Unsolved Number Theory Problem Attempt

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1 Problem Statement

Conjecture
Let k ≥ 2 a positive integer. The diophantine equation:

$$y = 2x_1x_2 \dots x_k + 1$$

has an infinity of solutions of primes. (For example: $571 = 2 \cdot 3 \cdot 5 \cdot 19 + 1, 691 = 2 \cdot 3 \cdot 5 \cdot 23 + 1$, or $647 = 2 \cdot 17 \cdot 19 + 1$, when $k = 4$, respectively, 3). (Gamma 2/1986).

2 Proof Attempt

For the purposes of this problem, let us restrict our solution set : $(x_1, x_2, x_3, ..., x_k, y)$ to the positive integers/natural numbers.

CLAIM 1 : ***y is an odd integer and $y > 2$.***

PROOF
Arguing by contradiction, if $y < 2$ then this would imply that :

$$2x_1x_2 \dots x_k + 1 < 2$$

$$2x_1x_2 \dots x_k < 1$$

Which is impossible since the LHS has to be ≥ 2 for all x_is in $\mathbf{N}$
Hence proved that $y > 2$.

Now coming to the first part of our claim,
We can safely conclude that y is an odd integer since it is a whole number which is not divisible by 2 and this is based on the observation that $y = 2m + 1$ where

$m \in \mathbf{N}$.

Now again for the purposes of our solution, we further restrict y from an odd integer to a prime number, implying that y is an odd prime.

CLAIM 2 : $\frac{y-1}{2}$ ***is always an integer for odd prime*** y***.***

PROOF :
Since y is an odd integer,
We can express y as $y = 2m + 1$ for $m \in \mathbf{N}$
Thus

$$\frac{y-1}{2} = \frac{(2m+1)-1}{2} = \frac{2m}{2} = m \in \mathbf{N}$$

This was also trivial to notice due to our assumption that $x_1, x_2, x_3, ..., x_k$ are all positive integers. Hence proved.

Now we have

$$\frac{y-1}{2} = x_1 x_2 \dots x_k = m$$

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SOLUTION CLAIM : ***for any value of*** m***, we can always have infinite prime number solutions of*** $(x_1, x_2, x_3, ..., x_k)$***.***

From here, we diverge to consider two cases :
CASE 1: ***When there is a possibility that*** $x_i = x_j$ ***for*** $i \neq j$

INVESTIGATION :
THEOREM : (THE FUNDAMENTAL THEOREM OF ARITHMETIC)
every positive integer (except the number 1) can be represented in exactly one way apart from rearrangement as a product of one or more primes **(Hardy and Wright 1979, pp. 2-3).**

Taking the above mentioned theorem into consideration, a suitable way to express m could be as a product of its primes.

Thus
$m = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3} \cdots p_n^{\alpha_n}$ where $p_i, \alpha_i \in \mathbf{N}$ and every p_i is a prime.

Now as per the equality that we found out above,

$$m = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}\cdots p_n^{\alpha_n} = x_1x_2\ldots x_k$$

Since all the p_i's are coprime to each other, we can certainly allot each of the p_i's to the x_i's and therefore get prime solutions for $(x_1, x_2, x_3, ..., x_k)$. Thus (WLOG $\alpha_1 < \alpha_2 < \cdots < \alpha_n$)

$$x_1 = x_2 = x_3 = \cdots = x_{\alpha_1} = p_1$$

$$x_{\alpha_1+1} = x_{\alpha_1+2} = x_{\alpha_1+3} = \cdots = x_{\alpha_2} = p_2$$

and so on and so forth, where $k = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n$.
Hence shown for this case

CASE 2 : ***When*** $x_i \neq x_j$ ***for*** $i \neq j$

INVESTIGATION :
Without Loss of Generality, let $p_i < p_j$ whenever $i < j$ where p has been defined as above.

Arguing casewise,
Let $(p_1, p_2, p_3 \cdots p_k, p_x)$ where $x > k$ be the solution set of primes that satisfy the given Diophantine equation.

Therefore,

$$p_x = 2p_1 \cdot p_2 \cdot p_3 \cdots p_k + 1$$

Consider $m \in N$ and :

$$p_x + 2m = 2(p_1 \cdot p_2 \cdot p_3 \cdots p_k + m) + 1$$

where $p_x + 2m$ is a prime (after even jumps from p_x).

Now we focus on $(p_1 \cdot p_2 \cdot p_3 \cdots p_k + m)$ and see if it can be of the form $x_1x_2\ldots x_k$ or not.
SUBCASE 1 : $(p_1 \cdot p_2 \cdot p_3 \cdots p_k + m)$ is a prime number.

INSIGHT :
If the given is a prime number then it is proven that there exist a pair of primes

that can satisfy the given equation.
Hence proved.

SUBCASE 2 : $(p_1 \cdot p_2 \cdot p_3 \cdots p_k + m)$ is not a prime number.

INSIGHT :
This suggests that $(p_1 \cdot p_2 \cdot p_3 \cdots p_k + m)$ can be expressed in terms of its prime factors (Due to Fundamental Theorem of Arithmetic above). Therefore :

$$p_1 \cdot p_2 \cdot p_3 \cdots p_k + m = p_a \cdot p_b \cdots p_r$$

where not all of $a, b, \cdots, r$ are $= 1, 2, \cdots, k$ (because $p_x \neq p_x + m$).
Thus each of $p_a, p_b, \cdots, p_r$ can be used as x_i's in the solution set.

Now we see,
There are infinite primes of the form $p_x + 2m$ (since all primes other than 2 are odd and $p_x \neq 2$) and as per subcases above, every one of those primes will bear prime solutions for x_i's and thus since there are infinite primes (Proof by Euclid), there are infinite solutions for the solution set.
Thereby proving the Solution Claim.

Hence shown
End of attempt.

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OPEN PROBLEM : Formulate the general lemma in the selection of x_is and y for which the given problem has only prime solutions in $\mathbf{N}$.