Some linear algebra background

CSE 250B

Matrix-vector notation

Vector $x \in \mathbb{R}^p$ and matrix $M \in \mathbb{R}^{r \times p}$:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_p \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1p} \\ M_{21} & M_{22} & \cdots & M_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & \cdots & M_{rp} \end{pmatrix}.$$

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Transpose x^T and $M^T \in \mathbb{R}^{p \times r}$:

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Properties of transpose: $(A^T)^T = A$ and $(AB)^T = B^T A^T$.

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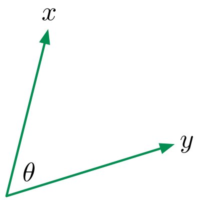

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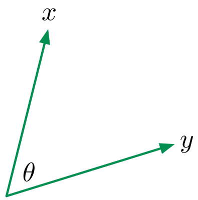

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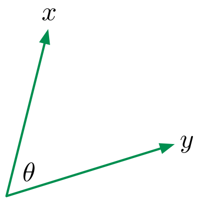

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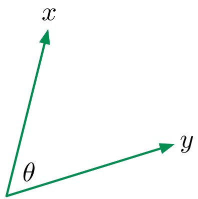

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What is $x \cdot x$?

Matrix-vector products

If $M \in \mathbb{R}^{r \times p}$ and $x \in \mathbb{R}^p$ then

$$Mx = \begin{pmatrix} \longleftarrow & M_1 & \longrightarrow \\ \longleftarrow & M_2 & \longrightarrow \\ & \vdots & \\ \longleftarrow & M_r & \longrightarrow \end{pmatrix} \begin{pmatrix} \big\uparrow \\ x \\ \big\downarrow \end{pmatrix} = \begin{pmatrix} M_1 \cdot x \\ M_2 \cdot x \\ \vdots \\ M_r \cdot x \end{pmatrix}$$

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If $M \in \mathbb{R}^{p \times p}$ and $x \in \mathbb{R}^p$ then $x \mapsto x^T Mx$ is a **quadratic function** from $\mathbb{R}^p$ to $\mathbb{R}$:

$$x^T Mx = \sum_{i,j=1}^{p} M_{ij} x_i x_j.$$

Quick quiz

1. Write the linear function $f(x_1, x_2) = 3x_1 + 2x_2$ using vector notation (here, $x_1, x_2 \in \mathbb{R}$).

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2. Write the quadratic function $f(x_1, x_2) = x_1^2 + 2x_1x_2 + 3x_2^2$ using matrices and vectors.
3. A linear function from $\mathbb{R}^3$ to $\mathbb{R}^3$ is given by the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}.$$

   As x varies, does Mx fill up all of $\mathbb{R}^3$?

A hierarchy of square matrices

Square $\qquad M \in \mathbb{R}^{p \times p}$

$\bigcup$

Symmetric $\qquad M = M^T$

$\bigcup$

Positive semidefinite $\qquad z^T M z \geq 0$ for all $z \in \mathbb{R}^p$

$\bigcup$

Positive definite $\qquad z^T M z > 0$ for all $z \neq 0$

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Symmetric matrix M is positive semidefinite (psd) if $z^T Mz \geq 0$ for all z.

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3. Show: If M, N are of the same size and PSD and $M + N$ is PSD.

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2. M is symmetric.
3. Pick any $z \in \mathbb{R}^r$. Then

$$\begin{aligned} z^T M z \;=\; z^T U U^T z &\;=\; (z^T U)(U^T z) \\ &\;=\; (U^T z)^T (U^T z) \;=\; \|U^T z\|^2 \;\geq\; 0. \end{aligned}$$

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Another useful fact: any covariance matrix is PSD. (Same argument, along with linearity of expectation.)

Eigenvalues and eigenvectors

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$$\underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 10 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 2x_1 \\ -x_2 \\ 10x_3 \end{pmatrix}}_{Mx}$$

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Let M be a $p \times p$ matrix.
We say $u \in \mathbb{R}^p$ is an **eigenvector** if M maps u onto the same direction, that is,

$$Mu = \lambda u$$

for some scaling constant λ. This λ is the **eigenvalue** associated with u.

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In both cases the eigenvectors form an orthonormal basis.

Eigenvectors of a real symmetric matrix

Theorem. Let M be any real symmetric $p \times p$ matrix. Then M has

- p eigenvalues $\lambda_1, \ldots, \lambda_p$
- corresponding eigenvectors $u_1, \ldots, u_p \in \mathbb{R}^p$ that are **orthonormal**:

$$u_i \cdot u_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

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Theorem. Let M be any real symmetric $p \times p$ matrix, and let $\lambda_1, \ldots, \lambda_p$ be its eigenvalues. Then:

- M is positive semidefinite iff every λ_i is ≥ 0.
- M is positive definite iff every λ is > 0.