9.3.11

EE24BTECH11024 - G. Abhimanyu Koushik

Question:
Solve the differential equation $\frac{d^2y}{dx^2} = y$ with initial conditions $y(0) = 1$ and $y'(0) = 0$
Solution:
Theoritical Solution:
Laplace Transform definition

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt \tag{0.1}$$

Properties of Laplace tranform

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) \tag{0.2}$$
$$\mathcal{L}(1) = \frac{1}{s} \tag{0.3}$$
$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{0.4}$$
$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}\left(e^{at}f(t)\right) = F(s-a) \tag{0.5}$$

Applying the properties to the given equation

$$y'' - y = 0 \tag{0.6}$$
$$\mathcal{L}(y'') - \mathcal{L}(y) = 0 \tag{0.7}$$
$$s^2\mathcal{L}(y) - sy(0) - y'(0) - \mathcal{L}(y) = 0 \tag{0.8}$$
$$\tag{0.9}$$

Substituting the initial conditions gives

$$\left(s^2 - 1\right)\mathcal{L}(y) = s \tag{0.10}$$
$$\mathcal{L}(y) = \frac{s}{s^2 - 1} \tag{0.11}$$
$$\mathcal{L}(y) = \frac{1}{2(s+1)} + \frac{1}{2(s-1)} \tag{0.12}$$
$$y = \frac{1}{2}\left(\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)\right) \tag{0.13}$$
$$y = \frac{1}{2}\left(e^{-x} + e^{x}\right)u(x) \tag{0.14}$$

The theoritical solution is

$$f(x) = \frac{1}{2}\left(e^{-x} + e^{x}\right)u(x) \tag{0.15}$$

We can arrive at a difference equation by applying Bilinear Z-transform on the Laplace equations. Take

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{0.16}$$

Here $T = h$

$$Y(z) = \frac{1}{2}\left(\frac{1}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1} + \frac{1}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)-1}\right) \tag{0.17}$$

$$Y(z) = \frac{1}{2}\left(\frac{T\left(1+z^{-1}\right)}{2\left(1-z^{-1}\right)+T\left(1+z^{-1}\right)} + \frac{T\left(1+z^{-1}\right)}{2\left(1-z^{-1}\right)-T\left(1+z^{-1}\right)}\right) \tag{0.18}$$

$$Y(z) = \frac{1}{2}\left(\frac{T\left(1+z^{-1}\right)}{(T-2)\,z^{-1}+(T+2)} - \frac{T\left(1+z^{-1}\right)}{(T+2)\,z^{-1}+(T-2)}\right) \tag{0.19}$$

$$\alpha_1 = -\frac{T-2}{T+2} \tag{0.20}$$

$$\alpha_2 = -\frac{T+2}{T-2} \tag{0.21}$$

$$Y(z) = \frac{T}{2(T+2)}\left(\frac{1}{1-\alpha_1 z^{-1}} + \frac{z^{-1}}{1-\alpha_1 z^{-1}}\right) - \frac{T}{2(T-2)}\left(\frac{1}{1-\alpha_2 z^{-1}} + \frac{z^{-1}}{1-\alpha_2 z^{-1}}\right) \tag{0.22}$$

The radius of convergence of $\frac{1}{1-\alpha_1 z^{-1}}$ and $\frac{z^{-1}}{1-\alpha_1 z^{-1}}$ is $|z| > |\alpha_1|$ and radius of convergence of $\frac{1}{1-\alpha_2 z^{-1}}$ and $\frac{z^{-1}}{1-\alpha_2 z^{-1}}$ is $|z| > |\alpha_2|$
The radius of convergence of $Y(z)$ is $max\left(|\alpha_1|, |\alpha_2|\right)$
Applying the inverse Z-transform with some rearrangement gives

$$\left(1-\alpha_1 z^{-1}\right)\left(1-\alpha_2 z^{-1}\right)Y(z) = \frac{T}{2}\left(1+z^{-1}\right)\left(\frac{1-\alpha_2 z^{-1}}{T+2} - \frac{1-\alpha_1 z^{-1}}{T-2}\right) \tag{0.23}$$

$$\begin{aligned}\left(1-(\alpha_1+\alpha_2)\,z^{-1}+z^{-2}\right)Y(z) = {} & \frac{T}{2}\left(\frac{1-(\alpha_2-1)\,z^{-1}-\alpha_2 z^{-2}}{T+2}\right) \\ & -\frac{T}{2}\left(\frac{1-(\alpha_1-1)\,z^{-1}-\alpha_1 z^{-2}}{T-2}\right)\end{aligned} \tag{0.24}$$

$$z^2 Y(z) - z(\alpha_1+\alpha_2)\,Y(z) + Y(z) = \frac{T}{2}\left(\left(\frac{z^2-(\alpha_2-1)\,z-\alpha_2}{T+2}\right) - \left(\frac{z^2-(\alpha_1-1)\,z-\alpha_1}{T-2}\right)\right) \tag{0.25}$$

$$z^2 Y(z) - z(\alpha_1+\alpha_2)\,Y(z) + Y(z) - z^2 y[0] - zy[1] - z(\alpha_1+\alpha_2)\,y[0]$$

$$+z^2 y[0] + zy[1] + z(\alpha_1+\alpha_2) y[0] = \frac{T}{2}\left(\left(\frac{z^2-(\alpha_2-1)z-\alpha_2}{T+2}\right)-\left(\frac{z^2-(\alpha_1-1)z-\alpha_1}{T-2}\right)\right) \tag{0.26}$$

$$y_{n+2}-(\alpha_1+\alpha_2)y_{n+1}+y_n+\delta[n+2]y[0]+\delta[n+1](y[1]+(\alpha_1+\alpha_2)y[0]) = \frac{T}{2}\left(\delta[n+2]\left(\frac{1}{T+2}-\frac{1}{T-2}\right)-\delta[n+1]\left(\frac{\alpha_1-1}{T-2}+\frac{\alpha_2-1}{T+2}\right)+\delta[n]\left(\frac{\alpha_1}{T-2}-\frac{\alpha_2}{T+2}\right)\right) \tag{0.27}$$

Since $n \geq 0$, $\delta[n+2] = 0$ and $\delta[n+1] = 0$

$$y_{n+2}-(\alpha_1+\alpha_2)y_{n+1}+y_n = \frac{T}{2}\left(\frac{\alpha_1}{T-2}-\frac{\alpha_2}{T+2}\right)\delta[n] \tag{0.28}$$

As $y[0] = 1$ from initial condition and

$$y[1] = y(0) + hy'(0) \tag{0.29}$$

$$\tag{0.30}$$

Hence $n \geq 2$ which gives the difference equation as

$$y_{n+2} = (\alpha_1+\alpha_2)y_{n+1} - y_n \tag{0.31}$$

Computational Solution:
The given differential equation is

$$y'' - y = 0 \tag{0.32}$$

Let

$$y' = y_1 \tag{0.33}$$

$$y = y_2 \tag{0.34}$$

Then

$$\frac{dy_1}{dx} = y_2 \tag{0.35}$$

$$\frac{dy_2}{dx} = y_1 \tag{0.36}$$

$$\int_{y_{1,k}}^{y_{1,k+1}} dy_1 = \int_{x_k}^{x_{k+1}} y_2 dx \tag{0.37}$$

$$\int_{y_{2,k}}^{y_{2,k+1}} dy_2 = \int_{x_k}^{x_{k+1}} y_1 dx \tag{0.38}$$

Discretizing the steps using trapezoidal rule gives us

$$y_{1,k+1} - y_{1,k} = \frac{h}{2}(y_{2,k}+y_{2,k+1}) \tag{0.39}$$

$$y_{2,k+1} - y_{2,k} = \frac{h}{2}(y_{1,k}+y_{1,k+1}) \tag{0.40}$$

Then solving for $y_{1,k+1}$ and $y_{2,k+1}$ in terms of $y_{1,k}$, $y_{2,k}$ and h will help us to calculate the value of function at x_{k+1}

$$y_{1,k+1} = y_{1,k} + \frac{h}{2}\left(y_{2,k} + \left(y_{2,k} + \frac{h}{2}\left(y_{1,k} + y_{1,k+1}\right)\right)\right) \tag{0.41}$$

$$y_{1,k+1} = y_{1,k}\left(1 + \frac{h^2}{4}\right) + y_{2,k}h + y_{1,k+1}\left(\frac{h^2}{4}\right) \tag{0.42}$$

$$y_{1,k+1}\left(1 - \frac{h^2}{4}\right) = y_{1,k}\left(1 + \frac{h^2}{4}\right) + y_{2,k}h \tag{0.43}$$

$$y_{1,k+1} = \frac{\left(y_{1,k}\right)\left(4 + h^2\right) + 4h\left(y_{2,k}\right)}{4 - h^2} \tag{0.44}$$

Similarly

$$y_{2,k+1} = \frac{\left(y_{2,k}\right)\left(4 + h^2\right) + 4h\left(y_{1,k}\right)}{4 - h^2} \tag{0.45}$$

The difference equations are

$$y_{1,k+1} = \frac{\left(y_{1,k}\right)\left(4 + h^2\right) + 4h\left(y_{2,k}\right)}{4 - h^2} \tag{0.46}$$

$$y_{2,k+1} = \frac{\left(y_{2,k}\right)\left(4 + h^2\right) + 4h\left(y_{1,k}\right)}{4 - h^2} \tag{0.47}$$

Using the above formula, recording the value of y at each value of $x_k = x_0 + kh$ and taking $y(0) = 1$ and $y'(0) = 0$ and plotting gives

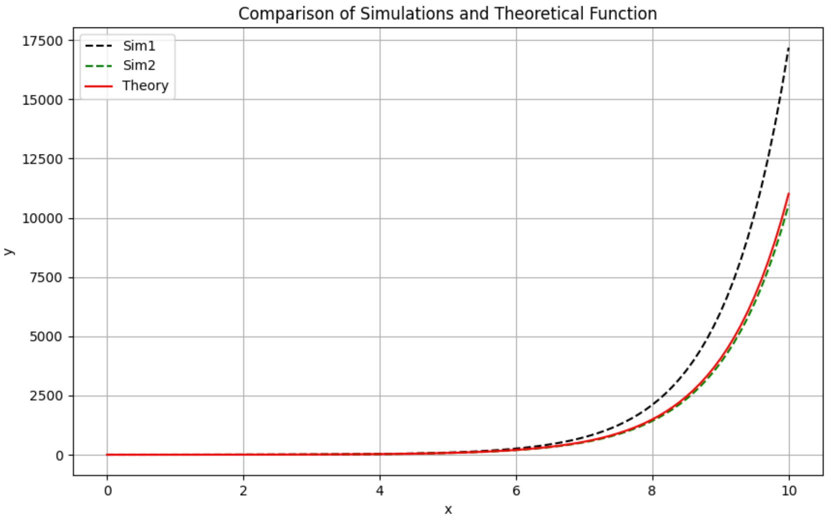

Fig. 0.1: Comparison between the Theoritical solution and Computational solutions, Red is theory, Black line is derived from difference equation from Trapezoidal method, while Green line is from Z-transform while taking stepsize to be 0.1