PS 7 Solutions:

1) $s \in S$ states
$i \in I$ consumers

$\forall s \in S$, $\sum_{i \in I} \omega_i^s = \bar{\omega}$

UMP: $\max_{c} \sum_{s \in S} \pi^s u_i(c_i^s) + u_i(c_i^0)$

$$\text{s.t. } p^0 c_i^0 + \sum_{s \in S} p^s c_i^s \leq p^0 \omega_i^0 + \sum_{s \in S} p^s \omega_i^s. \quad \text{----} (\lambda_i)$$

FOCs (ignoring non-negativity constraints):

$$\pi^s u_i'(c_i^s) = \lambda_i p^s.$$

Consequently for any i, j and t, s:

$$\frac{u_i'(c_i^s)}{u_j'(c_j^s)} = \frac{u_i'(c_i^t)}{u_j'(c_j^t)} = \frac{\lambda_i}{\lambda_j} \quad \text{----} (1)$$

And so $c_s^i = c_t^i$.
(If not, say, $c_s^i > c_t^i \Rightarrow$ by (1), $c_s^j > c_t^j$.
Consequently, market clearing fails for atleast s or t.)

Taking ratio of FOCs for consumption in different states:

$$\frac{\pi^s \cancel{u_i'(c_i^s)}}{\pi^t \cancel{u_i'(c_i^t)}} = \frac{p^s}{p^t}.$$

$$\Rightarrow \quad \underline{\underline{\frac{p^s}{p^t} = \frac{\pi^s}{\pi^t}}}$$

$\boxed{2}$ a) Planner's problem:

For some $\{\theta_i : i \in I\}$, $\sum_i \theta_i = 1$:

$$\max \sum_i \theta_i \sum_\omega \pi^\omega \cdot u_i(c_i(\omega))$$

$$\text{s.t.} \quad \sum_i c_i(\omega) \leq \sum_i e_i(\omega) = e(\omega) \quad \cdots\cdots \lambda^\omega$$

FOC:

$$\frac{\theta_i \cdot u_i'(c_i(\omega))}{\theta_j \cdot u_j'(c_j(\omega))} = \frac{\cancel{\lambda^\omega}}{\cancel{\lambda^\omega}} = 1 \quad \forall \omega$$

And so, $\forall \omega' \neq \tilde{\omega}$,

$$\frac{u_i'(c_i(\omega'))}{u_j'(c_j(\omega'))} = \frac{u_i'(c_i(\tilde{\omega}))}{u_j'(c_j(\tilde{\omega}))}$$

So, $c_i(\omega') = c_i(\tilde{\omega}) \quad \forall i$

in any $\omega', \tilde{\omega}$ such that $e(\omega') = e(\tilde{\omega}) = e_n$.

b) Since a competitive allocation satisfies UMP for all agents, consider the UMP for arbitrary agent i:

$$\max_{c_i} \sum_{\omega} \pi^{\omega} \cdot u_i(c_i(\omega))$$

$$\text{s.t.} \quad \sum_{\omega} p^{\omega} c_i(\omega) \leq \sum_{\omega} p^{\omega} e_i(\omega).$$

FOC gives us that: $\forall \omega' \neq \tilde{\omega}$

$$\frac{\pi^{\omega'} \cdot u_i'(c_i(\omega'))}{\pi^{\tilde{\omega}} \cdot u_i'(c_i(\tilde{\omega}))} = \frac{p^{\omega'}}{p^{\tilde{\omega}}}.$$

Similarly, $\forall i \neq j$ (with μ_i and μ_j as the lagrange multiplier to UMPs)

$$\frac{u_i'(c_i(\omega))}{u_i'(c_j(\omega))} = \frac{\mu_i}{\mu_j}$$

Since preferences are identical across consumers:

$$\frac{u'(c_i(\omega))}{u'(c_j(\omega))} = \frac{\mu_i}{\mu_j}.$$

$\Rightarrow \forall \omega, \; c_i(\omega) = c_j(\omega)$ (Else Walras law is violated)

Mkt. clr $\Rightarrow \sum_{i \leq N} c_i(\omega) = \sum_{i \leq N} e_i(\omega)$

$$\Rightarrow N\, c_i(\omega) = e_H \cdot r \cdot N + e_T \cdot (1-r) \cdot T$$

$\forall \omega$ such that $\bar{e}(\omega) = r$.

Thus, $\underline{c_i(\omega) = e_H \cdot r + e_T \cdot (1-r)}$.

(c) From the FOC's in (a):

$$\frac{\theta_i}{\theta_j} \cdot \frac{u'(c_i(\omega))}{u'(c_j(\omega))} = 1.$$

Plugging in CRRA specification:

$$\frac{c_i(\omega)}{c_j(\omega)} = \left(\frac{\theta_j}{\theta_i}\right)^{-1/\sigma}$$

The contract curve is then:

$\forall \omega', \tilde{\omega}$:

$$\frac{c_1(\omega') \cdot \sum_i (\theta_1/\theta_i)^{-1/\sigma}}{c_1(\tilde{\omega}) \cdot \sum_i (\theta_1/\theta_i)^{-1/\sigma}} = \frac{\bar{e}(\omega')}{\bar{e}(\tilde{\omega})}$$

$$\Rightarrow \frac{c_1(\omega')}{c_1(\tilde{\omega})} = \frac{\bar{e}(\omega')}{\bar{e}(\tilde{\omega})}.$$

$\boxed{3}$ UMP:

$$\max\ U_i(x_i^1, \ldots, x_i^S)$$

$$\text{s.t. } \sum_j q^j z_i^j \leq 0 \quad \cdots\cdots \lambda_i$$

$$\forall s, \quad p_s \cdot x_i^s \leq p_s \cdot \omega_i^s + (Az_i)^s \quad \cdots\cdots \mu_i^s$$

FOCs:

$$(x_i^s) \qquad \frac{\partial U_i}{\partial x_{il}^s} = \mu_i^s p_{sl}$$

$$(z_i^j) \quad -\lambda_i q^j + \sum_s \mu_i^s \cdot a_{sj} = 0$$

$$\oplus \qquad \sum_j q^j z_i^j = 0$$

$$p_s \cdot x_i^s = p_s \cdot \omega_i^s + (A \cdot z_i)^s, \quad \forall s$$

Social planner's problem:

$$\max \sum_i \theta^i \cdot U_i(x_i^1, \ldots, x_i^S) + \sum_s \gamma_s \cdot \left(\sum_i \omega_i^s - \sum_i x_i^s\right)$$

Efficient allocation satisfies:

$$\theta^i \frac{\partial U_i}{\partial x_{il}^s} = \gamma_{s,l}$$

$$\sum_i \omega_i^s = \sum_i x_i^s, \quad \forall s.$$

To see that the Radner equilibrium, set

$$\gamma_{s,\ell} = \mu_i^s \cdot p_{s,\ell}.$$

Market clearing is also a condition for Rander eqbm.

---

4) This corresponds to showing that the 2 budget constraints are identical, given objectives are identical.

$$B_i^{AD} = \left\{ x_i \in \mathbb{R}_+^{LS} : \sum_s \phi_s \left(x_i^s - \omega_i^s\right) \leq 0 \right\}$$

$$B_i^{R} = \left\{ \begin{array}{l} x_i \in \mathbb{R}_+^{LS} : \exists z_i \in \mathbb{R}^J \text{ s.t. } \sum_j q_j z_{ij} \leq 0 \\ \text{and } p^s \cdot (x_i^s - \omega_i^s) \leq \sum_j a_{sj} z_{ij}, \ \forall s \end{array} \right\}$$

(i) Suppose $x_i \in B_i^{AD}$:

Set $\pi^s = \phi_{s1}$ (good 1 is denoted as numeraire)

$$p_\ell^s = \frac{\phi_{s\ell}}{\pi_1}$$

$$q = \pi A.$$

Set $(Az_i)_s = p_s(x_i^s - \omega_i^s)$

$$\Rightarrow \quad q\, z_i = \Pi A \cdot A^{-1} \cdot \begin{bmatrix} p_i^1(x_i^1 - \omega_i^1) \\ \vdots \\ p_i^s(x_i^s - \omega_i^s) \end{bmatrix}$$

$$= \sum_s \phi_s \cdot (x_i^s - \omega_i^s)$$

Easy to now verify that $x_i \in B_i^R$.

(ii) Suppose $x_i \in B_i^R$:

Set $\phi_{sl} = \dfrac{\Pi_s\, p_{sl}}{p_{s1}}$.

Summing Rander B.C. across states:

$$\sum_s \phi_s \cdot x_i^s \leq \sum_s \phi_s \cdot \omega_i^s + \left[\sum_s (Az^i)^s - q\, z_i \right]$$

$$=$$