12.9.7.2.2

EE24BTECH11024 - G. Abhimanyu Koushik

Question:
Solve the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 0$
Solution:

Variable	**Description**
n	Order of given differential equation
a_i	Coeefficient of ith derivative of the function in the equation
c	constant in the equation
y^i	ith derivative of given function
$\mathbf{y}(t)$	$\begin{pmatrix} c \\ y(t) \\ y'(t) \\ \vdots \\ y^{n-1}(t) \end{pmatrix}$
h	stepsize, taken to be 0.001
$u(x)$	Unit step function

TABLE 0: Variables Used

Theoritical Solution:
Laplace Transform definition

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \tag{0.1}$$

Properties of Laplace tranform

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) \tag{0.2}$$
$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0) \tag{0.3}$$
$$\mathcal{L}(1) = \frac{1}{s} \tag{0.4}$$
$$\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = (\cos x)\, u(x) \tag{0.5}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = (\sin x)\, u(x) \tag{0.6}$$
$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{0.7}$$
$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}\left(e^{at} f(t)\right) = F(s-a) \tag{0.8}$$

Applying the properties to the given equation

$$y'' - 2y' + 2y = 0 \tag{0.9}$$

$$\mathcal{L}(y'') + \mathcal{L}(-2y') + \mathcal{L}(2y) = 0 \tag{0.10}$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) - 2s\mathcal{L}(y) + 2y(0) + 2\mathcal{L}(y) = 0 \tag{0.11}$$

Substituting the initial conditions gives

$$s^2\mathcal{L}(y) - s - 2s\mathcal{L}(y) + 2 + 2\mathcal{L}(y) = 0 \tag{0.12}$$

$$\mathcal{L}(y) = \frac{s-2}{s^2 - 2s + 2} \tag{0.13}$$

$$\mathcal{L}(y) = \frac{s-2}{(s-1)^2 + 1} \tag{0.14}$$

$$\mathcal{L}(y) = \frac{s-1}{(s-1)^2 + 1} + \frac{-1}{(s-1)^2 + 1} \tag{0.15}$$

Let

$$F(s) = \mathcal{L}(y) \tag{0.16}$$

then

$$F(s) = \frac{s-1}{(s-1)^2 + 1} + \frac{-1}{(s-1)^2 + 1} \tag{0.17}$$

$$F(s+1) = \frac{s}{(s)^2 + 1} + \frac{-1}{(s)^2 + 1} \tag{0.18}$$

$$\mathcal{L}^{-1}F(s+1) = \mathcal{L}^{-1}\left(\frac{s}{(s)^2 + 1}\right) + \mathcal{L}^{-1}\left(\frac{-1}{(s)^2 + 1}\right) \tag{0.19}$$

$$e^{-x}\mathcal{L}^{-1}F(s) = (\cos x - \sin x)\, u(x) \tag{0.20}$$

$$e^{-x}y = (\cos x - \sin x)\, u(x) \tag{0.21}$$

$$y = e^{x}(\cos x - \sin x)\, u(x) \tag{0.22}$$

The theoritical solution is

$$f(x) = e^{x}(\cos x - \sin x)\, u(x) \tag{0.23}$$

Computational Solution:
Consider the given linear differential equation

$$a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y' + a_0 y + c = 0 \tag{0.24}$$

Then

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \tag{0.25}$$

$$y(t+h) = y(t) + hy'(t) \tag{0.26}$$

Similarly

$$y^i(t+h) = y^i(t) + hy^{i+1}(t) \tag{0.27}$$
$$y^{n-1}(t+h) = y^{n-1}(t) + hy^n(t) \tag{0.28}$$
$$y^{n-1}(t+h) = y^{n-1}(t) + h\left(-\frac{a_{n-1}}{a_n}y^{n-1} - \frac{a_{n-2}}{a_n}y^{n-2} - \cdots - \frac{a_0}{a_n}y - \frac{c}{a_n}\right) \tag{0.29}$$

Where i ranges from 0 to $n-1$

$$\mathbf{y}(t+h) = \mathbf{y}(t) + \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -\frac{1}{a_n} & -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \dots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{pmatrix} (h\mathbf{y}(t)) \tag{0.30}$$

$$\mathbf{y}(t+h) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ -\frac{h}{a_n} & -\frac{a_0h}{a_n} & -\frac{a_1h}{a_n} & -\frac{a_2h}{a_n} & \dots & -\frac{a_{n-2}h}{a_n} & 1-\frac{a_{n-1}h}{a_n} \end{pmatrix} (\mathbf{y}(t)) \tag{0.31}$$

Discretizing the steps gives us

$$\mathbf{y}_{k+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ -\frac{h}{a_n} & -\frac{a_0h}{a_n} & -\frac{a_1h}{a_n} & -\frac{a_2h}{a_n} & \dots & -\frac{a_{n-2}h}{a_n} & 1-\frac{a_{n-1}h}{a_n} \end{pmatrix} (\mathbf{y}_k) \tag{0.32}$$

where k ranges from 0 to number of data points with y_0^i being the given initial condition and vector $\mathbf{y}_0 = \begin{pmatrix} c \\ y(0) \\ y'(0) \\ \vdots \\ y^{n-1}(0) \end{pmatrix}$

For the given question

$$\mathbf{y}_{k+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & h \\ -h & -2h & 1+2h \end{pmatrix} \mathbf{y}_k \tag{0.33}$$

Record the y_k for

$$x_k = lowerbound + kh \tag{0.34}$$

and then plot the graph. The result will be as given below.

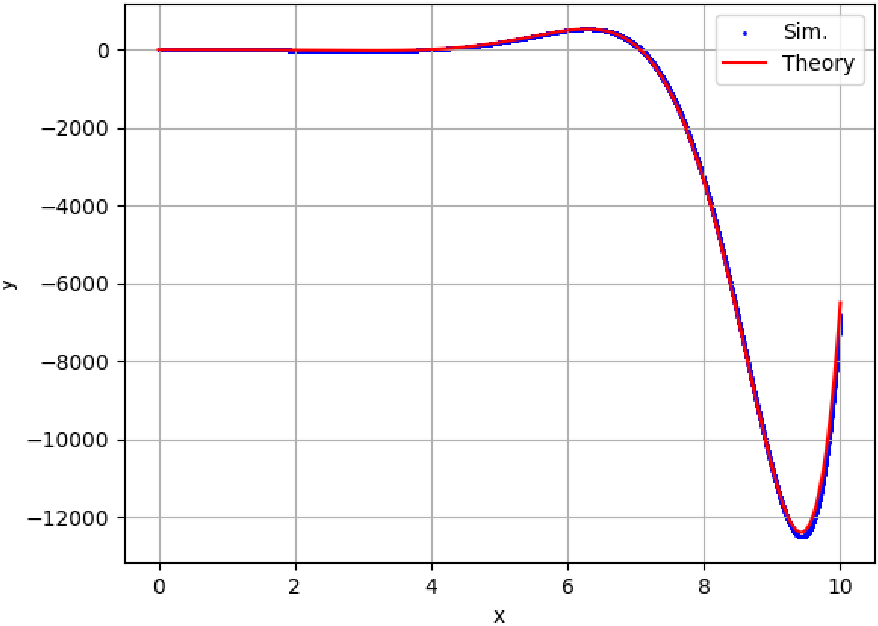

Fig. 0.1: Comparison between the Theoritical solution and Computational solution