

# 12.9.7.15

EE24BTECH11024 - G. Abhimanyu Koushik

## Question:

The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25,000 in the year 2004, what will be the population of the village in 2009?

## Solution:

Variable	Description
$n$	Order of given differential equation
$a_i$	Coefficient of $i$ th derivative of the function in the equation
$c$	constant in the equation
$y^i$	$i$ th derivative of given function
$\mathbf{y}(t)$	$\begin{pmatrix} c \\ y(t) \\ y'(t) \\ \vdots \\ y^{n-1}(t) \end{pmatrix}$
$h$	stepsize, taken to be 0.001
$u(x)$	Unit step function
$k_0$	proportionality constant

TABLE 0: Variables Used

Theoretical Solution:

Laplace Transform definition

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t) dt \quad (0.1)$$

Properties of Laplace tranform

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0) \quad (0.2)$$

$$\mathcal{L}(1) = \frac{1}{s} \quad (0.3)$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \quad (0.4)$$

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}(e^{at} f(t)) = F(s - a) \quad (0.5)$$

Applying the properties to the given equation

$$y' = k_0 y \quad (0.6)$$

$$\mathcal{L}(y') - \mathcal{L}(k_0 y) = 0 \quad (0.7)$$

$$s\mathcal{L}(y) - y(0) - k_0\mathcal{L}(y) = 0 \quad (0.8)$$

$$\mathcal{L}(y) = \frac{y(0)}{s - k_0} \quad (0.9)$$

$$y = y(0) e^{k_0 x} u(x) \quad (0.10)$$

Taking 1999 to be the initial year, we get  $y(0) = 20000$  and  $y(5) = 25000$   
Substituting the initial conditions gives

$$y(0) = 20000 \quad (0.11)$$

$$y(5) = 25000 \quad (0.12)$$

$$20000e^{5k_0} = 25000 \quad (0.13)$$

$$e^{5k_0} = \frac{5}{4} \quad (0.14)$$

$$5k_0 = \ln \frac{5}{4} \quad (0.15)$$

$$k_0 = \frac{1}{5} \ln \frac{5}{4} \quad (0.16)$$

The theoretical solution is

$$f(x) = 20000e^{\frac{1}{5}(\ln \frac{5}{4})x} u(x) \quad (0.17)$$

$$f(x) = 20000 \left( \frac{5}{4} \right)^{\frac{x}{5}} u(x) \quad (0.18)$$

Substituting  $x = 10$  gives 31250

Computational Solution:

First we have to find the  $k$  value in the differential equation, for that

$$y(t+h) = y(t) + hy'(t) \quad (0.19)$$

$$y(t+h) = y(t) + hk_0 y(t) \quad (0.20)$$

$$y(t+2h) = y(t+h) + hk_0 y(t+h) \quad (0.21)$$

$$y(t+2h) = y(t) + hk_0 y(t) + hk_0 (y(t) + hk_0 y(t)) \quad (0.22)$$

$$y(t+2h) = (1 + hk_0)^2 y(t) \quad (0.23)$$

Similarly

$$y(t+nh) = (1 + hk_0)^n y(t) \quad (0.24)$$

Substituting the initial condition and value of  $h$  gives

$$y(5) = (1 + 0.001k_0)^{5000} y(0) \quad (0.25)$$

$$25000 = 20000 (1 + 0.001k_0)^{5000} \quad (0.26)$$

$$\left(\frac{5}{4}\right)^{\frac{1}{5000}} - 1 = 0.001k_0 \quad (0.27)$$

$$k_0 = \frac{\left(\frac{5}{4}\right)^{\frac{1}{5000}} - 1}{0.001} \quad (0.28)$$

$$k_0 \approx 0.04462 \quad (0.29)$$

Consider the given linear differential equation

$$a_n y^n + a_{n-1} y^{n-1} + \dots + a_1 y' + a_0 y + c = 0 \quad (0.30)$$

Then

$$y'(t) = \lim_{h \rightarrow 0} \frac{y(t+h) - y(t)}{h} \quad (0.31)$$

$$y(t+h) = y(t) + hy'(t) \quad (0.32)$$

Similarly

$$y^i(t+h) = y^i(t) + hy^{i+1}(t) \quad (0.33)$$

$$y^{n-1}(t+h) = y^{n-1}(t) + hy^n(t) \quad (0.34)$$

$$y^{n-1}(t+h) = y^{n-1}(t) + h \left( -\frac{a_{n-1}}{a_n} y^{n-1} - \frac{a_{n-2}}{a_n} y^{n-2} - \dots - \frac{a_0}{a_n} y - \frac{c}{a_n} \right) \quad (0.35)$$

Where  $i$  ranges from 0 to  $n-1$

$$\mathbf{y}(t+h) = \mathbf{y}(t) + \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -\frac{1}{a_n} & -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \dots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{pmatrix} (h\mathbf{y}(t)) \quad (0.36)$$

$$\mathbf{y}(t+h) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ -\frac{h}{a_n} & -\frac{a_0 h}{a_n} & -\frac{a_1 h}{a_n} & -\frac{a_2 h}{a_n} & \dots & -\frac{a_{n-2} h}{a_n} & 1 - \frac{a_{n-1} h}{a_n} \end{pmatrix} (\mathbf{y}(t)) \quad (0.37)$$

Discretizing the steps gives us

$$\mathbf{y}_{k+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ -\frac{h}{a_n} & -\frac{a_0 h}{a_n} & -\frac{a_1 h}{a_n} & -\frac{a_2 h}{a_n} & \dots & -\frac{a_{n-2} h}{a_n} & 1 - \frac{a_{n-1} h}{a_n} \end{pmatrix} (\mathbf{y}_k) \quad (0.38)$$

where  $k$  ranges from 0 to number of data points with  $y_0^i$  being the given initial condition

and vector  $\mathbf{y}_0 = \begin{pmatrix} c \\ y(0) \\ y'(0) \\ \vdots \\ y^{n-1}(0) \end{pmatrix}$

for finding the value of  $a_0$ , we wil substitute the initial conditions as in theoritical solution

For the given question

$$\mathbf{y}_{k+1} = \begin{pmatrix} 1 & 0 \\ -h & 1 + hk_0 \end{pmatrix} \mathbf{y}_k \quad (0.39)$$

Record the  $y_k$  for

$$x_k = \text{lowerbound} + kh \quad (0.40)$$

and then plot the graph. The result will be as given below.

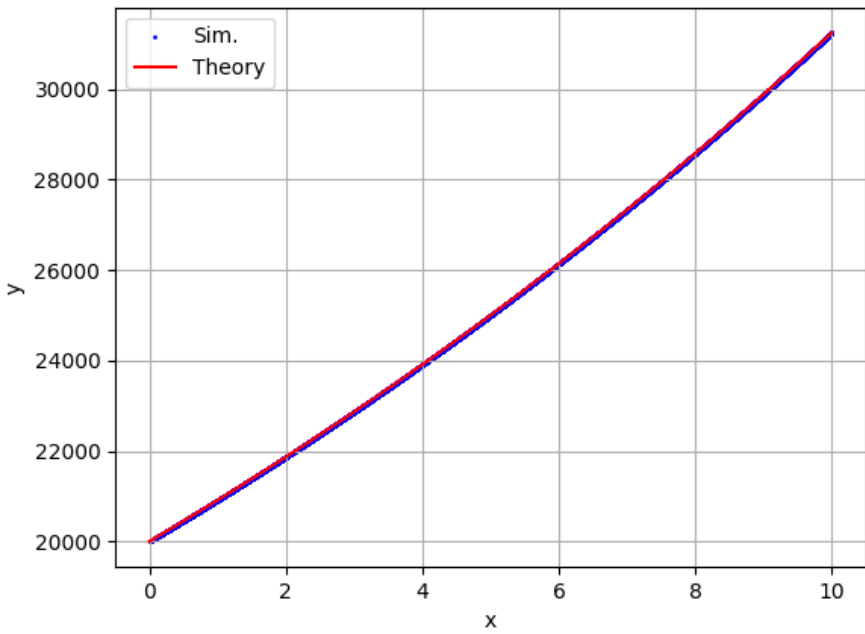


Fig. 0.1: Comparison between the Theoretical solution and Computational solution