FINAL JEE–MAIN EXAMINATION – JANUARY, 2024

(Held On Thursday 01st February, 2024) **TIME : 3 : 00 PM to 06 : 00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let $f(x) = |2x^2+5|x|-3|, x\in R$. If m and n denote the number of points where f is not continuous and not differentiable respectively, then m + n is equal to :

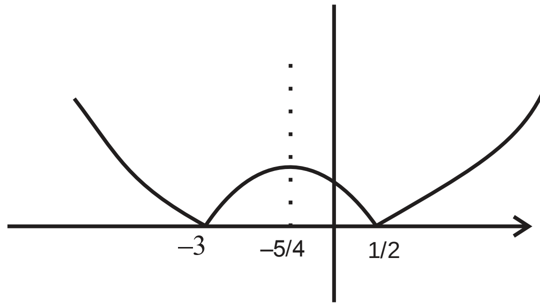
(1) 5 (2) 2

(3) 0 (4) 3

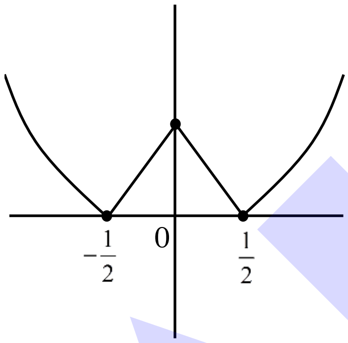
Ans. (4)

Sol. $f(x) = |2x^2+5|x|-3|$

Graph of $y = |2x^2 + 5x - 3|$



Graph of f(x)



Number of points of discontinuity = 0 = m

Number of points of non-differentiability = 3 = n

2. Let α and β be the roots of the equation $px^2 + qx - r = 0$, where $p \neq 0$. If p, q and r be the consecutive terms of a non-constant G.P and $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{3}{4}$, then the value of $(\alpha - \beta)^2$ is :

(1) $\frac{80}{9}$ (2) 9

(3) $\frac{20}{3}$ (4) 8

Ans. (1)

Sol. $px^2 + qx - r = 0 \,\genfrac{}{}{0pt}{}{\alpha}{\beta}$

$p = A,\ q = AR,\ r = AR^2$

$Ax^2 + ARx - AR^2 = 0$

$x^2 + Rx - R^2 = 0 \,\genfrac{}{}{0pt}{}{\alpha}{\beta}$

$\because \frac{1}{\alpha}+\frac{1}{\beta}=\frac{3}{4}$

$\therefore \frac{\alpha+\beta}{\alpha\beta}=\frac{3}{4} \Rightarrow \frac{-R}{-R^2}=\frac{3}{4} \Rightarrow R=\frac{4}{3}$

$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = R^2 - 4(-R^2) = 5\left(\frac{16}{9}\right)$

$= 80/9$

3. The number of solutions of the equation $4\sin^2x - 4\cos^3x + 9 - 4\cos x = 0$; $x \in [-2\pi, 2\pi]$ is :

(1) 1

(2) 3

(3) 2

(4) 0

Ans. (4)

Sol. $4\sin^2x - 4\cos^3x + 9 - 4\cos x = 0$; $x \in [-2\pi, 2\pi]$

$4 - 4\cos^2x - 4\cos^3x + 9 - 4\cos x = 0$

$4\cos^3x + 4\cos^2x + 4\cos x - 13 = 0$

$4\cos^3x + 4\cos^2x + 4\cos x = 13$

L.H.S. ≤ 12 can't be equal to 13.

4. The value of $\int_0^1 (2x^3 - 3x^2 - x + 1)^{\frac{1}{3}}\,dx$ is equal to:

(1) 0

(2) 1

(3) 2

(4) –1

Ans. (1)

Sol. $I = \int_0^1 \left(2x^3 - 3x^2 - x + 1\right)^{\frac{1}{3}} dx$

Using $\int_0^{2a} f(x)dx = 0$ where $f(2a-x) = -f(x)$

Here $f(1-x) = -f(x)$

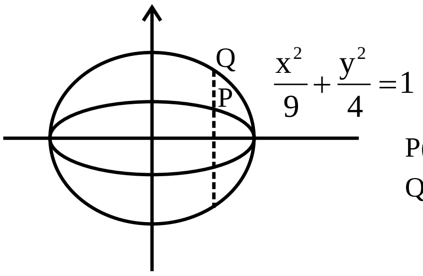
$\therefore\ I = 0$

5. Let P be a point on the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$. Let the line passing through P and parallel to y-axis meet the circle $x^2 + y^2 = 9$ at point Q such that P and Q are on the same side of the x-axis. Then, the eccentricity of the locus of the point R on PQ such that PR : RQ = 4 : 3 as P moves on the ellipse, is :

(1) $\frac{11}{19}$ (2) $\frac{13}{21}$

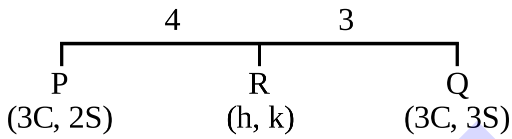
(3) $\frac{\sqrt{139}}{23}$ (4) $\frac{\sqrt{13}}{7}$

Ans. (4)



P(3cosθ, 2sinθ)
Q(3cosθ, 3sinθ)

Sol.



h = 3cosθ;

$k=\frac{18}{7}\sin\theta$

$\therefore$ locus $=\frac{x^2}{9}+\frac{49y^2}{324}=1$

$e=\sqrt{1-\frac{324}{49\times 9}}=\frac{\sqrt{117}}{21}=\frac{\sqrt{13}}{7}$

6. Let m and n be the coefficients of seventh and thirteenth terms respectively in the expansion of $\left(\frac{1}{3}x^{\frac{1}{3}}+\frac{1}{2x^{\frac{2}{3}}}\right)^{18}$. Then $\left(\frac{n}{m}\right)^{\frac{1}{3}}$ is :

(1) $\frac{4}{9}$ (2) $\frac{1}{9}$

(3) $\frac{1}{4}$ (4) $\frac{9}{4}$

Ans. (4)

Sol. $\left(\frac{x^{\frac{1}{3}}}{3}+2x^{\frac{-2}{3}}\right)^{18}$

$t_7={}^{18}c_6\left(\frac{x^{\frac{1}{3}}}{3}\right)^{12}\left(\frac{x^{\frac{-2}{3}}}{2}\right)^{6}={}^{18}c_6\frac{1}{(3)^{12}}\cdot\frac{1}{2^6}$

$t_{13}={}^{18}c_{12}\left(\frac{x^{\frac{1}{3}}}{3}\right)^{6}\left(\frac{x^{\frac{-2}{3}}}{2}\right)^{12}={}^{18}c_{12}\frac{1}{(3)^{6}}\cdot\frac{1}{2^{12}}.x^{-6}$

$m={}^{18}c_6.3^{-12}.2^{-6}\ :\ n={}^{18}c_{12}.2^{-12}.3^{-6}$

$\left(\frac{n}{m}\right)^{\frac{1}{3}}=\left(\frac{2^{-12}.3^{-6}}{3^{-12}.2^{-6}}\right)^{\frac{1}{3}}=\left(\frac{3}{2}\right)^{2}=\frac{9}{4}$

7. Let α be a non-zero real number. Suppose f : R → R is a differentiable function such that $f(0) = 2$ and $\lim\limits_{x\to-\infty} f(x)=1$. If $f'(x) = \alpha f(x) + 3$, for all x ∈ R, then $f(-\log_e 2)$ is equal to____.

(1) 3 (2) 5

(3) 9 (4) 7

Ans. (Bonus)

Sol. $f(0) = 2,\ \lim\limits_{x\to-\infty} f(x)=1$

$f'(x) - \alpha.f(x) = 3$

I.F $= e^{-\alpha x}$

$y(e^{-\alpha x}) = \int 3.e^{-\alpha x}dx$

$f(x).(e^{-\alpha x})=\frac{3e^{-\alpha x}}{-\alpha}+c$

$x=0 \Rightarrow 2=\frac{-3}{\alpha}+c \Rightarrow \frac{3}{\alpha}=c-2$ (1)

$f(x)=\frac{-3}{\alpha}+c.e^{\alpha x}$

Case-I $\alpha > 0$

$x\to-\infty \Rightarrow 1=\frac{-3}{\alpha}+c(0)$

$\alpha = -3$ (rejected)

Case-II $\alpha < 0$

as $\lim\limits_{x\to-\infty} f(x)=1 \Rightarrow c=0$ and $\frac{-3}{\alpha}=1 \Rightarrow \alpha=-3$

$\Rightarrow f(x) = 1$ (rejected)

as $f(0) = 2$

$\Rightarrow$ data is inconsistent

Ans. (Bonus)

8. Let P and Q be the points on the line $\frac{x+3}{8}=\frac{y-4}{2}=\frac{z+1}{2}$ which are at a distance of 6 units from the point R (1,2,3). If the centroid of the triangle PQR is (α, β, γ), then $\alpha^2+\beta^2+\gamma^2$ is:

(1) 26

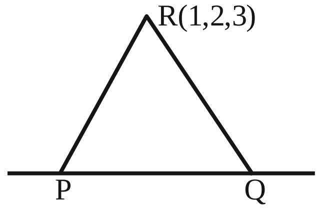
(2) 36

(3) 18

(4) 24

Ans. **(3)**

Sol.



$P(8\lambda-3, 2\lambda+4, 2\lambda-1)$

$PR = 6$

$(8\lambda-4)^2+(2\lambda+2)^2+(2\lambda-4)^2 = 36$

$\lambda = 0, 1$

Hence $P(-3, 4, -1)$ & $Q(5, 6, 1)$

Centroid of $\Delta PQR = (1, 4, 1) \equiv (\alpha, \beta, \gamma)$

$\alpha^2+\beta^2+\gamma^2 = 18$

9. Consider a ΔABC where A(1,3,2), B(–2,8,0) and C(3,6,7). If the angle bisector of $\angle BAC$ meets the line BC at D, then the length of the projection of the vector $\overrightarrow{AD}$ on the vector $\overrightarrow{AC}$ is:

(1) $\frac{37}{2\sqrt{38}}$

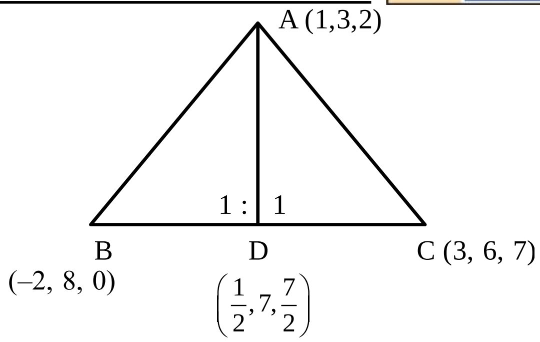
(2) $\frac{\sqrt{38}}{2}$

(3) $\frac{39}{2\sqrt{38}}$

(4) $\sqrt{19}$

Ans. **(1)**

Sol.



A(1, 3, 2); B(–2, 8, 0); C(3, 6, 7);

$\overrightarrow{AC} = 2\hat{i}+3\hat{j}+5\hat{k}$

$AB = \sqrt{9+25+4} = \sqrt{38}$

$AC = \sqrt{4+9+25} = \sqrt{38}$

$\overrightarrow{AD} = -\frac{1}{2}\hat{i}+4\hat{j}+\frac{3}{2}\hat{k} = -\frac{1}{2}\left(\hat{i}+8\hat{j}+3\hat{k}\right)$

Length of projection of $\overrightarrow{AD}$ on $\overrightarrow{AC}$

$= \left|\frac{\overrightarrow{AD}.\overrightarrow{AC}}{|\overrightarrow{AC}|}\right| = \frac{37}{2\sqrt{38}}$

10. Let S_n denote the sum of the first n terms of an arithmetic progression. If $S_{10} = 390$ and the ratio of the tenth and the fifth terms is 15 : 7, then $S_{15} - S_5$ is equal to:

(1) 800

(2) 890

(3) 790

(4) 690

Ans. **(3)**

Sol. $S_{10} = 390$

$\frac{10}{2}\left[2a+(10-1)d\right] = 390$

$\Rightarrow 2a + 9d = 78 \qquad (1)$

$\frac{t_{10}}{t_5} = \frac{15}{7} \Rightarrow \frac{a+9d}{a+4d} = \frac{15}{7} \Rightarrow 8a = 3d \qquad (2)$

From (1) & (2) $\qquad a = 3$ & $d = 8$

$S_{15} - S_5 = \frac{15}{2}(6+14\times 8) - \frac{5}{2}(6+4\times 8)$

$= \frac{15\times 118 - 5\times 38}{2} = 790$

11. If $\int_0^{\frac{\pi}{3}} \cos^4 x \, dx = a\pi + b\sqrt{3}$, where a and b are rational numbers, then 9a + 8b is equal to :

(1) 2 (2) 1

(3) 3 (4) $\frac{3}{2}$

Ans. (1)

Sol. $\int_0^{\pi/3} \cos^4 x dx$

$= \int_0^{\pi/3} \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$= \frac{1}{4}\int_0^{\pi/3} (1 + 2\cos 2x + \cos^2 2x) dx$

$= \frac{1}{4}\left[\int_0^{\pi/3} dx + 2\int_0^{\pi/3} \cos 2x \, dx + \int_0^{\pi/3} \frac{1+\cos 4x}{2} dx\right]$

$= \frac{1}{4}\left[\frac{\pi}{3} + (\sin 2x)_0^{\pi/3} + \frac{1}{2}\left(\frac{\pi}{3}\right) + \frac{1}{8}(\sin 4x)_0^{\pi/3}\right]$

$= \frac{1}{4}\left[\frac{\pi}{3} + (\sin 2x)_0^{\pi/3} + \frac{1}{2}\left(\frac{\pi}{3}\right) + \frac{1}{8}(\sin 4x)_0^{\pi/3}\right]$

$= \frac{1}{4}\left[\frac{\pi}{2} + \frac{\sqrt{3}}{2} + \frac{1}{8} \times \left(-\frac{\sqrt{3}}{2}\right)\right]$

$= \frac{\pi}{8} + \frac{7\sqrt{3}}{64}$

$\therefore a = \frac{1}{8}; b = \frac{7}{64}$

$\therefore 9a + 8b = \frac{9}{8} + \frac{7}{8} = 2$

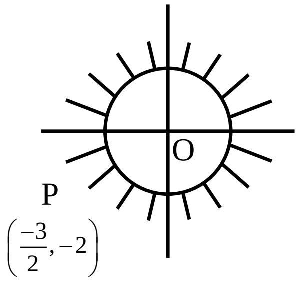
12. If z is a complex number such that $|z| \geq 1$, then the minimum value of $\left|z + \frac{1}{2}(3+4i)\right|$ is:

(1) $\frac{5}{2}$ (2) 2

(3) 3 (4) $\frac{3}{2}$

Ans. (Bonus)

Sol. $|z| \geq 1$



Min. value of $\left|z + \frac{3}{2} + 2i\right|$ is actually zero.

13. If the domain of the function $f(x) = \frac{\sqrt{x^2-25}}{(4-x^2)}$ $+ \log_{10}(x^2 + 2x - 15)$ is $(-\infty, \alpha) \cup [\beta, \infty)$, then $\alpha^2 + \beta^3$ is equal to :

(1) 140 (2) 175

(3) 150 (4) 125

Ans. (3)

Sol. $f(x) = \frac{\sqrt{x^2-25}}{4-x^2} + \log_{10}(x^2 + 2x - 15)$

Domain : $x^2 - 25 \geq 0 \Rightarrow x \in (-\infty, -5] \cup [5, \infty)$

$4 - x^2 \neq 0 \Rightarrow x \neq \{-2, 2\}$

$x^2 + 2x - 15 > 0 \Rightarrow (x+5)(x-3) > 0$

$\Rightarrow x \in (-\infty, -5) \cup (3, \infty)$

$\therefore x \in (-\infty, -5) \cup [5, \infty)$

$\alpha = -5; \beta = 5$

$\therefore \alpha^2 + \beta^3 = 150$

14. Consider the relations R_1 and R_2 defined as $aR_1b \Leftrightarrow a^2 + b^2 = 1$ for all a , b, $\in$ R and (a, b) R_2(c, d) $\Leftrightarrow$ a + d = b + c for all (a,b), (c,d) $\in N \times N$. Then

(1) Only R_1 is an equivalence relation

(2) Only R_2 is an equivalence relation

(3) R_1 and R_2 both are equivalence relations

(4) Neither R_1 nor R_2 is an equivalence relation

Ans. (2)

Sol. $aR_1 b \Leftrightarrow a^2 + b^2 = 1$; a, b $\in$ R

(a, b) R_2 (c, d) $\Leftrightarrow$ a + d = b + c; (a, b), (c, d) $\in$ N

for R_1 : Not reflexive symmetric not transitive

for R_2 : R_2 is reflexive, symmetric and transitive

Hence only R_2 is equivalence relation.

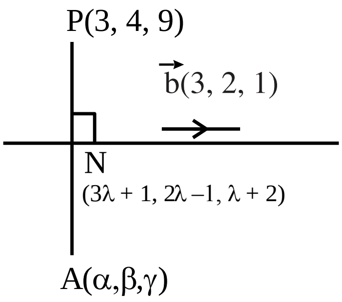
15. If the mirror image of the point P(3,4,9) in the line $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-2}{1}$ is (α, β, γ), then $14(\alpha+\beta+\gamma)$ is :

(1) 102 (2) 138

(3) 108 (4) 132

Ans. (3)

Sol.



$\overrightarrow{PN}.\vec{b}=0$?

$3(3\lambda-2)+2(2\lambda-5)+(\lambda-7)=0$

$14\lambda=23 \Rightarrow \lambda=\frac{23}{14}$

$N\left(\frac{83}{14},\frac{32}{14},\frac{51}{14}\right)$

$\therefore \frac{\alpha+3}{2}=\frac{83}{14} \Rightarrow \alpha=\frac{62}{7}$

$\frac{\beta+4}{2}=\frac{32}{14} \Rightarrow \beta=\frac{4}{7}$

$\frac{\gamma+9}{2}=\frac{51}{14} \Rightarrow \gamma=\frac{-12}{7}$

Ans. $14(\alpha+\beta+r)=108$

16. Let $f(x)=\begin{cases} x-1, & x \text{ is even,} \\ 2x, & x \text{ is odd,}\end{cases}$ $x \in N$. If for some $a \in N$, $f(f(f(a)))=21$, then $\lim\limits_{x\to a^-}\left\{\frac{|x|^3}{a}-\left[\frac{x}{a}\right]\right\}$, where [t] denotes the greatest integer less than or equal to t, is equal to :

(1) 121

(2) 144

(3) 169

(4) 225

Ans. (2)

Sol. $f(x)=\begin{cases} x-1; & x=\text{even} \\ 2x; & x=\text{odd}\end{cases}$

$f(f(f(a)))=21$

C–1: If a = even

$f(a)=a-1=\text{odd}$

$f(f(a))=2(a-1)=\text{even}$

$f(f(f(a)))=2a-3=21 \Rightarrow a=12$

C–2: If a = odd

$f(a)=2a=\text{even}$

$f(f(a))=2a-1=\text{odd}$

$f(f(f(a)))=4a-2=21$ (Not possible)

Hence a = 12

Now

$$\lim_{x\to 12^-}\left(\frac{|x|^3}{12}-\left[\frac{x}{12}\right]\right)$$

$$=\lim_{x\to 12^-}\frac{|x|^3}{12}-\lim_{x\to 12^-}\left[\frac{x}{12}\right]$$

$=144-0=144.$

17. Let the system of equations $x+2y+3z=5$, $2x+3y+z=9$, $4x+3y+\lambda z=\mu$ have infinite number of solutions. Then $\lambda+2\mu$ is equal to :

(1) 28 (2) 17

(3) 22 (4) 15

Ans. (2)

Sol. $x+2y+3z=5$

$2x+3y+z=9$

$4x+3y+\lambda z=\mu$

for infinite following $\Delta=\Delta_1=\Delta_2=\Delta_3=0$

$$\Delta=\begin{vmatrix}1&2&3\\2&3&1\\4&3&\lambda\end{vmatrix}=0 \Rightarrow \lambda=-13$$

$$\Delta_1=\begin{vmatrix}5&2&3\\9&3&1\\\mu&3&-13\end{vmatrix}=0 \Rightarrow \mu=15$$

$$\Delta_2=\begin{vmatrix}1&5&3\\2&9&1\\4&15&-13\end{vmatrix}=0$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 9 \\ 4 & 3 & 15 \end{vmatrix} = 0$$

for $\lambda = -13$, $\mu = 15$ system of equation has infinite solution hence $\lambda + 2\mu = 17$

18. Consider 10 observation $x_1, x_2, \ldots, x_{10}$. such that $\sum_{i=1}^{10} (x_i - \alpha) = 2$ and $\sum_{i=1}^{10} (x_i - \beta)^2 = 40$, where α, β are positive integers. Let the mean and the variance of the observations be $\frac{6}{5}$ and $\frac{84}{25}$ respectively. The $\frac{\beta}{\alpha}$ is equal to :

(1) 2 (2) $\frac{3}{2}$

(3) $\frac{5}{2}$ (4) 1

Ans. (1)

Sol. $x_1, x_2 \ldots\ldots x_{10}$

$$\sum_{i=1}^{10}(x_i - \alpha) = 2 \Rightarrow \sum_{i=1}^{10} x_i - 10\alpha = 2$$

$$\text{Mean } \mu = \frac{6}{5} = \frac{\sum x_i}{10}$$

$$\therefore \ \Sigma x_i = 12$$

$$10\alpha + 2 = 12 \ \therefore \ \alpha = 1$$

Now $\sum_{i=1}^{10}(x_i - \beta)^2 = 40$ Let $y_i = x_i - \beta$

$$\therefore \sigma_y^2 = \frac{1}{10}\sum y_i^2 - (\bar{y})^2$$

$$\sigma_x^2 = \frac{1}{10}\sum (x_i - \beta)^2 - \left(\frac{\sum_{i=1}^{10}(x_i - \beta)}{10}\right)^2$$

$$\frac{84}{25} = 4 - \left(\frac{12 - 10\beta}{10}\right)^2$$

$$\therefore \left(\frac{6 - 5\beta}{5}\right)^2 = 4 - \frac{84}{25} = \frac{16}{25}$$

$$6 - 5\beta = \pm 4 \Rightarrow \beta = \frac{2}{5} \text{ (not possible) or } \beta = 2$$

$$\text{Hence } \frac{\beta}{\alpha} = 2$$

19. Let Ajay will not appear in JEE exam with probability $p = \frac{2}{7}$, while both Ajay and Vijay will appear in the exam with probability $q = \frac{1}{5}$. Then the probability, that Ajay will appear in the exam and Vijay will not appear is :

(1) $\frac{9}{35}$

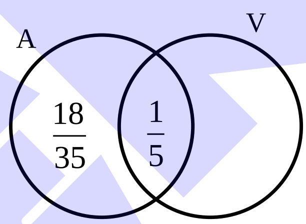
(2) $\frac{18}{35}$

(3) $\frac{24}{35}$

(4) $\frac{3}{35}$

Ans. (2)

Sol.



$$P(\bar{A}) = \frac{2}{7} = p$$

$$P(A \cap V) = \frac{1}{5} = q$$

$$P(A) = \frac{5}{7}$$

$$\text{Ans. } P(A \cap \bar{V}) = \frac{18}{35}$$

20. Let the locus of the mid points of the chords of circle $x^2+(y-1)^2=1$ drawn from the origin intersect the line $x+y = 1$ at P and Q. Then, the length of PQ is :

(1) $\frac{1}{\sqrt{2}}$

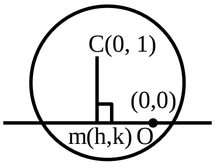
(2) $\sqrt{2}$

(3) $\frac{1}{2}$

(4) 1

Ans. (1)

Sol.

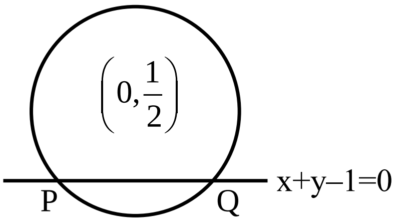


$m_{OM} \cdot m_{CM} = -1$

$\frac{k}{h} \cdot \frac{k-1}{h} = -1$

$\therefore$ locus is $x^2 + y(y-1) = 0$

$x^2 + y^2 - y = 0$



$p = \left|\frac{1/2}{\sqrt{2}}\right| \qquad p = \frac{1}{2\sqrt{2}}$

$PQ = 2\sqrt{r^2 - p^2}$

$= 2\sqrt{\frac{1}{4} - \frac{1}{8}} = \frac{1}{\sqrt{2}}$

SECTION-B

21. If three successive terms of a G.P. with common ratio $r(r > 1)$ are the lengths of the sides of a triangle and $[r]$ denotes the greatest integer less than or equal to r, then $3[r] + [-r]$ is equal to :

Ans. (1)

Sol. $a, ar, ar^2 \rightarrow$ G.P.

Sum of any two sides > third side

$a + ar > ar^2, \; a + ar^2 > ar, \; ar + ar^2 > a$

$r^2 - r - 1 < 0$

$r \in \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$ (1)

$r^2 - r + 1 > 0$

always true

$r^2 + r - 1 > 0$

$r \in \left(-\infty, \frac{-1-\sqrt{5}}{2}\right) \cup \left(\frac{-1+\sqrt{5}}{2}, \infty\right)$ (2)

Taking intersection of (1), (2)

$r \in \left(\frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$

As $r > 1$

$r \in \left(1, \frac{1+\sqrt{5}}{2}\right)$

$[r] = 1 \quad [-r] = -2$

$3[r] + [-r] = 1$

22. Let $A = I_2 - 2MM^T$, where M is real matrix of order 2×1 such that the relation $M^T M = I_1$ holds. If λ is a real number such that the relation $AX = \lambda X$ holds for some non-zero real matrix X of order 2×1, then the sum of squares of all possible values of λ is equal to :

Ans. (2)

Sol. $A = I_2 - 2MM^T$

$A^2 = (I_2 - 2MM^T)(I_2 - 2MM^T)$

$= I_2 - 2MM^T - 2MM^T + 4MM^TMM^T$

$= I_2 - 4MM^T + 4MM^T$

$= I_2$

$AX = \lambda X$

$A^2X = \lambda AX$

$X = \lambda(\lambda X)$

$X = \lambda^2 X$

$X(\lambda^2 - 1) = 0$

$\lambda^2 = 1$

$\lambda = \pm 1$

Sum of square of all possible values = 2

23. Let $f : (0, \infty) \to R$ and $F(x) = \int_0^x tf(t)dt$. If $F(x^2) = x^4 + x^5$, then $\sum_{r=1}^{12} f(r^2)$ is equal to :

Ans. (219)

Sol. $F(x) = \int_0^x t \cdot f(t)dt$

$F^1(x) = xf(x)$

Given $F(x^2) = x^4 + x^5$, let $x^2 = t$

$F(t) = t^2 + t^{5/2}$

$F'(t) = 2t + 5/2\, t^{3/2}$

$t \cdot f(t) = 2t + 5/2\, t^{3/2}$

$f(t) = 2 + 5/2\, t^{1/2}$

$$\sum_{r=1}^{12} f(r^2) = \sum_{r=1}^{12} 2 + \frac{5}{2} r$$

$$= 24 + 5/2 \left[\frac{12(13)}{2}\right]$$

$=219$

24. If $y = \dfrac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}} + \dfrac{1}{15}(3\cos^2 x - 5)\cos^3 x$, then $96y'\left(\dfrac{\pi}{6}\right)$ is equal to :

Ans. (105)

Sol. $y = \dfrac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}} + \dfrac{1}{15}(3\cos^2 x - 5)\cos^3 x$

$$y = \frac{(\sqrt{x}+1)(\sqrt{x})\left((\sqrt{x})^3-1\right)}{(\sqrt{x})\left((\sqrt{x})^2+(\sqrt{x})+1\right)} + \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$$

$$y = (\sqrt{x}+1)\ (\sqrt{x}-1) + \frac{1}{5}\ \cos^5 x - \frac{1}{3}\cos^3 x$$

$y' = 1 - \cos^4 x \cdot (\sin x) + \cos^2 x\ (\sin x)$

$$y'\left(\frac{\pi}{6}\right) = 1 - \frac{9}{16} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{2}$$

$$= \frac{32-9+12}{32} = \frac{35}{32}$$

$$= 96\ y'\left(\frac{\pi}{6}\right) = 105$$

25. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = -\hat{i} - 8\hat{j} + 2\hat{k}$ and $\vec{c} = 4\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three vectors such that $\vec{b} \times \vec{a} = \vec{c} \times \vec{a}$. If the angle between the vector $\vec{c}$ and the vector $3\hat{i} + 4\hat{j} + \hat{k}$ is θ, then the greatest integer less than or equal to $\tan^2\theta$ is :

Ans. (38)

Sol. $\vec{a} = \hat{i} + \hat{j} + k$

$\vec{b} = -\hat{i} - 8\hat{j} + 2\hat{k}$

$\vec{c} = 4\hat{i} + c_2\hat{j} + c_3 k$

$\vec{b} \times \vec{a} = \vec{c} \times \vec{a}$

$(\vec{b} - \vec{c}) \times \vec{a} = 0$

$\vec{b} - \vec{c} = \lambda\vec{\alpha}$

$\vec{b} = \vec{c} + \lambda\vec{\alpha}$

$-\hat{i} - 8\hat{j} + 2k = (4\hat{i} + c_2\hat{j} + c_3 k) + \lambda(\hat{i} + \hat{j} + k)$

$\lambda + 4 = -1 \Rightarrow \lambda = -5$

$\lambda + c_2 = -8 \Rightarrow c_2 = -3$

$\lambda + c_3 = 2 \Rightarrow c_3 = 7$

$\vec{c} = 4\hat{i} - 3\hat{j} + 7k$

$$\cos\theta = \frac{12-12+7}{\sqrt{26}\cdot\sqrt{74}} = \frac{7}{\sqrt{26}\cdot\sqrt{74}} = \frac{7}{2\sqrt{481}}$$

$$\tan^2\theta = \frac{625\times 3}{49}$$

$[\tan^2\theta] = 38$

26. The lines $L_1, L_2, \ldots, L_{20}$ are distinct. For $n = 1, 2, 3, \ldots, 10$ all the lines L_{2n-1} are parallel to each other and all the lines L_{2n} pass through a given point P. The maximum number of points of intersection of pairs of lines from the set $\{L_1, L_2, \ldots, L_{20}\}$ is equal to :

Ans. **(101)**

Sol. $L_1, L_3, L_5, \text{- -} L_{19}$ are Parallel

$L_2, L_4, L_6, \text{- -} L_{20}$ are Concurrent

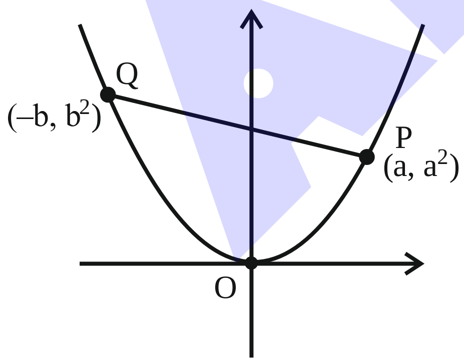
Total points of intersection $= {}^{20}C_2 - {}^{10}C_2 - {}^{10}C_2 + 1$

$= 101$

27. Three points O(0,0), P(a, a^2), Q(–b, b^2), $a > 0$, $b > 0$, are on the parabola $y = x^2$. Let S_1 be the area of the region bounded by the line PQ and the parabola, and S_2 be the area of the triangle OPQ. If the minimum value of $\frac{S_1}{S_2}$ is $\frac{m}{n}$, gcd(m, n) = 1, then m + n is equal to :

Ans. **(7)**

Sol.



$$S_2 = 1/2 \left| \begin{vmatrix} 0 & 0 & 1 \\ a & a^2 & 1 \\ -b & b^2 & 1 \end{vmatrix} \right| = 1/2\left(ab^2 + a^2b\right)$$

$$PQ :- \ y - a^2 = \frac{a^2 - b^2}{a+b}(x-a)$$

$$y - a^2 = (a-b)\,x - (a-b)a$$

$$y = (a-b)\,x + ab$$

$$S_1 = \int_{-b}^{a}\left((a-b)x + ab - x^2\right)dx$$

$$= (a-b)\frac{x^2}{2} + (ab)x - \frac{x^3}{3}\Bigg|_{-b}^{a}$$

$$= \frac{(a-b)^2(a+b)}{2} + ab(a+b) - \frac{(a^3+b^3)}{3}$$

$$\frac{S_1}{S_2} = \frac{\frac{(a-b)^2}{2} + ab - \frac{(a^2+b^2-ab)}{3}}{\frac{ab}{2}}$$

$$= \frac{3(a-b)^2 + 6ab - 2(a^2+b^2-ab)}{3ab}$$

$$= \frac{1}{3}\Bigg[\underbrace{\frac{a}{b} + \frac{b}{a}}_{\min=2} + 2\Bigg]$$

$$= \frac{4}{3} = \frac{m}{n} \qquad m + n = 7$$

28. The sum of squares of all possible values of k, for which area of the region bounded by the parabolas $2y^2 = kx$ and $ky^2 = 2(y - x)$ is maximum, is equal to :

Ans. **(8)**

Sol. $ky^2 = 2(y - x)$ $\qquad$ $2y^2 = kx$

Point of intersection →

$$ky^2 = 2\left(y - \frac{2y^2}{k}\right)$$

$y = 0$ $\qquad$ $ky = 2\left(1 - \frac{2y}{k}\right)$

$$ky + \frac{4y}{k} = 2$$

$$y = \frac{2}{k + \frac{4}{k}} = \frac{2k}{k^2 + 4}$$

$$A = \int_0^{\frac{2k}{k^2+4}} \left(\left(y - \frac{ky^2}{2}\right) - \left(\frac{2y^2}{k}\right)\right).dy$$

$$= \frac{y^2}{2} - \left(\frac{k}{2} + \frac{2}{k}\right) \cdot \frac{y^3}{3}\Bigg|_0^{\frac{2k}{k^2+4}}$$

$$= \left(\frac{2k}{k^2 + 4}\right)^2 \left[\frac{1}{2} - \frac{k^2 + 4}{2k} \times \frac{1}{3} \times \frac{2k}{k^2 + 4}\right]$$

$$= \frac{1}{6} \times 4 \times \left(\frac{1}{k + \frac{4}{k}}\right)^2$$

$$A \cdot M \geq G \cdot M \quad \frac{\left(k + \frac{4}{k}\right)}{2} \geq 2$$

$$k + \frac{4}{k} \geq 4$$

Area is maximum when $k = \frac{4}{k}$

$k = 2, -2$

29. If $\frac{dx}{dy} = \frac{1 + x - y^2}{y}$, $x(1) = 1$, then $5x(2)$ is equal to :

Ans. **(5)**

Sol. $\frac{dx}{dy} - \frac{x}{y} = \frac{1 - y^2}{y}$

Integrating factor $= e^{\int -\frac{1}{y}dy} = \frac{1}{y}$

$$x \cdot \frac{1}{y} = \int \frac{1 - y^2}{y^2} dy$$

$$\frac{x}{y} = \frac{-1}{y} - y + c$$

$x = -1 - y^2 + cy$

$x(1) = 1$

$1 = -1 - 1 + c \Rightarrow c = 3$

$x = -1 - y^2 + 3y$

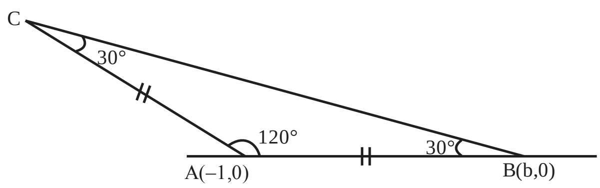
$5x(2) = 5(-1 - 4 + 6)$

$= 5$

30. Let ABC be an isosceles triangle in which A is at $(-1, 0)$, $\angle A = \frac{2\pi}{3}$, AB = AC and B is on the positive x-axis. If $BC = 4\sqrt{3}$ and the line BC intersects the line $y = x + 3$ at (α, β), then $\frac{\beta^4}{\alpha^2}$ is :

Ans. **(36)**

Sol.



$\frac{c}{\sin 30°} = \frac{4\sqrt{3}}{\sin 120°}$ [By sine rule]

$2c = 8 \Rightarrow c = 4$

$$AB = |(b+1)| = 4$$

$$b = 3,\ m_{AB} = 0$$

$$m_{BC} = \frac{-1}{\sqrt{3}}$$

$$BC:-\ y = \frac{-1}{\sqrt{3}}(x-3)$$

$$\sqrt{3}y + x = 3$$

Point of intersection : $y = x + 3,\ \sqrt{3}y + x = 3$

$$\left(\sqrt{3}+1\right)y = 6$$

$$y = \frac{6}{\sqrt{3}+1}$$

$$x = \frac{6}{\sqrt{3}+1} - 3$$

$$= \frac{6 - 3\sqrt{3} - 3}{\sqrt{3}+1}$$

$$= 3\frac{\left(1-\sqrt{3}\right)}{\left(1+\sqrt{3}\right)} = \frac{-6}{\left(1+\sqrt{3}\right)^2}$$

$$\frac{\beta^4}{\alpha^2} = 36$$