12.6.5.11 Presentation

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Problem Statement

It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of a.

Theoritical Solution

Given function

$$f\left(x\right) = x^4 - 62x^2 + ax + 9 \tag{3.1}$$

If a function has a local maxima at $x = x_0$ then $f'\left(x_0\right) = 0$ and $f''\left(x_0\right) \leq 0$

$$\begin{align} f'\left(x\right) &= 4x^3 - 124x + a \tag{3.2}\\ f''\left(x\right) &= 12x^2 - 124 \tag{3.3}\\ f'\left(1\right) &= 4 - 124 + a \tag{3.4}\\ f''\left(1\right) &= 12 - 124 \tag{3.5}\\ & \tag{3.6} \end{align}$$

$f''\left(1\right)$ is anyway negative so $f'\left(1\right)$ should be 0

$$\begin{align} a - 120 &= 0 \tag{3.7}\\ a &= 120 \tag{3.8} \end{align}$$

When $a = 120$, it satisfies all the conditions.

Computational Solution

Given function

$$f(x) = x^4 - 62x^2 + ax + 9 \tag{3.9}$$

At any critical point $x = x_0$, $(f'(x_0))^2$ is minimum. We need to minimize

$$L(a) = \left(f'(1)\right)^2 \tag{3.10}$$

$$L(a) = (a - 120)^2 \tag{3.11}$$

We use the method of gradient descent to find the minimum of the above function, since the objective function is convex.

$$a_{n+1} = a_n - \mu L'(a_n) \tag{3.12}$$

$$L'(a_n) = 2(a_n - 120) \tag{3.13}$$

$$\rightarrow a_{n+1} = a_n - 2\mu(a_n - 120) \tag{3.14}$$

Applying unilateral Z-transform,

$$zX(z) - zx_0 = (1 - 2\mu) X(z) + \frac{240\mu}{1 - z^{-1}} \tag{3.15}$$

The Unilateral Z-transform of a constant function $f(x) = c$ is $\frac{c}{1-z^{-1}}$ with Radius of convergence being $|z| > 1$

$$(z - (1 - 2\mu)) X(z) = zx_0 + \frac{240\mu}{1 - z^{-1}} \tag{3.16}$$

$$X(z) = \frac{zx_0}{z - (1 - 2\mu)} + \frac{240\mu}{(1 - z^{-1})(z - (1 - 2\mu))} \tag{3.17}$$

$$X(z) = \frac{x_0}{1 - (1 - 2\mu) z^{-1}} + \frac{240\mu}{z - 2(1 - \mu) + (1 - 2\mu) z^{-1}} \tag{3.18}$$

$$X(z) = \frac{x_0}{1 - (1 - 2\mu) z^{-1}} + \frac{240\mu z^{-1}}{1 - 2(1 - \mu) z^{-1} + (1 - 2\mu) z^{-2}} \tag{3.19}$$

$$= \sum_0^\infty (1 - 2\mu)^n z^{-n} + \sum_1^\infty \frac{1 - (1 - 2\mu)^n}{2\mu} z^{-n} \tag{3.20}$$

$$= \sum_0^\infty (1 - 2\mu)^n z^{-n} + \sum_1^\infty \frac{1}{2\mu} z^{-n} - \sum_1^\infty \frac{(1 - 2\mu)^n}{2\mu} z^{-n} \tag{3.21}$$

From the last equation, ROC is

$$
\begin{aligned}
|z| &> |1-2\mu| && (3.23)\\
|z| &> 1 && (3.24)\\
\rightarrow 0 < |1-2\mu| &< 1 && (3.25)\\
\rightarrow \mu &\in \left(0, \frac{1}{2}\right) && (3.26)
\end{aligned}
$$

Now, if μ satisfies the previous condition,

$$
\begin{aligned}
\lim_{n\to\infty}\left(a_{n+1}-a_n\right) &= 0 && (3.27)\\
\rightarrow \lim_{n\to\infty}\left(-2\mu\left(a_n-120\right)\right) &= 0 && (3.28)\\
\rightarrow -2\mu \lim_{n\to\infty}\left(a_n-120\right) &= 0 && (3.29)\\
\rightarrow \lim_{n\to\infty}\left(a_n\right) &= 120 && (3.30)\\
\rightarrow \lim_{n\to\infty} a_n &= 120 && (3.31)
\end{aligned}
$$

Taking initial guess = 50
step size = 0.1
tolerance(minimum value of gradient) = 1e-5
We get
$a_{min} = 119.99999528200934$

Quadratic Programming

The question minimum value of $L(a) = (a - 120)^2$ can be viewed as a Quadratic Programming problem as:

$$\min_{\mathbf{x}} \left| e_2^\top \mathbf{x} \right| \tag{3.32}$$

$$\text{s.t.} \tag{3.33}$$

$$\mathbf{x}^\top V \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.34}$$

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.35}$$

$$\mathbf{u} = \begin{pmatrix} -120 \\ -0.5 \end{pmatrix} \tag{3.36}$$

$$f = 14400 \tag{3.37}$$

The constraint here is non-convex since the constraint defines a set which is not convex, since points on the line joining any 2 points on the curve don't belong to the set. However, if we make the constraint

$$\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} \leq 0 \tag{3.38}$$

the constraint becomes convex. Using cvxpy to solve this convex optimization problem, we get

$$Optimalx : [[1.19999997e + 02] \tag{3.39}$$
$$[-3.74242669e - 05]] \tag{3.40}$$

Plot

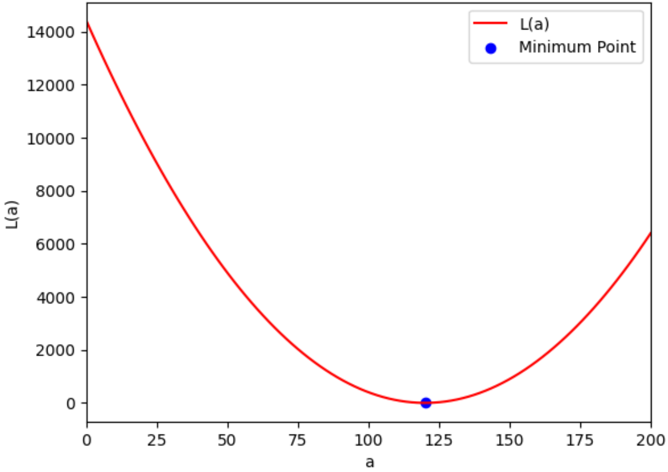

Figure: Graph of $L(a) = (a - 120)^2$ and the value of a which satisfies the given conditions

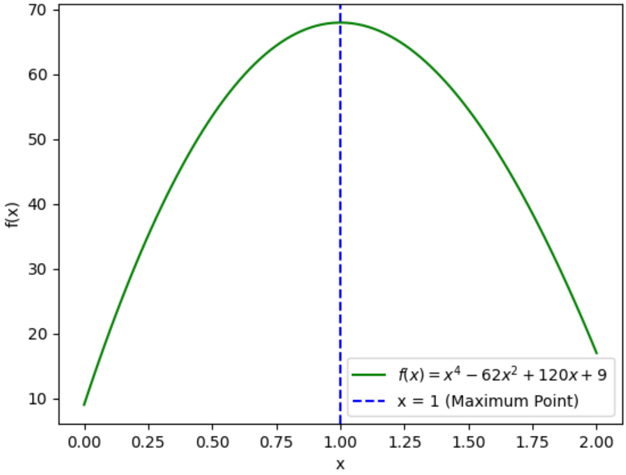

Figure: Graph of $f(x)$ when $a = 120$