FINAL JEE–MAIN EXAMINATION – JANUARY, 2024

(Held On Wednesday 31st January, 2024) TIME : 3 : 00 PM to 6 : 00 PM

MATHEMATICS | **TEST PAPER WITH SOLUTION**

SECTION-A

1. The number of ways in which 21 identical apples can be distributed among three children such that each child gets at least 2 apples, is

(1) 406
(2) 130
(3) 142
(4) 136

Ans. (4)

Sol. After giving 2 apples to each child 15 apples left now 15 apples can be distributed in ${}^{15+3-1}C_2 = {}^{17}C_2$ ways

$$= \frac{17 \times 16}{2} = 136$$

2. Let A (a, b), B(3, 4) and (–6, –8) respectively denote the centroid, circumcentre and orthocentre of a triangle. Then, the distance of the point P(2a + 3, 7b + 5) from the line 2x + 3y – 4 = 0 measured parallel to the line x – 2y – 1 = 0 is

(1) $\frac{15\sqrt{5}}{7}$

(2) $\frac{17\sqrt{5}}{6}$

(3) $\frac{17\sqrt{5}}{7}$

(4) $\frac{\sqrt{5}}{17}$

Ans. (3)

Sol. A(a,b), B(3,4), C(-6, -8)

2 : 1

C (-6, -8) — A (a, b) — B (3, 4)

$$\Rightarrow a = 0,\ b = 0 \qquad \Rightarrow P(3,5)$$

Distance from P measured along x – 2y – 1 = 0

$$\Rightarrow x = 3 + r\cos\theta,\quad y = 5 + r\sin\theta$$

Where $\tan\theta = \frac{1}{2}$

$$r(2\cos\theta + 3\sin\theta) = -17$$

$$\Rightarrow r = \left|\frac{-17\sqrt{5}}{7}\right| = \frac{17\sqrt{5}}{7}$$

3. Let z_1 and z_2 be two complex number such that $z_1 + z_2 = 5$ and $z_1^3 + z_2^3 = 20 + 15i$. Then $|z_1^4 + z_2^4|$ equals-

(1) $30\sqrt{3}$

(2) 75

(3) $15\sqrt{15}$

(4) $25\sqrt{3}$

Ans. (2)

Sol.- $z_1 + z_2 = 5$

$$z_1^3 + z_2^3 = 20 + 15i$$

$$z_1^3 + z_2^3 = (z_1 + z_2)^3 - 3z_1z_2(z_1 + z_2)$$

$$z_1^3 + z_2^3 = 125 - 3z_1.z_2(5)$$

$$\Rightarrow 20 + 15i = 125 - 15z_1z_2$$

$$\Rightarrow 3z_1z_2 = 25 - 4 - 3i$$

$$\Rightarrow 3z_1z_2 = 21 - 3i$$

$$\Rightarrow z_1.z_2 = 7 - i$$

$$\Rightarrow (z_1 + z_2)^2 = 25$$

$$\Rightarrow z_1^2 + z_2^2 = 25 - 2(7 - i)$$

$$\Rightarrow 11 + 2i$$

$$(z_1^2 + z_2^2)^2 = 121 - 4 + 44i$$

$$\Rightarrow \quad z_1^4 + z_2^4 + 2(7 - i)^2 = 117 + 44i$$

$$\Rightarrow \quad z_1^4 + z_2^4 = 117 + 44i - 2(49 - 1 - 14i)$$

$$\Rightarrow \quad |z_1^4 + z_2^4| = 75$$

4. Let a variable line passing through the centre of the circle $x^2 + y^2 - 16x - 4y = 0$, meet the positive co-ordinate axes at the point A and B. Then the minimum value of OA + OB, where O is the origin, is equal to

(1) 12

(2) 18

(3) 20

(4) 24

Ans. **(2)**

Sol.- $(y-2) = m(x-8)$

$\Rightarrow$ x-intercept

$\Rightarrow \left(\frac{-2}{m}+8\right)$

$\Rightarrow$ y-intercept

$\Rightarrow (-8m+2)$

$\Rightarrow OA + OB = \frac{-2}{m} + 8 - 8m + 2$

$f'(m) = \frac{2}{m^2} - 8 = 0$

$\Rightarrow m^2 = \frac{1}{4}$

$\Rightarrow m = \frac{-1}{2}$

$\Rightarrow f\left(\frac{-1}{2}\right) = 18$

$\Rightarrow$ Minimum = 18

5. Let $f, g : (0,\infty) \to R$ be two functions defined by $f(x) = \int_{-x}^{x} (|t| - t^2) e^{-t^2} dt$ and $g(x) = \int_0^{x^2} t^{1/2} e^{-t} dt$. Then the value of $\left(f\left(\sqrt{\log_e 9}\right) + g\left(\sqrt{\log_e 9}\right)\right)$ is equal to

(1) 6

(2) 9

(3) 8

(4) 10

Ans. (3)

Sol.-

$$f(x) = \int_{-x}^{x} (|t| - t^2) e^{-t^2} dt$$

$$\Rightarrow f'(x) = 2.(|x| - x^2) e^{-x^2} \text{............}(1)$$

$$g(x) = \int_0^{x^2} t^{\frac{1}{2}} e^{-t} dt$$

$$g'(x) = xe^{-x^2}(2x) - 0$$

$$f'(x) + g'(x) = 2xe^{-x^2} - 2x^2e^{-x^2} + 2x^2e^{-x^2}$$

Integrating both sides w.r.t.x

$$f(x) + g(x) = \int_0^{\alpha} 2xe^{-x^2} dx$$

$$x^2 = t$$

$$\Rightarrow \int_0^{\sqrt{\alpha}} e^{-t} dt = \left[-e^{-t}\right]_0^{\sqrt{\alpha}}$$

$$= -e^{\left(\log_e (9)^{-1}\right)+1}$$

$$\Rightarrow 9(f(x) + g(x)) = \left(1 - \frac{1}{9}\right)9 = 8$$

6. Let (α, β, γ) be mirror image of the point (2, 3, 5) in the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$.

Then $2\alpha + 3\beta + 4\gamma$ is equal to

(1) 32

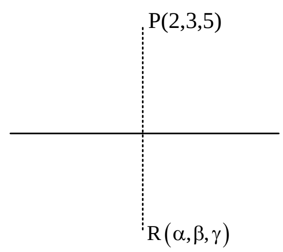
(2) 33

(3) 31

(4) 34

Ans. **(2)**

Sol.



$\because \overrightarrow{PR} \perp (2,3,4)$

$\therefore \overrightarrow{PR}.(2,3,4) = 0$

$(\alpha - 2, \beta - 3, \gamma - 5).(2,3,4) = 0$

$\Rightarrow 2\alpha + 3\beta + 4\gamma = 4 + 9 + 20 = 33$

7. Let P be a parabola with vertex (2, 3) and directrix $2x + y = 6$. Let an ellipse $E: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$ of eccentricity $\frac{1}{\sqrt{2}}$ pass through the focus of the parabola P. Then the square of the length of the latus rectum of E, is

(1) $\frac{385}{8}$

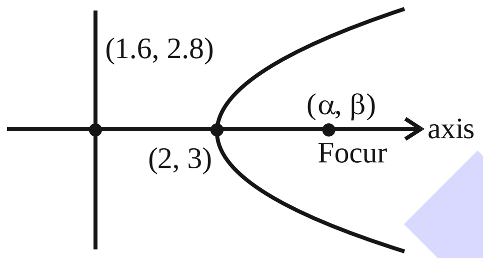
(2) $\frac{347}{8}$

(3) $\frac{512}{25}$

(4) $\frac{656}{25}$

Ans. (4)

Sol.-



Slope of axis $=\frac{1}{2}$

$y-3=\frac{1}{2}(x-2)$

$\Rightarrow 2y-6=x-2$

$\Rightarrow 2y-x-4=0$

$2x+y-6=0$

$4x+2y-12=0$

$\alpha+1.6=4 \Rightarrow \alpha=2.4$

$\beta+2.8=6 \Rightarrow \beta=3.2$

Ellipse passes through (2.4, 3.2)

$\Rightarrow \frac{\left(\frac{24}{10}\right)^2}{a^2}+\frac{\left(\frac{32}{10}\right)^2}{b^2}=1 \quad \ldots\ldots\ldots(1)$

Also $1-\frac{b^2}{a^2}=\frac{1}{2}=\frac{b^2}{a^2}=\frac{1}{2}$

$\Rightarrow a^2=2b^2$

Put in (1) $\Rightarrow b^2=\frac{328}{25}$

$\Rightarrow \left(\frac{2b^2}{a}\right)^2=\frac{4b^2}{a^2}\times b^2=4\times\frac{1}{2}\times\frac{328}{25}=\frac{656}{25}$

8. The temperature T(t) of a body at time t = 0 is 160° F and it decreases continuously as per the differential equation $\frac{dT}{dt}=-K(T-80)$, where K is positive constant. If T(15) = 120°F, then T(45) is equal to

(1) 85° F

(2) 95° F

(3) 90° F

(4) 80° F

Ans. (3)

Sol.-

$\frac{dT}{dt}=-k(T-80)$

$\int_{160}^{T}\frac{dT}{(T-80)}=\int_{0}^{t}-Kdt$

$\left[\ln|T-80|\right]_{160}^{T}=-kt$

$\ln|T-80|-\ln 80=-kt$

$\ln\left|\frac{T-80}{80}\right|=-kt$

$T=80+80e^{-kt}$

$120=80+80e^{-k.15}$

$\frac{40}{80}=e^{-k15}=\frac{1}{2}$

$\therefore T(45)=80+80e^{-k.45}$

$=80+80\left(e^{-k.15}\right)^3$

$=80+80\times\frac{1}{8}$

$=90$

9. Let 2^{nd}, 8^{th} and 44^{th}, terms of a non-constant A.P. be respectively the 1^{st}, 2^{nd} and 3^{rd} terms of G.P. If the first term of A.P. is 1 then the sum of first 20 terms is equal to-

(1) 980 (2) 960

(3) 990 (4) 970

Ans. **(4)**

Sol.- $1 + d,\ \ 1 + 7d,\ \ 1 + 43d$ are in GP

$(1 + 7d)^2 = (1 + d)(1 + 43d)$

$1 + 49d^2 + 14d = 1 + 44d + 43d^2$

$6d^2 - 30d = 0$

$d = 5$

$$S_{20} = \frac{20}{2}\left[2\times1+(20-1)\times5\right]$$

$$=10\left[2+95\right]$$

$$=970$$

10. Let $f :\to R \to (0,\infty)$ be strictly increasing function such that $\lim\limits_{x\to\infty}\frac{f(7x)}{f(x)}=1$. Then, the value of $\lim\limits_{x\to\infty}\left[\frac{f(5x)}{f(x)}-1\right]$ is equal to

(1) 4

(2) 0

(3) 7/5

(4) 1

Ans. **(2)**

Sol.- $f : R \to (0,\infty)$

$$\lim_{x\to\infty}\frac{f(7x)}{f(x)}=1$$

$\because$ f is increasing

$\therefore f(x) < f(5x) < f(7x)$

$$\because \frac{f(x)}{f(x)} < \frac{f(5x)}{f(x)} < \frac{f(7x)}{f(x)}$$

$$1 < \lim_{x\to\infty}\frac{f(5x)}{f(x)} < 1$$

$$\therefore \left[\frac{f(5x)}{f(x)}-1\right]$$

$$\Rightarrow 1-1=0$$

11. The area of the region enclosed by the parabola $y = 4x - x^2$ and $3y = (x - 4)^2$ is equal to

(1) $\frac{32}{9}$

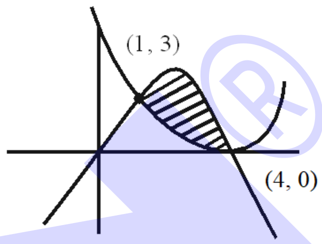
(2) 4

(3) 6

(4) $\frac{14}{3}$

Ans. **(3)**

Sol.-



$$\text{Area} = \left|\int_1^4\left[\left(4x-x^2\right)-\frac{(x-4)^2}{3}\right]\right|dx$$

$$\text{Area} = \left|\frac{4x^2}{2}-\frac{x^3}{3}-\frac{(x-4)^3}{9}\right|_1^4$$

$$=\left|\left(\frac{64}{2}-\frac{64}{3}-\frac{4}{2}+\frac{1}{3}-\frac{27}{9}\right)\right|$$

$$\Rightarrow (27-21)=6$$

12. Let the mean and the variance of 6 observation a, b, 68, 44, 48, 60 be 55 and 194, respectively if a > b, then a + 3b is

(1) 200

(2) 190

(3) 180

(4) 210

Ans. **(3)**

Sol.- a, b, 68, 44, 48, 60

Mean = 55 $\quad$ a > b

Variance = 194 $\quad$ a + 3b

$$\frac{a+b+68+44+48+60}{6}=55$$

$$\Rightarrow 220+a+b=330$$

$$\therefore\ a+b=110......(1)$$

Also,

$$\sum \frac{(x_i - \bar{x})^2}{n} = 194$$

$\Rightarrow (a-55)^2 + (b-55)^2 + (68-55)^2 + (44-55)^2$

$+(48-55)^2 + (60-55)^2 = 194 \times 6$

$\Rightarrow (a-55)^2 + (b-55)^2 + 169 + 121 + 49 + 25 = 1164$

$\Rightarrow (a-55)^2 + (b-55)^2 = 1164 - 364 = 800$

$a^2 + 3025 - 110a + b^2 + 3025 - 110b = 800$

$\Rightarrow a^2 + b^2 = 800 - 6050 + 12100$

$a^2 + b^2 = 6850$.......(2)

Solve (1) & (2);

a=75,b=35

$\therefore a + 3b = 75 + 3(35) = 75 + 105 = 180$

13. If the function $f:(-\infty,-1] \to (a,b]$ defined by $f(x) = e^{x^3-3x+1}$ is one-one and onto, then the distance of the point P(2b + 4, a + 2) from the line $x + e^{-3}y = 4$ is :

(1) $2\sqrt{1+e^6}$ (2) $4\sqrt{1+e^6}$

(3) $3\sqrt{1+e^6}$ (4) $\sqrt{1+e^6}$

Ans. **(1)**

Sol.- $f(x) = e^{x^3-3x+1}$

$f'(x) = e^{x^3-3x+1}.(3x^2-3)$

$= e^{x^3-3x+1}.3(x-1)(x+1)$

For $f'(x) \geq 0$

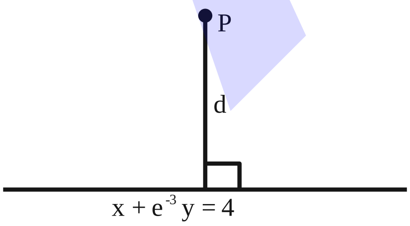
$\therefore f(x)$ is increasing function

$\therefore a = e^{-\infty} = 0 = f(-\infty)$

$b = e^{-1+3+1} = e^3 = f(-1)$

P(2b + 4, a + 2)

$\therefore P(2e^3 + 4, 2)$



$$d = \frac{(2e^3+4) + 2e^{-3} - 4}{\sqrt{1+e^{-6}}} = 2\sqrt{1+e^6}$$

14. Consider the function $f:(0,\infty) \to R$ defined by $f(x) = e^{-|\log_e x|}$. If m and n be respectively the number of points at which f is not continuous and f is not differentiable, then m + n is

(1) 0

(2) 3

(3) 1

(4) 2

Ans. **(3)**

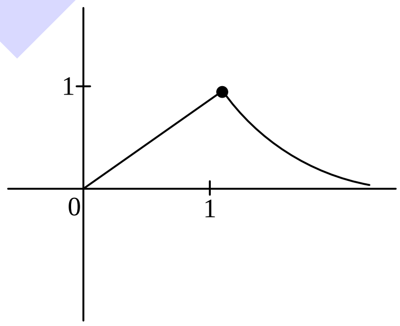
Sol.-

$f:(0,\infty) \to R$

$f(x) = e^{-|\log_e x|}$

$$f(x) = \frac{1}{e^{|\ln x|}} = \begin{cases} \frac{1}{e^{-\ln x}}; 0 < x < 1 \\ \frac{1}{e^{\ln x}}; x \geq 1 \end{cases}$$

$$\begin{cases} \frac{1}{\frac{1}{x}} = x; 0 < x < 1 \\ \frac{1}{x}, x \geq 1 \end{cases}$$



m = 0 (No point at which function is not continuous)

n = 1 (Not differentiable)

$\therefore$ m + n = 1

15. The number of solutions, of the equation $e^{\sin x} - 2e^{-\sin x} = 2$ is

(1) 2

(2) more than 2

(3) 1

(4) 0

Ans. **(4)**

Sol.- Take $e^{\sin x} = t(t > 0)$

$\Rightarrow t - \frac{2}{t} = 2$

$\Rightarrow \frac{t^2 - 2}{t} = 2$

$\Rightarrow t^2 - 2t - 2 = 0$

$\Rightarrow t^2 - 2t + 1 = 3$

$\Rightarrow (t-1)^2 = 3$

$\Rightarrow t = 1 \pm \sqrt{3}$

$\Rightarrow t = 1 \pm 1.73$

$\Rightarrow t = 2.73$ or -0.73 (rejected as $t > 0$)

$\Rightarrow e^{\sin x} = 2.73$

$\Rightarrow \log_e e^{\sin x} = \log_e 2.73$

$\Rightarrow \sin x = \log_e 2.73 > 1$

So no solution.

16. If $a = \sin^{-1}(\sin(5))$ and $b = \cos^{-1}(\cos(5))$, then $a^2 + b^2$ is equal to

(1) $4\pi^2 + 25$

(2) $8\pi^2 - 40\pi + 50$

(3) $4\pi^2 - 20\pi + 50$

(4) 25

Ans. **(2)**

Sol. $a = \sin^{-1}(\sin 5) = 5 - 2\pi$

and $b = \cos^{-1}(\cos 5) = 2\pi - 5$

$\therefore a^2 + b^2 = (5-2\pi)^2 + (2\pi - 5)^2$

$= 8\pi^2 - 40\pi + 50$

17. If for some m, n; ${}^6C_m + 2\left({}^6C_{m+1}\right) + {}^6C_{m+2} > {}^8C_3$ and ${}^{n-1}P_3 : {}^nP_4 = 1:8$, then ${}^nP_{m+1} + {}^{n+1}C_m$ is equal to

(1) 380

(2) 376

(3) 384

(4) 372

Ans. **(4)**

Sol.- ${}^6C_m + 2\left({}^6C_{m+1}\right) + {}^6C_{m+2} > {}^8C_3$

${}^7C_{m+1} + {}^7C_{m+2} > {}^8C_3$

${}^8C_{m+2} > {}^8C_3$

$\therefore m = 2$

And ${}^{n-1}P_3 : {}^nP_4 = 1:8$

$\frac{(n-1)(n-2)(n-3)}{n(n-1)(n-2)(n-3)} = \frac{1}{8}$

$\therefore n = 8$

$\therefore {}^nP_{m+1} + {}^{n+1}C_m = {}^8P_3 + {}^9C_2$

$= 8 \times 7 \times 6 + \frac{9 \times 8}{2}$

$= 372$

18. A coin is based so that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, then the probability of getting two tails and one head is-

(1) $\frac{2}{9}$

(2) $\frac{1}{9}$

(3) $\frac{2}{27}$

(4) $\frac{1}{27}$

Ans. **(1)**

Sol. Let probability of tail is $\frac{1}{3}$

$\Rightarrow$ Probability of getting head $= \frac{2}{3}$

$\therefore$ Probability of getting 2 tails and 1 head

$= \left(\frac{1}{3} \times \frac{2}{3} \times \frac{1}{3}\right) \times 3$

$= \frac{2}{27} \times 3$

$= \frac{2}{9}$

19. Let A be a 3×3 real matrix such that

$$A\begin{pmatrix}1\\0\\1\end{pmatrix}=2\begin{pmatrix}1\\0\\1\end{pmatrix},A\begin{pmatrix}-1\\0\\1\end{pmatrix}=4\begin{pmatrix}-1\\0\\1\end{pmatrix},A\begin{pmatrix}0\\1\\0\end{pmatrix}=2\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Then, the system $(A-3I)\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ has

(1) unique solution

(2) exactly two solutions

(3) no solution

(4) infinitely many solutions

Ans. (1)

Sol.- Let $A=\begin{bmatrix}x_1 & y_1 & z_1\\x_2 & y_2 & z_2\\x_3 & y_3 & z_3\end{bmatrix}$

Given $A\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}2\\0\\2\end{bmatrix}$ .... (1)

$\therefore\begin{bmatrix}x_1+z_1\\x_2+z_2\\x_3+z_3\end{bmatrix}=\begin{bmatrix}2\\0\\2\end{bmatrix}$

$\therefore x_1+z_1=2$ .... (2)

$x_2+z_2=0$ .... (3)

$x_3+z_3=0$ .... (4)

Given $A\begin{bmatrix}-1\\0\\1\end{bmatrix}=\begin{bmatrix}-4\\0\\4\end{bmatrix}$

$\therefore\begin{bmatrix}-x_1+z_1\\-x_2+z_2\\-x_3+z_3\end{bmatrix}=\begin{bmatrix}4\\0\\4\end{bmatrix}$

$\Rightarrow -x_1+z_1=-4$ .... (5)

$-x_2+x_2=0$ .... (6)

$-x_3+z_3=4$

Given $A\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\2\\0\end{bmatrix}$

$\therefore\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\2\\0\end{bmatrix}$

$\therefore y_1=0, y_2=2, y_3=0$

$\therefore$ from (2), (3), (4), (5), (6) and (7)

$x_1=3x, x_2=0, x_3=-1$

$y_1=0, y_2=2, y_3=0$

$z_1=-1, z_2=0, z_3=3$

$\therefore A=\begin{bmatrix}3 & 0 & -1\\0 & 2 & 0\\-1 & 0 & 3\end{bmatrix}$

$\therefore$ Now $(A-3I)\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}$

$\therefore\begin{bmatrix}0 & 0 & -1\\0 & -1 & 0\\-1 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}$

$\begin{bmatrix}-z\\-y\\-x\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$

$[z=-1],[y=-2],[x=-3]$

20. The shortest distance between lines L_1 and L_2, where $L_1:\dfrac{x-1}{2}=\dfrac{y+1}{-3}=\dfrac{z+4}{2}$ and L_2 is the line passing through the points $A(-4,4,3).B(-1,6,3)$ and perpendicular to the line $\dfrac{x-3}{-2}=\dfrac{y}{3}=\dfrac{z-1}{1}$, is

(1) $\dfrac{121}{\sqrt{221}}$ (2) $\dfrac{24}{\sqrt{117}}$

(3) $\dfrac{141}{\sqrt{221}}$ (4) $\dfrac{42}{\sqrt{117}}$

Ans. (3)

Sol.-

$$L_2 = \frac{x+4}{3} = \frac{y-4}{2} = \frac{z-3}{0}$$

$$\therefore S.D = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ 2 & -3 & 2 \\ 3 & 2 & 0 \end{vmatrix}}{\left|\vec{n}_1 \times \vec{n}_2\right|}$$

$$= \frac{\begin{vmatrix} 5 & -5 & -7 \\ 2 & -3 & 2 \\ 3 & 2 & 0 \end{vmatrix}}{\left|\vec{n}_1 \times \vec{n}_2\right|}$$

$$= \frac{141}{\left|-4\hat{i}+6\hat{j}+13\hat{k}\right|}$$

$$= \frac{141}{\sqrt{16+36+169}}$$

$$= \frac{141}{\sqrt{221}}$$

SECTION-B

21. $\left|\frac{120}{\pi^3}\int_0^{\pi}\frac{x^2\sin x\cos x}{\sin^4 x+\cos^4 x}dx\right|$ is equal to ________.

Ans. (15)

Sol.- $\int_0^{\pi}\frac{x^2 \sin x . \cos x}{\sin^4 x+\cos^4 x}dx$

$$= \int_0^{\frac{\pi}{2}}\frac{\sin x . \cos x}{\sin^4 x+\cos^4 x}\left(x^2-(\pi-x)^2\right)dx$$

$$= \int_0^{\frac{\pi}{2}}\frac{\sin x \cdot \cos x\left(2\pi x-\pi^2\right)}{\sin^4 x+\cos^4 x}$$

$$= 2\pi\int_0^{\frac{\pi}{2}}\frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx - \pi^2\int_0^{\frac{\pi}{2}}\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

$$= 2\pi\cdot\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx - \pi^2\int_0^{\frac{\pi}{2}}\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

$$= -\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

$$= -\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x\cos x dx}{1-2\sin^2 x\times\cos^2 x}$$

$$= -\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}}\frac{\sin 2x}{2-\sin^2 2x}dx$$

$$= -\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}}\frac{\sin 2x}{1+\cos^2 2x}dx$$

Let $\cos 2x = t$

22. Let a, b, c be the length of three sides of a triangle satisfying the condition $(a^2 + b^2)x^2 - 2b(a + c)$. $x + (b^2 + c^2) = 0$. If the set of all possible values of x is the interval (α,β), then $12\left(\alpha^2+\beta^2\right)$ is equal to ______.

Ans. (36)

Sol.- $\left(a^2+b^2\right)x^2-2b\left(a+c\right)x+b^2+c^2=0$

$$\Rightarrow a^2x^2-2abx+b^2+b^2x^2-2bcx+c^2=0$$

$$\Rightarrow \left(ax-b\right)^2+\left(bx-c\right)^2=0$$

$$\Rightarrow ax-b=0,\quad bx-c=0$$

$$\Rightarrow a+b>c \quad b+c>a \quad c+a>b$$

$a+ax>bx$	$ax+bx>a$	$ax^2+a>ax$
$a+ax>ax^2$	$ax+ax^2>a$	$x^2-x+1>0$
$x^2-x-1<0$	$x^2+x-1>0$	always true

$$\frac{1-\sqrt{5}}{2}<x<\frac{1+\sqrt{5}}{2}$$

$$x<\frac{-1-\sqrt{5}}{2},\quad \text{or } x>\frac{-1+\sqrt{5}}{2}$$

$$\Rightarrow \frac{\sqrt{5}-1}{2} < x < \frac{\sqrt{5}+1}{2}$$

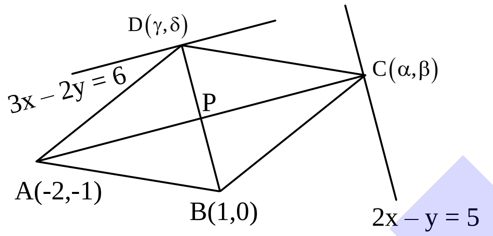
$$\Rightarrow \alpha = \frac{\sqrt{5}-1}{2}, \beta = \frac{\sqrt{5}+1}{2}$$

$$12\left(\alpha^2+\beta^2\right) = 12\left(\frac{\left(\sqrt{5}-1\right)^2+\left(\sqrt{5}+1\right)^2}{4}\right) = 36$$

23. Let A(–2, –1), B(1, 0), $C(\alpha,\beta)$ and $D(\gamma,\delta)$ be the vertices of a parallelogram ABCD. If the point C lies on 2x – y = 5 and the point D lies on 3x – 2y = 6, then the value of $|\alpha+\beta+\gamma+\delta|$ is equal to ________.

Ans. **(32)**

Sol.-



$$P \equiv \left(\frac{\alpha-2}{2}, \frac{\beta-1}{2}\right) \equiv \left(\frac{\gamma+1}{2}, \frac{\delta}{2}\right)$$

$$\frac{\alpha-2}{2} = \frac{\gamma+1}{2} \text{ and } \frac{\beta-1}{2} = \frac{\delta}{2}$$

$$\Rightarrow \alpha - \gamma = 3.....(1), \quad \beta - \delta = 1......(2)$$

Also, (γ,δ) lies on 3x – 2y = 6

$$3\gamma - 2\delta = 6 ......(3)$$

and (α,β) lies on 2x – y = 5

$$\Rightarrow 2\alpha - \beta = 5.........(4)$$

Solving (1), (2), (3), (4)

$$\alpha = -3, \ \beta = -11, \gamma = -6, \delta = -12$$

$$|\alpha+\beta+\gamma+\delta| = 32$$

24. Let the coefficient of x^r in the expansion of

$$(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + ... + (x+2)^{n-1}$$

be α_r. If $\sum_{r=0}^{n} \alpha_r = \beta^n - \gamma^n, \beta, \gamma \in N$, then the value of $\beta^2 + \gamma^2$ equals __________.

Ans. **(25)**

Sol.-

$$(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + ... + (x+2)^{n-1}$$

$$\sum \alpha_r = 4^{n-1} + 4^{n-2} \times 3 + 4^{n-3} \times 3^2 .... + 3^{n-1}$$

$$= 4^{n-1}\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 ..... + \left(\frac{3}{4}\right)^{n-1}\right]$$

$$= 4^{n-1} \times \frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}}$$

$$= 4^n - 3^n = \beta^n - \gamma^n$$

$$\beta = 4, \gamma = 3$$

$$\beta^2 + \gamma^2 = 16 + 9 = 25$$

25. Let A be a 3×3 matrix and det (A) = 2. If

$$n = \det(\underbrace{\text{adj}(\text{adj}(......(\text{adj}A)))}_{\text{2024-times}})$$

Then the remainder when n is divided by 9 is equal to ___________.

Ans. **(7)**

Sol.- $|A| = 2$

$$\underbrace{\text{adj}\left(\text{adj}\left(\text{adj}.....(a)\right)\right)}_{\text{2024 times}} = |A|^{(n-1)^{2024}}$$

$$= |A|^{2^{2024}}$$

$$= 2^{2^{2024}}$$

$2^{2024} = (2^2)2^{2022} = 4(8)^{674} = 4(9-1)^{674}$

$\Rightarrow 2^{2024} \equiv 4(\text{mod } 9)$

$\Rightarrow 2^{2024} \equiv 9m+4,\ m \leftarrow \text{even}$

$2^{9m+4} \equiv 16\cdot(2^3)^{3m} \equiv 16(\text{mod } 9)$

$\equiv 7$

26. Let $\vec{a} = 3\hat{i}+2\hat{j}+\hat{k}, \vec{b} = 2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}$ be a vector such that $(\vec{a}+\vec{b})\times\vec{c} = 2(\vec{a}\times\vec{b})+24\hat{j}-6\hat{k}$ and $(\vec{a}-\vec{b}+\hat{i}).\vec{c} = -3$. Then $|\vec{c}|^2$ is equal to ______.

Ans. **(38)**

Sol.- $(\vec{a}+\vec{b})\times\vec{c} = 2(\vec{a}\times\vec{b})+24\hat{j}-6\hat{k}$

$(5\hat{i}+\hat{j}+4\hat{k})\times\vec{c} = 2(7\hat{i}-7\hat{j}-7\hat{k})+24\hat{j}-6\hat{k}$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 1 & 4 \\ x & y & z \end{vmatrix} = 14\hat{i}+10\hat{j}-20\hat{k}$

$\Rightarrow \hat{i}(z-4y)-\hat{j}(5z-4x)+\hat{k}(5y-x) = 14\hat{i}+10\hat{j}-20\hat{k}$

$z-4y = 14, 4x-5z = 10, 5y-x = -20$

$(a-b+i).\vec{c} = -3$

$(2\hat{i}+3\hat{j}-2\hat{k}).\vec{c} = -3$

$2x+3y-2z = -3$

$\therefore x = 5, y = -3, z = 2$

$|\vec{c}|^2 = 25+9+4 = 38$

27. If $\lim\limits_{x\to 0}\dfrac{ax^2e^x - b\log_e(1+x)+cxe^{-x}}{x^2\sin x} = 1$, then $16(a^2+b^2+c^2)$ is equal to ______.

Ans. (81)

Sol.- $\lim\limits_{x\to 0}\dfrac{ax^2\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+.....\right)-b\left(x-\frac{x^2}{2}+\frac{x^3}{3}-.......\right)+cx\left(1-x+\frac{x^2}{x!}-\frac{x^3}{3!}+.......\right)}{x^3.\frac{\sin x}{x}}$

$= \lim\limits_{x\to\infty}\dfrac{(c-b)x+\left(\frac{b}{2}-c+a\right)x^2+\left(a-\frac{b}{3}+\frac{c}{2}\right)x^3+......}{x^3} = 1$

$c-b = 0,\quad \dfrac{b}{2}-c+a = 0$

$a-\dfrac{b}{3}+\dfrac{c}{2} = 1 \quad a = \dfrac{3}{4} \quad b = c = \dfrac{3}{2}$

$a^2+b^2+c^2 = \dfrac{9}{16}+\dfrac{9}{4}+\dfrac{9}{4}$

$16(a^2+b^2+c^2) = 81$

28. A line passes through A(4, –6, –2) and B(16, –2,4). The point P(a, b, c) where a, b, c are non-negative integers, on the line AB lies at a distance of 21 units, from the point A. The distance between the points P(a, b, c) and Q(4, –12, 3) is equal to ____.

Ans. **(22)**

Sol.-

$\dfrac{x-4}{12} = \dfrac{x+6}{4} = \dfrac{z+2}{6}$

$\dfrac{x-4}{\frac{6}{7}} = \dfrac{y+6}{\frac{2}{7}} = \dfrac{z+2}{\frac{3}{7}} = 21$

$\left(21\times\dfrac{6}{7}+4, \dfrac{2}{7}\times 21-6, \dfrac{3}{7}\times 21-2\right)$

$= (22, 0, 7) = (a, b, c)$

$\therefore \sqrt{324+144+16} = 22$

29. Let y = y(x) be the solution of the differential equation

$\sec^2 x dx + (e^{2y}\tan^2 x + \tan x)dy = 0,$

$0 < x < \dfrac{\pi}{2}, y\left(\dfrac{\pi}{4}\right) = 0$. If $y\left(\dfrac{\pi}{6}\right) = \alpha,$

Then $e^{8\alpha}$ is equal to ______.

Ans. **(9)**

Sol.-

$$\sec^2 x\frac{dx}{dy}+e^{2y}\tan^2 x+\tan x=0$$

$$\left(\text{Put } \tan x=t\Rightarrow \sec^2 x\frac{dx}{dy}=\frac{dt}{dy}\right)$$

$$\frac{dt}{dy}+e^{2y}\times t^2+t=0$$

$$\frac{dt}{dy}+t=-t^2.e^{2y}$$

$$\frac{1}{t^2}\frac{dt}{dy}+\frac{1}{t}=-e^{2y}$$

$$\left(\text{Put } \frac{1}{t}=u \quad \frac{-1}{t^2}\frac{dt}{dy}=\frac{du}{dy}\right)$$

$$\frac{-du}{dy}+u=-e^{2y}$$

$$\frac{du}{dy}-u=e^{2y}$$

$$\text{I.F.}=e^{-\int dy}=e^{-y}$$

$$ue^{-y}=\int e^{-y}\times e^{2y}dy$$

$$\frac{1}{\tan x}\times e^{-y}=e^{y}+c$$

$$x=\frac{\pi}{4},y=0,c=0$$

$$x=\frac{\pi}{6},\quad y=\alpha$$

$$\sqrt{3}e^{-\alpha}=e^{\alpha}+0$$

$$e^{2\alpha}=\sqrt{3}$$

$$e^{8\alpha}=9$$

30. Let A = {1, 2, 3, ...100}. Let R be a relation on A defined by (x, y) $\in$ R if and only if 2x = 3y. Let R_1 be a symmetric relation on A such that $R\subset R_1$ and the number of elements in R_1 is n. Then, the minimum value of n is ___________.

Ans. (66)

Sol.-

$$R=\{(3,2),(6,4),(9,6),(12,8),.........(99,66)\}$$

$$n(R)=33$$

$$\therefore 66$$