$$

$$\implies |H(j\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{1.21}$$

Symbol	**Value**	**Description**
L	$3.0\,\mathrm{H}$	Inductance
C	$27\,\mu\mathrm{F}$	Capacitance
R	$7.4\,\Omega$	Resistance
Q		Quality Factor: ratio of voltage across inductor or capacitor to that across the resistor at resonance
ω_0	$\dfrac{1}{\sqrt{LC}}$	Angular Resonant Frequency

Table 1.6: Given Parameters

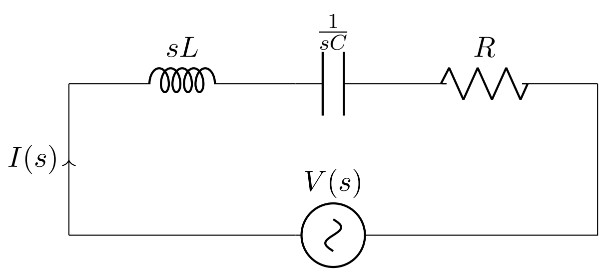

Figure 1.11: LCR Circuit

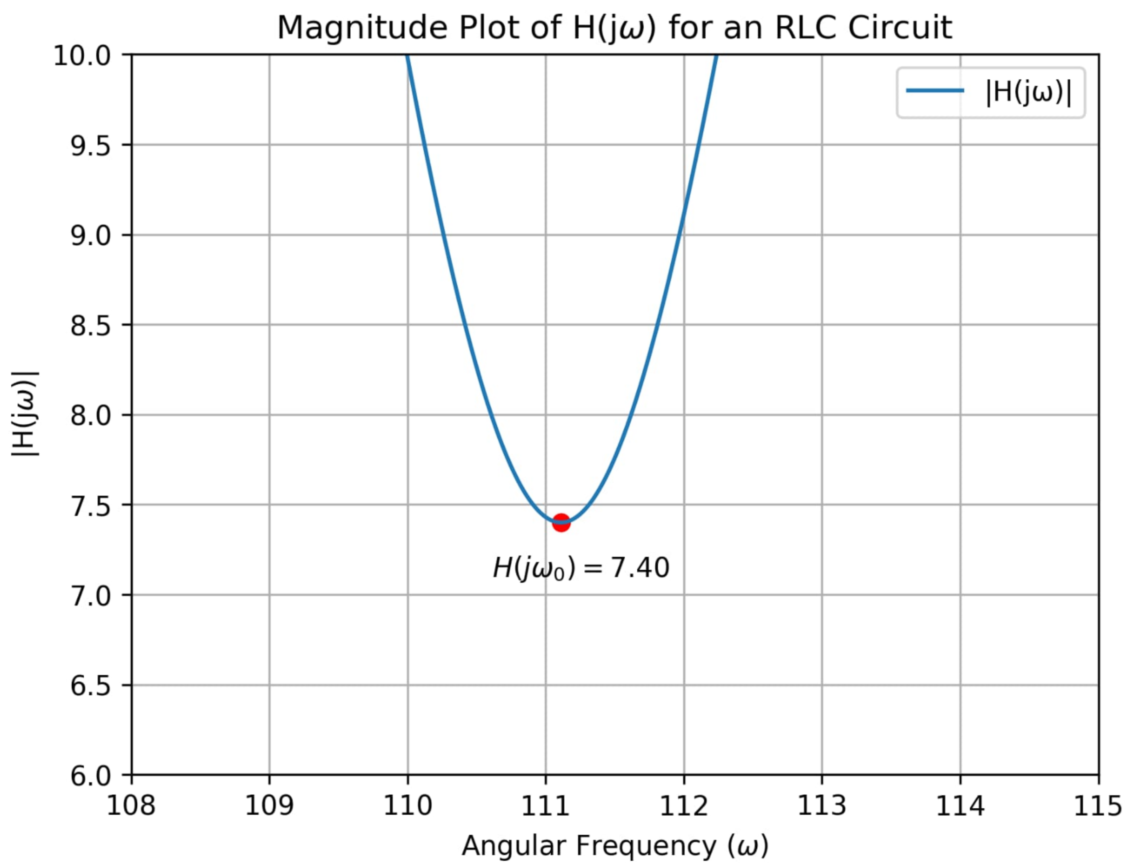

Figure 1.12: Impedance vs ω (using values in Table 1.6)

Chapter 2

Discrete

2.1. Z-transform

2.1.1 Write the five terms at n = 1, 2, 3, 4, 5 of the sequence and obtain the Z-transform of the series

$$
\begin{aligned}
x(n) &= -1, && n = 0 && (2.1)\\
&= \frac{x(n-1)}{n}, && n > 0 && (2.2)\\
&= 0, && n < 0 && (2.3)
\end{aligned}
$$

Solution:

$$x(1) = \frac{x(0)}{1} = -1 \tag{2.4}$$

$$x(2) = \frac{x(1)}{2} = -\frac{1}{2} \tag{2.5}$$

$$x(3) = \frac{x(2)}{3} = -\frac{1}{(2)(3)} = -\frac{1}{6} \tag{2.6}$$

$$x(4) = \frac{x(3)}{4} = -\frac{1}{(2)(3)(4)} = -\frac{1}{24} \tag{2.7}$$

$$x(5) = \frac{x(4)}{5} = -\frac{1}{(2)(3)(4)(5)} = -\frac{1}{120} \tag{2.8}$$

$$x(n) = \frac{-1}{n!}(u(n)) \tag{2.9}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{2.10}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{2.11}$$

using (2.9),

$$= \sum_{n=-\infty}^{\infty} \frac{-1}{n!} u(n) z^{-n} \tag{2.12}$$

$$= \sum_{n=0}^{\infty} \frac{-1}{n!} z^{-n} \tag{2.13}$$

$$= -e^{z^{-1}} \qquad \{z \in \mathbb{C} : z \neq 0\} \tag{2.14}$$

Symbol	**Value**	**Description**
$x(n)$	$\frac{-1}{n!}$	general term of the series
$X(z)$	$-e^{z^{-1}}$	Z-transform of x(n)
$u(n)$		unit step function

Table 2.1: Parameters

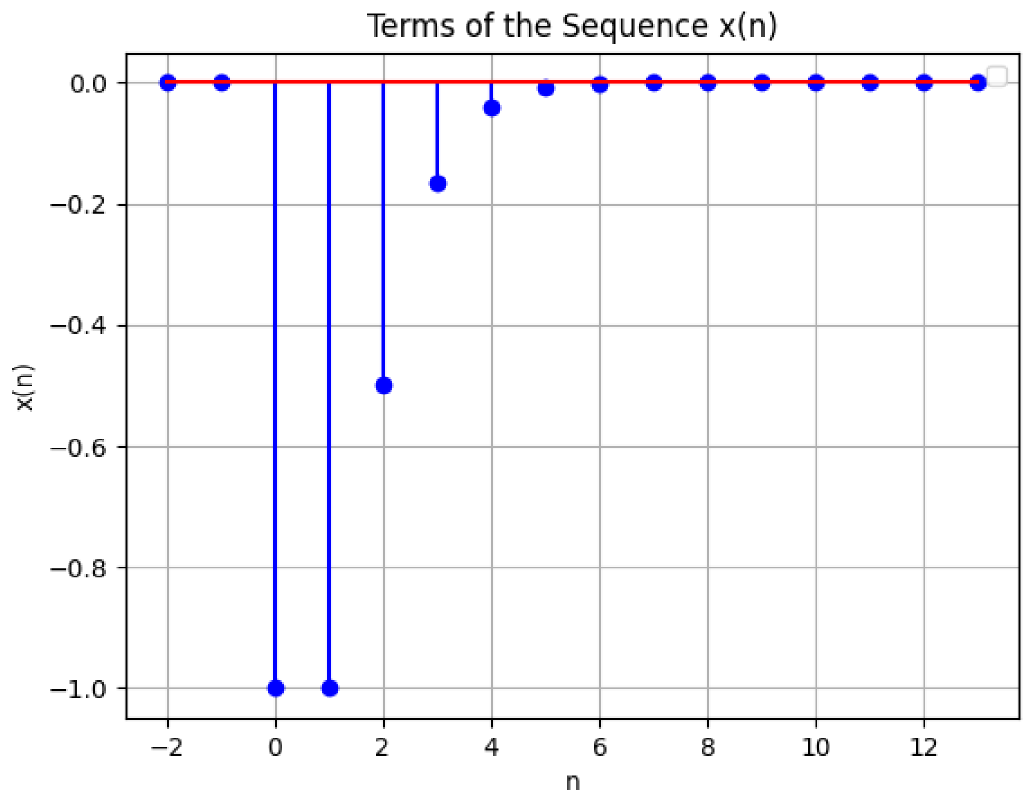

Figure 2.1: Plot of x(n) vs n

2.1.2 Subba Rao started work in 1995 at an annual salary of Rs. 5000 and received an increment of Rs. 200 each year. In which year did his income reach Rs. 7000?

Solution:

Parameter	Value	Description
$x(0)$	5000	Initial Income
d	200	Annual Increment (Common Difference)
$x(n)$	$(x(0)+nd)u(n)$	n^{th} term of the AP

Table 2.2: Input Parameters

From the values given in Table 2.2:

$$7000 = 5000 + 200n \tag{2.15}$$

$$\implies 2000 = 200n \tag{2.16}$$

$$\therefore n = 10 \tag{2.17}$$

Let Z-transform of $x(n)$ be $X(z)$.

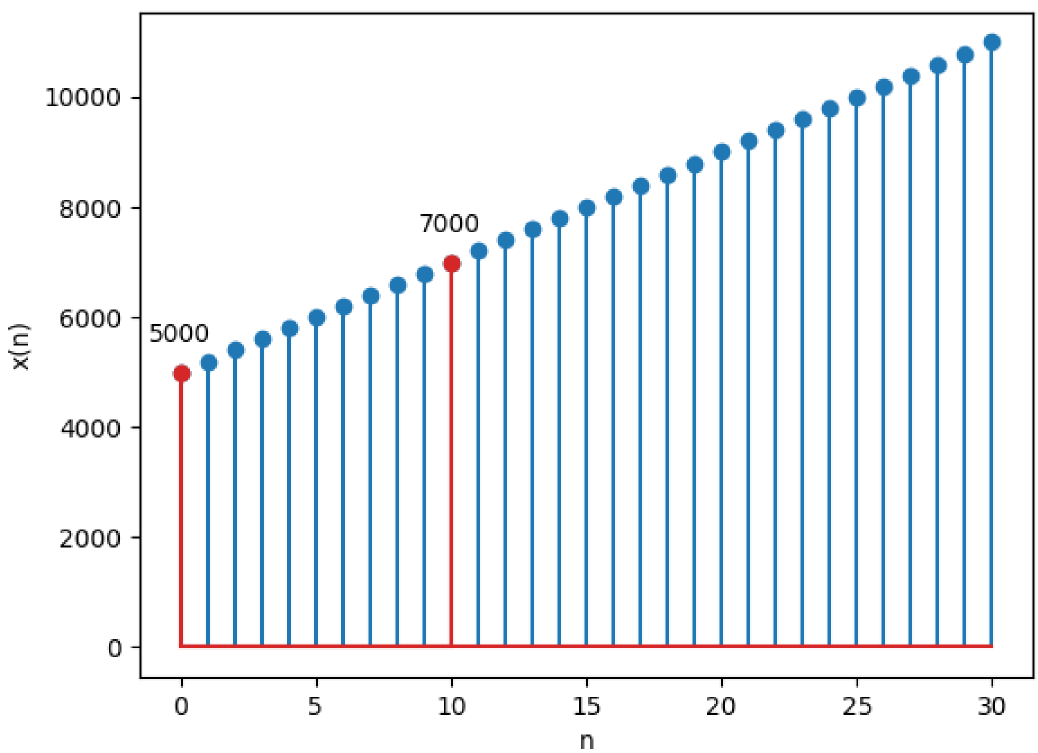

Figure 2.2: Plot of $x(n)$ vs n. See Table 2.2 for details.

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{2.18}$$

Using the values from Table 2.2:

$$X(z) = \frac{5000}{1 - z^{-1}} + \frac{200z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{2.19}$$

Appendix A

Convolution

A.1 The convolution sum is defined as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{A.1.1}$$

A.2 The unit step function is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{A.2.1}$$

A.3 If

$$x(n) = 0, \quad n < 0, \tag{A.3.1}$$

from (A.1.1),

$$x(n) * u(n) = \sum_{k=0}^{n} x(k) \tag{A.3.2}$$

Appendix B

Z-transform

B.1 The Z-transform of $p(n)$ is defined as

$$P(z) = \sum_{n=-\infty}^{\infty} p(n) z^{-n} \tag{B.1.1}$$

B.2 If

$$p(n) = p_1(n) * p_2(n), \tag{B.2.1}$$

$$P(z) = P_1(z) P_2(z) \tag{B.2.2}$$

B.3 For a Geometric progression

$$x(n) = x(0) r^n u(n), \tag{B.3.1}$$

$$\implies X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} x(0) r^n z^{-n} \tag{B.3.2}$$

$$= \sum_{n=0}^{\infty} x(0) \left(r z^{-1}\right)^n \tag{B.3.3}$$

$$= \frac{x(0)}{1 - r z^{-1}}, \quad |z| > |r| \tag{B.3.4}$$

B.4 Substituting $r = 1$ in (B.3.4),

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{B.4.1}$$

B.5 From (B.1.1) and (B.4.1),

$$U(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{B.5.1}$$

$$\implies \frac{dU(z)}{dz} = -z^{-1} \sum_{n=-\infty}^{\infty} n u(n) z^{-n} \tag{B.5.2}$$

$$\therefore n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{B.5.3}$$

B.6 For an AP,

$$x(n) = [x(0) + nd] \, u(n) = x(0) u(n) + d n u(n) \tag{B.6.1}$$

$$\implies X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{d z^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{B.6.2}$$

upon substituting from (B.4.1) and (B.5.3).

B.7 From (A.3.2), the sum to n terms of a GP can be expressed as

$$y(n) = x(n) * u(n) \tag{B.7.1}$$

where $x(n)$ is defined in (B.3.1). From (B.2.2), (B.3.4) and (B.4.1),

$$\begin{aligned} Y(z) &= X(z)\,U(z) && \text{(B.7.2)} \\ &= \left(\frac{x(0)}{1-rz^{-1}}\right)\left(\frac{1}{1-z^{-1}}\right) \quad |z|>|r|\cap|z|>|1| && \text{(B.7.3)} \\ &= \frac{x(0)}{(1-rz^{-1})(1-z^{-1})} \quad |z|>|r| && \text{(B.7.4)} \end{aligned}$$

which can be expressed as

$$Y(z) = \frac{x(0)}{r-1}\left(\frac{r}{1-rz^{-1}} - \frac{1}{1-z^{-1}}\right) \tag{B.7.5}$$

using partial fractions. Again, from (B.3.4) and (B.4.1), the inverse of the above can be expressed as

$$y(n) = x(0)\left(\frac{r^{n+1}-1}{r-1}\right)u(n) \tag{B.7.6}$$