Measurement based Quantum Computing and Error Correction*

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This paper provides a brief summary on an alternate way to implement Universal quantum computing namely Measurement based quantum computing. This consist entirely of 1 qubit measurement on a particular type of entangled states called cluster states. We discuss the universality and describe how it is computationally different from the gate based quantum computing. We also discuss its extension to 3D cluster state as well as its error correction capabilities.

I. INTRODUCTION

Various frameworks, like the quantum Turing machine and the quantum logic network model, have been used to formulate quantum computation.

We have demonstrated that one-qubit observations on a particular type of highly entangled multi-qubit states, known as cluster states, can serve as the foundation for universal quantum computation. A cluster state serves as a resource for quantum computation in this system, and the program is made up of the collection of measurements. Since the entanglement in a cluster state is destroyed by one-qubit measurements, this technique is known as the "one-way quantum computer." As a result, the cluster state can only be used once. In the following, we shall adopt the acronym QC_C, which stands for "one-way quantum computer," to emphasize the significance of the cluster state for the system.

Because it can effectively replicate any unitary quantum logic network, QC_C is universal. Thus, QC_C can be thought of as a quantum logic network simulator. Since the network model is the most popular way to describe a quantum computer, it is necessary to explain how it relates to QC_C.

The purpose of this report is three-fold. First, it will give a proof of universality of QC_C; second, it would be extension of the formulation of cluster states to 3-dimensional dimension, and lastly, it would be to study the error correction capability of 3-D cluster state.

II. CLUSTER STATE AND ITS FORMATION

Cluster states are pure quantum states of two-level systems (qubits) located on a cluster C. This cluster is a connected subset of a simple cubic lattice Z^d in d ≥ 1 dimension. The cluster states $|\phi\rangle_C$ obey the set of eigenvalue equations

$$K^{(a)}|\phi\rangle_C = (-1)^{\kappa_a}|\phi\rangle_C, \tag{1}$$

with the correlation operators

$$K^{(a)} = \sigma_x^{(a)} \bigotimes_{b\in \mathrm{nbgh}(a)} \sigma_z^{(b)}. \tag{2}$$

κ_a is a constant taking either 0 or 1 value. For our case study we take $\kappa_a = 0$ for all arbitrary sites **a** and and $nbgh(a)$ is the set of all neighboring lattice sites of **a**. Now $K^{(a)}$ acts as a stabilizer.

A. Formation of Cluster State

When a unitary gate U is applied to a state $|\psi\rangle$, the outcome is $U|\psi>$. The new state $U|\psi>$ can be expressed as $UMU^{\dagger}U|\psi\ >$, suggesting that $U|\psi>$ is an eigenstate of UMU, if $|\psi>$ is an eigenstate of M. Thus, we can simply manipulate a list of stabilizers to follow the effect of gates. The controlled-Z gate CZ, which satisfies $CZ^{\dagger} = CZ$, $CZ(I \bigotimes X)CZ = Z \bigotimes X$ and $CZ(X \bigotimes I)CZ = X \bigotimes Z$ will be of special interest. Multiplying these relations yields the action of CZ on any other stabilizer.

A step by step approach to construct a cluster state or more broadly a graph state is:

- Initialize all qubits to $|+>$ state. $|\psi>=|+>^{\bigotimes n}$ where n is the number of qubits.
- Stabilizers of the state $|\psi\ >$ are of the form XII......,IXI......,IIXII....... etc
- Apply the CZ gate between two neighboring qubits to generate the stabilizer of the form XZII.......,ZXZII......,IZXZII...... etc

Here we have generated a 1-D cluster state. The same procedure can be used to generate a higher-dimension cluster state. Here we note that each X is surrounded by a Z satisfying the form of the equation (2) mentioned above.

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* A footnote to the article title
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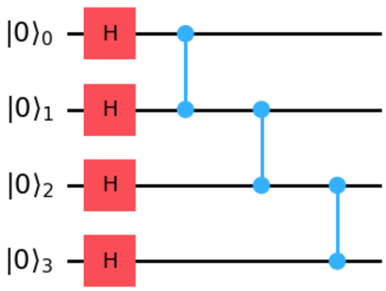


FIG. 1: State preparation of a 1-D 4 qubit cluster state.

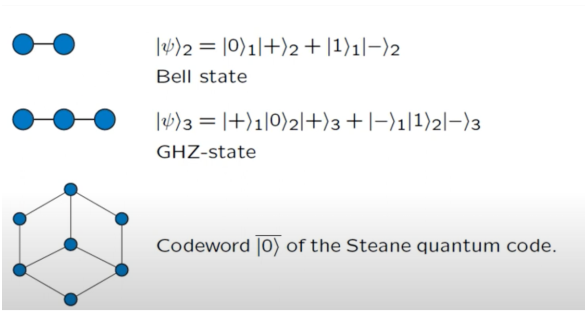


FIG. 2: Examples of cluster states

III. UNIVERSALITY PROOF OF QUANTUM COMPUTATION VIA ONE-QUBIT-MEASUREMENT

The approach taken to prove universality is simply showing that any quantum network or gate model can be easily implemented using QC_C. This is done by constructing a CNOT gate as well as showing the capability to produce any random 1-qubit unitary as these for a complete group to construct any unitary gate. We also show how idling as well as how unwanted qubit can be removed from a cluster state using 1 qubit measurements.

A. Removal of redundant qubits from Cluster state

While constructing a circuit on a cluster state there will always be some cluster qubits which are not required for the circuit realization. These cluster qubits are called redundant qubits.
In the description of the QCC as a quantum logic network, the first step of each computation will be to remove these redundant cluster qubits. Fortunately, the situation deosnot require us to physically remove these qubits from the lattice. To make them ineffective to the realized circuit, it suffices to measure each of them in the σ_z-eigenbasis. In this way, one is left with an entangled quantum state on the cluster C_N of the unmeasured

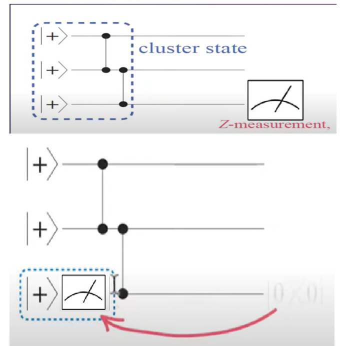


FIG. 3: Showcasing how a Z measurement removes the redundant qubits from cluster states

qubits and a product state on $C \backslash C_N$.

$$|\phi\rangle_C \longrightarrow |Z\rangle_{C\backslash C_N} \otimes |\phi'\rangle_{C_N}$$

where

$$|Z\rangle_{C\backslash C_N} = \left(\bigotimes_{i \in C\backslash C_N} |s_i\rangle_{i,z} \right)$$

and s_i are the results of the σ_z-measurements. Using the principle of deferred measurement we can see that applying measurement in $\sigma_z$basis on any site A would result in :

$$|\phi'\rangle = \bigotimes_{b \in nbgh(a)} (\sigma_z^{(b)})^{M_z(a)} |\phi\rangle_{C_N}$$

where $Mz(a) \in \{0, 1\}$ is the measurement outcome at the site A. We can see that post measurement the cluster state remains the same with some addition σ_Z operator acting on the neighboring cluster state qubits depending on the measurement outcome.

B. Teleportation/Identity in Cluster State

Idling or Identity in cluster state means leaving the cluster state as it is for future computation with disturbing the entanglement. What cluster state also allow us to do is teleport the state $|\psi >$ initialized on a cluster qubit to other cluster qubits by simply performing 1-qubit X measurements. This is analogous to a classical wire carrying the same information over a long distance connected to circuit wherever the information is required.

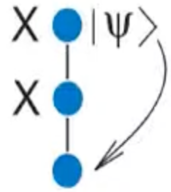
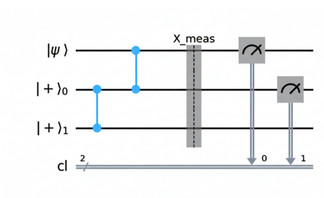
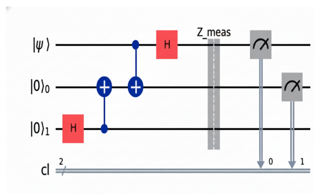


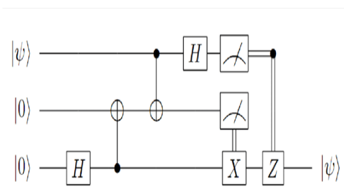
FIG. 4: A 1-D cluster state where the state $|\psi>$ has been teleported to the 3rd qubit



(a) Quantum teleportation circuit on cluster state



(b) Equivalent teleportation circuit



(c) Standard Quantum teleportation protocol

FIG. 5: An implementation of quantum teleportation performed on 1-D cluster. These circuits are equivalent with some subtle differences

Simple gate manipulations using the hadamard gate prove the equivalence of circuits (a) and (b) in FIG. 5. We note that circuit (b) and circuit (c) are nearly identical except an addition X and Z gate in the standard quantum teleportation circuit. These gates can be accounted for in QC_C given the measurement outcome we get after measuring the qubit $|\psi\rangle$ and qubit $|0\rangle_0$ in the Z basis.
From this we can conclude that the QC_C model can replicate a standard quantum teleportation circuit to an unitary equivalence by simply measuring the qubits in cluster state in X basis

C. Random Unitary Gate implementation

The penultimate step to prove the universality of QC_C would be to be able to implement any random unitary matrix. To do this we take following approach:

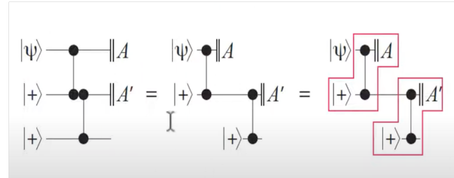


FIG. 6: 3 qubit cluster state acting as half teleportation concatenated circuit. The marked part represent a half teleportation circuit each

- We simply start with the cluster state and initialize one of the cluster qubit to a state $|\psi\rangle$. Without loss of generality we start with 3 qubits in cluster state.
- We measure the 1st in a random basis defined by A and the 2nd qubit in A' basis.
- We note that this circuit is unitary equivalent to 2 half teleportation circuit.
- The final state is obtained in the 3rd qubit.

Here we define A as $U_z^\dagger X U_z$ and A' as ${U'}_z^\dagger X U'_z$. Here $U_z = exp(i\alpha z)$ and $U'_z = exp(i\alpha' z)$ are random rotation about the the Z axis by an angle 2α and $2\alpha'$ respectively. We can now breakdown the working of these 2 half teleportation circuit. The upper teleportation circuit is equivalent to a simple measurement in X basis with the Unitary U_z acting on the state $|\psi\rangle$. Using a simple fact that U_Z gate commute with the CZ gate. The output obtained in the 2nd qubit is

$$|\psi'\rangle = (HZ_sU_z)|\psi\rangle$$

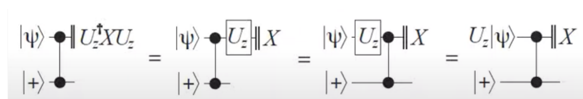


FIG. 7: Half teleportation circuit working in cluster state

Doing the same procedure with the bottom teleportation circuit we obtain the final output in the 3rd qubit as

$$|\psi''\rangle = (HZ'_sU'_z)(HZ_sU_z)|\psi\rangle$$

Using the fact the $I = HH$ and inputting it in between Z'_s and U'_z. We can simply the equation to

$$|\psi''\rangle = X'_sU'_xZ_sU_z|\psi\rangle$$

This addition X'_s and Z'_s gate are 1st and 2nd qubit measurement output dependent. Removing them will be covered later on. After removal of X and Z from the $|\psi''\rangle$ equation ,it is equivalent to any random unitary(by varying the parameter α and α')acting on the state $|\psi\rangle$. This proves that by measuring the qubits in cluster state in random basis A and A' is equivalent to implement a random unitary to the state $|\psi\rangle$.

D. Countering Randomness

While deriving the proof of the construction of any unitary gate in the QC_C formulation we encountered a problem, random X and Z gates which were output dependent.
As the QC_C formulation is entirely based on projective measurement, the output is not in our control. But what we can do is modify the basis we measure the next qubit in to counter this randomness.

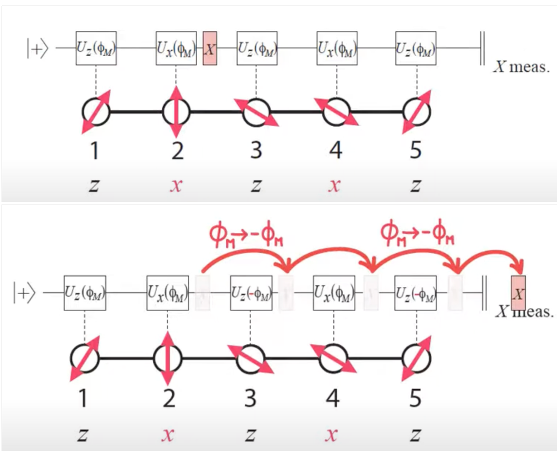


FIG. 8: Propagation of byproduct operators causes Unitary operator angles to flip in sign

Let us start with a simple case where the output of many multiple half teleportation circuit is of the form as shown in FIG. 8. What we observe is an extra X gate in the 2nd half teleportation circuit due to X measurement in the 2nd qubit.
What we wish to get rid of this extra X gate. To do this we use the fact that X and Z anti-commute. So interchanging them would introduce a -ve sign. Interchanging X and $U_z(\phi_m)$ result in $U_z(-\phi_m)$ followed by X. Keep following the same procedure wherever there is a $U_z(\phi_m)$ further down the circuit. A similar process can be done wherever we get an extra Z gate, just interchange the $U_x(\phi_m)$ by $U_x(-\phi_m)$. In essence propagating the byproduct X and Z gates down the circuit has the effect of flipping the angles of some of the unitary operators. Since we know the measurement outcomes beforehand and hence have info about the byproduct operators we can apriori flip the sign of those unitary gates to get our desired outcome.

E. CNOT gate

Implementing the CNOT gate in QC_C would finally prove its universality.

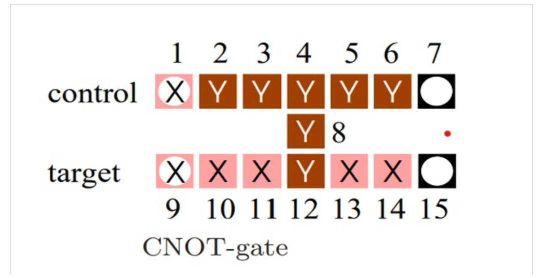


FIG. 9: CNOT gate implementation in QC_C framework

Fig. 9 show the QC_C implementation of a CNOT gate. It requires 15 qubits to implement a single CNOT gate. The procedure to realize the CNOT gate acting on the state $|\psi\rangle$ is as follows:

- Start with a state

$$|\psi_{in}\rangle_{C_{15}} = |\psi_{in}\rangle_{1,9} \otimes (\bigotimes_{i \in C_{15}\setminus\{1,9\}} + \; |+\rangle_i)$$

- Apply the CZ gate between all the qubits to form a cluster state
- Measure all the qubits of C_{15} except for the qubits 7,15(according to the Fig. 9). Qubits 1, 9, 10, 11, 13, 14 are measured in the σ_x eigen basis and qubits 2,6,8, 12 in the σ_y eigen basis.

According to the measurement result the following circuit is obtained:

$$U'_{\mathrm{CNOT}} = U_{\Sigma,\mathrm{CNOT}}\, \mathrm{CNOT}(c,t).$$

Therein the byproduct operator $U_{\Sigma,\mathrm{CNOT}}$ has the form

$$U_{\Sigma,\mathrm{CNOT}} = \sigma_x^{(c)\gamma_x^{(c)}} \sigma_x^{(t)\gamma_x^{(t)}} \sigma_z^{(c)\gamma_z^{(c)}} \sigma_z^{(t)\gamma_z^{(t)}}, \text{ with}$$

$$\begin{aligned}
\gamma_x^{(c)} &= s_2 + s_3 + s_5 + s_6 \\
\gamma_x^{(t)} &= s_2 + s_3 + s_8 + s_{10} + s_{12} + s_{14} \\
\gamma_z^{(c)} &= s_1 + s_3 + s_4 + s_5 + s_8 + s_9 + s_{11} + 1 \\
\gamma_z^{(t)} &= s_9 + s_{11} + s_{13}.
\end{aligned}$$

Here the s_i represent the measurement outcome obtained after measuring the ith qubit.

IV. 3D CLUSTER STATES

A cluster state can be expanded to a 3-D cubic lattice. A wire frame cubic lattice as shown in Fig 10 (a) can be superimposed over the cluster states qubits. The Lattice also called the "Primal Lattice" contains qubits in the edges and faces of this cubic cell. We can also define the "dual" lattice of the initial lattice as shown in Fig 10(b) which is also a cubic lattice wireframe with the difference that all edge qubits of primal lattice are face qubits of dual lattice and vice verse. The stabilizer operators are defined in 3D clusters using the same formula for 1D and 2D clusters. The formation of 3D cluster states also follows a procedure similar to that discussed in Section II.

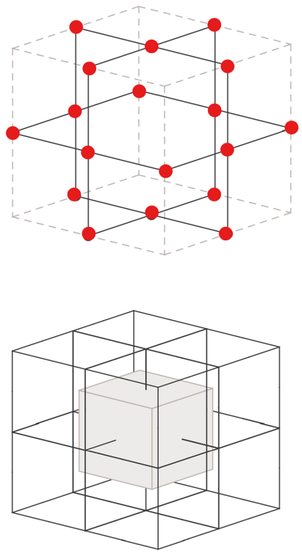


FIG. 10: 3D cluster state primal and dual cells

A. Defects and Logical qubits

Defects in cluster states are 1 chains of qubits measured on the Z basis, thus removing them from the cluster and leaving a hole in the cluster lattice.
Defects consisting of face qubits of the primal lattice are called primal defects, and those going from faces of dual qubits are called dual defects.
A pair of such non-intersecting defects of the same type constitutes a single logical qubit. Hence, logical qubits can also be primal or dual.

B. Logical States and Gates

We define a primal qubit in the $|+\rangle_L$ state if in a single even time step it is in the simultaneous +1 eigenstate of each of the two operators consisting of a ring of single qubit Z operators encircling and on the boundary of each defect.
Similarly, the $|-\rangle_L$ state is defined by simultaneous -1 eigenstate of the same.

This formulation allows us to define the Logical X operator as well. X_L is defined by just a ring of Z operators on either one of the defects. We can easily see that $X_L|-\rangle_L = (-1)|-\rangle_L$ and $X_L|+\rangle_L = |+\rangle_L$ as a single Z operator ring applies a phase of +1 or -1 on the qubit depending on the particular defect.

The Z_L operator is defined by Z operators on the 1 chain joining the primal side faces of both defects as shown in Fig.11

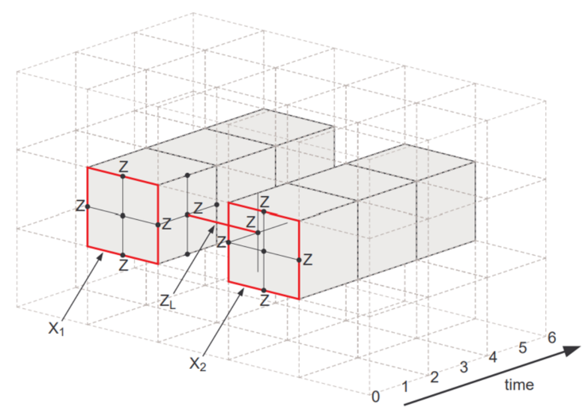


FIG. 11: X_L and Z_L operators on Logical qubit: X1 or X2 in isolation works as X_L operator and applying both acts as an identity on the logical qubit

To initialize a pair of defects in $|+\rangle_L$ state we use X basis measurements in the surrounding qubits enclosing the ring of Z operators we intend to apply as illustrated in Fig 10. This in effect creates a state in:
$|\psi\rangle = Z_1^{s_1} Z_2^{s_2} |+\rangle_L$
where s1 and s2 are the parity of the X measurement outcomes enclosing each of the defects.

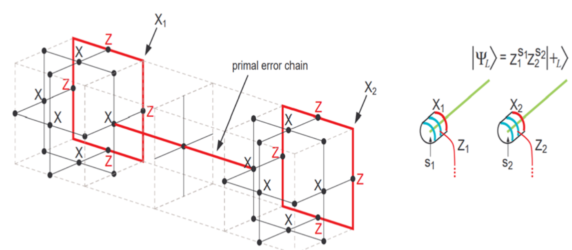


FIG. 12: Initialization to $|+\rangle_L$ state using X basis measurements on the qubits marked as black

V. EFFECT OF MEASUREMENTS ON A 3D CLUSTER

From the definition of cluster state we know that:
$\sigma_x^{(a)} \bigotimes_{b\in \mathrm{nbgh}(a)} \sigma_z^{(b)} |\psi\rangle_C = |\psi\rangle_C$

Now consider the case where we did a X basis measurement on site 'a' of $|\psi\rangle_c$ and get an outcome $M_x \in 0, 1$. After the measurement the following identity holds:
$(-1)^{M_x} \bigotimes_{b\in \mathrm{nbgh}(a)} \sigma_z^{(b)} |\psi\rangle_{C\backslash a} = |\psi\rangle_{C\backslash a}$

Hence we see that if we want to apply σ_z operators on the neighbors of site 'a' in the cluster state we can have an equivalent effect by measuring the site 'a' in the X basis and accounting for the measurement dependent phase $(-1)^{M_x}$. This idea could be extended to multiple X measurements as well where the phase depends on the parity of all the X measurements made and the equivalent σ_z operators act on the neighbors of all the sites of measurements.

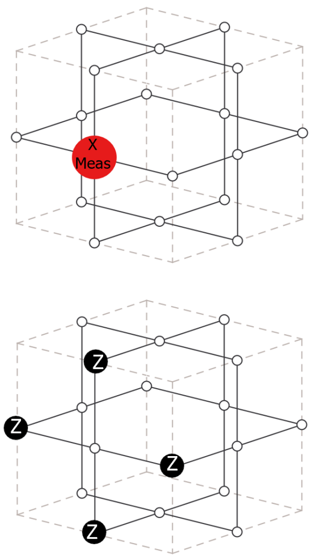


FIG. 13: Illustration of how X measurement is equivalent to applying σ_z on neighbors

A. Syndrome Measurement for Lattice cell

Using the previous discussion lets take a case where we measure each of the six face qubits of primal/dual cell in X basis. Since each edge qubit shares exactly two face qubits all the σ_z operators cancel out giving the final relation as:
$|\psi'\rangle_C = (-1)^S|\psi'\rangle_C$
Where S: parity of the six measurement outcomes of the six X measurements.

Hence from this we can conclude that if the cluster state has no error in either its qubits or its measurements for the above equation to hold S = 0
S acts as the syndrome of the entire lattice cell:
No error cell $\implies$ Even parity (S = 0)
Error cell $\implies$ Odd parity (S = 1)

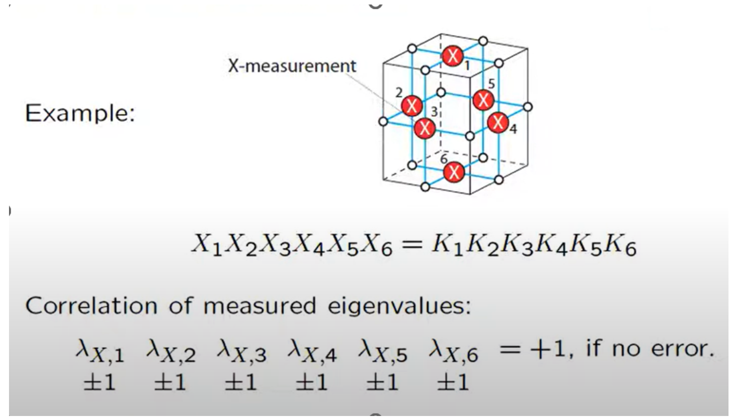


FIG. 14: Syndrome measurement on lattice cell

VI. ERROR DETECTION AND CORRECTION IN 3D CLUSTER STATES

A. Detection

From the syndrome measurements of the lattice cells we can identify cells which have odd parity and flag them as error cells. If two error cells share a common edge we can identify the qubit which must have contributed to the error in measurements as demonstrated in Fig.15

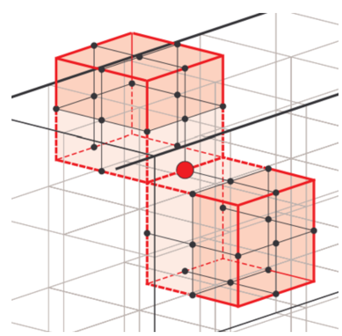


FIG. 15: Error qubit is demonstrated in red

In case of 1 chains of error qubits the cells on the ends of the chain are the one's which show odd parity as the cells with two error qubits show even parity and aren't flagged.
In such a case we use Minimum weight matching algorithm to find the minimum number of error qubits connecting the error cells. The minimum weight matching algorithm takes a weighted graph with an even number of vertices and produces a spanning list of disjoint edges with the property that no other list has lower total weight. The cells with odd parity become half the vertices we will feed into the algorithm. For every vertex in this list we add a vertex corresponding to the nearest point on the nearest primal boundary.

If we group all X measurement outcomes considering the primal cells it is possible to run into cases where the error cells aren't detectable as shown in Fig.17. To fix this we also group the measurement outcomes to find the parity of the dual lattice. This has the effect to make the edge qubits as the face qubits and face to edge qubits in the dual picture hence the undetectable error lattice cells in the primal picture become two or more detectable lattice cells in the dual picture.

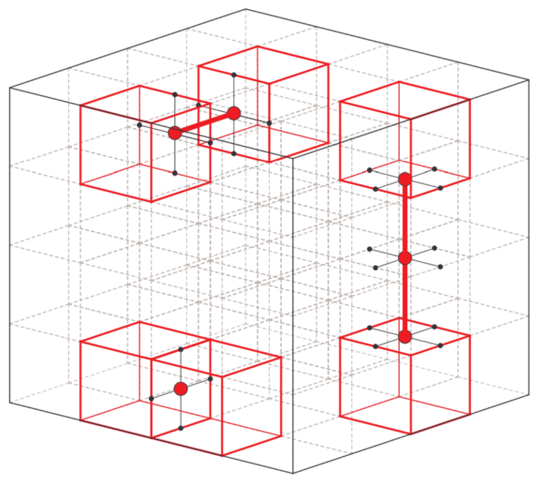


FIG. 16: Minimum weight matching of error cells

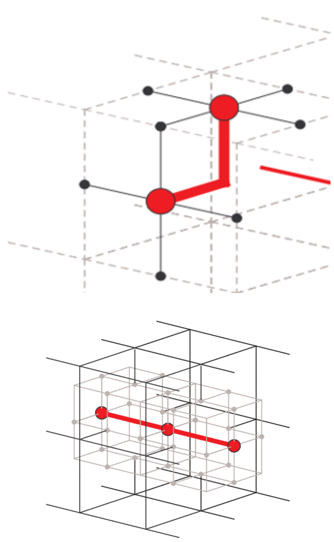


FIG. 17: Illustration of undetectable error in primal lattice which become detectable in dual lattice and vice versa

B. Correction

The error correction scheme for Cluster states is quite simple. The measurements collapses the qubits to one of the eigenstates of that measurement and takes the qubit out of the remaining cluster state. Hence once we detect the measurement outcomes the only effect they have is on the byproduct states during computation which depeneds on the parity of the measured qubits. Hence once we know which qubits had errors we can just flip their measurement outcomes that we recorded and restore the true parity information and the true byproduct states.

VII. CONCLUSION

We have provided a thorough explanation of the one-way quantum computer in this paper. In this manner, we provided the universality proof and explained how the QC_c relates to the network model of quantum computation. The correlations displayed by cluster states and states that can be formed from them under one-qubit measurements serve as the foundation for our explanation. This paper's primary goal has been to give a thorough explanation of the QC_C from a network standpoint. We have also described fault-tolerant initialization of logical states a and logical operators and demonstrated how a particular 3-D cluster state can be used to carry out general error correction even if it only directly detects Z and M_X mistakes.

VIII. SUMMARY OF CONTRIBUTION

Section 1,2,3 has been written by Purujeet Yadav and section 4,5,6 has been written by Abhijeet Bhatta. The content of all the sections has been compiled by both.

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