☆ Residual

$x \rightarrow$ ground truth homogenised 2D points (3×1)

$x\_est \rightarrow$ estimated homoginised 2D points (3×1)

$P\_est \rightarrow$ current estimated Projection matrix (3×4)

$X \rightarrow$ ground truth homoginised 3D points (4×1)

So we then have:

$$\text{residual}, \ r = \| x - x\_est \|^2$$

So, let $x = [x_1 \ \ x_2 \ \ x_3]^T$

And by ~~re~~projection : $x\_est = P\_est \cdot X$

let

$$P\_est = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{bmatrix}_{3\times4}$$

$$X = [X_1 \ \ X_2 \ \ X_3 \ \ X_4]^T$$

So $$x_{est} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{bmatrix}_{3\times 4} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}_{4\times 1}$$

$$x_{est} = \begin{bmatrix} P_{11}X_1 & P_{12}X_2 & P_{13}X_3 & P_{14}X_4 \\ P_{21}X_1 & P_{22}X_2 & P_{23}X_3 & P_{24}X_4 \\ P_{31}X_1 & P_{32}X_2 & P_{33}X_3 & P_{34}X_4 \end{bmatrix}_{3\times 1}$$

Now,

$$r = \| x - x_{est} \|^2$$

$$r = \left\| \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} P_{11}X_1 & P_{12}X_2 & P_{13}X_3 & P_{14}X_4 \\ P_{21}X_1 & P_{22}X_2 & P_{23}X_3 & P_{24}X_4 \\ P_{31}X_1 & P_{32}X_2 & P_{33}X_3 & P_{34}X_4 \end{bmatrix} \right\|^2$$

$$r = \left\| \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \begin{bmatrix} x_1 - P_{11}X_1 - P_{12}X_2 - P_{13}X_3 - P_{14}X_4 \\ x_2 - P_{21}X_1 - P_{22}X_2 - P_{23}X_3 - P_{24}X_4 \\ x_3 - P_{31}X_1 - P_{32}X_2 - P_{33}X_3 - P_{34}X_4 \end{bmatrix} \right\|^2$$

$$r = \left\| \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \right\|^2$$

$$r = (r_1)^2 + (r_2)^2 + (r_3)^2$$

Hence for one 2D - 3D correspondence, residual 'r' is a scalar.

In our case, ~~residu~~ we have m 2D-3D correspondences, so residual r will be:

$$r = \begin{bmatrix} r_{11}^2 + r_{12}^2 + r_{13}^2 \\ r_{21}^2 + r_{22}^2 + r_{23}^2 \\ \vdots \\ \vdots \\ r_{m1}^2 + r_{m2}^2 + r_{m3}^2 \end{bmatrix}_{\cancel{m \times 3}\; m \times 1}$$

$r \rightarrow m \times 1$ vector

where "$(r_{i1})^2 + (r_{i2})^2 + (r_{i3})^2$" is ~~equal~~ to given as defined previously for i^{th} point.

☆ Jacobian

In case of one 2D-3D correspondence, we have shown that :

$$r = \left\| \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \right\|^2 = r_1^2 + r_2^2 + r_3^2$$

where
$$r_1 = x_1 - P_{11}X_1 - P_{12}X_2 - P_{13}X_3 - P_{14}X_4$$
$$r_2 = x_2 - P_{21}X_1 - P_{22}X_2 - P_{23}X_3 - P_{24}X_4$$
$$r_3 = x_3 - P_{31}X_1 - P_{32}X_2 - P_{33}X_3 - P_{34}X_4$$

Now, as we are estimating P, and P has 12 parameters, our Jacobian for one 2D-3D correspondence will look like :

$$J = \left[\frac{\partial r}{\partial P_{11}}, \frac{\partial r}{\partial P_{12}}, \frac{\partial r}{\partial P_{13}}, \ldots\ldots, \frac{\partial r}{\partial P_{33}}, \frac{\partial r}{\partial P_{34}} \right]_{1\times 12}$$

i.e. differentiating r w.r.t. each element of P (row-wise).

So, having defined r, we need to compute these 12 derivatives.

Re - writing r :

$$r = (x_1 - P_{11}X_1 - P_{12}X_2 - P_{13}X_3 - P_{14}X_4)^2$$
$$+ (x_2 - P_{21}X_1 - P_{22}X_2 - P_{23}X_3 - P_{24}X_4)^2$$
$$+ (x_3 - P_{31}X_1 - P_{32}X_2 - P_{33}X_3 - P_{34}X_4)^2$$

So,

- $$\frac{\partial r}{\partial P_{11}} = 2(x_1 - P_{11}X_1 - P_{12}X_2 - P_{13}X_3 - P_{14}X_4).(-X_1)$$
- $$\frac{\partial r}{\partial P_{12}} = 2(x_1 - P_{11}X_1 - P_{12}X_2 - P_{13}X_3 - P_{14}X_4).(-X_2)$$
- $$\frac{\partial r}{\partial P_{13}} = 2(\quad " \qquad\qquad " \quad).(-X_3)$$
- $$\frac{\partial r}{\partial P_{14}} = 2(\quad " \qquad\qquad " \quad).(-X_4)$$

- $\dfrac{\partial r}{\partial P_{21}} = 2\left(x_2 - P_{21}X_1 - P_{22}X_2 - P_{23}X_3 - P_{24}X_4\right)\cdot(-X_1)$

- $\dfrac{\partial r}{\partial P_{22}} = 2\left(\quad \text{''} \qquad\qquad \text{''} \quad\right)\cdot(-X_2)$

- $\dfrac{\partial r}{\partial P_{23}} = 2\left(\quad \text{''} \qquad\qquad \text{''} \quad\right)\cdot(-X_3)$

- $\dfrac{\partial r}{\partial P_{24}} = 2\left(\quad \text{''} \qquad\qquad \text{''} \quad\right)\cdot(-X_4)$

- $\dfrac{\partial r}{\partial P_{31}} = 2\left(x_3 - P_{31}X_1 - P_{32}X_2 - P_{33}X_3 - P_{34}X_4\right)\cdot(-X_1)$

- $\dfrac{\partial r}{\partial P_{32}} = 2\left(\quad \text{''} \qquad\qquad \text{''} \quad\right)\cdot(-X_2)$

- $\dfrac{\partial r}{\partial P_{33}} = 2\left(\quad \text{''} \qquad\qquad \text{''} \quad\right)(-X_3)$

- $\dfrac{\partial r}{\partial P_{34}} = 2\left(\quad \text{''} \qquad\qquad \text{''} \quad\right)\cdot(-X_4)$

Hence we have:

$$J = \left[-2r_1X_1, -2r_1X_2, -2r_1X_3, -2r_1X_4, -2r_2X_1, -2r_2X_2, -2r_2X_3, -2r_2X_4, -2r_3X_1, -2r_3X_2, -2r_3X_3, -2r_3X_4 \right]_{1\times 12}$$

So, this was the case when we had 1 2D-3D correspondence. Here J is a 1×12 vector.

In case of m 2D-3D correspondences, J will be a $m\times12$ vector, where each row will be of the above type for each correspondence.

$$J = \begin{bmatrix} \partial r_1/\partial P_{11}, & \partial r_1/\partial P_{12}, & \cdots & \partial r_1/\partial P_{34} \\ \partial r_2/\partial P_{11}, & \partial r_2/\partial P_{12}, & \cdots & \partial r_2/\partial P_{34} \\ \vdots & \vdots & & \vdots \\ \partial r_m/\partial P_{11}, & \partial r_m/\partial P_{12}, & \cdots & \partial r_m/\partial P_{34} \end{bmatrix}_{m\times 12}$$

★ Residual - Normalised

In this case, we will normalise x_{est} before substracting from x. Similarly we will normalise x and X

So

$$x_{est} = \begin{bmatrix} P_{11}X_1 & P_{12}X_2 & P_{13}X_3 & P_{14}X_4 \\ P_{21}X_1 & P_{22}X_2 & P_{23}X_3 & P_{24}X_4 \\ P_{31}X_1 & P_{32}X_2 & P_{33}X_3 & P_{34}X_4 \end{bmatrix}_{3\times 4}$$

So

$$x_{est} = \begin{bmatrix} A^T X \\ B^T X \\ C^T X \end{bmatrix}_{3\times 1}$$

where

$$A^T = [P_{11} \quad P_{12} \quad P_{13} \quad P_{14}]$$

$$B^T = [P_{21} \quad P_{22} \quad P_{23} \quad P_{24}]$$

$$C^T = [P_{31} \quad P_{32} \quad P_{33} \quad P_{34}]$$

$$\cancel{X = [X_1 \quad X_2 \quad X_3 \quad 1X_4]^T} \quad \cancel{\rightarrow \text{normalised } X}$$

$$X = [X_1 \quad X_2 \quad X_3 \quad X_4]^T$$

So normalising x_{est} :

$$x_{est})_N = \begin{bmatrix} (A^T X / C^T X) \\ (B^T X / C^T X) \\ 1 \end{bmatrix}_{3\times 1}$$

Also

$$x)_N = \begin{bmatrix} x_1/x_3 \\ x_2/x_3 \\ 1 \end{bmatrix}$$

Now, we are defining residual as :

$$r = x)_N - x_{est})_N$$

$$r = \begin{bmatrix} \left(x_1/x_3 - A^T X/C^T X\right) \\ \left(x_2/x_3 - B^T X/C^T X\right) \\ 0 \end{bmatrix}_{3\times 1}$$

$$\Rightarrow \quad r = \begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix}_{3\times 1}$$

where,

$$r_1 = \left(\frac{x_1}{x_3} - \frac{A^T X}{C^T X}\right)$$

$$r_2 = \left(\frac{x_2}{x_3} - \frac{B^T X}{C^T X}\right)$$

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☆ Jacobian Normalised

Here, we proceed in similar manner as done before:

$$J = \left[\frac{\partial r}{\partial P_{11}}, \frac{\partial r}{\partial P_{12}}, \ldots\ldots, \frac{\partial r}{\partial P_{34}} \right]_{3\times 12}$$

$\because$ $r \rightarrow (3\times 1)$ vector, so $J \rightarrow (3 \times 12)$ matrix

Now, $r = \begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix}$, so:

w.r.t. $P_{11}, P_{12}, P_{13}, P_{14}$

$$\frac{\partial r}{\partial P_{11}} = \begin{bmatrix} \partial r_1/\partial P_{11} \\ \partial r_2/\partial P_{11} \\ \partial 0/\partial P_{11} \end{bmatrix} = \begin{bmatrix} -X_1/c^T X \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial r}{\partial P_{12}} = \begin{bmatrix} \partial r_1/\partial P_{12} \\ \partial r_2/\partial P_{12} \\ \partial 0/\partial P_{12} \end{bmatrix} = \begin{bmatrix} -X_2/c^T X \\ 0 \\ 0 \end{bmatrix}$$

$\vdots$

$\frac{\partial r}{\partial P_{13}}$ and $\frac{\partial r}{\partial P_{14}}$ can be obtained by symmetry.

w.r.t. P_{21}, P_{22}, P_{23} and P_{24}

- $$\frac{\partial \pi}{\partial P_{21}} = \begin{bmatrix} \partial \pi_1 / \partial P_{21} \\ \partial \pi_2 / \partial P_{21} \\ \partial 0 / \partial P_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ -X_1 / c^T X \\ 0 \end{bmatrix}$$

- $$\frac{\partial \pi}{\partial P_{22}} = \begin{bmatrix} \partial \pi_1 / \partial P_{22} \\ \partial \pi_2 / \partial P_{22} \\ \partial 0 / \partial P_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ -X_2 / c^T X \\ 0 \end{bmatrix}$$

$\vdots$

$\frac{\partial \pi}{\partial P_{23}}$, $\frac{\partial \pi}{\partial P_{24}}$ can be obtained by symmetry.

w.r.t. $P_{31}, P_{32}, P_{33}, P_{34}$

- $$\frac{\partial \pi}{\partial P_{31}} = \begin{bmatrix} \partial \pi_1 / \partial P_{31} \\ \partial \pi_2 / \partial P_{31} \\ \partial 0 / \partial P_{31} \end{bmatrix} = \begin{bmatrix} \frac{X_1 A^T X}{(c^T X)^2} \\ \frac{X_1 B^T X}{(c^T X)^2} \\ 0 \end{bmatrix}$$

- $$\frac{\partial \pi}{\partial P_{32}} = \begin{bmatrix} \partial \pi_1 / \partial P_{32} \\ \partial \pi_2 / \partial P_{32} \\ \partial 0 / \partial P_{32} \end{bmatrix} = \begin{bmatrix} \frac{X_2 A^T X}{(c^T X)^2} \\ \frac{X_2 B^T X}{(c^T X)^2} \\ 0 \end{bmatrix}$$

$\vdots$

$\frac{\partial \pi}{\partial P_{33}}$ and $\frac{\partial \pi}{\partial P_{34}}$ can be obtained by symmetry

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Proof : $i = \{1, 2, 3, 4\}$

- $$\frac{\partial r_1}{\partial P_{1i}} = \frac{\partial}{\partial P_{1i}}\left(x_1 - \frac{P_{11}X_1 + P_{12}X_2 + P_{13}X_3 + P_{14}X_4}{P_{31}X_1 + P_{32}X_2 + P_{33}X_3 + P_{34}X_4}\right)$$

$$= 0 - \frac{X_i \cdot 1}{(P_{31}X_1 + P_{32}X_2 + P_{33}X_3 + P_{34}X_4)}$$

$$= -\frac{X_i}{c^T X}$$

- $$\frac{\partial r_2}{\partial P_{2i}} = \frac{\partial}{\partial P_{2i}}\left(\frac{x_2}{x_3} - \frac{P_{21}X_1 + P_{22}X_2 + P_{23}X_3 + P_{24}X_4}{P_{31}X_1 + P_{32}X_2 + P_{33}X_3 + P_{34}X_4}\right)$$

$$= 0 - \frac{X_i \cdot 1}{P_{31}X_1 + P_{32}X_2 + P_{33}X_3 + P_{34}X_4}$$

$$= \frac{-X_i}{c^T X}$$

- $$\frac{\partial r_1}{\partial P_{3i}} = \frac{\partial}{\partial P_{3i}}\left(\frac{x_1}{x_3} - \frac{P_{11}X_1 + P_{12}X_2 + P_{13}X_3 + P_{14}X_4}{P_{31}X_1 + P_{32}X_2 + P_{33}X_3 + P_{34}X_4}\right)$$

$$= 0 - (P_{11}X_1 + P_{12}X_2 + P_{13}X_3 + P_{14}X_4) \cdot \frac{-1 \cdot X_i}{(P_{31}X_1 + P_{32}X_2 + P_{33}X_3 + P_{34}X_4)^2}$$

$$= \frac{X_i\,(A^T X)}{(c^T X)}$$

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- $\frac{\partial r_2}{\partial P_{3i}} \rightarrow$ similar proof to $\frac{\partial r_1}{\partial P_{31}}$

- $\frac{\partial r_2}{\partial P_{1i}} = 0 \rightarrow \therefore r_2$ does not contain P_{1i}

- $\frac{\partial r_1}{\partial P_{2i}} = 0 \rightarrow \therefore r_1$ does not contain P_{2i}

Hence, final Jacobian looks like :

$$J = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix}_{3\times 12}$$

where,

$$J_1 = \left[\frac{-x_1}{c^T x}, \frac{-x_2}{c^T x}, \frac{-x_3}{c^T x}, \frac{-x_4}{c^T x}, 0, 0, 0, 0, \frac{x_1(A^T x)}{(c^T x)^2}, \frac{x_2(A^T x)}{(c^T x)^2}, \frac{x_3(A^T x)}{(c^T x)^2}, \frac{x_4(A^T x)}{(c^T x)^2} \right]_{1\times 12}$$

$$J_2 = \left[0, 0, 0, 0, \frac{-x_1}{c^T x}, \frac{-x_2}{c^T x}, \frac{-x_3}{c^T x}, \frac{-x_4}{c^T x}, \frac{x_1(B^T x)}{(c^T x)^2}, \frac{x_2(B^T x)}{(c^T x)^2}, \frac{x_3(B^T x)}{(c^T x)^2}, \frac{x_4(B^T x)}{(c^T x)^2} \right]_{1\times 12}$$

$$J_3 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]_{1\times 12}$$