FINAL JEE–MAIN EXAMINATION – JANUARY, 2024

(Held On Wednesday 31st January, 2024) TIME : 9 : 00 AM to 12 : 00 NOON

MATHEMATICS **TEST PAPER WITH SOLUTION**

SECTION-A

1. For $0 < c < b < a$, let $(a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b) = 0$ and $\alpha \neq 1$ be one of its root. Then, among the two statements

(I) If $\alpha \in (-1,0)$, then b cannot be the geometric mean of a and c

(II) If $\alpha \in (0,1)$, then b may be the geometric mean of a and c

(1) Both (I) and (II) are true

(2) Neither (I) nor (II) is true

(3) Only (II) is true

(4) Only (I) is true

Ans. (1)

Sol. $f(x) = (a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b)$

$f(x) = a + b - 2c + b + c - 2a + c + a - 2b = 0$

$f(1) = 0$

$$\therefore \alpha \cdot 1 = \frac{c+a-2b}{a+b-2c}$$

$$\alpha = \frac{c+a-2b}{a+b-2c}$$

If, $-1 < \alpha < 0$

$$-1 < \frac{c+a-2b}{a+b-2c} < 0$$

$b + c < 2a$ and $b > \frac{a+c}{2}$

therefore, b cannot be G.M. between a and c.

If, $0 < \alpha < 1$

$$0 < \frac{c+a-2b}{a+b-2c} < 1$$

$b > c$ and $b < \frac{a+c}{2}$

Therefore, b may be the G.M. between a and c.

2. Let a be the sum of all coefficients in the expansion of $(1 - 2x + 2x^2)^{2023}(3 - 4x^2 + 2x^3)^{2024}$ and $b = \lim_{x\to 0}\left(\frac{\int_0^x \frac{\log(1+t)}{t^{2024}+1}dt}{x^2}\right)$. If the equations $cx^2 + dx + e = 0$ and $2bx^2 + ax + 4 = 0$ have a common root, where c, d, e $\in$ R, then d : c : e equals

(1) 2 : 1 : 4 (2) 4 : 1 : 4

(3) 1 : 2 : 4 (4) 1 : 1 : 4

Ans. (4)

Sol. Put x = 1

$\therefore a = 1$

$$b = \lim_{x\to 0}\frac{\int_0^x \frac{\ln(1+t)}{1+t^{2024}}dt}{x^2}$$

Using L' HOPITAL Rule

$$b = \lim_{x\to 0}\frac{\ln(1+x)}{(1+x^{2024})} \times \frac{1}{2x} = \frac{1}{2}$$

Now, $cx^2 + dx + e = 0$, $x^2 + x + 4 = 0$
(D < 0)

$$\therefore \frac{c}{1} = \frac{d}{1} = \frac{e}{4}$$

3. If the foci of a hyperbola are same as that of the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$ and the eccentricity of the hyperbola is $\frac{15}{8}$ times the eccentricity of the ellipse, then the smaller focal distance of the point $\left(\sqrt{2}, \frac{14}{3}\sqrt{\frac{2}{5}}\right)$ on the hyperbola, is equal to

(1) $7\sqrt{\frac{2}{5}} - \frac{8}{3}$ (2) $14\sqrt{\frac{2}{5}} - \frac{4}{3}$

(3) $14\sqrt{\frac{2}{5}} - \frac{16}{3}$ (4) $7\sqrt{\frac{2}{5}} + \frac{8}{3}$

Ans. (1)

Sol. $\frac{x^2}{9}+\frac{y^2}{25}=1$

$a = 3, b = 5$

$e=\sqrt{1-\frac{9}{25}}=\frac{4}{5}\quad \therefore \text{foci}=(0,\pm be)=(0,\pm 4)$

$\therefore e_H=\frac{4}{5}\times\frac{15}{8}=\frac{3}{2}$

Let equation hyperbola

$\frac{x^2}{A^2}-\frac{y^2}{B^2}=-1$

$\therefore B\cdot e_H=4\quad \therefore B=\frac{8}{3}$

$\therefore A^2=B^2\left(e_H^2-1\right)=\frac{64}{9}\left(\frac{9}{4}-1\right)\quad \therefore A^2=\frac{80}{9}$

$\therefore \frac{x^2}{\frac{80}{9}}-\frac{y^2}{\frac{64}{9}}=-1$

Directrix : $y=\pm\frac{B}{e_H}=\pm\frac{16}{9}$

$PS = e\cdot PM=\frac{3}{2}\left|\frac{14}{3}\cdot\sqrt{\frac{2}{5}}-\frac{16}{9}\right|$

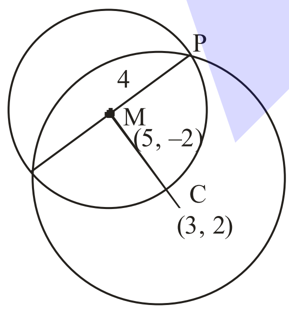
$=7\sqrt{\frac{2}{5}}-\frac{8}{3}$

4. If one of the diameters of the circle $x^2 + y^2 - 10x + 4y + 13 = 0$ is a chord of another circle C, whose center is the point of intersection of the lines $2x + 3y = 12$ and $3x - 2y = 5$, then the radius of the circle C is

(1) $\sqrt{20}$ (2) 4

(3) 6 (4) $3\sqrt{2}$

Ans. **(3)**



Sol.

$2x + 3y = 12$

$3x - 2y = 5$

$13x = 39$

$x = 3, y = 2$

Center of given circle is (5, –2)

Radius $\sqrt{25+4-13}=4$

$\therefore CM=\sqrt{4+16}=5\sqrt{2}$

$\therefore CP=\sqrt{16+20}=6$

5. The area of the region

$\left\{(x,y): y^2\le 4x, x<4, \frac{xy(x-1)(x-2)}{(x-3)(x-4)}>0, x\ne 3\right\}$

is

(1) $\frac{16}{3}$ (2) $\frac{64}{3}$

(3) $\frac{8}{3}$ (4) $\frac{32}{3}$

Ans. (4)

Sol. $y^2\le 4x, x<4$

$\frac{xy(x-1)(x-2)}{(x-3)(x-4)}>0$

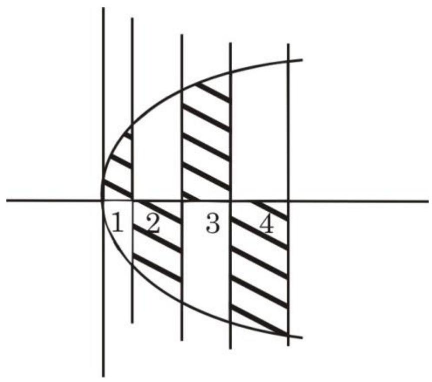
Case – I : $y>0$

$\frac{x(x-1)(x-2)}{(x-3)(x-4)}>0$

$x\in(0,1)\cup(2,3)$

Case – II : $y<0$

$\frac{x(x-1)(x-2)}{(x-3)(x-4)}<0,\ x\in(1,2)\cup(3,4)$



$\text{Area}=2\int_0^4\sqrt{x}\,dx$

$=2\cdot\frac{2}{3}\left[x^{3/2}\right]_0^4=\frac{32}{3}$

6. If $f(x)=\frac{4x+3}{6x-4}, x\neq\frac{2}{3}$ and (fof) (x) = g(x), where $g:\mathbb{R}-\left\{\frac{2}{3}\right\}\to\mathbb{R}-\left\{\frac{2}{3}\right\}$, then (gogog) (4) is equal to

(1) $-\frac{19}{20}$ (2) $\frac{19}{20}$

(3) – 4 (4) 4

Ans. (4)

Sol. $f(x)=\frac{4x+3}{6x-4}$

$$g(x)=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}=\frac{34x}{34}=x$$

$$g(x)=x \therefore g(g(g(4)))=4$$

7. $\lim\limits_{x\to0}\frac{e^{2|\sin x|}-2|\sin x|-1}{x^2}$

(1) is equal to – 1 (2) does not exist

(3) is equal to 1 (4) is equal to 2

Ans. (4)

Sol. $\lim\limits_{x\to0}\frac{e^{2|\sin x|}-2|\sin x|-1}{x^2}$

$$\lim_{x\to0}\frac{e^{2|\sin x|}-2|\sin x|-1}{|\sin x|^2}\times\frac{\sin^2 x}{x^2}$$

Let |sinx| = t

$$\lim_{t\to0}\frac{e^{2t}-2t-1}{t^2}\times\lim_{x\to0}\frac{\sin^2 x}{x^2}$$

$$=\lim_{t\to0}\frac{2e^{2t}-2}{2t}\times1=2\times1=2$$

8. If the system of linear equations

$x-2y+z=-4$

$2x+\alpha y+3z=5$

$3x-y+\beta z=3$

has infinitely many solutions, then 12α + 13β is equal to

(1) 60 (2) 64

(3) 54 (4) 58

Ans. (4)

Sol. $D=\begin{vmatrix}1&-2&1\\2&\alpha&3\\3&-1&\beta\end{vmatrix}$

$= 1(\alpha\beta + 3) + 2(2\beta - 9) + 1(-2 - 3\alpha)$

$= \alpha\beta + 3 + 4\beta - 18 - 2 - 3\alpha$

For infinite solutions $D = 0$, $D_1 = 0$, $D_2 = 0$ and $D_3 = 0$

$D = 0$

$\alpha\beta - 3\alpha + 4\beta = 17$ ….(1)

$$D_1=\begin{vmatrix}-4&-2&1\\5&\alpha&3\\3&-1&\beta\end{vmatrix}=0$$

$$D_2=\begin{vmatrix}1&-4&1\\2&5&3\\3&3&\beta\end{vmatrix}=0$$

$$\Rightarrow1(5\beta-9)+4(2\beta-9)+1(6-15)=0$$

$$13\beta-9-36-9=0$$

$13\beta=54, \beta=\frac{54}{13}$ put in (1)

$$\frac{54}{13}\alpha-3\alpha+4\left(\frac{54}{13}\right)=17$$

$$54\alpha-39\alpha+216=221$$

$15\alpha=5 \qquad \alpha=\frac{1}{3}$

Now, $12\alpha+13\beta=12.\frac{1}{3}+13.\frac{54}{13}$

$= 4 + 54 = 58$

9. The solution curve of the differential equation $y\frac{dx}{dy}=x\left(\log_e x-\log_e y+1\right)$, $x > 0$, $y > 0$ passing through the point (e, 1) is

(1) $\left|\log_e\frac{y}{x}\right|=x$ (2) $\left|\log_e\frac{y}{x}\right|=y^2$

(3) $\left|\log_e\frac{x}{y}\right|=y$ (4) $2\left|\log_e\frac{x}{y}\right|=y+1$

Ans. (3)

Sol. $\frac{dx}{dy} = \frac{x}{y}\left(\ln\left(\frac{x}{y}\right)+1\right)$

Let $\frac{x}{y} = t \Rightarrow x = ty$

$\frac{dx}{dy} = t + y\frac{dt}{dy}$

$t + y\frac{dt}{dy} = t(\ln(t)+1)$

$y\frac{dt}{dy} = t\ln(t) \Rightarrow \frac{dt}{t\ln(t)} = \frac{dy}{y}$

$\Rightarrow \int \frac{dt}{t.\ln(t)} = \int \frac{dy}{y}$

$\Rightarrow \int \frac{dp}{p} = \int \frac{dy}{y}$ let $\ln t = p$

$\frac{1}{t}dt = dp$

$\Rightarrow \ln p = \ln y + c$

$\ln(\ln t) = \ln y + c$

$\ln\left(\ln\left(\frac{x}{y}\right)\right) = \ln y + c$

at $x = e, y = 1$

$\ln\left(\ln\left(\frac{e}{1}\right)\right) = \ln(1) + c \Rightarrow c = 0$

$\ln\left|\ln\left(\frac{x}{y}\right)\right| = \ln y$

$\left|\ln\left(\frac{x}{y}\right)\right| = e^{\ln y}$

$\left|\ln\left(\frac{x}{y}\right)\right| = y$

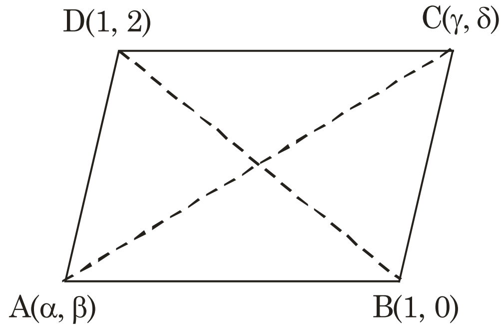
10. Let $\alpha, \beta, \gamma, \delta \in Z$ and let A (α, β), B $(1, 0)$, C (γ, δ) and D $(1, 2)$ be the vertices of a parallelogram ABCD. If AB = $\sqrt{10}$ and the points A and C lie on the line $3y = 2x + 1$, then $2(\alpha + \beta + \gamma + \delta)$ is equal to

(1) 10 (2) 5

(3) 12 (4) 8

Ans. (4)

Sol.



Let E is mid point of diagonals

$\frac{\alpha+\gamma}{2} = \frac{1+1}{2}$ & $\frac{\beta+\delta}{2} = \frac{2+0}{2}$

$\alpha+\gamma = 2$ $\beta+\delta = 2$

$2(\alpha+\beta+\gamma+\delta) = 2(2+2) = 8$

11. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} = \frac{(\tan x)+y}{\sin x(\sec x - \sin x\tan x)}$, $x \in \left(0, \frac{\pi}{2}\right)$ satisfying the condition $y\left(\frac{\pi}{4}\right) = 2$. Then, $y\left(\frac{\pi}{3}\right)$ is

(1) $\sqrt{3}\left(2+\log_e\sqrt{3}\right)$

(2) $\frac{\sqrt{3}}{2}(2+\log_e 3)$

(3) $\sqrt{3}(1+2\log_e 3)$

(4) $\sqrt{3}(2+\log_e 3)$

Ans. (1)

Sol. $\frac{dy}{dx} = \frac{\sin x + y\cos x}{\sin x.\cos x\left(\frac{1}{\cos x} - \sin x.\frac{\sin x}{\cos x}\right)}$

$= \frac{\sin x + y\cos x}{\sin x(1-\sin^2 x)}$

$\frac{dy}{dx} = \sec^2 x + y.2(\operatorname{cosec}2x)$

$\frac{dy}{dx} - 2\operatorname{cosec}(2x).y = \sec^2 x$

$\frac{dy}{dx} + p.y = Q$

$$\text{I.F.} = e^{\int p\,dx} = e^{\int -2\operatorname{cosec}(2x)dx}$$

Let $2x = t$

$$2\frac{dx}{dt} = 1$$

$$dx = \frac{dt}{2}$$

$$= e^{-\int \operatorname{cosec}(t)dt}$$

$$= e^{-\ln\left|\tan\frac{t}{2}\right|}$$

$$= e^{-\ln|\tan x|} = \frac{1}{|\tan x|}$$

$$y(\text{IF}) = \int Q(\text{IF})dx + c$$

$$\Rightarrow y\frac{1}{|\tan x|} = \int \sec^2 x \cdot \frac{1}{|\tan x|} + c$$

$$y.\frac{1}{|\tan x|} = \int \frac{dt}{|t|} + c \qquad \text{for } \tan x = t$$

$$y.\frac{1}{|\tan x|} = \ln|t| + c$$

$$y = |\tan x|(\ln|\tan x| + c)$$

Put $x = \frac{\pi}{4}$, $y = 2$

$2 = \ln 1 + c \Rightarrow c = 2$

$$y = |\tan x|(\ln|\tan x| + 2)$$

$$y\left(\frac{\pi}{3}\right) = \sqrt{3}\left(\ln\sqrt{3} + 2\right)$$

12. Let $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = 4\hat{i} + \hat{j} + 7\hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + 4\hat{k}$ be three vectors. If a vectors $\vec{p}$ satisfies $\vec{p} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{p} \cdot \vec{a} = 0$, then $\vec{p} \cdot (\hat{i} - \hat{j} - \hat{k})$ is equal to

(1) 24
(2) 36
(3) 28
(4) 32

Ans. (4)

Sol. $\vec{p} \times \vec{b} - \vec{c} \times \vec{b} = \vec{0}$

$(\vec{p} - \vec{c}) \times \vec{b} = \vec{0}$

$\vec{p} - \vec{c} = \lambda\vec{b} \Rightarrow \vec{p} = \vec{c} + \lambda\vec{b}$

Now, $\vec{p}.\vec{a} = 0$ (given)

So, $\vec{c}.\vec{a} + \lambda\vec{a}.\vec{b} = 0$

$(3 - 3 - 8) + \lambda(12 + 1 - 14) = 0$

$\lambda = -8$

$\vec{p} = \vec{c} - 8\vec{b}$

$\vec{p} = -31\hat{i} - 11\hat{j} - 52\hat{k}$

So, $\vec{p}.(\hat{i} - \hat{j} - \hat{k})$

$= -31 + 11 + 52$

$= 32$

13. The sum of the series $\frac{1}{1 - 3\cdot 1^2 + 1^4} + \frac{2}{1 - 3\cdot 2^2 + 2^4} + \frac{3}{1 - 3\cdot 3^2 + 3^4} + \ldots$ up to 10 terms is

(1) $\frac{45}{109}$ (2) $-\frac{45}{109}$

(3) $\frac{55}{109}$ (4) $-\frac{55}{109}$

Ans. (4)

Sol. General term of the sequence,

$$T_r = \frac{r}{1 - 3r^2 + r^4}$$

$$T_r = \frac{r}{r^4 - 2r^2 + 1 - r^2}$$

$$T_r = \frac{r}{(r^2 - 1)^2 - r^2}$$

$$T_r = \frac{r}{(r^2 - r - 1)(r^2 + r - 1)}$$

$$T_r = \frac{\frac{1}{2}\left[(r^2 + r - 1) - (r^2 - r - 1)\right]}{(r^2 - r - 1)(r^2 + r - 1)}$$

$$= \frac{1}{2}\left[\frac{1}{r^2 - r - 1} - \frac{1}{r^2 + r - 1}\right]$$

Sum of 10 terms,

$$\sum_{r=1}^{10} T_r = \frac{1}{2}\left[\frac{1}{-1} - \frac{1}{109}\right] = \frac{-55}{109}$$

14. The distance of the point Q(0, 2, –2) form the line passing through the point P(5, –4, 3) and perpendicular to the lines $\vec{r}=\left(-3\hat{i}+2\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+5\hat{k}\right),\ \lambda\in\mathbb{R}$ and $\vec{r}=\left(\hat{i}-2\hat{j}+\hat{k}\right)+\mu\left(-\hat{i}+3\hat{j}+2\hat{k}\right),\ \mu\in\mathbb{R}$ is

(1) $\sqrt{86}$

(2) $\sqrt{20}$

(3) $\sqrt{54}$

(4) $\sqrt{74}$

Ans. (4)

Sol. A vector in the direction of the required line can be obtained by cross product of

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ -1 & 3 & 2 \end{vmatrix}$$

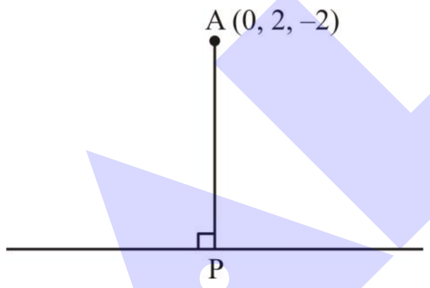
$=-9\hat{i}-9\hat{j}+9\hat{k}$

Required line,

$\vec{r}=\left(5\hat{i}-4\hat{j}+3\hat{k}\right)+\lambda'\left(-9\hat{i}-9\hat{j}+9\hat{k}\right)$

$\vec{r}=\left(5\hat{i}-4\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}+\hat{j}-\hat{k}\right)$

Now distance of (0, 2, –2)



P.V. of $P\equiv(5+\lambda)\hat{i}+(\lambda-4)\hat{j}+(3-\lambda)\hat{k}$

$\overrightarrow{AP}=(5+\lambda)\hat{i}+(\lambda-6)\hat{j}+(5-\lambda)\hat{k}$

$\overrightarrow{AP}\cdot\left(\hat{i}+\hat{j}-\hat{k}\right)=0$

$5+\lambda+\lambda-6-5+\lambda=0$

$\lambda=2$

$\left|\overrightarrow{AP}\right|=\sqrt{49+16+9}$

$\left|\overrightarrow{AP}\right|=\sqrt{74}$

15. For $\alpha,\beta,\gamma\neq0$. If $\sin^{-1}\alpha+\sin^{-1}\beta+\sin^{-1}\gamma=\pi$ and $(\alpha+\beta+\gamma)(\alpha-\gamma+\beta)=3\alpha\beta$, then γ equal to

(1) $\frac{\sqrt{3}}{2}$

(2) $\frac{1}{\sqrt{2}}$

(3) $\frac{\sqrt{3}-1}{2\sqrt{2}}$

(4) $\sqrt{3}$

Ans. (1)

Sol. Let $\sin^{-1}\alpha=A,\sin^{-1}\beta=B,\sin^{-1}\gamma=C$

$A+B+C=\pi$

$(\alpha+\beta)^2-\gamma^2=3\alpha\beta$

$\alpha^2+\beta^2-\gamma^2=\alpha\beta$

$\frac{\alpha^2+\beta^2-\gamma^2}{2\alpha\beta}=\frac{1}{2}$

$\Rightarrow\cos C=\frac{1}{2}$

$\sin C=\gamma$

$\cos C=\sqrt{1-\gamma^2}=\frac{1}{2}$

$\gamma=\frac{\sqrt{3}}{2}$

16. Two marbles are drawn in succession from a box containing 10 red, 30 white, 20 blue and 15 orange marbles, with replacement being made after each drawing. Then the probability, that first drawn marble is red and second drawn marble is white, is

(1) $\frac{2}{25}$

(2) $\frac{4}{25}$

(3) $\frac{2}{3}$

(4) $\frac{4}{75}$

Ans. (4)

Sol. Probability of drawing first red and then white

$=\frac{10}{75}\times\frac{30}{75}=\frac{4}{75}$

17. Let g(x) be a linear function and

$$f(x)=\begin{cases} g(x) & ,x\le 0 \\ \left(\dfrac{1+x}{2+x}\right)^{\frac{1}{x}} & ,x>0 \end{cases}$$, is continuous at x = 0.

If $f'(1)=f(-1)$, then the value of g(3) is

(1) $\frac{1}{3}\log_e\left(\frac{4}{9e^{1/3}}\right)$

(2) $\frac{1}{3}\log_e\left(\frac{4}{9}\right)+1$

(3) $\log_e\left(\frac{4}{9}\right)-1$

(4) $\log_e\left(\frac{4}{9e^{1/3}}\right)$

Ans. (4)

Sol. Let g(x) = ax + b

Now function f(x) in continuous at x = 0

$\therefore \lim\limits_{x\to 0^+} f(x)=f(0)$

$\Rightarrow \lim\limits_{x\to 0}\left(\frac{1+x}{2+x}\right)^{\frac{1}{x}}=b$

$\Rightarrow 0=b$

$\therefore g(x)=ax$

Now, for x > 0

$$f'(x)=\frac{1}{x}\cdot\left(\frac{1+x}{2+x}\right)^{\frac{1}{x}-1}\cdot\frac{1}{(2+x)^2}$$

$$+\left(\frac{1+x}{2+x}\right)^{\frac{1}{x}}\cdot\ln\left(\frac{1+x}{2+x}\right)\cdot\left(-\frac{1}{x^2}\right)$$

$\therefore f'(1)=\frac{1}{9}-\frac{2}{3}\cdot\ln\left(\frac{2}{3}\right)$

And f(–1) = g(–1) = –a

$\therefore a=\frac{2}{3}\ln\left(\frac{2}{3}\right)-\frac{1}{9}$

$\therefore g(3)=2\ln\left(\frac{2}{3}\right)-\frac{1}{3}$

$=\ln\left(\frac{4}{9\cdot e^{1/3}}\right)$

18. If $f(x)=\begin{vmatrix} x^3 & 2x^2+1 & 1+3x \\ 3x^2+2 & 2x & x^3+6 \\ x^3-x & 4 & x^2-2 \end{vmatrix}$

for all $x\in\mathbb{R}$, then 2f(0) + $f'(0)$ is equal to

(1) 48 (2) 24

(3) 42 (4) 18

Ans. (3)

Sol. $f(0)=\begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & 4 & -2 \end{vmatrix}=12$

$$f'(x)=\begin{vmatrix} 3x^2 & 4x & 3 \\ 3x^2+2 & 2x & x^3+6 \\ x^3-x & 4 & x^2-2 \end{vmatrix}+$$

$$\begin{vmatrix} x^3 & 2x^2+1 & 1+3x \\ 6x & 2 & 3x^2 \\ x^3-x & 4 & x^2-2 \end{vmatrix}+$$

$$\begin{vmatrix} x^3 & 2x^2+1 & 1+3x \\ 3x^2+2 & 2x & x^3+6 \\ 3x^2-1 & 0 & 2x \end{vmatrix}$$

$$\therefore f'(0)=\begin{vmatrix} 0 & 0 & 3 \\ 2 & 0 & 6 \\ 0 & 4 & -2 \end{vmatrix}+\begin{vmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & -2 \end{vmatrix}+\begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 6 \\ -1 & 0 & 0 \end{vmatrix}$$

$= 24 - 6 = 18$

$\therefore 2f(0)+f'(0)=42$

19. Three rotten apples are accidently mixed with fifteen good apples. Assuming the random variable x to be the number of rotten apples in a draw of two apples, the variance of x is

(1) $\frac{37}{153}$

(2) $\frac{57}{153}$

(3) $\frac{47}{153}$

(4) $\frac{40}{153}$

Ans. (4)

Sol. 3 bad apples, 15 good apples.

Let X be no of bad apples

Then $P(X=0) = \frac{{}^{15}C_2}{{}^{18}C_2} = \frac{105}{153}$

$P(X=1) = \frac{{}^{3}C_1 \times {}^{15}C_1}{{}^{18}C_2} = \frac{45}{153}$

$P(X=2) = \frac{{}^{3}C_2}{{}^{18}C_2} = \frac{3}{153}$

$E(X) = 0\times\frac{105}{153} + 1\times\frac{45}{153} + 2\times\frac{3}{153} = \frac{51}{153}$

$= \frac{1}{3}$

$Var(X) = E(X^2) - (E(X))^2$

$= 0\times\frac{105}{153} + 1\times\frac{45}{153} + 4\times\frac{3}{153} - \left(\frac{1}{3}\right)^2$

$= \frac{57}{153} - \frac{1}{9} = \frac{40}{153}$

20. Let S be the set of positive integral values of a for which $\frac{ax^2 + 2(a+1)x + 9a + 4}{x^2 - 8x + 32} < 0, \forall x \in \mathbb{R}$.

Then, the number of elements in S is :

(1) 1

(2) 0

(3) ∞

(4) 3

Ans. **(2)**

Sol. $ax^2 + 2(a+1)x + 9a + 4 < 0 \ \forall x \in R$

$\therefore a < 0$

SECTION-B

21. If the integral

$$525\int_0^{\frac{\pi}{2}} \sin 2x \cos^{\frac{11}{2}} x \left(1 + \cos^{\frac{5}{2}} x\right)^{\frac{1}{2}} dx$$ is equal to

$(n\sqrt{2} - 64)$, then n is equal to ______

Ans. (176)

Sol. $I = \int_0^{\frac{\pi}{2}} \sin 2x \cdot (\cos x)^{\frac{11}{2}} \left(1 + (\cos x)^{\frac{5}{2}}\right)^{\frac{1}{2}} dx$

Put $\cos x = t^2 \Rightarrow \sin x\, dx = -2t\, dt$

$\therefore I = 4\int_0^1 t^2 \cdot t^{11} \sqrt{(1+t^5)}(t)dt$

$I = 4\int_0^1 t^{14}\sqrt{1+t^5}dt$

Put $1 + t^5 = = k^2$

$\Rightarrow 5t^4 dt = 2k\, dk$

$\therefore I = 4\cdot\int_1^{\sqrt{2}} (k^2-1)^2 \cdot k\frac{2k}{5}dk$

$I = \frac{8}{5}\int_1^{\sqrt{2}} k^6 - 2k^4 + k^2\ dk$

$I = \frac{8}{5}\left[\frac{k^7}{7} - \frac{2k^5}{5} + \frac{k^3}{3}\right]_1^{\sqrt{2}}$

$I = \frac{8}{5}\left[\frac{8\sqrt{2}}{7} - \frac{8\sqrt{2}}{5} + \frac{2\sqrt{2}}{3} - \frac{1}{7} + \frac{2}{5} - \frac{1}{3}\right]$

$I = \frac{8}{5}\left[\frac{22\sqrt{2}}{105} - \frac{8}{105}\right]$

$\therefore 525 \cdot I = 176\sqrt{2} - 64$

22. Let $S = (-1, \infty)$ and $f : S \to \mathbb{R}$ be defined as

$f(x) = \int_{-1}^{x} (e^t - 1)^{11}(2t-1)^5(t-2)^7(t-3)^{12}(2t-10)^{61}dt$

Let p = Sum of square of the values of x, where f(x) attains local maxima on S. and q = Sum of the values of x, where f(x) attains local minima on S. Then, the value of $p^2 + 2q$ is ________

Ans. (27)

Sol.

$$f'(x)=\left(e^x-1\right)^{11}(2x-1)^5(x-2)^7(x-3)^{12}(2x-10)^{61}$$

$$\begin{array}{ccccccccccc} + & & - & & + & & - & & - & & + \\ & 0 & & \frac{1}{2} & & 2 & & 3 & & 5 & \end{array}$$

Local minima at $x=\frac{1}{2},\ x=5$

Local maxima at $x = 0,\ x = 2$

$\therefore p=0+4=4,\ q=\frac{1}{2}+5=\frac{11}{2}$

Then $p^2 + 2q = 16 + 11 = 27$

23. The total number of words (with or without meaning) that can be formed out of the letters of the word 'DISTRIBUTION' taken four at a time, is equal to _____

Ans. (3734)

Sol. We have III, TT, D, S, R, B, U, O, N

Number of words with selection (a, a, a, b)

$={}^8C_1\times\frac{4!}{3!}=32$

Number of words with selection (a, a, b, b)

$=\frac{4!}{2!2!}=6$

Number of words with selection (a, a, b, c)

$={}^2C_1\times{}^8C_2\times\frac{4!}{2!}=672$

Number of words with selection (a, b, c, d)

$={}^9C_4\times 4!=3024$

$\therefore$ total = 3024 + 672 + 6 + 32

= 3734

24. Let Q and R be the feet of perpendiculars from the point P(a, a, a) on the lines x = y, z = 1 and x = –y, z = –1 respectively. If $\angle QPR$ is a right angle, then $12a^2$ is equal to _____

Ans. (12)

Sol. $\frac{x}{1}=\frac{y}{1}=\frac{z-1}{0}=r\rightarrow Q(r,r,1)$

$\frac{x}{1}=\frac{y}{-1}=\frac{z+1}{0}=k\rightarrow R(k,-k,-1)$

$\overline{PQ}=(a-r)\hat{i}+(a-r)\hat{j}+(a-1)\hat{k}$

$a = r + a – r = 0.$

$2a = 2r \rightarrow a = r$

$\overrightarrow{PR}=(a-k)i+(a+k)\hat{j}+(a+1)\hat{k}$

$a-k-a-k=0 \Rightarrow k=0$

As, $PQ\perp PR$

$(a-r)(a-k)+(a-r)(a+k)+(a-1)(a+1)=0$

a = 1 or –1

$12a^2 = 12$

25. In the expansion of

$(1+x)\left(1-x^2\right)\left(1+\frac{3}{x}+\frac{3}{x^2}+\frac{1}{x^3}\right)^5$, $x\neq 0$, the sum of the coefficient of x^3 and x^{-13} is equal to ____

Ans. (118)

Sol. $(1+x)\left(1-x^2\right)\left(1+\frac{3}{x}+\frac{3}{x^2}+\frac{1}{x^3}\right)^5$

$=(1+x)\left(1-x^2\right)\left(\left(1+\frac{1}{x}\right)^3\right)^5$

$=\frac{(1+x)^2(1-x)(1+x)^{15}}{x^{15}}$

$=\frac{(1+x)^{17}-x(1+x)^{17}}{x^{15}}$

$=$ coeff $\left(x^3\right)$ in the expansion $\approx$ coeff $\left(x^{18}\right)$ in $(1+x)^{17}-x(1+x)^{17}$

= 0 – 1

= –1

coeff $\left(x^{-13}\right)$ in the expansion $\approx$ coeff $\left(x^2\right)$ in $(1 + x)^{17} – x(1 + x)^{17}$

$=\binom{17}{2}-\binom{17}{1}$

= 17 × 8 – 17

= 17 × 7

= 119

Hence Answer = 119 – 1 = 118

26. If α denotes the number of solutions of $|1 - i|^x = 2^x$ and $\beta = \left(\frac{|z|}{\arg(z)}\right)$, where

$$z = \frac{\pi}{4}(1+i)^4\left(\frac{1-\sqrt{\pi}i}{\sqrt{\pi}+i} + \frac{\sqrt{\pi}-i}{1+\sqrt{\pi}i}\right),\ i = \sqrt{-1}\text{, then}$$

the distance of the point (α, β) from the line $4x - 3y = 7$ is _____

Ans. (3)

Sol. $\left(\sqrt{2}\right)^x = 2^x \Rightarrow x = 0 \Rightarrow \alpha = 1$

$$z = \frac{\pi}{4}(1+i)^4\left[\frac{\sqrt{\pi}-\pi i - i - \sqrt{\pi}}{\pi+1} + \frac{\sqrt{\pi}-i-\pi i-\sqrt{\pi}}{1+\pi}\right]$$

$$= -\frac{\pi i}{2}\left(1+4i+6i^2+4i^3+1\right)$$

$$= 2\pi i$$

$$\beta = \frac{2\pi}{\frac{\pi}{2}} = 4$$

Distance from (1, 4) to $4x - 3y = 7$

Will be $\frac{15}{5} = 3$

27. Let the foci and length of the latus rectum of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$ be $(\pm 5, 0)$ and $\sqrt{50}$, respectively. Then, the square of the eccentricity of the hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2b^2} = 1$ equals

Ans. (51)

Sol. focii $\equiv (\pm 5, 0)$; $\frac{2b^2}{a} = \sqrt{50}$

$ae = 5 \qquad b^2 = \frac{5\sqrt{2}a}{2}$

$$b^2 = a^2\left(1-e^2\right) = \frac{5\sqrt{2}a}{2}$$

$$\Rightarrow a\left(1-e^2\right) = \frac{5\sqrt{2}}{2}$$

$$\Rightarrow \frac{5}{e}\left(1-e^2\right) = \frac{5\sqrt{2}}{2}$$

$$\Rightarrow \sqrt{2} - \sqrt{2}e^2 = e$$

$$\Rightarrow \sqrt{2}e^2 + e - \sqrt{2} = 0$$

$$\Rightarrow \sqrt{2}e^2 + 2e - e - \sqrt{2} = 0$$

$$\Rightarrow \sqrt{2}e\left(e+\sqrt{2}\right) - 1\left(1+\sqrt{2}\right) = 0$$

$$\Rightarrow \left(e+\sqrt{2}\right)\left(\sqrt{2}e - 1\right) = 0$$

$$\therefore e \neq -\sqrt{2}\ ;\ e = \frac{1}{\sqrt{2}}$$

$\frac{x^2}{b^2} - \frac{y^2}{a^2b^2} = 1 \qquad a = 5\sqrt{2}$

$b = 5$

$$a^2b^2 = b^2\left(e_1^2 - 1\right) \Rightarrow e_1^2 = 51$$

28. Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 1, |\vec{b}| = 4$ and $\vec{a}\cdot\vec{b} = 2$. If $\vec{c} = \left(2\vec{a}\times\vec{b}\right) - 3\vec{b}$ and the angle between $\vec{b}$ and $\vec{c}$ is α, then $192\sin^2\alpha$ is equal to______

Ans. (48)

Sol. $\vec{b}\cdot\vec{c} = \left(2\vec{a}\times\vec{b}\right)\cdot\vec{b} - 3|\vec{b}|^2$

$|b||c|\cos\alpha = -3|b|^2$

$|c|\cos\alpha = -12$, as $|b| = 4$

$\vec{a}\cdot\vec{b} = 2$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$|c|^2 = \left|\left(2\vec{a}\times\vec{b}\right) - 3\vec{b}\right|^2$$

$$= 64\times\frac{3}{4} + 144 = 192$$

$$|c|^2\cos^2\alpha = 144$$

$$192\cos^2\alpha = 144$$

$$192\sin^2\alpha = 48$$

29. Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 3), (1, 4)} be a relation on A. Let S be the equivalence relation on A such that $R \subset S$ and the number of elements in S is n. Then, the minimum value of n is ______

Ans. (16)

Sol. All elements are included

Answer is 16

30. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \frac{4^x}{4^x + 2} \text{ and}$$

$$M = \int_{f(a)}^{f(1-a)} x \sin^4(x(1-x))\,dx,$$

$$N = \int_{f(a)}^{f(1-a)} \sin^4(x(1-x))\,dx; a \neq \frac{1}{2}. \text{ If}$$

$\alpha M = \beta N, \alpha, \beta \in \mathbb{N}$, then the least value of $\alpha^2 + \beta^2$ is equal to ______

Ans. (5)

Sol. f(a) + f(1 – a) = 1.

$$M = \int_{f(a)}^{f(1-a)} (1-x).\sin^4 x(1-x)\,dx$$

$M = N - M$ $\qquad$ $2M = N$

$\alpha = 2; \beta = 1;$

Ans. 5