ENPM667 Project 1

Report on journal paper "Lateral Control of an Autonomous Vehicle"

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Contents

1 Abstract

A class of nonlinear under-actuated systems is studied and a solution is provided that is asymptotically stable. Other techniques such as back-stepping and forward control have also been utilized to control the nonlinear lateral dynamics of a vehicle. Even though the theoretical studies of the lateral control of autonomous vehicles are traditionally applied to lane keeping cases, the results can be applied to broader range of areas, such as lane changing cases. The report also compares the performance of the closed-loop systems controlled by the designed controller with a typical human driver and demonstrates the speediness and the effectiveness of the feedback controller.

2 Nomenclature

β	ratio of the lateral speed and the longitudinal speed
$\dot{\psi}$	yaw rate [rad/s]
y_L	heading error, relative yaw angle [rad]
δ	actual steering angle [rad]
δ_d	steering angle at the column system [rad]
ρ	road curvature $[m^{-1}]$
ηt	width of the tyre contact patch [m]
a_y	lateral acceleration of the car [m $\times$ s^{-2}]
B_u	damping coefficient of the steering system [Nm/(rad $\times$ s)]
$C_f(C_r)$	front (rear) tyre cornering stiffness [N/rad]
$F_f(F_r)$	front (rear) tyre lateral force [N]
I_z	moment of inertia of the car about the yaw-axis [kg $\times$ m^2]
J_s	moment of inertia of the steering system [kg $\times$ m^2]
K_a	visual anticipatory control of the driver
K_c	proportional gain of the transfer function representing the compensatory steering control of the driver
$l_f(l_r)$	distances of the front (rear) tyres to the mass center [m]
m	mass of the car [kg]
R_s	reduction ratio of the steering system,
T_c	torque generated by the controller [Nm]
$T_i(T_l)$	lag (lead) time constant of the transfer function representing the compensatory steering control of the driver
T_n	neuromuscular lag time constant of the driver
T_p	driver's preview time [s]
T_s	self-aligning moment of the steering system [Nm]
v_x	longitudinal speed of the car [m/s]
v_y	lateral speed of the car [m/s]
y_L	lateral deviation of the car [m]

3 Introduction

A lot of technology companies like Google, Tesla, Amazon, etc. are investing millions of dollars in the development of self-driving cars. Autonomous vehicles drastically reduce the number of car accidents by eliminating the human error. Thousands of people lose their lives around the globe in road accidents and thus, these lives can be saved by introducing autonomous vehicles on the road. Besides, traffic congestion which is mostly because of human factor can be significantly reduced as well. The autonomous vehicles can be designed in such a way that least amount of energy is consumed while accelerating or decelerating the car, thus making them more energy efficient.

A large part of the human population like children, elderly, disabled, etc cannot use cars. Introducing autonomous vehicles on the road can enable this part of the population to commute without having the driving skill. Following a predefined trajectory, such as a route planned by Google Map from source to destination, is an important aspect of autonomous vehicles. To make the autonomous cars to follow a predefined trajectory, the longitudinal and lateral dynamics of the car need to be studied and solved. The longitudinal and the lateral dynamics of the car are coupled. However, it is common assumption that the two are decoupled when the road curvature is small [1].

This report focuses on the control of the lateral dynamics of autonomous cars. In this report, the lateral dynamics control of an autonomous vehicle is analyzed as a non-linear system for which an asymptotically stable solution is provided. A Lyapunov-like analysis is used to prove the stability of the closed-loop system.

4 Organization

This report is organized as follows. In section 5, the model of car which is studied in this report is described and the lateral control problem for an autonomous car is formulated. Preliminary theorems which are essential to design the lateral controller are presented and derived in Section 6. The solution to the lateral control problem is given in Section 7, in which formal properties of the closed-loop system are presented. In section ,three examples are studied and how the controller works in different cases is shown. The effectiveness of the established controller is demonstrated by comparing the simulation results with typical driver performances. Finally, conclusions and suggestions for future work are given in Section 9.

5 System Modeling and Problem Statement

Assumption 1: The longitudinal speed(v_x) is constant and strictly positive.

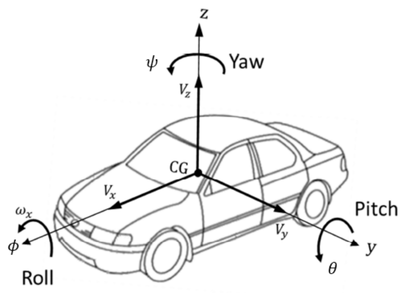

Fig 1: Axes of a car

Applying Newton's Second Law of Motion along the lateral axis (Y-axis) of the car,

$$ma_y = F_f + F_r \qquad (1)$$

where,
m $\rightarrow$ mass of vehicle in kilograms
a_y $\rightarrow$ acceleration of vehicle along Y direction
F_f $\rightarrow$ Lateral force acting on front wheels while taking a turn
F_r $\rightarrow$ Lateral force acting on rear wheels while taking a turn

The lateral acceleration (a_y) of the car is the sum of lateral motion of the car along Y-axis caused by the steering wheel and result of the cross product of the longitudinal velocity and the yaw rate.

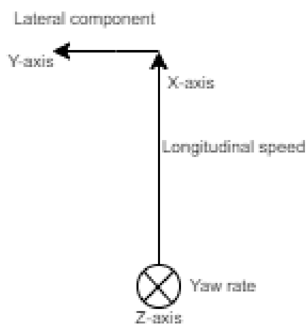

Fig 2: Velocity vector

$$\text{Lateral component} = \vec{\dot{\psi}} \text{ x } \vec{v_x}$$
$$\text{Lateral component} = \dot{\psi} v_x \sin(\theta)$$
$\because$ v_x is along X-axis and $\dot{\psi}$ is along Z-axis, the angle between them is 90°.
$$\text{Lateral component} = \dot{\psi} v_x \sin(90°)$$
$$\text{Lateral component} = \dot{\psi} v_x$$

$$a_y = \dot{v_y} + v_x \dot{\psi} \qquad (2)$$

where,
$\dot{v_y}$ $\rightarrow$ rate of change of velocity along Y direction
v_x $\rightarrow$ longitudinal speed
$\dot{\psi}$ $\rightarrow$ yaw rate

Since the lateral forces acting on a car are unequal, moment will act on the vehicle which is given by:

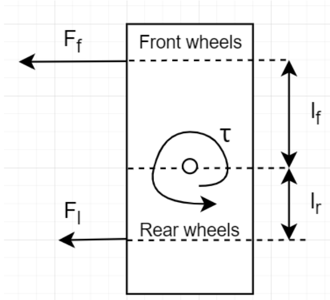

Fig. 3: Line diagram of a car

$$I_z\ddot{\psi} = \tau$$
$$I_z\ddot{\psi} = F_f l_f - F_r l_r \tag{3}$$

where,
I_z $\rightarrow$ Moment of inertia of car along the Z-axis
$\ddot{\psi}$ $\rightarrow$ Rate of change of yaw angle
l_f $\rightarrow$ distance between front wheels and center of vehicle
l_r $\rightarrow$ distance between rear wheels and center of vehicle

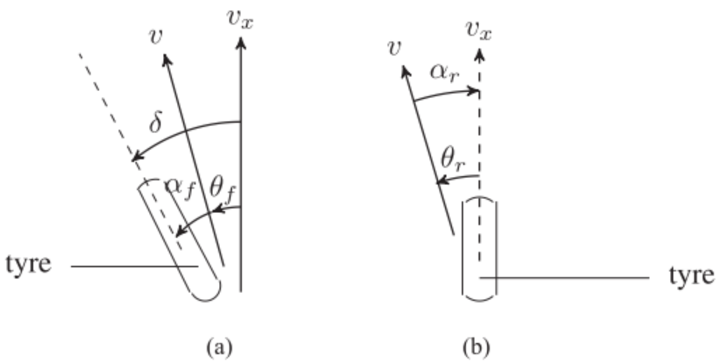

Fig 4: Line diagram of front and rear tyre

v $\rightarrow$ Resultant of longitudinal speed($\vec{v_x}$) and lateral speed ($\vec{v_y}$)
α_f $\rightarrow$ side slip angle for the front tyre(angle between $\vec{v}$ and the front tyre)
α_r $\rightarrow$ side slip angle for the rear tyre(angle between $\vec{v}$ and the rear tyre)
δ $\rightarrow$ steering angle
θ_f $\rightarrow$ angle between $\vec{v}$ and $\vec{v_x}$ for the front tyre
θ_r $\rightarrow$ angle between $\vec{v}$ and $\vec{v_x}$ for the front tyre

Lateral velocity of the front tyres:

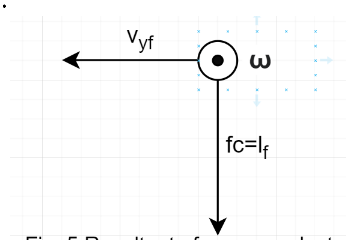

Fig: 5 Resultant of cross product

Note that in the above image $\omega = \dot{\psi}$

$$\vec{v}_{y_c} = \vec{v}_{y_f} + \vec{v}_{y_{f/c}}$$
$$\vec{v}_{y_c} = \vec{v}_{y_f} + \vec{\dot{\psi}} \text{ x } \vec{fc}$$

Applying Right Hand rule to get the direction of $\vec{v}_{y_f}$,

$$v_y = v_{y_f} + \dot{\psi}\ (fc)\sin(\theta)$$
$$v_y = v_{y_f} + \dot{\psi}\ l_f \sin(-90°)$$
$$v_y = v_{y_f} \text{ - } \dot{\psi}\ l_f$$
$$v_{y_f} = v_y + l_r\ \dot{\psi}$$

$$\tan(\theta_f) = \frac{v_{y_f}}{v_{x_f}}$$

The longitudinal speed of the front tyre (v_{x_f}) is same as the longitudinal speed of the car (v_x).

$$\therefore \tan(\theta)_f) = \frac{v_y + l_f\dot{\psi}}{v_x}$$

Lateral velocity of the rear tyres:

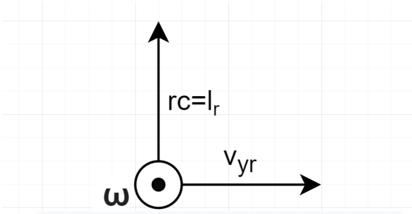

Fig: 6 Resultant of cross product

$$\vec{v}_{y_c} = \vec{v}_{y_r} + \vec{v}_{y_{r/c}}$$
$$\vec{v}_{y_c} = \vec{v}_{y_r} + \vec{\dot{\psi}} \text{ x } \vec{rc}$$
$$v_y = v_{y_r} + \dot{\psi}\ (rc)\sin(\theta)$$
$$v_y = v_{y_r} + \dot{\psi}\ l_r \sin(90°)$$
$$v_y = v_{y_r} + \dot{\psi}\ l_r$$
$$v_{y_r} = v_y \text{ - } l_r\ \dot{\psi}$$

$$\tan(\theta_r) = \frac{v_{yr}}{v_{xr}}$$

The longitudinal speed of the rear tyre (v_{x_r}) is same as the longitudinal speed of the car (v_x).

$$\therefore \tan(\theta_r = \frac{v_y - l_r\dot{\psi}}{v_x}$$

As per Page 402 of [2], the lateral tyre forces are proportional to the side-slip angle if the angles are small. Therefore, the lateral force is given by:

$$F_f = 2C_f\alpha_f$$
$$F_r = 2C_r\alpha_r$$

where C_f and C_r are the front and rear tyre cornering stiffness respectively.

From figure 4, $\alpha_f = \delta - \theta_f$ and $\alpha_r = -\theta_r$

$$\therefore F_f = 2C_f(\delta - \theta_f)$$
$$F_r = 2C_r(-\theta_r)$$

Let us consider $\beta = \frac{v_y}{v_x}$

$$\therefore \dot{\beta} = \frac{\dot{v_y}}{v_x}$$

Substituting equation (2) in (1),

$$\dot{v_y} + v_x\dot{\psi} = \frac{F_f}{m} + \frac{F_r}{m}$$

$$\dot{\beta}v_x + v_x\dot{\psi} = 2\frac{C_f(\delta-\theta_f)}{m} + 2\frac{C_r(-\theta_r)}{m}$$

$$\dot{\beta}v_x + v_x\dot{\psi} = 2\frac{C_f}{m}\delta - 2\frac{C_f\theta_f}{m} - 2\frac{C_r(\theta_r)}{m}$$

$$v_x(\dot{\beta} + \dot{\psi}) = \frac{2C_f}{m}\delta - \frac{2C_f}{m}\arctan(\beta + \frac{l_f\dot{\psi}}{v_x}) - \frac{2C_r}{m}\arctan(\beta - \frac{l_r\dot{\psi}}{v_x})$$

$$\dot{\beta} + \dot{\psi} = \frac{2C_f}{mv_x}\delta - \frac{2C_f}{mv_x}\arctan(\beta + \frac{l_f\dot{\psi}}{v_x}) - \frac{2C_r}{mv_x}\arctan(\beta - \frac{l_r\dot{\psi}}{v_x})$$

$$\dot{\beta} = \frac{2C_f}{mv_x}\delta - \dot{\psi} - \frac{2C_f}{mv_x}\arctan(\beta + \frac{l_f\dot{\psi}}{v_x}) - \frac{2C_r}{mv_x}\arctan(\beta - \frac{l_r\dot{\psi}}{v_x}) \qquad (4)$$

From equation (3),

$$\ddot{\psi} = \frac{F_f l_f}{I_z} - \frac{F_r l_r}{I_z}$$

$$\ddot{\psi} = \frac{2C_f l_f(\delta-\theta_f)}{I_z} - \frac{2C_r l_r(-\theta_r)}{I_z}$$

$$\ddot{\psi} = \frac{2C_f l_f\delta}{I_z} - \frac{2C_f l_f\theta_f}{I_z} + \frac{2C_r l_r\theta_r}{I_z}$$

$$\ddot{\psi} = \frac{2C_f l_f}{I_z} - \frac{2C_f l_f}{I_z}\arctan(\beta + \frac{l_f\dot{\psi}}{v_x}) + \frac{2C_r l_r}{I_z}\arctan(\beta - \frac{l_r\dot{\psi}}{v_x}) \qquad (5)$$

Steering angle of a car cannot be controlled directly. It is controlled by the steering torque. The dynamics of the steering system is given by the following equation:

$$J_s\ddot{\delta}_d + B_u\dot{\delta}_d = T_c - T_s \qquad (6)$$

where,

J_s → Moment of Inertia of the steering system
$\ddot{\delta}_d$ → Rate of change of velocity of steering angle
B_u → Damping coefficient of the steering system
$\dot{\delta}_d$ → Rate of change of steering angle
T_c → Input control torque
T_s → Self aligning moment of the steering system.

Note that the above equation is very similar to spring mass damper assemblage (m$\ddot{x}$ + $k\dot{x} = F$).

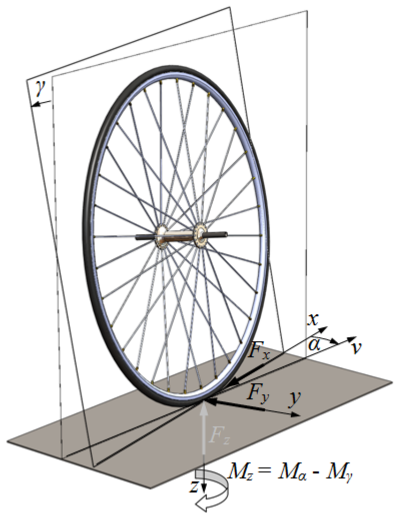

Fig 7: Self-aligning moment

T_s [1] is the self-aligning moment and it is the torque that a tyre generates as it rolls. This torque tends to steer the tyre about its vertical axis. If the slip angle is small and non-zero, this torque tends to steer the tyre towards the direction in which it is traveling and therefore, acts as a resistance to the control input torque. It is given by:

$$T_s = -\frac{2C_f\eta_t}{R_s} - \frac{2C_f l_f \eta_t}{R_s v_x}\dot{\psi} + \frac{2C_f\eta_t}{R_s^2}\delta_d \qquad (7)$$

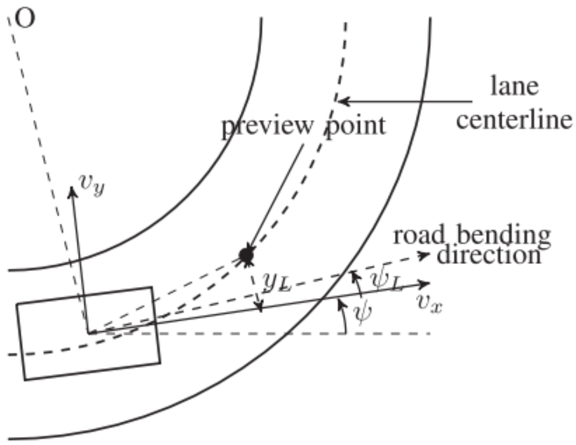

Fig 8: Graphical representation of variables y_L and ψ_L in lane keeping cases

Preview point is an arbitrary point on the road. The trajectory of the road or lane at the preview point is taken as a reference using which a controller is developed. Preview point is also applicable when a driver is driving a car. Whenever a driver drives a vehicle, he is using some location in front of the car as a reference to make the steering action. The preview point cannot be located directly in front of the car since the driver or controller cannot react immediately to the trajectory to be followed. In trajectory tracking or a lane keeping case, two additional variables are used to describe the relationship between the vehicle and the reference trajectory(central line of the lane). They are lateral deviation y_L and the heading error ψ_L. The dynamics of y_L is given below:

$$\dot{y}_L = v_x\beta + T_p v_x\dot{\psi} + v_x\psi_L, \qquad (8)$$

Consider the first term, $v_x\beta$,

$$v_x\beta = v_x\frac{v_y}{v_x}$$
$$v_x\beta = v_y$$

The first term, therefore, represents the lateral velocity (v_y) of the car.

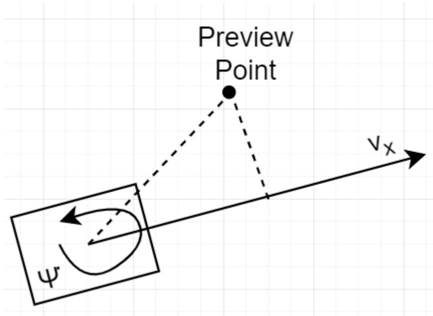

Fig 9: Lateral velocity component due to yaw rate

Consider the second term, $T_p v_x\dot{\psi}$.
$T_p v_x$ represents the distance between the car and the preview point along v_x.
$\therefore$ distancex$\dot{\psi}$ yields the lateral component along Y-direction due to rotation of the car.

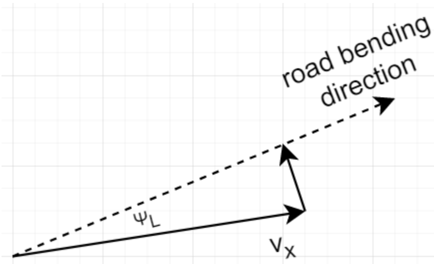

Fig 10: Lateral velocity component of v_x

Consider the third term, $v_x\psi_L$.
From figure (),

$$\tan(\psi_L) = \frac{lateralcomponent}{v_x}$$

Since ψ_L is small, $\tan(\psi_L) = \psi_L$.
$\therefore$ The lateral component of v_x along the road bending direction $= v_x\psi_L$

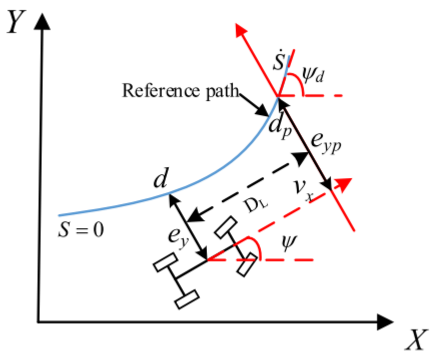

Fig 10: Path following model

ψ_L, the heading error, is the difference between ψ and ψ_d.
ψ_d, as defined in figure(), is the angle made by the tangent along the reference path with the horizontal axis.

$$\therefore \psi_L = \psi - \psi_d$$

Differentiating above equation with respect to time,

$$\therefore \dot{\psi}_L = \dot{\psi} - \dot{\psi}_d \qquad (9)$$

Since $\dot{\psi}_d$ is the rate of change of angle made by the tangent along the reference path with the horizontal axis, it is equal to ω.
where, ω is the angular velocity of the car along the curved path.

$$\therefore \omega = \frac{v_x}{R},$$

where, R is the radius of curvature of the road.
As per definition,

$$\rho = \frac{1}{R}$$

$$\therefore \omega = v_x \rho \qquad ,$$

Therefore, equation (9) becomes

$$\therefore \dot{\psi}_L = \dot{\psi} - v_x \rho \qquad (10)$$

Definition 1: The signal $\rho(t)$ is said to be feasible if and only if functions $\beta(t)$, $\dot{\psi}(t)$, $\delta(t)$, $\dot{\delta}(t)$, $y_L(t)$, $\psi_L(t)$, and $T_c(t)$ exist such that (4)-(5)-(6)-(7)-(8)-(10) hold true for all t$\geq$0.

Suppose $\rho(t)$ is feasible and v_x is a given positive constant, then the references for the variables $\beta(t)$, $\dot{\psi}(t)$, $\delta(t)$, $\dot{\delta}(t)$, $y_L(t)$, $\psi_L(t)$, $T_c(t)$ satisfy:

$$\dot{\beta}_r = \frac{2C_f}{mv_x}\delta_r - \dot{\psi}_r - \frac{2C_f}{mv_x}\arctan(\beta_r + \frac{l_f\dot{\psi}_r}{v_x}) - \frac{2C_r}{mv_x}\arctan(\beta - \frac{l_r\dot{\psi}}{v_x}),$$

$$\ddot{\psi}_r = \frac{2C_f l_f}{I_z}\delta_r - \frac{2C_f l_f}{I_z}\arctan(\beta + \frac{l_f\dot{\psi}_r}{v_x}) + \frac{2C_r l_r}{I_z}\arctan(\beta_r - \frac{l_f\dot{\psi}_r}{v_x}),$$

$$\ddot{\delta}_r = \frac{T_{cr} - T_s - B_u R_s \dot{\delta}_r}{J_s R_s},$$

$$\dot{y_{Lr}} = v_x\beta + T_p v_x \dot{\psi}_r + v_x \psi_L r,$$

$$\dot{\psi_{Lr}} = \dot{\psi}_r - v_x\rho, \qquad (11)$$

where,
$T_s = -\frac{2C_f\eta_t}{R_s}\beta_r - \frac{2C_f l_f \eta_t}{R_s v_x}\dot{\psi}_r + \frac{2C_f\eta_t}{R_s}\delta_r$

The control problem for the lateral motion can be formulated as follows.

Given the system (4)-(5)-(6)-(7)-(8)-(10) and the time history of the feasible road curvature $\rho(t)$, find a feedback controller $T_c(t)$ such that the closed-loop system has the following properties:

P1) The control effort is bounded, *i.e.* $\ni \beta > 0$ such that $|T_c(t)| \leq \beta$ for all t $\geq$ 0.

P2) The system converges to its reference value, *i.e.*

$$\lim_{x\to\infty}(\beta(t), \dot{\psi}(t), \delta(t), \dot{\delta}(t), y_L(t), \psi_L(t)) - (\beta_r(t), \dot{\psi}_r(t), \delta_r(t), \dot{\delta}_r(t), y_{Lr}(t), \psi_{Lr}(t)) = 0,$$

where $\beta_r, \dot{\psi}_r, \delta_r, y_{Lr}$ and ψ_{Lr} are feasible reference signals.

P3) $\lim_{x\to\infty} T_c(t)$ - $T_{c_r}(t) = 0$, where T_{c_r} is defined in (11).

6 Preliminary Theorems

In this section, the two basic theorems used to design the controller for the under-actuated nonlinear system, which is the torque to control the steering wheel in our case, are proved.

Theorem 1:

Consider a two dimensional nonlinear system. The dynamics of the system are given by following equations:

$$\dot{s_1} = q_{11}s_1 + p_1(s_2) \qquad (12)$$
$$\dot{s_2} = p_2(s_1, s_2) + gu \qquad (13)$$

where,
$s_1, s_2 \rightarrow$ states of the system
$u \rightarrow$ control input
p_1 : R $\rightarrow$ R, p_2 : $R^2 \rightarrow$ R denote two nonlinear mappings

Assume that:
A1) g $\neq 0$
A2) $q_{11} < 0$
A3) The function $\frac{p_1(s_2)}{s_2}$ is continuous at $s_2 = 0$.

Then there exits a feedback controller u(s_1, s_2) such that the zero equilibrium of the closed loop system is globally asymptotically stable. One such choice of feedback controller is given by:

$$\text{u}(s_1, s_2) = -\frac{ks_2 + s_1\frac{p_1(s_2)}{s_2} + p_2(s_1, s_2)}{g} \qquad (14)$$

for any k > 0.

Proof:

The stability of a nonlinear system is determined using Lyapunov function. A system is said to be asymptotically stable if the value of derivative of Lyapunov function with respect to time is 0 for every non-zero values of the states. Moreover, the derivative of Lyapunov function should be 0 if the zero-state system.

Consider the following standard Lyapunov function:

$$\text{L} = \frac{1}{2}(s_1^2 + s_2^2)$$

Taking its derivative with respect to time,

$$\dot{L} = \frac{2s_1\dot{s_1} + 2s_2\dot{s_2}}{2}$$
$$\dot{L} = s_1\dot{s_1} + s_2\dot{s_2}$$

Substituting values of $\dot{s_1}$ and $\dot{s_1}$ from system (12),

$$\dot{L} = s_1[q_{11}s_1 + p_1(s_2)] + s_2[p_2(s_1, s_2) + gu]$$

$$\dot{L} = q_{11}s_1^2 + s_1p_1(s_2) + s_2[p_2(s_1, s_2) + gu]$$

$$\dot{L} = q_{11}s_1^2 + \frac{s_1s_2p_1(s_2)}{s_2} + s_2[p_2(s_1, s_2) + gu]$$

$$\dot{L} = q_{11}s_1^2 + s_2[\frac{s_1p_1(s_2)}{s_2} + p_2(s_1, s_2) + gu]$$

Substituting the value of u as per equation (13), we get,

$$\dot{L} = q_{11}s_1^2 + s_2[\frac{s_1p_1(s_2)}{s_2} + p_2(s_1,s_2) - g\frac{ks_2+s_1\frac{p_1(s_2)}{s_2}+p_2(s_1,s_2)}{g}]$$

$$\dot{L} = q_{11}s_1^2 + s_2[\frac{s_1p_1(s_2)}{s_2} + p_2(s_1,s_2) - ks_2 - s_1\frac{p_1(s_2)}{s_2} - p_2(s_1,s_2)]$$

$$\dot{L} = q_{11}s_1^2 - ks_2^2$$

As per A2), q_{11} <0 and we know the fact that k>0, we conclude that,

$$\dot{L} < 0 \text{ for all } (s_1, s_2) \neq (0,0).$$

Furthermore,

$$\dot{L} = 0 \Longleftrightarrow s_1 = s_2 = 0$$

Since the value of $\dot{L}$ is less than or equal to zero for all non-negative values of s_1 and s_2, the given system is globally asymptotically stable for given choice of control input "u" as defined above.

Theorem 2:

Consider a system with two degrees of freedom. The dynamics of the system are given by following equations:

$$\dot{s_1} = a_{11}s_1 + a_{12}s_2 + f_1(s_2) + b_1u \qquad (15)$$

$$\dot{s_2} = a_{21}s_1 + a_{22}s_2 + f_2(s_2) + b_2u \qquad (16)$$

where,
$s_1(t), s_2(t), u(t) \in R$
$a_{11}, a_{12}, a_{21}, a_{22}$ are constant parameters
$f_1(s_2)$ and $f_2(s_2)$ are continuous functions of s_2

Assume that:
H1) $b_2(b_1a_{21}$ - $b_2a_{11}) > 0$
H2) Functions $\frac{f_1(s_2)}{s_2}$ and $\frac{f_2(s_2)}{s_2}$are continuous at $s_2 = 0$.

Then there exits a feedback controller u(s_1, s_2) such that the zero equilibrium of the closed loop system is globally asymptotically stable. One such choice is given by:

$$\text{u}(s_1, s_2) = \frac{-a_{21}s_1-a_{22}s_2-f_2(s_2)+b_1(b_1a_{21}-b_2a_{11})s_1}{b2}$$

$$- \frac{(b_1s_2-b_2s_1)[b_1a_{22}-b_2a_{12}+b_1\frac{f_2(s_2)}{s_2}]}{b_2}$$

$$- \frac{(b_1s_2-b_2s_1)b_2\frac{f_1(s_2)}{s_2}+ks_2}{b_2} \qquad (17)$$

for any k > 0.

Proof:

Let us define

$$\tilde{s_1} \triangleq (b_1s_2 - b_2s_1) \qquad (18)$$

Differentiating above equation with respect to time,

$$\dot{\tilde{s_1}} = b_1\dot{s_2} - b_2\dot{s_1}$$

Substituting values of $\dot{s_1}$ and $\dot{s_1}$ from system (14),

$$\dot{\tilde{s_1}} = b_1[a_{21}s_1 + a_{22}s_2 + f_2(s_2) + b_2u] \text{ - } b_2[a_{11}s_1 + a_{12}s_2 + f_1(s_2) + b_1u]$$

$$\dot{\tilde{s_1}} = b_1a_{21}s_1 + b_1a_{22}s_2 + b_1f_2(s_2) + b_1b_2u \text{ - } b_2a_{11}s_1 - b_2a_{12}s_2 - b_2f_1(s_2) - b_2b_1u$$

$$\dot{\tilde{s_1}} = (b_1a_{21} - b_2a_{11})s_1 + (b_1a_{22} - b_2a_{12})s_2 + [b_1f_2(s_2) - b_2f_1(s_2)]$$

$$\dot{\tilde{s_1}} = (b_1a_{21} - b_2a_{11})s_1 + (b_1a_{22} - b_2a_{12})s_2 + s_2\frac{[b_1f_2(s_2) - b_2f_1(s_2)]}{s_2}$$

$$\dot{\tilde{s_1}} = (b_1a_{21} - b_2a_{11})s_1 + [b_1a_{22} - b_2a_{12} + b_1\frac{f_2(s_2)}{s_2} - b_2\frac{f_1(s_2)}{s_2}]s_2 \qquad (19)$$

From equation (16):

$$\tilde{s_1} \triangleq (b_1s_2 - b_2s_1)$$

$$\therefore b_2s_1 = b_1s_2 - \tilde{s_1}$$

$$\therefore s_1 = \frac{b_1s_2 - \tilde{s_1}}{b_2} \qquad (20)$$

Substituting the value of s_1 in equation (17), we get,

$$\dot{\tilde{s_1}} = (b_1a_{21} - b_2a_{11})(\frac{b_1s_2 - \tilde{s_1}}{b_2}) + [b_1a_{22} - b_2a_{12} + b_1\frac{f_2(s_2)}{s_2} - b_2\frac{f_1(s_2)}{s_2}]s_2$$

$$\dot{\tilde{s_1}} = (b_1a_{21} - b_2a_{11})\frac{b_1}{b_2}s_2 - (b_1a_{21} - b_2a_{11})\frac{\tilde{s_1}}{b_2} + [b_1a_{22} - b_2a_{12} + b_1\frac{f_2(s_2)}{s_2} - b_2\frac{f_1(s_2)}{s_2}]s_2$$

$$\dot{\tilde{s_1}} = (a_{11} - \frac{b_1}{b_2}a_{21})\tilde{s_1} + (b_1a_{21} - b_2a_{11})\frac{b_1}{b_2}s_2 + [b_1a_{22} - b_2a_{12} + b_1\frac{f_2(s_2)}{s_2} - b_2\frac{f_1(s_2)}{s_2}]s_2$$

$$\dot{\tilde{s_1}} = (a_{11} - \frac{b_1}{b_2}a_{21})\tilde{s_1} + [b_1a_{22} - b_2a_{12} + b_1\frac{f_2(s_2)}{s_2} - b_2\frac{f_1(s_2)}{s_2} + (b_1a_{21} - b_2a_{11})\frac{b_1}{b_2}]s_2$$

Notice that the above equation is of the form:

$$\dot{\tilde{s_1}} = q_{11}\tilde{s_1} + p_1(s_2),$$

where,
$q_1 = (a_{11} - \frac{b_1}{b_2}a_{21})$
$p_1(s_2) = [b_1a_{22} - b_2a_{12} + b_1\frac{f_2(s_2)}{s_2} - b_2\frac{f_1(s_2)}{s_2} + (b_1a_{21} - b_2a_{11})\frac{b_1}{b_2}]s_2 \qquad (21)$

Also, substituting the value of s_1 from equation (20) in equation (16), we get,

$$\dot{s_2} = a_{21}\frac{b_1s_2 - \tilde{s_1}}{b_2} + a_{22}s_2 + f_2(s_2) + b_2u$$

Comparing the above equation with the form $\dot{s_2} = p_2(s_1, s_2) + gu$, we get,

$$p_2(\tilde{s}_1, s_2) = a_{21}\frac{b_1s_2 - \tilde{s_1}}{b_2} + a_{22}s_2 + f_2(s_2) \qquad (22)$$

$$g = b_2$$

We now exploit Theorem 1 to derive the state-feedback controller for system (15)-(16), for which we need to check if assumptions A1), A2), and A3) made for Theorem 1 hold. As per assumption H2) made for Theorem 2,

$$b_2(b_1a_{21} \text{ - } b_2a_{11}) > 0$$

Dividing the above equation by b_2^2, which is a positive value, yields:

$$\frac{b_2(b_1a_{21} - b_2a_{11})}{b_2^2} > 0$$

$$\Rightarrow \frac{b_1}{b_2}a_{21} - a_{11} > 0 \qquad (23)$$

$$\Rightarrow -(a_{11} + \frac{b_1}{b_2}a_{21}) > 0$$

$$\Rightarrow (a_{11} + \frac{b_1}{b_2}a_{21}) < 0$$

$$\Rightarrow q_{11} < 0$$

$\therefore$ The above state dynamics model satisfies assumption A2) made for Theorem 1.
As per equation (23)

$$\frac{b_1}{b_2}a_{21} - a_{11} > 0$$

$$\Rightarrow \frac{b_1}{b_2}a_{21} > a_{11}$$

$$\Rightarrow b_1a_{21} > b_2a_{11}$$

$$\therefore b_1a_{21} - b_2a_{11} > 0$$

As per assumption H2) of theorem 2,

$$b_2(b_1a_{21} - b_2a_{11}) > 0$$
$$\therefore b_2 > 0$$
$$\Rightarrow \text{g} > 0$$
$$\Rightarrow \text{g} \neq 0$$

This satisfies assumption A1)of Theorem 1.

From equation (20), $p_1(s_2)$ is a function of $f_1(s_2)$ and $f_2(s_2)$.

As per assumption H1) of Theorem 2,

$\frac{f_1(s_2)}{s_2}$ and $\frac{f_2(s_2)}{s_2}$ are continuous at $s_2 = 0$.

$\Rightarrow \frac{p_1(s_2)}{s_2}$ is also continuous at $s_2 = 0$

Thus, the above system dynamics model satisfies assumption A3) of Theorem 1.

Since the state dynamics model represented by equations (15) and (16) of Theorem 2 can be represented as per the state dynamics equations (12) and (13) of Theorem 1 and since they also satisfy all the assumptions for Theorem 1, the state-feedback control law derived in Theorem 1 can be utilized to find the control law for system(15)-(16).

According to Theorem 1, the state-feedback control law is of the form:

$$\text{u}(\tilde{s_1}, s_2) = -\frac{ks_2 + \tilde{s_1}\frac{p_1(s_2)}{s_2} + p_2(\tilde{s_1}, s_2)}{b_2}$$

for any k > 0.
The above control law globally asymptotically stabilizes the zero equilibrium of system (15)-(16)

From equation (),

$$p_1(s_2) = [b_1a_{22} - b_2a_{12} + b_1\frac{f_2(s_2)}{s_2} - b_2\frac{f_1(s_2)}{s_2} + (b_1a_{21} - b_2a_{11})\frac{b_1}{b_2}]s_2$$

$$\therefore \frac{p_1(s_2)}{s_2} = [b_1a_{22} - b_2a_{12} + b_1\frac{f_2(s_2)}{s_2} - b_2\frac{f_1(s_2)}{s_2} + (b_1a_{21} - b_2a_{11})\frac{b_1}{b_2}]$$

Substituting values of $\tilde{s_1}$, $\frac{p_1(s_2)}{s_2}$, and $p_2(\tilde{s_1}, s_2)$ from equations (18), (21), and (22) respectively, we get,

$$\text{u}(s_1, s_2) = -\frac{ks_2 + (b_1s_2 - b_2s_1)[b_1a_{22} - b_2a_{12} + b_1\frac{f_2(s_2)}{s_2} - b_2\frac{f_1(s_2)}{s_2} + (b_1a_{21} - b_2a_{11})\frac{b_1}{b_2}] + a_{21}\frac{b_1s_2 - \tilde{s_1}}{b_2} + a_{22}s_2 + f_2(s_2)}{b_2}$$

$$\mathrm{u}(s_1, s_2) = -\frac{ks_2+(b_1s_2-b_2s_1)[b_1a_{22}-b_2a_{12}+b_1\frac{f_2(s_2)}{s_2}-b_2\frac{f_1(s_2)}{s_2}+(b_1a_{21}-b_2a_{11})\frac{b_1}{b_2}]+a_{21}\frac{b_1s_2-(b_1s_2-b_2s_1)}{b_2}+a_{22}s_2+f_2(s_2)}{b_2}$$

Rearranging the terms, we get,

$$\mathrm{u}(s_1, s_2) = -\frac{ks_2+a_{21}\frac{b_1s_2-(b_1s_2-b_2s_1)}{b_2}+a_{22}s_2+f_2(s_2)}{b_2} + \frac{(b_1s_2-b_2s_1)[(b_1a_{22}-b_2a_{12}+b_1\frac{f_2(s_2)}{s_2})-(b_2\frac{f_1(s_2)}{s_2}+(b_1a_{21}-b_2a_{11})\frac{b_1}{b_2})]}{b_2}$$

$$\mathrm{u}(s_1, s_2) = -\frac{ks_2+a_{21}\frac{b_1s_2-(b_1s_2-b_2s_1)}{b_2}+a_{22}s_2+f_2(s_2)}{b_2} - \frac{(b_1s_2-b_2s_1)(b_1a_{22}-b_2a_{12}+b_1\frac{f_2(s_2)}{s_2})}{b_2} - \frac{(b_1s_2-b_2s_1)[b_2\frac{f_1(s_2)}{s_2}+(b_1a_{21}-b_2a_{11})\frac{b_1}{b_2})]}{b_2} \quad (24)$$

This is the desired control law for the system (15)-(16).
Hence, proved.

7 Lateral Design for Autonomous Vehicles

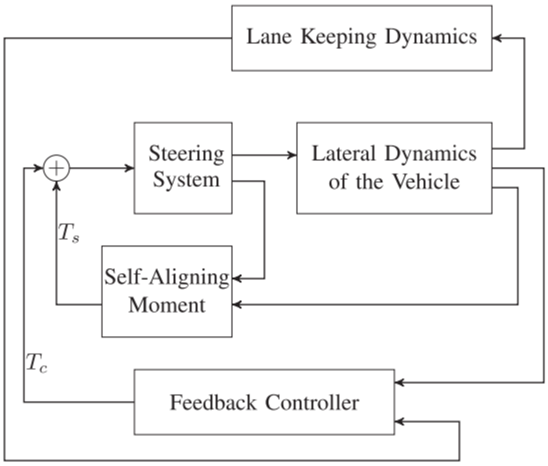

Fig 11: Block diagram of the overall system

The overall system is regarded as an interconnected system as depicted in Fig. 11. The open loop system contains three components, *i.e.* the 'Lateral Dynamics of the Vehicle', the 'Lane Keeping Dynamics' and the 'Steering System' together with the 'Self-Aligning Moment', of which the core subsystem is the "Lateral Dynamics of the Vehicle". This section gives a solution to the control problem stated in Section 5.

7.1 Control Design for the Lateral Dynamics of the Vehicle

Let us define the variables x_1 and x_2 as:

$$x_1 = \beta + \frac{l_f}{v_x}\dot{\psi} \qquad (25)$$

$$x_2 = \beta - \frac{l_r}{v_x}\dot{\psi} \qquad (26)$$

Subtracting above two equations to get the value of $\dot{\psi}$

$$x_1 - x_2 = \beta + \frac{l_f}{v_x}\dot{\psi} \text{ - } \beta + \frac{l_r}{v_x}\dot{\psi}$$

$$x_1 - x_2 = \frac{l_f + l_r}{v_x}\dot{\psi}$$

$$(x_1 - x_2)\frac{v_x}{l_f + l_r} = \dot{\psi}$$

$$x_1\frac{v_x}{l_f + l_r} - x_2\frac{v_x}{l_f + l_r} = \dot{\psi}$$

Differentiating above two equations with respect to time, we get,

$$\dot{x_1} = \dot{\beta} + \frac{l_f}{v_x}\ddot{\psi} \qquad (27)$$

$$\dot{x_2} = \dot{\beta} + \frac{l_r}{v_x}\ddot{\psi} \qquad (28)$$

Substituting values of $\dot{\beta}$ and $\ddot{\psi}$ from equation (4) and (5) in (27),

$$\dot{x_1} = \frac{2C_f}{mv_x}\delta - x_1\frac{v_x}{l_f + l_r} + x_2\frac{v_x}{l_f + l_r} - \frac{2C_f}{mv_x}\arctan x_1 - \frac{2C_r}{mv_x}\arctan x_2$$
$$+ \frac{l_f}{v_x}(\frac{2C_f l_f}{I_z}\delta - \frac{2C_f l_f}{I_z}\arctan x_1 + \frac{2C_r l_r}{I_z}\arctan x_2)$$

$$\dot{x_1} = \frac{2C_f}{mv_x}\delta - x_1\frac{v_x}{l_f + l_r} + x_2\frac{v_x}{l_f + l_r} - \frac{2C_f}{mv_x}\arctan x_1 - \frac{2C_r}{mv_x}\arctan x_2$$
$$+ \frac{2C_f l_f^2}{I_z v_x}\delta - \frac{2C_f l_f^2}{I_z v_x}\arctan x_1 + \frac{2C_r l_f l_r}{I_z v_x}\arctan x_2$$

$$\dot{x_1} = \frac{2C_f}{mv_x}\delta + \frac{2C_f l_f^2}{I_z v_x}\delta - \frac{v_x}{l_f + l_r}x_1 + \frac{v_x}{l_f + l_r}x_2 - \frac{2C_f}{mv_x}\arctan x_1$$
$$- \frac{2C_f l_f^2}{I_z v_x}\arctan(x_1) - \frac{2C_r}{mv_x}\arctan x_2 + \frac{2C_r l_f l_r}{I_z v_x}\arctan x_2$$

$$\dot{x_1} = (\frac{2C_f}{mv_x} + \frac{2C_f l_f^2}{I_z v_x})\delta - \frac{v_x}{l_f + l_r}x_1 + \frac{v_x}{l_f + l_r}x_2 - (\frac{2C_f}{mv_x} + \frac{2C_f l_f^2}{I_z v_x})\arctan x_1$$
$$+ (\frac{2C_r l_f l_r}{I_z v_x} - \frac{2C_r}{mv_x})\arctan x_2$$

$$\dot{x_1} = - \frac{v_x}{l_f + l_r}x_1 + \frac{v_x}{l_f + l_r}x_2 + (\frac{2C_r l_f l_r}{I_z v_x} - \frac{2C_r}{mv_x})\arctan x_2 + (\frac{2C_f}{mv_x} + \frac{2C_f l_f^2}{I_z v_x})(\delta - \arctan x_1) \qquad (29)$$

Similarly, substituting values of $\dot{\beta}$ and $\dot{\psi}$ from equation (4) and (5) in (28),

$$\dot{x_2} = \frac{2C_f}{mv_x}\delta - x_1\frac{v_x}{l_f + l_r} + x_2\frac{v_x}{l_f + l_r} - \frac{2C_f}{mv_x}\arctan x_1 - \frac{2C_r}{mv_x}\arctan x_2$$
$$- \frac{l_r}{v_x}(\frac{2C_f l_f}{I_z}\delta - \frac{2C_f l_f}{I_z}\arctan x_1 + \frac{2C_r l_r}{I_z}\arctan x_2)$$

$$\dot{x_2} = \frac{2C_f}{mv_x}\delta - x_1\frac{v_x}{l_f + l_r} + x_2\frac{v_x}{l_f + l_r} - \frac{2C_f}{mv_x}\arctan x_1 - \frac{2C_r}{mv_x}\arctan x_2$$
$$- \frac{2C_f l_f l_r}{I_z v_x}\delta + \frac{2C_f l_f l_r}{I_z v_x}\arctan x_1 - \frac{2C_r l_r^2}{I_z v_x}\arctan x_2$$

$$\dot{x_2} = (\frac{2C_f}{mv_x} - \frac{2C_f l_f l_r}{I_z v_x})\delta \text{ - } \frac{v_x}{l_f+l_r}x_1 + \frac{v_x}{l_f+l_r}x_2 - (\frac{2C_r l_r^2}{I_z v_x} + \frac{C_r}{mv_x})\text{arctan}x_2 - (\frac{2C_f}{mv_x} - \frac{2C_f l_f l_r}{I_z v_x})\text{arctan}x_1$$

$$\dot{x_2} = \text{- } \frac{v_x}{l_f+l_r}x_1 + \frac{v_x}{l_f+l_r}x_2 - (\frac{2C_r l_r^2}{I_z v_x} + \frac{2C_r}{mv_x})\text{arctan}x_2 + (\frac{2C_f}{mv_x} - \frac{2C_f l_f l_r}{I_z v_x})(\delta - \text{arctan}x_1) \qquad (30)$$

Considering $(\delta - \text{arctan}x_1)$ as an auxiliary input signal $\tilde{\delta}$.

The system (29)-(30) can be written in the form studied in Theorem 2 as:

$$\dot{x_1} = a_{11}x_1 + a_{12}x_2 + f_1(x_2) + b_1\tilde{\delta} \qquad (31)$$

$$\dot{x_2} = a_{21}x_1 + a_{22}x_2 + f_2(x_2) + b_2\tilde{\delta} \qquad (32)$$

where,

$a_{11} = a_{21} = \text{-}a_{12} = \text{-}a_{22} = \text{-}\frac{v_x}{l_f+l_r}$,

$b_1 = \frac{2C_f l_f^2}{I_z v_x} + \frac{2C_f}{mv_x}$

$f_1(x_2) = (\frac{2C_r l_r l_f}{I_z v_x} \text{ - } \frac{2C_r}{mv_x})\arctan x_2$

$f_2(x_2) = (\frac{2C_r l_r^2}{I_z v_x} + \frac{2C_r}{mv_x})\arctan x_2$

Note that the values of v_x, l_f, l_r, C_f, I_z, and m are positive.

$$\therefore a_{11} = a_{21} < 0 \text{ and } b_1 > 0$$

Assumption 2: For typical values of car parameters, $b_2 < 0$

As per assumption H1) of Theorem 2,

$$b_2(b_1a_{21} \text{ - } b_2a_{11}) > 0$$

By previous analysis on the signs of parameters a_{11}, a_{21}, b_1 for system (31)-(32) and by Assumption 2, the system (31)-(32) satisfies assumption H1) of Theorem 2.

Functions $f_1(x_2)$ and $f_2(x_2)$ are functions of $\arctan x_2$.

Function $\frac{\arctan x_2}{x}$ has a value at x = 0 and is therefore continuous at x = 0.
$\Rightarrow$ Functions $\frac{f_1(x_2)}{x_2}$ and $\frac{f_2(x_2)}{x_2}$ are continuous at x = 0.

Hence, the system (31)-(32) satisfies assumption H2) of Theorem 2.

As per Theorem 2, there exists a state-feedback controller $\tilde{\delta}(x_1, x_2)$ such that the zero equilibrium of the closed loop system is globally asymptotically stable and it is given by:

$$\tilde{\delta}(x_1, x_2) = \frac{-a_{21}x_1 - a_{22}x_2 - f_2(x_2) + b_1(b_1a_{21} - b_2a_{11})x_1}{b2} - \frac{(b_1x_2 - b_2x_1)[b_1a_{22} - b_2a_{12} + b_1\frac{f_2(x_2)}{x_2}]}{b_2}$$

$$- \frac{(b_1x_2 - b_2x_1)b_2\frac{f_1(x_2)}{x_2} + kx_2}{b_2} \qquad (33)$$

where a_{11}, a_{12}, a_{21}, b_1, b_2, $f_1(x_2)$, $f_2(x_2)$ are as defined above and $k_1 > 0$.

Lemma 1: Consider the system (4)-(5)-(6)-(7)-(8)-(10) with the feedback controller $\delta = \tilde{\delta} + \arctan x_1$, where $\tilde{\delta}$ is given by (33) and x_1 and x_2 are as defined in (25)-(26). The origin is a globally asymptotically stable equilibrium.

Proof:

Since $\tilde{\delta}(x_1, x_2)$ as defined in equation (33) is a state-feedback controller that makes the zero equilibrium of the the closed-loop system asymptotically stable, $\delta = \tilde{\delta} + \arctan x_1$ will also make the zero equilibrium of system (31)-(32) asymptotically stable.

7.2 Control Design for the Overall System

In this sub-section, a state-feedback controller is designed for the overall system (4)-(5)-(6)-(7)-(8)-(10) in which road curvature $\rho(t)$=0 for all t≥0. Since there is no change in the curvature of the road, the equilibrium of the system (4)-(5)-(6)-(7)-(8)-(10) will have values of $(\beta, \psi, \delta, y_L, \rho_L)$ = (0,0,0,0,0). Therefore, the reference trajectory that the system has to follow will have parameters as defined below:

$$\beta_r(t) = \dot{\psi}(t) = \delta_r(t) = \dot{\delta}_r(t) = y_{L_r}(t) = \psi_{L_r}(t) = 0, \text{ for all t } \geq 0.$$

Definition 2: Let $\phi(x)$: $\mathbb{R} \to \mathbb{R}$ be a function as defined below:

$$\phi(x) = \begin{cases} 1, & \text{if x} > 2, \\ \sqrt{1 - (\sqrt{2} - x)^2}, & \text{if } \frac{\sqrt{2}}{2} < x \leq \sqrt{2}, \\ x, & \text{if } -\frac{\sqrt{2}}{2} \leq x \leq \frac{\sqrt{2}}{2}, \\ -\sqrt{1 - (\sqrt{2} - x)^2}, & \text{if } -\sqrt{2} \leq x \leq -\frac{\sqrt{2}}{2}, \\ -1, & \text{if x} < -\sqrt{2}, \end{cases}$$

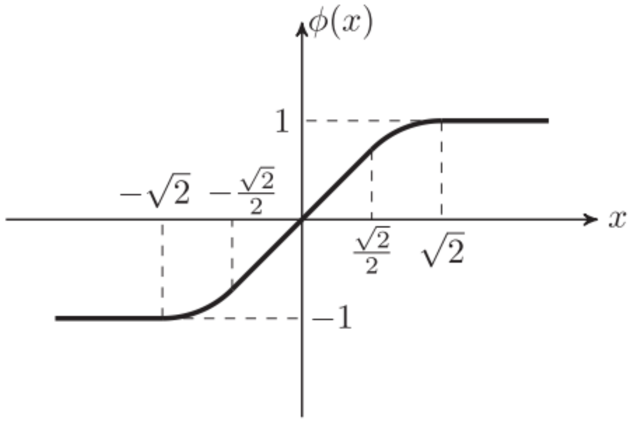

Fig 11: Graph of function $\phi(x)$

Reference [2] discusses about the control design for the feedforward systems. We can leverage this paper to prove Lemma 2 which is described below.

Lemma 2: Let us consider the lateral dynamics of the vehicle (4)-(6) together with the lane keeping dynamics (10). As per proposition [1] of reference [2], there exists $\kappa_1^*>0$, $\kappa_2^*>0$, $\epsilon_1^*>0$, and $\epsilon_2^*>0$ such that for any $\kappa_1 \in (0, \kappa_1^*)$, $\kappa_2 \in (0, \kappa_2^*)$, $\epsilon_1 \in (0, \epsilon_1^*)$, $\epsilon_1 \in (0, \epsilon_1^*)$ the zero equilibrium of the closed-loop system (31)-(32) with the controller

$$\delta = \delta^* = \tilde{\delta} + \arctan x_1 - \epsilon_1 \phi(\tfrac{\kappa_1}{\epsilon_1}\psi_L) - \epsilon_2 \phi(\tfrac{\kappa_2}{\epsilon_2} y_L) \tag{34}$$

is globally asymptotically stable. $\tilde{\delta}$ and x_1 are defined in (33) and (25) respectively.
Proof:

In this section, we are finding the state-feedback controller for the case in which road curvature $\rho = 0$. The lane keeping dynamics as described by system (8)-(10) are functions of variables β and $\dot{\psi}$ and is given below:

$$\dot{y_L} = v_x\beta + T_p v_x \dot{\psi} + v_x \psi_L, \tag{35}$$

$$\dot{\psi}_L = \dot{\psi} - v_x \rho, \tag{36}$$

Adding equations (25) and (26) to get the value of β,

$$x_1 + x_2 = 2\beta + \tfrac{l_f}{v_x}\dot{\psi} - \tfrac{l_r}{v_x}\dot{\psi}$$

$$x_1 + x_2 = 2\beta + (\tfrac{l_f}{v_x} - \tfrac{l_r}{v_x})\dot{\psi}$$

$$x_1 + x_2 = 2\beta + (\tfrac{l_f - l_r}{v_x})\ \dot{\psi}$$

Substituting the value of $\dot{\psi}$ as per equation (27),

$$x_1 + x_2 = 2\beta + (\tfrac{l_f - l_r}{v_x})\ (x_1 - x_2)\tfrac{v_x}{l_f + l_r}$$

$$x_1 + x_2 = 2\beta + (\tfrac{l_f - l_r}{l_f + l_r})\ (x_1 - x_2)$$

$$x_1 + x_2 - (\tfrac{l_f - l_r}{l_f + l_r})\ (x_1 - x_2) = 2\beta$$

$$x_1 - (\tfrac{l_f - l_r}{l_f + l_r})x_1 + x_2 + (\tfrac{l_f - l_r}{l_f + l_r})x_2 = 2\beta$$

$$(1 - \tfrac{l_f - l_r}{l_f + l_r})x_1 + (1 + \tfrac{l_f - l_r}{l_f + l_r})x_2 = 2\beta$$

$$(\tfrac{l_f + l_r - l_f + l_r}{l_f + l_r})x_1 + (\tfrac{l_f + l_r + l_f - l_r}{l_f + l_r})x_2 = 2\beta$$

$$(\tfrac{2l_r}{l_f + l_r})x_1 + (\tfrac{2l_f}{l_f + l_r})x_2 = 2\beta$$

$$(\tfrac{l_r}{l_f + l_r})x_1 + (\tfrac{l_f}{l_f + l_r})x_2 = \beta$$

Substituting values of β and $\dot{\psi}$ from above equations and in system (8)-(10),

$$\dot{y_L} = v_x(\tfrac{l_r}{l_f + l_r}x_1 + \tfrac{l_f}{l_f + l_r})x_2 + v_x\psi_L + T_p v_x(\tfrac{v_x}{l_f + l_r}x_1 - \tfrac{v_x}{l_f + l_r}x_2),$$

$$\dot{\psi}_L = \tfrac{v_x}{l_f + l_r}x_1 - \tfrac{v_x}{l_f + l_r}x_2 - v_x \cdot 0,$$

$$\dot{\psi}_L = \tfrac{v_x}{l_f + l_r}x_1 - \tfrac{v_x}{l_f + l_r}x_2,$$

System (29)-(30) as defined above can be written in the following form:

$$\dot{x_1} = a_{11}x_1 + a_{12}x_2 + f_1(x_2) + b_1\tilde{\delta}$$

$$\dot{x_2} = a_{21}x_1 + a_{22}x_2 + f_2(x_2) + b_2\tilde{\delta}$$

where, $f_1(x_2)$ and $f_2(x_2)$ are linear functions of $\arctan x_2$ and $x_2 = \beta - \frac{l_r}{v_x}\dot{\psi}$

In practical cases, the value of β (ratio of lateral speed and longitudinal speed) is less than 1 and quite small. Also, the value of yaw rate (rate of change of yaw angle) is small for most practical cases.

Therefore, the function $\arctan x_2$ can be approximated as 1.

Hence, the linear approximation of system (29)-(30) at the zero equilibrium point can be written as:

$$\dot{x} = \mathrm{Fx} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tilde{\delta},$$

where,

$$\mathrm{F} = \begin{bmatrix} a_{11} & a_{12} + \frac{2C_r l_r l_f}{I_z v_x} - \frac{2C_r}{mv_x} \\ a_{21} & a_{22} + \frac{2C_r l_r^1}{I_z v_x} - \frac{2C_r}{mv_x} \end{bmatrix}$$

We know that the zero equilibrium of the system (29)-(30) with the control law(33) is globally asymptotically stable. We will apply Proposition [1] of reference [2] twice to the closed-loop system (29)-(30)-(34)-(35)-(36). This will make the system globally asymptotically stable.

The signal δ^* calculated in (34) is smooth and twice differentiable.

Theorem 3: Let $\rho(t) = 0$. Consider the system (4)-(5)-(6)-(7)-(8)-(10) with the controller:

$T_c = T_s + J_s R_s[\ddot{\delta}^* - k_2(\dot{\delta} - \dot{\delta}^*)] + B_u R_s\dot{\delta}$ - $k_3[\dot{\delta} - \dot{\delta}^* + k_2(\dot{\delta} - \dot{\delta}^*)]$

$T_c = -\frac{2C_f\eta_t}{R_s}\beta$ - $\frac{2C_f l_f \eta_t}{R_s v_x}\dot{\psi} + \frac{2C_f\eta_t}{R_s}\delta + B_u R_s\dot{\delta} + J_s R_s[\ddot{\delta}^* - k_2(\dot{\delta} - \dot{\delta}^*)] - k_3[\dot{\delta} - \dot{\delta}^* + k_2(\dot{\delta} - \dot{\delta}^*)]$ (37)

where δ^* is calculated in (34). With this controller, the zero-equilibrium of the closed-loop system is globally asymptotically stable.

Proof: Using back-stepping principle of reference [3], the above controller can be designed.

7.3 State-Feedback Tracking Control Design for the Overall System

In this subsection, the state-feedback controller for the lateral control design of the overall system (4)-(5)-(6)-(7)-(8)-(10) with non-zero, but feasible, road curvature ρ(t) is discussed. ρ(t) is said to be feasible if the parameters β_r, $\dot{\psi}_r$, δ_r, y_{L_r}, $/psi_{L_r}$ and T_{c_r} exist, where T_{c_r} is the reference control input.

Taking the variables defined in equation (25)-(26) as reference, let us define the augmented reference signals x_{1_r} and x_{2_r} as:

$$x_{1_r} = \beta_r + \frac{l_f}{v_x}\dot{\psi}_r,$$

$x_{2_r} = \beta_r$ - $\frac{l_r}{v_x}\dot{\psi}_r,$

and the error signals x_{1_e} and x_{2_e} as:

$$x_{1_e} = x_1 - x_{1_r},$$
$$x_{2_e} = x_2 - x_{2_r}$$

Taking derivative with respect to time of x_{1_e} and x_{2_e} we get,

$$\dot{x}_{1_e} = \dot{x}_1 - \dot{x}_{1_r},$$
$$\dot{x}_{2_e} = \dot{x}_2 - \dot{x}_{2_r}$$

Taking system (31)-(32) as a reference, we can formulate:

$$\dot{x}_{1_r} = a_{11}x_{1_r} + a_{12}x_{2_r} + f_1(x_{2_r}) + b_1\tilde{\delta}_r, \quad (38)$$
$$\dot{x}_{2_r} = a_{21}x_{1_r} + a_{22}x_{2_r} + f_2(x_{2_r}) + b_2\tilde{\delta}_e, \quad (39)$$

Subtracting system (38)-(39) from system (31)-(32) we get,

$$\dot{x}_1 - \dot{x}_{1_r} = a_{11}(x_1 - x_{1_r}) + a_{12}(x_2 - x_{2_r}) + f_1(x_2) - f_1(x_{2_r}) + b_1(\tilde{\delta} - \tilde{\delta}_r),$$
$$\dot{x}_2 - \dot{x}_{2_r} = a_{21}(x_2 - x_{2_r}) + a_{22}(x_2 - x_{2_r}) + f_2(x_2) - f_2(x_{2_r}) + b_2(\tilde{\delta} - \tilde{\delta}_r),$$

From above relations,

$$\dot{x}_{1_e} = a_{11}x_{1_e} + a_{12}x_{2_e} + f_1(x_2) - f_2(x_{2_r}) + b_1\tilde{\delta}_e,$$
$$\dot{x}_{2_e} = a_{21}x_{1_e} + a_{22}x_{2_e} + f_2(x_2) - f_2(x_{2_r}) + b_2\tilde{\delta}_e,$$

where a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2, $f_1(x_2)$, and $f_2(x_2)$ have the same definitions as defined in ().

Also,

$$\tilde{\delta}_e = \tilde{\delta} - \tilde{\delta}_r$$
$$\tilde{\delta}_e = \delta - \arctan x_1 - \delta_r + \arctan x_{1_r}$$

Assumption 3: Assume that $x_2(t)x_{2_r}(t) >$ -1 for all t$\geq$0

Since in most of the driving scenarios the values of x_2 and x_{2_r} are small, the above assumption holds true.

In equation (),

$$f_1(x_2) = (\frac{2C_r l_r l_f}{I_z v_x} - \frac{2C_r}{m v_x})\arctan x_2 \quad (40)$$

$$f_1(x_{2_r}) = (\frac{2C_r l_r l_f}{I_z v_x} - \frac{2C_r}{m v_x})\arctan x_{2_r} \quad (41)$$

$$f_2(x_2) = (\frac{2C_r l_r^2}{I_z v_x} + \frac{2C_r}{m v_x})\arctan x_2 \quad (42)$$

$$f_2(x_{2_r}) = (\frac{2C_r l_r^2}{I_z v_x} + \frac{2C_r}{m v_x})\arctan x_{2_r} \quad (43)$$

Subtracting (41) from (40),

$$f_1(x_2) - f_1(x_{2_r}) = (\frac{2C_r l_r l_f}{I_z v_x} - \frac{2C_r}{m v_x})(\arctan x_2 - \arctan x_{2_r})$$

Subtracting (43) from (42),

$$f_2(x_2) - f_2(x_{2_r}) = (\frac{2C_r l_r^2}{I_z v_x} + \frac{2C_r}{m v_x})\ (\arctan x_2 - \arctan x_{2_r})$$

Consider ($\arctan x_2$ - $\arctan x_{2_r}$)

Let $\arctan x_2 =$ A and $\arctan x_{2_r} =$ B

$\therefore$ $x_2 = \tan(A)$ and $x_{2_r} = \tan(B)$

We know that,

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

$$\tan(A - B) = \frac{x_2 - x_{2_r}}{1 + x_2 x_{2_r}}$$

$$\text{A-B} = \arctan(\tfrac{x_2 - x_{2_r}}{1 + x_2 x_{2_r}})$$

$$\arctan x_2 \text{ - } \arctan x_{2_r} = \arctan(\tfrac{x_2 - x_{2_r}}{1 + x_2 x_{2_r}})$$

Let $h_1(x_{2_e}) \triangleq f_1(x_2)$ - $f_1(x_{2_r})$

$$= (\tfrac{2C_r l_r l_f}{I_z v_x} \text{ - } \tfrac{2C_r}{m v_x})\arctan(\tfrac{x_2 - x_{2_r}}{1 + x_2 x_{2_r}})$$

$$= (\tfrac{2C_r l_r l_f}{I_z v_x} \text{ - } \tfrac{2C_r}{m v_x})\arctan(\tfrac{x_{2e}}{1 + x_2 x_{2_r}})$$

Let $h_2(x_{2_e}) \triangleq f_2(x_2)$ - $f_2(x_{2_r})$

$$= (\tfrac{2C_r l_r^2}{I_z v_x} + \tfrac{2C_r}{m v_x})\arctan(\tfrac{x_2 - x_{2_r}}{1 + x_2 x_{2_r}})$$

$$= (\tfrac{2C_r l_r^2}{I_z v_x} + \tfrac{2C_r}{m v_x})\arctan(\tfrac{x_{2e}}{1 + x_2 x_{2_r}})$$

The function $\arctan x_2$ is continuous at x=0.
This implies that functions $h_1(x_{2_e})$ and $h_2(x_{2_e})$ are also continuous at x=0.
Also, the value of parameters a_{11}, a_{21}, b_1, and b_2 of system () are as same as system ().
Hence, the system () satisfies all the hypothesis of Theorem 2.
Therefore, a state-feedback controller "u" of the form as defined in () can be designed for the system () and is given below:

$$\tilde{\delta_e}(x_1, x_2) = \frac{-a_{21}x_{1_e} - a_{22}x_2 - h_2(x_{2_e}) + b_1(b_1 a_{21} - b_2 a_{11})x_{1_e}}{b2} - \frac{(b_1 x_{2_e} - b_2 x_{1_e})[b_1 a_{22} - b_2 a_{12} + b_1 \frac{h_2(x_{2_e})}{x_{2_e}}]}{b_2} - \frac{(b_1 x_{2_e} - b_2 x_{1_e}) b_2 \frac{h_1(x_{2_e})}{x_{2_e}} + k_1 x_{2_e}}{b_2}$$

where a_{11}, a_{12}, a_{21}, b_1, b_2, $f_1(x_2)$, $f_2(x_2)$ are as defined above and $k_1 > 0$.

Theorem 4: Consider the system (4)-(5)-(6)-(7-(8)-(10) controlled by the feedback controller

$$T_c = T_{c_r} \text{ - } \tfrac{2C_f \eta_t}{R_s}\beta_e \text{ - } \tfrac{2C_f l_f \eta_t}{R_s v_x}\dot{\psi}_e + \tfrac{2C_f \eta_t}{R_s}\delta_e + B_u R_s \dot{\delta}_e + J_s R_s[\ddot{\delta}_e^* - k_2(\dot{\delta}_e - \dot{\delta}_e^*)] - k_3[\dot{\delta}_e - \dot{\delta}_e^* + k_2(\dot{\delta}_e - \dot{\delta}_e^*)]$$

where,
$\delta_e^* = \tilde{\delta}_e + \arctan x_{1_e}$ - $\epsilon_1 \phi_e(\frac{\kappa_1}{\epsilon_1}\psi_{L_e})$ - $\epsilon_2 \phi(\frac{\kappa_2}{\epsilon_2} y_{L_e})$
$\delta_e = \delta$ - δ_r,
$\beta e = \beta$ - β_r,
$\dot{\psi}_e = \dot{\psi}$ - $\dot{\psi}_r$

8 Conclusion

The controller designed in the Theorem 4 leverages the concepts of back-stepping and feed-forward to design a controller for a class of non-linear under-actuated system. Even if the designed controller is for the lane keeping case, it can also be used for lane changing case. The designed controller can follow any feasible reference trajectories at a constant speed and lateral deviation is able to converge to zero within a short time. A driver model is used to compare the output of the controller. Since the model was quite difficult to implement in Simulink, we were not able to simulate the results due to knowledge and time constraint.

9 Bibliography

1) R. Rajamani, Vehicle Dynamics and Control. New York, NY, USA: Springer, 2012.
2) G. Kaliora and A. Astolfi, “Nonlinear control of feedforward systems with bounded signals,” IEEE Trans. Autom. Control, vol. 49, no. 11, pp. 1975–1990, Nov. 2004.
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10. Simulation results and analysis

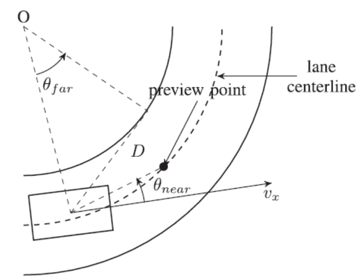
Considering constant forward speed of car [v_x = 10m/s] and parameters values for the car as mentioned below.

TABLE I
VEHICLE PARAMETERS

η	0.15	B_u	2.5	I_z	1500
C_f	170390	C_r	195940	J_s	0.05
l_f	1.48	l_r	1.12	m	1625
R_s	12				

The model is simulated in MATLAB/SIMULINK under these three separate case studies: uniform circular motion, tortuous path tracking and spiral tracking to perform and show effectiveness of the newly developed control system. Results prove that suggested feedback controller is fast and effective in tracking reference trajectories with time-varying curvature.

A. Driver Model



. Visual angle definitions in driving scenarios.

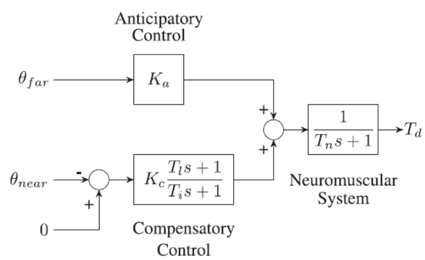


Fig. 6. Simplified two-level driver model.

TABLE II
TYPICAL PARAMETER VALUES FOR THE DRIVER MODEL

T_l	T_i	T_n	D	T_p	K_a	K_c
1.16	0.14	0.11	15	2	56.97	36.13

Tasks such as modelling of human perception and action are challenging. For advancements in vehicles and their dynamics, it is necessary to understand the interaction between driver and vehicle dynamics and simulation of whole driver-vehicle system.

The driver model combines a two-level visual strategy and high-frequency compensation based on kinesthetics feedback. It can accurately represent the driver's characteristics over a wide frequency range, while giving more insight on the internal driver behavior. The steering wheel angle represents the third input of the driver model and the interaction between driver and vehicle is modelled by considering the driver feedback torque feeling.

The model itself consists of three subblocks of control systems, namely anticipatory steering control, compensatory steering control and kinesthetics feedback and neuromuscular system.

Anticipatory steering control leans on the idea that drivers use both near and far regions of the road during steering. The self-steering behavior describes the steering properties of a vehicle independent from the steering influence of the driver. This steering characteristic can be felt by the driver using information from the far region such as the angle between the actual vehicle heading direction and the gaze direction to the far point.

In compensatory steering control, the driver uses his visual and kinesthetic perception to compensate the instantaneous variations of the trajectory. The information of the near region is used by the driver to maintain a central lane position and correct the vehicle's current position within the lane position.

Kinesthetics feedback and neuromuscular system is responsible for the high frequency compensation. The kinesthetic control of the driver can be divided into perception and action. The perceptual part represents how the feedback torque (self-aligning torque and other resistant torque) given by the steering wheel and felt by the driver through arm muscular and joint proprioception informs the driver. The neuromuscular system which represents the muscle activation dynamics consists of a time delay and a first order low-pass filter, where T_N is the neuromuscular lag time constant. The output of the neuromuscular system represents the torque provided by the driver.

The model is based on the analysis of the sensorimotor processes involved in driving, so that the elements of the model structure can be identified as known perceptual and motor components. The driver model integrates both anticipatory and compensatory visual strategies and considers the interaction between driver and vehicle through the sensation of torque feedback.

We use the above mentioned simplified two-level driver model to compare the performance with the feedback controller developed in the paper. The model is a single-output multi-input model on the observation of a near point [θ_{near}] and far point [θ_{far}]. The visual far angle can be approximated as:

$$\theta_{far} \approx D\rho.$$

Preview distance: $x_L = v_x T_p,$ T_p is driver's preview time.

The visual near angle can be calculated as:

$$\theta_{near} = \frac{y_L}{x_L} = \frac{y_L}{v_x T_p},$$

y_L is lateral deviation

T_l -> Lead time constant of the transfer function representing the compensatory steering control of the driver.

T_i -> Lag time constant of the transfer function representing the compensatory steering control of the driver.

T_n -> Constant neuromuscular lag time of the driver.

K_a -> Anticipatory control of the driver.

K_c -> Proportional gain of the driver w.r.t the visual near angular error.

B. Uniform circular motion

For uniform circular motion case study, the reference motion of the car should be in a uniform circular motion. Thus, the curvature of the path/trajectory is constant.

$$\rho(t) = 0.02 \text{ for all } t \geq 0.$$

Assume that the reference path changes suddenly from a straight line to a circle with 50m as radius, because the human driver would start turning steering wheel before starting the circular lane.

Performance of the feedback controller and the modeled human driver are compared and displayed together as simulation results. The feedback controller is represented by solid blue line, and human driver model as dashed red line. Lateral deviation, heading error, yaw rate and ratio between the lateral and longitudinal speed of the vehicle converges to steady-state or reference values. Results show that both human driver model and feedback controller can control the system. When the curvature of the path is constant, both the controllers were able to track the reference path. After comparing lateral deviation y_L for both systems, it is evident from the plot that the response of the closed loop system with the feedback controller is much quicker than that of the human driver. Plot shows with feedback controller, lateral deviation is almost zero well within 4s, on other hand human driver takes about 20s. It also states that the maximum lateral deviation is minimized to 0.3m from 3.2m after using feedback controller.

C. Tortuous path tracking

For this case, winding and tortuous path is the reference trajectory. Curvature of the reference path can be defined as:

$$\rho(t) = 0.02\sin(0.1t), \forall t \geq 0.$$

From the plot, it is clear that the car with human driver is not able to track the reference trajectory. The assumed initial lateral deviation and the initial yaw error are 4 m and 0.4 rad respectively. In this case, the feedback controller can make system settle down within 4 s, while the human driver model keeps lateral deviation nonzero even after 70 s. The differences between the time-history plot of the heading error, yaw rate, and ratio between lateral speed and longitudinal speed for two different closed loop system is small, but difference in lateral deviation is significant because of the following:

$$\dot{y}_L = 10\beta + 20\dot{\psi} + 10\psi_L.$$

The time histories of the variable y_L for two closed loop systems would have large differences, if constant longitudinal speed v_x is increased.

D. Spiral tracking

For this case study, the reference path is spiral, and its curvature is given by:

$$\rho(t) = 0.001t, \forall\, 0 \leq t \leq 40.$$

As a spiral, the radius of the reference path reduces uniformly and continuously from $+\infty$ (+ve infinity) to 25 m. The assumed initial lateral deviation and yaw error are 4 m and 0.4 rad respectively. From plots, it can be seen that human driver model does not settle down or converge lateral deviation values to reference values even after long time. Comparing settling time for all three cases, the closed loop system with feedback controller has settling time less than 4 s no matter what the reference is. Basically, regardless of the variation of the variable ρ (rho), the lateral deviation converges to zero.

ENPM667 Control of Robotic Systems

Lateral Control of an Autonomous Vehicle

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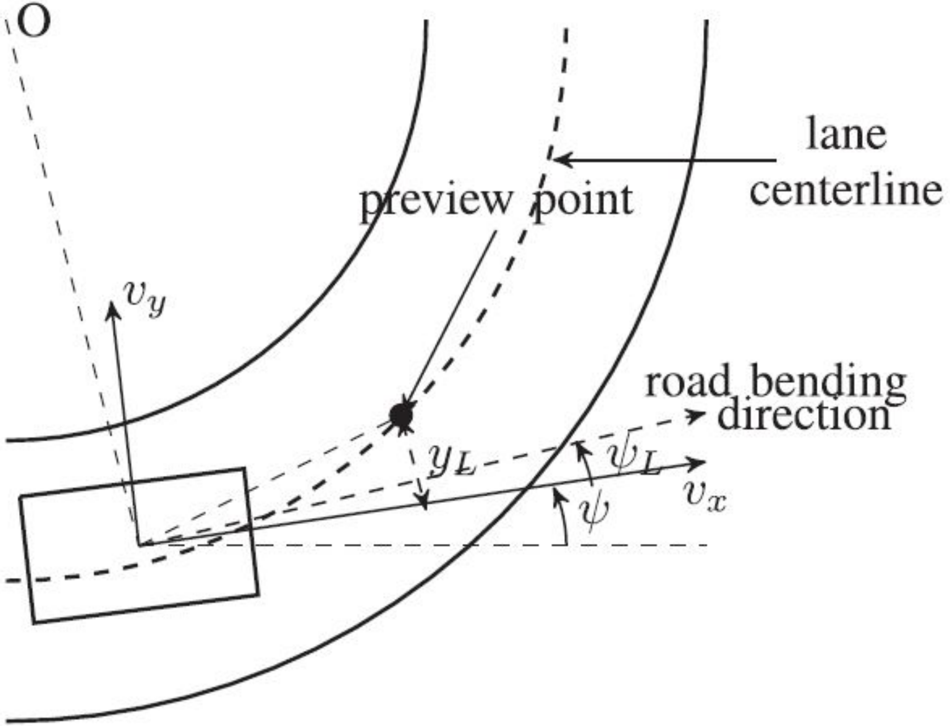
Concept of preview point



. Graphical definitions of variables y_L and ψ_L in lane keeping cases.

System Model

$$\dot{\beta} = \frac{2C_f\delta}{mv_x} - \dot{\psi} - \frac{2C_f}{mv_x}\arctan(\beta + \frac{l_f\dot{\psi}}{v_x}) - \frac{2C_r}{mv_x}\arctan(\beta - \frac{l_r\dot{\psi}}{v_x})$$

$$\ddot{\psi} = \frac{2C_f l_f}{I_z} - \frac{2C_f l_f}{I_z}\arctan(\beta + \frac{l_f\dot{\psi}}{v_x}) + \frac{2C_r l_r}{I_z}\arctan(\beta - \frac{l_r\dot{\psi}}{v_x})$$

$$\dot{y}_L = v_x\beta + T_p v_x\dot{\psi} + v_x\psi_L$$

$$\dot{\psi}_L = \dot{\psi} - v_x\rho,$$

Preliminary Theorems

Theorem 1:

$$\dot{s}_1 = q_{11}s_1 + p_1(s_2),$$
$$\dot{s}_2 = p_2(s_1, s_2) + gu,$$

$$u(s_1, s_2) = -\frac{ks_2 + s_1\frac{p_1(s_2)}{s_2} + p_2(s_1, s_2)}{g};$$

Theorem 2:

$$\dot{s}_1 = a_{11}s_1 + a_{12}s_2 + f_1(s_2) + b_1 u$$

$$\dot{s}_2 = a_{21}s_1 + a_{22}s_2 + f_2(s_2) + b_2 u$$

$$u(s_1, s_2) = \frac{-a_{21}s_1 - a_{22}s_2 - f_2(s_2) + b_1(b_1 a_{21} - b_2 a_{11})s_1}{b_2}$$

$$- \frac{(b_1 s_2 - b_2 s_1)\left[b_1 a_{22} - b_2 a_{12} + b_1 \frac{f_2(s_2)}{s_2}\right]}{b_2}$$

$$- \frac{-(b_1 s_2 - b_2 s_1)b_2 \frac{f_1(s_2)}{s_2} + k s_2}{b_2}, \qquad (13)$$

Applying Theorem 1 and 2 to system model:

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + f_1(x_2) + b_1\tilde{\delta},$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + f_2(x_2) + b_2\tilde{\delta},$$

$$\tilde{\delta} = \frac{-a_{21}x_1 - a_{22}x_2 - f_2(x_2) + b_1(b_1a_{21} - b_2a_{11})x_1}{b_2}$$

$$- \frac{(b_1x_2 - b_2x_1)\left[b_1a_{22} - b_2a_{12} + b_1\frac{f_2(x_2)}{x_2}\right]}{b_2}$$

$$- \frac{-(b_1x_2 - b_2x_1)b_2\frac{f_1(x_2)}{x_2} + k_1x_2}{b_2},$$

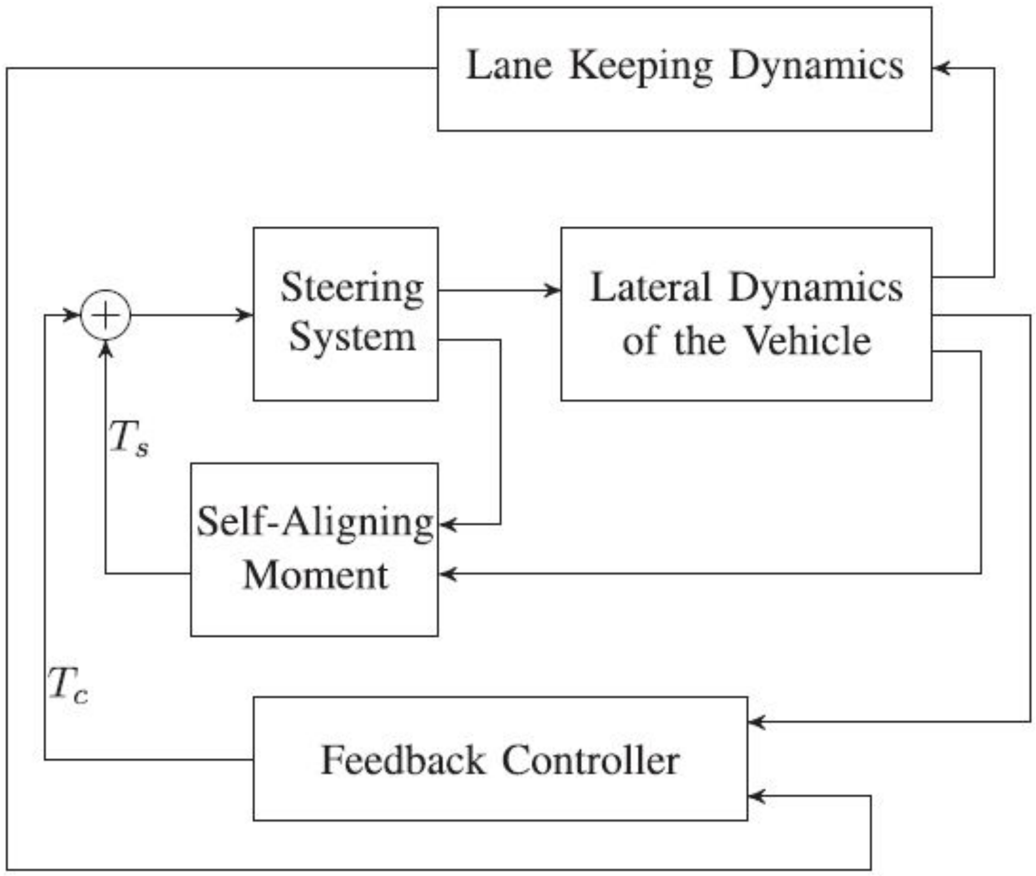
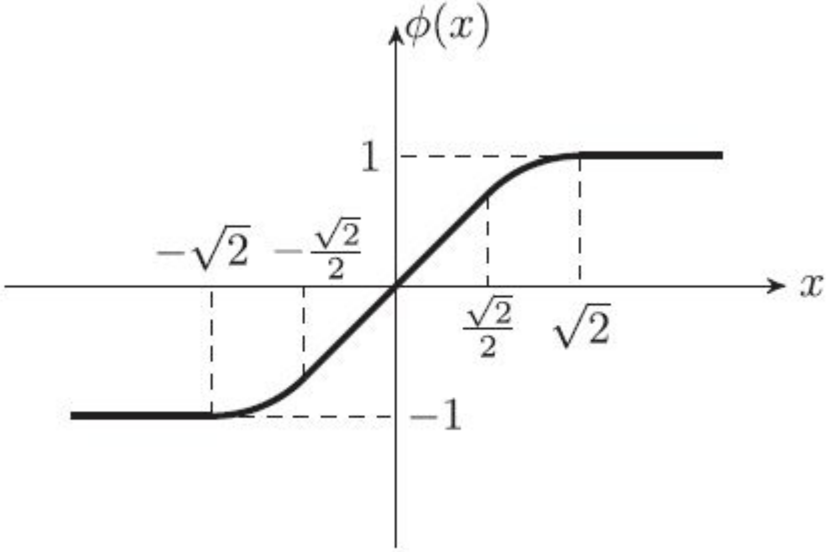


Fig. 3. Block diagram of the system.



The graph of the function $\phi(x)$ in Definition 2.

$$\phi(x) = \begin{cases} 1, & \text{if } x > \sqrt{2}, \\ \sqrt{1-(\sqrt{2}-x)^2}, & \text{if } \frac{\sqrt{2}}{2} < x \leq \sqrt{2}, \\ x, & \text{if } -\frac{\sqrt{2}}{2} \leq x \leq \frac{\sqrt{2}}{2}, \\ -\sqrt{1-(\sqrt{2}-x)^2}, & \text{if } -\sqrt{2} \leq x < -\frac{\sqrt{2}}{2} \\ -1, & \text{if } x < -\sqrt{2}. \end{cases}$$

Concept of feedforward system:

$$\delta = \delta^* = \tilde{\delta} + \arctan x_1 - \epsilon_1 \phi\left(\frac{\kappa_1}{\epsilon_1}\psi_L\right) - \epsilon_2 \phi\left(\frac{\kappa_2}{\epsilon_2} y_L\right)$$

Exploiting backstepping technique

$$
\begin{aligned}
T_c = {} & T_s + J_s R_s[\ddot{\delta}^* - k_2(\dot{\delta} - \dot{\delta}^*)] + B_u R_s \dot{\delta} \\
& - k_3[\dot{\delta} - \dot{\delta}^* + k_2(\dot{\delta} - \dot{\delta}^*)] \\
= {} & -\frac{2C_f \eta_t}{R_s}\beta - \frac{2C_f l_f \eta_t}{R_s v_x}\dot{\psi} + \frac{2C_f \eta_t}{R_s}\delta + B_u R_s \dot{\delta} \\
& + J_s R_s[\ddot{\delta}^* - k_2(\dot{\delta} - \dot{\delta}^*)] - k_3[\dot{\delta} - \dot{\delta}^* + k_2(\dot{\delta} - \dot{\delta}^*)],
\end{aligned}
$$

Applying the concept of feedforward system and backstepping technique:

$$\dot{x}_{1e} = a_{11}x_{1e} + a_{12}x_{2e} + f_1(x_2) - f_1(x_{2r}) + b_1\tilde{\delta}_e$$

$$\dot{x}_{2e} = a_{21}x_{1e} + a_{22}x_{2e} + f_2(x_2) - f_2(x_{2r}) + b_2\tilde{\delta}_e$$

Desired controller:

$$T_c = T_{c_r} - \frac{2C_f\eta_t}{R_s}\beta_e - \frac{2C_f l_f \eta_t}{R_s v_x}\dot{\psi}_e + \frac{2C_f\eta_t}{R_s}\delta_e + B_u R_s \dot{\delta}_e$$
$$+ J_s R_s[\ddot{\delta}_e^* - k_2(\dot{\delta}_e - \dot{\delta}_e^*)] - k_3[\dot{\delta}_e - \dot{\delta}_e^* + k_2(\dot{\delta}_e - \dot{\delta}_e^*)]$$

where

$$\delta_e^* = \tilde{\delta}_e + \arctan x_{1e} - \epsilon_1 \phi_e(\frac{\kappa_1}{\epsilon_1}\psi_{L_e}) - \epsilon_2 \phi(\frac{\kappa_2}{\epsilon_2} y_{L_e}),$$

$$\delta_e = \delta - \delta_r, \quad \beta_e = \beta - \beta_r, \quad \dot{\psi}_e = \dot{\psi} - \dot{\psi}_r,$$

$$\begin{aligned}\tilde{\delta}_e = & \frac{-a_{21}x_{1e} - a_{22}x_{2e} - h_2(x_{2_e}) + b_1(b_1 a_{21} - b_2 a_{11})x_{1e}}{b_2} \\ & - \frac{(b_1 x_{2e} - b_2 x_{1e})\left[b_1 a_{22} - b_2 a_{12} + b_1 \frac{h_2(x_{2e})}{x_{2e}}\right]}{b_2} \\ & - \frac{-(b_1 x_{2e} - b_2 x_{1e}) b_2 \frac{h_1(x_{2e})}{x_{2e}} + k_1 x_{2e}}{b_2},\end{aligned}$$

Driver model for comparison with the designed

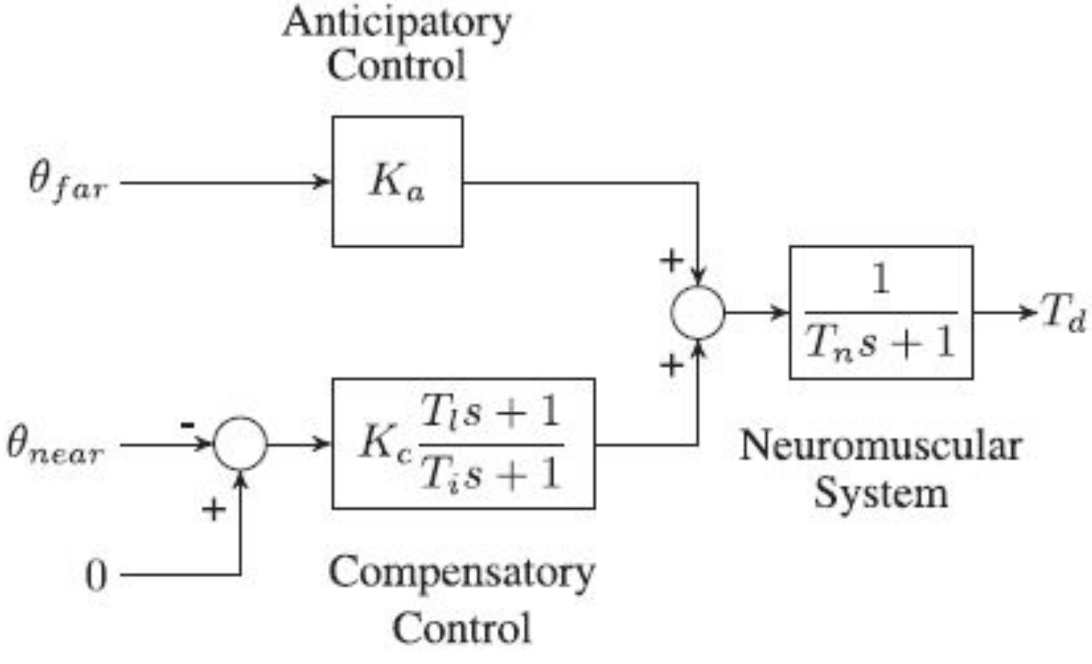


Fig. 6. Simplified two-level driver model.

TABLE II
TYPICAL PARAMETER VALUES FOR THE DRIVER MODEL

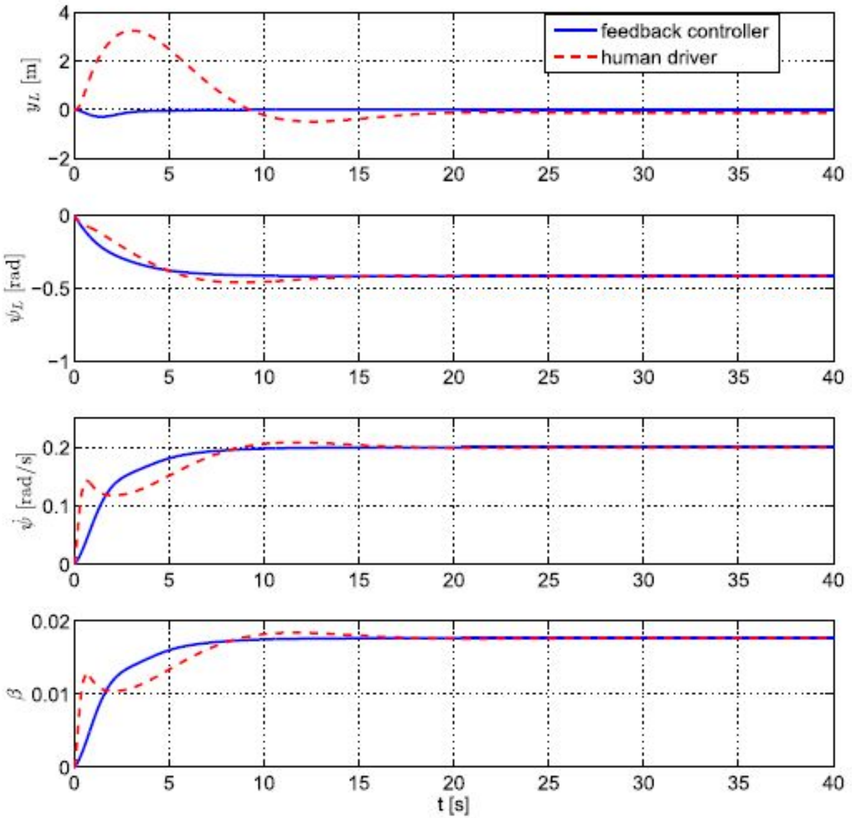
T_l	T_i	T_n	D	T_p	K_a	K_c
1.16	0.14	0.11	15	2	56.97	36.13

TABLE I
VEHICLE PARAMETERS

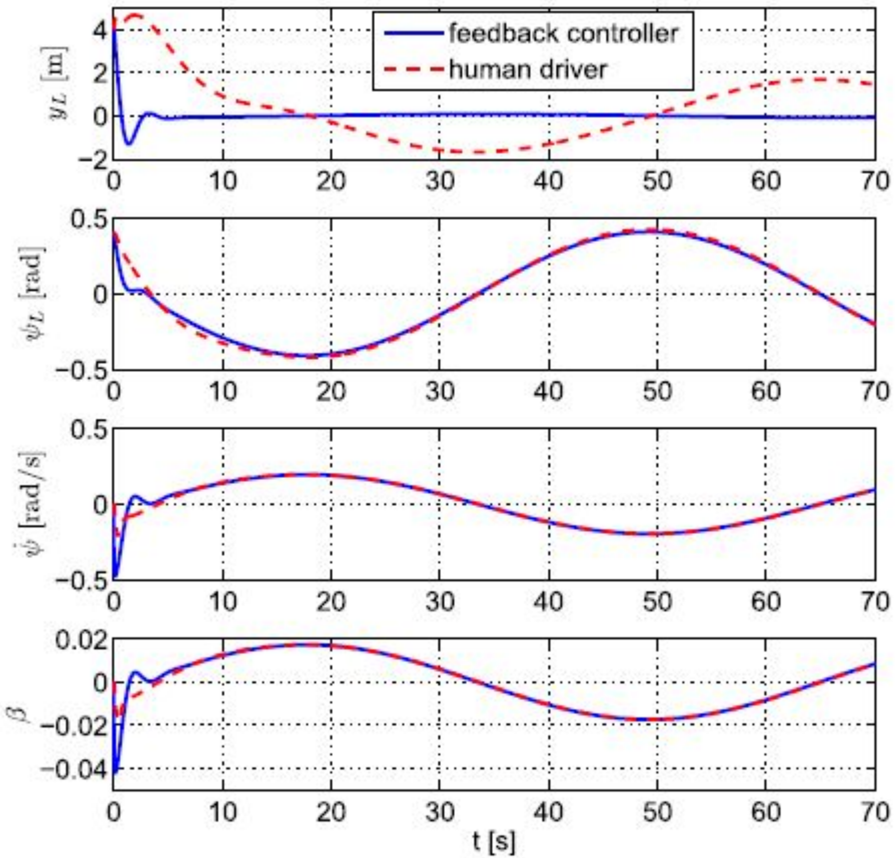
η	0.15	B_u	2.5	I_z	1500
C_f	170390	C_r	195940	J_s	0.05
l_f	1.48	l_r	1.12	m	1625
R_s	12				

Simulation and Results

1. Uniform Circular Motion



2. Tortuous path tracking



3. Spiral Tracking