SIGNAL PROCESSING FUNDAMENTALS Through NCERT

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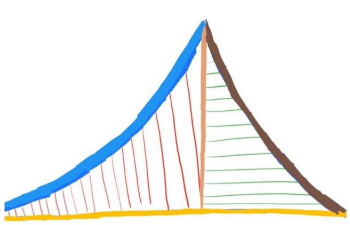


Contents

Introduction iii

1 Analog **1**

1.1 Harmonics 1

1.2 Filters 17

2 Discrete **23**

2.1 Z-transform 23

2.2 Sequences 33

3 Contour Integration **49**

A Convolution **53**

B Z-transform **55**

Introduction

This book introduces some concepts in signal processing through maths and physics problems in NCERT textbooks.

Chapter 1

Analog

1.1. Harmonics

1.1.1 Suppose that the electric field amplitude of an electromagnetic wave is $E_0 = 120\text{N/C}$ and that its frequency is $f = 50.0$ MHz.

(a) Determine, B_0, ω, k and λ

(b) Find expressions for $\mathbf{E}$ and $\mathbf{B}$

Solution:

Table 1.1: Input Parameters

Symbol	Description	value
f	frequency of source	50.0 MHz
E_0	Electric field amplitude	120 N/C
c	speed of light	3 x 10^8 m/s
$\mathbf{e_2}, \mathbf{e_3}$	Standard Basis vectors	N/A

Table 1.2: Formulae and Output

Symbol	Description	Formula	Value
$\mathbf{E}$	Electric field vector	$E_0 \sin(kx - 2\pi f t)\mathbf{e_2}$	$120 \sin[1.05x - 3.14\text{x}10^8 t]\mathbf{e_2}$
$\mathbf{B}$	Magnetic field vector	$B_0 \sin(kx - 2\pi f t)\mathbf{e_3}$	$(4x10^{-7}) \sin[1.05x - 3.14\text{x}10^8 t]\mathbf{e_3}$
B_0	Magnetic field strength	$\frac{E_0}{c}$	400nT
ω	Angular frequency	$2\pi f$	3.14 x 10^8m/s
k	Propagation constant	$\frac{2\pi f}{c}$	1.05rad/s
λ	Wavelength	$\frac{c}{f}$	6.0m

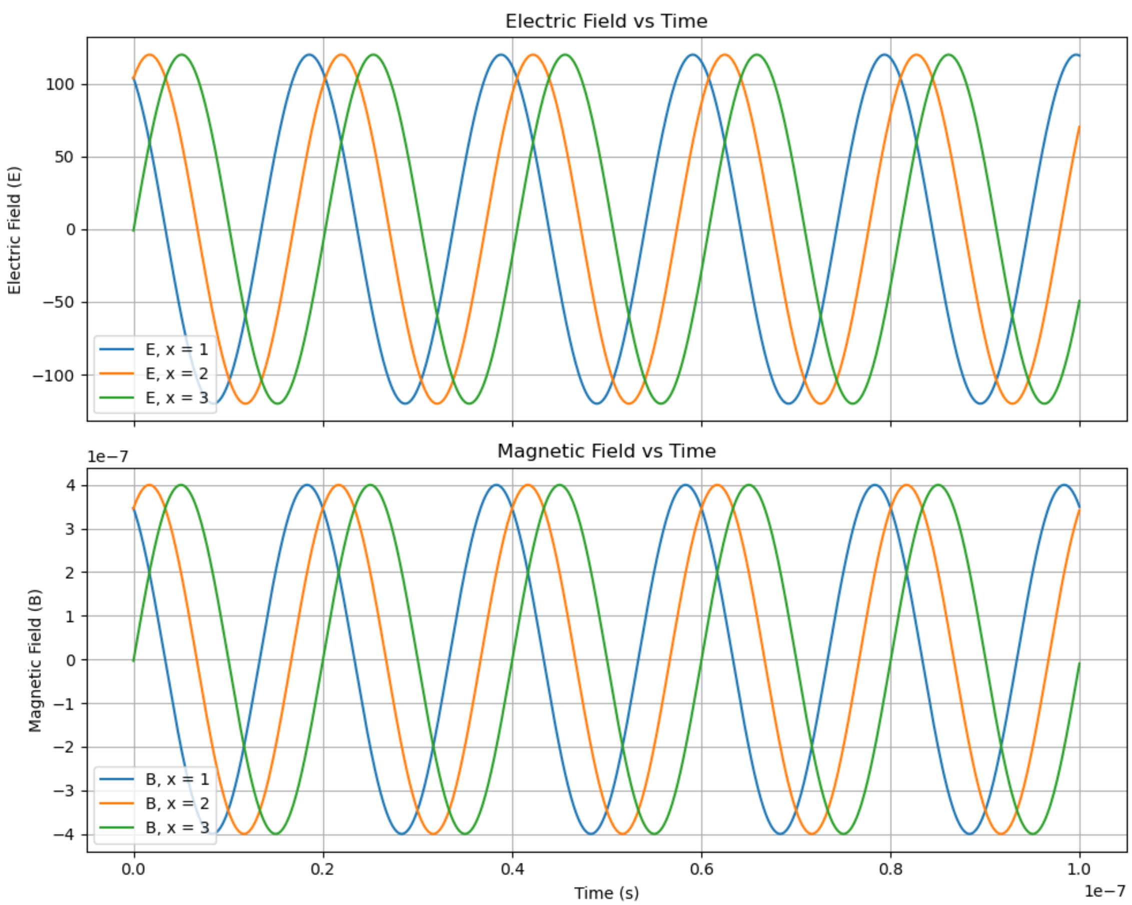


Figure 1.1.1: Graphs of **E** and **B**

1.1.2 A charged particle oscillates about its mean equilibrium position with a frequency of $10^9 Hz$. What is the frequency of the electromagnetic waves produced by the oscillator?

Solution:

Symbol	Value	Description
$y(t)$	$\cos(2\pi f_c t)$	Wave equation of electro-magnetic wave
f_c	10^9	Frequency of electromagnetic wave
t	seconds	Time

Table 1.1.2: Variable description

1.1.3 Given below are some functions of x and t to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represents (i) travelling wave, (ii) a stationary wave or (iii) none at all:

(a) $y = 2\cos(3x)\sin(10t)$

(b) $y = 2\sqrt{x - vt}$

(c) $y = 3\sin(5x - 0.5t) + 4\cos(5x - 0.5t)$

(d) $y = \cos x \sin t + \cos 2x \sin 2t$

Solution:

TRAVELLING WAVE	STATIONARY WAVE
$y(x,t) = A\sin(kx \pm \omega t)$	$y(x,t) = A\sin kx \cos \omega t$

PARAMETERS	DEFINITION
A	Amplitude
ω	Angular Velocity
x	Position
k	Wavenumber

Table 1.1.3: Travelling wave vs Stationary wave

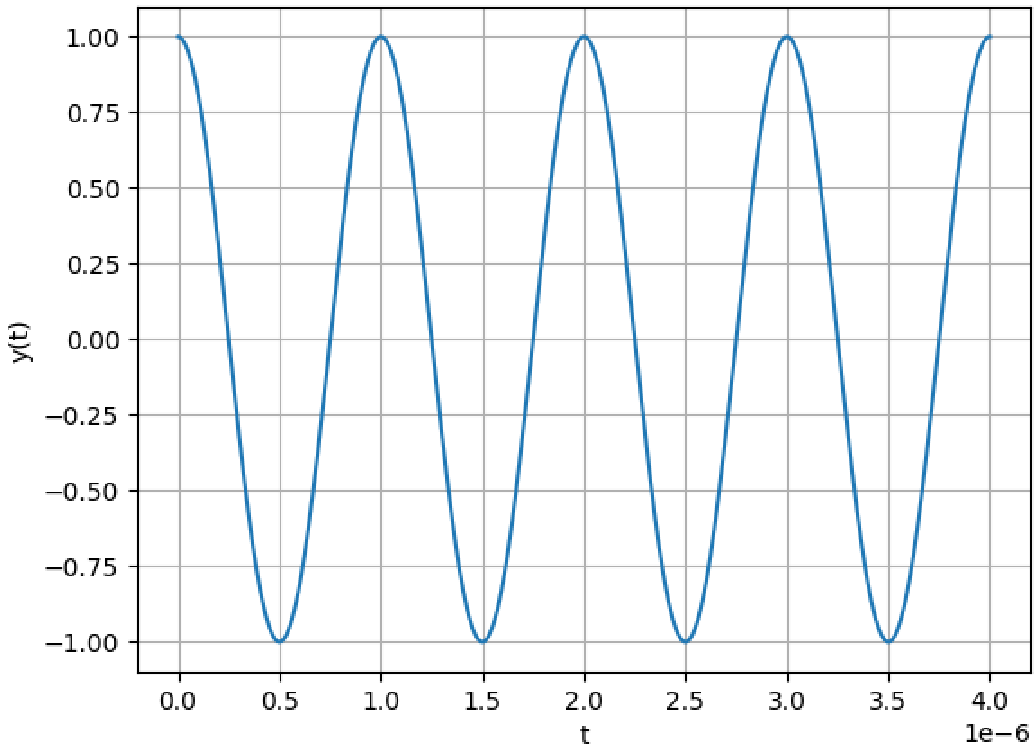


Figure 1.1.2: $y(t) = \cos\left(2\pi \times 10^9 t\right)$

Let us assume an equation:

$$y = A(x)\cos\left(\omega t + \phi\left(x\right)\right) \tag{1.1}$$

Fig. 1.1.3 and Fig. 1.1.3 are self explanatory for stationary and travelling waves.Fig. 1.1.3 and Fig. 1.1.3 are neither stationary nor travelling waves.

STATIONARY WAVE CONDITION	TRAVELLING WAVE CONDITION
(1) $A(x)$ should be a function of position x, and it can be expressed as $A(x) = A_0 cos(\omega t{+}\alpha)$ where A_0 is a constant, k is the wavenumber, x is the position and α is a phase constant.	(1) $A(x)$ should be a constant, and it can be expressed as $A(x) = A_0$ where A_0 is a constant number.
(2) $\phi(x)$ can be expressed as $\phi(x) = c$ where c is a constant.	(2) $\phi(x)$ represents a linear expression in x, and it can be expressed as $\phi(x) = kx{+}\theta$ where k is the wavenumber and θ is the phaseconstant.

Table 1.1.3: Travelling wave *vs* Stationary wave

1.1.4 For the travelling harmonic wave $y\,(x,t) = 2.0\cos 2\pi\,(10t - 0.0080x + 0.35)$ where x and y are in cm and t in s. Calculate the phase difference between oscillatory motion of two points separated by a distance of

(a) $4m$

(b) $0.5m$

(c) $\lambda/2$

(d) $3\lambda/4$

Solution:

$$(\Delta\theta) = (2\pi ft - kx_1 + \phi) - (2\pi ft - kx_2 + \phi) \tag{1.2}$$

$$= k\,(x_2 - x_1) \tag{1.3}$$

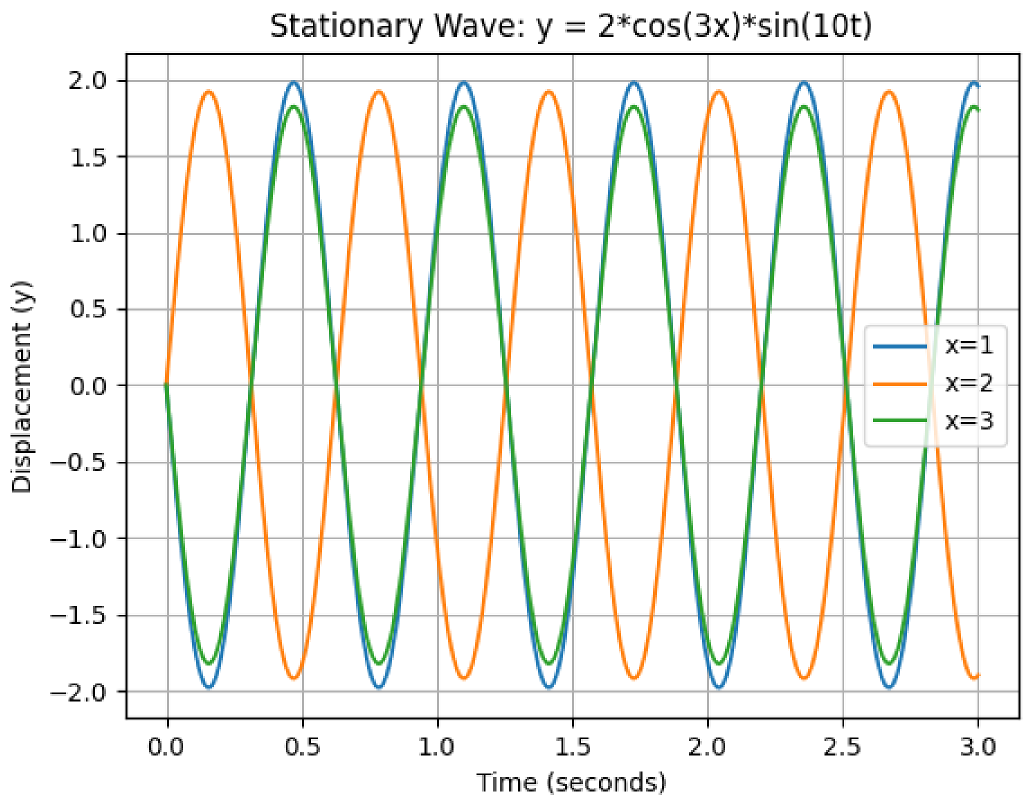


Figure 1.1.3: DIPLACEMENT *vs* TIME-graph1

1.1.5 (a) The peak voltage of an AC supply is 300 V. What is the rms voltage?

(b) The rms value of current in an AC circuit is 10 A. What is the peak current?

Solution:

Parameter	**Description**	**Value**
$y\left(x_i, t\right)$	equation of harmonic wave	$A \cos\left(2\pi f t - k x_i + \phi\right)$
k	angular wave number	$2\pi\,(0.008)$
$\lambda = \dfrac{2\pi}{k}$	wavelength	$125\,cm$
f	frequency	10
A	amplitude	2.0
ϕ	phase constant	$2\pi\,(0.35)$
θ_i	phase of i^{th} harmonic wave	$\left(2\pi f t - kx + \phi\right)$
x_i	position of i^{th} harmonic wave	
t	time	
$x_2 - x_1$	path difference	$400\,cm$
		$50\,cm$
		$\dfrac{\lambda}{2}$
		$\dfrac{3\lambda}{4}$

Table 1.1.4: Given parameters list

Parameter	**Description**	**subquestion**	**Value**
$\Delta\theta$	$\theta_1 - \theta_2$	(a)	6.4π radians
		(b)	0.8π radians
		(c)	π radians
		(d)	$\dfrac{3\pi}{2}$ radians

Table 1.1.4: Phase differences

parameter	value	description
$V(t)$	$V_0 \cdot \sin(2\pi f t + \phi)$	voltage in terms of time
$I(t)$	$I_0 \cdot \sin(2\pi f t + \phi)$	current in terms of time
V_0	$300\,\mathrm{V}$	peak voltage
V_{rms}	$\sqrt{\frac{1}{T}\int_0^T [V(t)]^2\,dt}$	rms value of Voltage
I_{rms}	$10\,\mathrm{A}$	rms value of current
I_0	$\sqrt{2} \times I_{\mathrm{rms}}$	peak current
f	$50\,\mathrm{Hz}$	frequency of the sinusoidal wave
T	$0.02\,\mathrm{s}$	time period of sinusoidal wave

Table 1.1.5: Input Parameter Table

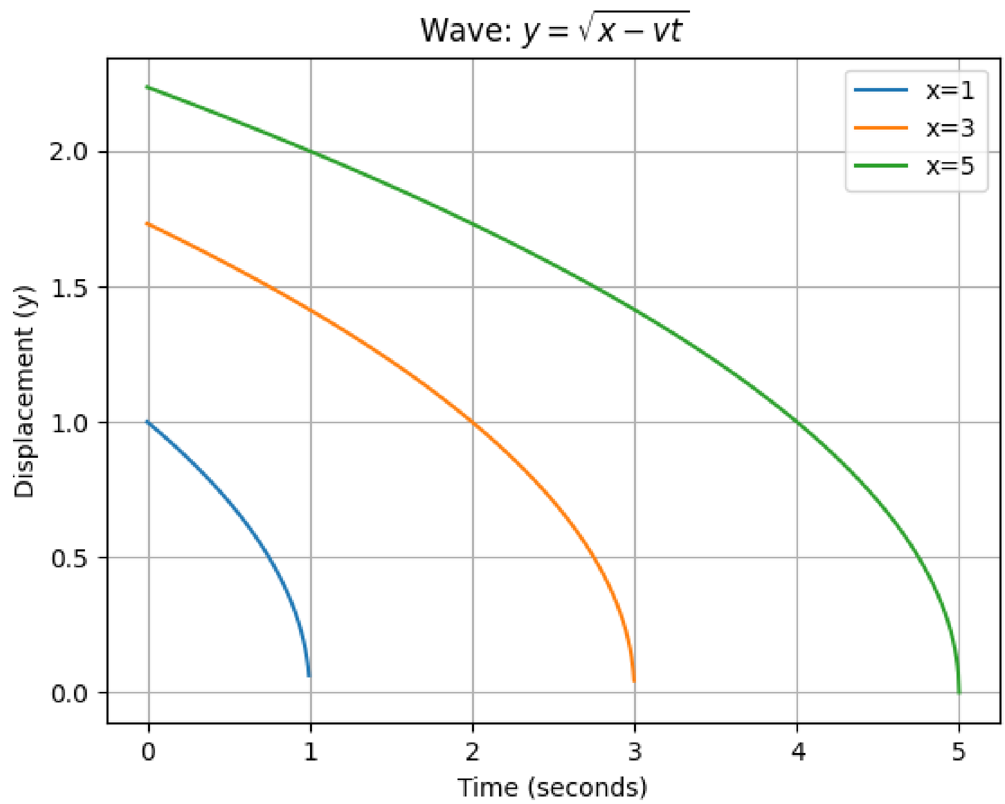


Figure 1.1.3: DIPLACEMENT vs TIME-graph2

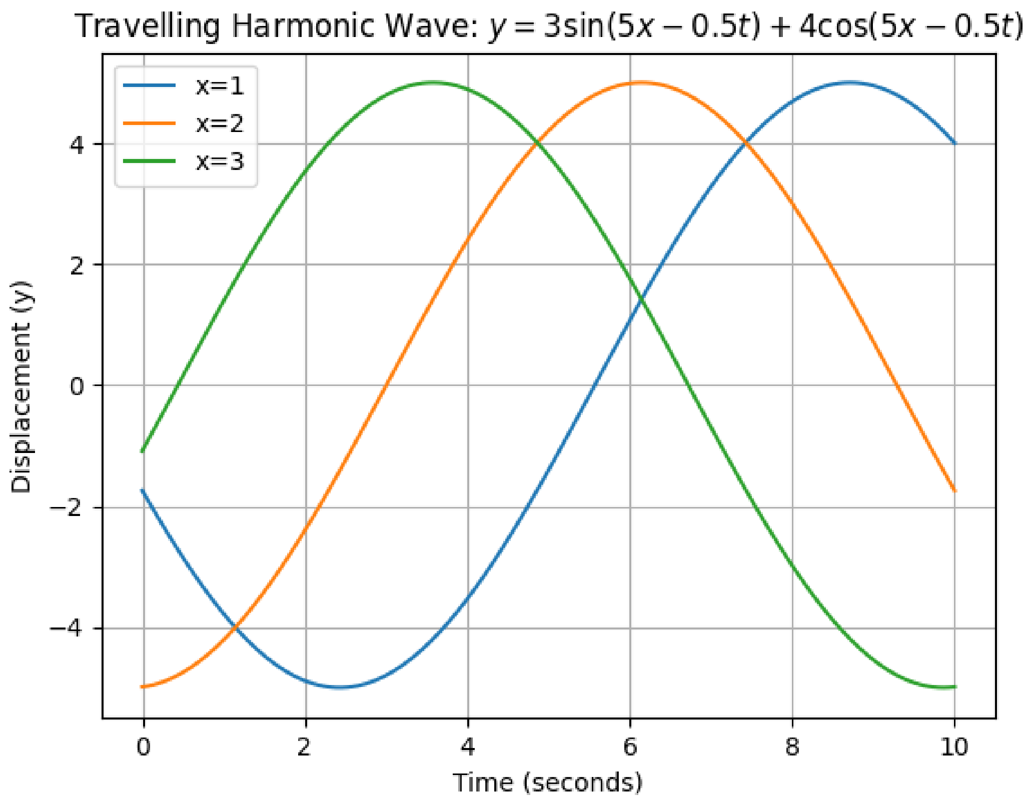


Figure 1.1.3: DIPLACEMENT vs TIME-graph3

(a)

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T [V(t)]^2\,dt \tag{1.4}$$

$$= f\int_0^{\frac{1}{f}} V_0^2 \cdot \sin^2(2\pi f t + \phi)\,dt \tag{1.5}$$

$$= \frac{1}{2}V_0^2\left(1 - \frac{1}{f}\int_0^{\frac{1}{f}} \cos(4\pi f t + 2\phi)\,dt\right) \tag{1.6}$$

$$= \frac{1}{2}V_0^2\left(1 - \frac{1}{f}\left[\frac{\sin(4\pi f t + 2\phi)}{4\pi f}\right]_0^{\frac{1}{f}}\right) \tag{1.7}$$

$$= \frac{1}{2}V_0^2\left(1 - \frac{1}{f}\cdot\frac{\sin(4\pi + 2\phi) - \sin(0 + 2\phi)}{4\pi f}\right) \tag{1.8}$$

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}} \tag{1.9}$$

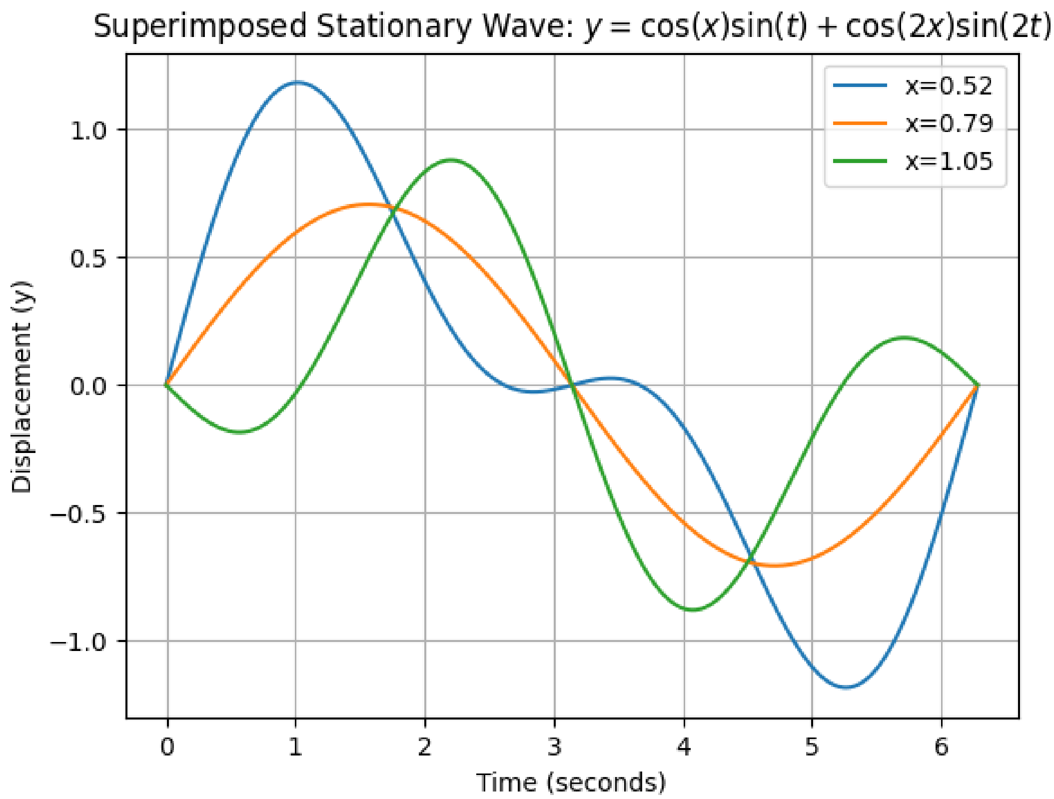


Figure 1.1.3: DIPLACEMENT *vs* TIME-graph4

To find the RMS voltage (V_{rms}) when the peak voltage (V_0) is 300V, you can use equation (1.9)

$$V_{\text{rms}} = \frac{300V}{\sqrt{2}} \approx 212.13V \tag{1.10}$$

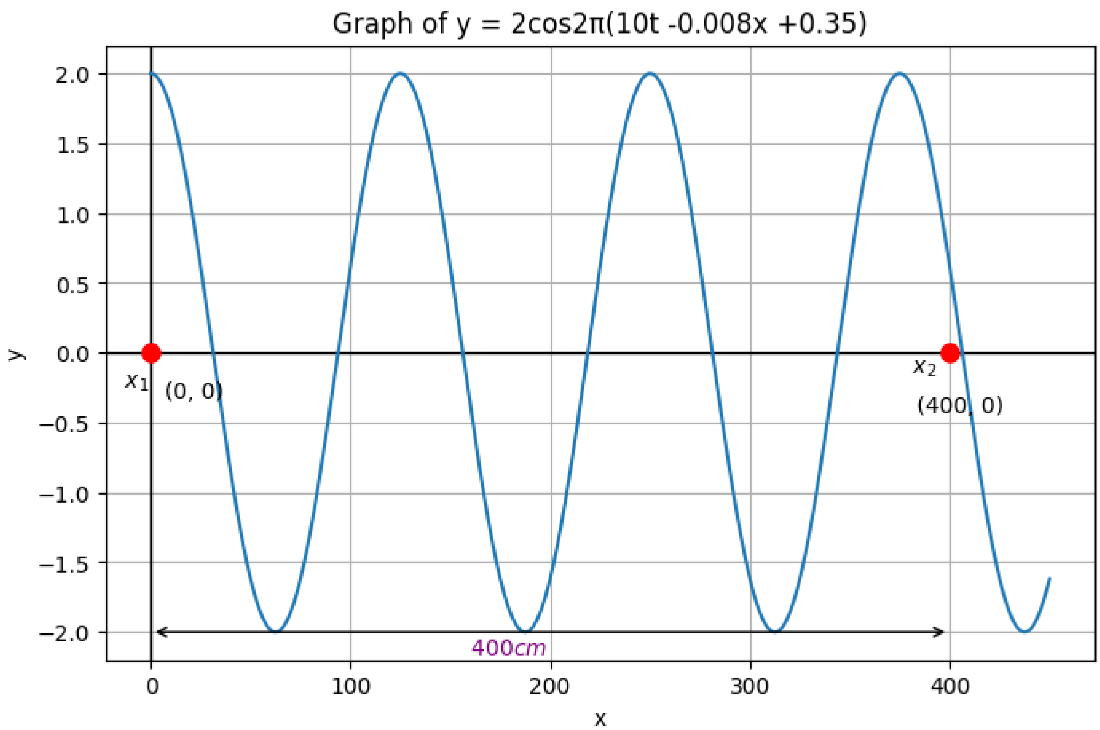


Figure 1.1.4:

(b)

$$I_{\text{rms}}^2 = \frac{1}{T}\int_0^T [I(t)]^2\,dt \tag{1.11}$$

$$= f\int_0^{\frac{1}{f}} I_0^2 \cdot \sin^2(2\pi ft+\phi)\,dt \tag{1.12}$$

$$= \frac{1}{2}I_0^2\left(1-\frac{1}{f}\left[\frac{\sin(4\pi ft+2\phi)}{4\pi f}\right]_0^{\frac{1}{f}}\right) \tag{1.13}$$

$$= \frac{1}{2}I_0^2\left(1-\frac{1}{f}\cdot\frac{\sin(4\pi+2\phi)-\sin(0+2\phi)}{4\pi f}\right) \tag{1.14}$$

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}} \tag{1.15}$$

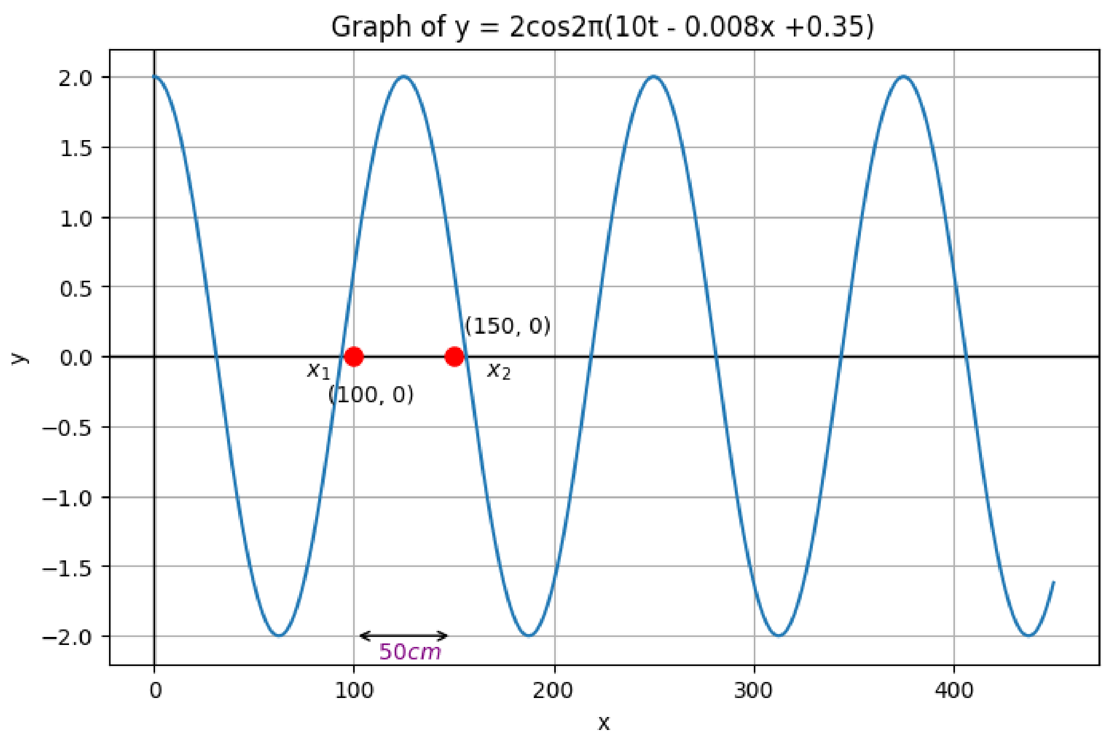
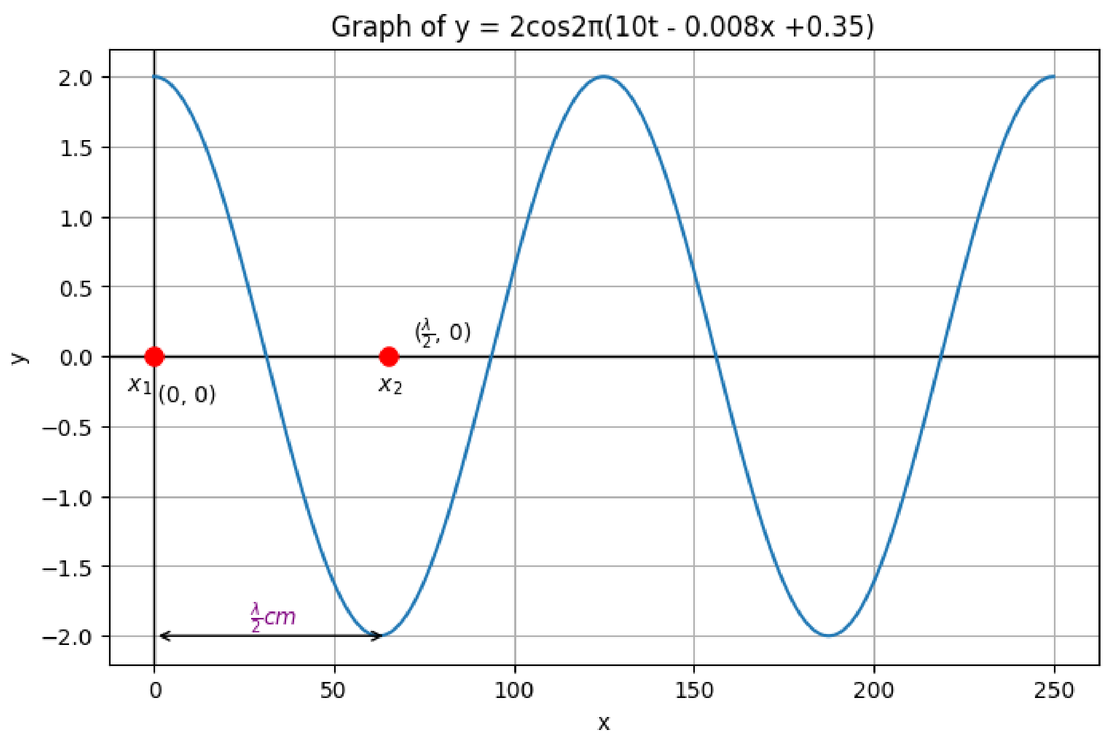


Figure 1.1.4:

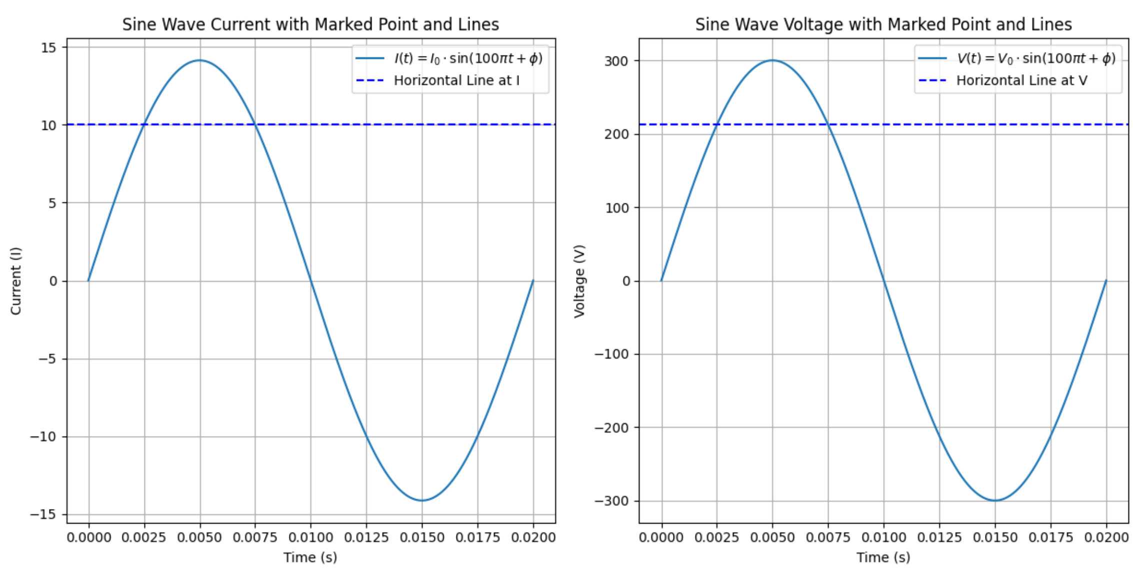
To find the peak current (I_0) when the RMS current (I_{rms}) is given, you can use equation (1.15)

$$I_0 \approx 10\,\text{A} \times 1.414 \approx 14.14\,\text{A} \tag{1.16}$$



Graph of y = 2cos2π(10t - 0.008x +0.35)
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
y
$(\frac{\lambda}{2}, 0)$
x_1
(0, 0)
x_2
$\frac{\lambda}{2}cm$
0
50
100
150
200
250
x

Figure 1.1.4:



Sine Wave Current with Marked Point and Lines
$I(t) = I_0 \cdot \sin(100\pi t + \phi)$
Horizontal Line at I
Current (I)
Time (s)
Sine Wave Voltage with Marked Point and Lines
$V(t) = V_0 \cdot \sin(100\pi t + \phi)$
Horizontal Line at V
Voltage (V)
Time (s)

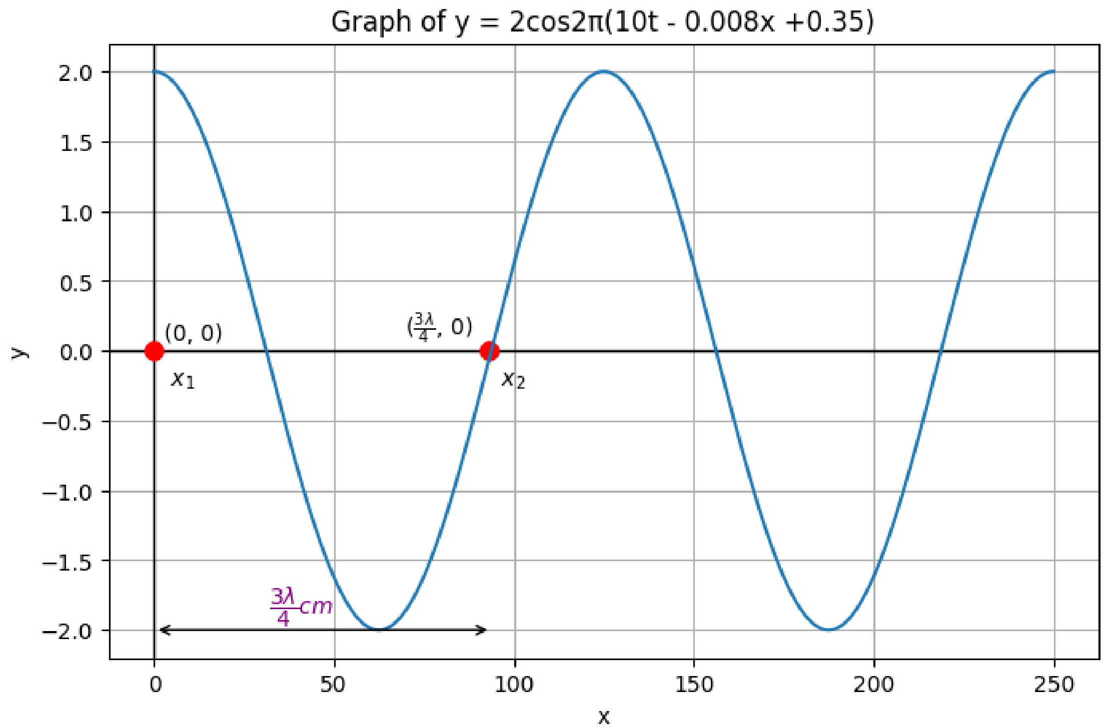


Figure 1.1.4:

1.1.6 In Young's double-slit experiment using monochromatic light of wavelength λ, the intensity of light at a point on the screen where path difference is λ, is K units. What is the intensity of light at a point where path difference is $\lambda/3$?

Solution:

From Table 1.1.6:

$$y\left(t\right) = A\sin(2\pi ft - kx_1) + A\sin(2\pi ft - kx_2) \tag{1.17}$$

$$y\left(t\right) = 2A\cos\left(\frac{k\Delta x}{2}\right)\sin\left(2\pi ft - \frac{k(x_1 + x_2)}{2}\right) \tag{1.18}$$

Parameter	**Description**	**Value**
$y_i\left(t\right)$	Equation of light from $S_{i^{\text{th}}}$	$A\sin(\omega t - kx_i)$
k	Wave number	$\frac{2\pi}{\lambda}$
I	Intensity of wave	$\propto A^2$
$\Delta x = x_1 - x_2$	Path difference	λ
		$\frac{\lambda}{3}$
K	Intensity of light at $\Delta x = \lambda$	
A	Amplitude of wave from source	
r	constant	$r \geq 0$

Table 1.1.6: Parameters

From Table 1.1.6 and equation (1.18):

$$\therefore I \propto 4A^2\cos^2\left(\frac{k\Delta x}{2}\right) \tag{1.19}$$

From Table 1.1.6 and equation (1.19):

$$\frac{K}{I_r} = \frac{4A^2\cos^2\left(\frac{2\pi}{2}\right)}{4A^2\cos^2\left(\frac{\pi}{3}\right)} \implies I_r = \frac{K}{4} \tag{1.20}$$

Hence, the Intensity of light at a point where path difference is $\frac{\lambda}{3}$ is $\frac{K}{4}$ units.

Parameter	**Description**	**Value**
I_r	Net Intensity of light at $\Delta x = \frac{\lambda}{3}$	$\frac{K}{4}$

Table 1.1.6:

Assuming $\Delta x = r\lambda$,

From equation (1.19):

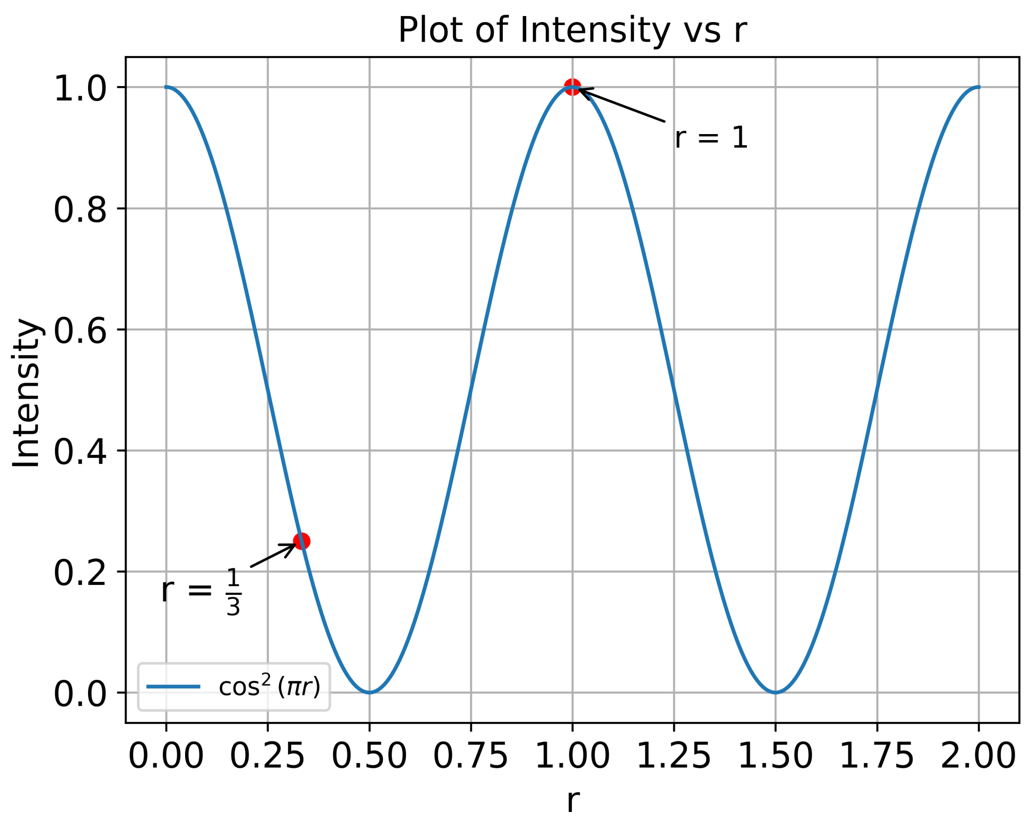


Figure 1.1.6:

1.2. Filters

1.2.1 Obtain the resonant frequency and Q-factor of a series LCR circuit with $L = 3.0\,H$, $C = 27\,\mu F$, and $R = 7.4\,\Omega$. It is desired to improve the sharpness of the resonance of the circuit by reducing its 'full width at half maximum' by a factor of 2. Suggest a suitable way.

Solution: Given parameters are:

Symbol	**Value**	**Description**
L	$3.0\,\mathrm{H}$	Inductance
C	$27\,\mu\mathrm{F}$	Capacitance
R	$7.4\,\Omega$	Resistance
Q		Quality Factor: ratio of voltage across inductor or capacitor to that across the resistor at resonance
ω_0	$\dfrac{1}{\sqrt{LC}}$	Angular Resonant Frequency

Table 1.11: Given Parameters

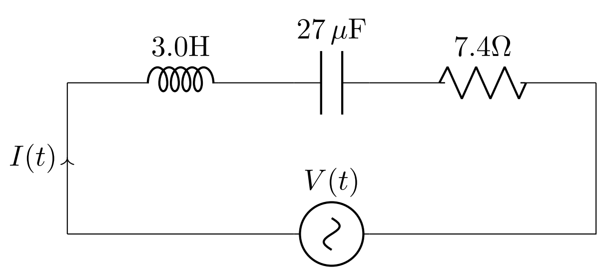


Figure 1.12: LCR Circuit

(a) Frequency Response of the Circuit

From Kirchhoff's Voltage Law (KVL):

$$V(t) = V_R + V_L + V_C \tag{1.21}$$

Using reactances from Fig. 1.13,

$$V(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s) \tag{1.22}$$
$$= I(s)\left(R + Ls + \frac{1}{sC}\right) \tag{1.23}$$
$$\implies I(s) = \frac{V(s)}{\left(R + Ls + \frac{1}{sC}\right)} \tag{1.24}$$

At resonance, the circuit becomes purely resistive. The reactances of capacitor

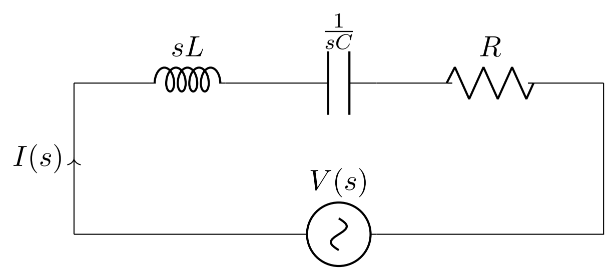


Figure 1.13: LCR Circuit

and inductor cancel out as follows:

$$Ls + \frac{1}{sC} = 0 \tag{1.25}$$
$$\implies s = j\frac{1}{\sqrt{LC}} \tag{1.26}$$

s can be expressed in terms of angular resonance frequency as

$$s = j\omega_0 \tag{1.27}$$

Comparing (1.26) and (1.27), we get

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{1.28}$$

(b) Quality Factor

i. Using voltage across inductor,

$$Q = \left(\frac{V_L}{V_R}\right)_{\omega_0} = \frac{|sLI(s)|}{|RI(s)|} \tag{1.29}$$

$$= \frac{1}{\sqrt{LC}}\frac{L}{R} \tag{1.30}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}} \tag{1.31}$$

ii. Using voltage across capacitor,

$$Q = \left(\frac{V_C}{V_R}\right)_{\omega_0} = \frac{\left|\frac{I(s)}{sC}\right|}{|RI(s)|} \tag{1.32}$$

$$= \frac{\sqrt{LC}}{RC} \tag{1.33}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}} \tag{1.34}$$

(c) Plot of Impedance vs Angular Frequency

Impedance is defined as

$$H(s) = \frac{V(s)}{I(s)} \tag{1.35}$$

Using (1.24),

$$H(s) = R + sL + \frac{1}{sC} \tag{1.36}$$

$$\implies H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \tag{1.37}$$

$$\implies |H(j\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{1.38}$$

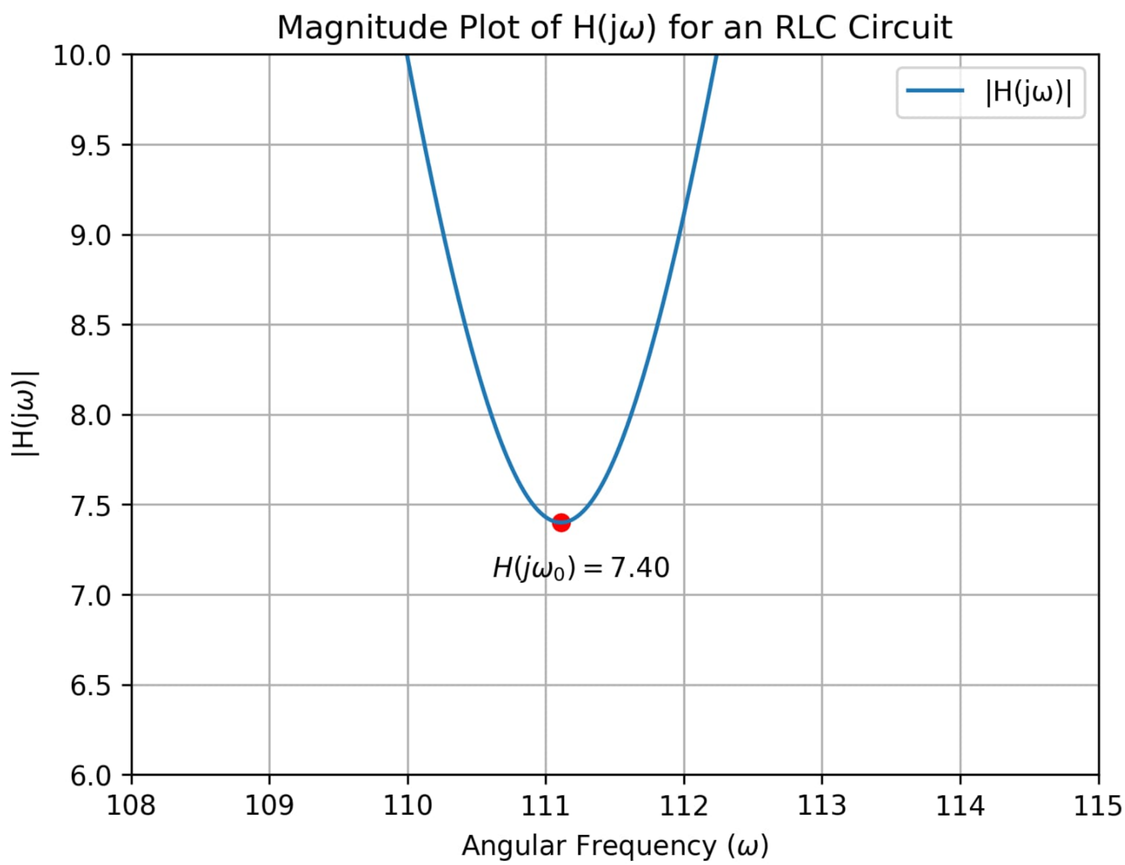


Figure 1.14: Impedance vs ω (using values in Table 1.11)

Chapter 2

Discrete

2.1. Z-transform

2.1.1 Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \cdots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \cdots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

Solution:

Parameter	**Description**	**Value**
n	Integer	.... -2,-1,0,1, 2, ...
$x_1(n)$	General term of Numerator	$\left(n^3+5n^2+8n+4\right) \cdot u(n)$
$x_2(n)$	General Term of Denominator	$\left(n^3+4n^2+5n+2\right) \cdot u(n)$
$y_1(n)$	Sum of terms of numerator	?
$y_2(n)$	Sum of terms of denominator	?
$U(z)$	z-transform of $u(n)$	$\frac{1}{1-z^{-1}}, \{z \in \mathbb{C} : \lvert z\rvert > 1\}$
ROC	Region of convergence	$\left\{z : \left\lvert\sum_{n=-\infty}^{\infty} x(n)z^{-n}\right\rvert < \infty\right\}$

Table 1: Parameter Table

1. Analysis of Numerator:

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} \tag{2.1}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 5n^2 + 8n + 4\right) u(n) z^{-n} \tag{2.2}$$

Using results of equations (B.3.2) to (B.3.5) we get:

$$\therefore X_1(z) = \frac{4 + 2z^{-1}}{(1 - z^{-1})^4}, |z| > 1 \tag{2.3}$$

From (A.3.2)

$$y_1(n) = x_1(n) * u(n) \tag{2.4}$$

$$Y_1(z) = X_1(z) U(z) \tag{2.5}$$

$$= \frac{4 + 2z^{-1}}{(1 - z^{-1})^5}, |z| > 1 \tag{2.6}$$

Using partial fractions:

$$Y_1(z) = \frac{22z^{-1}}{(1 - z^{-1})} + \frac{48z^{-2}}{(1 - z^{-1})^2} + \frac{52z^{-3}}{(1 - z^{-3})^3}, \tag{2.7}$$

$$+ \frac{28z^{-4}}{(1 - z^{-1})^4} + \frac{6z^{-5}}{(1 - z^{-1})^5} + 4, |z| > 1$$

Substituting results of equation (B.4.6) to (B.4.9) in equation (2.7):

$$y_1(n) = \frac{3n^4 + 26n^3 + 81n^2 + 106n + 48}{12} u(n) \tag{2.8}$$

$$= \frac{(3n + 8)(n + 1)(n + 2)(n + 3)}{12} u(n) \tag{2.9}$$

2. <u>Analysis of Denominator:</u>

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \tag{2.10}$$

$$= \sum_{n=-\infty}^{\infty} \left(n^3 + 4n^2 + 5n + 2\right) u(n) z^{-n} \tag{2.11}$$

Using results of equation (B.3.2) to (B.3.5) we get:

$$\therefore X_2(z) = \frac{2 + 4z^{-1}}{(1 - z^{-1})^4}, |z| > 1 \tag{2.12}$$

From (A.3.2)

$$y_2(n) = x_2(n) * u(n) \tag{2.13}$$

$$Y_2(z) = X_2(z) U(z) \tag{2.14}$$

$$= \frac{2 + 4z^{-1}}{(1 - z^{-1})^5}, |z| > 1 \tag{2.15}$$

Using partial fractions:

$$\begin{aligned} Y_2(z) = & \frac{14z^{-1}}{(1 - z^{-1})} + \frac{36z^{-2}}{(1 - z^{-1})^2} + \frac{44z^{-3}}{(1 - z^{-3})^3} \\ & + \frac{26z^{-4}}{(1 - z^{-1})^4} + \frac{6z^{-5}}{(1 - z^{-1})^5} + 2, |z| > 1 \end{aligned} \tag{2.16}$$

Substituting results of equation (B.4.6) to (B.4.9) in equation (2.16):

$$y_2(n) = \frac{3n^4 + 22n^3 + 57n^2 + 62n + 24}{12} u(n) \tag{2.17}$$

$$= \frac{(3n + 4)(n + 1)(n + 2)(n + 3)}{12} u(n) \tag{2.18}$$

As the sequence start from $n = 0$, in RHS of question n should be replaced by $n + 1$:

$$\frac{y_1(n)}{y_2(n)} = \frac{3n+8}{3n+4} \tag{2.19}$$

Hence Prooved.

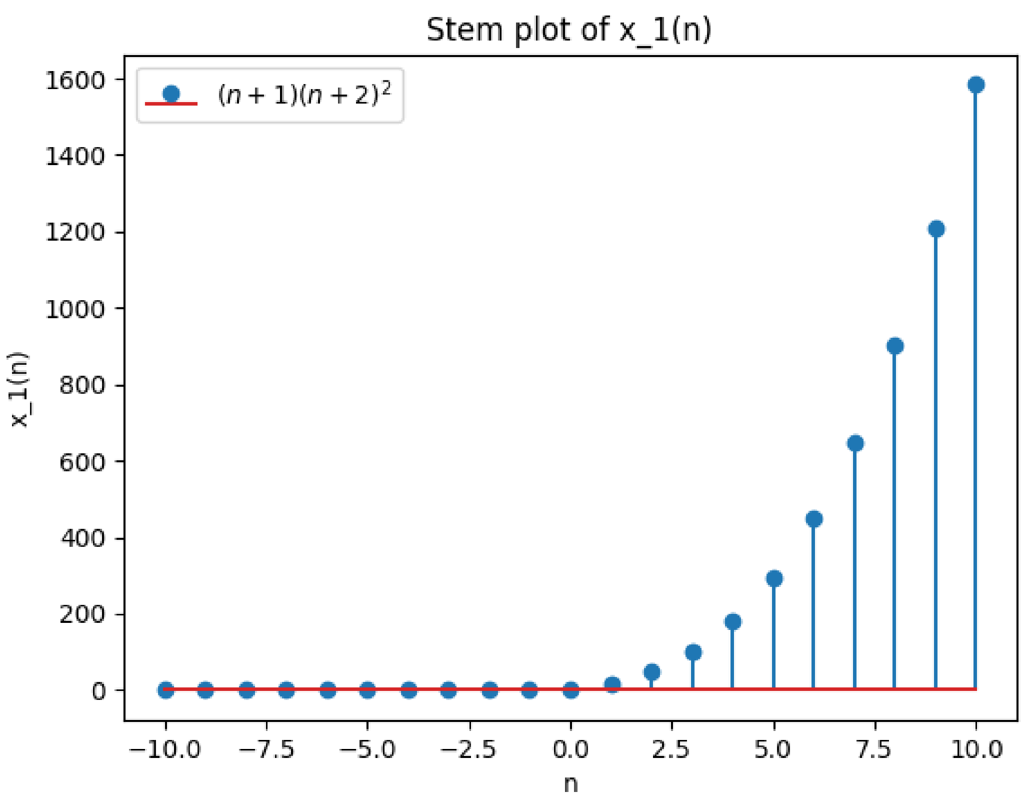


Figure 2.1: Stem Plot of $x_1(n)$

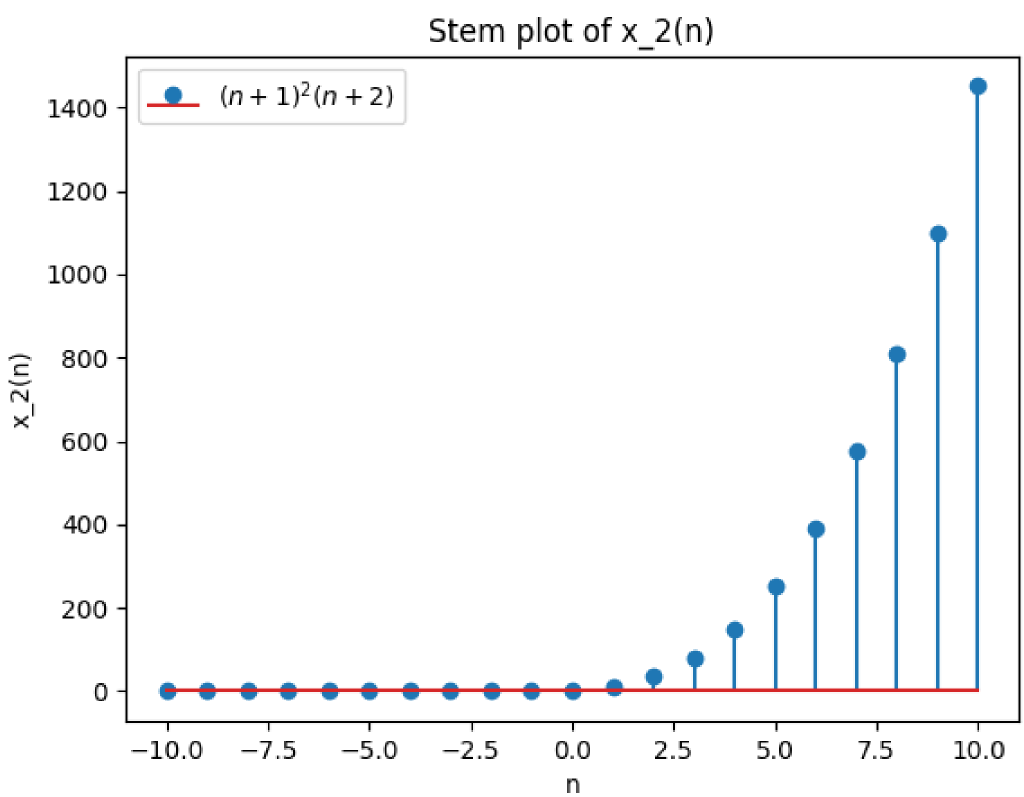


Figure 2.2: Stem Plot of $x_2(n)$

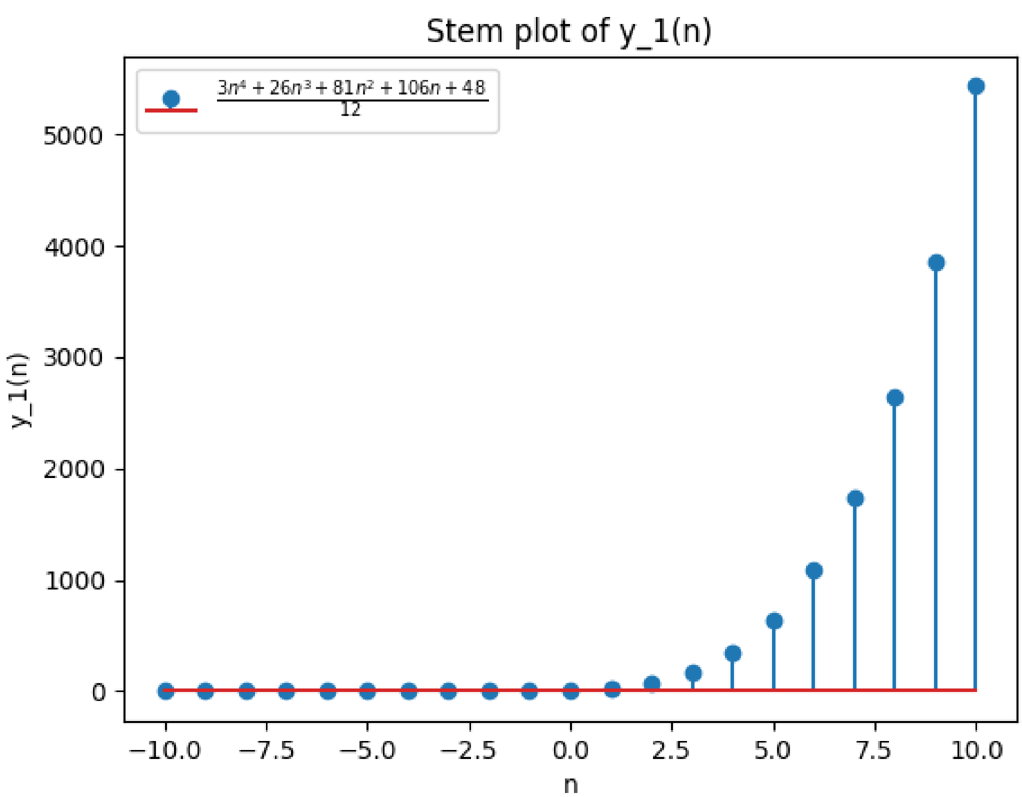


Figure 2.3: Stem Plot of $y_1(n)$

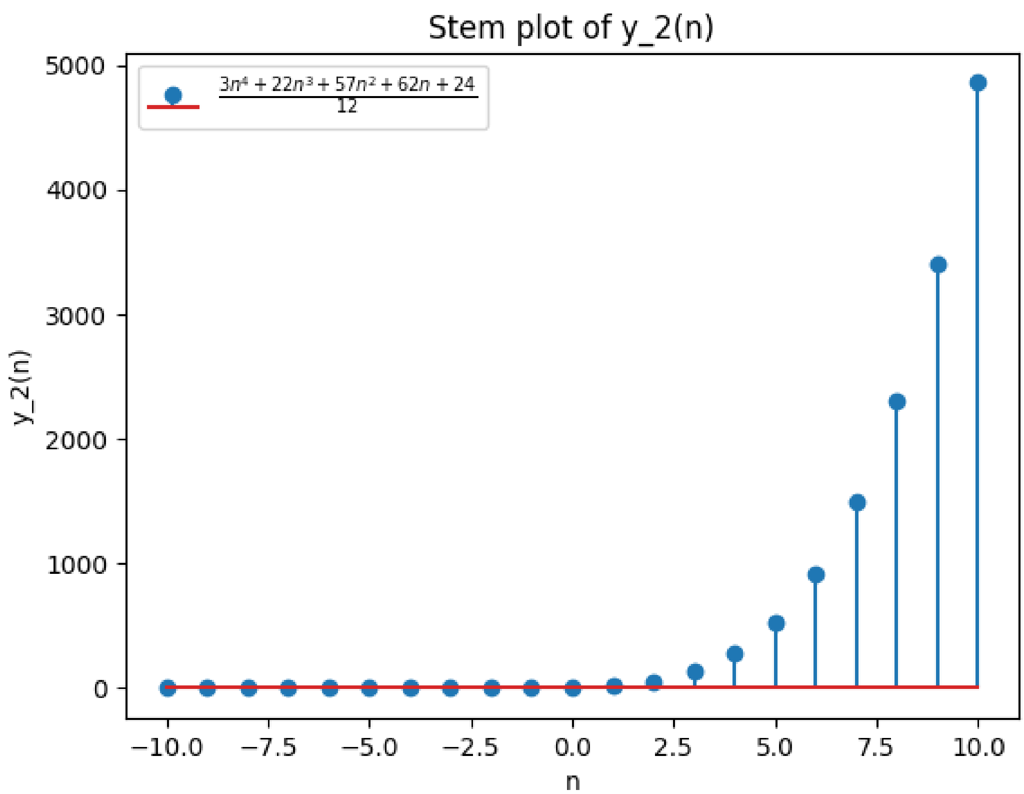


Figure 2.4: Stem Plot of $y_2(n)$

2.1.2 Write the five terms at n = 1, 2, 3, 4, 5 of the sequence and obtain the Z-transform of the series

$$x(n) = -1, \qquad n = 0 \tag{2.20}$$

$$= \frac{x(n-1)}{n}, \qquad n > 0 \tag{2.21}$$

$$= 0, \qquad n < 0 \tag{2.22}$$

Solution:

$$x(1) = \frac{x(0)}{1} = -1 \tag{2.23}$$
$$x(2) = \frac{x(1)}{2} = -\frac{1}{2} \tag{2.24}$$
$$x(3) = \frac{x(2)}{3} = -\frac{1}{(2)(3)} = -\frac{1}{6} \tag{2.25}$$
$$x(4) = \frac{x(3)}{4} = -\frac{1}{(2)(3)(4)} = -\frac{1}{24} \tag{2.26}$$
$$x(5) = \frac{x(4)}{5} = -\frac{1}{(2)(3)(4)(5)} = -\frac{1}{120} \tag{2.27}$$
$$x(n) = \frac{-1}{n!}(u(n)) \tag{2.28}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{2.29}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{2.30}$$

using (2.28),

$$= \sum_{n=-\infty}^{\infty} \frac{-1}{n!} u(n) z^{-n} \tag{2.31}$$
$$= \sum_{n=0}^{\infty} \frac{-1}{n!} z^{-n} \tag{2.32}$$
$$= -e^{z^{-1}} \qquad \{z \in \mathbb{C} : z \neq 0\} \tag{2.33}$$

Symbol	**Value**	**Description**
$x(n)$	$\frac{-1}{n!}$	general term of the series
$X(z)$	$-e^{z^{-1}}$	Z-transform of x(n)
$u(n)$		unit step function

Table 2.2: Parameters

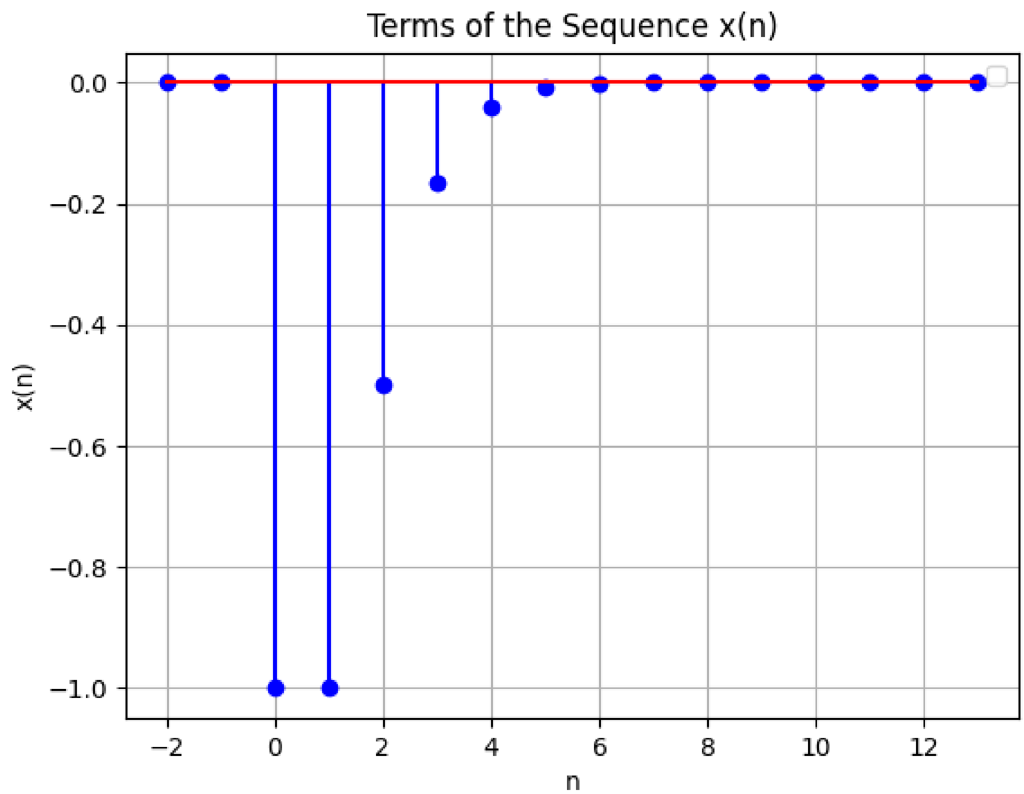


Figure 2.5: Plot of x(n) vs n

2.1.3 Subba Rao started work in 1995 at an annual salary of Rs. 5000 and received an increment of Rs. 200 each year. In which year did his income reach Rs. 7000?

Solution:

Parameter	Value	Description
$x(0)$	5000	Initial Income
d	200	Annual Increment (Common Difference)
$x(n)$	$(x(0) + nd)u(n)$	n^{th} term of the AP

Table 2.3: Input Parameters

From the values given in Table 2.3:

$$7000 = 5000 + 200n \tag{2.34}$$

$$\implies 2000 = 200n \tag{2.35}$$

$$\therefore n = 10 \tag{2.36}$$

Let Z-transform of $x(n)$ be $X(z)$.

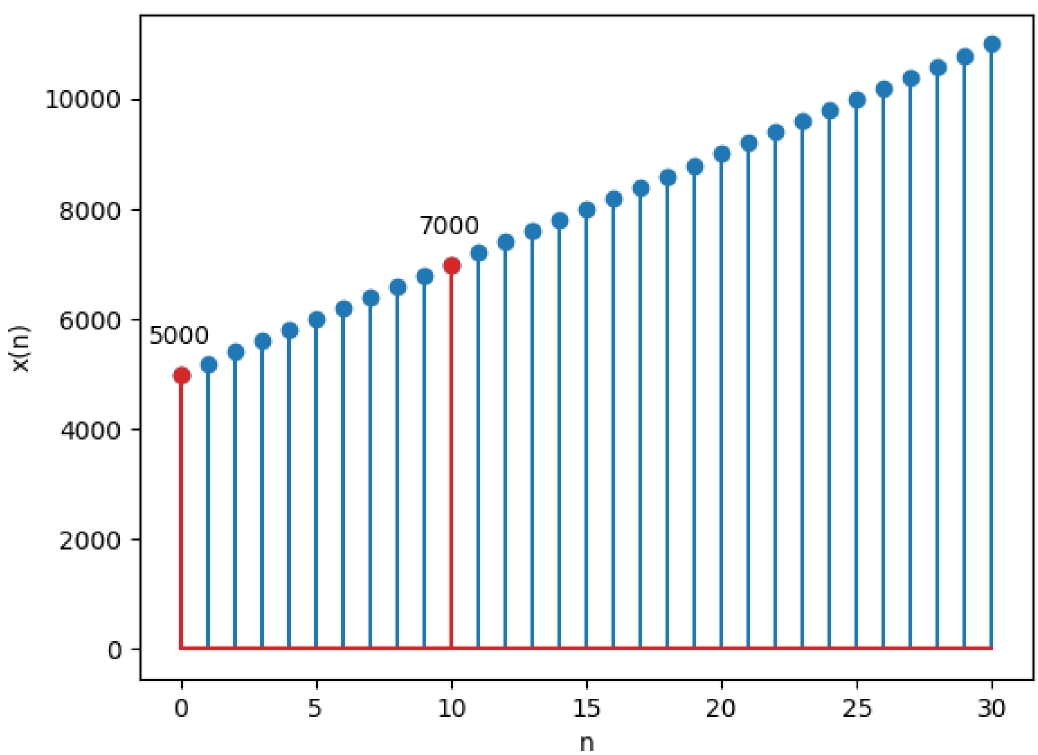


Figure 2.6: Plot of $x(n)$ vs n. See Table 2.3 for details.

$$X(z) = \frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{2.37}$$

Using the values from Table 2.3:

$$X(z) = \frac{5000}{1-z^{-1}} + \frac{200z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{2.38}$$

2.2. Sequences

2.2.1 For what value of n, are the nth terms of two A.Ps: 63, 65, 67,... and 3, 10, 17,... equal?

Solution:

Parameter	**Sub-question**	**Description**	**Value**
$x_i(0)$	$x_1(0)$	1^{st} term of 1^{st} A.P.	63
	$x_2(0)$	1^{st} term of 2^{nd} A.P.	3
d_i	d_1	Common difference of 1^{st} A.P.	2
	d_2	Common difference of 2^{nd} A.P.	7

Table 2.4: input values

$$x_i(n) = x(0)u(n) + dnu(n) \tag{2.39}$$

$$X(z) = \frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{2.40}$$

(a)

$$x_1(n) = 63u(n) + 2nu(n) \tag{2.41}$$

$$X_1(z) = \frac{63}{1-z^{-1}} + \frac{2z^{-1}}{(1-z^{-1})^2} \quad |z| > 1 \tag{2.42}$$

(b)

$$x_2(n) = 3u(n) + 7nu(n) \tag{2.43}$$

$$X_2(z) = \frac{3}{1 - z^{-1}} + \frac{7z^{-1}}{\left(1 - z^{-1}\right)^2} \quad |z| > 1 \tag{2.44}$$

(c) given,

$$x_1(n) = x_2(n) \tag{2.45}$$

$$\therefore 63 + 2n = 7n + 3 \tag{2.46}$$

$$\implies n = 12 \tag{2.47}$$

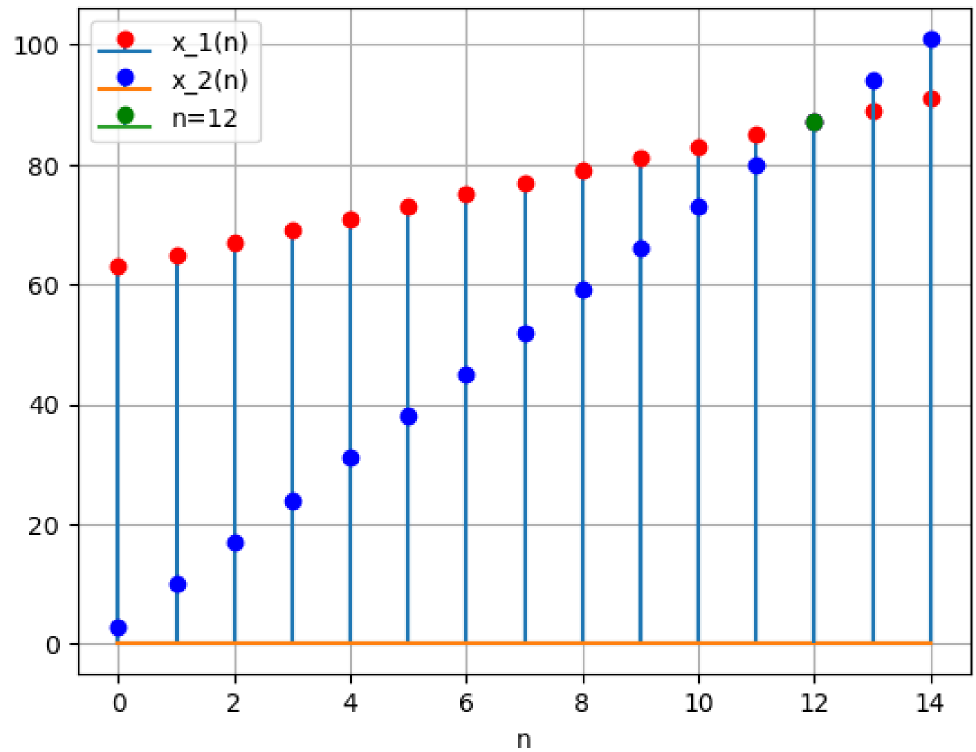


Figure 2.7: Graphs of $x_1\left(n\right)$ and $x_2\left(n\right)$ and both are equal at $n = 12$

2.2.2 Two APs have the same common difference.The difference between their 100th terms is 100,what is the difference between their 1000th terms?

Solution:

$$x(n) = \{x(0) + nd\}u(n) \tag{2.48}$$

$$x(99) - y(99) = 100 \tag{2.49}$$

$$\implies (x(0) + 99d) - (y(0) + 99d) = 100 \tag{2.50}$$

$$\implies x(0) - y(0) = 100 \tag{2.51}$$

$$x(n) - y(n) = (x(0) + nd) - (y(0) + nd) \tag{2.52}$$

$$= x(0) - y(0) \tag{2.53}$$

$$= 100 \tag{2.54}$$

$$\implies x(999) - y(999) = 100 \tag{2.55}$$

Variable	**Description**	**Value**
$x(n)$	n^{th} term of X	none
$y(n)$	n^{th} term of Y	none
d	common difference between the terms of AP	none
$x(99) - y(99)$	difference of 99^{th} terms of X and Y	100

Table 2.5: input parameters

Let

$$x(n) = \{101, 106, 111, ...\} \tag{2.56}$$

$$y(n) = \{1, 6, 11, ...\} \tag{2.57}$$

2.2.3 Check whether -150 is a term of the AP: 11,8,5,2,....

Solution:

$$x(n) = x(0) + nd \tag{2.58}$$

$$n = \frac{x(n) - x(0)}{d} \tag{2.59}$$

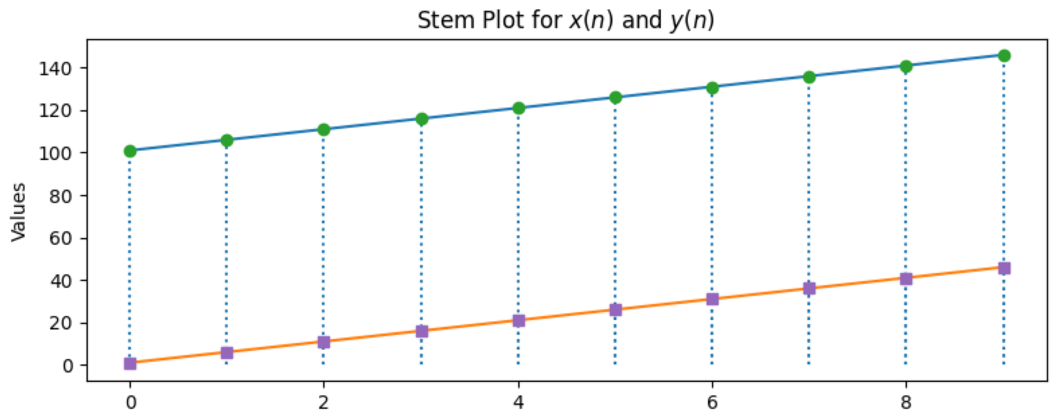


Figure 2.8:

$$x(n) - x(0) \equiv 0 \pmod{d} \tag{2.60}$$

On substitutings values

$$-161 \equiv 2 \pmod{-3} \tag{2.61}$$

Thus -150 is not a term of the given AP.

$$\boxed{x(n) = (11 - 3n) \times u(n)} \tag{2.62}$$

$$X(z) = \frac{11}{1 - z^{-1}} - \frac{3z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{2.63}$$

Variable	Description	Value
$x(0)$	First term of AP	11
d	Common difference	-3
$x(n)$	General term of given AP	None

Table 2.6: Input parameters

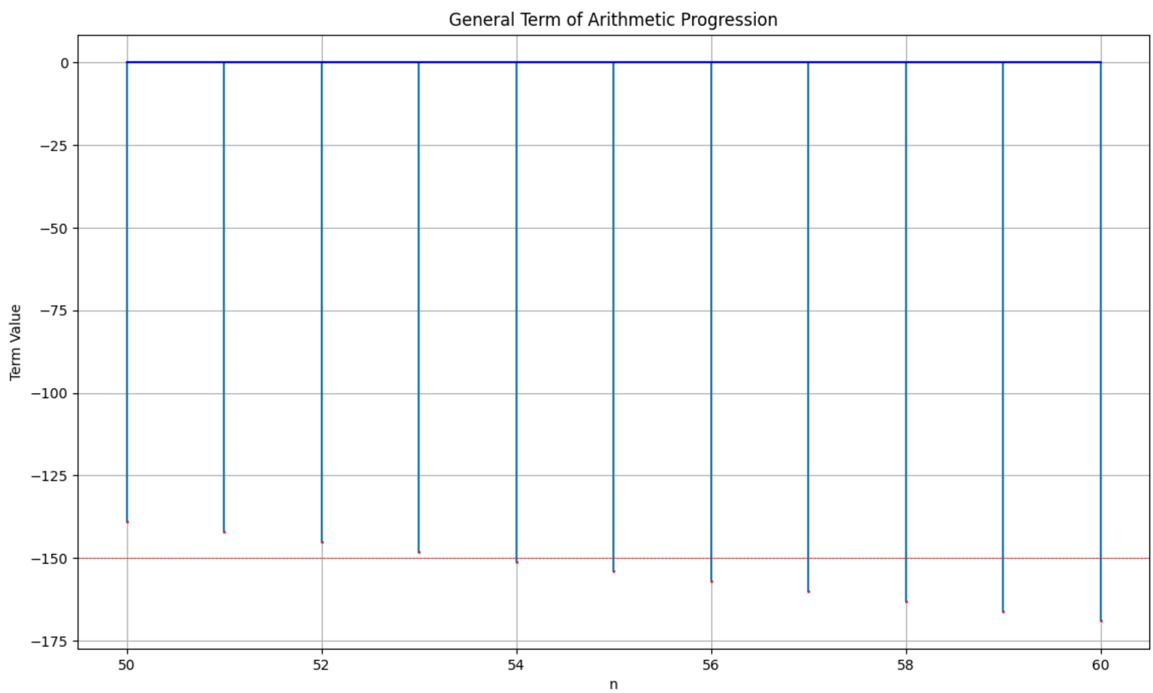


Figure 2.9: Representation of x(n)

2.2.4 Write the first five terms of the sequence $a_n = \frac{n(n^2+5)}{4}$.

Solution:

$$x(n) = \left(\frac{n^3 + 3n^2 + 8n + 6}{4}\right) u(n) \tag{2.64}$$

$$n^k u(n) \overset{\mathcal{Z}}{\longleftrightarrow} (-1)^k z^k \frac{d^k}{dz^k} U(z) \tag{2.65}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{2.66}$$

$$n^2 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1 + z^{-1})}{(1 - z^{-1})^3} \quad |z| > 1 \tag{2.67}$$

$$n^3 u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4} \quad |z| > 1 \tag{2.68}$$

Referencing the equations from (2.66), (2.67), and (2.68).

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{(z^{-1})(1 + 4z^{-1} + z^{-2})}{4(1 - z^{-1})^4} + \frac{3(z^{-1})(1 + z^{-1})}{4(1 - z^{-1})^3} + \frac{2z^{-1}}{(1 - z^{-1})^2} + \frac{3}{2(1 - z^{-1})} \quad |z| > 1 \tag{2.69}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{3}{2(1 - z^{-1})^3} + \frac{3z^{-2}}{2(1 - z^{-1})^4} \quad |z| > 1 \tag{2.70}$$

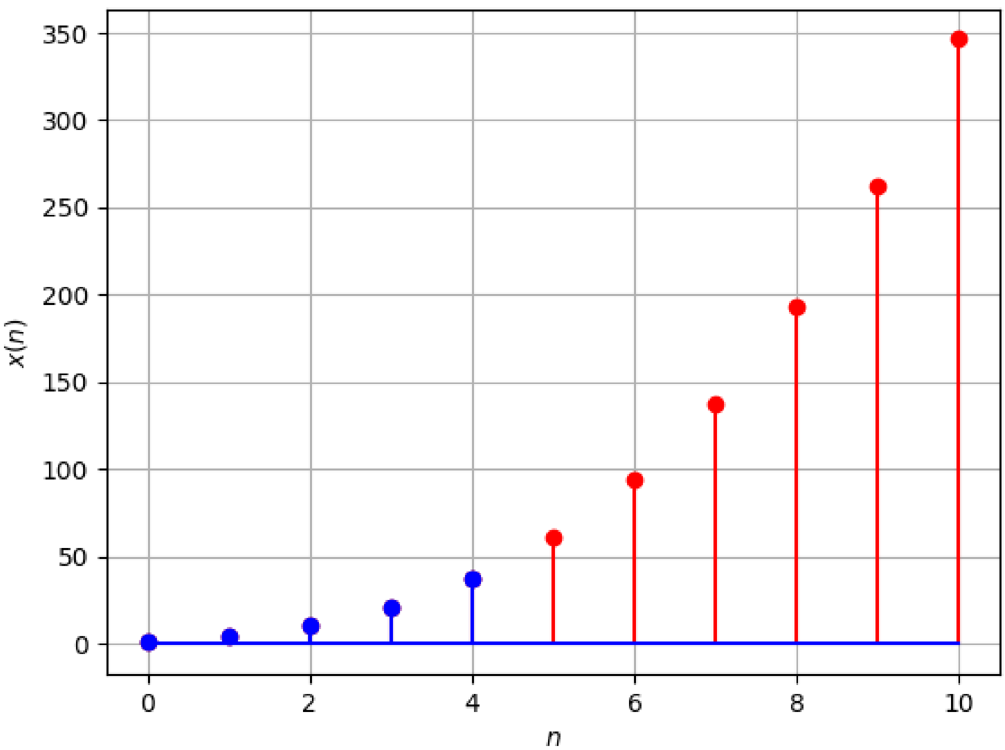


Figure 2.10: Plot of equation(2.64)

2.2.5 (a) 30th term of the AP: 10, 7, 4, . . . is

(b) 11th term of the AP: $-3, -\frac{1}{2}, 2, \ldots$ is

Solution:

$$x_i(n) = [x_i(0) + nd_i]\, u(n) \tag{2.71}$$

<table>
<tr><th>Parameter</th><th>value</th><th>Description</th></tr>
<tr><td rowspan="2">$x_i(0)$</td><td>10</td><td rowspan="2">First term</td></tr>
<tr><td>-3</td></tr>
<tr><td rowspan="2">d_i</td><td>-3</td><td rowspan="2">Common difference</td></tr>
<tr><td>$\frac{5}{2}$</td></tr>
<tr><td>$x_1(29)$</td><td>?</td><td>30th term</td></tr>
<tr><td>$x_2(10)$</td><td>?</td><td>11th term</td></tr>
</table>

Table 2.7: Input Parameters

(a) From (2.71) Table 2.7 :

$$x_1(n) = [10 - 3n]\, u(n) \tag{2.72}$$

$$x_1(29) = -77 \tag{2.73}$$

$$X_1(z) = \frac{10 - 13z^{-1}}{(1 - z^{-1})^2} \quad |z| > 1 \tag{2.74}$$

(b) From (2.71) and Table 2.7 :

$$x_2(n) = \left[-3 + \frac{5}{2}n\right] u(n) \tag{2.75}$$

$$x_2(10) = 22 \tag{2.76}$$

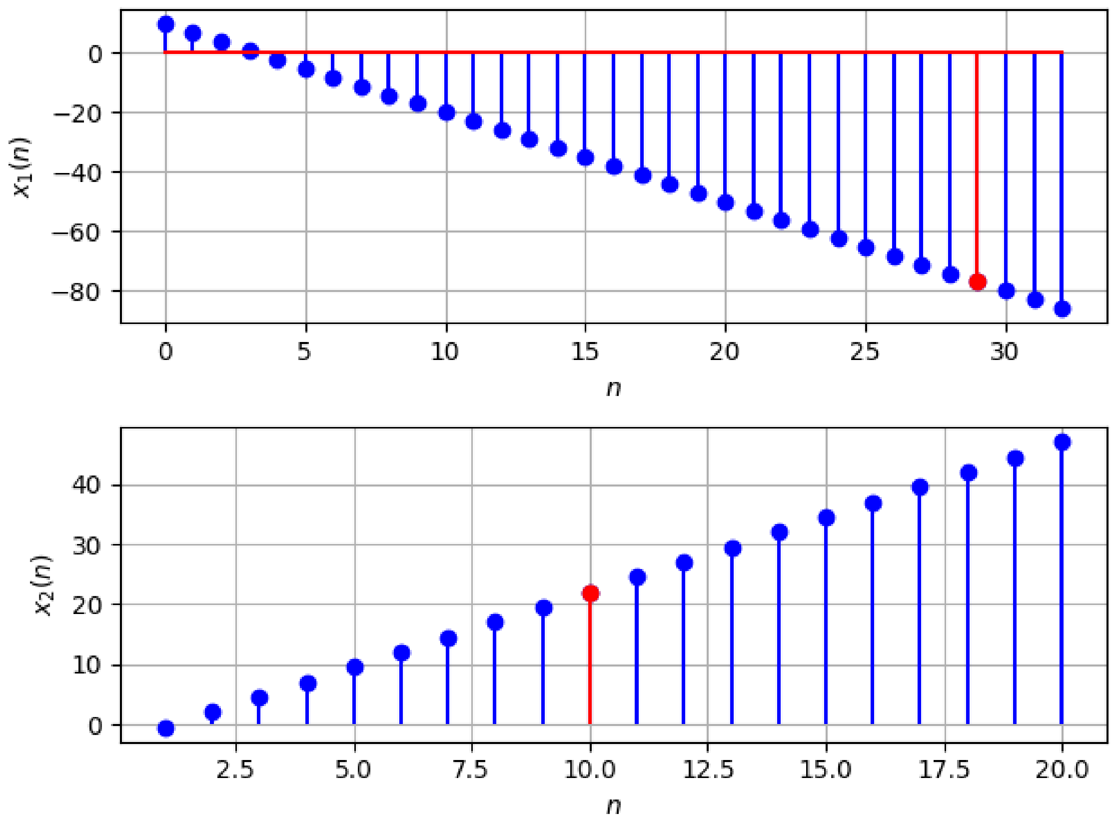


Figure 2.11: stem plots of $x_1(n)$ and $x_2(n)$

2.2.6 Write the first five terms of the sequence whose nth term is $\frac{2n-3}{6}$ and obtain the Z transform of the series **Solution:**

$$x\left(n\right) = \frac{2n-1}{6}\left(u\left(n\right)\right) \tag{2.78}$$

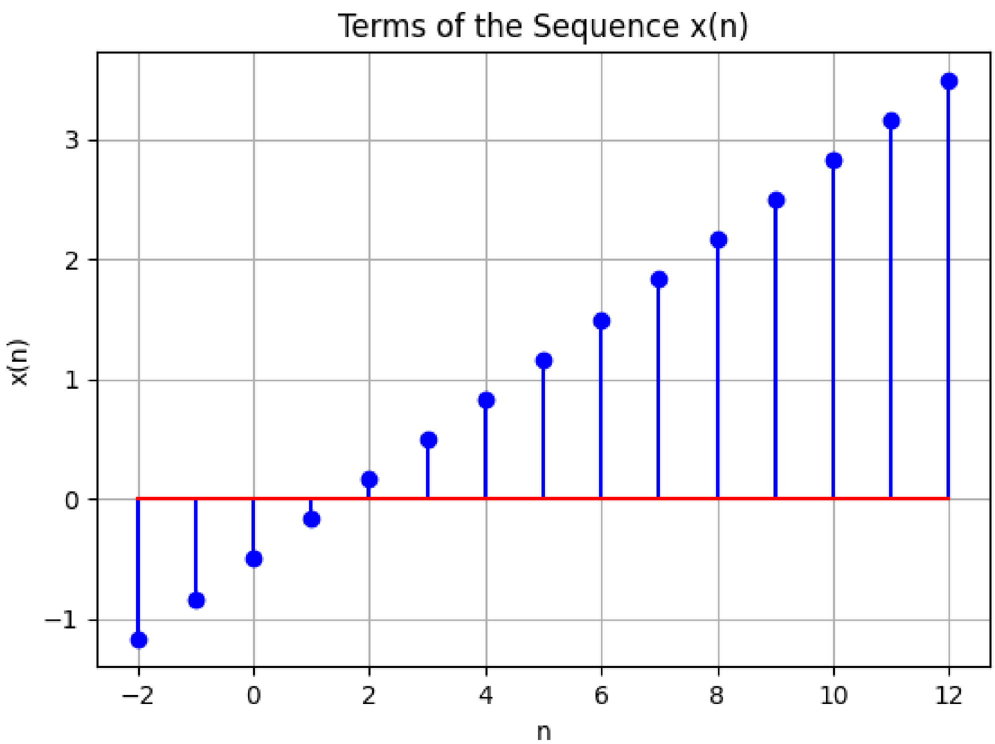


Figure 2.12: Plot of x(n) vs n

$$X(z) = \frac{3z^{-1} - 1}{6(1 - z^{-1})^2} \quad |z| > 1 \tag{2.79}$$

2.2.7 For what values of x, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P ?

Solution: Let r be the common ratio

Variable	Description	Value
$x(0)$	First term of the GP	$-\left(\frac{2}{7}\right)$
$x(1)$	Second term of the GP	x
$x(2)$	Third term of the GP	$-\left(\frac{7}{2}\right)$
r	Common ratio of the GP	
$x(n)$	General term	$x(0)\, r^n\, u(n)$

Table 2.8: Variables Used

From Table 2.8:

$$\implies \frac{x}{\left(-\frac{2}{7}\right)} = \frac{\left(-\frac{7}{2}\right)}{x} = r \tag{2.80}$$

$$x^2 = \left(-\frac{2}{7}\right) \cdot \left(-\frac{7}{2}\right) \tag{2.81}$$

$$x = \pm 1 \tag{2.82}$$

$$\implies r = \pm\frac{7}{2} \tag{2.83}$$

The signal corresponding to this is

$$x(n) = \left(-\frac{2}{7}\right)\left(\pm\frac{7}{2}\right)^n u(n) \tag{2.84}$$

Applying z-Transform :

$$\implies X_1(z) = \left(\frac{1}{7}\right)\left(\frac{4}{7z^{-1}+2}\right) \quad |z| > \frac{7}{2} \tag{2.85}$$

$$\implies X_2(z) = \left(\frac{1}{7}\right)\left(\frac{4}{7z^{-1}-2}\right) \quad |z| > \frac{7}{2} \tag{2.86}$$

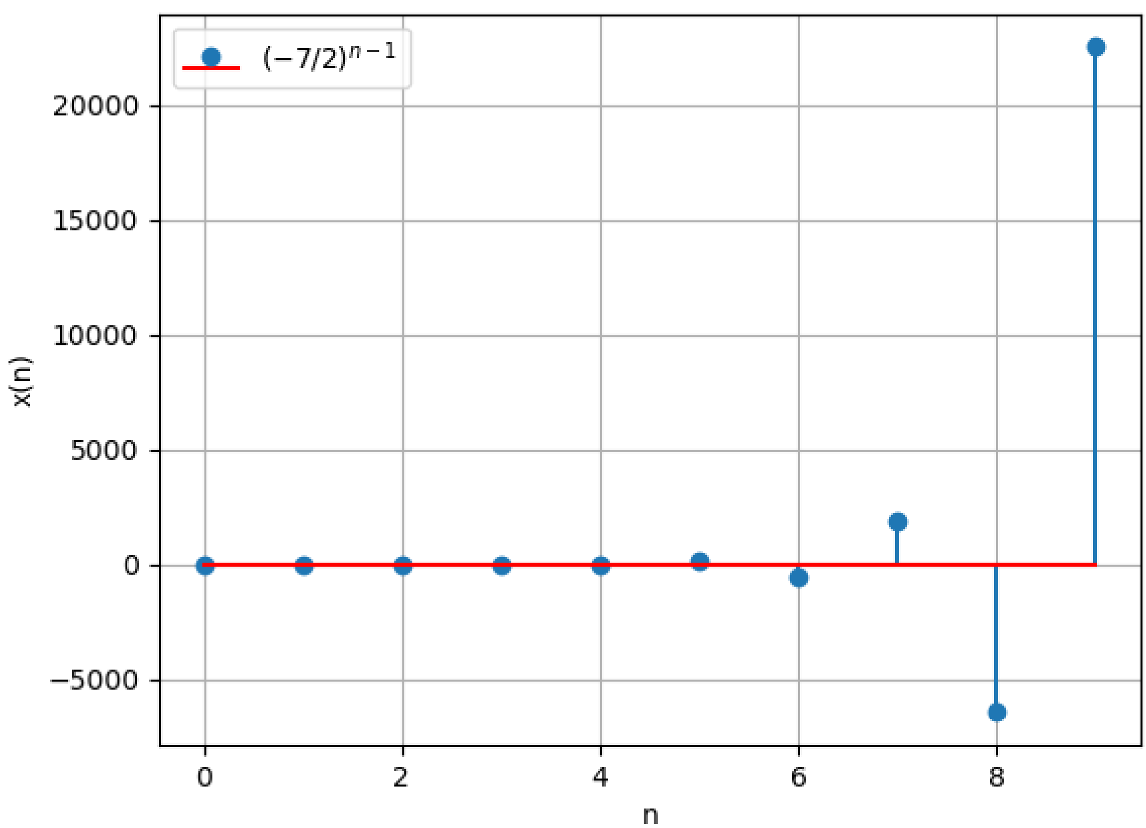


Figure 2.13: Stem Plot of x_1(n)

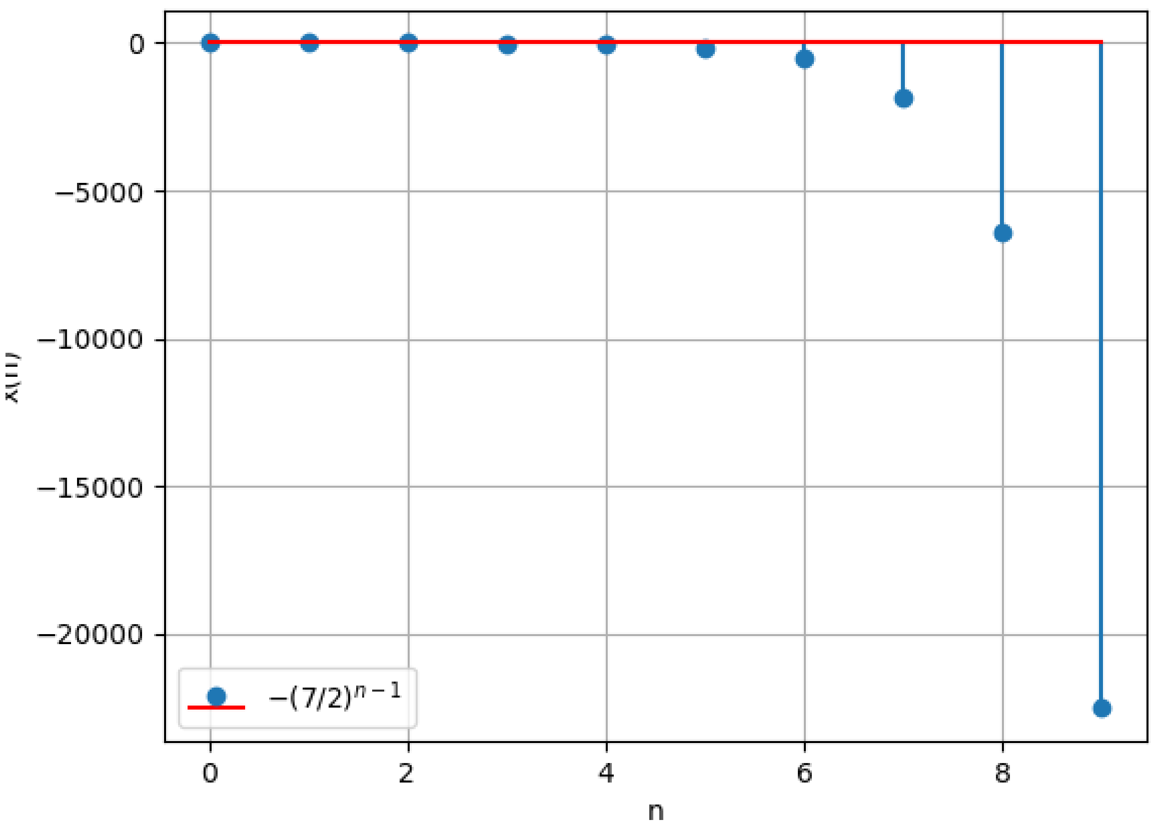


Figure 2.14: Stem Plot of $x_2(n)$

2.2.8 Find the 20^{th} and n^{th} terms of the G.P $\frac{5}{2}$, $\frac{5}{4}$, $\frac{5}{8}$,.....

Solution:

From Table 2.9: Z-Transform of $x(n)$:

$$\implies X(z) = \frac{5}{2}\left(\frac{1}{1 - \frac{z^{-1}}{2}}\right); \left\{z \in \mathbb{C} : |z| > \frac{1}{2}\right\} \tag{2.87}$$

Parameter	**Description**	**Value**
$x(0)$	First Term	$\frac{5}{2}$
$r = \frac{x(1)}{x(0)}$	Common Ratio	$\frac{1}{2}$
$x(n)$	n^{th} Term	$\frac{5}{2}\left(\frac{1}{2}\right)^{n} \cdot u(n)$
$x(19)$	20^{th} Term	$\frac{5}{2}\left(\frac{1}{2}\right)^{19}$
$u(n)$	Unit step function	

Table 2.9: Parameters

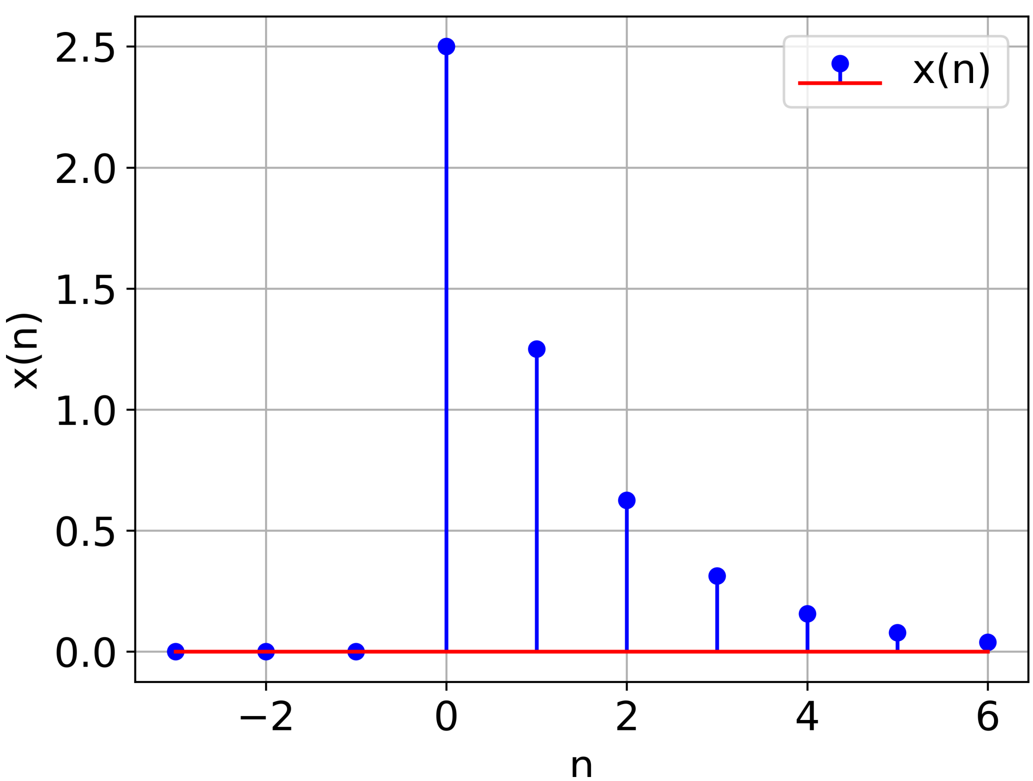


Figure 2.15:

Chapter 3

Contour Integration

3.1 Find the sum of the first 15 multiples of 8.

Solution:

PARAMETER	VALUE	DESCRIPTION
$x(0)$	8	First term
d	8	common difference
$x(n)$	$[8+8n]u(n)$	General term of the series

Table 3.1: Parameter Table1

For an AP,

$$X(z) = \frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2} \tag{3.1}$$

$$\implies X(z) = \frac{8}{1-z^{-1}} + \frac{8z^{-1}}{(1-z^{-1})^2} \tag{3.2}$$

$$= \frac{8}{(1-z^{-1})^2}, \quad |z| > 1 \tag{3.3}$$

$$y(n) = x(n) * u(n) \tag{3.4}$$

$$\implies Y(z) = X(z)\,U(z) \tag{3.5}$$

$$Y(z) = \left(\frac{8}{(1-z^{-1})^2}\right)\left(\frac{1}{1-z^{-1}}\right) \tag{3.6}$$

$$= \frac{8}{(1-z^{-1})^3}, \quad |z| > 1 \tag{3.7}$$

Using Contour Integration to find the inverse Z-transform,

$$y(14) = \frac{1}{2\pi j}\oint_C Y(z)\, z^{13}\, dz \tag{3.8}$$

$$= \frac{1}{2\pi j}\oint_C \frac{8z^{13}}{(1-z^{-1})^3}\, dz \tag{3.9}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{3.10}$$

$$= \frac{1}{(2)!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3\frac{8z^{16}}{(z-1)^3}\right) \tag{3.11}$$

$$= 4\lim_{z\to 1}\frac{d^2}{dz^2}(z^{16}) \tag{3.12}$$

$$= 960 \tag{3.13}$$

$$\therefore \boxed{y(14) = 960} \tag{3.14}$$

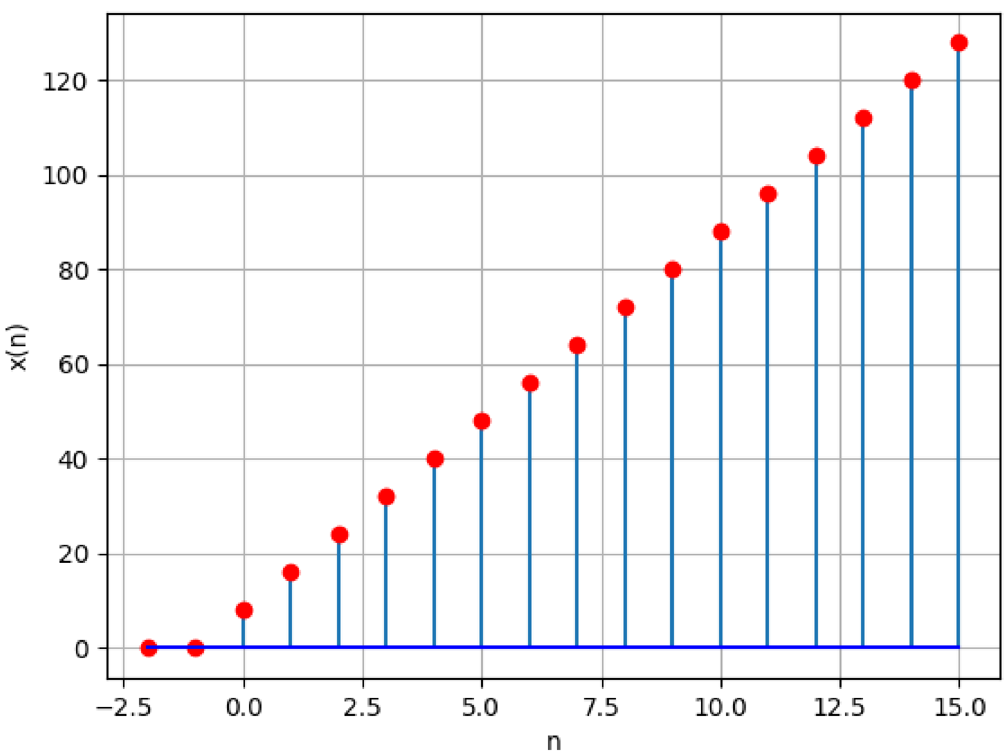


Figure 3.1: Plot of x(n) *vs* n

Appendix A

Convolution

A.1 The convolution sum is defined as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{A.1.1}$$

A.2 The unit step function is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{A.2.1}$$

A.3 If

$$x(n) = 0, \quad n < 0, \tag{A.3.1}$$

from (A.1.1),

$$x(n) * u(n) = \sum_{k=0}^{n} x(k) \tag{A.3.2}$$

Appendix B

Z-transform

B.1 The Z-transform of $p(n)$ is defined as

$$P(z) = \sum_{n=-\infty}^{\infty} p(n) z^{-n} \tag{B.1.1}$$

B.2 If

$$p(n) = p_1(n) * p_2(n), \tag{B.2.1}$$

$$P(z) = P_1(z) P_2(z) \tag{B.2.2}$$

B.3

$$nx(n) \overset{\mathcal{Z}}{\longleftrightarrow} -zX'(z) \tag{B.3.1}$$

From (B.3.1)

$$\Longrightarrow \; nu\left(n\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{\left(1-z^{-1}\right)^{2}}, |z|>1 \tag{B.3.2}$$

$$\Longrightarrow \; n^{2}u\left(n\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^{3}}, |z|>1 \tag{B.3.3}$$

$$\Longrightarrow \; n^{3}u\left(n\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^{4}}, |z|>1 \tag{B.3.4}$$

$$\Longrightarrow \; n^{4}u\left(n\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{\left(1-z^{-1}\right)^{5}} \tag{B.3.5}$$

$$\text{where}\, |z|>1$$

B.4

$$x(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k}X(z) \tag{B.4.1}$$

Using (B.4.1):

$$nu\left(n-1\right) \overset{\mathcal{Z}}{\longleftrightarrow} z\frac{2z^{-2}}{\left(1-z^{-1}\right)^{2}} \tag{B.4.2}$$

Now ,

$$\frac{\left(n-1\right)}{2}u\left(n-2\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-2}}{\left(1-z^{-1}\right)^{2}} \tag{B.4.3}$$

$$\frac{\left(n-1\right)\left(n-2\right)}{6}u\left(n-3\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-3}}{\left(1-z^{-1}\right)^{3}} \tag{B.4.4}$$

$$\vdots$$

$$\frac{\left(n-1\right)\left(n-2\right)\ldots\left(n-k+1\right)}{(k-1)!}u\left(n-k\right) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-k}}{\left(1-z^{-1}\right)^{k}} \tag{B.4.5}$$

$$\implies Z^{-1}\left[\frac{z^{-2}}{\left(1-z^{-1}\right)^{2}}\right]=(n-1)\,u\,(n-1) \tag{B.4.6}$$

$$\implies Z^{-1}\left[\frac{z^{-3}}{\left(1-z^{-1}\right)^{3}}\right]=\frac{(n-1)\,(n-2)}{2}u\,(n-1) \tag{B.4.7}$$

$$\implies Z^{-1}\left[\frac{z^{-4}}{\left(1-z^{-1}\right)^{4}}\right]=\frac{(n-1)\,(n-2)\,(n-3)}{6}u\,(n-1) \tag{B.4.8}$$

$$\implies Z^{-1}\left[\frac{z^{-5}}{\left(1-z^{-1}\right)^{5}}\right]=\frac{(n-1)\,(n-2)\,(n-3)\,(n-4)}{24} \tag{B.4.9}$$

$$u\,(n-1)$$

B.5 For a Geometric progression

$$x\,(n)=x\,(0)\,r^{n}u\,(n)\,, \tag{B.5.1}$$

$$\implies X\,(z)=\sum_{n=-\infty}^{\infty}x\,(n)\,z^{-n}=\sum_{n=0}^{\infty}x\,(0)\,r^{n}z^{-n} \tag{B.5.2}$$

$$=\sum_{n=0}^{\infty}x\,(0)\left(rz^{-1}\right)^{n} \tag{B.5.3}$$

$$=\frac{x\,(0)}{1-rz^{-1}},\quad |z|>|r| \tag{B.5.4}$$

B.6 Substituting $r=1$ in (B.5.4),

$$u(n)\overset{\mathcal{Z}}{\longleftrightarrow}U(z)=\frac{1}{1-z^{-1}},\quad |z|>1 \tag{B.6.1}$$

B.7 From (B.3.1) and (B.6.1),

$$nu(n)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}}{(1-z^{-1})^{2}},\quad |z|>1 \tag{B.7.1}$$

B.8 For an AP,

$$x(n) = [x(0) + nd]\,u(n) = x(0)u(n) + dnu(n) \tag{B.8.1}$$

$$\implies X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{B.8.2}$$

upon substituting from (B.6.1) and (B.7.1).

B.9 From (A.3.2), the sum to n terms of a GP can be expressed as

$$y(n) = x(n) * u(n) \tag{B.9.1}$$

where $x(n)$ is defined in (B.5.1). From (B.2.2), (B.5.4) and (B.6.1),

$$Y(z) = X(z)\,U(z) \tag{B.9.2}$$

$$= \left(\frac{x(0)}{1 - rz^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right) \quad |z| > |r| \cap |z| > |1| \tag{B.9.3}$$

$$= \frac{x(0)}{(1 - rz^{-1})(1 - z^{-1})} \quad |z| > |r| \tag{B.9.4}$$

which can be expressed as

$$Y(z) = \frac{x(0)}{r - 1}\left(\frac{r}{1 - rz^{-1}} - \frac{1}{1 - z^{-1}}\right) \tag{B.9.5}$$

using partial fractions. Again, from (B.5.4) and (B.6.1), the inverse of the above can be expressed as

$$y(n) = x(0)\left(\frac{r^{n+1} - 1}{r - 1}\right)u(n) \tag{B.9.6}$$

B.10 For the AP $x(n)$, the sum of first $n+1$ terms can be expressed as

$$y(n) = \sum_{k=0}^{n} x(k) \tag{B.10.1}$$

$$\implies y(n) = \sum_{k=-\infty}^{\infty} x(k)u(n-k) \tag{B.10.2}$$

$$= x(n) * u(n) \tag{B.10.3}$$

Taking the Z-transform on both sides, and substituting (B.8.2) and (B.6.1),

$$Y(z) = X(z)U(z) \tag{B.10.4}$$

$$\implies Y(z) = \left(\frac{x(0)}{1-z^{-1}} + \frac{dz^{-1}}{(1-z^{-1})^2}\right)\frac{1}{1-z^{-1}} \quad |z| > 1 \tag{B.10.5}$$

$$= \frac{x(0)}{(1-z^{-1})^2} + \frac{dz^{-1}}{(1-z^{-1})^3}, \quad |z| > 1 \tag{B.10.6}$$

B.11 From (B.4.1) and (B.7.1),

$$(n+1)u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-z^{-1})^2}, \quad |z| > 1, \tag{B.11.1}$$

From (B.11.1) and (B.3.1),

$$n(n+1)u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^3}, \quad |z| > 1, \tag{B.11.2}$$

B.12 Taking the inverse Z-transform of (B.10.6),

$$y(n) = x(0)\left[(n+1)u(n)\right] + \frac{d}{2}\left[n(n+1)u(n)\right] \tag{B.12.1}$$

$$= \frac{n+1}{2}\left\{2x(0) + nd\right\}u(n) \tag{B.12.2}$$