Richer output spaces

DSE 220

Multiclass classification

We have mostly discussed binary classification problems, with $|\mathcal{Y}| = 2$.
Do the methods we've studied generalize to cases with $k > 2$ labels?

- Nearest neighbor?
- Generative models?
- Linear classifiers?

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Linear classifiers seem inherently binary: there are just two sides of the boundary! How can they be extended to multiple classes?

Multiclass logistic regression

Binary logistic regression: for $\mathcal{X} = \mathbb{R}^p$, the classifier is given by $w \in \mathbb{R}^p$:

$$\Pr(y = 1|x) = \frac{e^{w \cdot x}}{1 + e^{w \cdot x}}.$$

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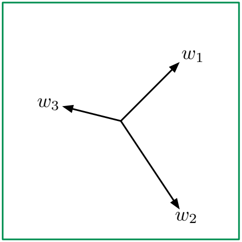
Learning: given data $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathbb{R}^p \times \mathcal{Y}$, find vectors $w_1, \ldots, w_k \in \mathbb{R}^p$ that maximize the likelihood

$$\prod_{i=1}^n \Pr(y^{(i)}|x^{(i)}).$$

Taking negative log gives a convex minimization problem.

Multiclass prediction with linear functions

- $\mathcal{X} = \mathbb{R}^p$ and $\mathcal{Y} = \{1, 2, \ldots, k\}$.
- **Model:** $w_1, \ldots, w_k \in \mathbb{R}^p$, one per class.
- **Prediction:** On instance x, predict label $\arg\max_j w_j \cdot x$.



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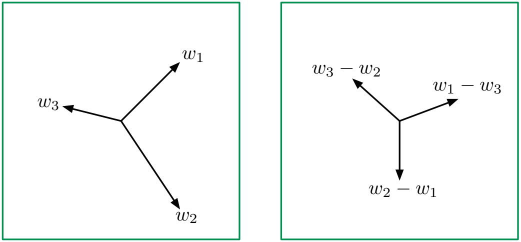

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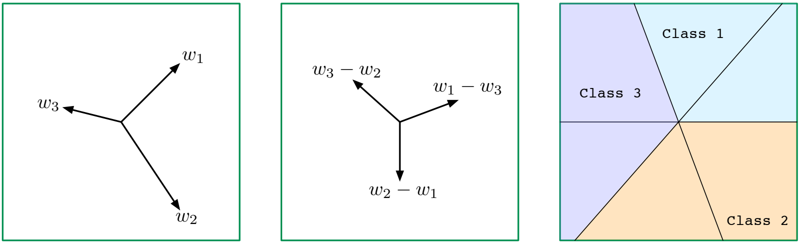

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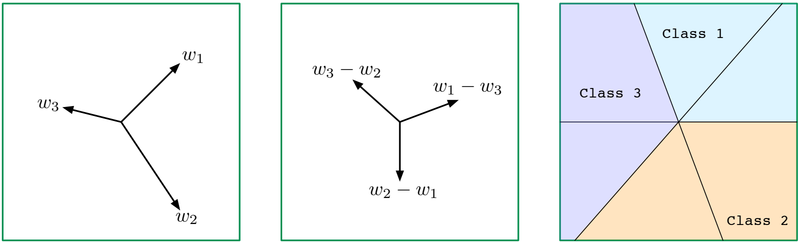

Each class is the intersection of half-spaces through the origin.

Multiclass Perceptron

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Learning. Given training set $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$:

- Initialize $w_1 = \cdots = w_k = 0$
- Repeat while some training point (x, y) is misclassified:

$$\begin{aligned} \text{for correct label } y\text{:} \quad & w_y = w_y + x \\ \text{for predicted label } \widehat{y}\text{:} \quad & w_{\widehat{y}} = w_{\widehat{y}} - x \end{aligned}$$

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Guarantee: Suppose all $\|x^{(i)}\| \leq R$ and that there exist unit-length $u_1, \ldots, u_k \in \mathbb{R}^p$ and "margin" $\gamma > 0$ such that for all i and all $y \neq y^{(i)}$,

$$u_{y^{(i)}} \cdot x^{(i)} - u_y \cdot x^{(i)} \;\geq\; \gamma.$$

Then the multiclass perceptron algorithm makes at most $2kR^2/\gamma^2$ updates.

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$$
\boxed{
\begin{aligned}
&\min_{w_1, \ldots, w_k \in \mathbb{R}^p, \xi \in \mathbb{R}^n} \quad \frac{1}{2} \sum_{j=1}^{k} \|w_j\|^2 + C \sum_{i=1}^{n} \xi_i \\
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Once again, a convex optimization problem.

Quick quiz

Suppose we have input space $\mathcal{X} = \mathbb{R}^p$ and label space $\mathcal{Y} = \{1, 2, \ldots, k\}$, and we have a training set of size n.

1. If we use multiclass SVM, how many variables does the primal program have?

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1. If we use multiclass SVM, how many variables does the primal program have?
2. How many constraints does it have?

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Part-of-speech tagging.

the/D cat/N bit/V the/D dog/N

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- Other such component scores

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To tag a given sentence x: find the tagging y with maximum score.
Can be done efficiently by dynamic programming.

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Here are two possible parse trees y for the sentence $x =$ "I shot an elephant in my pajamas".

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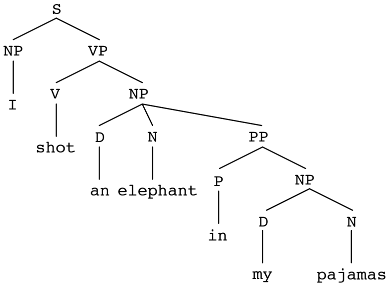
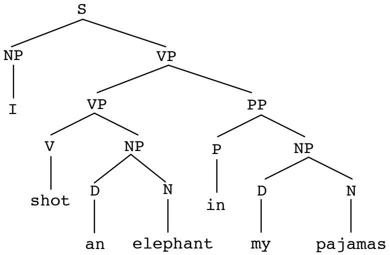
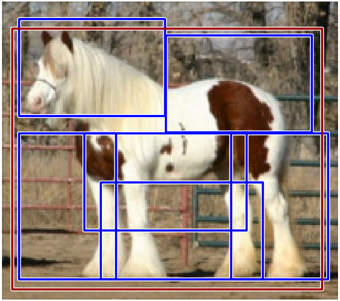




Structured output spaces: examples

Parts-based object recognition.



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- Features based on both the input and output.
  For any instance (e.g. sentence) x and candidate output (e.g. part-of-speech tagging) y, let

  $$\phi_1(x, y), \phi_2(x, y), \ldots, \phi_k(x, y)$$

  be features that give a sense of whether y is a desirable output for x. For instance: all word-tag pairs and tag trigrams.
  Package these features into a vector:

  $$\Phi(x, y) = (\phi_1(x, y), \phi_2(x, y), \ldots, \phi_k(x, y))$$

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  The score for output $y \in \mathcal{Y}$ is $w \cdot \Phi(x, y)$, where $w \in \mathbb{R}^k$.

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Learning task: given data, find a suitable weight vector w.

Structured-output Perceptron

Given training data $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)}) \in \mathcal{X} \times \mathcal{Y}$:

- Initialize $w = 0$
- Repeat until satisfied:
  - For $i = 1$ to n:

$$\text{Prediction:} \quad \widehat{y} = \arg\max_{y} w \cdot \Phi(x^{(i)}, y)$$

$$\text{If } y^{(i)} \neq \widehat{y}\text{:} \quad w = w + \Phi(x^{(i)}, y^{(i)}) - \Phi(x^{(i)}, \widehat{y})$$

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Convergence guarantee under a margin condition, as before.

Quick quiz

How does structured-output perceptron generalize multiclass perceptron?

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Loss function.
Not all errors are equal, especially when the outputs have many parts.
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Clever optimization tricks are needed to solve this efficiently.