Discreet 12.9.5.24

HIBA MUHAMMED
EE23BTECH11026

Problem Statement

If S_1, S_2, S_3 are the sum of the first n natural numbers, their squares, and their cubes, respectively, show that

$$9(S\,2)^2{=}(S\,3)(1{+}8(S\,1))$$

Solution

Sequence	Expression	Description
s_1	$\frac{n(n+1)}{2}$	sum of n natural numbers
s_2	$\frac{n(n+1)(2n+1)}{6}$	sum of squares
s_3	$\left(\frac{n(n+1)}{2}\right)^2$	sum of cubes
x_1	$x_1(n) = nu(n)$	
x_2	$x_2(n) = n^2u(n)$	
x_3	$x_3(n) = n^3u(n)$	

TABLE I
Input Equations

$$nu(n) \stackrel{Z}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \tag{1}$$

$$n^2u(n) \stackrel{Z}{\longleftrightarrow} \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3}, \quad |z| > 1 \tag{2}$$

$$n^3u(n) \stackrel{Z}{\longleftrightarrow} \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}, \quad |z| > 1 \tag{3}$$

$$n^4u(n) \stackrel{Z}{\longleftrightarrow} \frac{z^{-1}(1+11z^{-1}+11z^{-2}+z^{-3})}{(1-z^{-1})^5} \tag{4}$$

$$x(n) \stackrel{Z}{\longleftrightarrow} X(z) \tag{5}$$

$$y(x) = x(n) * u(n) \tag{6}$$

$$Y(z) = X(z) \cdot u(z) \tag{7}$$

from (1) to (7)

1)

$$X_1(z) = \frac{z^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \tag{8}$$

$$Y_1(z) = \frac{z^{-1}}{(1-z^{-1})^3} \tag{9}$$

$$Y_1(z) = \frac{-1}{(1-z^{-1})^2} + \frac{1}{(1-z^{-1})^3} \tag{10}$$

$$y_1(n) = n\frac{(n+1)}{2}u(n) \tag{11}$$

$$\tag{12}$$

2)

$$X_2(z) = \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3}, \quad |z| > 1 \tag{13}$$
$$Y_2(z) = \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^4} \tag{14}$$
$$Y_2(z) = \frac{1}{(1-z^{-1})^2} - \frac{3}{(1-z^{-1})^3} + \frac{2}{(1-z^{-1})^4} \tag{15}$$
$$y_2(n) = \frac{(n)(n+1)(2n+1)}{6}u(n) \tag{16}$$
$$\tag{17}$$

3)

$$X_3(z) = \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}, \quad |z| > 1 \tag{18}$$
$$Y_3(z) = \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^5} \tag{19}$$
$$Y_3(z) = \frac{-1}{(1-z^{-1})^2} + \frac{7}{(1-z^{-1})^3} - \frac{12}{(1-z^{-1})^4} + \frac{6}{(1-z^{-1})^5} \tag{20}$$
$$y_3(n) = \left(\frac{(n)(n+1)}{2}\right)^2 u(n) \tag{21}$$

$y_2^2 = (y_3)(1 + 8(y_1))$

$$\text{LHS} = 9(y_2)^2 = 9\left(\frac{(n+1)(n)(2n+1)}{6}\right)^2 u(n) \tag{22}$$
$$= n^6u(n)^2 + 3n^5u(n)^2 + \frac{13}{4}n^4u(n)^2 + \frac{3}{2}n^3u(n)^2 + \frac{1}{4}n^2u(n)^2 \tag{23}$$
$$= n^6 + 3n^5 + \frac{13}{4}n^4 + \frac{3}{2}n^3 + \frac{1}{4}n^2 \qquad n \geq 0 \tag{24}$$
$$\text{RHS} = (y_3)(1 + 8(y_1)) = \left(\frac{(n+1)(n+2)}{2}\right)^2 u(n)(1 + 8\left(\frac{(n+1)(n+2)}{2}\right)u(n)) \tag{25}$$
$$= n^6u(n)^3 + 3n^5u(n)^3 + 3n^4u(n)^3 + \frac{1}{4}n^4u(n)^2 + n^3u(n)^3 + \frac{1}{2}n^3u(n)^2 + \frac{1}{4}n^2u(n)^2 \tag{26}$$
$$= n^6 + 3n^5 + \frac{13}{4}n^4 + \frac{3}{2}n^3 + \frac{1}{4}n^2 \qquad n \geq 0 \tag{27}$$

LHS=RHS

$$9(S_2)^2 = (S_3)(1 + 8(S_1))$$

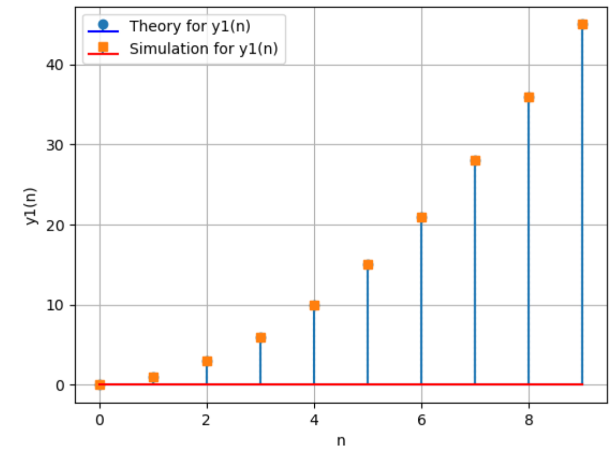

Fig. 1. Simulation vs Theory for y1(n)

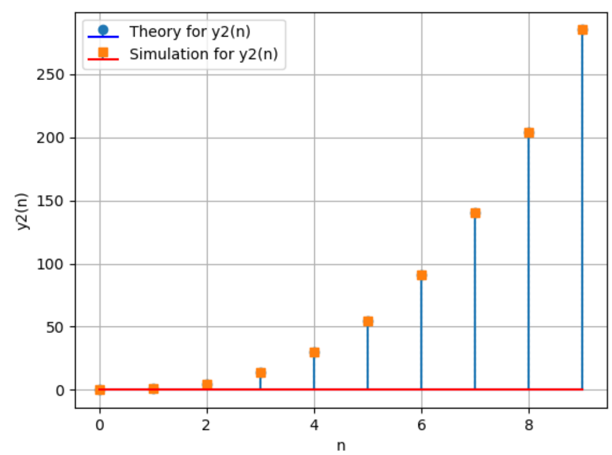

Fig. 2. Simulation vs Theory for y2(n)

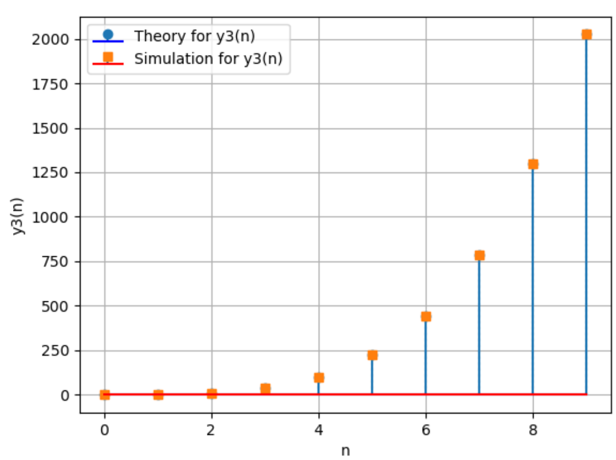

Fig. 3. Simulation vs Theory for y3(n)