```
syms L1 L2 L3 L4 theta1 theta2 theta3 theta_dot_1 theta_dot_2 theta_dot_3 real
```

```
[T10,R10]=DH(0,0,theta1,L1+L2)
```

T10 =

$$\begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & L_1+L_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

R10 =

$$\begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
[T21,R21]=DH(90,0,theta2,0)
```

T21 =

$$\begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

R21 =

$$\begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 \\ 0 & 0 & -1 \\ \sin(\theta_2) & \cos(\theta_2) & 0 \end{pmatrix}$$

```
[T32,R32]=DH(0,L3,theta3,0)
```

T32 =

$$\begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & L_3 \\ \sin(\theta_3) & \cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

R32 =

$$\begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 \\ \sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
[T43,R43]=DH(0,L4,0,0)
```

T43 =

$$\begin{pmatrix} 1 & 0 & 0 & L_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

R43 =

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
w00=[0;0;0]; %intializing w and v , w=0 and v=0 because they are the fixed frame
v00=[0;0;0];
```

```
% wab means omeaga of frame 'a' written in frame of 'b'
% vab means velocity of frame 'a' written in frame of 'b'
[w11,v11] = omega_and_vel_next(R10',w00,[0;0;theta_dot_1],v00,[0;0;L1+L2]);
[w22,v22] = omega_and_vel_next(R21',w11,[0;0;theta_dot_2],v11,[0;0;0]);
[w33,v33] = omega_and_vel_next(R32',w22,[0;0;theta_dot_3],v22,[L3;0;0]);
[w44,v44] = omega_and_vel_next(R43',w33,[0;0;0],v33,[L4;0;0]);
```

```
w11=simplify(w11);
v11=simplify(v11);
w22=simplify(w22);
v22=simplify(v22);
w33=simplify(w33);
v33=simplify(v33);
w44=simplify(w44) % w44 means omeaga of frame '4' written in frame of '4'
```

w44 =

$$\begin{pmatrix} \dot{\theta}_1 \sin(\theta_2+\theta_3) \\ \dot{\theta}_1 \cos(\theta_2+\theta_3) \\ \dot{\theta}_2+\dot{\theta}_3 \end{pmatrix}$$

```
v44=simplify(v44) % v44 means velocity of frame '4' written in frame of '4'
```

v44 =

$$\begin{pmatrix} L_3 \dot{\theta}_2 \sin(\theta_3) \\ L_4 \left(\dot{\theta}_2+\dot{\theta}_3\right) + L_3 \dot{\theta}_2 \cos(\theta_3) \\ -L_4 \dot{\theta}_1 \cos(\theta_2+\theta_3) - L_3 \dot{\theta}_1 \cos(\theta_2) \end{pmatrix}$$

```
v40=simplify(R10*R21*R32*R43*v44) % v40 means velocity of frame '4' written in frame of '0'
```

v40 =

$$\begin{pmatrix} L_3 \dot{\theta}_2 \cos(\theta_2+\theta_3) \cos(\theta_1) \sin(\theta_3) - \dot{\theta}_1 \sin(\theta_1)\, \sigma_1 - \sin(\theta_2+\theta_3) \cos(\theta_1)\, \sigma_2 \\ \dot{\theta}_1 \cos(\theta_1)\, \sigma_1 - \sin(\theta_2+\theta_3) \sin(\theta_1)\, \sigma_2 + L_3 \dot{\theta}_2 \cos(\theta_2+\theta_3) \sin(\theta_1) \sin(\theta_3) \\ L_4 \dot{\theta}_2 \cos(\theta_2+\theta_3) + L_4 \dot{\theta}_3 \cos(\theta_2+\theta_3) + L_3 \dot{\theta}_2 \cos(\theta_2) \end{pmatrix}$$

where

$$\sigma_1 = L_4 \cos(\theta_2+\theta_3) + L_3 \cos(\theta_2)$$

$$\sigma_2 = L_4 \dot{\theta}_2 + L_4 \dot{\theta}_3 + L_3 \dot{\theta}_2 \cos(\theta_3)$$

```
w40=simplify(R10*R21*R32*R43*w44) % w40 means omeaga of frame '4' written in frame of '0'
```

w40 =

$$\begin{pmatrix} \sin(\theta_1) \left(\dot{\theta}_2+\dot{\theta}_3\right) \\ -\cos(\theta_1) \left(\dot{\theta}_2+\dot{\theta}_3\right) \\ \dot{\theta}_1 \end{pmatrix}$$

```
J1=subs(v40, [theta_dot_1 ,theta_dot_2 ,theta_dot_3], [1,0,0]);
J2=subs(v40, [theta_dot_1 ,theta_dot_2 ,theta_dot_3], [0,1,0]);
J3=subs(v40, [theta_dot_1 ,theta_dot_2 ,theta_dot_3], [0,0,1]);
Jv0=[J1,J2,J3];
Jv0=simplify(Jv0) % linear velocity Jacobian written in frame of 0
```

Jv0 =

$$\begin{pmatrix} -\sin(\theta_1)\, \sigma_1 & -\cos(\theta_1)\, \sigma_2 & -L_4 \sin(\theta_2+\theta_3) \cos(\theta_1) \\ \cos(\theta_1)\, \sigma_1 & -\sin(\theta_1)\, \sigma_2 & -L_4 \sin(\theta_2+\theta_3) \sin(\theta_1) \\ 0 & \sigma_1 & L_4 \cos(\theta_2+\theta_3) \end{pmatrix}$$

where

$$\sigma_1 = L_4 \cos(\theta_2+\theta_3) + L_3 \cos(\theta_2)$$

$$\sigma_2 = L_4 \sin(\theta_2+\theta_3) + L_3 \sin(\theta_2)$$

```
check = simplify(v40-Jv0*[theta_dot_1 ;theta_dot_2 ;theta_dot_3]) %verifies v=J*theta_dot
```

check =

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

```
J4=subs(w40, [theta_dot_1 ,theta_dot_2 ,theta_dot_3], [1,0,0]);
J5=subs(w40, [theta_dot_1 ,theta_dot_2 ,theta_dot_3], [0,1,0]);
J6=subs(w40, [theta_dot_1 ,theta_dot_2 ,theta_dot_3], [0,0,1]);
Jw0=[J4,J5,J6];
Jw0=simplify(Jw0) % angular velocity Jacobian written in frame of 0
```

Jw0 =

$$\begin{pmatrix} 0 & \sin(\theta_1) & \sin(\theta_1) \\ 0 & -\cos(\theta_1) & -\cos(\theta_1) \\ 1 & 0 & 0 \end{pmatrix}$$

```
check = simplify(w40-Jw0*[theta_dot_1 ;theta_dot_2 ;theta_dot_3]) %verifies w=J*theta_dot
```

check =

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

```
J0=[Jv0;Jw0] %because [v;w]=J*[theta_dot]
```

J0 =

$$\begin{pmatrix} -\sin(\theta_1)\, \sigma_1 & -\cos(\theta_1)\, \sigma_2 & -L_4 \sin(\theta_2+\theta_3) \cos(\theta_1) \\ \cos(\theta_1)\, \sigma_1 & -\sin(\theta_1)\, \sigma_2 & -L_4 \sin(\theta_2+\theta_3) \sin(\theta_1) \\ 0 & \sigma_1 & L_4 \cos(\theta_2+\theta_3) \\ 0 & \sin(\theta_1) & \sin(\theta_1) \\ 0 & -\cos(\theta_1) & -\cos(\theta_1) \\ 1 & 0 & 0 \end{pmatrix}$$

where

$$\sigma_1 = L_4 \cos(\theta_2+\theta_3) + L_3 \cos(\theta_2)$$

$$\sigma_2 = L_4 \sin(\theta_2+\theta_3) + L_3 \sin(\theta_2)$$

```
J1=subs(v44, [theta_dot_1 ,theta_dot_2 ,theta_dot_3], [1,0,0]);
J2=subs(v44, [theta_dot_1 ,theta_dot_2 ,theta_dot_3], [0,1,0]);
J3=subs(v44, [theta_dot_1 ,theta_dot_2 ,theta_dot_3], [0,0,1]);
Jv4=[J1,J2,J3];
Jv4=simplify(Jv4) % linear velocity Jacobian written in frame of 4
```

Jv4 =

$$\begin{pmatrix} 0 & L_3 \sin(\theta_3) & 0 \\ 0 & L_4 + L_3 \cos(\theta_3) & L_4 \\ -L_4 \cos(\theta_2+\theta_3) - L_3 \cos(\theta_2) & 0 & 0 \end{pmatrix}$$

```
check = simplify(v44-Jv4*[theta_dot_1 ;theta_dot_2 ;theta_dot_3]) %verifies v=J*theta_dot
```

check =

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

```
J4=subs(w44, [theta_dot_1 ,theta_dot_2 ,theta_dot_3], [1,0,0]);
J5=subs(w44, [theta_dot_1 ,theta_dot_2 ,theta_dot_3], [0,1,0]);
J6=subs(w44, [theta_dot_1 ,theta_dot_2 ,theta_dot_3], [0,0,1]);
Jw4=[J4,J5,J6];
Jw4=simplify(Jw4) % angular velocity Jacobian written in frame of 4
```

Jw4 =

$$\begin{pmatrix} \sin(\theta_2+\theta_3) & 0 & 0 \\ \cos(\theta_2+\theta_3) & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

```
check = simplify(w44-Jw4*[theta_dot_1 ;theta_dot_2 ;theta_dot_3]) %verifies w=J*theta_dot
```

check =

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

```
J4=[Jv4;Jw4] %because [v;w]=J*[theta_dot]
```

J4 =

$$\begin{pmatrix} 0 & L_3 \sin(\theta_3) & 0 \\ 0 & L_4 + L_3 \cos(\theta_3) & L_4 \\ -L_4 \cos(\theta_2+\theta_3) - L_3 \cos(\theta_2) & 0 & 0 \\ \sin(\theta_2+\theta_3) & 0 & 0 \\ \cos(\theta_2+\theta_3) & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

```
Rot=sym(zeros(6,6));
Rot(1:3,1:3)=R10*R21*R32*R43;
Rot(4:6,4:6)=R10*R21*R32*R43;
Rot % matrix that converts J4 to J0
```

Rot =

$$\begin{pmatrix} \sigma_4 & \sigma_3 & \sin(\theta_1) & 0 & 0 & 0 \\ \sigma_2 & \sigma_1 & -\cos(\theta_1) & 0 & 0 & 0 \\ \sigma_6 & \sigma_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4 & \sigma_3 & \sin(\theta_1) \\ 0 & 0 & 0 & \sigma_2 & \sigma_1 & -\cos(\theta_1) \\ 0 & 0 & 0 & \sigma_6 & \sigma_5 & 0 \end{pmatrix}$$

where

$$\sigma_1 = -\cos(\theta_2) \sin(\theta_1) \sin(\theta_3) - \cos(\theta_3) \sin(\theta_1) \sin(\theta_2)$$

$$\sigma_2 = \cos(\theta_2) \cos(\theta_3) \sin(\theta_1) - \sin(\theta_1) \sin(\theta_2) \sin(\theta_3)$$

$$\sigma_3 = -\cos(\theta_1) \cos(\theta_2) \sin(\theta_3) - \cos(\theta_1) \cos(\theta_3) \sin(\theta_2)$$

$$\sigma_4 = \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) - \cos(\theta_1) \sin(\theta_2) \sin(\theta_3)$$

$$\sigma_5 = \cos(\theta_2) \cos(\theta_3) - \sin(\theta_2) \sin(\theta_3)$$

$$\sigma_6 = \cos(\theta_2) \sin(\theta_3) + \cos(\theta_3) \sin(\theta_2)$$

```
simplify(J0-(Rot*J4)) % verifies that J0 == [[R 0];[0 R]] * J4
```

ans =

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
max_rankk = min(size(J0)) % maximum possible rank or highest degree of maneuverability
```

max_rankk = 3

```
q40=simplify(T10*T21*T32*T43*eye(4));
q=q40(1:3,4); %position of frame 4 wrt frame 0
```

```
%verify v44
q_differentiate40 = diff(q,theta1)*theta_dot_1+diff(q,theta2)*theta_dot_2+diff(q,theta3)*theta_dot_3;
q_differentiate44 = simplify((R10*R21*R32*R43)'*(q_differentiate40));
simplify(q_differentiate44-v44) % verifies our v44 is correct and thus v40 is also correct
```

ans =

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

```
%verify w44
r=q40(1:3,1:3); % rotational matrix to convert from frame 4 to frame 0
rdot=simplify(diff(r,theta1)*theta_dot_1+diff(r,theta2)*theta_dot_2+diff(r,theta3)*theta_dot_3);
omega40=simplify(rdot*r');
omega40=simplify(invert_skew(omega40)); %maps the skew omega to omega vector
omega44=simplify((R10*R21*R32*R43)'*omega40);
simplify(omega44-w44) % verifies our w44 is correct and thus w40 is also correct
```

ans =

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

```
eqn = det(Jv0)==0;
assume(L1~= 0 & L2~= 0 & L3~= 0 & L4~= 0 & L4<L3) % L1=L2=L4!=0 and L4<L3
S = solve(eqn,[theta1,theta2,theta3],'ReturnConditions',true,'Real',true);
subs(S.conditions,S.parameters,[theta1,theta2,theta3]) % workspace boundary characterizing condition
```

ans = $\theta_1 \in \mathbb{R} \wedge \theta_2 \in \mathbb{R} \wedge \theta_3 \in \mathbb{R} \wedge \cos(\theta_2+\theta_3) \sin(\theta_2) = \sin(\theta_2+\theta_3) \cos(\theta_2)$

```
J0_eval_1=subs(J0,[L1,L2,L3,L4,theta1,theta2,theta3],[1,1,1.5,1,0,0,0]) %evaluating J0 at workspace boundary
```

J0_eval_1 =

$$\begin{pmatrix} 0 & 0 & 0 \\ \frac{5}{2} & 0 & 0 \\ 0 & \frac{5}{2} & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

```
rank(J0_eval_1) % rank of J0 at worksapce boundary
```

ans = 3

```
J0_eval_2=subs(J0,[L1,L2,L3,L4,theta1,theta2,theta3],[1,1,1.5,1,0,0,pi/4]) %evaluating J0 inside workspace boundary
```

J0_eval_2 =

$$\begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}+\frac{3}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2}+\frac{3}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

```
rank(J0_eval_2) % rank of J0 inside worksapce boundary
```

ans = 3

```
max_rank=min(size(J0)) % maximum possible rank
```

max_rank = 3

```
J4_eval_1=subs(J4,[L1,L2,L3,L4,theta1,theta2,theta3],[1,1,1.5,1,0,0,0]) %evaluating J4 at workspace boundary
```

J4_eval_1 =

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{5}{2} & 1 \\ -\frac{5}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

```
rank(J4_eval_1) % rank of J4 at worksapce boundary
```

ans = 3

```
J4_eval_2=subs(J4,[L1,L2,L3,L4,theta1,theta2,theta3],[1,1,1.5,1,0,0,pi/4]) %evaluating J4 inside workspace boundary
```

J4_eval_2 =

$$\begin{pmatrix} 0 & \frac{3\sqrt{2}}{4} & 0 \\ 0 & \frac{3\sqrt{2}}{4}+1 & 1 \\ -\frac{\sqrt{2}}{2}-\frac{3}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

```
rank(J4_eval_2) % rank of J4 inside worksapce boundary
```

ans = 3

```
max_rank=min(size(J4)) % maximum possible rank
```

max_rank = 3

```
Jv0_eval_1=subs(Jv0,[L1,L2,L3,L4,theta1,theta2,theta3],[1,1,1.5,1,0,0,0]) %evaluating Jv0 at workspace boundary
```

Jv0_eval_1 =

$$\begin{pmatrix} 0 & 0 & 0 \\ \frac{5}{2} & 0 & 0 \\ 0 & \frac{5}{2} & 1 \end{pmatrix}$$

```
rank(Jv0_eval_1) % rank of Jv0 at worksapce boundary
```

ans = 2

```
Jv0_eval_2=subs(Jv0,[L1,L2,L3,L4,theta1,theta2,theta3],[1,1,1.5,1,0,0,pi/4]) %evaluating Jv0 inside workspace boundary
```

Jv0_eval_2 =

$$\begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}+\frac{3}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2}+\frac{3}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

```
rank(Jv0_eval_2) % rank of Jv0 inside worksapce boundary
```

ans = 3

```
max_rank=min(size(Jv0)) % maximum possible rank
```

max_rank = 3

```
Jv4_eval_1=subs(Jv4,[L1,L2,L3,L4,theta1,theta2,theta3],[1,1,1.5,1,0,0,0]) %evaluating Jv4 at workspace boundary
```

Jv4_eval_1 =

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{5}{2} & 1 \\ -\frac{5}{2} & 0 & 0 \end{pmatrix}$$

```
rank(Jv4_eval_1) % rank of Jv4 at worksapce boundary
```

ans = 2

```
Jv4_eval_2=subs(Jv4,[L1,L2,L3,L4,theta1,theta2,theta3],[1,1,1.5,1,0,0,pi/4]) %evaluating Jv4 inside workspace boundary
```

Jv4_eval_2 =

$$\begin{pmatrix} 0 & \frac{3\sqrt{2}}{4} & 0 \\ 0 & \frac{3\sqrt{2}}{4}+1 & 1 \\ -\frac{\sqrt{2}}{2}-\frac{3}{2} & 0 & 0 \end{pmatrix}$$

```
rank(Jv4_eval_2) % rank of Jv4 inside worksapce boundary
```

ans = 3

```
max_rank=min(size(Jv4)) % maximum possible rank
```

max_rank = 3