c. roll die n times & take avg. of all the rolls (A)

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$$E(A) = \frac{1}{n} E\left[\sum_{i=1}^{n} Z_i\right] = \frac{\sum_{i=1}^{n} E(Z_i)}{n} = \frac{4n}{n} = 4$$

$$E(A^2) = E\left[\left[\frac{1}{n}\left(\sum_{i=1}^{n} Z_i\right)\right]^2\right] = \frac{1}{n^2} E\left[\sum_{i=1}^{n} (Z_i)^2 + \sum_{i\neq j} Z_i Z_j\right] = \frac{1}{n^2}\sum_{i=1}^{n} E(Z_i^2) + \frac{1}{n^2}\sum_{i\neq j} E(Z_i Z_j) = \frac{9+16n-16}{n} = \frac{3+16n}{n}$$

$$var(A) = E(A^2) - [E(A)]^2 = \frac{3+16n}{n} - 16 = \frac{3}{n} + \frac{16n}{n} - 16 = \frac{3}{n}$$

10. m balls n bins $m \geq n$

a. $X_i \Rightarrow$ # of balls that fall w/in bin i; what is $Pr(X_i = 0)$

$$Pr(X_i = 0) = \left(\frac{n-1}{n}\right)^m$$

b. $$Pr(X_i = 1) = \binom{m}{1}\left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right)^{m-1}$$

m poss. tosses that ball can be in i

P of tossing ball into ith bin

P that other balls do not go into ith bin

c. $A_j = 1$ if the j^{th} balls falls into bin i

$$E(A_j) = Pr(A_j = 1) = 1/n$$

$$E(X_i) = \sum_{j=1}^{m} E(A_j) = m/n$$

d. $$var(A_j) = E(A_j^2) - [E(A_j)]^2 = \frac{1}{n} - \frac{1}{n^2}$$

$$var(X_i) = \sum_{j=1}^{m} var(A_j) = m\left(\frac{1}{n} - \frac{1}{n^2}\right)$$

12. fair coin is tossed repeatedly until same outcome occurs twice in a row

↳ expected # of tosses

$$P(X=k) = \left(\frac{1}{2}\right)^{k-1} \quad k \geq 2 \text{ (minimum 2 rolls)}$$

$$E(n) = \sum_{k=2}^{\infty} k\left(\frac{1}{2}\right)^{k-1} = 2\left[\sum_{k=2}^{\infty} k\left(\frac{1}{2}\right)^k\right] = 2\left[\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k - \frac{1}{2}\right] = 2\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k - 1$$

consider $\sum_{k=1}^{\infty} kr^k = \frac{r}{1-r^2}$ $|r| < 1$

$\therefore$

$$= 2(2) - 1 = 4 - 1 = 3$$

DSE 210: Worksheet 4

Orysya Stus

2. fair dice rolled until 6 is seen
↳ expected # of rolls

$Pr(X=6) = 1/6 \quad E(X) = \sum x_i p_i$ $\quad x_i$ = value of the ith outcome, p_i = IP of the ith outcome

$$E(x) = \frac{1}{Pr(X=6)} = \frac{1}{(1/6)} = 6$$

4. 10 floors n ppl on elevator => each go to some floor 1->10

a. exactly 1 person gets out @ the ith floor => use

$$Pr(x_i) = n\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^{n-1} \Rightarrow n\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{n-1}$$

b. expected # of floors in which 1 person gets out

$$x = \begin{cases} 1 & \text{exactly 1 person gets out} \\ 0 & \text{otherwise} \end{cases}$$

↳ linearity of expectation

$$E(X_i) = Pr(X_i=1) = \sum_{i=1}^{10} E(X_i) = 10n\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{n-1} = n\left(\frac{9}{10}\right)^{n-1}$$

6. n beds for n students each student gets into a bed chosen uniformly @ random
↳ expected # of students who end up in their bed

$$Pr(X=1) = \frac{1}{n} \quad E(X_i) = Pr(X=1) = \sum_{i=1}^{n} \frac{1}{n} = 1$$

1 student is expected to get into their bed

8. 6 sides $\quad Pr(1) = Pr(2) = Pr(3) = Pr(4) = 1/8 \quad Pr(5) = Pr(6) = 1/4$

a. $E(Z) = 1\left(\frac{1}{8}\right) + 2\left(\frac{1}{8}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{8}\right) + \frac{1}{4}(5+6) = 4$

$$E(Z^2) = \frac{1}{8}(1^2+2^2+3^2+4^2) + \frac{1}{4}(5^2+6^2) = 19$$

$$var(Z) = E(Z-\mu)^2 = E(Z^2) - \mu^2 = E(Z^2) - (E(Z))^2 = 19 - 4^2 = 3$$

b. roll the dice 10x, independently X is the sum of all the rolls

$$E(X) = E\left[\sum_{i=1}^{10} Z_i\right] = \sum_{i=1}^{10} E(Z_i) = \sum_{i=1}^{10} 4 = 4 \cdot 10 = 40$$

$$E(X^2) = E\left[\left(\sum_{i=1}^{10} Z_i\right)^2\right] = E\left[\sum_{i=1}^{10}(Z_i)^2 + \sum_{i \neq j} Z_i Z_j\right] = \sum_{i=1}^{10} E(Z_i^2) + \sum_{i \neq j} E(Z_i Z_j) = 19 \cdot 10 + 10 \cdot 9 \cdot 4 \cdot 4 = 1630$$

$$var(X) = E(X^2) - [E(X)]^2 = 1630 - 40^2 = 1630 - 1600 = 30$$