FINAL JEE–MAIN EXAMINATION – JANUARY, 2024

(Held On Thursday 01st February, 2024) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. A bag contains 8 balls, whose colours are either white or black. 4 balls are drawn at random without replacement and it was found that 2 balls are white and other 2 balls are black. The probability that the bag contains equal number of white and black balls is:

(1) $\frac{2}{5}$ (2) $\frac{2}{7}$

(3) $\frac{1}{7}$ (4) $\frac{1}{5}$

Ans. (2)

Sol.

P(4W4B/2W2B) =

$$\frac{P(4W4B)\times P(2W2B/4W4B)}{P(2W6B)\times P(2W2B/2W6B)+P(3W5B)\times P(2W2B/3W5B)+\ldots\ldots\ldots\ldots+P(6W2B)\times P(2W2B/6W2B)}$$

$$=\frac{\frac{1}{5}\times\frac{{}^4C_2\times{}^4C_2}{{}^8C_4}}{\frac{1}{5}\times\frac{{}^2C_2\times{}^6C_2}{{}^8C_4}+\frac{1}{5}\times\frac{{}^3C_2\times{}^5C_2}{{}^8C_4}+\ldots+\frac{1}{5}\times\frac{{}^6C_2\times{}^2C_2}{{}^8C_4}}$$

$$=\frac{2}{7}$$

2. The value of the integral

$$\int_0^{\frac{\pi}{4}}\frac{xdx}{\sin^4(2x)+\cos^4(2x)}\ equals:$$

(1) $\frac{\sqrt{2}\pi^2}{8}$ (2) $\frac{\sqrt{2}\pi^2}{16}$

(3) $\frac{\sqrt{2}\pi^2}{32}$ (4) $\frac{\sqrt{2}\pi^2}{64}$

Ans. (3)

Sol. $\displaystyle\int_0^{\frac{\pi}{4}}\frac{xdx}{\sin^4(2x)+\cos^4(2x)}$

Let $2x=t$ then $dx=\frac{1}{2}dt$

$$I=\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{tdt}{\sin^4 t+\cos^4 t}$$

$$I=\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-t\right)dt}{\sin^4\left(\frac{\pi}{2}-t\right)+\cos^4\left(\frac{\pi}{2}-t\right)}$$

$$I=\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{\frac{\pi}{2}dt}{\sin^4 t+\cos^4 t}-I$$

$$2I=\frac{\pi}{8}\int_0^{\frac{\pi}{2}}\frac{dt}{\sin^4 t+\cos^4 t}$$

$$2I=\frac{\pi}{8}\int_0^{\frac{\pi}{2}}\frac{\sec^4 tdt}{\tan^4 t+1}$$

Let tant = y then $\sec^2$t dt = dy

$$2I=\frac{\pi}{8}\int_0^{\infty}\frac{(1+y^2)dy}{1+y^4}$$

$$=\frac{\pi}{16}\int_0^{\infty}\frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy$$

Put $y-\frac{1}{y}=p$

$$I=\frac{\pi}{16}\int_{-\infty}^{\infty}\frac{dp}{p^2+\left(\sqrt{2}\right)^2}$$

$$=\frac{\pi}{16\sqrt{2}}\left[\tan^{-1}\left(\frac{p}{\sqrt{2}}\right)\right]_{-\infty}^{\infty}$$

$$I=\frac{\pi^2}{16\sqrt{2}}$$

3. If A = $\begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}$, B = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, C = ABA^T and X = A^TC^2A, then det X is equal to :

(1) 243

(2) 729

(3) 27

(4) 891

Ans. (2)

Sol.

$$A = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix} \Rightarrow \det(A) = 3$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \det(B) = 1$$

Now C = ABA^T $\Rightarrow$det(C) = (dct (A))^2 x det(B)

$$|C| = 9$$

Now |X| = $|A^TC^2A|$

= $|A^T|\,|C|^2\,|A|$

= $|A|^2\,|C|^2$

= 9 x 81

= 729

4. If tanA = $\dfrac{1}{\sqrt{x(x^2+x+1)}}$, $\tan B = \dfrac{\sqrt{x}}{\sqrt{x^2+x+1}}$

and

$$\tan C = \left(x^{-3} + x^{-2} + x^{-1}\right)^{\frac{1}{2}}, 0 < A, B, C < \frac{\pi}{2}, \textit{then}$$

A + B is equal to :

(1) C

(2) $\pi - C$

(3) $2\pi - C$

(4) $\dfrac{\pi}{2} - C$

Ans. (1)

Sol.

Finding tan (A + B) we get

$\Rightarrow$ tan (A + B) =

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\dfrac{1}{\sqrt{x(x^2+x+1)}} + \dfrac{\sqrt{x}}{\sqrt{x^2+x+1}}}{1 - \dfrac{1}{x^2+x+1}}$$

$\Rightarrow$ tan (A + B) = $\dfrac{(1+x)\left(\sqrt{x^2+x+1}\right)}{(x^2+x)\left(\sqrt{x}\right)}$

$$\frac{(1+x)\left(\sqrt{x^2+x+1}\right)}{(x^2+x)\left(\sqrt{x}\right)}$$

$$\tan(A+B) = \frac{\sqrt{x^2+x+1}}{x\sqrt{x}} = \tan C$$

$$A + B = C$$

5. If n is the number of ways five different employees can sit into four indistinguishable offices where any office may have any number of persons including zero, then n is equal to:

(1) 47

(2) 53

(3) 51

(4) 43

Ans. (3)

Sol.

Total ways to partition 5 into 4 parts are :

5, 0, 0, 0 $\Rightarrow$1 way

4, 1, 0, 0 $\Rightarrow \dfrac{5!}{4!} = 5$ ways

3, 2, 0, 0, $\Rightarrow \dfrac{5!}{3!2!} = 10$ ways

$2,2,0,1 \Rightarrow \dfrac{5!}{2!2!2!} = 15$ ways

$2,1,1,1 \Rightarrow \dfrac{5!}{2!(1!)^3 3!} = 10$ ways

$3,1,1,0 \Rightarrow \dfrac{5!}{3!2!} = 10$ ways

Total $\Rightarrow$ 1+5+10+15+10+10 = 51 ways

6. Let $S=\{z \in C : |z-1|=1$ and $(\sqrt{2}-1)(z+\bar{z})-i(z-\bar{z})=2\sqrt{2}\}$. Let z_1, z_2 $\in S$ be such that $|z_1|=\max\limits_{z\in s}|z|$ and $|z_2|=\min\limits_{z\in s}|z|$.

Then $\left|\sqrt{2}z_1-z_2\right|^2$ equals :

(1) 1 (2) 4

(3) 3 (4) 2

Ans. (4)

Sol. Let Z = x + iy

Then $(x-1)^2+y^2=1 \quad \to (1)$

& $(\sqrt{2}-1)(2x)-i(2iy)=2\sqrt{2}$

$\Rightarrow (\sqrt{2}-1)x+y=\sqrt{2} \to (2)$

Solving (1) & (2) we get

Either x = 1 or $x=\dfrac{1}{2-\sqrt{2}} \to (3)$

On solving (3) with (2) we get

For x = 1 $\Rightarrow$ y = 1 $\Rightarrow Z_2 = 1 + i$

& for

$x=\dfrac{1}{2-\sqrt{2}} \Rightarrow y=\sqrt{2}-\dfrac{1}{\sqrt{2}} \Rightarrow Z_1=\left(1+\dfrac{1}{\sqrt{2}}\right)+\dfrac{i}{\sqrt{2}}$

Now

$\left|\sqrt{2}z_1-z_2\right|^2$

$=\left|\left(\dfrac{1}{\sqrt{2}}+1\right)\sqrt{2}+i-(1+i)\right|^2$

$=\left(\sqrt{2}\right)^2$

$=2$

7. Let the median and the mean deviation about the median of 7 observation 170, 125, 230, 190, 210, a, b be 170 and $\dfrac{205}{7}$ respectively. Then the mean deviation about the mean of these 7 observations is :

(1) 31

(2) 28

(3) 30

(4) 32

Ans. (3)

Sol. Median = 170 $\Rightarrow$ 125, a, b, 170, 190, 210, 230

Mean deviation about

Median =

$$\frac{0+45+60+20+40+170-a+170-b}{7}=\frac{205}{7}$$

$\Rightarrow$ a + b = 300

Mean = $\dfrac{170+125+230+190+210+a+b}{7}=175$

Mean deviation

About mean =

$$\frac{50+175-a+175-b+5+15+35+55}{7}=30$$

8. Let $\vec{a}=-5\hat{i}+\hat{j}-3\hat{k}, \vec{b}=\hat{i}+2\hat{j}-4\hat{k}$ and $\vec{c}=\left(\left(\left(\vec{a}\times\vec{b}\right)\times\hat{i}\right)\times\hat{i}\right)\times\hat{i}$. Then $\vec{c}\cdot\left(-\hat{i}+\hat{j}+\hat{k}\right)$ is equal to

(1) –12 (2) –10

(3) –13 (4) –15

Ans. (1)

Sol. $\vec{a}=-5\hat{i}+j-3\hat{k}$

$\vec{b}=\hat{i}+2\hat{j}-4\hat{k}$

$(\vec{a}\times\vec{b})\times\hat{i}=\left(\vec{a}\cdot\hat{i}\right)\vec{b}-\left(\vec{b}\cdot\hat{i}\right)\vec{a}$

$=-5\vec{b}-\vec{a}$

$=\left(\left(\left(-5\vec{b}-\vec{a}\right)\times\hat{i}\right)\times\hat{i}\right)$

$=\left(\left(-11\hat{j}+23\hat{k}\right)\times\hat{i}\right)\times\hat{i}$

$\Rightarrow\left(11\hat{k}+23\hat{j}\right)\times\hat{i}$

$\Rightarrow\left(11\hat{j}-23\hat{k}\right)$

$\vec{c}.\left(-\hat{i}+\hat{j}+\hat{k}\right)=11-23=-12$

9. Let S = $\{x \in R : (\sqrt{3}+\sqrt{2})^x + (\sqrt{3}-\sqrt{2})^x = 10\}$.

Then the number of elements in S is :

(1) 4 (2) 0

(3) 2 (4) 1

Ans. (3)

Sol. $\left(\sqrt{3}+\sqrt{2}\right)^x + \left(\sqrt{3}-\sqrt{2}\right)^x = 10$

Let $\left(\sqrt{3}+\sqrt{2}\right)^x = t$

$t + \frac{1}{t} = 10$

$t^2 - 10t + 1 = 0$

$t = \frac{10 \pm \sqrt{100-4}}{2} = 5 \pm 2\sqrt{6}$

$\left(\sqrt{3}+\sqrt{2}\right)^x = \left(\sqrt{3}\pm\sqrt{2}\right)^2$

x = 2 or x = –2

Number of solutions = 2

10. The area enclosed by the curves xy + 4y = 16 and x + y = 6 is equal to :

(1) $28 - 30 \log_e 2$ (2) $30 - 28 \log_e 2$

(3) $30 - 32 \log_e 2$ (4) $32 - 30 \log_e 2$

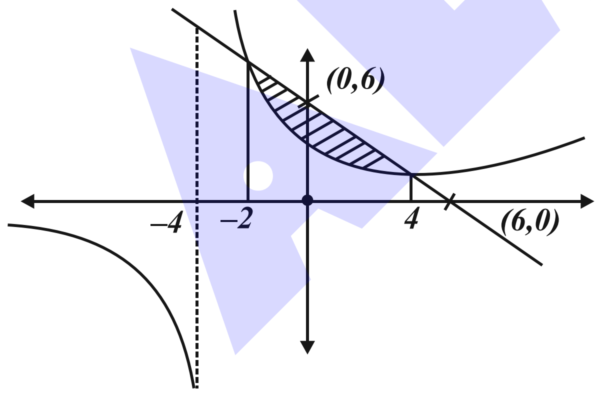
Ans. (3)

Sol. xy + 4y = 16 , x + y = 6

y(x + 4) = 16 ____(1) , x + y = 6___(2)

on solving, (1) & (2)

we get x = 4, x = –2



Area = $\int_{-2}^{4} \left((6-x) - \left(\frac{16}{x+4} \right) \right) dx$

$= 30 - 32 \ln 2$

11. Let f : **R** → **R** and g : **R** → **R** be defined as

$f(x) = \begin{cases} \log_e x & , x > 0 \\ e^{-x} & , x \le 0 \end{cases}$ and

$g(x) = \begin{cases} x & , x \ge 0 \\ e^x & , x < 0 \end{cases}$. Then, gof : **R** → **R** is :

(1) one-one but not onto

(2) neither one-one nor onto

(3) onto but not one-one

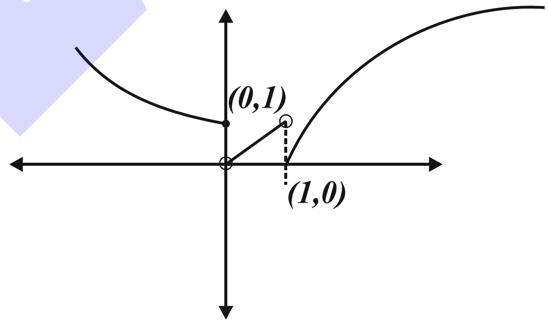
(4) both one-one and onto

Ans. (2)

Sol.

$g(f(x)) = \begin{cases} f(x), f(x) \ge 0 \\ e^{f(x)}, f(x) < 0 \end{cases}$

$g(f(x)) = \begin{cases} e^{-x}, (-\infty, 0] \\ e^{\ln x}, (0,1) \\ \ln x, [1, \infty) \end{cases}$



Graph of g(f(x))

g(f(x)) ⇒ Many one into

12. If the system of equations

2x + 3y – z = 5

x + αy + 3z = –4

3x – y + βz = 7

has infinitely many solutions, then 13 αβ is equal to

(1) 1110 (2) 1120

(3) 1210 (4) 1220

Ans. (2)

Sol. Using family of planes

$2x + 3y - z - 5 = k_1 (x + \alpha y + 3z + 4) + k_2 (3x - y + \beta z - 7)$

$2 = k_1 + 3k_2,\ 3 = k_1\alpha - k_2,\ -1 = 3k_1 + \beta k_2,\ -5 = 4k_1 - 7k_2$

On solving we get

$$k_2 = \frac{13}{19}, k_1 = \frac{-1}{19}, \alpha = -70, \beta = \frac{-16}{13}$$

$$13\alpha\beta = 13(-70)\left(\frac{-16}{13}\right)$$

$$= 1120$$

13. For $0 < \theta < \pi/2$, if the eccentricity of the hyperbola $x^2 - y^2\operatorname{cosec}^2\theta = 5$ is $\sqrt{7}$ times eccentricity of the ellipse $x^2 \operatorname{cosec}^2\theta + y^2 = 5$, then the value of θ is :

(1) $\frac{\pi}{6}$ (2) $\frac{5\pi}{12}$

(3) $\frac{\pi}{3}$ (4) $\frac{\pi}{4}$

Ans. (3)

Sol.

$$e_h = \sqrt{1+\sin^2\theta}$$

$$e_c = \sqrt{1-\sin^2\theta}$$

$$e_h = \sqrt{7}e_c$$

$$1+\sin^2\theta = 7(1-\sin^2\theta)$$

$$\sin^2\theta = \frac{6}{8} = \frac{3}{4}$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{3}$$

14. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} = 2x(x+y)^3 - x(x+y) - 1$, $y(0) = 1$.

Then, $\left(\frac{1}{\sqrt{2}} + y\left(\frac{1}{\sqrt{2}}\right)\right)^2$ equals :

(1) $\frac{4}{4+\sqrt{e}}$ (2) $\frac{3}{3-\sqrt{e}}$

(3) $\frac{2}{1+\sqrt{e}}$ (4) $\frac{1}{2-\sqrt{e}}$

Ans. (4)

Sol. $$\frac{dy}{dx} = 2x(x+y)^3 - x(x+y) - 1$$

$$x+y = t$$

$$\frac{dt}{dx} - 1 = 2xt^3 - xt - 1$$

$$\frac{dt}{2t^3 - t} = xdx$$

$$\frac{tdt}{2t^4 - t^2} = xdx$$

Let $t^2 = z$

$$\int \frac{dz}{2(2z^2 - z)} = \int xdx$$

$$\int \frac{dz}{4z\left(z - \frac{1}{2}\right)} = \int xdx$$

$$\ln\left|\frac{z - \frac{1}{2}}{z}\right| = x^2 + k$$

$$z = \frac{1}{2-\sqrt{e}}$$

15. Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} \frac{a - b\cos 2x}{x^2} & ; \quad x < 0 \\ x^2 + cx + 2 & ; \quad 0 \le x \le 1 \\ 2x + 1 & ; \quad x > 1 \end{cases}$$

If f is continuous everywhere in **R** and m is the number of points where f is **NOT** differential then m + a + b + c equals :

(1) 1 (2) 4

(3) 3 (4) 2

Ans. (4)

Sol. At x = 1, f(x) is continuous therefore,

$f(1^-) = f(1) = f(1^+)$

$f(1) = 3 + c$ ....(1)

$f(1^+) = \lim_{h\to 0} 2(1 + h) + 1$

$f(1^+) = \lim_{h\to 0} 3 + 2h = 3$ ....(2)

from (1) & (2)

c = 0

at x = 0, f(x) is continuous therefore,

$f(0^-) = f(0) = f(0^+)$ ....(3)

$f(0) = f(0^+) = 2$ ....(4)

$f(0^-)$ has to be equal to 2

$$\lim_{h\to 0}\frac{a-b\cos(2h)}{h^2}$$

$$\lim_{h\to 0}\frac{a-b\left\{1-\frac{4h^2}{2!}+\frac{16h^4}{4!}+\ldots\right\}}{h^2}$$

$$\lim_{h\to 0}\frac{a-b+b\left\{2h^2-\frac{2}{3}h^4\ldots\right\}}{h^2}$$

for limit to exist a – b = 0 and limit is 2b ....(5)

from (3), (4) & (5)

a = b = 1

checking differentiability at x = 0

$$\text{LHD}: \lim_{h\to 0}\frac{\frac{1-\cos 2h}{h^2}-2}{-h}$$

$$\lim_{h\to 0}\frac{1-\left(1-\frac{4h^2}{2!}+\frac{16h^4}{4!}\ldots\right)-2h^2}{-h^3}=0$$

$$\text{RHD}: \lim_{h\to 0}\frac{(0+h)^2+2-2}{h}=0$$

Function is differentiable at every point in its domain

$\therefore$ m = 0

m + a + b + c = 0 + 1 + 1 + 0 = 2

16. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a > b be an ellipse, whose eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is $\sqrt{14}$. Then the square of the eccentricity of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is :

(1) 3 (2) 7/2

(3) 3/2 (4) 5/2

Ans. (3)

Sol.

$$e=\frac{1}{\sqrt{2}}=\sqrt{1-\frac{b^2}{a^2}} \Rightarrow \frac{1}{2}=1-\frac{b^2}{a^2}$$

$$\frac{2b^2}{a}=14$$

$$e_H=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{1}{2}}=\sqrt{\frac{3}{2}}$$

$$(e_H)^2=\frac{3}{2}$$

17. Let 3, a, b, c be in A.P. and 3, a – 1, b + 1, c + 9 be in G.P. Then, the arithmetic mean of a, b and c is :

(1) –4 (2) –1

(3) 13 (4) 11

Ans. (4)

Sol.

3, a, b, c $\to$ A.P $\Rightarrow$ 3, 3+d, 3+2d, 3+3d

3, a–1, b+1, c + 9 $\to$ G.P $\Rightarrow$ 3, 2+d, 4+2d, 12+3d

a = 3 + d $\quad (2+d)^2=3(4+2d)$

b = 3 + 2d $\quad$ d = 4, -2

c = 3 + 3d

If d = 4 G.P $\Rightarrow$ 3, 6, 12, 24

a = 7

b = 11

c = 15

$$\frac{a+b+c}{3}=11$$

18. Let C : $x^2 + y^2 = 4$ and C' : $x^2 + y^2 - 4\lambda x + 9 = 0$ be two circles. If the set of all values of λ so that the circles C and C' intersect at two distinct points, is **R**– [a, b], then the point (8a + 12, 16b – 20) lies on the curve :

(1) $x^2 + 2y^2 - 5x + 6y = 3$

(2) $5x^2 - y = -11$

(3) $x^2 - 4y^2 = 7$

(4) $6x^2 + y^2 = 42$

Ans. (4)

Sol. $x^2 + y^2 = 4$

C (0, 0) $\quad r_1 = 2$

C' $(2\lambda, 0)$ $\quad r_2 = \sqrt{4\lambda^2 - 9}$

$|r_1 - r_2| < CC' < |r_1 + r_2|$

$$\left|2 - \sqrt{4\lambda^2 - 9}\right| < |2\lambda| < 2 + \sqrt{4\lambda^2 - 9}$$

$$4 + 4\lambda^2 - 9 - 4\sqrt{4\lambda^2 - 9} < 4\lambda^2$$

True $\lambda \in$ R…. (1)

$$4\lambda^2 < 4 + 4\lambda^2 - 9 + 4\sqrt{4\lambda^2 - 9}$$

$$5 < 4\sqrt{4\lambda^2 - 9} \text{ and } \quad \lambda^2 \ge \frac{9}{4}$$

$$\frac{25}{16} < 4\lambda^2 - 9 \qquad \lambda \in \left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{3}{2}, \infty\right)$$

$$\frac{169}{64} < \lambda^2$$

$$\lambda \in \left(-\infty, -\frac{13}{8}\right) \cup \left(\frac{13}{8}, \infty\right) \qquad \ldots(2)$$

from (1) and (2) $\lambda \in$

$$\lambda \in \left(-\infty, -\frac{13}{8}\right) \cup \left(\frac{13}{8}, \infty\right) \Rightarrow R - \left[-\frac{13}{8}, \frac{13}{8}\right]$$

as per question $a = -\dfrac{13}{8}$ and $b = \dfrac{13}{8}$

$\therefore$ required point is (–1, 6) with satisfies option (4)

19. If $5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 2, \forall\, x \neq 0$ and $y = 9x^2 f(x)$, then y is strictly increasing in :

(1) $\left(0, \frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$

(2) $\left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$

(3) $\left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(0, \frac{1}{\sqrt{5}}\right)$

(4) $\left(-\infty, \frac{1}{\sqrt{5}}\right) \cup \left(0, \frac{1}{\sqrt{5}}\right)$

Ans. (2)

Sol. $5\,f(x) + 4\,f\left(\frac{1}{x}\right) = x^2 - 2, \ \forall x \neq 0 \ldots(1)$

Substitute $x \to \dfrac{1}{x}$

$$5f\left(\frac{1}{x}\right) + 4f(x) = \frac{1}{x^2} - 2 \qquad \ldots(2)$$

On solving (1) and (2)

$$f(x) = \frac{5x^4 - 2x^2 - 4}{9x^2}$$

$y = 9x^2 f(x)$

$y = 5x^4 - 2x^2 - 4 \quad \ldots(3)$

$$\frac{dy}{dx} = 20x^3 - 4x$$

for strictly increasing

$$\frac{dy}{dx} > 0$$

$4x(5x^2 - 1) > 0$

$$x \in \left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$$

20. If the shortest distance between the lines $\dfrac{x-\lambda}{-2} = \dfrac{y-2}{1} = \dfrac{z-1}{1}$ and $\dfrac{x-\sqrt{3}}{1} = \dfrac{y-1}{-2} = \dfrac{z-2}{1}$ is 1, then the sum of all possible values of λ is :

(1) 0 $\qquad$ (2) $2\sqrt{3}$

(3) $3\sqrt{3}$ $\qquad$ (4) $-2\sqrt{3}$

Ans. (2)

Sol. Passing points of lines L_1 & L_2 are

$$(\lambda,2,1)\&\left(\sqrt{3},1,2\right)$$

$$\text{S.D} = \frac{\begin{vmatrix} \sqrt{3}-\lambda & -1 & 1 \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}}{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}}$$

$$1 = \left|\frac{\sqrt{3}-\lambda}{\sqrt{3}}\right|$$

$$\lambda = 0, \lambda = 2\sqrt{3}$$

SECTION-B

21. If x = x(t) is the solution of the differential equation $(t + 1)dx = (2x + (t + 1)^4)\, dt$, $x(0) = 2$, then, x(1) equals __________.

Ans. (14)

Sol. $(t+1)dx = (2x+(t+1)^4)dt$

$$\frac{dx}{dt} = \frac{2x+(t+1)^4}{t+1}$$

$$\frac{dx}{dt} - \frac{2x}{t+1} = (t+1)^3$$

$$I\cdot F = e^{-\int\frac{2}{t+1}dt} = e^{-2\ln(t+1)} = \frac{1}{(t+1)^2}$$

$$\frac{x}{(t+1)^2} = \int\frac{1}{(t+1)^2}(t+1)^3dt + c$$

$$\frac{x}{(t+1)^2} = \frac{(t+1)^2}{2} + c$$

$$\Rightarrow c = \frac{3}{2}$$

$$x = \frac{(t+1)^4}{2} + \frac{3}{2}(t+1)^2$$

put, $t = 1$

$x = 2^3 + 6 = 14$

22. The number of elements in the set

$S = \{(x, y, z) : x, y, z \in \mathbf{Z},\ x + 2y + 3z = 42,\ x, y, z \geq 0\}$ equals __________.

Ans. (169)

Sol. $x + 2y + 3z = 42$, $\quad x, y, z \geq 0$

$z = 0 \quad x + 2y = 42 \Rightarrow 22$

$z = 1 \quad x + 2y = 39 \Rightarrow 20$

$z = 2 \quad x + 2y = 36 \Rightarrow 19$

$z = 3 \quad x + 2y = 33 \Rightarrow 17$

$z = 4 \quad x + 2y = 30 \Rightarrow 16$

$z = 5 \quad x + 2y = 27 \Rightarrow 14$

$z = 6 \quad x + 2y = 24 \Rightarrow 13$

$z = 7 \quad x + 2y = 21 \Rightarrow 11$

$z = 8 \quad x + 2y = 18 \Rightarrow 10$

$z = 9 \quad x + 2y = 15 \Rightarrow 8$

$z = 10 \quad x + 2y = 12 \Rightarrow 7$

$z = 11 \quad x + 2y = 9 \Rightarrow 5$

$z = 12 \quad x + 2y = 6 \Rightarrow 4$

$z = 13 \quad x + 2y = 3 \Rightarrow 2$

$z = 14 \quad x + 2y = 0 \Rightarrow 1$

Total : 169

23. If the Coefficient of x^{30} in the expansion of $\left(1+\frac{1}{x}\right)^6 (1 + x^2)^7 (1 - x^3)^8$; $x \neq 0$ is α, then |α| equals __________.

Ans. (678)

Sol. coeff of x^{30} in $\dfrac{(x+1)^6(1+x^2)^7(1-x^3)^8}{x^6}$

coeff. of x^{36} in $(1+x)^6(1+x^2)^7(1-x^3)^8$

General term

$^6C_{r_1}\,{}^7C_{r_2}\,{}^8C_{r_3}(-1)^{r_3}x^{r_1+2r_2+3r_3}$

$r_1+2r_2+3r_3=36$

Case-I : $r_1+2r_2=12$ (Taking r_3 = 8)

r_1	r_2	r_3
0	6	8
2	5	8
4	4	8
6	3	8

Case-II : $r_1+2r_2=15$ (Taking r_3 = 7)

r_1	r_2	r_3
1	7	7
3	6	7
5	5	7

Case-III : $r_1+2r_2=18$ (Taking r_3 = 6)

r_1	r_2	r_3
4	7	6
6	6	6

Coeff. = 7 + (15 × 21) + (15 × 35) + (35)

–(6 × 8) – (20 × 7 × 8) – (6 × 21 × 8) + (15 × 28)

+ (7 × 28) = –678 = α

|α| = 678

24. Let 3, 7, 11, 15, ..., 403 and 2, 5, 8, 11, . . ., 404 be two arithmetic progressions. Then the sum, of the common terms in them, is equal to _________.

Ans. (6699)

Sol. 3, 7, 11, 15, ….., 403

2, 5, 8, 11, …., 404

LCM (4, 3) = 12

11, 23, 35,….. let (403)

403 = 11 + (n - 1) x 12

$\dfrac{392}{12}=n-1$

$33\cdot 66=n$

$n=33$

Sum $\dfrac{33}{2}(22+32\times 12)$

=6699

25. Let {x} denote the fractional part of x and $f(x)=\dfrac{\cos^{-1}(1-\{x\}^2)\sin^{-1}(1-\{x\})}{\{x\}-\{x\}^3}$, $x \neq 0$. If L and R respectively denotes the left hand limit and the right hand limit of f(x) at x = 0, then $\dfrac{32}{\pi^2}(L^2+R^2)$ is equal to _____________.

Ans. (18)

Sol. Finding right hand limit

$$\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)$$

$$=\lim_{h\to 0} f(h)$$

$$=\lim_{h\to 0}\frac{\cos^{-1}(1-h^2)\sin^{-1}(1-h)}{h(1-h^2)}$$

$$=\lim_{h\to 0}\frac{\cos^{-1}(1-h^2)}{h}\left(\frac{\sin^{-1}1}{1}\right)$$

Let $\cos^{-1}(1-h^2)=\theta \Rightarrow \cos\theta=1-h^2$

$$=\frac{\pi}{2}\lim_{\theta\to 0}\frac{\theta}{\sqrt{1-\cos\theta}}$$

$$=\frac{\pi}{2}\lim_{\theta\to 0}\frac{1}{\sqrt{\frac{1-\cos\theta}{\theta^2}}}$$

$$=\frac{\pi}{2}\frac{1}{\sqrt{1/2}}$$

$$R=\frac{\pi}{\sqrt{2}}$$

Now finding left hand limit

$$L = \lim_{x\to 0^-} f(x)$$

$$= \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} \frac{\cos^{-1}\left(1-\{-h\}^2\right)\sin^{-1}\left(1-\{-h\}\right)}{\{-h\}-\{-h\}^3}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}\left(1-(-h+1)^2\right)\sin^{-1}\left(1-(-h+1)\right)}{(-h+1)-(-h+1)^3}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}\left(-h^2+2h\right)\sin^{-1} h}{(1-h)\left(1-(1-h)^2\right)}$$

$$= \lim_{h\to 0}\left(\frac{\pi}{2}\right)\frac{\sin^{-1} h}{\left(1-(1-h)^2\right)}$$

$$= \frac{\pi}{2}\lim_{h\to 0}\left(\frac{\sin^{-1} h}{-h^2+2h}\right)$$

$$= \frac{\pi}{2}\lim_{h\to 0}\left(\frac{\sin^{-1} h}{h}\right)\left(\frac{1}{-h+2}\right)$$

$$L = \frac{\pi}{4}$$

$$\frac{32}{\pi^2}\left(L^2+R^2\right) = \frac{32}{\pi^2}\left(\frac{\pi^2}{2}+\frac{\pi^2}{16}\right)$$

$= 16 + 2$

$= 18$

26. Let the line L : $\sqrt{2}\,x + y = \alpha$ pass through the point of the intersection P (in the first quadrant) of the circle $x^2 + y^2 = 3$ and the parabola $x^2 = 2y$. Let the line L touch two circles C_1 and C_2 of equal radius $2\sqrt{3}$. If the centres Q_1 and Q_2 of the circles C_1 and C_2 lie on the y-axis, then the square of the area of the triangle PQ_1Q_2 is equal to __________.

Ans. (72)

Sol. $x^2 + y^2 = 3$ and $x^2 = 2y$

$$y^2 + 2y - 3 = 0 \Rightarrow (y+3)(y-1) = 0$$

$$y = -3 \text{ or } y = 1$$

$$y = 1\ x = \sqrt{2} \Rightarrow P(\sqrt{2},1)$$

p lies on the line

$$\sqrt{2}x + y = \alpha$$

$$\sqrt{2}(\sqrt{2}) + 1 = \alpha$$

$$\alpha = 3$$

For circle C_1

Q_1 lies on y axis

Let Q_1 $(0,\alpha)$ coordinates

$R_1 = 2\sqrt{3}$ (Given

Line L act as tangent

Apply P = r (condition of tangency)

$$\Rightarrow \left|\frac{\alpha-3}{\sqrt{3}}\right| = 2\sqrt{3}$$

$$\Rightarrow |\alpha-3| = 6$$

$\alpha - 3 = 6$ or $\alpha - 3 = -6$

$\Rightarrow \alpha = 9$ $\quad \alpha = -3$

$$\Delta PQ_1Q_2 = \frac{1}{2}\begin{vmatrix}\sqrt{2} & 1 & 1\\ 0 & 9 & 1\\ 0 & -3 & 1\end{vmatrix}$$

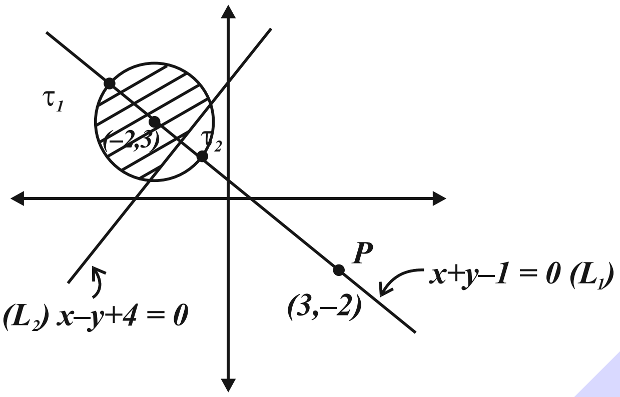
$$= \frac{1}{2}\left(\sqrt{2}(12)\right) = 6\sqrt{2}$$

$$\left(\Delta PQ_1Q_2\right)^2 = 72$$

27. Let $P = \{z \in \mathbb{C} : |z + 2 - 3i| \leq 1\}$ and $Q = \{z \in \mathbb{C} : z(1 + i) + \bar{z}(1 - i) \leq -8\}$. Let in $P \cap Q$, $|z - 3 + 2i|$ be maximum and minimum at z_1 and z_2 respectively. If $|z_1|^2 + 2|z|^2 = \alpha + \beta\sqrt{2}$, where α, β are integers, then $\alpha + \beta$ equals ________.

Ans. (36)

Sol.



Clearly for the shaded region z_1 is the intersection of the circle and the line passing through P (L_1) and z_2 is intersection of line L_1 & L_2

Circle : $(x + 2)^2 + (y - 3)^2 = 1$

$L_1 : x + y - 1 = 0$

$L_2 : x - y + 4 = 0$

On solving circle & L_1 we get

$z_1 : \left(-2 - \frac{1}{\sqrt{2}}, 3 + \frac{1}{\sqrt{2}}\right)$

On solving L_1 and z_2 is intersection of line L_1 & L_2

we get $z_2 : \left(\frac{-3}{2}, \frac{5}{2}\right)$

$$|z_1|^2 + 2|z_2|^2 = 14 + 5\sqrt{2} + 17$$

$$= 31 + 5\sqrt{2}$$

So $\alpha = 31$

$\beta = 5$

$\alpha + \beta = 36$

28. If $\int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2}\cos x\,dx}{(1+e^{\sin x})(1+\sin^4 x)} = \alpha\pi + \beta \log_e(3 + 2\sqrt{2})$, where α, β are integers, then $\alpha^2 + \beta^2$ equals ________.

Ans. (8)

Sol. $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x}{\left(1+e^{\sin x}\right)\left(1+\sin^4 x\right)}dx$

Apply king

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x\left(e^{\sin x}\right)}{\left(1+e^{\sin x}\right)\left(1+\sin^4 x\right)}dx \quad ....(2)$$

adding (1) & (2)

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x}{1+\sin^4 x}dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{8\sqrt{2}\cos x}{1+\sin^4 x}dx,$$

$\sin x = t$

$$I = \int_0^1 \frac{8\sqrt{2}}{1+t^4}dx$$

$$I = 4\sqrt{2}\int_0^1 \left(\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}} - \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\right)dt$$

$$I = 4\sqrt{2}\int_0^1 \frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2} - \frac{\left(1-\frac{1}{t^2}\right)}{\left(t+\frac{1}{t}\right)^2-2}dt$$

$$Let\ t - \frac{1}{t} = z \ \&\ t + \frac{1}{t} = k$$

$$= 4\sqrt{2}\left[\int_{-\infty}^{0}\frac{dz}{z^2+2} - \int_{\infty}^{2}\frac{dk}{k^2-2}\right]$$

$$= 4\sqrt{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}\right]_{-\infty}^{0} - \left[\frac{1}{2\sqrt{2}}\ln\left(\frac{k-\sqrt{2}}{k+\sqrt{2}}\right)\right]_{\infty}^{2}$$

$$= 4\sqrt{2}\left[\frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\left[\ln\frac{2-\sqrt{2}}{2+\sqrt{2}}\right]\right]$$

$$= 2\pi + 2\ln(3+2\sqrt{2})$$

$$\alpha = 2$$

$$\beta = 2$$

29. Let the line of the shortest distance between the lines

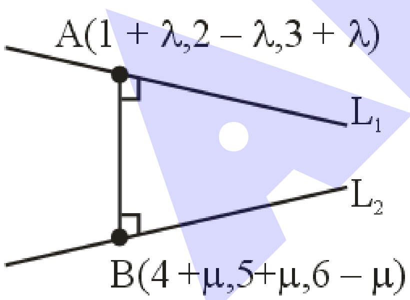
$L_1 : \vec{r} = \left(\hat{i}+2\hat{j}+3\hat{k}\right) + \lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and

$L_2 : \vec{r} = \left(4\hat{i}+5\hat{j}+6\hat{k}\right) + \mu\left(\hat{i}+\hat{j}-\hat{k}\right)$

intersect L_1 and L_2 at P and Q respectively. If (α, β, γ) is the midpoint of the line segment PQ, then $2(\alpha + \beta + \gamma)$ is equal to ____________.

Ans. (21)

Sol.



$\vec{b} = \hat{i} - \hat{j} + \hat{k}$ (DR's of L_1)

$\vec{d} = \hat{i} + \hat{j} - \hat{k}$ (DR's of L_2)

$$\vec{b}\times\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 0\hat{i} + 2\hat{j} + 2\hat{k}$ (DR's of Line perpendicular to L_1 and L_2)

DR of AB line

$= (0,2,2) = (3+\mu-\lambda, 3+\mu+\lambda, 3-\mu-\lambda)$

$$\frac{3+\mu-\lambda}{0} = \frac{3+\mu+\lambda}{2} = \frac{3-\mu-\lambda}{2}$$

Solving above equation we get $\mu = -\frac{3}{2}$ and $\lambda = \frac{3}{2}$

point $A = \left(\frac{5}{2}, \frac{1}{2}, \frac{9}{2}\right)$

$B = \left(\frac{5}{2}, \frac{7}{2}, \frac{15}{2}\right)$

Point of AB $= \left(\frac{5}{2}, 2, 6\right) = (\alpha, \beta, \gamma)$

$2(\alpha + \beta + \gamma) = 5 + 4 + 12 = 21$

30. Let A= {1, 2, 3, . . . 20}. Let R_1 and R_2 two relation on A such that

$R_1 = \{(a, b) : b \text{ is divisible by } a\}$

$R_2 = \{(a, b) : a \text{ is an integral multiple of } b\}$.

Then, number of elements in $R_1 - R_2$ is equal to _________.

Ans. (46)

Sol. $n(R_1) = 20 + 10 + 6 + 5 + 4 + 3 + 2 + 2 + 2$

$$+2 + \underbrace{1 + \ldots + 1}_{10\,\text{times}}$$

$n(R_1) = 66$

$R_1 \cap R_2 = \{(1,1),(2,2),\ldots(20,20)\}$

$n(R_1 \cap R_2) = 20$

$n(R_1 - R_2) = n(R_1) - n(R_1 \cap R_2)$

$= n(R_1) - 20$

$= 66 - 20$

$R_1 - R_2 = 46$ Pair