GATE 2023-ME-50

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EE23BTECH11050

Question GATE 23 ME 50:
The initial value problem $\frac{dy}{dt} + 2y = 0, y(0) = 1$ is solved numerically using the forward Euler's method with a constant and positive time step of δ.
Let y_n represent the numerical solution obtained after n steps. The condition $|y_{n+1}| \leq |y_n|$is satisfied if and only if δ does not exceed
Solution:
Numerical solution: -
By forward Euler's method formula

$$y(n+1) = y(n) + \delta f(x, y) \tag{1}$$

From question we get

$$\frac{dy}{dx} = -2y = f(x, y) \tag{2}$$

From (2) in (1)

$$y(n+1) - y(n) = -2\delta y(n) \tag{3}$$
$$y(n+1) = (1 - 2\delta)y(n) \tag{4}$$
$$y(n) \xleftrightarrow{Z} Y(z) \tag{5}$$
$$y(n+1) \xleftrightarrow{Z} zY(z) - y(0) \tag{6}$$
$$\implies zY(z) - y(0) = (1 - 2\delta)Y(z) \tag{7}$$
$$Y(z) = \frac{1}{z - 1 + 2\delta} \tag{8}$$
$$\frac{1}{z - (1 - 2\delta)} \xleftrightarrow{Z} (1 - 2\delta)^n u(n) \tag{9}$$

For good approximation we choose δ = 0.4

$$y(n) = (0.2)^n u(n) \tag{10}$$

Now using the condition given in question

$$|y(n+1)| \leq |y(n)| \tag{11}$$
$$|(1 - 2\delta)^2| \leq |1 - 2\delta| \tag{12}$$
$$|1 - 2\delta| \leq 1 \tag{13}$$
$$0 \leq \delta \leq 1 \tag{14}$$

From this we can say that the maximum value of δ is 1

Theoritical solution: -
By properties of Laplace transform: -

$$Y(s) = \mathcal{L}y(s) \tag{15}$$
$$\mathcal{L}y' = sY(s) - y(0) \tag{16}$$

Given equation: -

$$y' + 2y = 0 \tag{17}$$
$$\mathcal{L}(y' + 2y) = 0 \tag{18}$$

From (15) and (16)

$$sY(s) - 1 + 2Y(s) = 0 \tag{19}$$
$$\frac{1}{s+2} = Y(s) \tag{20}$$
$$y(t) = \mathcal{L}^{-1}Y(s) \tag{21}$$
$$\implies y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) \tag{22}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s+k}\right) = e^{-kt}u(t) \tag{23}$$
$$\implies y(t) = e^{-2t}u(t) \tag{24}$$

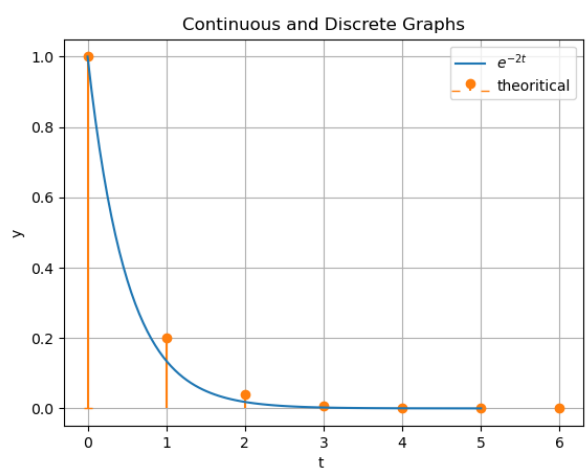

Fig. 1. simulation vs analysis