NCERT 12.7 12Q

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Question: An LC circuit contains a $50\mu H$ inductor and a $50\mu F$ capacitor with an initial charge of $10mC$. The resistance of the circuit is negligible. Let the instant the circuit is closed by $t = 0$.

a) What is the total energy stored initially? Is it conserved during LC oscillations?

b) What is the natural frequency of the circuit?

c) At what time is the energy stored **(i)** completely electrical (i.e., stored in the capacitor)? **(ii)** completely magnetic (i.e., stored in the inductor)?

d) At what times is the total energy shared equally between the inductor and the capacitor?

e) If a resistor is inserted in the circuit, how much energy is eventually dissipated as heat?
(NCERT 12.7 12Q)
Solution:

Parameter	Description	Value
L	Inductance	$50\mu H$
C	Capacitance	$50\mu F$
$E(0)$	Initial Energy of Capacitor	?
$q(0)$	Initial Charge on Capacitor	$10mC$
$v(0^-)$	Initial Voltage on Capacitor	$200V$
ω_o	Angular Resonant frequency	?

TABLE I
PARAMETER TABLE

(a) Initial energy stored :

$$E(0) = \frac{1}{2}C\left(v\left(0^-\right)\right)^2 \tag{1}$$
$$= 1J \tag{2}$$

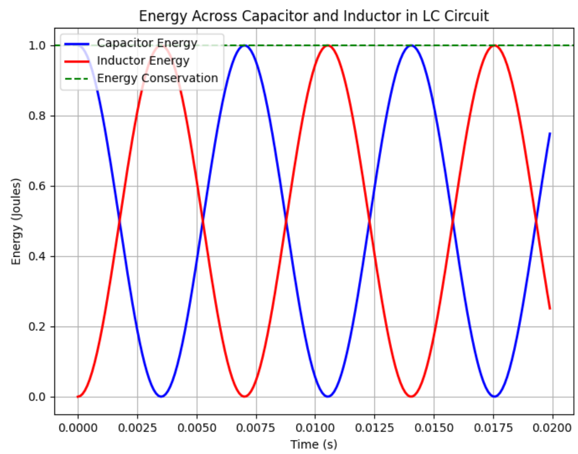

Fig. 1. Energy is Conserved During Oscillations total energy being limited to the initial energy

(b) The Laplace domain circuit :

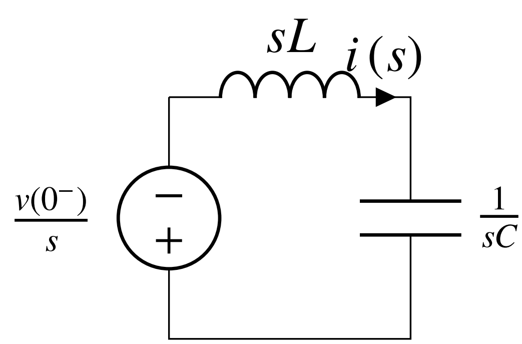

Fig. 2. LC Circuit in lapalace domain

Writing KVL in Fig. 2

$$\frac{-200}{s} - I(s)Ls - I(s)\frac{1}{sC} = 0 \tag{3}$$
$$I(s) = \frac{10^8}{25s^2 + 10^{10}} \tag{4}$$

Simplifying ,

$$I(s) = 200\left(\frac{2\times 10^4}{s^2 + \left(2\times 10^4\right)^2}\right) \tag{5}$$

Now,

$$\sin\left(at\right)u\left(t\right) \overset{\mathcal{L}}{\longleftrightarrow} \frac{a}{s^2+a^2} \tag{6}$$

$$\sin\left(2\times10^4 t\right)u\left(t\right) \overset{\mathcal{L}}{\longleftrightarrow} \frac{2\times10^4}{s^2+\left(2\times10^4\right)^2} \tag{7}$$

Uisng (7) on (4)and taking inverse-Laplace transform:

$$i\left(t\right) = -200\sin(2\times10^4 t)u\left(t\right) \tag{8}$$

From Fig. 2 voltage across capacitor

$$V\left(s\right) = -\left(I\left(s\right)\frac{1}{sC} + \frac{V\left(0^-\right)}{s}\right) \tag{9}$$

$$= \frac{5\times10^3 s}{25s^2+10^{10}} \tag{10}$$

Simplifying,

$$V\left(s\right) = 200\left(\frac{s}{s^2+\left(2\times10^4\right)^2}\right) \tag{11}$$

$$\cos\left(at\right)u\left(t\right) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+a^2} \tag{12}$$

$$\cos\left(2\times10^4 t\right)u\left(t\right) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+\left(2\times10^4\right)^2} \tag{13}$$

Using (13) on (10)and taking inverse-Laplace transform:

$$v\left(t\right) = 200\cos(2\times10^4 t)u\left(t\right) \tag{14}$$

The impedance of the circuit :

$$Z = \frac{V\left(s\right)}{I\left(s\right)} = sL + \frac{1}{sC} \tag{15}$$

s can be expressed in angular frequency as :

$$s = j\omega \tag{16}$$

$$Z = \left(\omega L - \frac{1}{\omega C}\right)j \tag{17}$$

$$|Z| = \omega L - \frac{1}{\omega C} \tag{18}$$

For resonant angular frequency imaginary part of impedance is zero :

$$\omega_o = \frac{1}{\sqrt{LC}} \tag{19}$$

$$= 20,000\text{rad/sec} \tag{20}$$

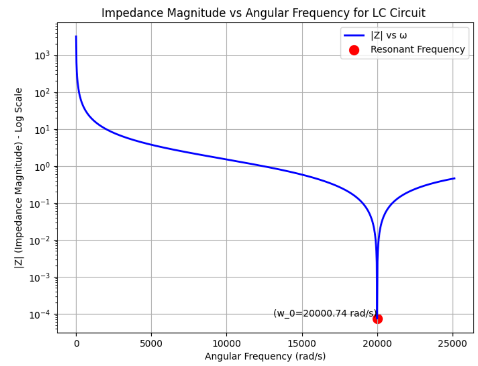

Fig. 3. The frequency at which impedance is minimum is resonant frequency which is at 20000 rad/sec

(c) The energy stored is completely electrical when $i\left(t\right) = 0$.
The energy stored is completely magnetic when $v\left(t\right) = 0$

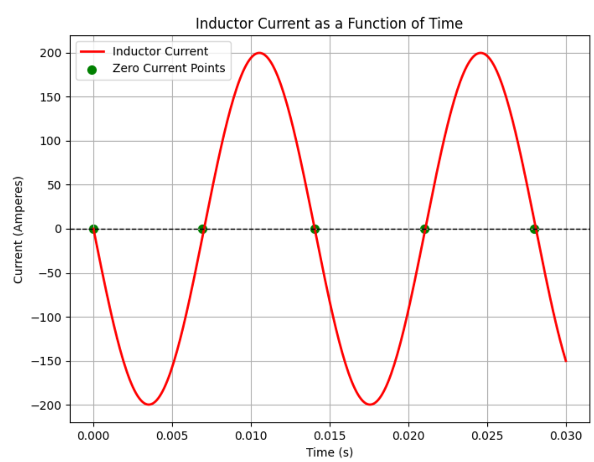

Fig. 4. Energy is completely electrical at the marked points.

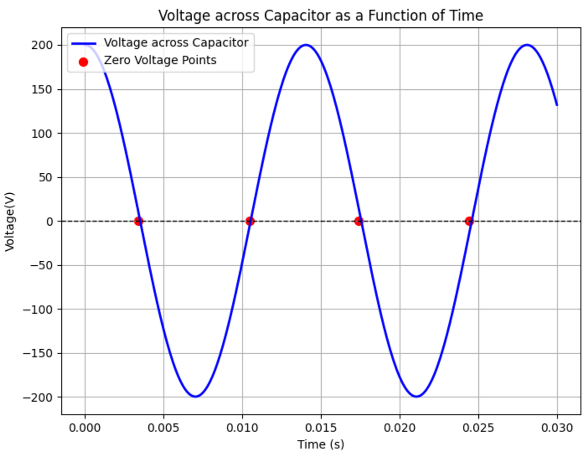

Fig. 5. Energy is completely magnetic at the marked points.

(d) When energy is equally shared then the capacitor has half of the maximum energy.

$$\frac{1}{2}C\left(v\left(t\right)\right)^2 = \frac{1}{2} \tag{21}$$

$$\cos\omega_o t = \frac{1}{\sqrt{2}} \tag{22}$$

$$t = \frac{(2n+1)T}{8}, T = \frac{2\pi}{\omega_o} \tag{23}$$

Hence, the total energy is equally shared between the inductor and capacitor at the time,$t = \frac{T}{8}, \frac{3T}{8}, \frac{5T}{8} \dots$

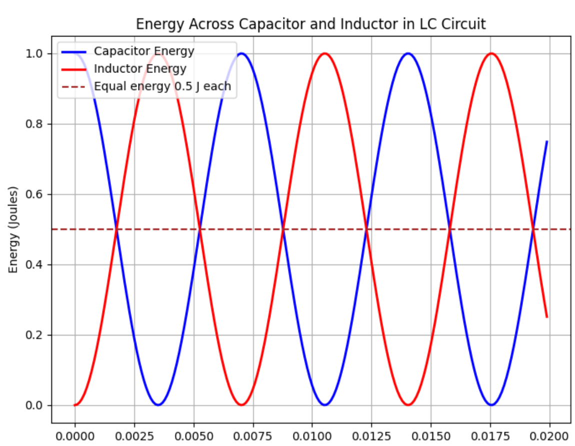

Fig. 6. At the intersection points with 0.5 J horizontal line both capacitor and inductor have equal energy.

(e) Once the resistor is added to the LC circuit, it starts dissipating energy in the form of heat.

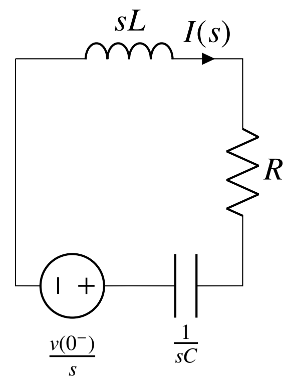

Fig. 7. Laplace domain LCR Circuit with $R = 1\Omega$

Applying KVL in Fig. 7:

$$\left(\frac{1}{sC} + sL + R\right)I\left(s\right) = \frac{v\left(0^-\right)}{s} \tag{24}$$

$$I\left(s\right) = \frac{10^{10}}{2500s^2 + 50\times 10^6 s + 10^{12}} \tag{25}$$

$$= -\frac{400}{\sqrt{3}}\left(\frac{\sqrt{3}\times 10^4}{\left(s+10^4\right)^2 + \left(\sqrt{3}\times 10^4\right)^2}\right) \tag{26}$$

By frequency-shifting property:

$$e^{-\alpha t}x\left(t\right) \stackrel{\mathcal{L}}{\longleftrightarrow} X\left(s+\alpha\right) \tag{27}$$

Applying (27) on (7)

$$e^{-10^4 t}\sin(10^4\sqrt{3}t)u\left(t\right) \stackrel{\mathcal{L}}{\longleftrightarrow} \left(\frac{\sqrt{3}\times 10^4}{\left(s+10^4\right)^2 + \left(\sqrt{3}\times 10^4\right)^2}\right) \tag{28}$$

Using (28) and Taking inverse Laplace transform:

$$i\left(t\right) = -\frac{400}{\sqrt{3}}e^{-10^4 t}\sin(10^4\sqrt{3}t) \tag{29}$$

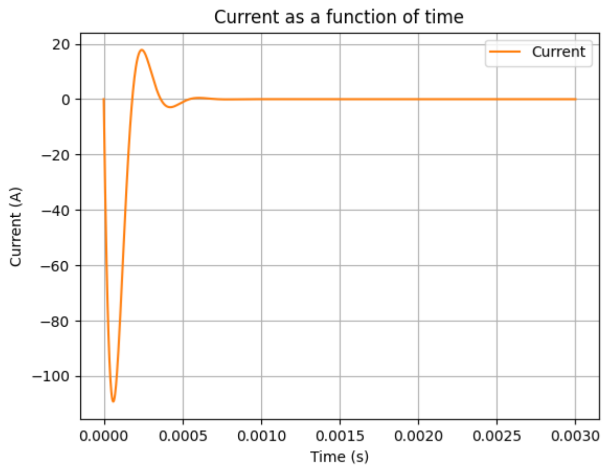

Fig. 8. Graph of current in the circuit. Becomes zero after all the energy is dissipated, $R = 1\Omega$

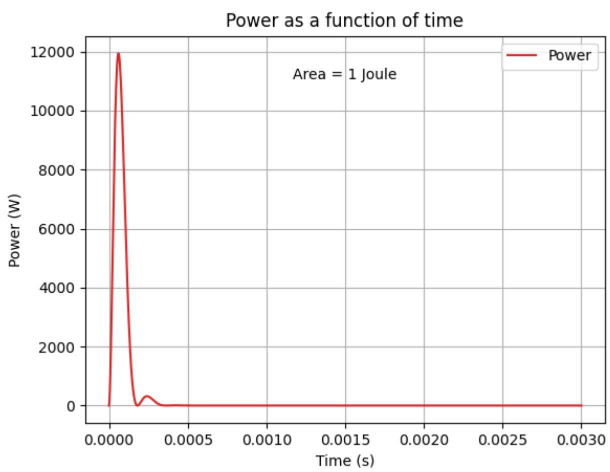

Fig. 9. Power Dissipated across resistor, $R = 1\Omega$. The area under the curve is 1J.