

Today's Content

1. Modular Arithmetic
2. Problems based on %

1. Modular Arithmetic Introduction.

$$\begin{aligned} A \% M &= \text{Remainder when } A \text{ is divided by } M \\ n \% 4 &= \{0, 1, 2, 3\} \quad n \% 5 = \{0, 1, 2, 3, 4\} \\ \text{Range} &= [0 \dots M-1] \end{aligned}$$

Why do we need % : It can compress range to expected range.

$$\begin{array}{c} -\infty \\ \updownarrow \\ \text{Min Max} \\ \downarrow \\ +\infty \end{array} \rightarrow \% M = [0 \dots M-1]$$

Rules for % Arithmetic : $\{+, -, *, /\}$

$$\frac{a}{\{0..2\}} + \frac{b}{\{0..4\}} = \frac{a+b}{\{0..6\}}$$

1. $(a+b) \% M = (a \% M + b \% M) \% M$

$$[0 \dots M-1] \quad [0 \dots M-1] + [0 \dots M-1] = [0 \dots 2M-2] \% M = [0 \dots M-1]$$

Ex: $a=9, b=8, m=5$

$$(9+8) \% 5 = (9 \% 5 + 8 \% 5) \% 5$$

$$(17) \% 5 = (4+3) \% 5 =$$

$$2 = (7) \% 5 = 2$$

2. $(a \% M) \% M = (a \% M) \% M = a \% M$
 $[0 \dots M-1] \% M = [0 \dots M-1]$

3. $(a+m) \% M = (a \% M + \underline{m \% M}) \% M$
 $= (a \% M + 0) \% M$
 $= (a \% M) \% M$

$$(a+m) \% M = a \% M$$

Note: If we have %M outside brackets, taking +m inside brackets won't effect your answer.

4. $(a-b) \% m = (a \% m - b \% m + m) \% m$ → -ve range to make it +ve

$[0..m-1]$ a b $= (-[m-1] \dots m-1) + m = (1 \dots 2m-1) \% m = [0..m-1]$

$[0..m-1] - [0..m-1]$

Ex:

$\begin{matrix} a & - & b & = & a-b \\ [4] & [10] & [6] & [12] & [-8] & [4] \end{matrix}$

Conceptually:

$-\infty \rightarrow \infty \rightarrow \% m = [0..m-1]$. In python works

Practically: In C/C++/Java: $\% m$ will give us -ve remainder, to avoid it we are adding m $\% m$

4. If $a < 0$:

$a \% m = [0..m-1]$ Only python

$a \% m = (a \% m + m) \% m$: In C/C++/Java

$[-(m-1) \dots 0] + m = [1 \dots m] \% m = [0..m-1]$

5. $(a+b) \% m = (a \% m + b \% m) \% m$

$[0..m-1]$ $[0..m-1] + [0..m-1] \rightarrow \{0 \dots (m-1)^2\} \% m = [0..m-1]$

6. $(a^2) \% m = (a * a) \% m$

$= (a \% m * a \% m) \% m$

$= ((a \% m)^2) \% m$

6. $(a^2) \% m = (a \% m)^2 \% m$

7. $(a^b) \% m = (a \% m)^b \% m$

Properties.

$$(a+b)\%m = (a\%m + b\%m)\%m$$

$$(a*b)\%m = (a\%m * b\%m)\%m$$

$$(a+m)\%m = a\%m$$

$$(a\%m)\%m = a\%m$$

$$(a-b)\%m = (a\%m - b\%m + m)\%m$$

$$(a^b)\%m = (a\%m)^b\%m$$

$$\text{For -ve numbers } a\%m = (a\%m + m)\%m$$

Quizes:

$$Q_1: (37^{103} - 1)\%12 = (\underline{37^{103}\%12} - 1\%12 + 12)\%12$$

$$= (\underline{(37\%12)^{103}}\%12 - 1\%12 + 12)\%12$$

$$= (1^{103}\%12 - 1\%12 + 12)\%12$$

$$= (\cancel{1\%12} - \cancel{1\%12} + 12)\%12$$

$$= (12)\%12 = 0$$

Q1: Given $arr[N]$ we find count of pairs such that

$(arr[i] + arr[j]) \% M = 0$ Note: $i \neq j$ and pair(i, j) same as pair(j, i)

Constraints:

$$1 \leq N \leq 10^5$$

$$0 \leq arr[i] \leq 10^9$$

Ex: $arr[] = \{4, 3, 6, 3, 8, 12\}$ ans =

$$M = 6 \quad (arr[i] + arr[j]) \% M = 0.$$

$$(arr[0] + arr[4]) \% 6 = (4 + 8) \% 6 = 0 \checkmark$$

$$(arr[1] + arr[3]) \% 6 = (3 + 3) \% 6 = 0 \checkmark$$

$$(arr[2] + arr[5]) \% 6 = (6 + 12) \% 6 = 0 \checkmark$$

Idea1: Generate all pairs & check if their sum $\% M = 0$.

Given $arr[]$ & N ,

int $c = 0$; TC: $O(N^2)$ SC: $O(1)$

for (int $i = 0$; $i < N$; $i++$) { $\neq i = N-1$

for (int $j = i+1$; $j < N$; $j++$) { $j = i+1, N$; $j < N$

if $((arr[i] + arr[j]) \% M == 0)$ {

$c++$

}

return c ;

Idea: $(arr[i] + arr[j]) \% M = (\underline{arr[i] \% M} + \underline{arr[j] \% M}) \% M$

Hint: Iterate & calculate $\% M$ on all array values

Ex:

$M = 6$ $A[N] = \{ 2 \ 6 \ 4 \ 8 \ 14 \ 13 \ 5 \ 12 \ 24 \ 16 \ 19 \}$

$A[N] = \{ 2 \ 0 \ 4 \ 2 \ 2 \ 1 \ 5 \ 0 \ 0 \ 4 \ 1 \}$

$arr \% M = [0..M-1]$

obs: $(\underline{arr[i] \% M} + \underline{arr[j] \% M}) \% M = 0$

→ Ex $M = 6$: $(2 + 4) \% 6 = 0$

: $(3 + 3) \% 6 = 0$

: $(1 + 5) \% 6 = 0$

obs: $(k + M - k) \% M = 0$

Edge Case

obs: $(\underline{arr[i] \% M} + \underline{arr[j] \% M}) \% M = 0$

$M = 6$: $(0 + 0) \% 6 = 0$

obs2: $(0 + 0) \% M = 0$

Conclusion:

for Given $arr[N]$ & M

Case 1: $arr[i] \% M = 0 \Leftrightarrow arr[j] \% M = 0$

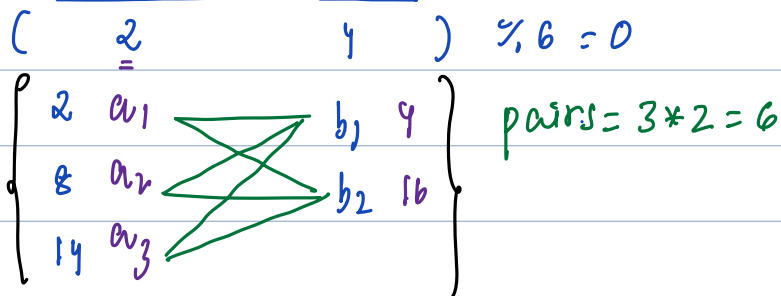
Case 1: $arr[i] \% M = k \Leftrightarrow arr[j] \% M = M - k$

0 1 2 3 4 5 6 7 8 9 10
 $M = 6$ $A[11] = \{2, 6, 4, 8, 14, 13, 5, 12, 24, 16, 19\}$

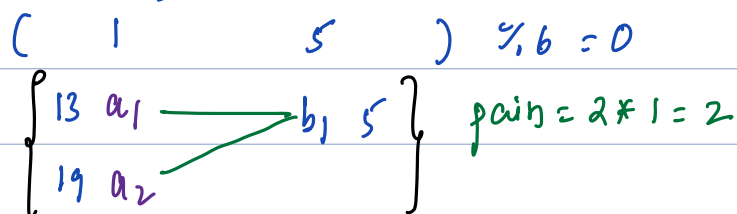
$$arr[i] = arr[i] \% 6$$

$A[11] = \{2, 0, 4, 2, 2, 1, 5, 0, 0, 4, 1\}$

Ex $M = 6$ obs: $(arr[i] \% M + arr[j] \% M) \% M = 0$



Ex $M = 6$ obs: $(arr[i] \% M + arr[j] \% M) \% M = 0$



Hint: Store count of $arr[i] \% M$, using that we can optimize
 Why we?

$$arr[i] \% M = \{0, 1, \dots, M-1\}$$

Need to store count of each remainder

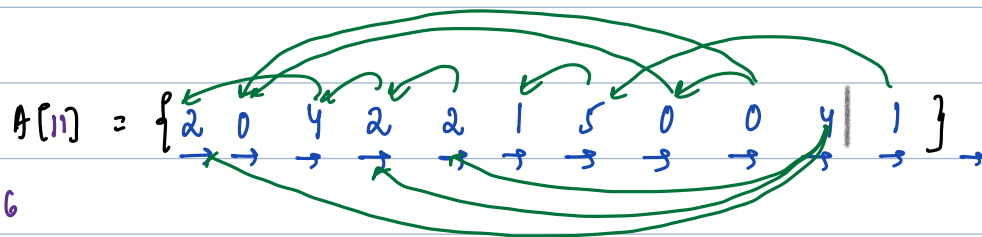
To do above, create array $[M]$, store frequency of each remainder

Index: 0 1 2 ... M-1

Idea:

0 1 2 3 4 5 6 7 8 9 10
 $M = 6$ $A[i] = \{ 2, 6, 4, 8, 14, 13, 5, 12, 24, 16, 19 \}$

$$ar[i] = ar[i] \% 6$$



$ar[i] \% M$

$ar[j] \% M$

$c = 0;$

Remainder

Pair for it

Count Frequency

2

4

$c = c + 0$

0

0

$c = c + 0$

4

2

$c = c + 1$

2

4

$c = c + 1$

2

4

$c = c + 1$

1

5

$c = c + 0$

5

1

$c = c + 1$

0

0

$c = c + 1$

0

0

$c = c + 2$

4

2

$c = c + 3$

1

5

$c = c + 1$

$cnt[6] = \{ 3, 2, 3, 0, 2, 1 \}$

return c;
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int pairs(vector<int> &ar, int M) { TC: $O(N+N) = O(N)$ SC: $O(M)$

#Step 1: Replace $ar[i] = ar[i] \% M$

int N = ar.size();

for (int i = 0; i < N; i++) {

$ar[i] = ar[i] \% M$ → if -ve: $(ar[i] \% M + M) \% M$

return <int> cnt[M, 0];

int c = 0;

for (int i = 0; i < N; i++) {

 int k = ar[i];

 int tar = M - k;

 if (k == 0) {

 tar = 0;

 c = c + cnt[tar]; # Add how many target elements present.

 cnt[k]++; # Increase freq of k

return c;