FINAL JEE–MAIN EXAMINATION – JANUARY, 2024

(Held On Tuesday 30th January, 2024) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

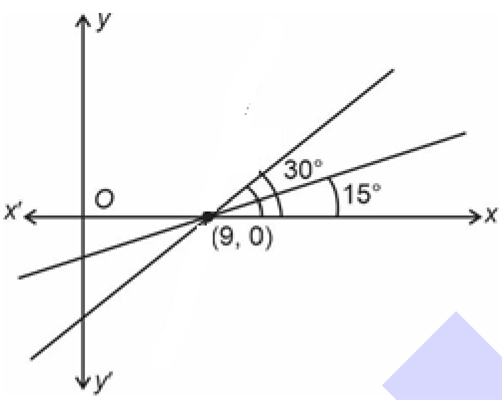
1. A line passing through the point A(9, 0) makes an angle of 30º with the positive direction of x-axis. If this line is rotated about A through an angle of 15º in the clockwise direction, then its equation in the new position is

(1) $\frac{y}{\sqrt{3}-2}+x=9$ (2) $\frac{x}{\sqrt{3}-2}+y=9$

(3) $\frac{x}{\sqrt{3}+2}+y=9$ (4) $\frac{y}{\sqrt{3}+2}+x=9$

Ans. (1)

Sol.



$\text{Eq}^n : y-0=\tan 15^\circ (x-9) \Rightarrow y=(2-\sqrt{3})(x-9)$

2. Let S_a denote the sum of first n terms an arithmetic progression. If S_{20} = 790 and S_{10} = 145, then S_{15} – S_5 is :

(1) 395

(2) 390

(3) 405

(4) 410

Ans. (1)

Sol. $S_{20}=\frac{20}{2}[2a+19d]=790$

2a + 19d = 79 .....(1)

$S_{10}=\frac{10}{2}[2a+9d]=145$

2a + 9d = 29 .....(2)

From (1) and (2) a = -8, d = 5

$S_{15}-S_5=\frac{15}{2}[2a+14d]-\frac{5}{2}[2a+4d]$

$=\frac{15}{2}[-16+70]-\frac{5}{2}[-16+20]$

= 405-10

= 395

3. If z = x + iy, xy $\neq 0$, satisfies the equation $z^2+i\bar{z}=0$, then $|z^2|$ is equal to :

(1) 9

(2) 1

(3) 4

(4) $\frac{1}{4}$

Ans. (2)

Sol. $z^2=-i\bar{z}$

$|z^2|=|i\bar{z}|$

$|z^2| = |z|$

$|z|^2 - |z| = 0$

$|z|(|z| - 1) = 0$

$|z| = 0$ (not acceptable)

$\therefore |z| = 1$

$\therefore |z|^2 = 1$

4. Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ be two vectors such that $|\vec{a}|=1$; $\vec{a}.\vec{b}=2$ and $|\vec{b}|=4$. If $\vec{c}=2(\vec{a}\times\vec{b})-3\vec{b}$, then the angle between $\vec{b}$ and $\vec{c}$ is equal to :

(1) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(2) $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

(3) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

(4) $\cos^{-1}\left(\frac{2}{3}\right)$

Ans. (3)

Sol. Given $|\vec{a}|=1, |\vec{b}|=4, \vec{a}.\vec{b}=2$

$$\vec{c}=2\left(\vec{a}\times\vec{b}\right)-3\vec{b}$$

Dot product with $\vec{a}$ on both sides

$\vec{c}.\vec{a}=-6$ .....(1)

Dot product with $\vec{b}$ on both sides

$\vec{b}.\vec{c}=-48$ .....(2)

$$\vec{c}.\vec{c}=4\left|\vec{a}\times\vec{b}\right|^2+9\left|\vec{b}\right|^2$$

$$|\vec{c}|^2=4\left[|a|^2|b|^2-\left(a.b\right)^2\right]+9\left|\vec{b}\right|^2$$

$$|\vec{c}|^2=4\left[(1)(4)^2-(4)\right]+9(16)$$

$$|\vec{c}|^2=4[12]+144$$

$$|\vec{c}|^2=48+144$$

$$|\vec{c}|^2=192$$

$$\therefore \cos\theta=\frac{\vec{b}.\vec{c}}{|\vec{b}||\vec{c}|}$$

$$\therefore \cos\theta=\frac{-48}{\sqrt{192}.4}$$

$$\therefore \cos\theta=\frac{-48}{8\sqrt{3}.4}$$

$$\therefore \cos\theta=\frac{-3}{2\sqrt{3}}$$

$$\therefore \cos\theta=\frac{-\sqrt{3}}{2}\Rightarrow\theta=\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$$

5. The maximum area of a triangle whose one vertex is at (0, 0) and the other two vertices lie on the curve $y = -2x^2 + 54$ at points (x, y) and (-x, y) where $y > 0$ is :

(1) 88

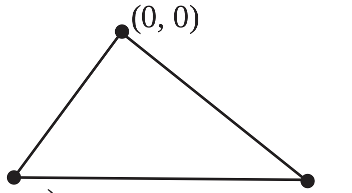
(2) 122

(3) 92

(4) 108

Ans. (4)

Sol.



Area of Δ

$$=\frac{1}{2}\begin{vmatrix}0 & 0 & 1\\ x & y & 1\\ -x & y & 1\end{vmatrix}$$

$$\Rightarrow\left|\frac{1}{2}(xy+xy)\right|=|xy|$$

$$\text{Area}(\Delta)=|xy|=\left|x\left(-2x^2+54\right)\right|$$

$$\frac{d(\Delta)}{dx}=\left|\left(-6x^2+54\right)\right|\Rightarrow\frac{d\Delta}{dx}=0 \text{ at } x=3$$

Area = 3 (-2 × 9 + 54) = 108

6. The value of $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{n^3}{\left(n^2+k^2\right)\left(n^2+3k^2\right)}$ is :

(1) $\frac{\left(2\sqrt{3}+3\right)\pi}{24}$

(2) $\frac{13\pi}{8\left(4\sqrt{3}+3\right)}$

(3) $\frac{13\left(2\sqrt{3}-3\right)\pi}{8}$

(4) $\frac{\pi}{8\left(2\sqrt{3}+3\right)}$

Ans. (2)

Sol. $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n^3}{n^4\left(1+\frac{k^2}{n^2}\right)\left(1+\frac{3k^2}{n^2}\right)}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{n^3}{\left(1+\frac{k^2}{n^2}\right)\left(1+\frac{3k^2}{n^2}\right)}$$

$$=\int_0^1\frac{dx}{3\left(1+x^2\right)\left(\frac{1}{3}+x^2\right)}$$

$$=\int_0^1 \frac{1}{3}\times\frac{3}{2}\frac{\left(x^2+1\right)-\left(x^2+\frac{1}{3}\right)}{\left(1+x^2\right)\left(x^2+\frac{1}{3}\right)}dx$$

$$=\frac{1}{2}\int_0^1\left[\frac{1}{x^2+\left(\frac{1}{\sqrt{3}}\right)^2}-\frac{1}{1+x^2}\right]dx$$

$$=\frac{1}{2}\left[\sqrt{3}\tan^{-1}\left(\sqrt{3}x\right)\right]_0^1-\frac{1}{2}\left(\tan^{-1}x\right)_0^1$$

$$=\frac{\sqrt{3}}{2}\left(\frac{\pi}{3}\right)-\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{\pi}{2\sqrt{3}}-\frac{\pi}{8}$$

$$=\frac{13\pi}{8.\left(4\sqrt{3}+3\right)}$$

7. Let g : R→R be a non constant twice differentiable such that $g'\left(\frac{1}{2}\right)=g'\left(\frac{3}{2}\right)$. If a real valued function f is defined as $f(x)=\frac{1}{2}\left[g(x)+g(2-x)\right]$, then

(1) f''(x) = 0 for atleast two x in (0, 2)

(2) f''(x) = 0 for exactly one x in (0, 1)

(3) f''(x) = 0 for no x in (0, 1)

(4) $f'\left(\frac{3}{2}\right)+f'\left(\frac{1}{2}\right)=1$

Ans. (1)

Sol. $f'(x)=\frac{g'(x)-g'(2-x)}{2}, f'\left(\frac{3}{2}\right)=\frac{g'\left(\frac{3}{2}\right)-g'\left(\frac{1}{2}\right)}{2}=0$

Also $f'\left(\frac{1}{2}\right)=\frac{g'\left(\frac{1}{2}\right)-g'\left(\frac{3}{2}\right)}{2}=0,\ f'\left(\frac{1}{2}\right)=0$

$\Rightarrow f'\left(\frac{3}{2}\right)=f'\left(\frac{1}{2}\right)=0$

$\Rightarrow$ roots in $\left(\frac{1}{2},1\right)$ and $\left(1,\frac{3}{2}\right)$

$\Rightarrow f''(x)$ is zero at least twice in $\left(\frac{1}{2},\frac{3}{2}\right)$

8. The area (in square units) of the region bounded by the parabola $y^2 = 4(x - 2)$ and the line y = 2x - 8

(1) 8

(2) 9

(3) 6

(4) 7

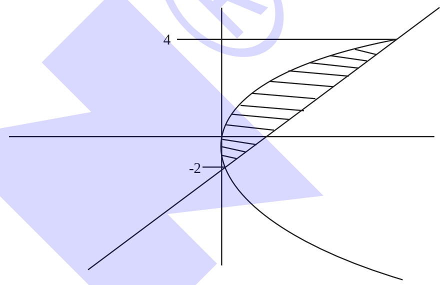
Ans. (2)

Sol. Let X = x – 2

$y^2 = 4x,\quad y = 2(x + 2) - 8$

$y^2 = 4x,\quad y = 2x - 4$

$$A=\int_{-2}^{4}\frac{y^2}{4}-\frac{y+4}{2}$$



= 9

9. Let y = y (x) be the solution of the differential equation sec x dy + {2(1 – x) tan x + x(2 – x)} dx = 0 such that y(0) = 2.Then y(2) is equal to :

(1) 2

(2) 2{1 – sin (2)}

(3) 2{sin (2) + 1}

(4) 1

Ans. (1)

Sol. $\frac{dy}{dx}=2(x-1)\sin x+\left(x^2-2x\right)\cos x$

Now both side integrate

$y(x)=\int 2(x-1)\sin x\,dx+\left[\left(x^2-2x\right)(\sin x)-\int(2x-2)\sin x\,dx\right]$

$y(x)=\left(x^2-2x\right)\sin x+\lambda$

$y(0)=0+\lambda\Rightarrow 2=\lambda$

$y(x)=\left(x^2-2x\right)\sin x+2$

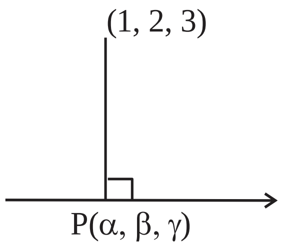
$y(2)=2$

10. Let (α,β,γ) be the foot of perpendicular from the point (1, 2, 3) on the line $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$. then $19(\alpha+\beta+\gamma)$ is equal to :

(1) 102
(2) 101
(3) 99
(4) 100

Ans. (2)

Sol.



Let foot P $(5k-3, 2k+1, 3k-4)$

$DR's \rightarrow AP: 5k-4,\ 2k-1, 3k-7$

$DR's \rightarrow Line: 5,\ 2, 3$

Condition of perpendicular lines $(25k\text{-}20)+(4k\text{-}2)+(9k-21)=0$

Then $k=\frac{43}{38}$

Then $19(\alpha+\beta+\gamma)$**=101**

11. Two integers x and y are chosen with replacement from the set $\{0, 1, 2, 3, \ldots, 10\}$. Then the probability that $|x-y|>5$ is :

(1) $\frac{30}{121}$

(2) $\frac{62}{121}$

(3) $\frac{60}{121}$

(4) $\frac{31}{121}$

Ans. (1)

Sol. If $x=0, y=6, 7, 8, 9, 10$
If $x=1, y=7, 8, 9, 10$
If $x=2, y=8, 9, 10$
If $x=3, y=9, 10$
If $x=4, y=10$
If $x=5, y=$ no possible value
Total possible ways $=(5+4+3+2+1)\times 2$
$=30$

Required probability $=\frac{30}{11\times 11}=\frac{30}{121}$

12. If the domain of the function $f(x)=\cos^{-1}\left(\frac{2-|x|}{4}\right)+\left(\log_e(3-x)\right)^{-1}$ is $[-\alpha,\beta)-\{y\}$, then $\alpha+\beta+\gamma$ is equal to :

(1) 12
(2) 9
(3) 11
(4) 8

Ans. (3)

Sol. $-1\le\left|\frac{2-|x|}{4}\right|\le 1$

$\Rightarrow \left|\frac{2-|x|}{4}\right|\le 1$

$-4 \le 2-|x| \le 4$
$-6 \le -|x| \le 2$
$-2 \le |x| \le 6$
$|x| \le 6$
$\Rightarrow\ x \in [-6, 6]$ …(1)
Now, $3-x \ne 1$
And $x \ne 2$ …(2)
and $3-x>0$
$x<3$ …(3)
From (1), (2) and (3)
$\Rightarrow\ x \in [-6, 3)-\{2\}$
$\alpha=6$
$\beta=3$
$\gamma=2$
$\alpha+\beta+\gamma=11$

13. Consider the system of linear equation $x+y+z=4\mu$, $x+2y+2\lambda z=10\mu$, $x+3y+4\lambda^2 z=\mu^2+15$, where $\lambda, \mu \in R$. Which one of the following statements is NOT correct ?

(1) The system has unique solution if $\lambda\ne\frac{1}{2}$ and $\mu\ne 1, 15$

(2) The system is inconsistent if $\lambda = \frac{1}{2}$ and $\mu \neq 1$

(3) The system has infinite number of solutions if $\lambda = \frac{1}{2}$ and $\mu = 15$

(4) The system is consistent if $\lambda \neq \frac{1}{2}$

Ans. (2)

Sol. $x + y + z = 4\mu$, $x + 2y + 2\lambda z = 10\mu$, $x + 3y + 4\lambda^2 z = \mu^2 + 15$,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2\lambda \\ 1 & 3 & 4\lambda^2 \end{vmatrix} = (2\lambda - 1)^2$$

For unique solution $\Delta \neq 0,\ 2\lambda - 1 \neq 0,\ \left(\lambda \neq \frac{1}{2}\right)$

Let $\Delta = 0,\ \lambda = \frac{1}{2}$

$$\Delta_y = 0,\ \Delta_x = \Delta_z = \begin{vmatrix} 4\mu & 1 & 1 \\ 10\mu & 2 & 1 \\ \mu^2 + 15 & 3 & 1 \end{vmatrix}$$

$= (\mu - 15)(\mu - 1)$

For infinite solution $\lambda = \frac{1}{2},\ \mu = 1$ or 15

14. If the circles $(x+1)^2 + (y+2)^2 = r^2$ and $x^2 + y^2 - 4x - 4y + 4 = 0$ intersect at exactly two distinct points, then

(1) $5 < r < 9$

(2) $0 < r < 7$

(3) $3 < r < 7$

(4) $\frac{1}{2} < r < 7$

Ans. (3)

Sol. If two circles intersect at two distinct points

$\Rightarrow |r_1 - r_2| < C_1C_2 < r_1 + r_2$

$|r - 2| < \sqrt{9 + 16} < r + 2$

$|r - 2| < 5$ and $r + 2 > 5$

$-5 < r - 2 < 5 \qquad r > 3$ ....(2)

$-3 < r < 7$ ....(1)

From (1) and (2)

$3 < r < 7$

15. If the length of the minor axis of ellipse is equal to half of the distance between the foci, then the eccentricity of the ellipse is :

(1) $\frac{\sqrt{5}}{3}$

(2) $\frac{\sqrt{3}}{2}$

(3) $\frac{1}{\sqrt{3}}$

(4) $\frac{2}{\sqrt{5}}$

Ans. (4)

Sol. $2b = ae$

$\frac{b}{a} = \frac{e}{2}$

$e = \sqrt{1 - \frac{e^2}{4}}$

$e = \frac{2}{\sqrt{5}}$

16. Let M denote the median of the following frequency distribution.

Class	0-4	4-8	8-12	12-16	16-20
Frequency	3	9	10	8	6

Then 20 M is equal to :

(1) 416

(2) 104

(3) 52

(4) 208

Ans. (4)

Sol.

Class	Frequency	Cumulative frequency
0-4	3	**3**
4-8	9	**12**
8-12	10	**22**
12-16	8	**30**
16-20	6	**36**

$$M = l + \left(\frac{\frac{N}{2} - C}{f}\right)h$$

$$M = 8 + \frac{18-12}{10} \times 4$$

M = 10.4

20M = 208

17. If $f(x) = \begin{vmatrix} 2\cos^4 x & 2\sin^4 x & 3+\sin^2 2x \\ 3+2\cos^4 x & 2\sin^4 x & \sin^2 2x \\ 2\cos^4 x & 3+2\sin^4 x & \sin^2 2x \end{vmatrix}$ then $\frac{1}{5} f'(0)$ is equal to ______

(1) 0

(2) 1

(3) 2

(4) 6

Ans. (1)

Sol. $\begin{vmatrix} 2\cos^4 x & 2\sin^4 x & 3+\sin^2 2x \\ 3+2\cos^4 x & 2\sin^4 x & \sin^2 2x \\ 2\cos^4 x & 3+2\sin^2 4x & \sin^2 2x \end{vmatrix}$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\begin{vmatrix} 2\cos^4 x & 2\sin^4 x & 3+\sin^2 2x \\ 3 & 0 & -3 \\ 0 & 3 & -3 \end{vmatrix}$$

f(x) = 45

$f'(x) = 0$

18. Let A (2, 3, 5) and C(-3, 4, -2) be opposite vertices of a parallelogram ABCD if the diagonal $\overrightarrow{BD} = \hat{i} + 2\hat{j} + 3\hat{k}$ then the area of the parallelogram is equal to

(1) $\frac{1}{2}\sqrt{410}$

(2) $\frac{1}{2}\sqrt{474}$

(3) $\frac{1}{2}\sqrt{586}$

(4) $\frac{1}{2}\sqrt{306}$

Ans. (2)

Sol. Area = $\left|\overrightarrow{AC} \times \overrightarrow{BD}\right|$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -1 & 7 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= \frac{1}{2}\left|-17\hat{i} - 8\hat{j} + 11\hat{k}\right| = \frac{1}{2}\sqrt{474}$$

19. If $2\sin^3 x + \sin 2x \cos x + 4\sin x - 4 = 0$ has exactly 3 solutions in the interval $\left[0, \frac{n\pi}{2}\right]$, $n \in N$, then the roots of the equation $x^2 + nx + (n-3) = 0$ belong to :

(1) $(0, \infty)$

(2) $(-\infty, 0)$

(3) $\left(-\frac{\sqrt{17}}{2}, \frac{\sqrt{17}}{2}\right)$

(4) Z

Ans. (2)

Sol. $2\sin^3 x + 2\sin x.\cos^2 x + 4\sin x - 4 = 0$

$2\sin^3 x + 2\sin x.(1 - \sin^2 x) + 4\sin x - 4 = 0$

$6\sin x - 4 = 0$

$\sin x = \frac{2}{3}$

n = 5 (in the given interval)

$x^2 + 5x + 2 = 0$

$x = \frac{-5 \pm \sqrt{17}}{2}$

Required interval $(-\infty, 0)$

20. Let $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to R$ be a differentiable function such that $f(0) = \frac{1}{2}$, If the $\lim_{x \to 0} \frac{x\int_0^x f(t)dt}{e^{x^2} - 1} = \alpha$, then $8\alpha^2$ is equal to :

(1) 16

(2) 2

(3) 1

(4) 4

Ans. (2)

Sol. $\lim\limits_{x\to 0}\dfrac{x\int_0^x f(t)dt}{\left(\dfrac{e^{x^2}-1}{x^2}\right)\times x^2}$

$\lim\limits_{x\to 0}\dfrac{\int_0^x f(t)dt}{x}$ $\qquad \left(\lim\limits_{x\to 0}\dfrac{e^{x^2}-1}{x^2}=1\right)$

$=\lim\limits_{x\to 0}\dfrac{f(x)}{1}$ (using L Hospital)

$f(0)=\dfrac{1}{2}$

$\alpha=\dfrac{1}{2}$

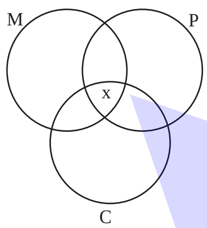
$8\alpha^2=2$

SECTION-B

21. A group of 40 students appeared in an examination of 3 subjects – Mathematics, Physics & Chemistry. It was found that all students passed in at least one of the subjects, 20 students passed in Mathematics, 25 students passed in Physics, 16 students passed in Chemistry, at most 11 students passed in both Mathematics and Physics, at most 15 students passed in both Physics and Chemistry, at most 15 students passed in both Mathematics and Chemistry. The maximum number of students passed in all the three subjects is ________.

Ans. **(10)**

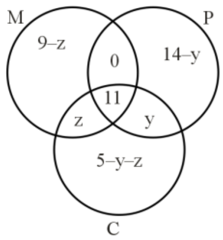
Sol.



$11-x\geq 0$ (Maths and Physics)

$x\leq 11$

x =11 does not satisfy the data.



$11+z\leq 15\Rightarrow z\leq 4$

$11+y\leq 15\Rightarrow y\leq 4$

Now

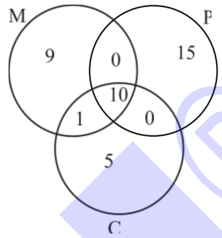
$9-z+0+14-y+z+11+y+5-y-z=40$

$\Rightarrow y+z=-1$

Not possible

$\Rightarrow x\leq 10$

For x = 10



Hence maximum number of students passed in all the three subjects is 10.

22. If d_1 is the shortest distance between the lines $x + 1 = 2y = -12z$, $x = y + 2 = 6z - 6$ and d_2 is the shortest distance between the lines $\dfrac{x-1}{2}=\dfrac{y+8}{-7}=\dfrac{z-4}{5}$, $\dfrac{x-1}{2}=\dfrac{y-2}{1}=\dfrac{z-6}{-3}$, then the value of $\dfrac{32\sqrt{3}d_1}{d_2}$ is :

Ans. **(16)**

Sol. $L_1:\dfrac{x+1}{1}=\dfrac{y}{1/2}=\dfrac{z}{-1/12}$, $L_2:\dfrac{x}{1}=\dfrac{y+2}{1}=\dfrac{z-1}{\frac{1}{6}}$

d_1 = shortest distance between L_1 & L_2

$=\left|\dfrac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|(\vec{b}_1\times\vec{b}_2)|}\right|$

$d_1=2$

$L_3:\dfrac{x-1}{2}=\dfrac{y+8}{-7}=\dfrac{z-4}{5}$, $L_4:\dfrac{x-1}{2}=\dfrac{y-2}{1}=\dfrac{z-6}{-3}$

d_2 = shortest distance between L_3 & L_4

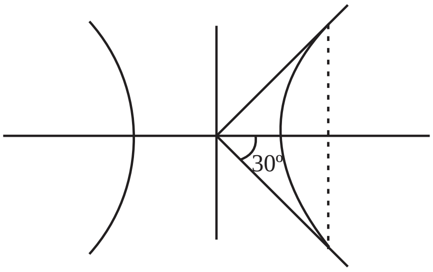
$d_2=\dfrac{12}{\sqrt{3}}$ Hence

$=\dfrac{32\sqrt{3}d_1}{d_2}=\dfrac{32\sqrt{3}\times 2}{\frac{12}{\sqrt{3}}}=16$

23. Let the latus rectum of the hyperbola $\frac{x^2}{9}-\frac{y^2}{b^2}=1$ subtend an angle of $\frac{\pi}{3}$ at the centre of the hyperbola. If b^2 is equal to $\frac{l}{m}\left(1+\sqrt{n}\right)$, where l and m are co-prime numbers, then $l^2 + m^2 + n^2$ is equal to ________

Ans. (182)

Sol. LR subtends 60º at centre



$$\Rightarrow \tan 30° = \frac{b^2/a}{ae} = \frac{b^2}{a^2e} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow e = \frac{\sqrt{3}b^2}{9}$$

Also, $e^2 = 1+\frac{b^2}{9} \Rightarrow 1+\frac{b^2}{9} = \frac{3b^4}{81}$

$\Rightarrow b^4 = 3b^2 + 27$

$\Rightarrow$ $b^4 - 3b^2 - 27 = 0$

$\Rightarrow b^2 = \frac{3}{2}(1+\sqrt{13})$

$\Rightarrow \ell = 3, m = 2, n = 13$

$\Rightarrow \ell^2 + m^2 + n^2 = 182$

24. Let A = {1, 2, 3,….7} and let P(1) denote the power set of A. If the number of functions $f: A \to P(A)$ such that $a \in f(a), \forall a \in A$ is m^n, m and $n \in N$ and m is least, then m + n is equal to ________.

Ans. (44)

Sol. $f: A \to P(A)$

$a \in f(a)$

That means 'a' will connect with subset which contain element 'a'.

Total options for 1 will be 2^6. (Because 2^6 subsets contains 1)

Similarly, for every other element

Hence, total is $2^6 \times 2^6 \times 2^6 \times 2^6 \times 2^6 \times 2^6 \times 2^6 = 2^{42}$

Ans. 2+42 = 44

25. The value $9\int_0^9\left[\sqrt{\frac{10x}{x+1}}\right]dx$, where [t] denotes the greatest integer less than or equal to t, is _____.

Ans. (155)

Sol. $\frac{10x}{x+1} = 1 \qquad \Rightarrow x = \frac{1}{9}$

$\frac{10x}{x+1} = 4 \qquad \Rightarrow x = \frac{2}{3}$

$\frac{10x}{x+1} = 9 \qquad \Rightarrow x = 9$

$$I = 9\left(\int_0^{1/9} 0dx + \int_{1/9}^{2/3} 1.dx + \int_{2/3}^{9} 2dx\right)$$

$= 155$

26. Number of integral terms in the expansion of $\left\{7^{\left(\frac{1}{2}\right)} + 11^{\left(\frac{1}{6}\right)}\right\}^{824}$ is equal to ________.

Ans. (138)

Sol. General term in expansion of $\left((7)^{1/2} + (11)^{1/6}\right)^{824}$ is

$t_{r+1} = {}^{824}C_r\ (7)^{\frac{824-r}{2}}\ (11)^{r/6}$

For integral term, r must be multiple of 6.

Hence r = 0, 6, 12, ……….822

27. Let $y = y(x)$ be the solution of the differential equation $(1 - x^2)\,dy = \left[xy + (x^3+2)\sqrt{3(1-x^2)}\right]dx$, $-1 < x < 1, y(0) = 0$. If $y\left(\frac{1}{2}\right) = \frac{m}{n}$, m and n are co-prime numbers, then m + n is equal to ________.

Ans. (97)

Sol. $\frac{dy}{dx} - \frac{xy}{1-x^2} = \frac{(x^3+2)\sqrt{3(1-x^2)}}{1-x^2}$

$\text{IF} = e^{-\int \frac{x}{1-x^2}dx} = e^{+\frac{1}{2}\ln(1-x^2)} = \sqrt{1-x^2}$

$y\sqrt{1-x^2} = \sqrt{3}\int (x^3+2)\,dx$

$y\sqrt{1-x^2} = \sqrt{3}\left(\frac{x^4}{4} + 2x\right) + c$

$\Rightarrow y(0) = 0 \qquad \therefore c = 0$

$y\left(\frac{1}{2}\right) = \frac{65}{32} = \frac{m}{n}$

$m + n = 97$

28. Let $\alpha, \beta \in N$ be roots of equation $x^2 - 70x + \lambda = 0$, where $\frac{\lambda}{2}, \frac{\lambda}{3} \notin N$. If λ assumes the minimum possible value, then $\frac{\left(\sqrt{\alpha-1} + \sqrt{\beta-1}\right)(\lambda+35)}{|\alpha-\beta|}$ is equal to :

Ans. (60)

Sol. $x^2 - 70x + \lambda = 0$

$\alpha + \beta = 70$

$\alpha\beta = \lambda$

$\therefore \alpha(70 - \alpha) = \lambda$

Since, 2 and 3 does not divide λ

$\therefore \alpha = 5, \beta = 65, \lambda = 325$

By putting value of α, β, λ we get the required value 60.

29. If the function $f(x) = \begin{cases} \frac{1}{|x|} & , |x| \geq 2 \\ ax^2 + 2b, & |x| < 2 \end{cases}$ is differentiable on R, then 48 (a + b) is equal to ____.

Ans. (15)

Sol. $f(x)\begin{cases} \frac{1}{x};\ x \geq 2 \\ ax^2 + 2b;\ -2 < x < 2 \\ -\frac{1}{x};\ x \leq -2 \end{cases}$

Continuous at x = 2 $\quad \Rightarrow \frac{1}{2} = \frac{a}{4} + 2b$

Continuous at x = -2 $\quad \Rightarrow \frac{1}{2} = \frac{a}{4} + 2b$

Since, it is differentiable at x = 2

$-\frac{1}{x^2} = 2ax$

Differentiable at x = 2 $\quad \Rightarrow \frac{-1}{4} = 4a \Rightarrow a = \frac{-1}{16}$, b $= \frac{3}{8}$

30. Let $\alpha = 1^2 + 4^2 + 8^2 + 13^2 + 19^2 + 26^2 + \ldots\ldots$ upto 10 terms and $\beta = \sum_{n=1}^{10} n^4$. If $4\alpha - \beta = 55k + 40$, then k is equal to __________.

Ans. (353)

Sol. $\alpha = 1^2 + 4^2 + 8^2 \ldots.$

$t_n = an^2 + bn + c$

$1 = a + b + c$

$4 = 4a + 2b + c$

$8 = 9a + 3b + c$

On solving we get, $a = \frac{1}{2}$, $b = \frac{3}{2}$, $c = -1$

$$\alpha = \sum_{n=1}^{10}\left(\frac{n^2}{2} + \frac{3n}{2} - 1\right)^2$$

$$4\alpha = \sum_{n=1}^{10}\left(n^2 + 3n - 2\right)^2, \quad \beta = \sum_{n=1}^{10} n^4$$

$$4\alpha - \beta = \sum_{n=1}^{10}(6n^3 + 5n^2 - 12n + 4) = 55(353) + 40$$