ASSIGNMENT- I

ADVANCED COMPUTATIONAL FLUID DYNAMICS (ME670)

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Given, A lid-driven cavity.

Flow can be Assumed 2D, incompressible, steady and isothermal.

- Non-dimensionalized Navier-Stokes equation for 2D, incompressible, steady condition obtained from dimensional N-S eq^n

$$\boxed{(u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u}$$

- Length(L) $r^* = \frac{r}{L}$ and $\nabla^* = L\nabla$

- Flow velocity U $u^* = \frac{u}{U}$

- Pressure $p^* = \frac{P}{\rho U^2}$

Substituting the scales the non-dimensionalized eq^n obtain is

$$\boxed{(u^* \cdot \nabla^*) u^* = -\nabla^* p^* + \frac{1}{Re} \nabla^{*2} u^*}$$

For me taking Non-dimensionalized eq^n As main eq^n.

Hence removing Stars and Re-writing

$$\boxed{(u \cdot \nabla) u = -\nabla p + \frac{1}{Re} \nabla^2 u}$$

$\underline{u} \to$ velocity vector

$\underline{u} \to ui + vj$

$p \to$ Scalar

Governing eqⁿs → ① Continuity eqⁿ
② Momentum eqⁿ (i.e Navier Stokes eqⁿ)

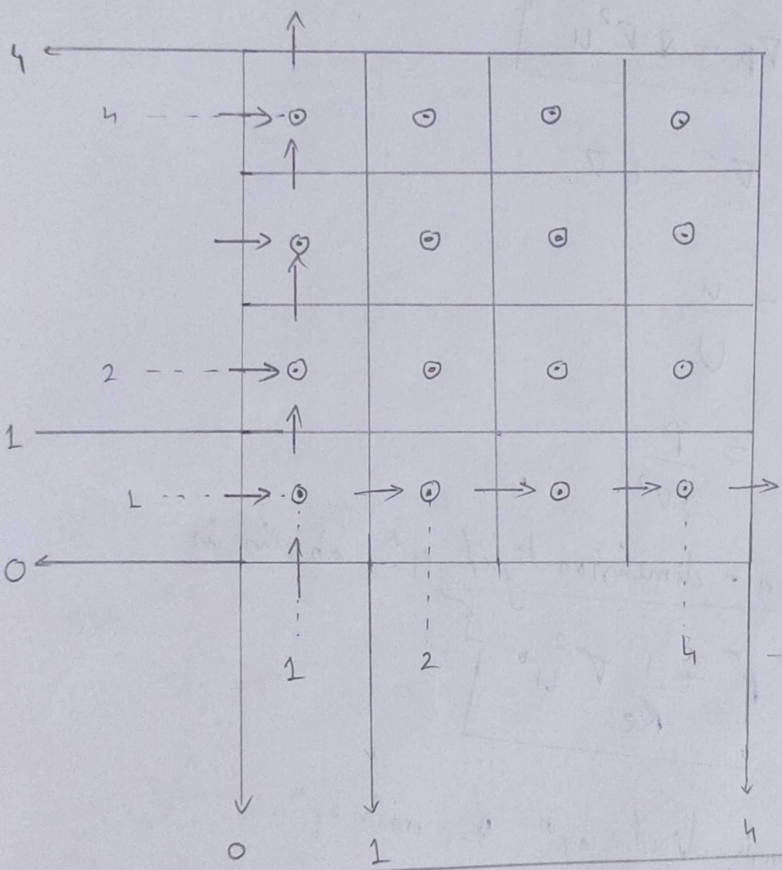
Grid → 129 × 129 $n = 128$, ~~indices~~ $x \to 0$ to 128
$y \to 0$ to 128

③ control volume

(I) U control volume   (II) V control volume   (III) P control volume

Shown below 5×5 grid (n=4)

$x \to j$, $y \to i$



⊙ → Scalar
x: 1 − n , y: 1 − n

(→) → u velocity
x (j): 0 − n   y (i): 1 − n

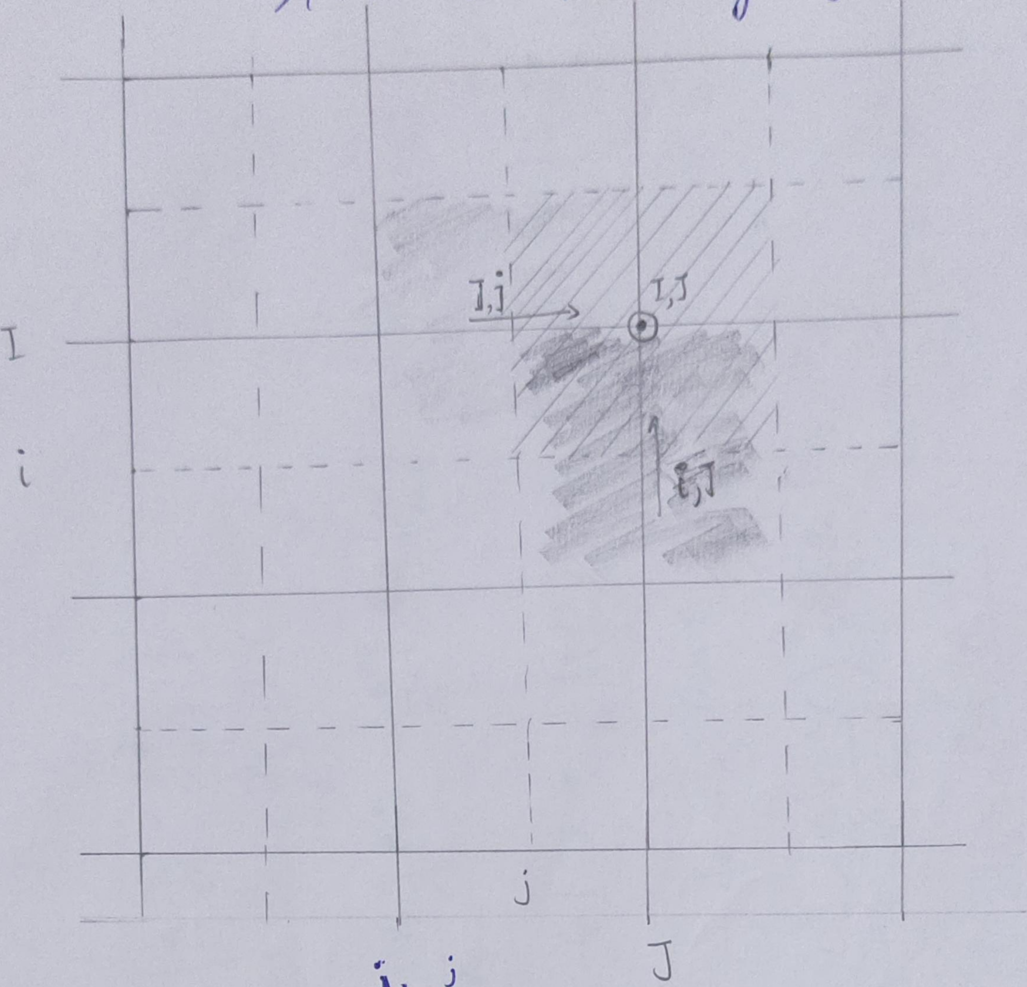
(↑) → v velocity
x(j): 0 − n   y(i): 1 − n

interior points for u → (2,2) − (n−1, n−1)   [(i,j) – i, j]

interior points for v → (2,2) − (n−2, n−1)   [(i,j) – i, j]

interior points for P → (2,2) − (n−1, n−1)   [i,j – i,j]

STAGGARED GRID (for J=3, I=3)



i, j

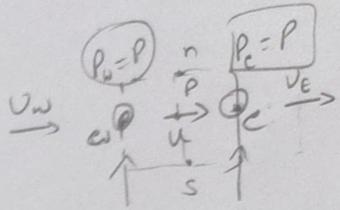
→ u-> control volume (3, 2)

→ v-> control volume (2, 3)

→ P -> control volume (3, 3)

Momentum eq^n x - momentum eq^n (2D)

$$\frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho u v) = -\frac{dP}{dx} + \frac{\partial}{\partial x}\mu\frac{\partial u}{\partial x} + \frac{\partial\mu}{\partial y}\frac{du}{dy} + S_u$$



Non-dimensionalizing

$$\frac{\partial}{\partial x}(u u) + \frac{\partial}{\partial y}(u v) = -\frac{dP}{dx} + \frac{1}{Re}\left[\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{du}{dy}\right] + S_u$$

integrating differential momentum eq^n over control volume of (u) (cancelling A as uniform grids $A_s = A_n = A_e = A_e$)

$$\cancel{A} F_e u_e - \cancel{A}F_w u_w + \cancel{A}F_n u_n - \cancel{A}F_s u_s = -\left(\frac{P_E - P_P}{\cancel{\delta x}}\right)\cancel{A}\delta x + \frac{\cancel{A}}{Re}\left[\left(\frac{\partial u}{\partial x}\right)_e - \left(\frac{\partial u}{\partial x}\right)_w + \left(\frac{\partial u}{\partial y}\right)_n - \left(\frac{\partial u}{\partial y}\right)_s\right] + \cancel{S}\Delta V$$

$$F_e u_e - F_w u_w + F_n u_n - F_s u_s = -(P_E - P_P)\,\delta x + \frac{1}{Re}\left[\left(\frac{\partial u}{\partial x}\right)_e - \left(\frac{\partial u}{\partial x}\right)_w\right] + \frac{1}{Re}\left[\left(\frac{\partial u}{\partial y}\right)_n - \left(\frac{\partial u}{\partial y}\right)_s\right]$$

, $\boxed{h = \delta x = \delta y}$

central differencing scheme

$F_e = \frac{U_E + U_P}{2}$ $F_n = \frac{V_w + V_e}{2}$ (above at n)

$F_w = \frac{U_P + U_W}{2}$ $F_s = \frac{V_w + V_e}{2}$ (below at s)

$D_e = D_w = D_n = D_s = \frac{1}{h\,Re}$

$$\frac{1}{Re}\left(\frac{\partial u}{\partial x}\right)_e = \frac{1}{Re}\left(\frac{U_E - U_P}{\delta x}\right) \Rightarrow D_e\left[U_E - U_P\right]$$

$$\frac{1}{Re}\left(\frac{\partial u}{\partial x}\right)_w = D_w\left(U_P - U_W\right)$$

Similarly

$$\frac{1}{Re}\left(\frac{\partial u}{\partial y}\right)_n = D_n\left(U_N - U_P\right)$$

$$\frac{1}{Re}\left(\frac{du}{dy}\right)_s = D_s\left(U_P - U_S\right)$$

$$\boxed{a_P U_P = \sum a_{nb} U_{nb} + (P_P - P_E)\,\delta x + b}$$, $b = \bar{S}\Delta V \rightarrow$ Source term

$\boxed{b = S_P + S_u}$

General form of the momentum eq^n ~~after~~ over a control volume.

$$\boxed{a_P = a_W + a_E + a_S + a_N + (F_e - F_w) + (F_n - F_s) - S_P}$$

$\boxed{a_E = D_e - \frac{F_e}{2}}$, $\boxed{a_W = D_w + \frac{F_w}{2}}$, $\boxed{a_N = D_n - \frac{F_n}{2}}$, $\boxed{a_S = D_s + \frac{F_s}{2}}$

$S_P \Rightarrow$ Source term containing U_P. ,

Now, coefficients & source term changes depending upon the location of the cell.

(I) Interior points

(II) corner points

(III) Boundary points Top, Bottom (wall)

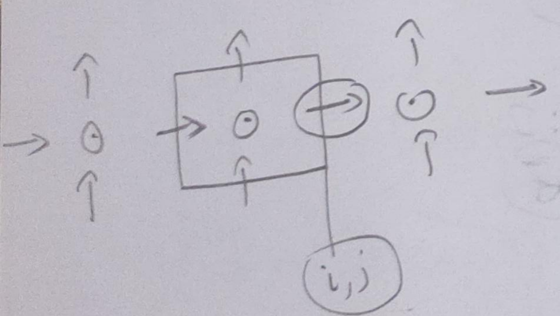
(IV) Boundary points Left & Right (wall).

(6)

U → Momentum eq^n rearranging, solving for $U_{i,j}^*$.

$$(U^*)_{ij} = \Big[\left(P^*_{ij} - P^*_{i,j+1}\right) h + (a_w)_{ij} U^*_{i,j-1} + (a_e)_{ij} U^*_{i,j+1} + (a_s)_{ij} U^*_{i-1,j} + (a_n)_{ij} U^*_{i+1,j} + (S_u)_{ij} \Big] \Big/ (a_p)_{ij}$$

① Interior points



$\left(i \in [2, n-1],\ j \in [2, n-2]\right)$ → indices.

Note $\boxed{S_u = 0\ ,\ S_p = 0}$ all other coefficients are non-zero.

Interior Points $[(i \in (2, n-1), j \in (2, n-2))]$ V momentum
y-axis x-axis

grid size
$$\boxed{h = dx = dy = \frac{L}{n}}$$

$$D_e = D_w = D_s = D_n = \frac{1}{(h)(Re)}$$

Coefficients (Central difference)

$$(f_w)_{ij} = \frac{1}{2}(u_{ij} + u_{ij-1})$$

$$(f_e)_{ij} = \frac{1}{2}(u_{i\,j+1} + u_{ij})$$

$$(f_s)_{ij} = \frac{1}{2}(V_{i-1,j+1} + V_{i-1,j})$$

$$(f_n)_{ij} = \frac{1}{2}(V_{i,j+1} + V_{ij})$$

$$(a_w)_{ij} = (D_w)_{ij} + \frac{1}{2} F_{w\,ij}$$

$$(a_e)_{ij} = (D_e)_{ij} - \frac{1}{2}(F_e)_{ij}$$

$$(a_s)_{ij} = (D_s)_{ij} + \frac{1}{2}(F_s)_{ij}$$

$$(a_n)_{ij} = (D_n)_{ij} - \frac{1}{2}(F_n)_{ij}$$

$$(S_P)_{ij} = 0 \; , \quad (S_u)_{ij} = 0$$

$$\boxed{(a_P)_{ij} = (a_w)_{ij} + (a_e)_{ij} + (a_s)_{ij} + (a_n)_{ij} + \underline{(F_e)_{ij} - (F_w)_{ij} + (F_n)_{ij} - (F_s)_{ij}} - \underline{(S_P)_{ij}}}$$

source term having u_{ij}.

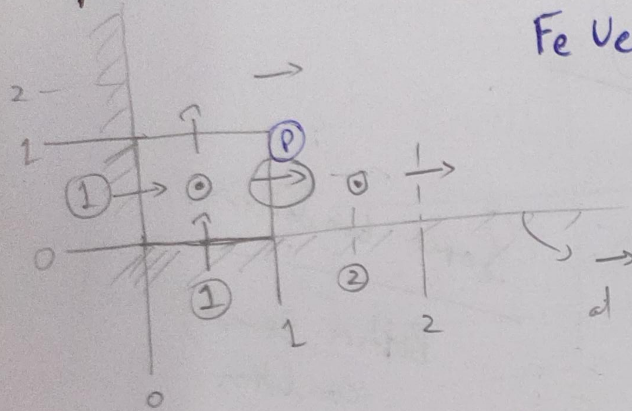
as guess value of U (u^*) will not satisfy the continuity condition initially

$(F_n - F_s)$ & $(F_e - F_w) \neq 0$

for u^*

Left Bottom Corner :→ $\boxed{U \text{ momentum}}$

$$F_e U_e - F_w U_w + F_n U_n - F_s U_s = (P_{12} - P_{11})\,\delta x + \frac{1}{Re}\left[\left(\frac{\partial u}{\partial x}\right)_e - \left(\frac{\partial u}{\partial x}\right)_w + \left(\frac{\partial u}{\partial y}\right)_n - \left(\frac{\partial u}{\partial y}\right)_s\right]$$



→ u is zero at Boundary

$$F_e\frac{(U_{12} + U_{11})}{2} - F_w\frac{(U_{11} + U_{10})}{2} + F_n\frac{(U_{21} + U_{11})}{2} - (S_u)_1 \cancel{+ S_u\,(\text{Bottom boundary})}$$

$$= (P_{12} - P_{11})\,\delta x + \frac{1}{Re}\left[\frac{U_{12} - U_{11}}{\delta x} - \frac{U_{11} - U_{10}}{\delta x} + \frac{U_{21} - U_{11}}{\delta y} - \frac{U_{11} - U_{boundary}}{\delta y/2}\right]$$

$$(P_{12} - P_{11})\,\delta x + D_e U_{12} + (S_u)_2 + D_n (U_{21}) - (U_{11})(D_e + D_w + D_s + D_n - S_p)$$

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from above → Left Bottom Corner

(I) $\boxed{F_s = 0 \;,\; D_s = 0 \;,\; S_p = -2D \;,\; S_u = 0}$

Similarly, solving for other Boundary conditions

(II) Bottom Boundary excluding corners

$\boxed{F_s = 0 \;,\; D_s = 0 \;,\; S_p = -2D \;,\; S_u = 0}$

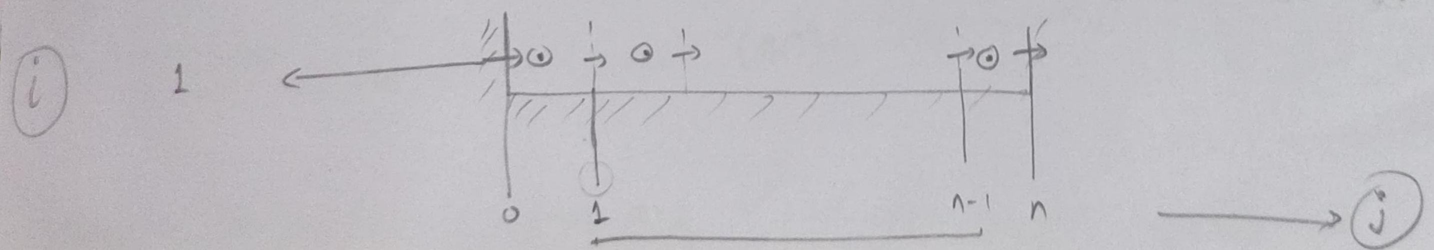
(III) Bottom Right corner

$\boxed{F_s = 0 \;,\; D_s = 0 \;,\; S_p = -2D \;,\; S_u = 0}$

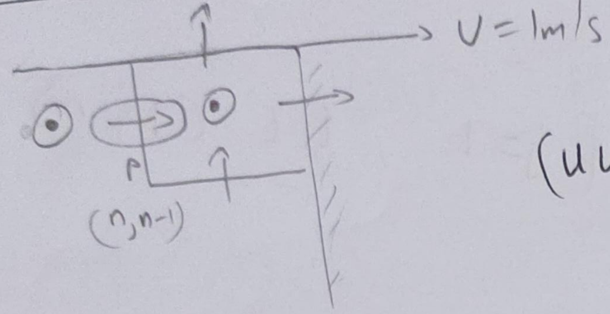
Same conditions.

Bottom Boundary condition

indices $(i = 1,\; j \in [1, n-1])$



Right Top Corner :- This Boundary condition Holds good for Top Boundary including both side corners.



$$(UU)_e - (UU)_w + (UV)_n - (UV)_s = (\Delta P)\delta x + \frac{1}{Re}\left[\left(\frac{du}{dx}\right)_e - \left(\frac{du}{dx}\right)_w\right] + \frac{1}{Re}\left[\left(\frac{du}{dy}\right)_n - \left(\frac{du}{dy}\right)_s\right].$$

$$F_e\left(\frac{U_{nn} + U_{nn+1}}{2}\right) - F_w\left(\frac{U_{nn-1} + U_{nn}}{2}\right) + \cancel{F_n(V} (Su)_1 - F_s\left(\frac{U_{n-1n} + U}{2}\right)$$

$$= (\Delta P)\delta x + D_e\left(\underline{U_{nn+1} - U_{nn}}\right) - D_w\left(\underline{U_{nn} - U_{nn-1}}\right) + S_p U_{nn} + (Su)_2 - D_s\left(\underline{U_{nn-1} - U_{n-1n-1}}\right)$$

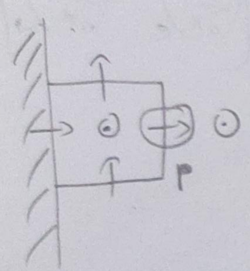
$$\boxed{F_n = 0,\quad D_n = 0,\quad S_p = -2D,\quad (Su)_2 = 2D,\quad (Su)_1 = 0}$$ (as V is zero at the top boundary $F_n = 0$)

Top Boundary coefficients.

$\left[(VU) = 0 \text{ at top boundary} \quad (VU)_n = 0\right]$

indices $\rightarrow$ $(i = n,\ j \in [1, n-1])$

Left Boundary (excluding corners)



$$\boxed{a_P U_P = \sum a_{nb} U_{nb} - \Delta P \, dx}$$

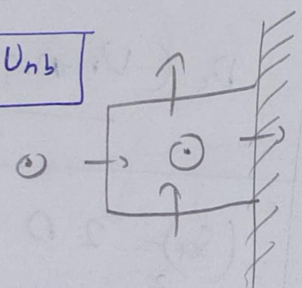
$\longrightarrow$ $[\, j = 1 \,,\; i \in (1, n) \,]$

no coefficient is zero

$S_u = 0 \quad , \quad S_p = 0$

Right Boundary (excluding corners)

$$\boxed{a_P U_P = -\Delta P \, dx + \sum a_{nb} U_{nb}}$$



$\longrightarrow$ $[\, j = n-1 \,,\; i \in (1, n) \,]$

no coefficient is zero

$$\boxed{S_p = 0 \quad , \quad S_u = 0}$$

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Momentum eq^n y-momentum eq^n (2D)

$$\frac{\partial}{\partial x}(\rho v u) + \frac{\partial}{\partial y}(\rho v v) = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\mu\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}\mu\frac{\partial v}{\partial y} + S_u$$

Non dimensionalizing

$$\frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(vv) = -\frac{\partial P}{\partial x} + \frac{1}{Re}\left(\frac{\partial}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}\frac{\partial v}{\partial y}\right) + S_u$$

Integrating differential non-dimensional momentum eq^n over control volume (CV)

(considering Area term as uniform ~~...~~ grids with same no. of grids)

($A_e = A_w = A_s = A_n = A$)

$$\cancel{A}\left[F_e v_e - F_w v_w + F_n v_n - F_s v_s\right] = -\frac{(P_N - P_P)}{8}\cancel{A}\,\delta y + \frac{A}{Re}\left[\left(\frac{dv}{\delta x}\right)_e - \left(\frac{dv}{\delta x}\right)_w + \left(\frac{\partial v}{\partial y}\right)_n - \left(\frac{dv}{dy}\right)_s\right]$$

$$+ \bar{S}_u \Delta V$$

$$\boxed{a_p v_p = -(P_N - P_P)\,\delta y + \sum a_{nb} v_{nb} + S_u}$$

$$\boxed{a_p = a_w + a_e + a_s + a_n + (F_e - F_w) + (F_n - F_s) - S_p},$$

$a_e = D_e - \frac{F_e}{2}$

$a_w = D_w + \frac{F_w}{2}$

$a_n = D_n - \frac{F_n}{2}$

$a_s = D_s + \frac{F_s}{2}$

$D_e = \frac{1}{Re\,dx}$; $D_w = \frac{1}{Re\,dx}$

$D_n = D_s = \frac{1}{Re\,dy}$

$$\boxed{dx = dy = h = \frac{1}{n}}$$

Interior points $\left[(i \in (2, n-2)), \; j \in (2, n-1) \right]$ y-momentum

y-axis x-axis

Coefficients (Central difference)

$$(f_w)_{ij} = \frac{1}{2}\left[u_{i+1,j-1} + u_{i,j-1} \right]$$

$$(f_e)_{ij} = \frac{1}{2}\left[u_{i+1,j} + u_{ij} \right]$$

$$(f_s)_{ij} = \frac{1}{2}\left(V_{ij} + V_{i-1j} \right]$$

$$(f_n)_{ij} = \frac{1}{2}\left(V_{i+1j} + V_{ij} \right]$$

$$(a_w)_{ij} = (D_w)_{ij} + \frac{1}{2}(F_w)_{ij}$$

$$(a_e)_{ij} = (D_e)_{ij} - \frac{1}{2}(F_e)_{ij}$$

$$(a_s)_{ij} = (D_s)_{ij} + \frac{1}{2}(F_s)_{ij}$$

$$(a_n)_{ij} = (D_n)_{ij} - \frac{1}{2}(F_n)_{ij}$$

$$(S_P)_{ij} = 0 \quad , \quad (S_u)_{ij} = 0$$

$$D_e = D_w = D_n = D_s = \frac{1}{h * Re}$$

$$\boxed{h = \frac{1}{n}}$$ n → no. of cells of scalar (P) (n=128) given

$$\boxed{\delta x = \delta y = h}$$

$$\boxed{(a_P)_{ij} V^*_{ij}} = (a_w)_{ij} V^*_{i-1j} + (a_e)_{ij} V^*_{i+1j} + (a_s)_{ij} V^*_{ij-1} + (a_n)_{ij} (V^*)_{ij+1} + (S_u)_{ij} + (P^*_{ij} - P^*_{i+1j})\,\delta y$$

$$(a_P)_{ij} = (a_w)_{ij} + (a_e)_{ij} + (a_s)_{ij} + (a_n)_{ij} + (F_e)_{ij} - (F_w)_{ij} + (F_n)_{ij} - (F_s)_{ij} - (S_P)_{ij}$$

Now we define p' as difference b/w correct pressure field p and guessed pressure field p^*, so that.

$$p = p^* + p'$$

Similarly, u' & v'

$$u = u^* + u'$$

$$v = v^* + v'$$

Substituting of the corrected pressure eqn into the momentum equation and Dropping $\Sigma a_{nb} u_{nb}$ terms we get

$$(a_p)_1 u' = (A)(p'_{i,j-1} - p_{i,j})$$

$$\Rightarrow u'_{ij} = d_{ij}(p'_{i,j-1} - p_{i,j}) \quad , \quad d_{ij} = \frac{A}{(a_p)_1}$$

$$(a_p)_2 v' = (A)(p_{i-1,j} - p_{i,j})$$

$$v' = d_{ij}(p_{i-1,j} - p_{i,j}) \quad , \quad d_{ij} = \frac{A}{(a_p)_2}$$

so $u_{i,j} = u^*_{ij} + d_{i,j}(p'_{i,j-1} - p'_{i,j})$

$$v_{i,j} = v^*_{i,j} + d_{i,j}(p'_{i-1,j} - p_{i,j})$$

Similarly,

$$u_{i,j+1} = u^*_{i,j+1} + d_{i,j+1}(p'_{i,j} - p_{i,j+1})$$

$$v_{i+1,j} = v^*_{i+1,j} + d_{i+1,j}(p_{i,j} - p_{i+1,j})$$

The velocity field is also subjected to the constraint that it should satisfy continuity equation.

$$\left((\rho u A)_{i,j+1} - (\rho u A)_{i,j}\right) + \left((\rho v A)_{i+1,j} - (\rho v A)_{i,j}\right) = 0$$

Solving this we get

$$a_p P'_p = a_w P'_w + a_e P'_e + a_n P'_n + a_s P'_s + b'$$

where

$$a_p = a_w + a_e + a_n + a_s$$

$$a_w = d = \frac{A}{a} \quad , \quad b = u^*_{i,j} - u^*_{i,j+1} + v^*_{i,j} - v^*_{i+1,j}$$

after solving for p'

$p^{new} = p^* + \alpha_p p'$, $\alpha_p \rightarrow$ relaxation factor

$u^{new} = \alpha_u u + (1-\alpha_u) u^{n-1}$, $\alpha_u \rightarrow$ under relaxation factor

$v^{new} = \alpha_v V + (1-\alpha_v) v^{n-1}$, $\alpha_v \rightarrow$ under relaxation factor

#

P_dash[i][j] = [aw(p_dash[i][j-1]) + ae(P_dash[i-1][j]) + an(P_dash[i+1][j]) + bij]/aa

a_w = h/a[0][i][j-1];

a_e = h/a[0][i][j];

a_s = h/a[1][i-1][j];

a_n = h/a[1][i][j];

bij = (u_star[i][j-1] - u_star[i][j] + v_star[i-1][j] - v_star[i][j])

aa = aw + ae + as + an

a_w, a_e, a_s, a_n will be zero according to different boundaries