FINAL JEE–MAIN EXAMINATION – JANUARY, 2024

(Held On Tuesday 30th January, 2024) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. Consider the system of linear equations

$x + y + z = 5$, $x + 2y + \lambda^2 z = 9$,

$x + 3y + \lambda z = \mu$, where $\lambda, \mu \in R$. Then, which of the following statement is NOT correct?

(1) System has infinite number of solution if $\lambda = 1$ and $\mu = 13$

(2) System is inconsistent if $\lambda = 1$ and $\mu \neq 13$

(3) System is consistent if $\lambda \neq 1$ and $\mu = 13$

(4) System has unique solution if $\lambda \neq 1$ and $\mu \neq 13$

Ans. (4)

Sol. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & \lambda^2 \\ 1 & 3 & \lambda \end{vmatrix} = 0$

$\Rightarrow 2\lambda^2 - \lambda - 1 = 0$

$\lambda = 1, -\frac{1}{2}$

$\begin{vmatrix} 1 & 1 & 5 \\ 2 & \lambda^2 & 9 \\ 3 & \lambda & \mu \end{vmatrix} = 0 \Rightarrow \mu = 13$

Infinite solution $\lambda = 1$ & $\mu = 13$

For unique sol^n $\lambda \neq 1$

For no sol^n $\lambda = 1$ & $\mu \neq 13$

If $\lambda \neq 1$ and $\mu \neq 13$

Considering the case when $\lambda = -\frac{1}{2}$ and $\mu \neq 13$ this will generate no solution case

2. For $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$, let $3\sin(\alpha+\beta) = 2\sin(\alpha-\beta)$ and a real number k be such that $\tan\alpha = k\tan\beta$. Then the value of k is equal to :

(1) $-\frac{2}{3}$ (2) -5

(3) $\frac{2}{3}$ (4) 5

Ans. Bonus

Sol. $3\sin\alpha\cos\beta + 3\sin\beta\cos\alpha$

$= 2\sin\alpha\cos\beta - 2\sin\beta\cos\alpha$

$5\sin\beta\cos\alpha = -\sin\alpha\cos\beta$

$\tan\beta = -\frac{1}{5}\tan\alpha$

$\tan\alpha = -5\tan\beta$

Not possible as $\tan\alpha, \tan\beta$ are positive

$\Rightarrow$ Data inconsistent

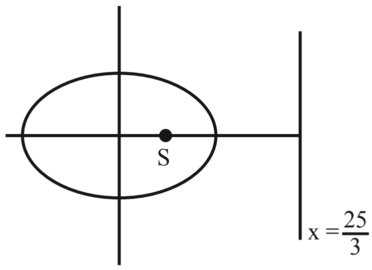
3. Let $A(\alpha, 0)$ and $B(0, \beta)$ be the points on the line $5x + 7y = 50$. Let the point P divide the line segment AB internally in the ratio 7 : 3. Let $3x - 25 = 0$ be a directrix of the ellipse $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the corresponding focus be S. If from S, the perpendicular on the x-axis passes through P, then the length of the latus rectum of E is equal to

(1) $\frac{25}{3}$ (2) $\frac{32}{9}$

(3) $\frac{25}{9}$ (4) $\frac{32}{5}$

Ans. (4)

Sol. $\left.\begin{matrix} A = (10, 0) \\ B = \left(0, \frac{50}{7}\right) \end{matrix}\right\} P = (3, 5)$



$ae = 3$

$\frac{a}{e} = \frac{25}{3}$

$a = 5$

$b = 4$

Length of $LR = \frac{2b^2}{a} = \frac{32}{5}$

4. Let $\vec{a} = \hat{i} + \alpha\hat{j} + \beta\hat{k}$, $\alpha, \beta \in R$. Let a vector $\vec{b}$ be such that the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$ and $|\vec{b}|^2 = 6$, If $\vec{a}.\vec{b} = 3\sqrt{2}$, then the value of $(\alpha^2 + \beta^2)|\vec{a} \times \vec{b}|^2$ is equal to

(1) 90 (2) 75
(3) 95 (4) 85

Ans. (1)

Sol. $|\vec{b}|^2 = 6$; $|\vec{a}||\vec{b}|\cos\theta = 3\sqrt{2}$

$|\vec{a}|^2 |\vec{b}|^2 \cos^2\theta = 18$

$|\vec{a}|^2 = 6$

Also $1 + \alpha^2 + \beta^2 = 6$

$\alpha^2 + \beta^2 = 5$

to find

$(\alpha^2 + \beta^2)\ |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$

$= (5)(6)(6)\left(\frac{1}{2}\right)$

$= 90$

5. Let $f(x) = (x+3)^2(x-2)^3, x \in [-4, 4]$. If M and m are the maximum and minimum values of f, respectively in [–4, 4], then the value of M – m is :

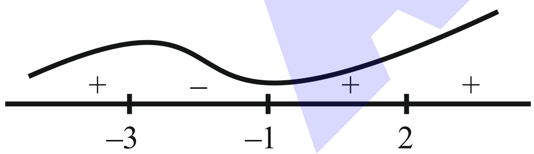
(1) 600 (2) 392
(3) 608 (4) 108

Ans. (3)

Sol. $f'(x) = (x + 3)^2 . 3(x - 2)^2 + (x - 2)^3\, 2(x + 3)$

$= 5(x + 3)(x - 2)^2(x + 1)$

$f'(x) = 0, x = -3, -1, 2$



$f(-4) = -216$

$f(-3) = 0, f(4) = 49 \times 8 = 392$

$M = 392, m = -216$

$M - m = 392 + 216 = 608$

Ans = '3'

6. Let a and b be be two distinct positive real numbers. Let 11^{th} term of a GP, whose first term is a and third term is b, is equal to p^{th} term of another GP, whose first term is a and fifth term is b. Then p is equal to

(1) 20 (2) 25
(3) 21 (4) 24

Ans. (3)

Sol. 1^{st} GP $\Rightarrow t_1 = a, t_3 = b = ar^2 \Rightarrow r^2 = \frac{b}{a}$

$t_{11} = ar^{10} = a(r^2)^5 = a \cdot \left(\frac{b}{a}\right)^5$

2^{nd} G.P. $\Rightarrow T_1 = a, T_5 = ar^4 = b$

$\Rightarrow r^4 = \left(\frac{b}{a}\right) \Rightarrow r = \left(\frac{b}{a}\right)^{1/4}$

$T_p = ar^{p-1} = a\left(\frac{b}{a}\right)^{\frac{p-1}{4}}$

$t_{11} = T_p \Rightarrow a\left(\frac{b}{a}\right)^5 = a\left(\frac{b}{a}\right)^{\frac{p-1}{4}}$

$\Rightarrow\ 5 = \frac{p-1}{4} \Rightarrow p = 21$

7. If $x^2 - y^2 + 2hxy + 2gx + 2fy + c = 0$ is the locus of a point, which moves such that it is always equidistant from the lines $x + 2y + 7 = 0$ and $2x - y + 8 = 0$, then the value of $g + c + h - f$ equals

(1) 14 (2) 6
(3) 8 (4) 29

Ans. (1)

Sol. Cocus of point P(x, y) whose distance from Gives

$X + 2y + 7 = 0$ & $2x - y + 8 = 0$ are equal is

$\frac{x + 2y + 7}{\sqrt{5}} = \pm\frac{2x - y + 8}{\sqrt{5}}$

$(x + 2y + 7)^2 - (2x - y + 8)^2 = 0$

Combined equation of lines

$(x-3y+1)(3x+y+15)=0$

$3x^2-3y^2-8xy+18x-44y+15=0$

$x^2-y^2-\frac{8}{3}xy+6x-\frac{44}{3}y+5=0$

$x^2-y^2+2h\,xy+2gx\,2+2fy+c=0$

$h=\frac{4}{3},\ g=3,\ f=-\frac{22}{3},\ c=5$

$g+c+h-f=3+5-\frac{4}{3}+\frac{22}{3}=8+6=14$

8. Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{b}|=1$ and $|\vec{b}\times\vec{a}|=2$. Then $\left|(\vec{b}\times\vec{a})-\vec{b}\right|^2$ is equal to

(1) 3

(2) 5

(3) 1

(4) 4

Ans. (2)

Sol. $|\vec{b}|=1\ \&\ |\vec{b}\times\vec{a}|=2$

$\left(\vec{b}\times\vec{a}\right)\cdot\vec{b}=\vec{b}\cdot\left(\vec{b}\times\vec{a}\right)=0$

$\left|(\vec{b}\times\vec{a})-\vec{b}\right|^2=\left|\vec{b}\times\vec{a}\right|^2+\left|\vec{b}\right|^2$

$=4+1=5$

9. Let $y=f(x)$ be a thrice differentiable function in $(-5,5)$. Let the tangents to the curve $y=f(x)$ at $(1,f(1))$ and $(3,f(3))$ make angles $\frac{\pi}{6}$ and $\frac{\pi}{4}$, respectively with positive x-axis. If $27\int_1^3\left((f'(t))^2+1\right)f''(t)dt=\alpha+\beta\sqrt{3}$ where α, β are integers, then the value of $\alpha+\beta$ equals

(1) –14

(2) 26

(3) –16

(4) 36

Ans. (2)

Sol. $y=f(x)\Rightarrow\frac{dy}{dx}=f'(x)$

$\left.\frac{dy}{dx}\right)_{(1,f(1))}=f'(1)=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}\Rightarrow f'(1)=\frac{1}{\sqrt{3}}$

$\left.\frac{dy}{dx}\right)_{(3,f(3))}=f'(3)=\tan\frac{\pi}{4}=1\Rightarrow f'(3)=1$

$27\int_1^3\left((f'(t))^2+1\right)f''(t)dt=\alpha+\beta\sqrt{3}$

$I=\int_1^3\left((f'(t))^2+1\right)f''(t)dt$

$f'(t)=z\Rightarrow f''(t)\,dt=dz$

$z=f'(3)=1$

$z=f'(1)=\frac{1}{\sqrt{3}}$

$I=\int_{1/\sqrt{3}}^{1}(z^2+1)dz=\left(\frac{z^3}{3}+z\right)_{1/\sqrt{3}}^{1}$

$=\left(\frac{1}{3}+1\right)-\left(\frac{1}{3}\cdot\frac{1}{3\sqrt{3}}+\frac{1}{\sqrt{3}}\right)$

$=\frac{4}{3}-\frac{10}{9\sqrt{3}}=\frac{4}{3}-\frac{10}{27}\sqrt{3}$

$\alpha+\beta\sqrt{3}=27\left(\frac{4}{3}-\frac{10}{27}\sqrt{3}\right)=36-10\sqrt{3}$

$\alpha=36,\ \beta=-10$

$\alpha+\beta=36-10=26$

10. Let P be a point on the hyperbola $H:\frac{x^2}{9}-\frac{y^2}{4}=1$, in the first quadrant such that the area of triangle formed by P and the two foci of H is $2\sqrt{13}$. Then, the square of the distance of P from the origin is

(1) 18

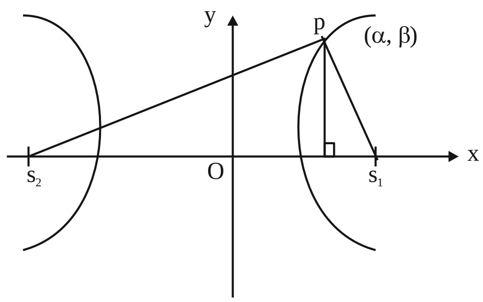
(2) 26

(3) 22

(4) 20

Ans. (3)

Sol.



$$\frac{x^2}{9}-\frac{y^2}{4}=1$$

$a^2 = 9, b^2 = 4$

$$b^2 = a^2(e^2-1) \Rightarrow e^2 = 1+\frac{b^2}{a^2}$$

$$e^2 = 1+\frac{4}{9}=\frac{13}{9}$$

$$e=\frac{\sqrt{13}}{3} \Rightarrow s_1s_2 = 2ae = 2\times 3\times\sqrt{\frac{13}{3}} = 2\sqrt{13}$$

$$\text{Area of } \Delta PS_1S_2 = \frac{1}{2}\times\beta\times s_1s_2 = 2\sqrt{13}$$

$$\Rightarrow \frac{1}{2}\times\beta\times(2\sqrt{13}) = 2\sqrt{13} \Rightarrow \beta = 2$$

$$\frac{\alpha^2}{9}-\frac{\beta^2}{4}=1 \Rightarrow \frac{\alpha^2}{9}-1=1 \Rightarrow \alpha^2 = 18 \Rightarrow \alpha = 3\sqrt{2}$$

Distance of P from origin $= \sqrt{\alpha^2+\beta^2}$

$$= \sqrt{18+4} = \sqrt{22}$$

11. Bag A contains 3 white, 7 red balls and bag B contains 3 white, 2 red balls. One bag is selected at random and a ball is drawn from it. The probability of drawing the ball from the bag A, if the ball drawn in white, is :

(1) $\frac{1}{4}$ (2) $\frac{1}{9}$

(3) $\frac{1}{3}$ (4) $\frac{3}{10}$

Ans. (3)

Sol. E_1 : A is selected

A	B
3 W 7 R	3 W 2 R

E_2 : B is selected

E : white ball is drawn

P (E_1/E) =

$$\frac{P(E).P(E/E_1)}{P(E_1).\,P(E/E_1)+P(E_2).\,P(E/E_2)} = \frac{\frac{1}{2}\times\frac{3}{10}}{\frac{1}{2}\times\frac{3}{10}+\frac{1}{2}\times\frac{3}{5}}$$

$$= \frac{3}{3+6} = \frac{1}{3}$$

12. Let $f : R \to R$ be defined $f(x) = ae^{2x}+be^{x}+cx$. If $f(0) = -1$, $f'(\log_e 2) = 21$ and $\int_0^{\log_e 4}(f(x)-cx)\,dx = \frac{39}{2}$, then the value of $|a+b+c|$ equals :

(1) 16 (2) 10

(3) 12 (4) 8

Ans. (4)

Sol. $f(x) = ae^{2x} + be^{x} + cx$ $\quad f(0) = -1$

$a + b = -1$

$f'(x) = 2ae^{2x} + be^{x} + c$ $\quad f'(\ln 2) = 21$

$8a + 2b + c = 21$

$$\int_0^{\ln 4}(ae^{2x}+be^{x})\,dx = \frac{39}{2}$$

$$\left[\frac{ae^{2x}}{2}+be^{x}\right]_0^{\ln 4} = \frac{39}{2} \Rightarrow 8a+4b-\frac{a}{2}-b = \frac{39}{2}$$

$15a + 6b = 39$

$15a - 6a - 6 = 39$

$9a = 45 \Rightarrow a = 5$

$b = -6$

$c = 21 - 40 + 12 = -7$

$a + b + c = -8$

$|a + b + c| = 8$

13. Let $L_1 : \vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(\hat{i} - \hat{j} + 2\hat{k}),\ \lambda \in R$

$L_2 : \vec{r} = (\hat{j} - \hat{k}) + \mu(3\hat{i} + \hat{j} + p\hat{k}),\ \mu \in R$ and

$L_3 : \vec{r} = \delta(\ell\hat{i} + m\hat{j} + n\hat{k})\ \delta \in R$

Be three lines such that L_1 is perpendicular to L_2 and L_3 is perpendicular to both L_1 and L_2. Then the point which lies on L_3 is

(1) (–1, 7, 4) (2) (–1, –7, 4)
(3) (1, 7, –4) (4) (1, –7, 4)

Ans. (1)

Sol. $L_1 \perp L_2$ $\quad L_3 \perp L_1,\ L_2$

$3 - 1 + 2P = 0$

$P = -1$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix} = -\hat{i} + 7\hat{j} + 4\hat{k}$$

$\therefore\ (-\delta, 7\delta, 4\delta)$ will lie on L_3

For $\delta = 1$ the point will be (-1, 7, 4)

14. Let a and b be real constants such that the function f defined by $f(x) = \begin{cases} x^2 + 3x + a, & x \le 1 \\ bx + 2, & x > 1 \end{cases}$ be differentiable on R. Then, the value of $\int_{-2}^{2} f(x)\,dx$ equals

(1) $\frac{15}{6}$ (2) $\frac{19}{6}$
(3) 21 (4) 17

Ans. (4)

Sol. f is continuous $\quad f'(x) = 2x + 3\ ,\ x < 1$

$\therefore\ 4 + a = b + 2 \quad\quad b\ ,\ x > 1$

$a = b - 2 \quad\quad$ f is differentiable

$\therefore\ b = 5$

$\therefore\ a = 3$

$$\int_{-2}^{1}(x^2 + 3x + 3)dx + \int_{1}^{2}(5x + 2)dx$$

$$= \left[\frac{x^3}{3} + \frac{3x^2}{2} + 3x\right]_{-2}^{1} + \left[\frac{5x^2}{2} + 2x\right]_{1}^{2}$$

$$= \left(\frac{1}{3} + \frac{3}{2} + 3\right) - \left(\frac{-8}{3} + 6 - 6\right) + \left(10 + 4 - \frac{5}{2} - 2\right)$$

$$= 6 + \frac{3}{2} + 12 - \frac{5}{2} = 17$$

15. Let $f : \mathbb{R} - \{0\} \to \mathbb{R}$ be a function satisfying $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ for all x, y, $f(y) \ne 0$. If $f'(1) = 2024$, then

(1) $xf'(x) - 2024\,f(x) = 0$
(2) $xf'(x) + 2024f(x) = 0$
(3) $xf'(x) + f(x) = 2024$
(4) $xf'(x) - 2023f(x) = 0$

Ans. (1)

Sol. $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ $\quad f'(1) = 2024$, $f(1) = 1$

Partially differentiating w. r. t. x

$f'\left(\frac{x}{y}\right) \cdot \frac{1}{y} = \frac{1}{f(y)} f'(x)$

$y \to x$

$f'(1) \cdot \frac{1}{x} = \frac{f'(x)}{f(x)}$

$2024f(x) = xf'(x) \Rightarrow xf'(x) - 2024\,f(x) = 0$

16. If z is a complex number, then the number of common roots of the equation $z^{1985} + z^{100} + 1 = 0$ and $z^3 + 2z^2 + 2z + 1 = 0$, is equal to :

(1) 1 (2) 2
(3) 0 (4) 3

Ans. (2)

Sol. $z^{1985} + z^{100} + 1 = 0\ \ \&\ \ z^3 + 2z^2 + 2z + 1 = 0$

$(z + 1)(z^2 - z + 1) + 2z(z + 1) = 0$

$(z + 1)(z^2 + z + 1) = 0$

$\Rightarrow z = -1,\ z = w, w^2$

Now putting $z = -1$ not satisfy

Now put $z = w$

$\Rightarrow w^{1985} + w^{100} + 1$

$\Rightarrow w^2 + w + 1 = 0$

Also, $z = w^2$

$\Rightarrow w^{3970} + w^{200} + 1$

$\Rightarrow w + w^2 + 1 = 0$

Two common root

17. Suppose $2-p$, p, $2-\alpha$, α are the coefficient of four consecutive terms in the expansion of $(1+x)^n$. Then the value of $p^2-\alpha^2+6\alpha+2p$ equals

(1) 4 (2) 10

(3) 8 (4) 6

Ans. (Bonus)

Sol. $2-p, p, 2-\alpha, \alpha$

Binomial coefficients are

${}^nC_r$, ${}^nC_{r+1}$, ${}^nC_{r+2}$, ${}^nC_{r+3}$ respectively

$\Rightarrow$ ${}^nC_r + {}^nC_{r+1} = 2$

$\Rightarrow$ ${}^{n+1}C_{r+1} = 2$ ……(1)

Also, ${}^nC_{r+2} + {}^nC_{r+3} = 2$

$\Rightarrow$ ${}^{n+1}C_{r+3} = 2$ …….(2)

From (1) and (2)

${}^{n+1}C_{r+1} = {}^{n+1}C_{r+3}$

$\Rightarrow$ $2r+4 = n+1$

$n = 2r+3$

${}^{2r+4}C_{r+1} = 2$

Data Inconsistent

18. If the domain of the function $f(x) = \log_e\left(\frac{2x+3}{4x^2+x-3}\right)+\cos^{-1}\left(\frac{2x-1}{x+2}\right)$ is $(\alpha,\beta]$, then the value of $5\beta-4\alpha$ is equal to

(1) 10 (2) 12

(3) 11 (4) 9

Ans. (2)

Sol. $\frac{2x+3}{4x^2+x-3} > 0$ and $-1 \le \frac{2x-1}{x+2} \le 1$

$\frac{2x+3}{(4x-3)(x+1)} > 0$ $\quad \frac{3x+1}{x+2} \ge 0$ & $\frac{x-3}{x+2} \le 0$

– + – +

–3/2 –1 3/4

$(-\infty,-2)\cup\left[\frac{-1}{3},\infty\right)$ …..(1)

$(-2,3]$ …….(2)

$\left[\frac{-1}{3},3\right]$ ……(3) $\quad$ (1) $\cap$ (2) $\cap$ (3)

$\left(\frac{3}{4},3\right]$

$\alpha = \frac{3}{4}$ $\beta = 3$

$5\beta - 4\alpha = 15-3 = 12$

19. Let $f: R \to R$ be a function defined

$f(x)=\frac{x}{(1+x^4)^{1/4}}$ and $g(x)=f(f(f(f(x))))$ then

$18\int_0^{\sqrt{2\sqrt{5}}} x^2 g(x)dx$

(1) 33 (2) 36

(3) 42 (4) 39

Ans. (4)

Sol. $f(x) = \frac{x}{(1+x^4)^{1/4}}$

$$fof(x) = \frac{f(x)}{(1+f(x)^4)^{1/4}} = \frac{\frac{x}{(1+x^4)^{1/4}}}{\left(1+\frac{x^4}{1+x^4}\right)^{1/4}} = \frac{x}{(1+2x^4)^{1/4}}$$

$$f(f(f(f(x)))) = \frac{x}{(1+4x^4)^{1/4}}$$

$$18\int_0^{\sqrt{2\sqrt{5}}} \frac{x^3}{(1+4x^4)^{1/4}}dx$$

Let $1+4x^4 = t^4$

$16x^3\,dx = 4t^3\,dt$

$$\frac{18}{4}\int_1^3 \frac{t^3 dt}{t}$$

$$= \frac{9}{2}\left(\frac{t^3}{3}\right)_1^3$$

$$= \frac{3}{2}[26] = 39$$

20. Let $R = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$ be a non-zero 3×3 matrix, where $x\sin\theta = y\sin\left(\theta+\frac{2\pi}{3}\right) = z\sin\left(\theta+\frac{4\pi}{3}\right) \ne 0, \theta\in(0,2\pi)$. For a square matrix M, let trace (M) denote the sum of all the diagonal entries of M. Then, among the statements:

(I) Trace (R) = 0

(II) If trace (adj(adj(R)) = 0, then R has exactly one non-zero entry.

(1) Both (I) and (II) are true

(2) Neither (I) nor (II) is true

(3) Only (II) is true

(4) Only (I) is true

Ans. (3)

Sol. $x\sin\theta = y\sin\left(\theta+\frac{2\pi}{3}\right) = z\sin\left(\theta+\frac{4\pi}{3}\right) = \lambda$ (say), $\lambda \neq 0$

$\Rightarrow x, y, z \neq 0$ and $\sin\theta$, $\sin\left(\theta+\frac{2\pi}{3}\right)$, $\sin\left(\theta+\frac{4\pi}{3}\right) \neq 0$

Also,

$$\sin\theta + \sin\left(\theta+\frac{2\pi}{3}\right) + \sin\left(\theta+\frac{4\pi}{3}\right) = 0 \ \forall\ \theta \in R$$

$$\Rightarrow x+y+z = \frac{-\lambda}{2}\frac{\left(\sin^2\theta + \sin^2\left(\theta+\frac{2\pi}{3}\right) + \sin^2\left(\theta+\frac{4\pi}{3}\right)\right)}{\sin\theta\sin\left(\theta+\frac{2\pi}{3}\right)\sin\left(\theta+\frac{4\pi}{3}\right)} \neq 0$$

(i) Trace (R) = $x + y + z \neq 0$

$\Rightarrow$ Statement (i) is False

(ii) Adj(Adj(R)) = |R| R

Trace (Adj(Adj(R)))

$= xyz(x + y + z) \neq 0$

$\Rightarrow$ Hypothesis of conditional statement (ii) is false

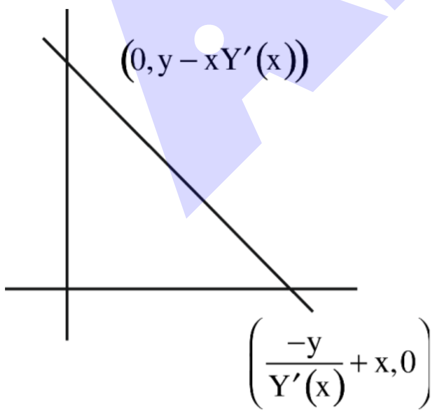
$\Rightarrow$ Conditional statement (ii) is vacuously true !!

SECTION-B

21. Let Y = Y(X) be a curve lying in the first quadrant such that the area enclosed by the line $Y - y = Y'(x)(X - x)$ and the co-ordinate axes, where (x, y) is any point on the curve, is always $\frac{-y^2}{2Y'(x)} + 1$, $Y'(x) \neq 0$. If Y(1) = 1, then 12Y(2) equals ______ .

Ans. (20)

Sol. $A = \frac{1}{2}\left(\frac{-y}{Y'(x)} + x\right)(y - xY/x) = \frac{-y^2}{2Y'(x)} + 1$



$\Rightarrow (-y + xY'(x))(y - xY'(x)) = -y^2 + 2Y'(x)$

$-y^2 + xyY'(x) + xyY'(x) - x^2[Y'(x)]^2$

$= -y^2 + 2Y'(x)$

$2xy - x^2\,Y'(x) = 2$

$\frac{dy}{dx} = \frac{2xy - 2}{x^2}$

$\frac{dy}{dx} - \frac{2}{x}y = \frac{-2}{x^2}$

I.F. $= e^{-2\ln x} = \frac{1}{x^2}$

$y \cdot \frac{1}{x^2} = \frac{2}{3}x^{-3} + c$

Put x = 1, y = 1

$1 = \frac{2}{3} + c \Rightarrow c = \frac{1}{3}$

$Y = \frac{2}{3}\cdot\frac{1}{X} + \frac{1}{3}X^2$

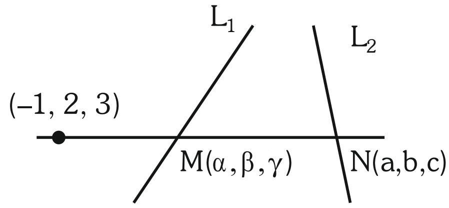
$\Rightarrow 12Y(2) = \frac{5}{3} \times 12 = 20$

22. Let a line passing through the point (–1, 2, 3) intersect the lines $L_1 : \frac{x-1}{3} = \frac{y-2}{2} = \frac{z+1}{-2}$ at $M(\alpha,\beta,\gamma)$ and $L_2 : \frac{x+2}{-3} = \frac{y-2}{-2} = \frac{z-1}{4}$ at N(a, b, c). Then the value of $\frac{(\alpha+\beta+\gamma)^2}{(a+b+c)^2}$ equals ____.

Ans. (196)

Sol. $M(3\lambda+1, 2\lambda+2, -2\lambda-1) \ \therefore\ \alpha+\beta+\gamma = 3\lambda+2$

$N(-3\mu-2, -2\mu+2, 4\mu+1) \ \therefore\ a+b+c = -\mu+1$



$\frac{3\lambda+2}{-3\mu-1} = \frac{2\lambda}{-2\mu} = \frac{-2\lambda-4}{4\mu-2}$

$3\lambda\mu + 2\mu = 3\lambda\mu + \lambda$

$2\mu = \lambda$

$2\lambda\mu - \lambda = \lambda\mu + 2\mu$

$\lambda\mu = \lambda + 2\mu$

$\Rightarrow \lambda\mu = 2\lambda$

$\Rightarrow \quad \mu = 2 \quad (\lambda \neq 0)$

$\therefore \quad \lambda = 4$

$\alpha + \beta + \gamma = 14$

$a + b + c = -1$

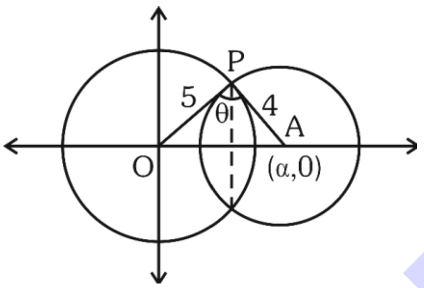
$$\frac{(\alpha+\beta+\gamma)^2}{(a+b+c)^2} = 196$$

23. Consider two circles $C_1 : x^2 + y^2 = 25$ and $C_2 : (x - \alpha)^2 + y^2 = 16$, where $\alpha \in (5, 9)$. Let the angle between the two radii (one to each circle) drawn from one of the intersection points of C_1 and C_2 be $\sin^{-1}\left(\frac{\sqrt{63}}{8}\right)$. If the length of common chord of C_1 and C_2 is β, then the value of $(\alpha\beta)^2$ equals ______ .

Ans. (1575)

Sol. $C_1 : x^2 + y^2 = 25$, $C_2 : (x-\alpha)^2 + y^2 = 16$

$5 < \alpha < 9$



$$\theta = \sin^{-1}\left(\frac{\sqrt{63}}{8}\right)$$

$$\sin\theta = \frac{\sqrt{63}}{8}$$

Area of $\Delta OAP = \frac{1}{2} \times \alpha\left(\frac{\beta}{2}\right) = \frac{1}{2} \times 5 \times 4\sin\theta$

$$\Rightarrow \quad \alpha\beta = 40 \times \frac{\sqrt{63}}{8}$$

$$\alpha\beta = 5 \times \sqrt{63}$$

$$(\alpha\beta)^2 = 25 \times 63 = 1575$$

24. Let $\alpha = \sum_{k=0}^{n}\left(\frac{({}^nC_k)^2}{k+1}\right)$ and $\beta = \sum_{k=0}^{n-1}\left(\frac{{}^nC_k\ {}^nC_{k+1}}{k+2}\right)$.

If $5\alpha = 6\beta$, then n equals __________ .

Ans. (10)

Sol. $\alpha = \sum_{k=0}^{n} \frac{{}^nC_k \cdot {}^nC_k}{k+1} \cdot \frac{n+1}{n+1}$

$$= \frac{1}{n+1}\sum_{k=0}^{n} {}^{n+1}C_{k+1} \cdot {}^nC_{n-k}$$

$$\alpha = \frac{1}{n+1} \cdot {}^{2n+1}C_{n+1}$$

$$\beta = \sum_{k=0}^{n-1} {}^nC_k \cdot \frac{{}^nC_{k+1}}{k+2}\,\frac{n+1}{n+1}$$

$$\frac{1}{n+1}\sum_{k=0}^{n-1} {}^nC_{n-k} \cdot {}^{n+1}C_{k+2}$$

$$= \frac{1}{n+1} \cdot {}^{2n+1}C_{n+2}$$

$$\frac{\beta}{\alpha} = \frac{{}^{2n+1}C_{n+2}}{{}^{2n+1}C_{n+1}} = \frac{2n+1-(n+2)+1}{n+2}$$

$$\frac{\beta}{\alpha} = \frac{n}{n+2} = \frac{5}{6}$$

$n = 10$

25. Let S_n be the sum to n-terms of an arithmetic progression 3, 7, 11, ...

If $40 < \left(\frac{6}{n(n+1)}\sum_{k=1}^{n} S_k\right) < 42$, then n equals ___ .

Ans. (9)

Sol. $S_n = 3 + 7 + 11 + \ldots\ldots$ n terms

$$= \frac{n}{2}(6+(n-1)4) = 3n + 2n^2 - 2n$$

$$= 2n^2 + n$$

$$\sum_{k=1}^{n} S_k = 2\sum_{k=1}^{n} K^2 + \sum_{k=1}^{n} K$$

$$= 2 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= n(n+1)\left[\frac{2n+1}{3} + \frac{1}{2}\right]$$

$$= \frac{n(n+1)(4n+5)}{6}$$

$$\Rightarrow \quad 40 < \frac{6}{n(n+1)}\sum_{k=1}^{n} S_k < 42$$

$40 < 4n + 5 < 42$

$35 < 4n < 37$

$n = 9$

26. In an examination of Mathematics paper, there are 20 questions of equal marks and the question paper is divided into three sections : A, B and C . A student is required to attempt total 15 questions taking at least 4 questions from each section. If section A has 8 questions, section B has 6 questions and section C has 6 questions, then the total number of ways a student can select 15 questions is ________ .

Ans. (11376)

Sol. If 4 questions from each section are selected

Remaining 3 questions can be selected either in (1, 1, 1) or (3, 0, 0) or (2, 1, 0)

$\therefore$ Total ways = ${}^8c_5 \cdot {}^6c_5 \cdot {}^6c_5 + {}^8c_6\,{}^6c_5 \cdot {}^6c_4 \times 2 + {}^8c_5 \cdot {}^6c_6 \cdot {}^6c_4 \times 2 + {}^8c_4 \cdot {}^6c_6 \cdot {}^6c_5 \times 2 + {}^8c_7 \cdot {}^6c_4 \cdot {}^6c_4$

$= 56 . 6 . 6 + 28 . 6 . 15 . 2 + 56 . 15 . 2 + 70 . 6 . 2 + 8 . 15 . 15$

$= 2016 + 5040 + 1680 + 840 + 1800 = 11376$

27. The number of symmetric relations defined on the set {1, 2, 3, 4} which are not reflexive is _____ .

Ans. (960)

Sol. Total number of relation both symmetric and reflexive $= 2^{\frac{n^2-n}{2}}$

Total number of symmetric relation $= 2^{\left(\frac{n^2+n}{2}\right)}$

$\Rightarrow$ Then number of symmetric relation which are not reflexive

$\Rightarrow 2^{\frac{n(n+1)}{2}} - 2^{\frac{n(n-1)}{2}}$

$\Rightarrow 2^{10} - 2^6$

$\Rightarrow 1024 - 64$

$= 960$

28. The number of real solutions of the equation $x\left(x^2+3|x|+5|x-1|+6|x-2|\right)=0$ is ______ .

Ans. (1)

Sol. $x = 0$ and $x^2+3|x|+5|x-1|+6|x-2|=0$

Here all terms are +ve except at x = 0

So there is no value of x

Satisfies this equation

Only solution x = 0

No of solution 1.

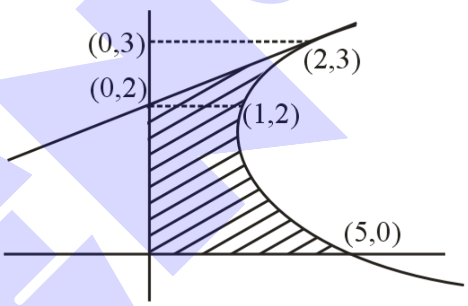
29. The area of the region enclosed by the parabola $(y-2)^2 = x-1$, the line $x - 2y + 4 = 0$ and the positive coordinate axes is ______.

Ans. (5)

Sol. Solving the equations

$(y-2)^2 = x - 1$ and $x - 2y + 4 = 0$

$X = 2(y-2)$



$\frac{x^2}{4} = x - 1$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x = 2$

Exclose area (w.r.t. y-axis) $= \int_0^3 x\,dy$ – Area of Δ.

$= \int_0^3 \left((y-2)^2+1\right)dy - \frac{1}{2}\times 1 \times 2$

$= \int_0^3 \left(y^2-4y+5\right)dy - 1$

$= \left[\frac{y^3}{3} - 2y^2 + 5y\right]_0^3 - 1$

$= 9 - 18 + 15 - 1 = 5$

30. The variance σ^2 of the data

x_i	0	1	5	6	10	12	17
f_i	3	2	3	2	6	3	3

Is ________ .

Ans. (29)

Sol.

x_i	f_i	f_ix_i	$f_ix_i^2$
0	3	0	0
1	2	2	2
5	3	15	75
6	2	12	72
10	6	60	600
12	3	36	432
17	3	51	867
	$\Sigma f_i = 22$		$\Sigma f_ix_i^2 = 2048$

$\therefore \quad \Sigma f_ix_i = 176$

So $\bar{x} = \dfrac{\sum f_ix_i}{\sum f_i} = \dfrac{176}{22} = 8$

for $\sigma^2 = \dfrac{1}{N}\sum f_ix_i^2 - (\bar{x})^2$

$= \dfrac{1}{22} \times 2048 - (8)^2$

$= 93.090964$

$= 29.0909$