ECON 6100	3/26/2021
Section 6	
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1 Review

An equilibrium can be though to be comprised of three components:

1. What gets consumed?
2. What gets produced?
   (a) **What goods are produced and how much?**
   (b) What factors are used and how much?
3. What prices make this exchange work?

1.1 Two sector models

The economy is endowed with two production processes (sectors) f_A and f_B that produce goods A and B respectively. It is a two sector model because each sector produces a unique good.

We typically assume there are two factors of production, capital (k) and labor (l) that move freely between the two sectors.

The production functions are assumed to satisfy:

A.1 The production function f_j is twice continuously differentiable with $f_j' > 0$ and $f_j'' < 0$.

A.2 The production function satisfies Inada condition (this is important because ... *who has the time to check for corner solutions?*)

$$\lim_{k \to 0} \frac{\partial f_j(k,l)}{\partial k} = \lim_{l \to 0} \frac{\partial f_j(k,l)}{\partial l} = +\infty \tag{1}$$

A.3 f_j is homogenous of degree 1 (constant returns to scale).

Min. cost of prod 5 units of j = 5x min. cost of 1 unit of j.

1.2 HOV model

Consider a small open economy trading goods A and B in a large world market. Consequently, prices p_A and p_B are determined independently of production here. The economy is endowed with endowments of factor inputs K and L.

1.2.1 Producer feasibility and efficiency

The PPS can be written as:

$$PPS = \{(y_A, y_B) : \; y_j \leq f_j(k_j, l_j) \; \forall j \in \{A, B\}, \widetilde{k_A + k_B \leq K}, \widetilde{l_A + l_B \leq L}\} \tag{2}$$

The production possibility frontier (PPF) are the set of all (y_A, y_B) pairs that simultaneously solves:

$$\begin{aligned} \phi(y_B) = \max f_A(k_A, l_A) \\ \text{s.t. } f_B(k_B, l_B) \geq y_B \\ k_A + k_B \leq K \\ l_A + l_B \leq L \end{aligned}$$

and

$$\begin{aligned} \phi(y_A) = \max f_B(k_B, l_B) \\ \text{s.t. } f_A(k_A, l_A) \geq y_A \\ k_A + k_B \leq K \\ l_A + l_B \leq L \end{aligned}$$

$\longleftrightarrow$ condition on directing l_A & k_A to produce atleast y_A, what's the max. amount of y_B I can make.

1.2.2 Equilibrium

Define any equilibrium as $\left(\underbrace{w^*, r^*}_{\text{factor prices}}, (l_j^*, k_j^*, y_j^*)_{j \in \{A,B\}}\right)$ such that:

1. $(l_j^*, k_j^*, y_j^*)_{j \in \{A,B\}}$ maximizes profit:

$$l_j^*, k_j^*, y_j^* = \arg\max p_j y_j - r^* k_j - w^* l_j \text{ s.t. } y_j \leq f(k_i, l_i) \tag{3}$$

2. $\sum_j (l_j^*, k_j^*) = (L, K)$

We know from lecture that there are two classes of equilibria here:

- Diversified

- Specialized

But first, recall that Shepard's lemma gives us the factor use as the gradient of the unit cost functions:

$$l_j = \frac{\partial c_j(w,r)}{\partial w}, \; k_j = \frac{\partial c_j(w,r)}{\partial r} \tag{4}$$

Cost function:

$$C_j(\underbrace{w, r}_{\text{factor prices}}, \underset{\substack{\uparrow \\ \text{units to be produce}}}{q}) = \min_{l,k} \; wl + rK$$

$$\text{s.t.} \quad f_j(k, l) \geq q$$

$$k, l \geq 0$$

Unit output cost function:

$$c_j(w, r) \equiv C_j(w, r, 1).$$

By HOD 1 of f_j: $\quad C_j(w, r, y) = y \, c_j(w, r).$

Any eqbm satisfies: "=" should be '=' by complementary slackness

$$\left.\begin{aligned} (f) \quad & y_A \left(P_A - c_A(w, r)\right) \boxed{\geq} 0 \\ \xrightarrow{w,r} \quad & y_B \left(P_B - c_B(w, r)\right) \boxed{\geq} 0 \end{aligned}\right\}$$

Q: What happens when $y_A, y_B > 0$?

$P_A = c_A$ & $P_B = c_B$

$$\underset{l_A, l_B, k_A, k_B}{\nearrow} (+) \quad y_A \begin{bmatrix} \partial c_A(w, r) / \partial r \\ \partial c_A(w, r) / \partial w \end{bmatrix} + y_B \begin{bmatrix} \partial c_B(w, r) / \partial r \\ \partial c_B(w, r) / \partial w \end{bmatrix} = \begin{bmatrix} K \\ L \end{bmatrix}.$$

$\rightarrow k_A$ $\quad \rightarrow k_B$ $\quad \rightarrow k_B$

Shepard: k_A

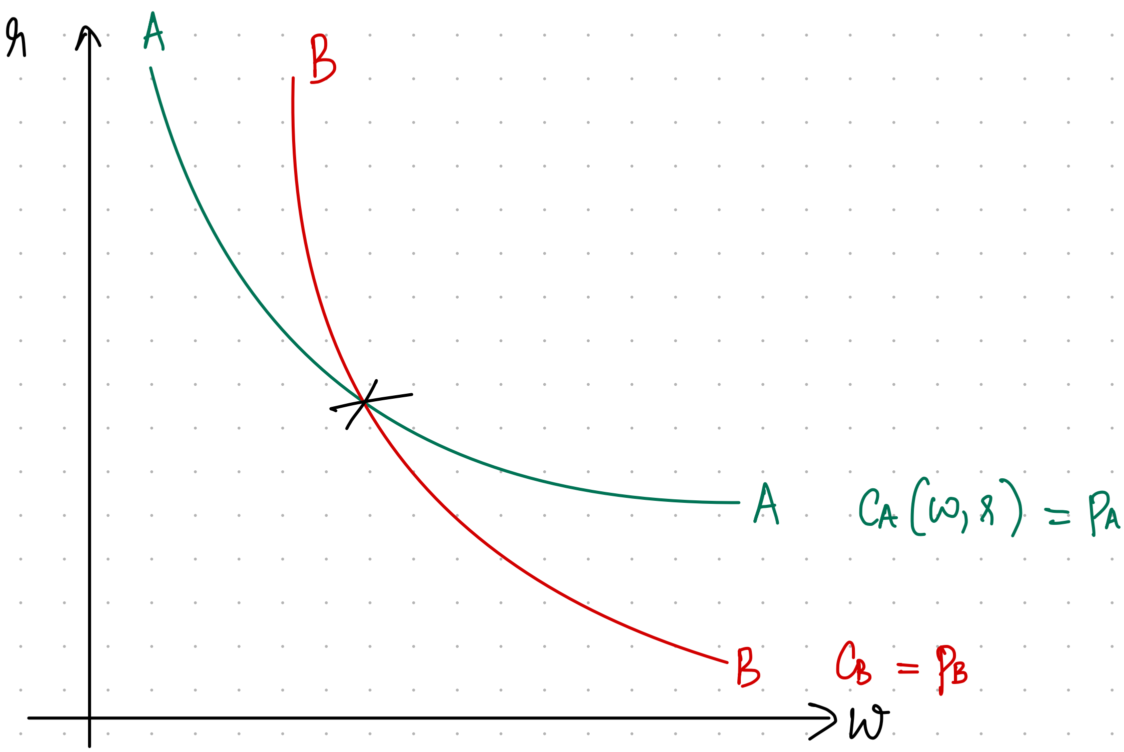
There are 2 types of specialized eqbm.

① $c_A(w,r) = p_A$ but $c_B(w,r) > p_B$.

$\Rightarrow y_B = 0$

Note that even when $y_A > 0$, $\pi_A = 0$.

② The interesting kind:



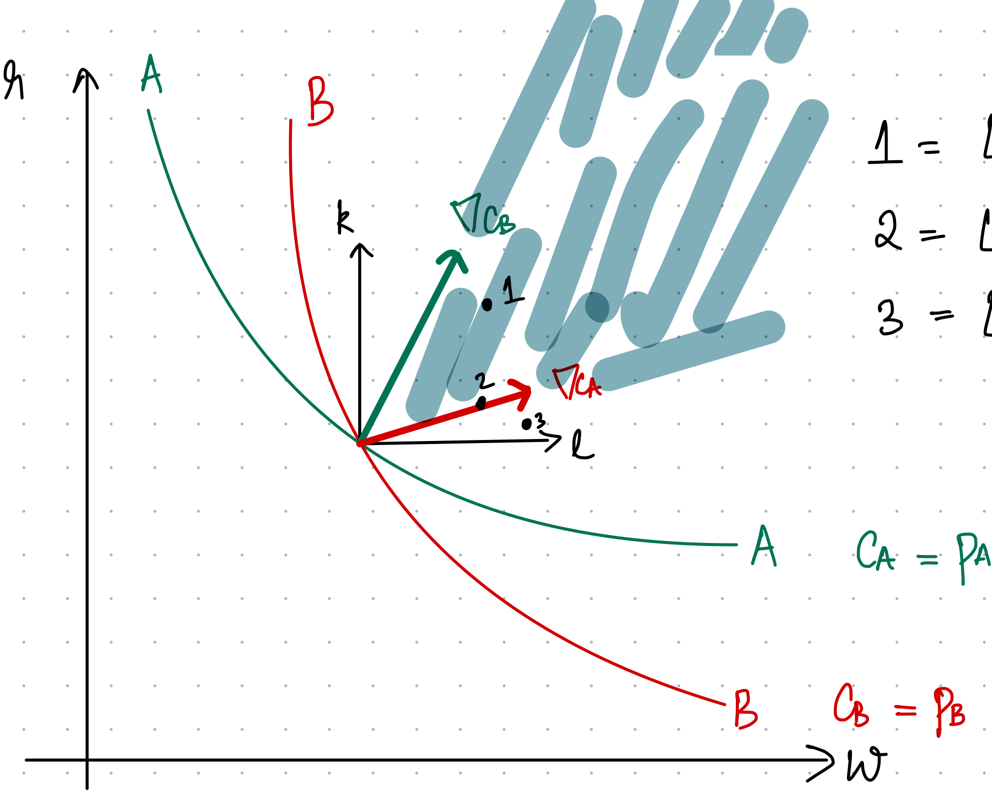
* These are the unit isoquant lines for A, B.

By Shepard's lemma, we know:

$$\frac{\partial c_j(w,r)}{\partial w} = l_j \;, \quad \frac{\partial c_j(w,r)}{\partial r} = k_j .$$

w, r $\boxed{c_A(w,r) = p_A \text{ and } c_B(w,r) = p_B}$ happens where?

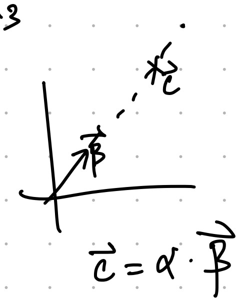
$\rightsquigarrow$ Notice that this determines w^*, r^*.



$1 = L_1, K_1$

$2 = L_2, K_2$

$3 = L_3, K_3$



(1) $\exists\ y_A, y_B > 0 :$

$$y_A \nabla C_A + y_B \nabla C_B = (L_1, K_1)$$ → Diversified Eqbm

(2) $\exists\ y_A > 0,\ y_B = 0 :$

$$y_A \nabla C_A = (L_2, K_2)$$ → Boundary Eqbm.

CORRECTED

(3) $\nexists\ y_A, y_B > 0 : y_A \nabla C_A + y_B \nabla C_B = (L_3, K_3)$.

Clearly $y_A^* = 0$.

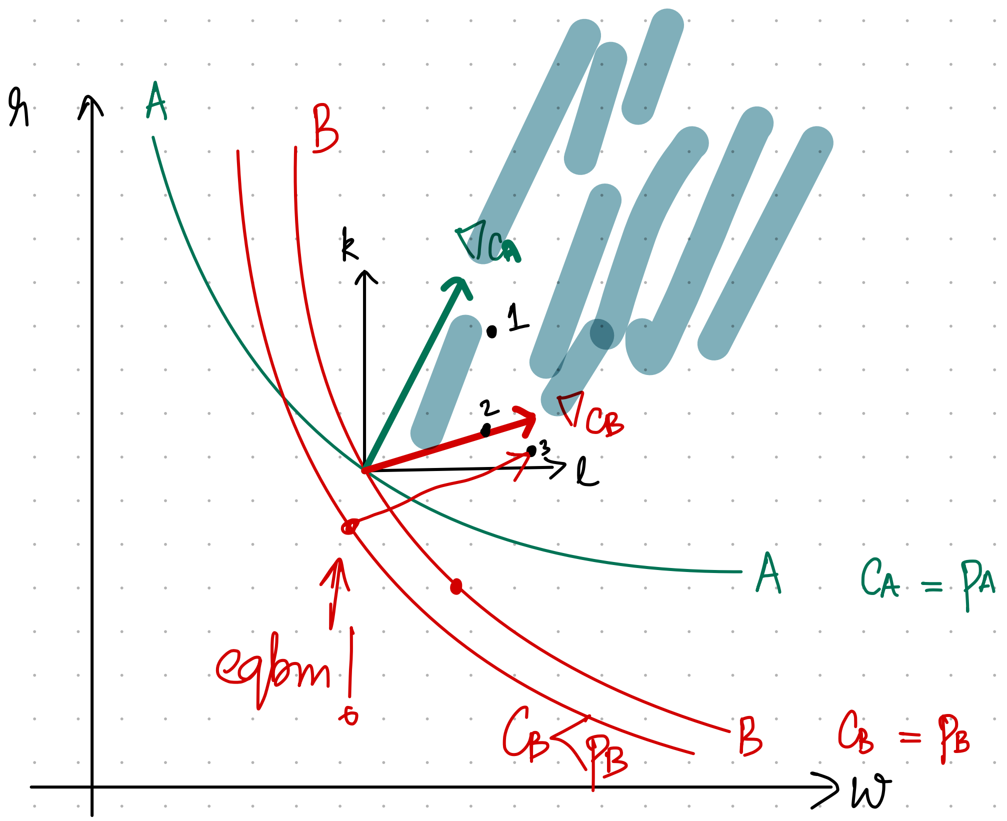
But $y_B^* > 0$ as w^*, r^* such that $c_B(w^*, r^*) = p_B$

cannot satisfy $y_B^* \nabla_B^* = (L_3, K_3)$.

So, lower w^*, r^* until $c_B(w^*, r^*) < p_B$

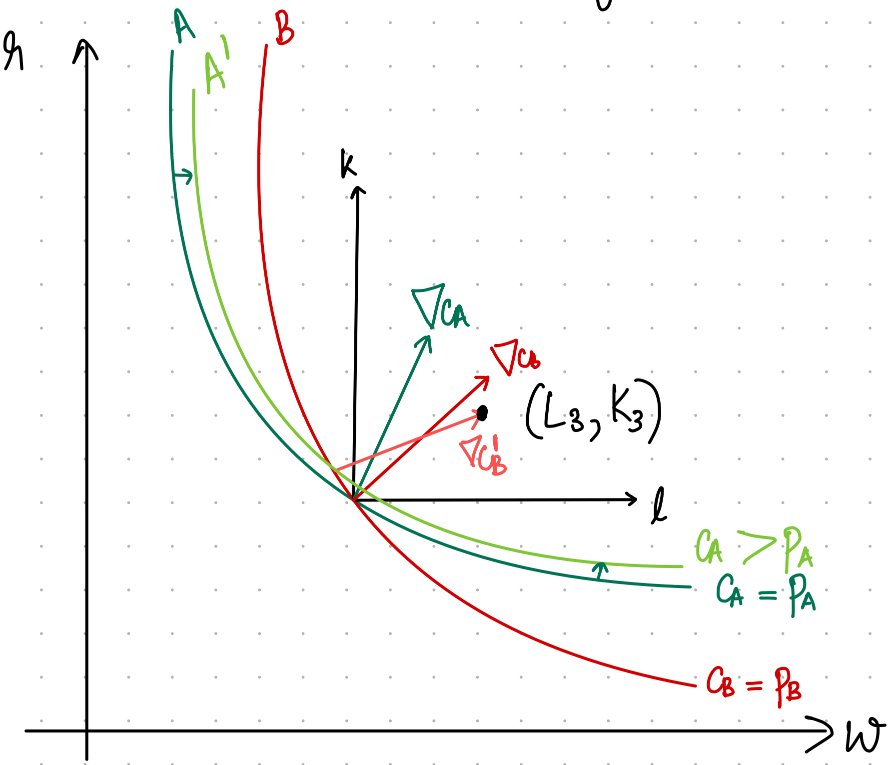
$\Rightarrow$ B earns positive profit.

@ SuengWha: This was incorrect!



See correction in next slide!

Correction: Thanks Yaling for the question!



My error during section was not paying attention to complementary slackness.

By complementary slackness, in any equilibrium:

$$y_A\left[p_A - c_A(w,r)\right] = 0 \quad , \quad y_B\left[p_B - c_B(w,r)\right] = 0$$

i.e. Any industry with positive production must have 0 profits.

So, for endowments (L_3, K_3) which cannot be expressed as a conical combination of ∇c_A and ∇c_B, i.e: $\nexists\ y_A, y_B > 0$ such that:

$$y_A \nabla c_A + y_B \nabla c_B = \begin{bmatrix} L_3 \\ K_3 \end{bmatrix}.$$

at the point:
$$\left.\begin{aligned} c_A(w,r) &= p_A \\ c_B(w,r) &= p_B \end{aligned}\right\} (*)$$

Consequently, in any equilibrium, it must be that equation system (*) is violated.

Since ∇c_B is closer to (L_3, K_3), in eqbm $y_B > 0$ and $y_A = 0$.

Thus, by complementary slackness:

$$c_A(w, r) \geq p_A$$
$$c_B(w, r) = p_B$$

At this point, $\exists\, y_B > 0$:

$$y_B \nabla c_B = \begin{bmatrix} L_3 \\ K_3 \end{bmatrix}$$

Intuition:

When K/L ratio is small, it must be that the capital intensive industry does not produce:

$$\frac{k_A}{l_A} = \frac{\partial c_A / \partial r}{\partial c_A / \partial w} > \frac{\partial c_B / \partial r}{\partial c_B / \partial w} = \frac{k_B}{l_B}$$

$$\Rightarrow \quad y_A = 0 \quad \text{and} \quad y_B > 0!$$

2 Problems

Problem 1. Suppose in a small open economy, world output prices for good 1 and 2 are p and 1, respectively. The production functions are:

$$q_1 = k^{1/2} l^{1/2}$$
$$q_2 = k^{3/4} l^{1/4}$$

$\to C_A = P_A,\ C_B = P_B$

(a) Compute the equilibrium factor prices. $\longleftarrow$ $C_A, C_B, \nabla C_A, \nabla C_B$

(b) Compute $\frac{\partial w}{\partial p}$ when both goods are produced.

(c) Suppose $p = 1$. If the endowment of capital and labor are both 100, do both firms operate?

If (L,K) is in conic combin of $\nabla C_A, \nabla C_B \to$ diversified eqbm, else not

(d) Suppose $p = 1$. If the endowment of capital and labor are 100 and 400 respectively, do both firms operate?

Problem 2. A small open economy produces two goods, A and B, using two inputs, capital (k) and labor (l). The production function for the two goods are:

$$f_A(l_A, k_A) = \min\{\alpha_A l_A, \beta_A k_A\}$$
$$f_B(l_B, k_B) = \min\{\alpha_B l_B, \beta_B k_B\}$$

Let p_A and p_B be the world output prices of good A and B respectively. Also, let the country's stock of capital and labor be $(K, L) \gg 0$. Denote the prices of capital and labor by r and w respectively. Suppose $K/L \in (1/4, 1/2)$.

(a) Let $p_A = \alpha_A = \alpha_B = 1$ and $\beta_A = 4$, $\beta_B = 2$. Suppose we are in the situation where both A and B are produced in positive quantities. Solve for the competitive equilibrium when both wage rate and rental rate are positive.

(b) Suppose the economy specified above is in a diversified competitive equilibrium with $w, r > 0$. What can you say regarding the factor intensity of industry A compared to industry B? What is the effect of an increase in p_B on the equilibrium input prices?

(a) $C_A(w,r) = w + \frac{r}{4}$; $\nabla C_A(w,r) = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$

$C_B(w,r) = w + \frac{r}{2}$; $\nabla C_B = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$

$C_A = P_A$; $C_B = P_B = ?$

$w + \frac{r}{4} = 1$; $w + \frac{r}{2} = P_B \longrightarrow w^* = 2 - P_B$; $r^* = 4(P_B - 1)$.

$\frac{l_A}{k_A} = \frac{\partial C_A(w,r)/\partial w}{\partial C_A(w,r)/\partial r} = 4 \ \boxed{\text{vs}} > \frac{\partial C_B/\partial w}{\partial C_B/\partial r} = 2 = \frac{l_B}{k_B}$

4

$\boxed{1}$ Notice that the eqbm is characterized in terms of unit cost function, so let's find it:

$$c(r,w) = \min_{k,l} \; wl + rk$$
$$\text{s.t.} \quad k^\alpha l^{1-\alpha} \geq 1 \;, \; k,l \geq 0$$

$$\mathcal{L} = wl + rk + \lambda\left[1 - k^\alpha l^{1-\alpha}\right]$$

FOC(k): $r = \lambda \alpha k^{\alpha-1} l^{1-\alpha}$ + Const: $k^\alpha l^{1-\alpha} = 1$.

FOC(l): $w = \lambda(1-\alpha) k^\alpha l^{-\alpha}$

$$c(r,w) = \left(\frac{r}{\alpha}\right)^\alpha \left(\frac{w}{1-\alpha}\right)^{1-\alpha} \checkmark$$

$\boxed{\begin{matrix} C_A = P_A \\ C_B = P_B \end{matrix}} \rightarrow r^*, w^*$

So, $c_1(r,w) = 2\sqrt{rw}$ $\quad \alpha = 1/2$

$c_2(r,w) = \dfrac{4}{3^{3/4}} r^{3/4} w^{1/4}$ $\quad \alpha = 3/4$

Recall (w,r) are completely characterized by $P_1 = c_1(w,r)$ and $P_2 = c_2(w,r)$.

$$\therefore \quad p = 2\sqrt{wr}$$
$$1 = \frac{4}{3^{3/4}} r^{3/4} w^{1/4}$$

$$\therefore \quad r^* = \left(3/4\right)^{3/2} \frac{1}{p}$$

$$w^* = \left(\frac{4}{27}\right)^{1/2} p^3$$

$$\Rightarrow \frac{\partial w^*}{\partial p} = ? = \sqrt{\frac{4}{27}}\, 3p^2$$

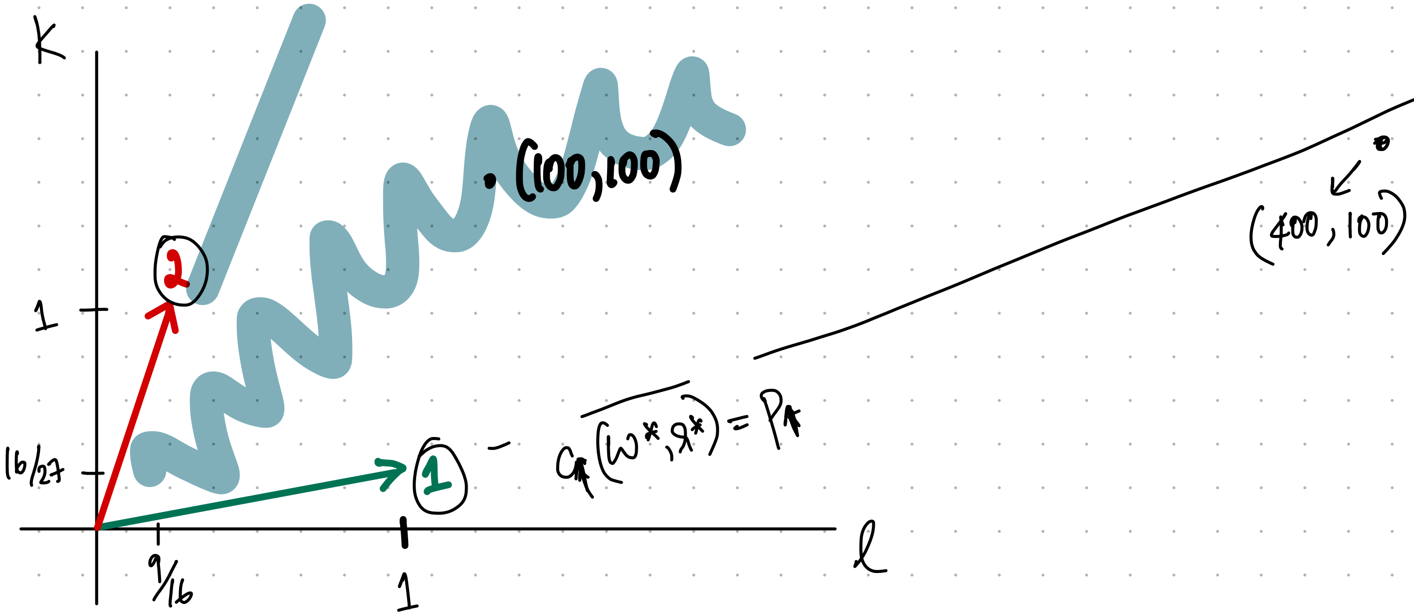
$\boxed{c}$ To answer this, I must compute

$$\nabla c_1(r^*, w^*),\ \nabla c_2(r^*, w^*).$$

After careful algebra:

$$\underset{(w,r)}{\nabla c_1} = \begin{bmatrix} \sqrt{27}/4 \\ 4/\sqrt{27} \end{bmatrix}$$

$$\nabla c_2 = \begin{bmatrix} 3^{-3/4} \left(\frac{27}{16}\right)^{3/4} \\ 3^{1/4} \left(16/27\right)^{1/4} \end{bmatrix}$$



$$\underset{>0}{\gamma_A} \cdot \nabla c_A + \underset{>0}{\gamma_B} \nabla c_B = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

$\boxed{2}$ $c(w,r) = \min \; w\cdot l + rk$

s.t. $\underbrace{\min\{\alpha l, \beta k\} \geq 1}$.

$\Leftrightarrow$ $\alpha l \geq 1$

$\beta k \geq 1$.

$\mathcal{L} = wl + rk + \lambda_1(1-\alpha l) + \lambda_2(1-\beta k)$.

FOC (l): $w = \lambda_1 \alpha$

(k): $r = \lambda_2 \beta$

Consts: $l^* = 1/\alpha \; ; \; k^* = 1/\beta$.

$c(w,r) = wl^* + rk^* = \frac{w}{\alpha} + \frac{r}{\beta}$.

(b) Stopler-Samuelson: In a diversified eqbm if $p_A \uparrow$, the input that A is intensive in has its price $\uparrow$ & other $\downarrow$.

$r \uparrow$ (B is intensive in k)

$w \downarrow$