

# Introduction to Optimization

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# Outline

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# Introduction to Optimization

- Imagine a situation where there is only 1 chocolate and you have to share it with your sibling... what will you do? You will break it into 2 exactly equal pieces and share it. This is a very simple example of optimization.
- Optimization is making the most effective use of a situation or resource.
- An optimization problem consists of maximizing or minimizing a function.
- We do so by choosing values from a set and computing the value of the function.



# Representation of An Optimization Problem

An optimization problem, has the form

$$\text{minimize } f_0(x) \tag{1}$$

$$\text{subject to } f_i(x) \leq b_i, i = 1, \dots, m \tag{2}$$

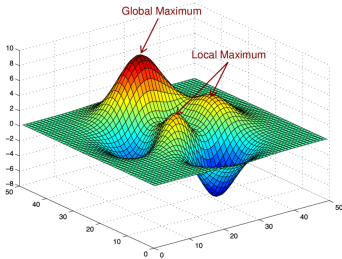
- $x \in \mathbb{R}$
- $b_i \in \mathbb{R}$
- $m$  represents the number of constraints

## Where do we optimize? (Applications)

- Portfolio optimization: we seek the best way to invest some capital in a set of  $n$  assets.
- Device sizing: In electronic design, it is the task of choosing the width and length of each device in an electronic circuit.
- Data fitting: We find a mathematical model, from a family of potential mathematical models, that best fits some observed data and prior information.

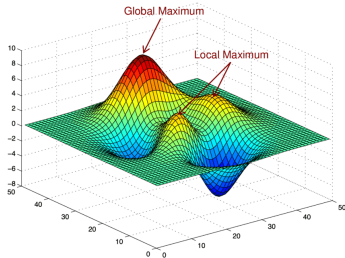
# Local Optimization

- A point which minimizes the objective function among feasible points that are near it,
- It may not be the lowest point in the overall function.



# Global Optimization

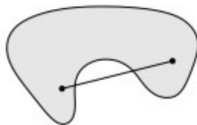
A point where the function value is smaller than or equal to the value at all other feasible points.



# Convex Set

A set  $C$  is convex if

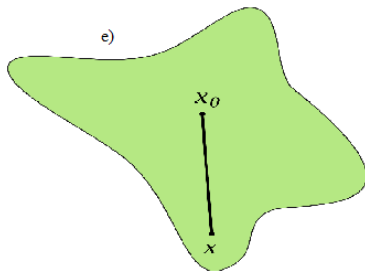
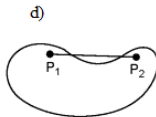
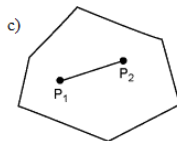
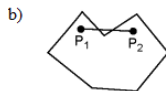
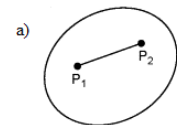
- the line segment between any two points in  $C$  lies in  $C$ ,
- i.e., if for any  $x_1, x_2 \in C$  and any  $\theta$  with  $0 \leq \theta \leq 1$ , we have  $\theta x_1 + (1 - \theta)x_2 \in C$ .





# Exercise

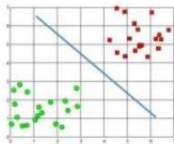
Identify the convex sets



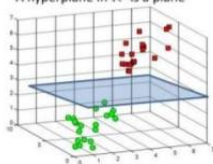
# Examples: Hyperplanes and Halfspaces

- A hyperplane is a set of the form  $\{x|a^T x = b\}$  where  $a \in \mathbb{R}^n$ ,  $a \neq 0$  and  $b \in \mathbb{R}$
- One halfspace is determined by  $a^T x \geq b$
- Another halfspace is determined by  $a^T x \leq b$
- A hyperplane divides  $\mathbb{R}^n$  into two halfspaces.

A hyperplane in  $\mathbb{R}^2$  is a line



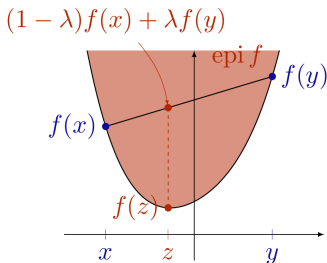
A hyperplane in  $\mathbb{R}^3$  is a plane



A hyperplane in  $\mathbb{R}^n$  is an  $n-1$  dimensional subspace

# Convex Function

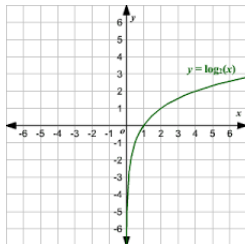
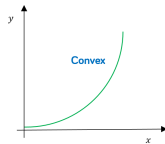
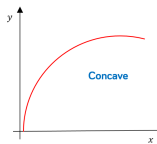
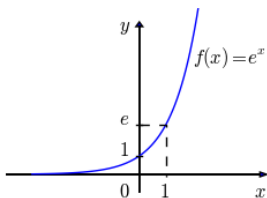
- A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex if **dom**  $f$  is a convex set
- if for all  $x, y \in \mathbf{dom} f$ ,  $0 \leq \lambda \leq 1$ , we have
$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$
- This inequality is also called *Jensen's Inequality*.
- Geometrically, line segment between  $(x, f(x))$  and  $(y, f(y))$ , which is the chord from  $x$  to  $y$ , lies above the graph of  $f$



# Convex Function Examples

- Exponential :  $e^{ax}$ ,  $a \in \mathbb{R}$  is convex on  $\mathbb{R}$
- Powers of absolute values  $|x|^p$ , for  $p \geq 1$  is convex on  $\mathbb{R}$

Differentiate and see if they are convex !



# What is Convex optimization?

- A convex optimization problem is one of the form
$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, i = 1, \dots, m\end{array}$$
- where the functions  $f_0, \dots, f_m : \mathbb{R}^n \rightarrow \mathbb{R}$  are convex, i.e., satisfy

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

- $\forall x, y \in \mathbb{R}^n$
- all  $\alpha, \beta \in \mathbb{R}$
- with  $\alpha + \beta = 1$  ,  $\alpha \geq 0$ ,  $\beta \geq 0$

.

# Convexity Conditions: First Order Convexity

- If  $f$  is differentiable (i.e., its gradient exists at each point in **dom**  $f$ )
- $f$  satisfies the following condition:  $f(y) \geq f(x) + f(x)'(y - x)$   
(univariate case)
- $f$  satisfies the following condition:  $f(y) \geq f(x) + \nabla f(x)^T(y - x)$   
(multivariate case)
- such that, **dom**  $f$  is convex and the inequality holds for all  $x, y \in \text{dom } f$ .

# Convexity Conditions: Second Order Convexity

For a function in  $\mathbb{R}$ , the condition is

$$f''(x) \geq 0$$

( and **dom**  $f$  convex ),

which means that the derivative is nondecreasing.



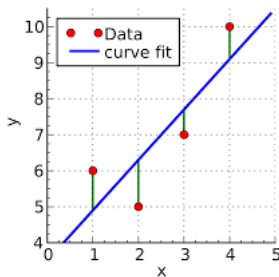
# Examples of optimization

## Least Squares

- No constraints
- Objective function is sum of squares :

$$\text{minimize } f_0(x) = \|Ax - b\|_2^2 = \sum_{i=1}^k (a_i^T x - b_i)^2$$

Here  $A \in \mathbb{R}^n$  (with  $k \geq n$ ),  $a_i^T$  are the rows of  $A$  and the vector  $x \in \mathbb{R}^n$



is the optimization variable



# Examples of Optimization

## Linear Programming

- Objective function linear
- All constraint functions linear

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, i = 1, \dots, m\end{array}$$

Here the vectors  $c, a_1, \dots, a_m \in \mathbb{R}^n$  and scalars  $b_1, \dots, b_m \in \mathbb{R}$  are problem parameters that specify the objective and constraint functions.

- No analytical solution
- A number of methods available for solving it

# Convex Optimization Using R

Linear Programs(LP) are problems that can be expressed as

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{such that } Ax \leq b \\ &\quad x \geq 0 \end{aligned}$$

where  $x$  represents the vector of variables (to be determined),  
 $c$  and  $b$  are vectors of (known) coefficients,  
 $A$  is a (known) matrix of coefficients

## Convex Optimization Using R

A company wants to maximize the profit for two products A and B which are sold at Rs.25 and Rs.20 respectively. There are 1800 resource units available every day and product A requires 20 units while B requires 12 units. Both of these products require a production time of 4 minutes and total available working hours are 8 in a day. What should be the production quantity for each of the products to maximize profits.

# Convex Optimization using R

Solution :

The objective function in the above problem will be:

$$\max(\text{Sales}) = \max(25x_1 + 20x_2)$$

where,

$x_1$  is the units of Product A produced

$x_2$  is the units of Product B produced

The constraints (resource and time):

$$20x_1 + 12x_2 \leq 1800 \text{ (Resource Constraint)}$$

$$4x_1 + 4x_2 \leq 8 * 60 \text{ (Time Constraint)}$$

# Convex Optimization using R

Use `lp` package and `lp()` function to find the optimal solution.

The syntax for `lp()` function is:

`lp(direction="min", objective.in, const.mat, const.dir, const.rhs)`

- `direction` controls whether to minimize or maximize
- Coefficients `c` are encoded a vector `objective.in`
- Constraints `A` are given as a matrix `const.mat` with directions `const.dir`
- Constraints `b` are inserted as a vector `const.rhs`

# Convex Optimization Using R

```
## Load the package lpSolve
library(lpSolve)

## Set the coefficients of the decision variables
objective.in <- c(25, 20)

## Create constraint matrix
const.mat <- matrix(c(20, 12, 4, 4), nrow=2, byrow=TRUE)

## define constraints
time_constraint <- (8*60)
resource_constraint <- 1800

## RHS for the constraints
const.rhs <- c(resource_constraint, time_constraint)

## Constraints direction
const.dir <- c("<=", "<=")

## Find the optimal solution
optimum <- lp(direction="max", objective.in, const.mat, const.dir, const.rhs)
```

R Code

```
## Display the optimum values for x1 and x2
optimum$solution

## [1] 45 75

## Check the value of objective function at optimal point
optimum$objval

## [1] 2625
```

Output

## LP Exercise : Solve using R

A carpenter makes tables and chairs. Each table can be sold for a profit of 30 and each chair for a profit of 10. The carpenter can afford to spend up to 40 hours per week working and takes six hours to make a table and three hours to make a chair. Customer demand requires that he makes at least three times as many chairs as tables. Tables take up four times as much storage space as chairs and there is room for at most four tables each week.

Formulate this problem as a linear programming problem and solve it

*Thank you*