Probability spaces

DSE 210

Probability spaces

How to interpret a statement like:

> *The chance of getting a flush in a five-card poker hand is about* 0.20%. *(Flush = five of the same suit.)*

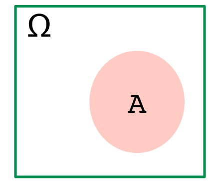
The underlying **probability space** has two components:

1. The **sample space** (the space of outcomes).
   In the example, $\Omega = \{$all possible five-card hands$\}$.
2. The **probabilities of outcomes**.
   In the example, all hands are equally likely: probability $1/|\Omega|$.

Note: $\sum_{\omega \in \Omega} \Pr(\omega) = 1$.

Event of interest: the set of outcomes $A = \{\omega : \omega \text{ is a flush}\} \subset \Omega$.

$$\Pr(A) = \sum_{\omega \in A} \Pr(\omega) = \frac{|A|}{|\Omega|}$$



Examples

Roll a die. What is the chance of getting a number > 3?

Sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Probabilities of outcomes: $\Pr(\omega) = \frac{1}{6}$.

Event of interest: $A = \{4, 5, 6\}$

$$\Pr(A) = \Pr(4) + \Pr(5) + \Pr(6) = \frac{1}{2}.$$

Roll three dice. What is the chance that their sum is 3?

Sample space

$$\begin{aligned}\Omega &= \{(1,1,1), (1,1,2), (1,1,3), \ldots, (6,6,6)\} \\ &= \Omega_o \times \Omega_o \times \Omega_o\end{aligned}$$

where $\Omega_o = \{1, 2, 3, 4, 5, 6\}$.

Probabilities of outcomes:

$$\Pr(\omega) = \frac{1}{|\Omega|} = \frac{1}{216}$$

Event of interest: $A = \{(1,1,1)\}$. $\Pr(A) = \frac{1}{216}$.

Roll n dice.

Then $\Omega = \Omega_o \times \cdots \times \Omega_o = \Omega_o^n$, where $\Omega_o = \{1, 2, 3, 4, 5, 6\}$.

What is $|\Omega|$? 6^n.

Probability of an outcome: $\Pr(\omega) = \frac{1}{6^n}$.

Socks in a drawer. A drawer has three blue socks and three red socks. You put your hand in and pull out two socks at random. What is the probability that they match?

Think of grabbing one sock first, then another.

$\Omega = \{(B,B),(B,R),(R,B),(R,R)\} = \{B,R\}^2$.

Probabilities:

$$\begin{aligned}
\Pr((B,B)) &= \frac{1}{2}\cdot\frac{2}{5} = \frac{1}{5}\\
\Pr((B,R)) &= \frac{1}{2}\cdot\frac{3}{5} = \frac{3}{10}\\
\Pr((R,B)) &= \frac{1}{2}\cdot\frac{3}{5} = \frac{3}{10}\\
\Pr((R,R)) &= \frac{1}{2}\cdot\frac{2}{5} = \frac{1}{5}
\end{aligned}$$

Event of interest: $A = \{(B,B),(R,R)\}$. $\Pr(A) = \frac{2}{5}$.

Shuffle a pack of cards.

Sample space $\Omega = \{\text{all possible orderings of 52 cards}\}$.

What is $|\Omega|$?

$$52! = 52\cdot 51\cdot 50\cdot 49\cdots 3\cdot 2\cdot 1$$

Socks in a drawer, cont'd. This time the drawer has three blue socks and two red socks. You put your hand in and pull out two socks at random. What is the probability that they match?

Sample sample space, $\Omega = \{(B,B),(B,R),(R,B),(R,R)\} = \{B,R\}^2$.

Different probabilities:

$$\begin{aligned}
\Pr((B,B)) &= \frac{3}{5}\cdot\frac{2}{4} = \frac{3}{10}\\
\Pr((B,R)) &= \frac{3}{5}\cdot\frac{2}{4} = \frac{3}{10}\\
\Pr((R,B)) &= \frac{2}{5}\cdot\frac{3}{4} = \frac{3}{10}\\
\Pr((R,R)) &= \frac{2}{5}\cdot\frac{1}{4} = \frac{1}{10}
\end{aligned}$$

Event of interest: $A = \{(B,B),(R,R)\}$. $\Pr(A) = \frac{2}{5}$.

Toss a fair coin 10 times. What is the chance none are heads?

Sample space $\Omega = \{H,T\}^{10}$. It includes, for instance, (H,T,H,T,H,T,H,T,H,T).

What is $|\Omega|$? $2^{10} = 1024$.

For any sequence of coin tosses $\omega \in \Omega$, we have $\Pr(\omega) = \frac{1}{1024}$.

Event of interest: $A = \{(T,T,T,T,T,T,T,T,T,T)\}$. $\Pr(A) = \frac{1}{1024}$.

What is the probability of exactly one head?

Event of interest: $A = \{\omega \in \Omega : \omega \text{ has exactly one } H\}$.

What is $|A|$? 10.
Each sequence in A can be specified by the location of the one H, and there are 10 choices for this.

What is $\Pr(A)$? $\frac{10}{1024}$.

Toss a fair coin 10 times. What is the chance of exactly two heads?

Again, sample space $\Omega = \{H, T\}^{10}$, with $|\Omega| = 2^{10} = 1024$.

For any sequence of coin tosses $\omega \in \Omega$, we have $\Pr(\omega) = \frac{1}{1024}$.

Event of interest: $A = \{\omega \in \Omega : \omega \text{ has exactly two } H\text{'s}\}$.

What is $|A|$? $\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$.
Each sequence in A can be specified by the locations of the two H's and there are $\binom{10}{2}$ choices for these locations.

What is $\Pr(A)$? $\frac{45}{1024}$.

What is the probability of exactly k heads?

Event of interest: $A = \{\omega \in \Omega : \omega \text{ has exactly } k \ H\text{'s}\}$.

What is $|A|$? $\binom{10}{k}$.

What is $\Pr(A)$? $\binom{10}{k}/1024$.

Five-card poker. You are dealt 5 cards from a deck of 52.

Sample space $\Omega = \{\text{all possible hands}\}$.

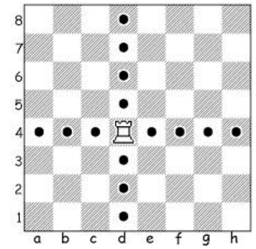
Probabilities: each hand is equally likely, $\Pr(\omega) = 1/|\Omega|$.

What is $|\Omega|$? $\binom{52}{5}$.

What is the probability of getting a flush (five of the same suit)?
Event of interest: $F = \{\text{flush hands}\}$. Then $|F| = 4 \times \binom{13}{5}$
Therefore $\Pr(F) = |F|/|\Omega| = 4 \times \binom{13}{5}/\binom{52}{5}$.

What is the chance of a straight flush (flush, and in sequence)?
Let $S = \{\text{straight flush hands}\}$. Then $|S| = 4 \cdot 9 = 36$.
And $\Pr(S) = |S|/|\Omega|$.

Rooks on a chessboard.



What is the maximum number of rooks you can place so that no rook is attacking any other? 8.

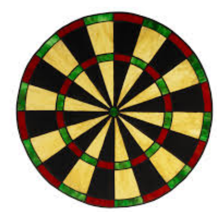
How many ways are there to place 8 rooks on the board, attacking or not? $\binom{64}{8}$

How many non-attacking placements of 8 rooks are there?
$8 \cdot 7 \cdot 6 \cdot 5 \cdots = 8!$

Randomly place 8 rooks on the board. What is the probability that it is a non-attacking placement?

$$\frac{8!}{\binom{64}{8}}.$$

Dartboard. A dartboard has radius 1 and its central bullseye has radius 0.1. You throw a dart and it lands at a random location on the board.



Sample space $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$.
All points are equally likely.

What is the probability of hitting the bullseye?

Event of interest: $B = \{(x, y) : x^2 + y^2 \leq (0.1)^2\}$.
$\Pr(B) = \text{area}(B)/\text{area}(\Omega) = 0.01$.

What is the probability of hitting the exact center? 0.

Birthday paradox. A room contains k people. What is the chance that they all have different birthdays?

Number the people $1, 2, \ldots, k$.
Number the days of the year $1, 2, \ldots, 365$.

Let $\omega = (\omega_1, \ldots, \omega_k)$, where $\omega_i \in \{1, 2, \ldots, 365\}$ is the birthday of person i. Thus $\Omega = \{1, 2, \ldots, 365\}^k$.

What is $|\Omega|$? 365^k.

Event of interest: $A = \{(\omega_1, \ldots, \omega_k) : \text{all } \omega_i \text{ different}\}$.
What is $|A|$? $365 \cdot 364 \cdot 363 \cdots (365 - k + 1)$.

Therefore,

$$\Pr(A) = \frac{365 \cdot 364 \cdots (365 - k + 1)}{365^k} = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365 - k + 1}{365}$$

For $k = 23$, this is less than $1/2$. In other words, **in a group of 23 random people, chances are some pair of them have a common birthday!**