Todays Content

1. Modular Arthmetics
2. Problems based on %

1. Modular Arithmetic Introduction.

$A \% M$ = Remainder when A is divided by M

$n \% 4 = \{0\ 1\ 2\ 3\}$ $\quad n \% 5 = \{0\ 1\ 2\ 3\ 4\}$

Range : $[0 \dots M-1]$

Why do we need % : It can compress range to expected range.

$$\left.\begin{matrix}-\infty \\ \\ +\infty\end{matrix}\right] \rightarrow \% M = \overset{\text{Min}}{[0} .. \overset{\text{Max}}{M-1]}$$

Rules for % Arithmetic : $\{+, -, *, /\}$

$$\underset{\{0..2\}}{a} + \underset{\{0..4\}}{b} = \underset{\{0..6\}}{a+b}$$

1. $(a+b)\%m = (a\%m + b\%m)\%m$

$[0..M-1] \quad [0..M-1] + [0..M-1] = \{0 .. 2M-2\}\%M = \{0..M-1\}$

Ex: $a=9,\ b=8,\ m=5$

$(9+8)\%5 = (9\%5 + 8\%5)\%5$

$(17)\%5 = (4+3)\%5 =$

$2 = (7)\%5 = 2$

2. $(a\%m)\%M = (a\%m)\%M = a\%M$

$[0..M-1]\%M = [0..M-1]$

3. $(a+m)\%m = (a\%m + m\%m)\%m$

$= (a\%m + 0)\%m$

$= (a\%m)\%m$

$$\boxed{(a+m)\%m = a\%m}$$

Note: If we have %m outside brackets, taking %m inside brackets won't effect your answer.

-ve range to make it +ve

4. $\boxed{(a-b)\%m = (a\%m - b\%m + m)\%m}$

$[0..M-1]$

$a \quad b = (-(M-1) \quad M-1) + M = (1...2M-1)\%M = (0..M-1)$

$[0..M-1] - [0..M-1]$

Ex: $\underline{a} - \underline{b} = \underline{a-b}$

$[4 \quad 10] \quad [6 \quad 12] \quad [-8 \quad 4]$

Conceptually:

$\left.\begin{matrix}-\infty \\ \infty\end{matrix}\right\} \rightarrow \%M = [0..M-1]$. In python works

Practically: In C/C++/ Java : %M in -ve will give use -ve remainder, to avoid it we are adding M & %M

4. If $a < 0$:

$a\%M = [0..M-1]$ Only python

$\boxed{a\%M = (a\%m + m)\%m \;:\; \text{In C/C++/Java}}$

$[-(M-1)..0] + M = [1..M]\%M = (0..M-1)$

5. $(a*b)\%m = (a\%m * b\%m)\%m$

$[0..M-1]$ $\quad [0..M-1] * [0..M-1] \rightarrow \{0 \quad (M-1)^2\}\%M = \{0..M-1\}$

6. $(a^2)\%m = (a*a)\%m$

$= (a\%m * a\%m)\%m$

$= ((a\%m)^2)\%m$

6. $\boxed{(a^2)\%m = (a\%m)^2\%m}$

7. $\boxed{(a^b)\%m = (a\%m)^b\%m}$

Properties.

$(a+b)\%m = (a\%m + b\%m)\%m$

$(a*b)\%M = (a\%M * b\%M)\%M$

$(a+m)\%m = a\%m$

$(a\%m)\%m = a\%m$

$(a-b)\%m = (a\%m - b\%m + m)\%m$

$(a^b)\%m = (a\%m)^b\%m$

For -ve numbers $a\%M = (a\%M + M)\%M$

Quizes:

Q1: $(37^{103}-1)\%12 = (\underline{\underline{37^{103}}}\%12 - 1\%12 + 12)\%12$

$= (\underline{\underline{(37\%12)}}^{103}\%12 - 1\%12 + 12)\%12$

$= (1^{103}\%12 - 1\%12 + 12)\%12$

$= (\cancel{1\%12} - \cancel{1\%12} + 12)\%12$

$= (12)\%12 = 0$

Q1: Given ar[N] ele find count of pairs such that

$(ar[i] + ar[j]) \% M = 0$ Note: $i != j$ and pair(i, j) same as pair(j, i)

Constraints:

$1 <= N <= 10^5$

$0 <= ar[i] <= 10^9$

0 1 2 3 4 5

Ex: ar[] = { 4 3 6 3 8 12 } ans =

M = 6 $\quad (ar[i] + ar[j]) \% M = 0.$

$(ar[0] + ar[4]) \% 6 = (4 + 8) \% 6 = 0$ ✓

$(ar[1] + ar[3]) \% 6 = (3 + 3) \% 6 = 0$ ✓

$(ar[2] + ar[5]) \% 6 = (6 + 12) \% 6 = 0$ ✓

Idea1: Generate all pairs & check if their sum % M = 0.

Given ar[] & N.

```
int c=0;          TC: O(N^2)  SC: O(1)
for(int i=0; i<N; i++){          # i = N-1
    for( int j=i+1; j<N; j++){   j = i+1, N; j<N;
        if( (ar[i] + ar[j]) % M == 0){
            c++;
        }
    }
}

return c;
```

Ideas: $(arr[i] + arr[j]) \% M = (\underline{arr[i] \% M} + \underline{arr[j] \% M}) \% M$

Hint: Iterate & calculate % M on all array values

Ex:

		0	1	2	3	4	5	6	7	8	9	10
M = 6	A[N] =	2	6	4	8	14	13	5	12	24	16	19
	A[N] =	2	0	4	2	2	1	5	0	0	4	1

ele % M = [0..M-1]

obs: $(\underline{arr[i] \% M} + \underline{arr[j] \% M}) \% M = 0$

Eg M = 6: $(2 + 4) \% 6 = 0$

: $(3 + 3) \% 6 = 0$

: $(1 + 5) \% 6 = 0$

$\boxed{\text{obs}: (k + M-k) \% M = 0}$

Edge Case

% 6 = [0..5]

obs: $(\underline{arr[i] \% M} + \underline{arr[j] \% M}) \% M = 0$

M = 6 : $(0 + 0) \% 6 = 0$

$\boxed{\text{obs2}: (0 + 0) \% M = 0}$

Conclusion:

For Given arr[N] & M

$\boxed{\text{Case 1}: arr[i] \% M = 0 \rightleftarrows arr[j] \% M = 0}$

$\boxed{\text{Case 1}: arr[i] \% M = k \rightleftarrows arr[j] \% M = M-k}$

0 1 2 3 4 5 6 7 8 9 10

M = 6 A[11] = {2 6 4 8 14 13 5 12 24 16 19}

ar[i] = ar[i] % 6

A[11] = {2 0 4 2 2 1 5 0 0 4 1}

Eg M = 6 obs: $(\underline{arr[i]\%M} + \underline{arr[j]\%M})\%M = 0$

(2 4) % 6 = 0

2	a_1	b_1	4
8	a_2	b_2	16
14	a_3		

pairs = 3*2 = 6

Eg M = 6 obs: $(\underline{arr[i]\%M} + \underline{arr[j]\%M})\%M = 0$

(1 5) % 6 = 0

13	a_1	b_1	5
19	a_2		

pairs = 2*1 = 2

Hint: Store count of $\underline{arr[i]\%M}$, using that we can optimize

Why we?

arr[i]%M = {0 1 .. M-1}

Need to store count of each remainder

To do above, create $\underline{array[M]}$, store frequency of each remainder

Index : $\underline{0\ 1\ 2\ ..\ M-1}$

Idea:

	0	1	2	3	4	5	6	7	8	9	10
M = 6 A[11] =	2	6	4	8	14	13	5	12	24	16	19

ar[i] = ar[i] % 6

A[11] = { 2 0 4 2 2 1 5 0 0 4 1 }

M = 6

ar[i] % M   ar[j] % M   c = 0;

Remainder	Pair for it	Count Frequency
2	4	c = c + 0
0	0	c = c + 0
4	2	c = c + 1
2	4	c = c + 1
2	4	c = c + 1
1	5	c = c + 0
5	1	c = c + 1
0	0	c = c + 1
0	0	c = c + 2
4	2	c = c + 3
1	5	c = c + 1

return c;   11

	0	1	2	3	4	5
cnt[6] =	3	2	3	0	2	1

```
int pairs(vector<int> &arr, int M){   TC: O(N+N) = O(N)   SC: O(M)

    #Step1: Replace arr[i] = arr[i]%M
    int N = arr.size();
    for( int i=0; i<N; i++){
        arr[i] = arr[i] %M;   →  If -ve : (arr[i]%M + M) %M
    }

    vector<int> cnt(M,0);
    int c=0;
    for(int i=0; i<N; i++){
        int k= arr[i];
        int tar= M-k;
        if( k==0){
            tar =0;
        }
        c = c + cnt[tar]; # Add how many target elements present.
        cnt[k]++; # Increase freq of k
    }
    return c;
}
```