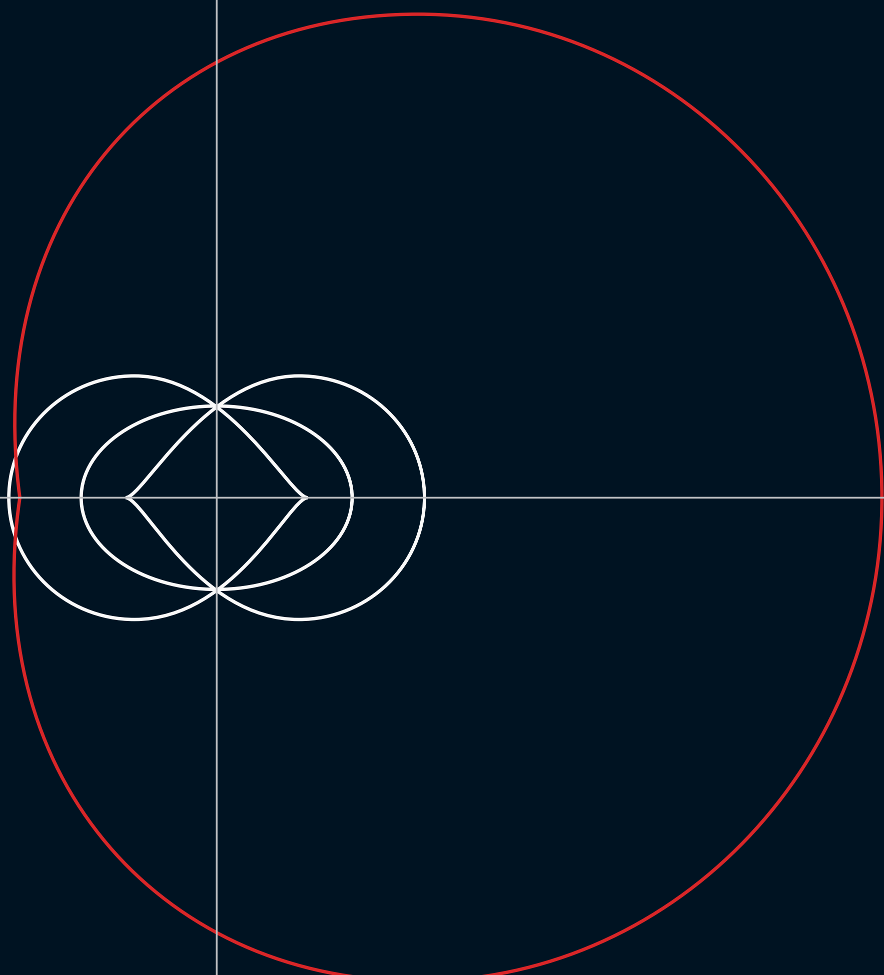
Solution of Mathisson-Papapetrou-Dixon Equations

for Spinning Test Particles in a Kerr Metric

Nelson Velandia-Heredia, S. J.



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Introduction

Historically one of the problems that the General Theory of Relativity has studied is the description of the movement of mass distribution in a gravitational field. The importance of this topic is heightened when dealing with astrophysical phenomena such as accretion discs in rotating black holes [1], gravitomagnetic effects [2], or gravitational waves induced by spinning particles orbiting a rotating black hole [3]. Our approach for this mass distribution, which has small dimensions compared with the central massive body [4], is to study the set of equations of motion for a spinning particle in a rotating gravitational field. Thereafter, we calculate the trajectory numerically for a spinning test particle when it is orbiting around a rotating massive body. This thesis aims to give a numerical solution to the full set of equations for a spinning test particle orbiting around a rotating field.

To study the problem of the motion of spinning test particles in an axially symmetric metric, we will use two different formulations that had been worked: The formulation of the Mathisson-Papapetrou-Dixon equations (MPD equations) [5] and Carter's equations [6]. The MPD equations describe the motion of a spinning body which is moving in a rotating gravitational field. This set of equations is deduced by the multipole expansion of the energy-momentum tensor of a distribution of mass in the middle of a gravitational field with a density of angular momentum (a). For the momenta of lower order, the equations relate the linear and angular momentum of an extended body around a central mass with rotation. For Carter's equations, the equations are deduced by the first integrals of motion which relate the constants of motion such as the energy (E), the angular momentum (J), and the mass (M). Carter uses another constant of motion (Q), which relates the angular momentum of the test body with the latitudinal motion of a body around a rotating mass.

With regard to the MPD equations, these equations study the problem of motion in a distribution of mass in a rotating gravitational field and were researched initially by Papapetrou [7] and Mathisson [8]. They, to study this motion, calculated a multipole expansion of symmetric momentum-energy tensor ($T^{\alpha\beta}$), which represents the distribution of mass in a rotating gravitational field [9], and from this expansion, they yielded a set of equations that includes both the monopole and dipole momentum for studying the motion of a spinning test body in a gravitational field [8]. Then, Dixon took up this system of equations [10] and defined the total momentum vector and the spin tensor for an extended body in an arbitrary gravitational field. In a paper, Dixon studied the particular case of the extended body in a Sitter universe and used the spin supplementary condition $u_\beta S^{\alpha\beta} = 0$ to define the center of mass [10].

The majority of works that one finds are centered on the description of orbits around a rotating massive body in the equatorial plane. Tanaka *et al.* [1] using the Teukolsky formalism for the perturbations around a Kerr black hole calculated the energy flux of gravitational waves induced by a spinning test particle moving in orbits near the equatorial plane of the rotating central mass [3]. They used the equations of motion for a spinning particle derived by Papapetrou, introduce the tetrad frame, and evaluate the equations of motion in the linear order of the spin to calculate the waveform and the energy flux of gravitational waves by a spinning particle orbiting a rotating black hole. In the analytical solution given by Tanaka *et al.,* the spin value is fixed and orthogonal to the equatorial plane ($S_\perp$). Another work that uses an approximation method for describing the influence of spin on the motion of extended spinning test particles in a rotating gravitational field was done by Mashhoon and Singh [11]. In this paper, they study the case for circular equatorial motion in the exterior Kerr spacetime and compare numerically their calculations with the numerical solution of the extended pole-dipole system in Kerr spacetime given by the MPD equations.

In the MPD equations, the solution of the motion of spinning test particles, it is necessary to consider a spin supplementary condition (SSC) which determines the center of mass of the particle for obtaining the evolution equation. Kyrian and Semerek, in their paper [12], consider different spin conditions and compare the different trajectories obtained for various spin magnitudes and conclude that the behavior of a spinning test particle with different supplementary conditions fixing different representative worldlines. For the numerical integration, they take the case when the spin is orthogonal to the equatorial plane. In a previous paper, they integrated the MPD equations with the Pirani condition ($P_\sigma S^{\mu\sigma} = 0$) [13], and studied the effect of the spin-curvature interaction in the deviations from geodesic motion when the spinning test particles are ejected from the horizon of events of central mass in a meridional plane.

To study the influence of the spin in the gravitomagnetic clock effect, Faruque [14] calculates the first-order correction of the angular velocity analytically intending to find the orbital period both for the prograde period and for the retrograde period of the two spinning test particles. He found that the spin value of the particle reduces the magnitude value of the clock effect. The spin value (S) is fixed and does not change in time.

In the last decades, authors such as Kyrian [12], Semerák [13], Plyatsko [15] and Mashhoon [11] worked in the numerical solution of the equations of motion for spinning test particles orbiting around a rotating gravitational field given by the MPD equations. In each case, they performed numerical calculations for a particular case, such as the particle in an equatorial plane or the spin value constant in time. For the scope of this work, the most important contribution will be the numerical solution of the full set of MPD equations without any restrictions on spin orientation. These results will be material for studying the gravitomagnetic effects and for the influence of the spin in Michelson-Morley type experiments.

The formulation of Carter's equations in order to calculate the system of equations of motion for a test particle in a rotating field derived directly from the Kerr metric using the symmetries of the geometry of a rotating massive body and from the conserved quantities of energy (E), angular momentum (J) and a constant fourth given by Carter (Q).

In the literature, some papers use Carter's equations to study the motion of spinless test particles in the equatorial plane in a Kerr metric [16],[17]. In particular, for Carter's equations, when the spinless test particle is orbiting in non-equatorial planes, there are two particular situations. First, authors as Kheng [18], Teo [19] and Wilkins [20] study the case where the particle is out of the equatorial plane and does not have spin, while Tsoubelis [21], [22] and Stog [23] work on the case where the spinless test particles start from one of the poles of the rotating central mass. We make a numerical comparison from the results obtained by Carter's equations with our results given by the numerical solution to the full MPD equations.

If one knows the conserved constants, the Killing vector and the covariant derivative of this Killing vector in a point, one can establish a constant relation between the associated momentum to the conserved quantity and the spin of the test particle in the case when the spinning test particles are in the equatorial plane. With this relation and Carter's equations, we will study the particular case when the spin of the particle is parallel to the symmetric axis of the massive rotating body and is orbiting in an equatorial plane. The majority of papers consider the particular case when the value spin is fixed and orthogonal to the equatorial plane [24], [25].

One of the purposes of this thesis is to study the gravitomagnetic effects. These effects are derived by the analogy between Coulomb's law and Newton's gravitation law. There is a relationship between Maxwell's equations and the linearized Einstein equations. Therefore, our first step will be to linearize the Einstein field equations and compare them with some electromagnetic phenomena. Then, we will take the MPD equations given by Plyastsko *et al.* for a spinning test particle orbiting around a rotating massive body [15]. Since it is not possible to find an analytical solution for the set of eleven coupled differential equations, we will give a numerical solution for the case when the spinning test particle orbits in a Kerr metric. The main contribution of this work is to yield the numerical solution for the case of spinning particles around a rotating gravitational field. On the other hand, one finds that the majority of works give the analytical solution for particular cases such as spinless test particles in the Schwarzschild metric and in the equatorial planes or the spin values constricted in the time. We calculate the trajectories of spinning test particles in rotating gravitational fields without restrictions on its velocity and spin orientation. From this work, we will study the gravitomagnetism effects and give an exact numerical solution for the clock effect [26].

After that, we study the effects of spin when a test particle travels in the field of a rotating massive body to describe the trajectories of test particles in Michelson and Morley type experiments.

The present thesis is structured as follows. It begins with a theoretical chapter that synthesizes the basic elements that we will work for studying the motion of spinning test particles around a rotating massive body. In this same chapter, we give an overview of the MPD equations and Carter's equations that we will work in this thesis for our numerical calculations.

In the third chapter, we will give the basic structure for describing the trajectories of test particles in a Kerr metric both in the Mathisson-Papapetrou-Dixon equations and in Carter's equations. We yield the set of equations of motion for the spinning test particles when they are orbiting around a rotating massive body.

The fourth chapter is centered on the study of the gravitomagnetism regarding the study of trajectories of spinning test particles. First of all, we give an overview of the *Gravity Probe B* experiment whose objective was to detect both the Lense-Thirring effect and the precession of a gyroscope when it is orbiting the meridional plane of Earth [27]. NASA launched a satellite which transported four gyroscopes with the aim of measuring the drag of inertial systems and the geodetic effect produced by the gravitational field [28]. Then we study the gravitomagnetic effects given by the rotation of massive bodies and the relationship with spinning test particles. Also, we study the effects by spin in the description of the trajectories of test particles around rotating fields. Finally, we study the Michelson-Morley type experiments to introduce the influence of stablishing the spin. For this famous experiment, we study the consequences of introducing the spin for the test particle and considering its behavior.

The last chapter is dedicated to the conclusions and outlook of work. We will give a numerical solution both for the MPD equations and for Carter's equation to compare these methods when the two test particles are orbiting around a rotating body in opposite directions. We will compare our results with the literature and will give our conclusions [14]. Also, we will present a future work about the description of trajectories of spinning test particles when traveling in non-equatorial planes.

1 Formulation for the Equations of Motion

1.1 Introduction

In this chapter, we will consider the effects of the spin for the test particles. In particular, we will study the motion of spinning test particles in symmetric axial gravitational fields in the weak field approach. We will use the two different standard formulations by the Mathisson-Papapetrou-Dixon equations (MPD) [10] and Carter's equations [6] as a starting point of our specific problem. We will extend the MPD formulation by including the spin, obtaining the equations of motion with an explicit spin dependency. On the other hand, we will use the Carter formulation to obtain the specific values for some constants and also compare the contributions of the final results with respect to the already accepted calculations.

In the gravitational field, the free particles follow a geodesic. The geodesic is defined as the curve which its tangent vector ($X^\alpha = dx^\alpha/d\lambda$) is parallel transported along the curve, which in terms of its covariant derivative can be written [30] as:

$$\frac{D}{\partial\lambda}\left(\frac{\partial}{\partial\lambda}\right) = \nabla_x X = 0.$$

Using the coordinate basis, $\{\partial/\partial x^\alpha\}$ and $\{dx^\alpha\}$ one obtains the equation for the geodesics:

$$\frac{d^2x^\alpha}{d\lambda^2} = \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}, \tag{1.1}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the coefficients of connection and λ is an affine parameter. These coefficients are defined in terms of the metric $g_{\mu\nu}$ as

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\rho}\left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right). \tag{1.2}$$

The test particle follows a geodesic which represents the trajectory in a gravitation field without taking into account any kind of force.

There are two cases in consideration: first, the test particle has mass; that is, the affine parameter λ is the proper time (τ), and the four velocity $dx^\mu/d\tau$ is normalized as $g_{\mu\nu}\,(dx^\mu/d\tau)\,(dx^\nu/d\tau) = c^2$. For the second case, the particle does not have mass so that the tangent vector k^μ is null, therefore $g_{\mu\nu}k^\mu k^\nu = 0$.

We will solve the equations of motion for a spinning test particle in a Kerr metric. For this chapter, we define the signature of the metric as (-, -, -,+). This metric in Boyer Lindquist (r, θ, ϕ, t) coordinates is given by [31]

$$\begin{aligned} ds^2 &= -\frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \left[r^2 + a^2 + \frac{2GMr}{c^2\rho^2}a^2\sin^2\theta\right]\sin^2\theta d\phi^2 \\ &\quad + \frac{4GMr}{c^2\rho^2}ac\sin^2\theta dt d\phi + \left(1 - \frac{2GMr}{c^2\rho^2}\right)c^2dt^2, \end{aligned} \tag{1.3}$$

where:

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{1.4}$$

$$\Delta = r^2 - \frac{2GMr}{c^2} + a^2, \tag{1.5}$$

a is the angular momentum of the distribution for a unit of mass:

$$a = \frac{J}{Mc}. \tag{1.6}$$

1.2 *Equation of Linear Field*

For a given mass distribution and test particles far away from this distribution, the gravitational field is asymptotically Minkowskian; in other words, when one is far away from this distribution, the gravitational force is weak and the gravitational space is a Minkowskian space. Therefore, one can consider under certain conditions that the metric for this mass distribution deviates a little from the Minkowskian metric and one would speak of a relativistic weak gravitational field.

In the approximation of weak gravitational field, we find the approximated solution for the Einstein field equations of the General Relativity Theory; that means, we linearize the gravitational field equations.

NEWTONIAN MECHANICS

The Newtonian gravitational theory is included in the General Relativity Theory for the conditions of low velocities and weak field:

The velocities are small relative to the speed of light[1]

$$\frac{|\mathbf{u}|}{c} \ll 1 \tag{1.7}$$

1. Bold characters correspond to vectors in R3

and under this condition (1.7) when a particle travels from a point where the gravitational potential is equal zero to distribution of mass, we have that the gravitational potential Φ is given by

$$\Phi = -\frac{GM}{r}, \tag{1.8}$$

and is called the condition of weak field. This potential must satisfice

$$\frac{2\,|\Phi|}{c^2} \ll 1. \tag{1.9}$$

In this limit, we are in the Newtonian gravity [32].

The second condition is the weak field and independent of time; that is, the temporal variations of the field are negligible. In other words, the mass that produces the field is moving slowly.

GENERAL RELATIVITY FOR A WEAK FIELD

Now, under the approximation of weak field (1.9) the metric can be written as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \left|h_{\mu\nu}\right| \ll 1. \tag{1.10}$$

In General Relativity and the approximation of weak field, one can approach a finite distribution of matter with a small deviation of plane space.

From (1.10) the contravariant components of the metric tensor are given by

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}. \tag{1.11}$$

Thus replacing (1.11) in the symbol of Christoffel (1.2) and retaining terms of first order in $h^{\alpha\beta}$ we obtain

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\sigma}\left(h_{\alpha\mu,\nu} + h_{\alpha\nu,\mu} - h_{\mu\nu,\alpha}\right). \tag{1.12}$$

We consider a test particle in the gravitational field and free from external forces, so the geodesic equation for the spatial components ($i = 1, 2, 3$) is given by

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{\alpha\beta}\frac{dx^a}{d\tau}\frac{dx^\beta}{d\tau} = 0. \tag{1.13}$$

In the approach of low velocities and weak field independent of time, the second term of the left of (1.13) is reduced to

$$\Gamma^i_{\alpha\beta}\frac{dx^a}{dt}\frac{dx^\beta}{dt} \simeq c^2\Gamma^i_{44}. \tag{1.14}$$

From (1.13) we have

$$\frac{d^2x^i}{dt^2} \simeq -c^2\Gamma^i_{44}. \tag{1.15}$$

The connection is showed as

$$\Gamma^i_{44} = -\frac{1}{2}h_{44,i}, \tag{1.16}$$

then the geodesic equation is

$$\frac{d^2x^i}{d^2t} = \frac{c^2}{2}\frac{\partial h_{44}}{\partial x^i}. \tag{1.17}$$

Comparing (1.17) with the equation of motion (1.1) we have

$$h_{44} = -\frac{2}{c^2}\Phi + \text{cte}. \tag{1.18}$$

In a finite distribution of mass $\Phi \longrightarrow 0$, when $r \longrightarrow \infty$, the metric is asymptotically flat, i.e., $h \longrightarrow 0$ for $r \longrightarrow \infty$. So in the equation (1.18), the constant $cte = 0$.

The g_{44} component of the metric is

$$g_{44} \simeq -1 - 2\frac{\Phi}{c^2}. \tag{1.19}$$

The next step is to write the Einstein field equations in the approximation of weak field and low velocities. For that case, the equations are written in terms of h and given by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.20}$$

and the Ricci tensor is given by

$$R_{\lambda\mu} = \Gamma^\sigma_{\sigma\lambda,\mu} - \Gamma^\sigma_{\lambda\mu,\sigma} + \Gamma^\tau_{\lambda\sigma}\Gamma^\sigma_{\mu\tau} - \Gamma^\tau_{\lambda\mu}\Gamma^\sigma_{\tau\sigma}, \tag{1.21}$$

where the comma represents the partial derivative.

We write the Christoffel symbols (1.2) in terms of $h_{\mu\nu}$ as:

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2}\left[h_\nu{}^\gamma{}_{,\mu} + h^\gamma{}_{\mu,\nu} - h_{\mu\nu}{}^{,\gamma}\right]. \tag{1.22}$$

So the Ricci tensor (1.21), in the approximation of weak field, using the expression (1.22), is given by:

$$R_{\mu\nu} = \frac{1}{2}\left[h_{,\mu\nu} - h_\nu{}^\alpha{}_{,\mu\alpha} - h_\mu{}^\alpha{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^\alpha\right] \tag{1.23}$$

and the curvature scalar becomes:

$$R = \left[h_{,\alpha}{}^\alpha - h^{\alpha\beta}{}_{,\alpha\beta}\right]. \tag{1.24}$$

So we replace (1.10), (1.23), and (1.24) in the Einstein tensor (1.20) and we obtain

$$\frac{1}{2}\left[h_{,\mu\nu} - h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^{\alpha}\right] - \frac{1}{2}\left(\eta_{\mu\nu} + h_{\mu\nu}\right)\left[h_{,\alpha}{}^{a} - h^{\alpha\beta}{}_{,\alpha\beta}\right]$$
$$= \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1.25}$$

We define

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \tag{1.26}$$

where h is defined as: $h = h_{\alpha}{}^{\alpha}$. The equation (1.25) can be expressed as

$$-\overline{h}_{\mu\nu,\alpha}{}^{\alpha} + \eta_{\mu\nu}\overline{h}^{\alpha\beta}{}_{,\alpha\beta} - \overline{h}_{\nu}{}^{\alpha}{}_{,\mu\alpha} - \overline{h}_{\mu}{}^{\alpha}{}_{,\nu\alpha} = \frac{16\pi G}{c^4} T_{\mu\nu}, \tag{1.27}$$

in this equation, the first term corresponds to D'lambertian operator which is defined as

$$\overline{h}_{\mu\nu,\alpha}{}^{\alpha} = \Box \overline{h}_{\mu\nu} = \left(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2\right)\overline{h}_{\mu\nu}. \tag{1.28}$$

If one uses the Hilbert gauge or Lorentz gauge, the equation (1.27) is reduced to

$$\overline{h}^{\mu\alpha}{}_{,\alpha} = 0. \tag{1.29}$$

From this gauge the linearized field equations (1.27) take the form of a non-homogeneous wave equation:

$$\Box \overline{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}. \tag{1.30}$$

After finding the relationship between the metric tensor and the potential in the newtonian limit, we obtain a particular solution of (1.30) for a spherical uniform mass distribution with a constant angular velocity. With the help of this wave equation, in the approximation of low velocities and weak field, the components of tensor $T_{\mu\nu}$ give the expressions analogous to Maxwell's equations.

The delayed solution of the non-homogenous wave, the equation (1.30) can be written as

$$\overline{h}_{\mu\nu}\left(ct, \mathbf{x}\right) = -\frac{4G}{c^4}\int \frac{T_{\mu\nu}\left(ct - \left|\mathbf{x} - \mathbf{x}\prime\right|, \mathbf{x}\prime\right)}{\left|\mathbf{x} - \mathbf{x}\prime\right|} d^3\mathbf{x}\prime, \tag{1.31}$$

where $x^{\mu} = (ct, \mathbf{x})$. Now for the case stationary, that is, $T^{\mu\nu}{}_{,0} = 0$. In other words, the momentum-energy tensor is constant in time; the solution (1.31) is reduced to:

$$\overline{h}^{\mu\nu}\left(\mathbf{x}\right) = -\frac{4G}{c^4}\int \frac{T^{\mu\nu}\left(\mathbf{x}\prime\right)}{\left|\mathbf{x} - \mathbf{x}\prime\right|} d^3\mathbf{x}\prime. \tag{1.32}$$

A particular case is the non-relativistic stationary source, where the velocity u of any particle is small compared with the density of energy [35].

In any coordinated system x^{μ}, where the four velocity is u^{μ}, the contravariant components of the tensor are given by

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu}, \tag{1.33}$$

where ρ is the proper density of fluid and u^{μ} is the four velocity of fluid which is defined as: $u^{\mu} = \gamma_u (c, \mathbf{u})$. In the limit of low velocities, the Lorentz factor $\gamma_u = \left(1 - u^2/c^2\right)^{-1/2} \approx 1$. Therefore, the components of the metric tensor are

$$T^{ij} = \rho u^i u^j, \quad T^{4i} = cj^i, \quad T^{44} = \rho c^2. \tag{1.34}$$

From the relation $\left|T^{ij}\right| / \left|T^{44}\right| \sim u^2/c^2$, thus $T^{ij} \approx 0$ up to the order of approximation written above. Let be defined the scalar gravitational potential Φ and the potential gravitational vector A^i, independent of time as

$$\Phi(\mathbf{x}) \equiv -G \int \frac{\rho(\acute{\mathbf{x}})}{|\mathbf{x} - \mathbf{x}\prime|} d^3\acute{\mathbf{x}}, \tag{1.35}$$

$$A^i(\mathbf{x}) \equiv -\frac{4G}{c^2} \int \frac{\rho(\acute{\mathbf{x}})\, u^i(\acute{\mathbf{x}})}{|\mathbf{x} - \mathbf{x}\prime|} d^3\acute{\mathbf{x}}. \tag{1.36}$$

Thus the solution for (1.32) in the linearized equations, can be written as

$$\overline{h}^{ij} = 0, \quad \overline{h}^{4i} = \frac{A^i}{c}, \quad \overline{h}^{44} = \frac{4\Phi}{c^2}. \tag{1.37}$$

The corresponding components of $h^{\mu\nu}$ are given by $h^{\mu\nu} = \overline{h}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\overline{h}$. The result (1.37) implies that $\overline{h} = \overline{h}^{44}$ and the components will be

$$h_{11} = h_{22} = h_{33} = h_{44} = \frac{2\Phi}{c^2}, \quad h_{4i} = \frac{A_i}{c}. \tag{1.38}$$

From the line element

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

and in the approximation of weak field, where the metric is given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ we have

$$g_{ij} = -1 + \frac{2\Phi}{c^2} g_{4i} = \frac{A_i}{c}, \quad g_{44} = 1 + \frac{2\Phi}{c^2},$$

thus the line element is

$$ds^2 = -\left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij} dx^i dx^j + \frac{4}{c} (\mathbf{A} \cdot d\mathbf{x})\, dt + c^2 \left(1 + \frac{2\Phi}{c^2}\right) dt^2. \tag{1.39}$$

Now the aim is to find the analogy between the linearized Einstein field equations and electromagnetism. We already found that the elements of tensor h are

$$h^{11} = h^{22} = h^{33} = h^{44} = \frac{2\Phi}{c^2}, \qquad h^{4i} = \frac{A^i}{c}, \qquad h^{ij} = 0 \text{ if } i \neq j, \tag{1.40}$$

where Φ and $\mathbf{A}$ are defined as the gravitational scalar and the potential vector, respectively. Now let us consider the time-independent Maxwell equations

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \qquad y \qquad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}$$

and take into account the following identifications [44]

$$\epsilon_0 \longleftrightarrow -\frac{1}{4\pi G} \qquad \text{y} \qquad \mu_0 \longleftrightarrow -\frac{16\pi G}{c^2}, \tag{1.41}$$

we can obtain the linearized field equations:

$$\nabla^2 \Phi = 4\pi G \rho \qquad \text{y} \qquad \nabla^2 \mathbf{A} = \frac{16\pi G}{c^2}\mathbf{j}, \tag{1.42}$$

where $\mathbf{j} \equiv \rho \mathbf{v}$ is the density (or density of mass current). These equations, time-independent, have the solutions (1.35 and 1.36).

So we have the gravitomagnetic and gravitoelectric fields [33] :

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{y} \qquad \mathbf{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{A}). \tag{1.43}$$

Using the equations (1.42), we verify $\mathbf{E}$ and $\mathbf{B}$ fields are related to Maxwell's equations as

$$\begin{aligned} \nabla \cdot \mathbf{E} &= -4\pi G \rho & \qquad \nabla \cdot \frac{1}{2}\mathbf{B} = 0 \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B}\right) & \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} - \frac{4\pi G}{c}\mathbf{j}. \end{aligned} \tag{1.44}$$

The same as in electromagnetism, we must postulate, in addition, the Lorentz force for describing this motion. In the last section, we wrote the equation of motion for the test particle in a gravitational field is the geodesic equation:

$$\ddot{x}^\sigma + \Gamma^\sigma_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0, \tag{1.45}$$

the points indicate the differentiation in regard to the proper time τ of the particle. It is taken a small velocity v and the relation $\gamma_v = \left(1 - v^2/c^2\right)^{-\frac{1}{2}} \approx 1$. Writing the position $x^\mu = (ct, \mathbf{x})$, the four velocity of the particle is given by [35]

$$\dot{x}^\mu = \gamma_v (c, \mathbf{v}) \approx (c, \mathbf{v}).$$

Now we replace the derivatives with respect to t. Therefore, the spatial components of (1.45) are written as

$$\begin{aligned} \frac{d^2x^i}{dt^2} &\approx -\left(c^2\Gamma^i_{00} + 2c\Gamma^i_{oj}v^j + \Gamma^i_{ij}v^iv^j\right) \\ &\approx -\left(c^2\Gamma^i_{00} + 2c\Gamma^i_{oj}v^j\right), \end{aligned} \tag{1.46}$$

where we had canceled spatial terms because their reason with respect to temporal term $c^2\Gamma^i_{44}$ is in order to v^2/c^2. To the first order of gravitational field $h_{\mu\nu}$, the connection coefficients are given by (1.22). So we take (1.46) and remembering that $h^{ij}=0$, we obtain

$$\begin{aligned} \frac{d^2x^i}{dt^2} &\approx c\left(h_{4j},^i - h^i_{4,j}\right)v^j - c^2h^i_{4},_4 + \frac{1}{2}c^2h_{44},^i \\ &= -c\delta^{ik}\left(h_{4j,k} - h_{4k,j}\right)v^j - c^2\delta^{i\sigma}h_{i4},_4 - \frac{1}{2}c^2\delta^{ij}h_{44,j}. \end{aligned} \tag{1.47}$$

Replacing the values of (1.40) in the previous equation, the expression for the motion is written as

$$\frac{d^2\mathbf{x}}{dt^2} \approx -\nabla\Phi - \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{A}) + \mathbf{v}\times(\nabla\times\mathbf{A}).$$

Then, we use (1.43) and obtain the Lorentz force for the case gravitational as:

$$\frac{d^2\mathbf{x}}{dt^2} \approx \mathbf{E} + \mathbf{v}\times\mathbf{B}, \tag{1.48}$$

for particles that are moving very slowly in the gravitational field of a stationary mass distribution.

Now then, we can conclude that a mass current generates the Gravitomagnetic Effects (GM), in analogy with the features of magnetism made for a mass current. In the literature [33], [34] can be found a parallel between the problems characterized by Maxwell's Equations and the linearized Einstein field Equations. We will discuss this issue in the next sections.

1.3 Linearized Kerr Metric

We take the Kerr metric in the Boyer Lindquist coordinates (r, θ, ϕ, t) as in eq. (1.3). This metric describes the spacetime geometry outside a rotating body. Also, we can approach the geometry far away from the source with the linearization of this metric. Let us define the lengths, a and MG/c^2, which are small compared to the distance (r) from the central body to the spinning test particle, that is, $a/r \ll 1$ and $MG/c^2r \ll 1$. Then Kerr Metric can be linearized in a/r and MG/c^2r [35] and be written in the following way:

$$g_{\mu\nu} \approx \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} +$$

$$\begin{pmatrix} -\frac{2MG}{c^2 r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2GMa\sin^2\theta}{cr} \\ 0 & 0 & \frac{2GMa\sin^2\theta}{cr} & -\frac{2MG}{c^2 r} \end{pmatrix}. \tag{1.49}$$

This linearization divides the metric two parts, in a flat part and an additive perturbation part, allowing the interpretation to distinguish for the flat part, an extra like effective potential

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} \tag{1.50}$$

where $\eta_{\mu\nu}$ are the components of flat spacetime, and $\left|h_{\mu\nu}\right| \ll 1$ are the perturbative elements. Therefore, the line element for the Kerr Metric in the limit of weak field [35] will be given by

$$\begin{aligned} ds^2 &= -\left(1+\frac{2GM}{c^2 r}\right)dr^2 - \left(r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \\ &+\frac{4GMa}{cr}\sin^2\theta d\phi dt + c^2\left(1-\frac{2GM}{c^2 r}\right)dt^2. \end{aligned} \tag{1.51}$$

where the $g_{t\phi}$ component is called the gravitomagnetic potential.

This metric is useful for calculating the General Relativity effects due to the rotation of the Earth, or in astrophysical situations, where the gravitational field is weak.

The gravitomagnetic term generally is referred to as the set of gravitational phenomena with relation to the orbiting test particles, the precession of gyroscopes, the motion of clocks and atoms, and the propagation of electromagnetic waves which in the system of General Relativity Einstein Theory comes from distributions of matter and energy no static [34]. In the approximation of weak field and low velocities, the Einstein field equations (1.20), are linearized and it is found the analogy with Maxwell's equations for electromagnetism. As a consequence, a gravitomagnetic field $\vec{B}_g$, induced for the components no diagonal g_{4i}, $i = 1, 2, 3$ of the metric of space-time related with mass-energy currents are present. A particular case is given when the particle is far away from a rotating body with angular momentum $\vec{J}$, in consequence, the gravitomagnetic field can be written as

$$\vec{B}_g\left(\vec{r}\right) = \frac{G}{cr^3}\left[\vec{J} - 3\left(\vec{J}\cdot\hat{r}\right)\hat{r}\right], \tag{1.52}$$

where G is the Newtonian gravitational constant. This concerns, for instance, to a test particle that is moving with a velocity $\vec{v}$. The acceleration is given by

$$\vec{A}_{GM} = \left(\frac{\vec{v}}{c}\right) \times \vec{B}_g, \tag{1.53}$$

which is the cause of two gravitomagnetic effects: Lense-Thirring Effect and the geodesic precession.

Now we will present the basics for the motion of spinning test particles around a rotating massive body, according to the formulations by MPD equations and Carter's equations.

1.4 *Mathisson-Papapetrou-Dixon Equations*

In order to obtain the MPD equations of motion, we take the momentum energy symmetric tensor for a mass distribution ($T^{\alpha\beta}$), which satisfies the equation of continuity

$$\nabla_\beta T^{\alpha\beta} = 0. \tag{1.54}$$

Using the geometrized units ($c = G = 1$), and the greek indices running from $1 - 4$.

This approximation works out for very small bodies, allowing for the neglect of the influence of other bodies on the body of interest. This body is usually called *a test particle*. As a consequence, the dimensions of the test particle are tiny compared to the characteristic length of the gravitational field. In this way, the particle describes a narrow world tube (M) in a four-dimensional space-time (Figure 1.1). Inside this tube, the line l represents the motion of the particle [7]. In this thesis, we take the particular spin condition, which fixes a center of mass and establishes an interaction between the intrinsic angular momentum and the gravitational field. A world tube is formed by all possible centroids [12].

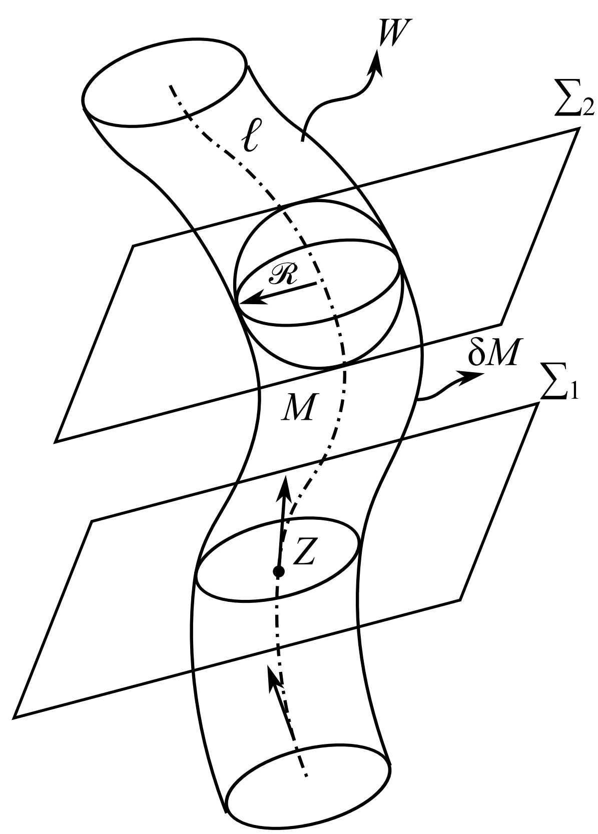

Figure 1.1. World tube line of the spinning test particle. Source: own elaboration

Now we define the linear and angular momentum for the test particle, which is described for a momentum-energy symmetric tensor. For a test particle, described by a tensor $T^{\alpha\beta}$, the radius of a world tube W is not zero (Figure 1.1). This tube is spread in time, both in the past and in the future, but it is bounded spatially. It is assumed that spacetime accepts isometries that are described by the Killing vector ξ_β such that

$$\nabla_{(\alpha}\xi_{\beta)} = 0, \tag{1.55}$$

and the equation (1.55) shows that

$$\nabla_\alpha\left(\xi_\beta T^{\alpha\beta}\right) = \left(\nabla_\alpha\xi_\beta\right)T^{\alpha\beta} + \xi_\beta\left(\nabla_\alpha T^{\alpha\beta}\right) = 0. \tag{1.56}$$

Integrating the expression (1.56) on a volume M which includes a part of world tube W, and considering an arbitrary like time surface, and two spacelike hypersurfaces Σ_1 and Σ_2, the integral takes the form [9]

$$\int_M \nabla_\alpha\left(\xi_\beta T^{\alpha\beta}\right)\sqrt{-g}d^4x = 0, \tag{1.57}$$

which can be written as

$$\int_M \partial_\alpha\left(\sqrt{-g}\xi_\beta T^{\alpha\beta}\right)d^4x = 0. \tag{1.58}$$

Now writing each part that defines the surface M

$$\int_{\Sigma_2\cap M}\xi_\beta T^{\alpha\beta}d\Sigma_\alpha + \int_{\Sigma_1\cap M}\xi_\beta T^{\alpha\beta}d\Sigma_\alpha + \int_{\partial M}\xi_\beta T^{\alpha\beta}d\Sigma_\alpha = 0. \tag{1.59}$$

The last term vanishes because $T^{\alpha\beta}$ is zero in ∂M and the others two terms can be restricted to $\Sigma_\alpha \cap W$ for the same reason. So,

$$\int_{\Sigma_2\cap M}\xi_\beta T^{\alpha\beta}d\Sigma_\alpha = -\int_{\Sigma_1\cap M}\xi_\beta T^{\alpha\beta}d\Sigma_\alpha. \tag{1.60}$$

Therefore,

$$\int_\Sigma \xi_\beta T^{\alpha\beta}d\Sigma_\alpha = C, \tag{1.61}$$

is a constant of motion, independent of the hypersurface.

Consider a general space-time $\widetilde{M}$ and let be $x(\lambda, \gamma)$ a family parametric of geodesics in this space-time, where γ classifies the geodesics and λ is the affine parameter along of each geodesic. One has

$$\dot{x}^\alpha := \frac{\partial x^\alpha}{\partial\lambda} \quad \text{and} \quad V^\alpha := \frac{\partial x^\alpha}{\partial\gamma}, \tag{1.62}$$

where $\dot{x}^{\alpha}$ is the tangent vector to the geodesic and V^{α} is the deviation vector. Then, $V^{\alpha}(\lambda,\gamma)$ satisfies the deviation equation of the geodesic for each value γ [10]

$$\frac{D^2\xi^{\alpha}}{d\lambda^2}+R^{\alpha}_{\beta\gamma\delta}\dot{x}^{\beta}\dot{x}^{\gamma}\xi^{\delta}=0. \tag{1.63}$$

A solution for this equation is determined by the value of ξ^{α} and $D\xi^{\alpha}/d\lambda$ in any fixed value of λ. Now one chooses any fixed point of z and supposes that the values of ξ_{α} and $\nabla_{[\alpha}\xi_{\beta]}$ are given in z. One obtains

$$\frac{D\xi_{\alpha}}{d\lambda}=\dot{x}^{\beta}\nabla_{[\beta}\xi_{\alpha]}. \tag{1.64}$$

Now, in the definition of world function, let us have $z\equiv x(\lambda_0=0)$ and $x\equiv x(\lambda_0=\lambda)$. But for the reduced expression given by Dixon [10], σ^{κ} is the derivative of the world function σ at the point $z(\gamma)$, as

$$\sigma^{\kappa}=-\lambda\dot{x}^{\kappa}\qquad \text{y}\qquad \sigma^{\alpha}=\lambda\dot{x}^{\alpha}, \tag{1.65}$$

one obtains

$$\sigma^{\kappa}\left(z\left(v\right),x\left(\lambda,\gamma\right)\right)=-\lambda\dot{x}^{\kappa}\left(0,\gamma\right), \tag{1.66}$$

where $z(v):=x(0,\gamma)$. We derive (1.66) respect to γ and as the differentiation works in each term separately, we have

$$\sigma^{\kappa}{}_{\varphi}V^{\varphi}+\sigma^{\kappa}{}_{\alpha}V^{\alpha}=-\lambda\frac{D\dot{x}^{\kappa}}{d\gamma} \tag{1.67}$$

and from (1.62) we obtain

$$\frac{D\dot{x}^{\kappa}}{d\gamma}=\frac{DV^{\kappa}}{d\lambda}. \tag{1.68}$$

It defined $\left(\sigma^{\alpha}_{.\kappa}\right)^{-1}$ as the inverse of the matrix $\sigma^{\kappa}_{.\alpha}$, therefore

$$\left(\sigma^{\alpha}{}_{\kappa}\right)^{-1}\sigma^{\kappa}{}_{\beta}=A^{\alpha}_{\beta}, \tag{1.69}$$

where A^{α}_{β} is the unit tensor. Then (1.67) with (1.68) one has

$$V^{\alpha}=\left(\sigma^{\alpha}{}_{\varphi}\right)^{-1}\sigma^{\varphi}{}_{\kappa}V^{\kappa}-\lambda\left(\sigma^{\alpha}{}_{\kappa}\right)^{-1}\frac{DV^{\kappa}}{d\lambda}. \tag{1.70}$$

The last equation is the formal solution of the deviation equation of the geodesic (1.63). Now we define the bitensors as

$$K^{\alpha}_{.\kappa}=\left(\sigma^{\alpha}{}_{\varphi}\right)^{-1}\sigma^{\varphi}{}_{\kappa}\qquad\text{and}\qquad H^{\alpha}{}_{\kappa}=-\left(\sigma^{\alpha}{}_{\kappa}\right)^{-1}; \tag{1.71}$$

therefore, the equation (1.70) can be expressed as

$$V^{\alpha} = K^{\alpha}{}_{\kappa} V^{\kappa} - \lambda H^{\alpha}{}_{\kappa} \frac{DV^{\kappa}}{d\lambda}. \tag{1.72}$$

If we apply the last equation to the case where ξ_{α} is a Killing field vector using (1.64) and (1.66) we obtain

$$\xi_{\alpha} = K_{\alpha}{}^{\kappa} \xi_{\kappa} + H_{\alpha}{}^{\kappa} \sigma^{\varphi} \nabla_{[\kappa} \xi_{\varphi]}. \tag{1.73}$$

This expression is acceptable for all x in the neighborhood of z, and is explicit the setting of ξ_{α} for the values of the Killing field vector ξ_{α} and the covariant derivative $\nabla_{[\kappa}\xi_{\gamma]}$ in a point. If one integrates the last Killing vector (1.73), with the expression (1.61) one obtains

$$\int_{\Sigma} \left(K_{\alpha}{}^{\kappa} \xi_{\kappa} + H_{\alpha}{}^{\kappa} \sigma^{\lambda} \nabla_{[\kappa} \xi_{\lambda]} \right) T^{\alpha\beta} d\Sigma_{\beta} = C. \tag{1.74}$$

Here $\xi_{\kappa} = \xi_{\kappa}(z)$ and $\nabla_{[\kappa}\xi_{\lambda]} = \nabla_{[\kappa}\xi_{\lambda]}(z)$, z is a fixed arbitrary point. We define the linear and angular momentum as

$$p^{\kappa}(z, \Sigma) \equiv \int_{\Sigma} K_{\alpha}{}^{\kappa} T^{\alpha\beta} d\Sigma_{\beta}, \tag{1.75}$$

$$S^{\kappa\lambda}(z, \Sigma) \equiv 2\int_{\Sigma} H_{\alpha}{}^{[\kappa} \sigma^{\lambda]} T^{\alpha\beta} d\Sigma_{\beta}, \tag{1.76}$$

where p^{κ} is the linear momentum and $S^{\kappa\lambda}$ is the spin tensor. There exists in each point z a only four vector u such that u and $p(x, u)$ are collinear [36]

$$u^{[\mu} p^{\nu]}(z, u) = 0, \tag{1.77}$$

where [ ] means antisymmetrization. On the other hand, there exists an only world line like time $z^{\mu}(\lambda)$ that satisfies [36]

$$p_{\mu}(z) S^{\mu\nu}(z) = 0,$$

this world line is called the center of mass of the body.

The constant (1.74) can be written as

$$C = p^{\kappa}(z, \Sigma) \xi_{\kappa} + \frac{1}{2} S^{\kappa\gamma}(z, \Sigma) \nabla_{[\kappa}\xi_{\gamma]}. \tag{1.78}$$

Since the definitions p^{κ} and $S^{\kappa\lambda}$ do not depend on the Killing vector fields, the definitions can be used for an arbitrary space-time without any symmetry. However, when there exist isometries in the space-time, C gives a linear combination of linear momentum and angular moment, which is constant. So we have [37]

$$\frac{D}{d\lambda}\left[p^{\kappa}(z,\Sigma)\,\xi_{\kappa}+\frac{1}{2}S^{\kappa\gamma}(z,\Sigma)\,\nabla_{[\kappa}\xi_{\gamma]}\right]=0, \tag{1.79}$$

which can be explicit as

$$\frac{Dp^{\kappa}}{d\lambda}\xi_{\kappa}+p^{\kappa}\frac{D}{d\lambda}\xi_{\kappa}+\frac{1}{2}\left(\frac{DS^{\kappa\gamma}}{d\lambda}\nabla_{[\kappa}\xi_{\gamma]}+S^{\kappa\lambda}\frac{D}{d\lambda}\nabla_{[\kappa}\xi_{\gamma]}\right)=0. \tag{1.80}$$

From (1.55) we have $\frac{D}{d\lambda}\xi_{\kappa}=v^{\mu}\nabla_{[\mu}\xi_{\kappa]}$ with $v^{\mu}\equiv\frac{dz^{\mu}}{d\lambda}$, which it defines the tangent vector to world line $z^{\mu}(\lambda)$; and given that the Killing field vectors satisfy

$$\nabla_{\alpha}\nabla_{\beta}\xi_{\gamma}=R_{\beta\gamma\alpha\delta}\xi^{\delta}, \tag{1.81}$$

so the expression (1.80) can be written as

$$\frac{D}{d\lambda}C=\xi_{\kappa}\left[\frac{Dp^{\kappa}}{d\lambda}+\frac{1}{2}S^{\delta\gamma}u^{\mu}R_{\delta\gamma\mu}{}^{\kappa}\right]+\frac{1}{2}\nabla_{[\kappa}\xi_{\gamma]}\left[\frac{DS^{\kappa\gamma}}{d\lambda}-2p^{[\kappa}u^{\gamma]}\right]=0. \tag{1.82}$$

This equation has a solution for all Killing field vector if each term in the brackets vanishes separately. These terms are defined both the total force and total torque acting on a body [5],

$$F^{\kappa}\equiv\frac{Dp^{\kappa}}{d\lambda}+\frac{1}{2}S^{\delta\gamma}u^{\mu}R^{\kappa}_{\delta\gamma\mu} \tag{1.83}$$

$$L^{\kappa\gamma}\equiv\frac{DS^{\kappa\gamma}}{d\lambda}-2p^{[\kappa}u^{\gamma]}. \tag{1.84}$$

With these definitions, one can write (1.82) as

$$\xi_{\kappa}F^{\kappa}+\frac{1}{2}\nabla_{[\kappa}\xi_{\gamma]}L^{\kappa\gamma}=0. \tag{1.85}$$

These two definitions (1.83) and (1.84) can be generalized to arbitrary space-time since they do not depend on Killing vectors. However, in general space-time, the higher multipolar momenta contribute to the force and the torque. The expression (1.85) expresses the connection between the integrals of motion and the isometries of space-time.

We take the particular case for a test particle, in this case, the force and the torque are zero in the equations (1.83) and (1.84); therefore these equations are reduced to

$$\frac{Dp^{\kappa}}{d\lambda}=-\frac{1}{2}S^{\delta\gamma}u^{\mu}R^{\kappa}_{\delta\gamma\mu} \tag{1.86}$$

$$\frac{DS^{\kappa\gamma}}{d\lambda}=2p^{[\kappa}u^{\gamma]}. \tag{1.87}$$

For our study, we take the pole-dipole approximation, which deals with the equations of motion of a spinning test particle, only including the mass monopole and spin dipole. Multipoles of higher orders and non-gravitational effects are ignored. First, when the analysis is restricted to particles whose dynamics are only affected by the monopole moments, the motion is simply a geodesic. Second, if it is the dipole moment, the motion corresponds to a test particle with spin and is no longer a geodesic. In this case, the monopole and dipole moments give the kinematic momentum p^μ and the spin tensor $S^{\mu\nu}$ of the body as measured by an observer moving along the reference worldline with velocity V^μ [38].

The set of equations (1.86) and (1.87) has more unknown variables than equations, so the system is undermined. Therefore, a spin supplementary condition (SSC) has to be imposed in order to solve the set of equations. This condition is related to the choice of a center of mass whose evolution is described by an observer and where the mass dipole vanishes [39]. When the spinning test particle moves with a constant velocity v, the part which moves faster appears to be heavier, and the one that moves more slowly appears to be lighter. Therefore, there is a shift of the center of mass Δx compared to an observer with zero-3-momentum. Inside of body size, it is possible to find an observer for whom the reference worldline coincides with the center of mass. All the possible centroids set up a worldtube whose size is the Möller radius.

In the description of the motion of a spinning test particle, the tangent vector to the worldline (u^μ) is no longer parallel to the linear momentum p^μ as we know it from geodesic motion. The choice of a supplementary condition is related to the ability to find an expression between u^μ and p^μ [40]. In this case, the rest mass m is not a constant, so the kinematical mass is redefined by

$$p_\mu u^\mu = -m \tag{1.88}$$

with respect to the kinematical four velocity u^μ. Then, the dynamical mass is denoted with regard to the four-momentum p_μ by M which satisfies

$$p_\mu p^\mu = -M^2. \tag{1.89}$$

In this context, a dynamical velocity is defined by

$$v^\mu = \frac{p^\mu}{M}. \tag{1.90}$$

In the case, $m = M$, because the tangent vector u^ν is parallel to dynamical four velocity v^ν when it is the motion of a geodesic.

In general, two conditions are usually imposed. The Mathisson-Pirani supplementary condition is [8], [41]

$$u_\sigma S^{\mu\sigma} = 0. \tag{1.91}$$

In this condition, the observer is comoving with the particle and is in the rest frame of the particle. There is not a unique representative worldline, and therefore it is dependent on the observer's velocity and the initial conditions. Further, this condition exhibits helical motion in contrast to a straight line in flat spacetime. In the works by Costa *et al.* show that these helical motions are physical motions and have a hidden momentum [42].

Another supplementary condition, it is the Tulczyjew-Dixon condition [43]

$$p_\sigma S^{\mu\sigma} = 0 \tag{1.92}$$

where

$$p^\sigma = m u^\sigma + u_\lambda \frac{DS^{\sigma\lambda}}{ds} \tag{1.93}$$

is the four momentum.

In addition to the MPD equations, we take the Tulczyjew's condition as a supplementary condition (1.92), which implies that $dm/d\tau = 0$. The motion effects induced by this condition must be confined to the worldtube of the centroid, that is, the worldtube formed by all the possible positions of the center of mass, as measured by every possible observer [45]. In the flat spacetime case, it is a tube of radius S/M centered around the center of mass measured in the zero 3-momentum frame.

We contract $S_{\mu\nu}$ in the equation (1.84) and use the condition (1.92), we obtain the magnitude S of spin which is defined as

$$S^2 = S_\mu S^\mu = \frac{1}{2m^2} S_{\mu\nu} S^{\mu\nu}, \tag{1.94}$$

this is a constant of motion. We obtain the constant too [43]

$$m^2 = p_\mu p^\mu, \tag{1.95}$$

where m is interpreted as the mass of the particle.

In general, the four momentum p^μ and the tangent vector u^μ are not colinear. In fact, from the set of equations (1.83), (1.84) and the supplementary condition (1.92), we deduce that [10]

$$p^{[\mu} u^{\nu]} = -\frac{1}{4m} \sqrt{-g} \epsilon^{\mu\nu\lambda\rho} R_{\lambda\alpha\beta\gamma} u^\alpha S^{\beta\gamma} S_\rho, \tag{1.96}$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the antisymmetric tensor and S_ρ is the spin vector which is defined by

$$S_\rho = \frac{1}{2m} \sqrt{-g} \epsilon_{\mu\nu\lambda\rho} p^\mu S^{\nu\lambda}. \tag{1.97}$$

The next step is to parametrize the four vector of velocity u^μ and v^μ, with the parameter of proper time τ, as

$$u^\mu(\tau) v_\mu(\tau) = 1 \tag{1.98}$$

where v^μ is the four velocity of the center of the mass, parallel to the line of world l and Dixon calls "dynamic velocity" [43] (1.1). u^μ is called "kinematical velocity" and is perpendicular to hypersurface (Σ).

Now we derive the equation of evolution of $v^{\mu}(\tau)$ in terms of $u^{\mu}(\tau)$. For this, we take the definition of total four momentum as

$$p^{\mu} = mu^{\mu} - u_{\sigma}\dot{S}^{\mu\sigma}. \tag{1.99}$$

We multiply each one of the sides of the equation for m

$$\begin{aligned} mp^{\mu} &= m^2u^{\mu} - p_{\sigma}\dot{S}^{\mu\sigma}, \\ m^2u^{\mu} - mp^{\mu} &= p_{\sigma}\dot{S}^{\mu\sigma}. \end{aligned} \tag{1.100}$$

We resolve u^{μ}

$$u^{\mu} = \frac{m}{m^2}p^{\mu} + \frac{p_{\sigma}}{m^2}\dot{S}^{\mu\sigma}.$$

It is imposed the restriction $u_{\sigma}S^{\mu\sigma} = 0$; therefore the right part of the equation can be written (1.100) as

$$p_{\sigma}\dot{S}^{\mu\sigma} = -\dot{p}S^{\mu\sigma}. \tag{1.101}$$

We replace this definition in the expression of u^{μ}

$$\begin{aligned} u^{\mu} &= \frac{m}{m^2}mv^{\mu} - \frac{\dot{p}}{m^2}S^{\mu\sigma}, \\ u^{\mu} &= v^{\mu} - \frac{\dot{p}}{m^2}S^{\mu\sigma}. \end{aligned}$$

With the help of the tensor $S^{\mu\nu}$, we find that

$$\left(4m^2 + R_{\rho\alpha\beta\gamma}S^{\alpha\rho}S^{\beta\gamma}\right)\dot{p}_{\sigma}S^{\mu\sigma} = -2mS^{\mu\sigma}R_{\sigma\alpha\beta\gamma}p^{\alpha}S^{\beta\gamma}. \tag{1.102}$$

From the equations (1.100), (1.101), and (1.102) we obtain the following result

$$\begin{aligned} v^{\mu} &= u^{\mu} - \frac{2mS^{\mu\sigma}R_{\sigma\alpha\beta\gamma}p^{\alpha}S^{\beta\gamma}}{m^2\left(4m^2 + R_{\rho\alpha\beta\gamma}S^{\alpha\rho}S^{\beta\gamma}\right)} \\ v^{\mu} &= u^{\mu} - \frac{2mS^{\mu\sigma}R_{\sigma\alpha\beta\gamma}mu^{\alpha}S^{\beta\gamma}}{m^2\left(4m^2 + R_{\rho\alpha\beta\gamma}S^{\alpha\rho}S^{\beta\gamma}\right)} \\ v^{\mu} &= u^{\mu} - \frac{2S^{\mu\sigma}R_{\sigma\alpha\beta\gamma}u^{\alpha}S^{\beta\gamma}}{4m^2 + R_{\rho\alpha\beta\gamma}S^{\alpha\rho}S^{\beta\gamma}} \end{aligned} \tag{1.103}$$

$$v^{\mu} - u^{\mu} = -\frac{1}{2}\left(\frac{S^{\mu\nu}R_{\nu\rho\sigma\kappa}u^{\rho}S^{\sigma\kappa}}{m^2 + \frac{1}{4}R_{\chi\xi\zeta\eta}S^{\chi\xi}S^{\zeta\eta}}\right). \tag{1.104}$$

With this equation (1.104), the equations (1.86) and (1.87), we determine the evolution of the orbit and the spin for a small spinning test particle completely.

Sometimes it is more useful to work with a spin four-vector S^{μ} than the tensor $S^{\mu\nu}$. The antisymmetry of the spin tensor only allows six independent spin values to be reduced to a four vector. Of course, this four vector S^{μ} depends on the SSC [46] and is defined as (1.97). The measure of the spin divided by the dynamical rest mass, i.e., S/M defines the minimal radius or Möller radius.

When the space-time admits a Killing vector ξ^{υ}, there is a property that includes the covariant derivative and the spin tensor, which gives a constant and is given by the expression [43]

$$p^{\nu}\xi_{\nu} + \frac{1}{2}\xi_{\nu,\mu}S^{\nu\mu} = \text{ constant,} \tag{1.105}$$

where p^{ν} is the linear momentum, $\xi_{\nu,\mu}$ is the covariant derivative of Killing vector, and $S^{\nu\mu}$ is the spin tensor of the particle. In the case of the Kerr metric, there are two Killing vectors, owing to its stationary and axisymmetric nature. In consequence, Eq. (1.105) yields two constants of motion: E, the total energy and J_z, the component of its angular momentum along the axis of symmetry [47].

The next section presents other possible formulation for solving the equation of motion of a test particle around to a rotating body. This method is called Carter's equation [6].

1.5 *Carter's Equations*

Now we present, in a brief form, Carter's equations for a particle around a massive rotating body. In a Kerr type metric, the symmetries provide three constant of motion: Energy (E), the angular momentum (J) and the mass (M). In addition, there is another constant which is due to the separability of the Hamilton-Jacobi Equation and is called Q. The Lagrange equation for a Kerr metric gives the first integrals of t and φ immediately. For the other two integrals for (r) and (θ) are obtained for a separable solution of the Hamilton-Jacobi equation [48]. The set of equations is given by [23]

$$\begin{aligned} \Sigma\dot{t} &= a\left(J - aE\sin^2\theta\right) \\ &\quad + \frac{\left(r^2+a^2\right)\left[E\left(r^2+a^2-aJ\right)\right]}{\Delta}, \end{aligned} \tag{1.106}$$

$$\Sigma\dot{r} = \pm R = \pm\left\{\begin{array}{c} \left[E\left(r^2+a^2\right) \mp aJ\right]^2 \\ -\Delta\left[r^2+Q+\left(J \mp aE\right)^2\right] \end{array}\right\}^{1/2}, \tag{1.107}$$

$$\Sigma\dot{\theta} = \pm\Theta = \pm\left\{Q - \cos^2\theta\left[a^2\left(1-E^2\right) + \frac{J^2}{\sin^2\theta}\right]\right\}^{1/2}, \tag{1.108}$$

$$\Sigma\dot{\phi} = \frac{J}{\sin^2\theta} - aE + \frac{a}{\Delta}\left[E\left(r^2+a^2\right) - aJ\right], \tag{1.109}$$

where J, E and Q are constants and

$$\begin{aligned} \Sigma &:= r^2 + a^2\cos^2\theta, \\ \Delta &:= r^2 + a^2 - 2Mr, \end{aligned} \tag{1.110}$$

M and $a = J/M$ are the mass and specific angular momentum of the central source.

The Carter's constant (Q) is a conserved quantity of the particle in free fall around of a rotating massive body. This quantity affects the latitudinal motion of the particle and is related to the angular momentum in the direction θ. From (1.108) one analyzes that in the equatorial plane, the relation between Q and the motion in θ is given by

$$\Sigma \dot{\theta}^2 = Q \tag{1.111}$$

When $Q = 0$ corresponds to equatorial orbit and for the case when $Q \neq 0$, one has a non-equatorial orbit.

In the next section, we find that when there are isommetries in the space-time, there exist two constants (1.78) that relate the linear and angular momenta.