

Today's Content

1. Modular Arithmetic
2. Problems based on %

1. Modular Arithmetic Introduction.

$$\begin{cases} A \% M = \text{Remainder when } A \text{ divided by } M \\ n \% 4 = \{0, 1, 2, 3\} \quad n \% 5 = \{0, 1, 2, 3, 4\} \\ \text{Range} = [0 \dots M-1] \end{cases}$$

Why do we need %? To compress range to our use case

$$\left. \begin{array}{c} -\infty \\ \\ \infty \end{array} \right\} \xrightarrow{\text{Min Max}} \% M = [0 \dots M-1]$$

Rules for % Arithmetic: $\{+, -, \times, /\}$

$$\frac{a}{\{0, 2\}} + \frac{b}{\{0, 5\}} = \frac{a+b}{\{0, 7\}}$$

1. $(a+b) \% m = (a \% m + b \% m) \% m$

$$[0 \dots M-1] \quad [0 \dots M-1] + [0 \dots M-1] = \{0 \dots 2M-2\} \% M = [0 \dots M-1]$$

Ex: $a=9, b=8, m=5$

$$(9+8) \% 5 = (9 \% 5 + 8 \% 5) \% 5$$

$$(17) \% 5 = (4+3) \% 5$$

$$2 = (7) \% 5 = 2$$

2. $(a \% m) \% m = (a \% m) \% m = a \% m$

$$[0 \dots M-1] \% M = [0 \dots M-1]$$

3. $(a+m) \% m = (a \% m + m \% m) \% m$

$$= (a \% m + 0) \% m$$

$$= (a \% m) \% m$$

$$(a+m) \% m = a \% m$$

Note: if we have %m outside, adding +m inside bracket won't effect my value.

Concepts..

$$\left. \begin{matrix} -\infty \\ +\infty \end{matrix} \right\} \rightarrow \%M = [0..M-1]$$

In C/C++/Java:

$\%M$ m-be numbers will give -ve remainders

In C/C++/Java:

↳ Later:

Say $a < 0$:

$$a \% M = (a \% M + M) \% M$$

This part should work for every

C++ Python

$$\{0..M-1\} \quad \{- (M-1) .. 0\} + M = \{1..M\} \% M = \{0..M-1\}$$

In C/C++/Java

In Python

Say $a < 0$: $a \% M = [0..M-1]$

$-8 \% 6$
 $-10 \% 7$

$$5. \quad (a - b) \% M = (a \% M - b \% M + M) \% M$$

$$[0..M-1] \quad [0..M-1] - [0..M-1] = [- (M-1) .. M-1] + M = [1 .. 2M-1] \% M = [0..M-1]$$

Ex: $\frac{a}{[4 \ 10]} - \frac{b}{[6 \ 12]} = \frac{a-b}{[-8 \ 4]}$

$$5. \quad (a * b) \% M = (a \% M * b \% M) \% M$$

$$[0..M-1] \quad [0..M-1] * [0..M-1] \rightarrow \{0..(M-1)^2\} \% M = \{0..M-1\}$$

$$6. \quad (a^2) \% M = (a * a) \% M$$

$$= (a \% M * a \% M) \% M$$

$$= (a \% M)^2 \% M$$

$$6. \quad (a^2) \% M = (a \% M)^2 \% M$$

$$7. \quad (a^b) \% M = (a \% M)^b \% M$$

Properties.

$$(a+b)\%m = (a\%m + b\%m)\%m$$

$$(a*b)\%m = (a\%m * b\%m)\%m$$

$$(a+m)\%m = a\%m$$

$$(a\%m)\%m = a\%m$$

$$(a-b)\%m = (a\%m - b\%m + m)\%m$$

$$(a^b)\%m = (a\%m)^b\%m$$

$$\text{For -ve numbers } a\%m = (a\%m + m)\%m$$

Quizes:

$$Q_1: (37^{103} - 1)\%12 = (37^{103}\%12 - 1\%12 + 12)\%12$$

$$= ((37\%12)^{103}\%12 - 1\%12 + 12)\%12$$

$$= (\underline{10^{103}}\%12 - 1\%12 + 12)\%12$$

$$= (1\cancel{\%12} - 1\cancel{\%12} + 12)\%12$$

$$= (0)$$

Divisibility rule : 3 : Sum of digit's divisible by 3. \Rightarrow Plur Inded 3w

$$(789)\%3 = (700 + 80 + 9)\%3$$

$$= (700\%3 + 80\%3 + 9\%3)\%3$$

$$= ((7*100)\%3 + (8*10)\%3 + 9\%3)\%3$$

$$= ((7\%3 * \underline{100\%3})\%3 + (8\%3 * \underline{10\%3})\%3 + 9\%3)\%3$$

$$= ((7\%3)\%3 + (8\%3)\%3 + 9\%3)\%3$$

$$= (7\%3 + 8\%3 + 9\%3)\%3$$

$$= (7+8+9)\%3$$

Q1: Given $arr[N]$ we find count of pairs (i, j) such that

$(arr[i] + arr[j]) \% M = 0$ Note: $i \neq j$ and pair(i, j) same as pair(j, i)

Constraints

$$1 \leq N \leq 10^5$$

$$1 \leq arr[i] \leq 10^9$$



Ex: $arr[] = \{4, 3, 6, 3, 8, 12\}$ ans =

$$M = 6 \quad (arr[i] + arr[j]) \% M = 0.$$

$$(arr[1] + arr[3]) \% 6 = (3 + 3) \% 6 = 0$$

$$(arr[0] + arr[4]) \% 6 = (4 + 8) \% 6 = 0$$

$$(arr[2] + arr[5]) \% 6 = (6 + 12) \% 6 = 0$$

Idea: Generate all pairs & check if $sum \% M = 0$

$C = 0$

TC: $O(N^2)$ SC: $O(1)$

$i = 0; i < N; i++ \{$

$j = i+1; j < N; j++ \{$

$\text{if } (arr[i] + arr[j]) \% M == 0 \{$

$C++;$

$\}$

$\}$

$\text{return } C;$

Idea: $(arr[i] + arr[j]) \% M \rightarrow (\underline{arr[i]\%M} + \underline{arr[j]\%M}) \% M$

Hint: Calculate $arr[i]\%M$ values

Ex:

$M=6$ $A[] = \{ 2 \ 5 \ 4 \ 8 \ 14 \ 13 \ 6 \ 12 \ 24 \ 16 \ 19 \}$

$A[] = \{ 2 \ 5 \ 4 \ 2 \ 2 \ 1 \ 0 \ 0 \ 0 \ 4 \ 1 \}$

$ck \% M = [0..M-1]$, $M=6$, $[0..5]$ ←

obs: $(\underline{arr[i]\%M} + \underline{arr[j]\%M}) \% M = 0$

Ex $M=6$: $(2 + 4) \% 6 = 0$

: $(1 + 5) \% 6 = 0$

: $(3 + 3) \% 6 = 0$

obs1: $(k + M-k) \% M = 0$

Answer $M=6$: $(0 + 0, \cancel{6}) \% 6 = 0$

obs2: $(0 + 0) \% M = 0$

Conclusion

Case 1: $arr[i]\%M = 0 \Rightarrow arr[j]\%M = 0$

Case 1: $arr[i]\%M = k \Rightarrow arr[j]\%M = M-k$

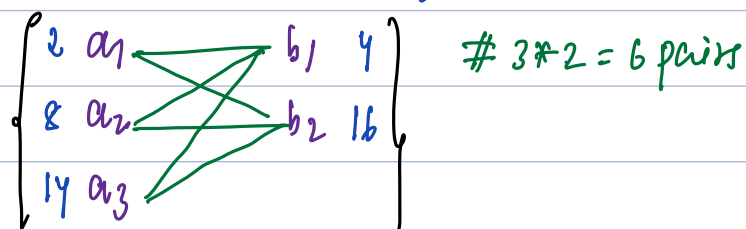
$$M=6 \quad A[] = \{ 2 \ 5 \ 4 \ 8 \ 14 \ 13 \ 6 \ 12 \ 24 \ 16 \ 19 \}$$

$$A[i] \% M$$

$$A[] = \{ 2 \ 5 \ 4 \ 2 \ 2 \ 1 \ 0 \ 0 \ 0 \ 4 \ 1 \}$$

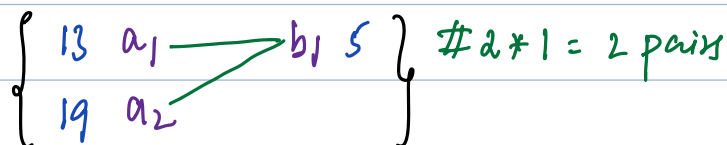
$$\text{eg } M=6 \quad \text{obs: } (a_i \% M + a_j \% M) \% M = 0$$

$$(\quad 2 \quad \text{AND} \quad 4 \quad) \% 6 = 0$$



$$\text{eg } M=6 \quad \text{obs: } (a_i \% M + a_j \% M) \% M = 0$$

$$M=6 \quad (\quad 1 \quad \quad \quad 5 \quad) \% 6 = 0$$



Hint: store count of all remainders, to get count of pairs where?

1. cnt[M] \rightarrow when we do $\% M = [0 \dots M-1]$ & for each remainder store count.

0 1 2 3 4 5 6 7 8 9 10
 $M = 6$ $A[] = \{2, 5, 4, 8, 14, 13, 6, 12, 24, 16, 19\}$

$A[i] \% M$

$A[] = \{2, 5, 4, 2, 2, 1, 0, 0, 0, 4, 1\}$

Wait d'it
 hashing is
 better
 than in count
 problem

$M = 6$

$arr[i] \% M$

$arr[j] \% M$

$C = 0;$

0 1 2 3 4 5

Remainder

Pair for it

Count Frequency

$cnt[6] = \{3, 2, 3, 0, 2, 1\}$

2

4

$C = C + 0$

5

1

$C = C + 0$

4

2

$C = C + 1$

2

4

$C = C + 1$

2

4

$C = C + 1$

1

5

$C = C + 1$

0

0

$C = C + 0$

0

0

$C = C + 1$

0

0

$C = C + 2$

→ 4

→ 2

$C = C + 3$

1

5

$C = C + 1$

return C;

||

int pairs(vector<int> &ar, int M) { Tc: $O(N+N) = O(N)$ Sc: $O(M)$

int N = ar.size();

for (int i = 0; i < N; i++) {
 ar[i] = ar[i] % M;
}

vector<int> cnt(M, 0);

int pair = 0;

for (int i = 0; i < N; i++) {

 int t = M - ar[i];

 if (ar[i] == 0) {

 t = 0;

 pair = pair + cnt[t]; # how many times target is present.

 cnt[ar[i]]++; # insert array element

}

return pair;

}