10.3.2.4.4

EE24BTECH11024 - G. Abhimanyu Koushik

Question:
Check is the pair of linear equations $2x - 2y - 2 = 0$, $4x - 4y - 5 = 0$. And if consistent, obtain the solution
Solution:
Given

$$2x - 2y - 2 = 0 \tag{0.1}$$

$$4x - 4y - 5 = 0 \tag{0.2}$$

Simplifying and using matrix notation,

$$\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{0.3}$$

The matrix A can be decomposed into:

$$A = L \cdot U, \tag{0.4}$$

where:

$$L = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \tag{0.5}$$

$$U = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}. \tag{0.6}$$

Factorization of LU:
Given a matrix $\mathbf{A}$ of size $n \times n$, LU decomposition is performed row by row and column by column. The update equations are as follows:
1. Start by initializing $\mathbf{L}$ as the identity matrix $\mathbf{L} = \mathbf{I}$ and $\mathbf{U}$ as a copy of $\mathbf{A}$.
2. For each column $j \geq k$, the entries of U in the k-th row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j} \quad \forall \quad j \geq k \tag{0.7}$$

3. For each row $i > k$, the entries of L in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right) \quad \forall \quad i > k \tag{0.8}$$

The system $A\mathbf{x} = \mathbf{b}$ is transformed into $L \cdot U \cdot \mathbf{x} = \mathbf{b}$. Let $\mathbf{y}$ satisfy $L\mathbf{y} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}. \tag{0.9}$$

Using forward substitution:

$$y_1 = 1 \tag{0.10}$$
$$4y_1 + y_2 = 5 \tag{0.11}$$
$$y_2 = 1 \tag{0.12}$$

Thus:

$$\mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{0.13}$$

Next, solve $U\mathbf{x} = \mathbf{y}$:

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{0.14}$$

Using back substitution:

$$0x + 0y = 1 \tag{0.15}$$

As it is not possible for any value of x and y, the solution does not exist.

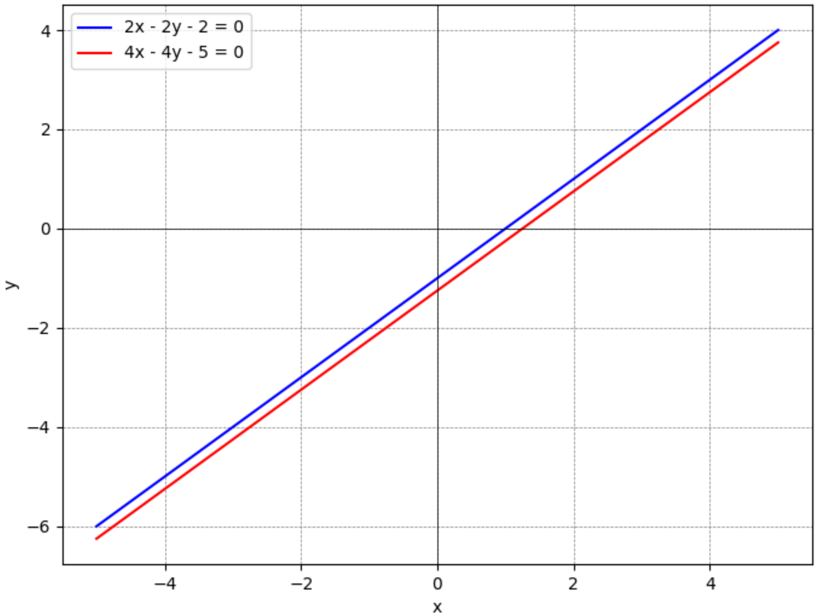

Fig. 0.1: Solution to set of linear equations