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If $R_n(z)$ is the remainder after n terms then

7.1.24

$$R_n(z)=(-1)^n \frac{1\cdot 3 \ldots (2n-1)}{(2z^2)^n}\,\theta,$$

$$\theta=\int_0^\infty e^{-t}\left(1+\frac{t}{z^2}\right)^{-n-\frac{1}{2}} dt \qquad \left(|\arg z|<\frac{\pi}{2}\right)$$

$$|\theta|<1 \qquad \left(|\arg z|<\frac{\pi}{4}\right)$$

For x real, $R_n(x)$ is less in absolute value than the first neglected term and of the same sign.

Rational Approximations[2] $(0\leq x<\infty)$

7.1.25

$$\operatorname{erf} x=1-(a_1t+a_2t^2+a_3t^3)\,e^{-x^2}+\epsilon(x), \qquad t=\frac{1}{1+px}$$

$$|\epsilon(x)|\leq 2.5\times 10^{-5}$$

$p=.47047 \qquad a_1=.34802\ 42 \qquad a_2=-.09587\ 98 \qquad a_3=.74785\ 56$

7.1.26

$$\operatorname{erf} x=1-(a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5)\,e^{-x^2}+\epsilon(x),$$

$$t=\frac{1}{1+px}$$

$$|\epsilon(x)|\leq 1.5\times 10^{-7}$$

$p=.32759\ 11 \qquad a_1=.25482\ 9592$
$a_2=-.28449\ 6736 \qquad a_3=1.42141\ 3741$
$a_4=-1.45315\ 2027 \qquad a_5=1.06140\ 5429$

7.1.27

$$\operatorname{erf} x=1-\frac{1}{[1+a_1x+a_2x^2+a_3x^3+a_4x^4]^4}+\epsilon(x)$$

$$|\epsilon(x)|\leq 5\times 10^{-4}$$

$a_1=.278393 \qquad a_2=.230389$
$a_3=.000972 \qquad a_4=.078108$

7.1.28

$$\operatorname{erf} x=1-\frac{1}{[1+a_1x+a_2x^2+\cdots+a_6x^6]^{16}}+\epsilon(x)$$

$$|\epsilon(x)|\leq 3\times 10^{-7}$$

$a_1=.07052\ 30784 \qquad a_2=.04228\ 20123$
$a_3=.00927\ 05272 \qquad a_4=.00015\ 20143$
$a_5=.00027\ 65672 \qquad a_6=.00004\ 30638$

Infinite Series Approximation for Comple[x] Function [7.19]

7.1.29

$$\operatorname{erf}(x+iy)=\operatorname{erf} x+\frac{e^{-x^2}}{2\pi x}[(1-\cos 2xy)+i\ \text{si}$$

$$+\frac{2}{\pi}e^{-x^2}\sum_{n=1}^\infty \frac{e^{-\frac{1}{4}n^2}}{n^2+4x^2}[f_n(x,y)+ig_n(x,y)$$

where

$$f_n(x,y)=2x-2x\cosh ny\cos 2xy+n\sinh n$$

$$g_n(x,y)=2x\cosh ny\sin 2xy+n\sinh ny\ \text{co}$$

$$|\epsilon(x,y)|\approx 10^{-16}|\operatorname{erf}(x+iy)|$$

7.2. Repeated Integrals of the Error F

Definition

7.2.1

$$i^n\operatorname{erfc} z=\int_z^\infty i^{n-1}\operatorname{erfc} t\,dt \qquad (n=0,1,$$

$$i^{-1}\operatorname{erfc} z=\frac{2}{\sqrt{\pi}}e^{-z^2},\ i^0\operatorname{erfc} z=\operatorname{erfc}$$

Differential Equation

7.2.2 $$\frac{d^2y}{dz^2}+2z\frac{dy}{dz}-2ny=0$$

$$y=Ai^n\operatorname{erfc} z+Bi^n\operatorname{erfc}(-z)$$

(A and B are constants.)

Expression as a Single Integral

7.2.3 $$i^n\operatorname{erfc} z=\frac{2}{\sqrt{\pi}}\int_z^\infty \frac{(t-z)^n}{n!}e^{-t^2}dt$$

Power Series[3]

7.2.4 $$i^n\operatorname{erfc} z=\sum_{k=0}^\infty \frac{(-1)^k z^k}{2^{n-k}k!\,\Gamma\left(1+\frac{n-k}{2}\right)}$$

Recurrence Relations

7.2.5

$$i^n\operatorname{erfc} z=-\frac{z}{n}i^{n-1}\operatorname{erfc} z+\frac{1}{2n}i^{n-2}\operatorname{erfc} z$$

$$(n=1,$$

7.2.6

$$2(n+1)(n+2)i^{n+2}\operatorname{erfc} z$$

$$=(2n+1+2z^2)i^n\operatorname{erfc} z-\frac{1}{2}i^{n-2}\operatorname{erfc}$$

[2] Approximations **7.1.25–7.1.28** are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N. J., 1955 (with permission).

$(n=1, 2$

[3] The terms in this series corresponding to $n+4$, $n+6$, . . . are understood to be zero.