DSE 210: Probability and statistics **Winter 2016**

Worksheet 4 — Random variable, expectation, and variance

1. A die is thrown twice. Let X_1 and X_2 denote the outcomes, and define random variable X to be the minimum of X_1 and X_2. Determine the distribution of X.

2. A fair die is rolled repeatedly until a six is seen. What is the expected number of rolls?

3. On any given day, the probability it will be sunny is 0.8, the probability you will have a nice dinner is 0.25, and the probability that you will get to bed early is 0.5. Assume these three events are independent. What is the expected number of days before all three of them happen together?

4. An elevator operates in a building with 10 floors. One day, n people get into the elevator, and each of them chooses to go to a floor selected uniformly at random from 1 to 10.

   (a) What is the probability that exactly one person gets out at the ith floor? Give your answer in terms of n.

   (b) What is the expected number of floors in which exactly one person gets out? *Hint:* let X_i be 1 if exactly one person gets out on floor i, and 0 otherwise. Then use linearity of expectation.

5. You throw m balls into n bins, each independently at random. Let X be the number of balls that end up in bin 1.

   (a) Let X_i be the event that the ith ball falls in bin 1. Write X as a function of the X_i.

   (b) What is the expected value of X?

6. There is a dormitory with n beds for n students. One night the power goes out, and because it is dark, each student gets into a bed chosen uniformly at random. What is the expected number of students who end up in their own bed?

7. In each of the following cases, say whether X and Y are independent.

   (a) You randomly permute $(1, 2, \ldots, n)$. X is the number in the first position and Y is the number in the second position.

   (b) You randomly pick a sentence out of *Hamlet*. X is the first word in the sentence and Y is the second word.

   (c) You randomly pick a card from a pack of 52 cards. X is 1 if the card is a nine, and is 0 otherwise. Y is 1 if the card is a heart, and is 0 otherwise.

   (d) You randomly deal a ten-card hand from a pack of 52 cards. X is 1 if the hand contains a nine, and is 0 otherwise. Y is 1 if *all* cards in the hand are hearts, and is 0 otherwise.

8. A die has six sides that come up with different probabilities:

$$\Pr(1) = \Pr(2) = \Pr(3) = \Pr(4) = 1/8, \ \Pr(5) = \Pr(6) = 1/4.$$

   (a) You roll the die; let Z be the outcome. What is $\mathbb{E}(Z)$ and $\text{var}(Z)$?

(b) You roll the die 10 times, independently; let X be the *sum* of all the rolls. What is $\mathbb{E}(X)$ and $\mathrm{var}(X)$?

(c) You roll the die n times and take the average of all the rolls; call this A. What is $\mathbb{E}(A)$? What is $\mathrm{var}(A)$?

9. Let $X_1, X_2, \ldots, X_{100}$ be the outcomes of 100 independent rolls of a fair die.

   (a) What are $\mathbb{E}(X_1)$ and $\mathrm{var}(X_1)$?

   (b) Define the random variable X to be $X_1 - X_2$. What are $\mathbb{E}(X)$ and $\mathrm{var}(X)$?

   (c) Define the random variable Y to be $X_1 - 2X_2 + X_3$. What is $\mathbb{E}(Y)$ and $\mathrm{var}(Y)$?

   (d) Define the random variable $Z = X_1 - X_2 + X_3 - X_4 + \cdots + X_{99} - X_{100}$. What are $\mathbb{E}(Z)$ and $\mathrm{var}(Z)$?

10. Suppose you throw m balls into n bins, where $m \geq n$. For the following questions, give answers in terms of m and n.

    (a) Let X_i be the number of balls that fall into bin i. What is $\Pr(X_i = 0)$?

    (b) What is $\Pr(X_i = 1)$?

    (c) What is $\mathbb{E}(X_i)$?

    (d) What is $\mathrm{var}(X_i)$?

11. Give an example of random variables X and Y such that $\mathrm{var}(X + Y) \neq \mathrm{var}(X) + \mathrm{var}(Y)$.

12. Suppose a fair coin is tossed repeatedly until the same outcome occurs twice in a row (that is, two heads in a row or two tails in a row). What is the expected number of tosses?

13. In a sequence of coin tosses, a *run* is a series of consecutive heads or consecutive tails. For instance, the longest run in $HTHHHTTHHTHH$ consists of three heads. We are interested in the following question: when a fair coin is tossed n times, how long a run is the resulting sequence likely to contain?

    To study this, pick any k between 1 and n, and let R_k denote the number of runs of length exactly k (for instance, a run of length $k+1$ doesn't count). In order to figure out $\mathbb{E}(R_k)$, we define the following random variables: $X_i = 1$ if a run of length exactly k begins at position i, where $i \leq n - k + 1$.

    (a) What are $\mathbb{E}(X_1)$ and $\mathbb{E}(X_{n-k+1})$?

    (b) What is $\mathbb{E}(X_i)$ for $1 < i < n - k + 1$?

    (c) What is $\mathbb{E}(R_k)$?

    (d) What is, roughly, the largest k for which $\mathbb{E}(R_k) \geq 1$?