Bounded above / Bounded below:

- Let $\{a_n\}$ be a sequence.

  Number U is known as the upper bound of $\{a_n\}$ if for any $n \in \mathbb{N}$,

  $$\boxed{a_n \leq U}$$

- Let $\{a_n\}$ be an increasing sequence.

  Let U be an upper ound of $\{a_n\}$ with following property.

  Property: for any infinitesimly small number ϵ, $U-\epsilon$ is not an upper bound of $\{a_n\}$.

  [In other words, for any number even fractionally or infinitesimly smaller than U, typically expressed as $U-\epsilon$, you can find natural number N_0, such that

  for all $N \geq N_0$

  $$U - \epsilon < a_N \leq U$$]

  then U is called as a limit of sequence $\{a_n\}$

$$\boxed{\lim \{a_n\} = 0}$$

Let $\{b_n\}$ be a decreasing sequence.
Let L be a lowerbound
[for all $n \in \mathbb{N}$, $a_n \geq L$]
If L has a following property,
for any $\epsilon > 0$, there exists n_0
such that $n \geq n_0 \Rightarrow$.

$$L \leq b_n < L + \epsilon.$$

then

$$\lim \{b_n\} = L.$$

$a_n = \frac{1}{n}$ decreasing seq.
0 = lower bound
$\underline{1}$ = lower bound

$\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}$,

$$0 \leq \frac{1}{n} < 0 + \epsilon, \quad \forall n \geq n_0$$

$\lim \left\{\frac{1}{n}\right\} = 0.$

$$\frac{n^2}{2+n^2}$$

$n=1 \quad \frac{1^2}{2+1^2} = \frac{1}{3}$

$n=2 \quad \frac{2^2}{2+2^2} = \frac{4}{6} = \frac{2}{3}$

$\frac{n^2}{2+n^2}$ increasing!

$n=3 - \frac{3^2}{2+3^2} = \frac{9}{11}$

$n=4 \quad \frac{4^2}{2+4^2} = \frac{16}{18} = \frac{8}{9}$

$\frac{9}{11} \quad \frac{8}{9} \qquad \frac{n^2}{2+n^2}$

$8 \times 9 \quad \underline{88}$

$$\frac{}{99} \quad \frac{}{99}$$

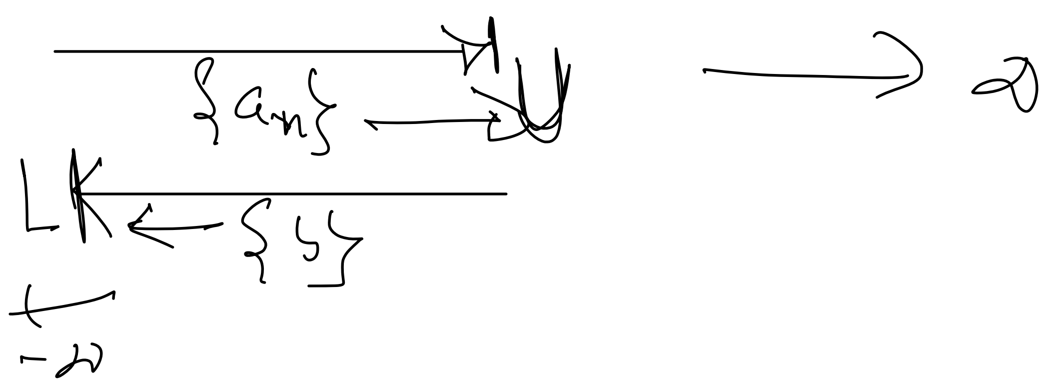
(1)

$$\frac{81}{99} \quad \frac{88}{99}$$

$$1-\epsilon < \frac{n^2}{1+n^2} <$$

$$\boxed{\lim \frac{n^2}{1+n^2} = 1}$$

Let $\{a_n\}$ be a seq.
if $\lim \{a_n\}$ exists in real number set then $\{a_n\}$ is a convergent seq. else $\{a_n\}$ is a divergent seq.



Timing Complexity

$T(n)$ where $n \in \mathbb{N}$.

$\{T_n\} \to$ Seq.

Continuously increasing seq. with no upper bound

$\lim \{T_n\} \to \infty$ $O(1)$

$\leq (1)$

How can we talk about the upper or lower bounds of such seq.?

$\boxed{T(N)}$

$\longrightarrow \infty$

$\longrightarrow$ function!

$\boxed{\text{(standard function)}}$.

$$\boxed{T(n) = An^2 + Bn + C.}$$

$X(n) =$ Something

$T(n) \longrightarrow \infty.$

Define upper bound of $T(n)$ w.r.t. some other function $X(n)$.

Let $f(n)$ be a function. Domain $\equiv \mathbb{N}$.
Let $g(n)$ be another function Domain $\equiv \mathbb{N}$.

if there exists $n_0 \in \mathbb{N}$, and $c \in \mathbb{R}$ such that

$$n \geq n_0 \Rightarrow f(n) \leq c \cdot g(n).$$

then $f(n)$ is bounded above w.r.t. $g(n)$.
or $g(n)$ is asymptotically upper bound of $f(n)$.

$$3n^2 + 7n - 2 = T(n).$$

$C = 4.$

$g(n) = 4n^2.$

n	$4n^2$	$3n^2+7n-2.$
1	4	8
2	16	$12+14-2=24.$

3	36	$27+21-2=44$
4	64	$48+28-2=74$
5	100	$75+35-2=108$
6	144	$108+42-2=148$
7 (NS)	196	$196-2=192$
8	256	$192+56-2=246$
	42	

$n \geq 7, \quad 4n^2 \geq 3n^2+7n-2$

$\boxed{n_0=7, \quad c=4}$

$4n^2$ is asymptotically U.B. of $3n^2+7n-2$

- Let $f(n)$ and $g(n)$ be functions from $\mathbb{N} \to \mathbb{R}$.
- $g(n)$ is an asymptotic upper bound

of $f(n)$ if we can find n_0 and c such that

$f(n) \leq c \cdot g(n)$ for all $n \geq n_0$

Asymptotic lower bound.

$g(n)$ is asymptotic lower bound of $f(n)$.

---

n^2 is asymptotic U.B. of $3n^2+7n-2$.

$n_0 = 7$, $c = 4$.

$4 \cdot n^2 \geq 3n^2 + 7n - 2$, for all $n \geq 7$

Let $f(n)$ and $g(n)$ be two functions from $\mathbb{N} \to \mathbb{R}$. then $g(n)$ is asymptotic lower bound of $f(n)$ if there exist c and n_0 such that:

$c \cdot g(n) \leq f(n)$ for all $n \geq n_0$.

$f(n) = 3n^2 + 7n - 2.$

$g(n) = \underline{\underline{n^2}}.$

$\boxed{c = 2}$

n	$f(n)$	$2 \cdot g(n)$
1	8	$2 \cdot 1^2 = 2.$
2	24	$2 \cdot 2^2 = 8$
3	44	$2 \cdot 3^2 = 18$
4	74	$2 \cdot 4^2 = 32$
		etc

$n = 1, \; c = 2.$

$f(n) = 3n^2 + 7n - 2.$

$\geq 2 \cdot g(n) = 2 \cdot n^2.$

whenever $n \geq 1 \; (= n_0).$

$\boxed{n_0 = 1}$ $\boxed{c = 2}$

$O(g(n)) = \{ f(n)$: there exists c and n_0 such that

$0 \leq f(n) \leq c \cdot g(n)$

whenever $n \geq n_0$

$\}$

$\Omega(g(n)) = \{ f(n)$: there exist c and n_0 such that

$0 \leq c \cdot g(n) \leq f(n)$

whenever $n \geq n_0$

$\}$

$O(g(n))$ $\qquad$ $g(n) = n^3$

$g(n) = n^2$

$O(n^2)$ $\qquad$ $O(n^3)$

$O(n^2)$ is a collection of all functions $f(n)$ whose asymptotic upper bound is $\underline{n^2}$.

i.e. if $f(n)$ is such that

there exist c, n_0, for which

$0 \leq f(n) \leq c \cdot n^2$ whenever $n \geq n_0$

is true then,

$f(n) \in O(n^2)$

$O(g(n))$ = Set of all functions whose asymptotic upper bound is $g(n)$.

$\boxed{3n^2 + 7n - 2}$ $\boxed{n^2}$. n^3 n^4.

$\boxed{C \cdot n^2}$

$4n^2$

for any C cn^3

$\boxed{C = 1}$

$\boxed{3n^2 + 7n - 2}$ $\boxed{1 \cdot n^3}$

27

$c = 1, \; n_0 = 5$

$$f(n) = 3n^2 + 7n - 2 \begin{cases} \to \underline{\underline{n}}^2 = g_1(n) \\ \to \underline{\underline{n}}^3 = g_2(n) \end{cases}$$

for some c, we can find n_0.

$0 \leq f(n) \leq c \cdot n^2$ whenever $n \geq n_0$.

for any c, we can find n_0

$0 \leq f(n) \leq c \cdot n^3$ whenever $n \geq n_0$.

$f(n)$ is U.B. $\underline{\underline{g(n)}}$,

for some c, $f(n) \leq c \cdot g(n)$,

$g(n)$ asymptotic Tight U.B.

$g(n)$ asymptotic U.B.

$g(n) = n^2$ Asymptotic tight U.B.
of $f(n) = 3n^2 + 7n - 2$.

---

for $f(n) = 3n^2 + 7n - 2$, $g(n) = n^2$ is
lower bound.
for some c, we can find n_0,
$0 \leq c \cdot n^2 < f(n) = 3n^2 + 7n - 2$
whenever $n \geq n_0$

$g(n) = n$.

for any c, $(c = 50)$:
$3n^2 + 7n - 2 \geq 50 \cdot n$, after some

n_0.

$f(n) = 3n^2+7n-2$

$g_1(n) = n^2$, for some c, there exists n_0,

$0 \le c n^2 < 3n^2+7n-2$

$g_2(n) = n$.

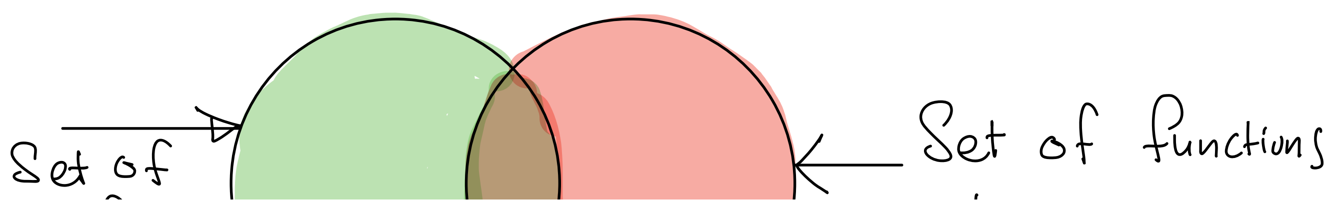
for any c, there exists n_0 such that

$0 \le c \cdot n < 3n^2+7n-2$

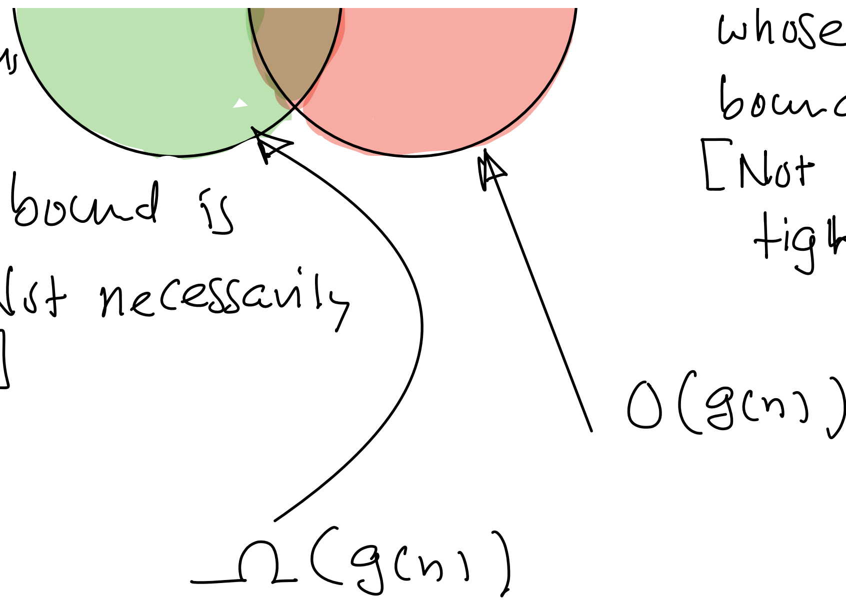
$g_1(n)$ is Asymptotic Tight L.B. of $f(n)$

$g_2(n)$ is Not Asymptotic Tight but only Asymptotic L.B.

Let $g(n)$ be a function



all functions whose lower bound is $g(n)$. [Not necessarily tight]



whose upper bound is $g(n)$ [Not necessarily tight]

Big-oh$(g(n))$ = Set of all functions whose upper bound is $g(n)$ [Not necessarily tight]

Big-omega$(g(n))$ = Set of all functions whose lower bound is $g(n)$ [Not necessarily tight]

$g(n) = n^2$.

$f(n) = 3n^2 + 7n - 2$.

$f(n) \in O(n^2)$ and $f(n) \in \Omega(n^2)$.

$f_1(n) = 3n + 2$.

$f_1(n) \in O(n^2)$

$f_1(n) \notin \Omega(n^2)$.

$f_2(n) = 4n^3 - 7$. $\quad f_2(n) \in \Omega(n^2)$

for any c.

$0 \le cn^2 \le 4n^3 - 7$, for all $n \ge n_0$

n^2 is a lower bound of $4n^3 - 7$.

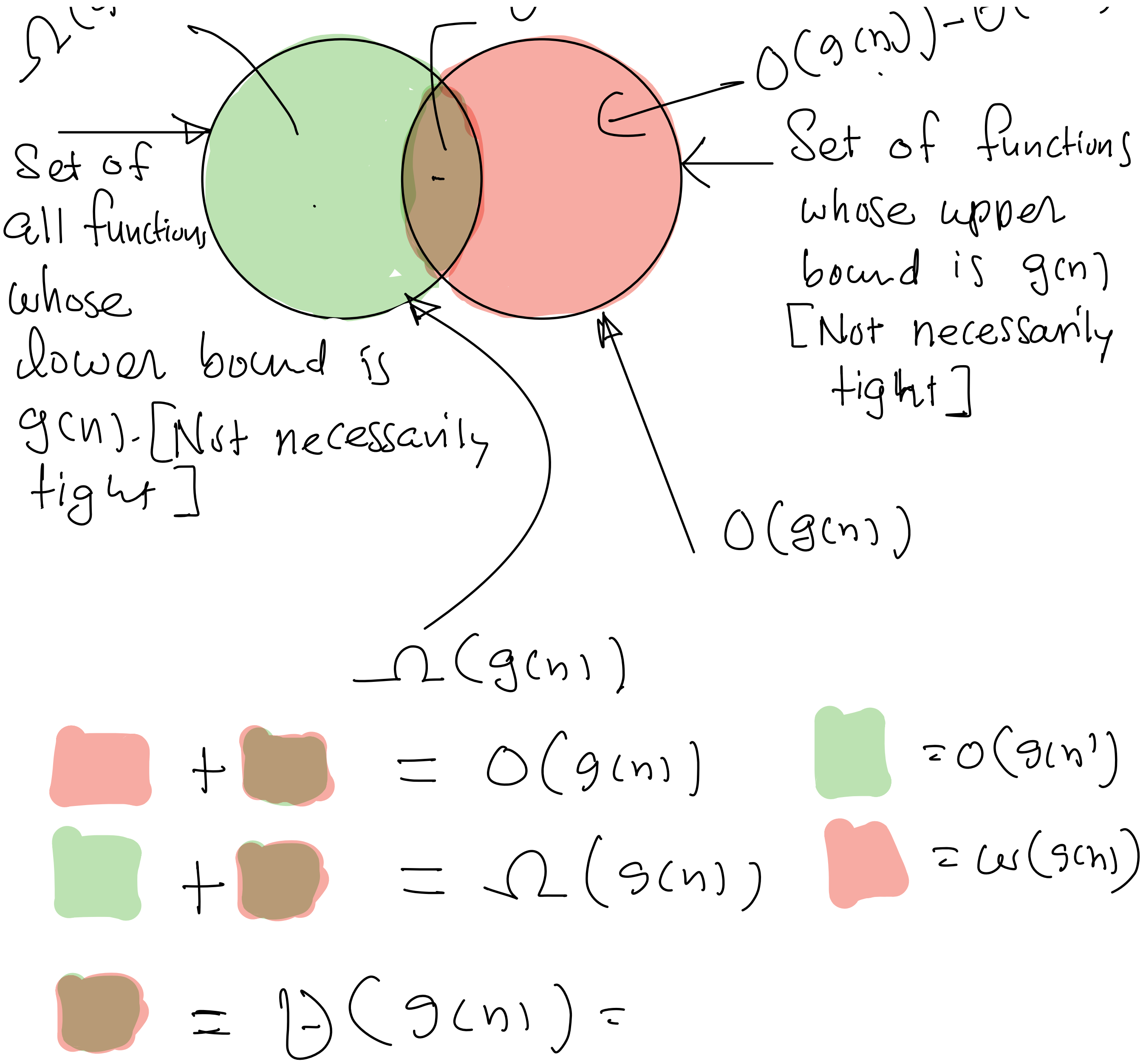
n^2 is not an upper bound of $4n^3 - 7$.

$f_2(n) \in \Omega(n^2)$

$f_2(n) \notin O(n^2)$

$g(n) = n^2$

$f_3(n) = 4n^3 - 7$	$f_1(n) = 3n^2 + 7n - 2$	$f_2(n) = 3n + 2$
$f_3(n) \notin O(g(n))$	$f_1(n) \in O(g(n))$	$f_2(n) \in O(g(n))$
$f_3(n) \in \Omega(g(n))$	$f_1(n) \in \Omega(g(n))$	$f_2(n) \notin \Omega(g(n))$
	$f_1(n) \in O(g(n))$ and $f_1(n) \in \Omega(g(n))$ $\equiv$ $f_1(n) \in \Theta(g(n))$	
$\omega(g(n)) - O(g(n))$	$\Theta(g(n))$	$\Omega(g(n))$



$\Theta(g(n))$ = Set of all functions whose asymptotic upper & lower bound is $g(n)$

$f_2(n) = 3n+2$.

$f_2(n) \in O(n^2)$ but $f_2(n) \notin \Omega(n^2)$

$o(g(n))$ = small oh$(g(n))$
= Set of all functions whose asymptotic upper bound is $g(n)$ but asymptotic lower bound is Not $g(n)$

$f_3(n) = 4n^3 - 7$.

$f_3(n) \in \Omega(n^2)$ but $f_3(n) \notin O(n^2)$

$f_3(n) \in \omega(n^2)$

Small omega (n^2) = Set of all functions whose asymptotic lower bound is n^2 but asymptotic upper bound IS NOT n^2.

$3n^2 + 7n - 2.$

$O(n^2)$ $\Omega(n^2)$

for some c.

$0 \le 3n^2 + 7n^2 - 2 \le c \cdot n^2.$

for some c.

$0 \le cn^2 \le 3n^2 + 7n - 2$

n^2

$3n^2 + 7n - 2$

Asymptotic tight U

Asymptotic tight L.

$f_3(n) = 3n+2$

$\boxed{f_3 \in O(n^2)}$

$f_3 \notin \boxed{\Omega(n^2)}$

X

$f_3(n) \in \boxed{O(n^2)}$

any
for some c.

$3n+2 \le cn^2$

$f_2($

$f(n) = 4n^3 - 7$

$f(n) \notin O(n^2)$

$f(n) \in \Omega(n^2)$

$0 \leq Cn^2 \leq 4n^3 - 7$

any C

$O(g(n))$

$f(n)$

$\Omega(g(n))$

$O(g(n)) = \cup$

$f(n)$.

or

$\Omega(g(n))$

$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$

$o(g(n)) = O(g(n)) - \Theta(g(n))$

$\omega(g(n)) = \Omega(g(n)) - \Theta(g(n))$

Asymptotic ~ s lim fav's

Combine