Midsem

CS-F351 Theory of Computation

October 31, 2021

1. DFA:

   (a) Let $\Sigma = \{0, 1\}$ be an alphabet and $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ are states where 0 is initial as well as final state and for $q \in \mathbb{Z}_5$ and $i \in \Sigma$, $\delta(q, i) = q^2 - i \pmod 5$. Prove by using induction that this DFA accepts exactly the set of binary strings with even number of 1's.

   Solution: We will prove the following logical formula together.

   $\hat{\delta}(0, x) = 0 \Leftrightarrow x$ has even number of $1'$s

   $\hat{\delta}(0, x) \in \{1, 4\} \Leftrightarrow x$ has odd number of $1'$s

   States 2 and 3 are unrecachable from initial state 0.

   Apply induction of the length of binary string $|x|$.

   Base case: $|x| = 1$ then $x \in \{0, 1\}$ and both the above statemens will be true.

   Induction hypothesis: Assume that above both the statements are true up to length $|x| = n$ binary strings.

   [1 marks]

   Induction step: If x is followed by 1 and $\hat{\delta}(0, x) = 0 \Leftrightarrow \hat{\delta}(0, x1) = 4$ and $x1$ contain odd number of $1's$.

   If x is followed by 0 and $\hat{\delta}(0, x) = 0$ then $\hat{\delta}(0, x0) = 0$ and $x0$ contains even number of $1's$.

   If x is followed by 1 and $\hat{\delta}(0, x) = 4$ then $\hat{\delta}(0, x1) = 0$ and even number of $1's$ in $x1$.

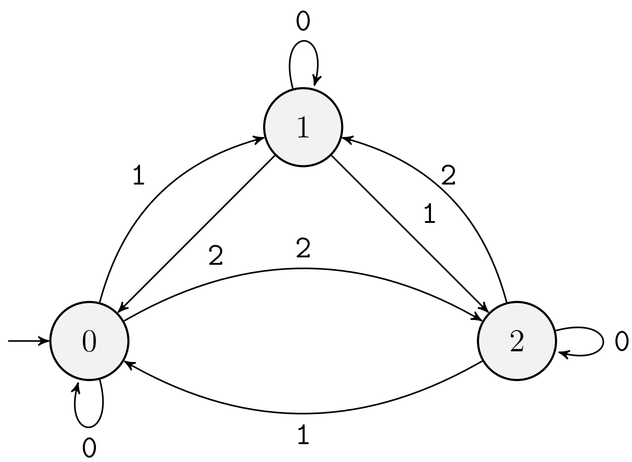
   If x is followed by 0 and $\hat{\delta}(0, x) = 4$ then $\hat{\delta}(0, x0) = 1$ and $x0$ does not contain even number of $1's$.

   Similar arguments can be given when $\hat{\delta}(0, x) = 1$.

   [1 marks]

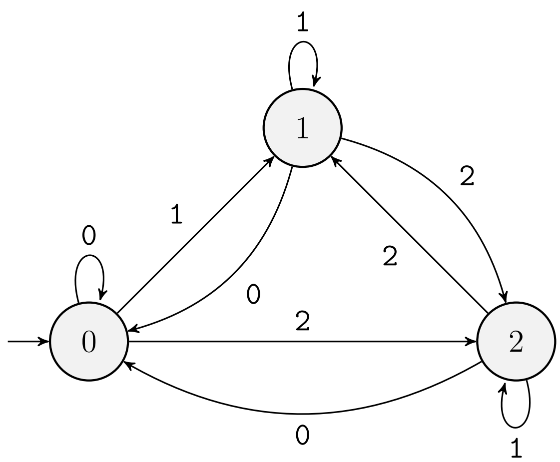
   (b) Let $\Sigma = \{0, 1, 2\}$ be an input alphabet. For $a_1a_2 \ldots a_n \in \Sigma^+$, define $A_3(a_1a_2 \ldots a_n) = a_1 +_3 a_2 +_3 \ldots +_3 a_n$ and $A_3(a_1a_2 \ldots a_n) = a_1 \times_3 a_2 \times_3 \ldots \times_3 a_n$ where $+_3$ is addition modulo 3 and $\times_3$ is multiplication modulo 3. Prove that $\{x \in \Sigma^+ \mid A_3(x) = M_3(x)\}$ is regular.

Solution:



$\boxed{A_3}$

[1 marks]



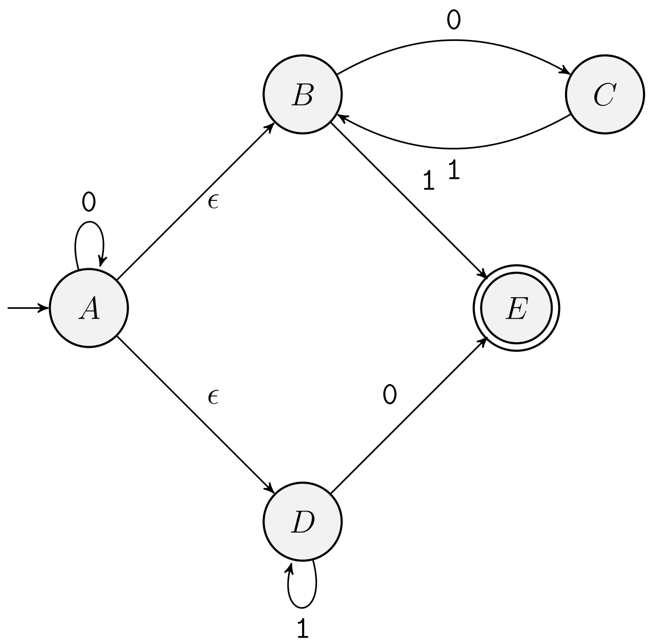
$\boxed{M_3}$

[1 marks]

Let Q_1 be the states of machine A_3 and Q_2 be the states of the machine M_3 and the input alphabet be $\Sigma = \{0, 1, 2\}$. Construct the required machine $M = (Q_1 \times Q_2, \Sigma, \delta, \{(0, 0)\}, \{(0, 0), (1, 1), (2, 2)\})$ where $\delta((q_i, q_j), a) = (\delta_{A_3}(q_i, a), \delta_{M_3}(q_j, a))$ for all $q_i \in Q_1$, $q_j \in Q_2$ and $a \in \Sigma$.

[1 marks]

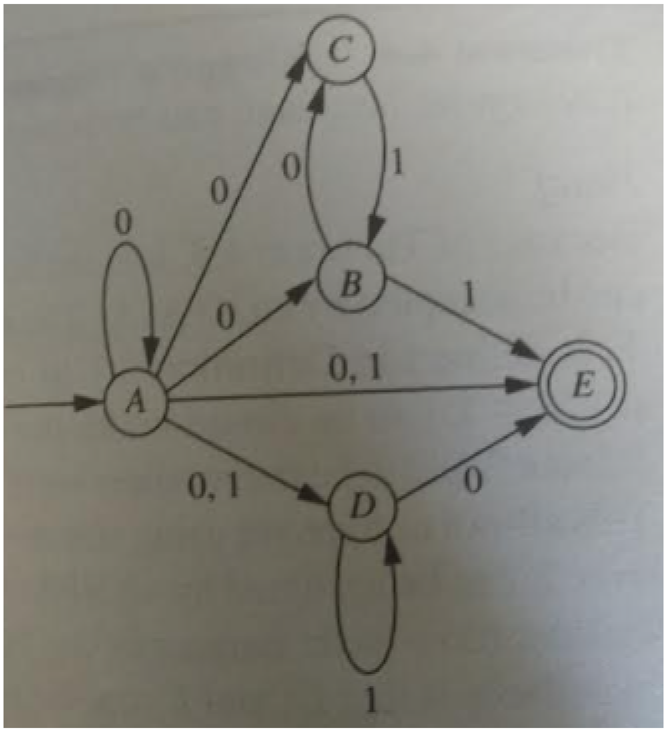
2. Convert the following ϵ-NFA to a DFA.



Solution:

If got correct NFA then [2 marks]. Otherwise [0 marks]

If got correct DFA then additional [3 marks]. Otherwise [0 marks]


C
0
0
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1
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0, 1
A
E
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D
0
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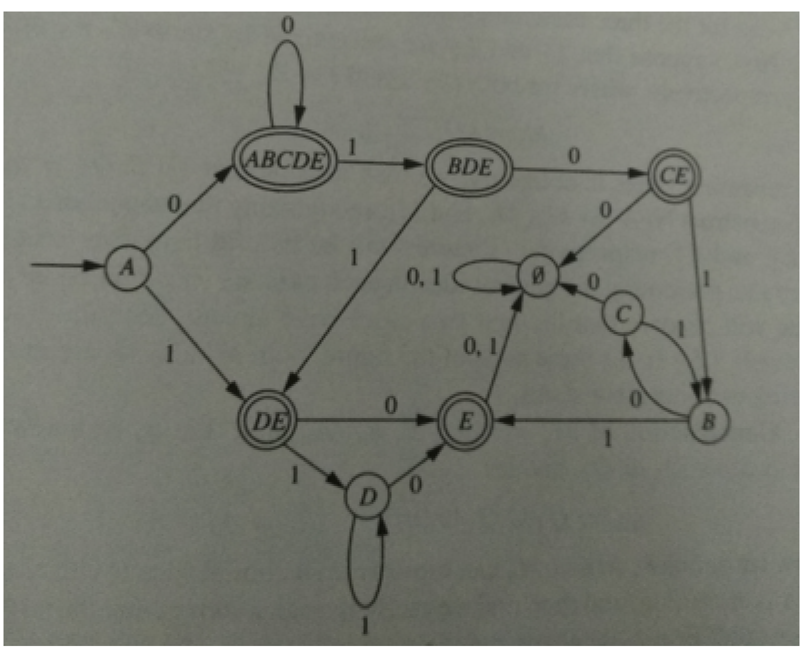
0
ABCDE
1
BDE
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CE
0
0
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1
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C
1
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DE
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E
0
B
1
1
D
0
1

3. Prove that $L = \{0^i 1^j \mid gcd(i, j) = 1\}$ is not regular.

   Solution: Let L be a regular language. Pick an n as $n = p > 0$ where p is a prime number, and then pick a $0^p 1^{p-1!} \in L$.

   [1 marks]

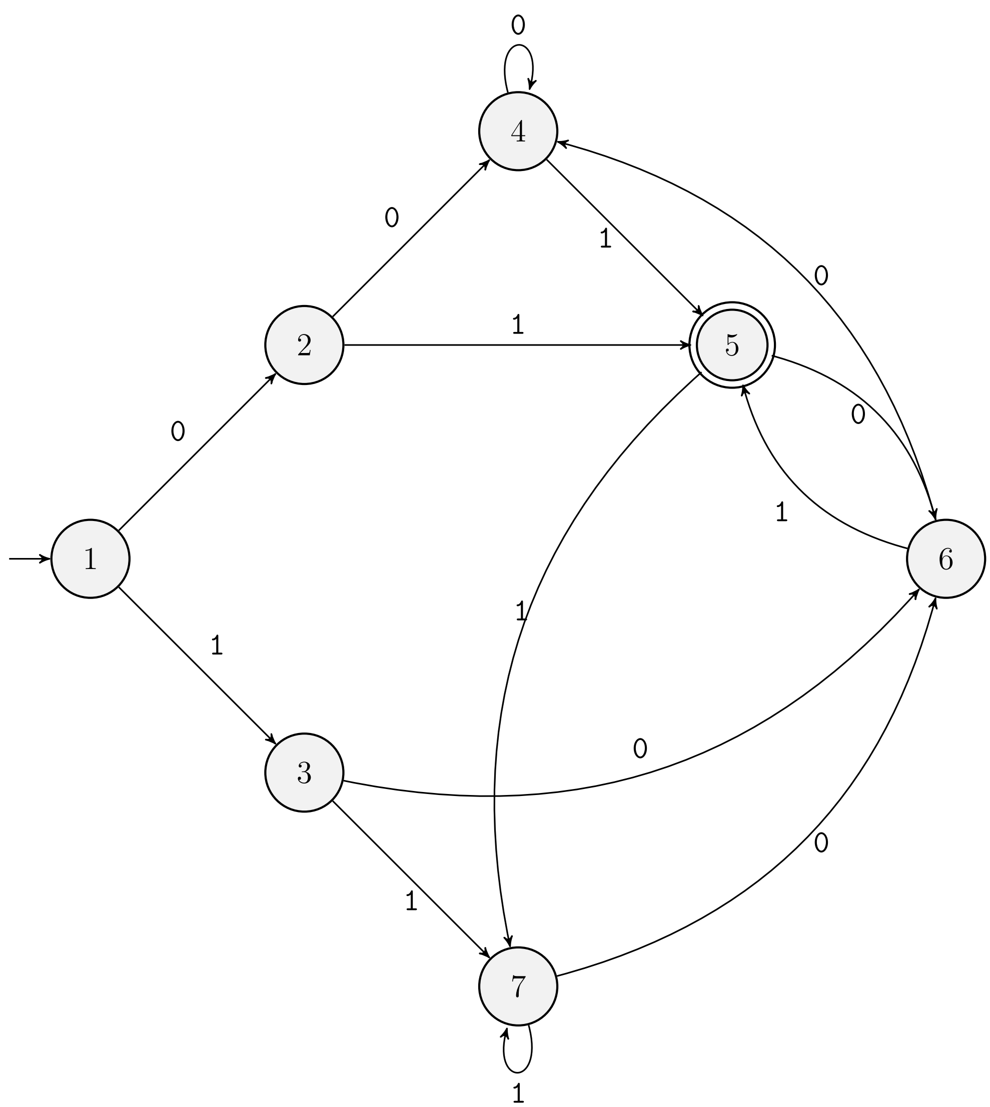
   The decomposition of $0^p 1^{p-1!}$ as $0^{p-k}0^k 1^{p-1!} \in L$ where $k \geq 1$ is the only decompostion which satisfies the conditions of pumping lemma. The pumped string $0^{p-k}0^{mk}1^{p-1!} \in L$ for all $m \geq 0$.

   [2 marks]

   But for $m = 0$, $0^{p-k}1^{p-1!} \notin L$. Hence L is not regular.

   [2 marks]

4. Minimize the number of states in the following DFA.



Solution:

$y - axis = 7, 6, 5, 4, 3, 2$

$x - axis = 1, 2, 3, 4, 5, 6$

Non final states: $\{1, 2, 3, 4, 6, 7\}$

Final states: $\{5\}$

Entries of the table will be defined as follows. For all $1 \leq i \leq 7$, $(i, 5)$ or $(5, i)$ will get $\times$.

[1 marks]

Entries like $(1, 3), (1, 7)$ will wait for $(2, 6)$. But the entries $(2, 6), (2, 4), (3, 7), (4, 6)$ will be defined as empty. Hence the entries $(1, 3)$ and $(1, 7)$ will also be defined as empty. Rest of the entries will be defined as $\times$.

[1 marks]

In the minimized DFA we get the following states $q_0 = \{1, 3, 7\}, q_1 = \{2, 4, 6\}$ and $q_2 = \{5\}$.

[2 marks]

The transitions in the minimized DFA will be as follows.

$\delta(q_0, 0) = q_1, \delta(q_0, 1) = q_1$ $\delta(q_1, 0) = q_1, \delta(q_1, 1) = q_2$ $\delta(q_2, 0) = q_1, \delta(q_2, 1) = q_0$

[1 marks]

5. Let $L = \{uu^R v \mid u, v \in \Sigma^+\}$ be a language. By using Myhill Nerode theorem prove that L is not regular.

Solution: Let $k \neq m$ and $ab^{2k+1}aR_L ab^{2m+1}a$.

[1 marks]

We know that R_L is right invariant. By right congruence $ab^{2k+1}aab^{2k+1}aaR_L ab^{2m+1}aab^{2k+1}aa$.

[1 marks]

Note that the first string in L (take $u = ab^{2k+1}a$ and $v = a$) but the second string can not be because a prefix of the second string has to end with ba to qualify for uu^R. It happens for $ab^{2m+1}a$, which is not valid since it is not even, and $ab^{2m+1}aab^{2k+1}a$, which is not palindrome. Hence there are infinite equivalence classes for $\{ab^{2k+1}a \mid k \geq 0\}$.

[3 marks]

6. Show using mathematical induction that the strings produced by the following context free grammar with productions

$$S \to 0 \mid S0 \mid 0S \mid 1SS \mid SS1 \mid S1S$$

has more 0's than 1's. Clearly mention, induction parameter, base case, induction hypothesis and induction step. Converse part is hard to proof. Please do not try here.

Solution: Induction on the number of derivation steps which derives x.

Base case: $S \to 0$ is the only production which produces string in a single derivation.

Induction hypothesis: Assume that if S derives x in one or more steps then $n_0(x) > n_1(x)$.

[2 marks]

Induction step: Let x' be the string which is derivable from S in one or more steps and it uses at exactly one more derivation step than the number of derivation steps is used to derive x.

To derive x' from S if we may use any of the productions given above. We will prove that in all cases $n_0(x') > n_1(x')$.

$S \to 0S \mid S0$ has been used first then by induction hypothesis we can say that right side S derive string in which $n_0(x) > n_1(x)$ after appending oe prepending 0 in x this inequality will hold. [1 marks]

$S \to S1S \mid 1SS \mid SS1$ has been used first then by induction hypothesis we can say that each S present at right side of S derive string in which $n_0(x) > n_1(x)$. We have two S on the right side of production and by induction hypothesis left S and right S will derive the strings in which $n_0(x) \geq n_1(x) + 1$ and $n_0(x') \geq n_1(x') + 1$. Hence $n_0(x1x') > n_1(x1x')$, $n_0(xx'1) > n_1(xx'1)$ and $n_0(1xx') > n_1(1xx')$.

[2 marks]