Equations for Non-Newtonian Fluid

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1 Basic equations

Momentum equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\nu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\nu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(2\nu\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial y}\right)$$

Pressure poisson equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)$$

2 Discretization of Viscous terms

First viscous term for u

$$\begin{aligned}\frac{\partial}{\partial x}\left(2\nu\frac{\partial u}{\partial x}\right) &= 2\frac{\partial \nu}{\partial x}\frac{\partial u}{\partial x} + 2\nu\frac{\partial^2 u}{\partial x^2} \\ &= 2\left(\frac{\nu^n_{i+1,j}-\nu^n_{i-1,j}}{2\Delta x}\right)\left(\frac{u^n_{i+1,j}-u^n_{i-1,j}}{2\Delta x}\right) \\ &\quad + 2\nu^n_{i,j}\left(\frac{u^n_{i+1,j}-2u^n_{i,j}+u^n_{i-1,j}}{\Delta x^2}\right)\end{aligned} \tag{1}$$

Second viscous term for u

$$\begin{aligned}\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right) &= \frac{\partial \nu}{\partial y}\frac{\partial u}{\partial y} + \nu\frac{\partial^2 u}{\partial y^2} \\ &= \left(\frac{\nu^n_{i,j+1}-\nu^n_{i,j-1}}{2\Delta y}\right)\left(\frac{u^n_{i,j+1}-u^n_{i,j-1}}{2\Delta y}\right) \\ &\quad + \nu^n_{i,j}\left(\frac{u^n_{i,j+1}-2u^n_{i,j}+u^n_{i,j-1}}{\Delta y^2}\right)\end{aligned} \tag{2}$$

Third viscous term for u

$$\begin{aligned}\frac{\partial}{\partial y}\left(\nu\frac{\partial v}{\partial x}\right) &= \frac{\partial \nu}{\partial y}\frac{\partial v}{\partial x} + \nu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) \\ &= \frac{\partial \nu}{\partial y}\frac{\partial v}{\partial x} + \nu^n_{i,j}\frac{\partial}{\partial y}\left(\frac{v^n_{i+1,j}-v^n_{i-1,j}}{2\Delta x}\right) \\ &= \frac{\partial \nu}{\partial y}\frac{\partial v}{\partial x} + \frac{\nu^n_{i,j}}{2\Delta x}\left(\frac{\partial v^n_{i+1,j}}{\partial y} - \frac{\partial v^n_{i-1,j}}{\partial y}\right) \\ &= \frac{\partial \nu}{\partial y}\frac{\partial v}{\partial x} + \frac{\nu^n_{i,j}}{2\Delta x}\left[\left(\frac{v^n_{i+1,j+1}-v^n_{i+1,j-1}}{2\Delta y}\right) - \left(\frac{v^n_{i-1,j+1}-v^n_{i-1,j-1}}{2\Delta y}\right)\right] \\ &= \left(\frac{\nu^n_{i,j+1}-\nu^n_{i,j-1}}{2\Delta y}\right)\left(\frac{v^n_{i+1,j}-v^n_{i-1,j}}{2\Delta x}\right) \\ &\quad + \frac{\nu^n_{i,j}}{2\Delta x}\left[\left(\frac{v^n_{i+1,j+1}-v^n_{i+1,j-1}}{2\Delta y}\right) - \left(\frac{v^n_{i-1,j+1}-v^n_{i-1,j-1}}{2\Delta y}\right)\right]\end{aligned} \tag{3}$$

First viscous term for v

$$\begin{aligned}\frac{\partial}{\partial x}\left(\nu\frac{\partial v}{\partial x}\right) &= \frac{\partial \nu}{\partial x}\frac{\partial v}{\partial x} + \nu\frac{\partial^2 v}{\partial x^2} \\ &= \left(\frac{\nu^n_{i+1,j} - \nu^n_{i-1,j}}{2\Delta x}\right)\left(\frac{v^n_{i+1,j} - v^n_{i-1,j}}{2\Delta x}\right) + \nu^n_{i,j}\left(\frac{v^n_{i+1,j} - 2v^n_{i,j} + v^n_{i-1,j}}{\Delta x^2}\right)\end{aligned} \tag{4}$$

Second Viscous term for v

$$\begin{aligned}\frac{\partial}{\partial y}\left(2\nu\frac{\partial v}{\partial y}\right) &= 2\frac{\partial \nu}{\partial y}\frac{\partial v}{\partial y} + 2\nu\frac{\partial^2 v}{\partial y^2} \\ &= 2\left(\frac{\nu^n_{i,j+1} - \nu^n_{i,j-1}}{2\Delta y}\right)\left(\frac{v^n_{i,j+1} - v^n_{i,j-1}}{2\Delta y}\right) + 2\nu^n_{i,j}\left(\frac{v^n_{i,j+1} - 2v^n_{i,j} + v^n_{i,j-1}}{\Delta y^2}\right)\end{aligned} \tag{5}$$

Third viscous term for v

$$\begin{aligned}\frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial y}\right) &= \frac{\partial \nu}{\partial x}\frac{\partial u}{\partial y} + \nu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \\ &= \frac{\partial \nu}{\partial x}\frac{\partial u}{\partial y} + \nu^n_{i,j}\frac{\partial}{\partial x}\left(\frac{u^n_{i,j+1} - u^n_{i,j-1}}{2\Delta y}\right) \\ &= \frac{\partial \nu}{\partial x}\frac{\partial u}{\partial y} + \frac{\nu^n_{i,j}}{2\Delta y}\left(\frac{\partial u^n_{i,j+1}}{\partial x} - \frac{\partial u^n_{i,j-1}}{\partial x}\right) \\ &= \frac{\partial \nu}{\partial x}\frac{\partial u}{\partial y} + \frac{\nu^n_{i,j}}{2\Delta y}\left[\left(\frac{u^n_{i+1,j+1} - u^n_{i-1,j+1}}{2\Delta x}\right) - \left(\frac{u^n_{i+1,j-1} - u^n_{i-1,j-1}}{2\Delta x}\right)\right] \\ &= \left(\frac{\nu^n_{i+1,j} - \nu^n_{i-1,j}}{2\Delta x}\right)\left(\frac{u^n_{i,j+1} - u^n_{i,j-1}}{2\Delta y}\right) \\ &\quad + \frac{\nu^n_{i,j}}{2\Delta y}\left[\left(\frac{u^n_{i+1,j+1} - u^n_{i-1,j+1}}{2\Delta x}\right) - \left(\frac{u^n_{i+1,j-1} - u^n_{i-1,j-1}}{2\Delta x}\right)\right]\end{aligned} \tag{6}$$

3 Complete Discretization of the Equations

Momentum Equation for u

$$\begin{aligned}
\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}+u_{i,j}^{n}\frac{u_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta x}+v_{i,j}^{n}\frac{u_{i,j}^{n}-u_{i,j-1}^{n}}{\Delta y}= \\
-\frac{1}{\rho}\frac{p_{i+1,j}^{n}-p_{i-1,j}^{n}}{2\Delta x}+\left[2\left(\frac{\nu_{i+1,j}^{n}-\nu_{i-1,j}^{n}}{2\Delta x}\right)\left(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n}}{2\Delta x}\right)\right. \\
\left.+2\nu_{i,j}^{n}\left(\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{\Delta x^{2}}\right)\right] \\
+\left[\left(\frac{\nu_{i,j+1}^{n}-\nu_{i,j-1}^{n}}{2\Delta y}\right)\left(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n}}{2\Delta y}\right)\right. \\
\left.+\nu_{i,j}^{n}\left(\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{\Delta y^{2}}\right)\right] \\
+\left[\left(\frac{\nu_{i,j+1}^{n}-\nu_{i,j-1}^{n}}{2\Delta y}\right)\left(\frac{v_{i+1,j}^{n}-v_{i-1,j}^{n}}{2\Delta x}\right)\right. \\
\left.+\frac{\nu_{i,j}^{n}}{2\Delta x}\left[\left(\frac{v_{i+1,j+1}^{n}-v_{i+1,j-1}^{n}}{2\Delta y}\right)-\left(\frac{v_{i-1,j+1}^{n}-v_{i-1,j-1}^{n}}{2\Delta y}\right)\right]\right]
\end{aligned}$$

Momentum Equation for v

$$\begin{aligned}
\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{\Delta t}+u_{i,j}^{n}\frac{v_{i,j}^{n}-v_{i-1,j}^{n}}{\Delta x}+v_{i,j}^{n}\frac{v_{i,j}^{n}-v_{i,j-1}^{n}}{\Delta y}= \\
-\frac{1}{\rho}\frac{p_{i,j+1}^{n}-p_{i,j-1}^{n}}{2\Delta y} \\
+\Bigg[\left(\frac{\nu_{i+1,j}^{n}-\nu_{i-1,j}^{n}}{2\Delta x}\right)\left(\frac{v_{i+1,j}^{n}-v_{i-1,j}^{n}}{2\Delta x}\right) \\
+\nu_{i,j}^{n}\left(\frac{v_{i+1,j}^{n}-2v_{i,j}^{n}+v_{i-1,j}^{n}}{\Delta x^{2}}\right)\Bigg] \\
+\Bigg[2\left(\frac{\nu_{i+1,j}^{n}-\nu_{i-1,j}^{n}}{2\Delta y}\right)\left(\frac{v_{i+1,j}^{n}-v_{i-1,j}^{n}}{2\Delta y}\right) \\
+2\nu_{i,j}^{n}\left(\frac{v_{i+1,j}^{n}-2v_{i,j}^{n}+v_{i-1,j}^{n}}{\Delta y^{2}}\right)\Bigg] \\
+\Bigg[\left(\frac{\nu_{i+1,j}^{n}-\nu_{i-1,j}^{n}}{2\Delta x}\right)\left(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n}}{2\Delta y}\right) \\
+\frac{\nu_{i,j}^{n}}{2\Delta y}\Bigg[\left(\frac{u_{i+1,j+1}^{n}-u_{i-1,j+1}^{n}}{2\Delta x}\right)- \\
\left(\frac{u_{i+1,j-1}^{n}-u_{i-1,j-1}^{n}}{2\Delta x}\right)\Bigg]\Bigg]
\end{aligned}$$

Pressure Poisson Equation

$$\begin{aligned}
\frac{p_{i+1,j}^{n}-2p_{i,j}^{n}+p_{i-1,j}^{n}}{\Delta x^{2}}+\frac{p_{i,j+1}^{n}-2p_{i,j}^{n}+p_{i,j-1}^{n}}{\Delta y^{2}}= \\
\rho\Big[\frac{1}{\Delta t}\left(\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}+\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}\right) \\
-\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} \\
-2\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta y}\frac{v_{i+1,j}-v_{i-1,j}}{2\Delta x} \\
-\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}\Big]
\end{aligned}$$

4 Explicit Equations for the unknown

Solving for u^{n+1}

$$
\begin{aligned}
u_{i,j}^{n+1} = u_{i,j}^{n} &- u_{i,j}^{n}\frac{\Delta t}{\Delta x}\left(u_{i,j}^{n} - u_{i-1,j}^{n}\right) - v_{i,j}^{n}\frac{\Delta t}{\Delta y}\left(u_{i,j}^{n} - u_{i,j-1}^{n}\right) \\
&- \frac{\Delta t}{\rho 2\Delta x}\left(p_{i+1,j}^{n} - p_{i-1,j}^{n}\right) \\
&+ \left[\left(\frac{\Delta t}{2\Delta x^2}\left(\nu_{i+1,j}^{n} - \nu_{i-1,j}^{n}\right)\left(u_{i+1,j}^{n} - u_{i-1,j}^{n}\right)\right)\right. \\
&+ \left(\frac{2\Delta t}{\Delta x^2}\nu_{i,j}^{n}\left(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right)\right) \\
&+ \left(\frac{\Delta t}{4\Delta y^2}\left(\nu_{i,j+1}^{n} - \nu_{i,j-1}^{n}\right)\left(u_{i,j+1}^{n} - u_{i,j-1}^{n}\right)\right) \\
&+ \left(\frac{\Delta t}{\Delta y^2}\nu_{i,j}^{n}\left(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right)\right) \\
&+ \left(\frac{\Delta t}{4\Delta y\Delta x}\left(\nu_{i,j+1}^{n} - \nu_{i,j-1}^{n}\right)\left(v_{i+1,j}^{n} - v_{i-1,j}^{n}\right)\right) \\
&+ \left.\left(\frac{\nu_{i,j}^{n}\Delta t}{4\Delta x\Delta y}\left[\left(v_{i+1,j+1}^{n} - v_{i+1,j-1}^{n}\right) - \left(v_{i-1,j+1}^{n} - v_{i-1,j-1}^{n}\right)\right]\right)\right]
\end{aligned}
$$

Solving for v^{n+1}

$$
\begin{aligned}
v_{i,j}^{n+1} = v_{i,j}^{n} &- u_{i,j}^{n}\frac{\Delta t}{\Delta x}\left(v_{i,j}^{n} - v_{i-1,j}^{n}\right) - v_{i,j}^{n}\frac{\Delta t}{\Delta y}\left(v_{i,j}^{n} - v_{i,j-1}^{n}\right) \\
&- \frac{\Delta t}{\rho 2\Delta y}\left(p_{i,j+1}^{n} - p_{i,j-1}^{n}\right) \\
&+ \left[\left(\frac{\Delta t}{4\Delta x^2}\left(\nu_{i+1,j}^{n} - \nu_{i-1,j}^{n}\right)\left(v_{i+1,j}^{n} - v_{i-1,j}^{n}\right)\right)\right. \\
&+ \left(\frac{\Delta t}{\Delta x^2}\nu_{i,j}^{n}\left(v_{i+1,j}^{n} - 2v_{i,j}^{n} + v_{i-1,j}^{n}\right)\right) \\
&+ \left(\frac{\Delta t}{2\Delta y^2}\left(\nu_{i,j+1}^{n} - \nu_{i,j-1}^{n}\right)\left(v_{i,j+1}^{n} - v_{i,j-1}^{n}\right)\right) \\
&+ \left(\frac{2\nu_{i,j}^{n}\Delta t}{\Delta y^2}\left(v_{i,j+1}^{n} - 2v_{i,j}^{n} + v_{i,j-1}^{n}\right)\right) \\
&+ \left(\frac{\Delta t}{4\Delta x\Delta y}\left(\nu_{i+1,j}^{n} - \nu_{i-1,j}^{n}\right)\left(u_{i,j+1}^{n} - u_{i,j-1}^{n}\right)\right) \\
&\left.+ \left(\frac{\nu_{i,j}^{n}\Delta t}{4\Delta y\Delta x}\left[\left(u_{i+1,j+1}^{n} - u_{i-1,j+1}^{n}\right) - \left(u_{i+1,j-1}^{n} - u_{i-1,j-1}^{n}\right)\right]\right)\right]
\end{aligned}
$$

Solving for p^{n+1}

$$
\begin{aligned}
p_{i,j}^{n} = &\frac{\left(p_{i+1,j}^{n} + p_{i-1,j}^{n}\right)\Delta y^2 + \left(p_{i,j+1}^{n} + p_{i,j-1}^{n}\right)\Delta x^2}{2\left(\Delta x^2 + \Delta y^2\right)} \\
&- \frac{\rho\Delta x^2\Delta y^2}{2\left(\Delta x^2 + \Delta y^2\right)} \\
&\times \left[\frac{1}{\Delta t}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right)\right. \\
&- \left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right)\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right) \\
&\left.- 2\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right]
\end{aligned}
$$