12.8.3.11

EE24BTECH11024 - G. Abhimanyu Koushik

Question: Using the method of integration find the area bounded by the curve $|x|+|y| = 1$
Solution:
Theoretical Solution:
The curve $|x| + |y| = 1$ consists of 4 lines

$$x + y = 1 \tag{1}$$
$$-x + y = 1 \tag{2}$$
$$x - y = 1 \tag{3}$$
$$-x - y = 1 \tag{4}$$

Solving the line equations
Let the line equations be $\mathbf{x} = \mathbf{h}_1 + \kappa_1\mathbf{m}_1$ and $\mathbf{x} = \mathbf{h}_2 + \kappa_2\mathbf{m}_2$ then

$$\mathbf{h}_2 + \kappa_2\mathbf{m}_2 = \mathbf{h}_1 + \kappa_1\mathbf{m}_1 \tag{5}$$
$$\mathbf{h}_1 - \mathbf{h}_2 = \kappa_2\mathbf{m}_2 - \kappa_1\mathbf{m}_1 \tag{6}$$
$$\mathbf{h}_1 - \mathbf{h}_2 = \begin{pmatrix} \mathbf{m}_2 & -\mathbf{m}_1 \end{pmatrix}\begin{pmatrix} \kappa_2 \\ \kappa_1 \end{pmatrix} \tag{7}$$

Solving the equation using row reduction gives the value of κ_1 and κ_2 which can be subsituted in line equation to get the point
For the given lines, the values of $\mathbf{m}$ and $\mathbf{h}$ are

$$\mathbf{m}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{8}$$
$$\mathbf{m}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{9}$$
$$\mathbf{m}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10}$$
$$\mathbf{m}_4 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{11}$$
$$\mathbf{h}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{12}$$
$$\mathbf{h}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{13}$$
$$\mathbf{h}_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{14}$$
$$\mathbf{h}_4 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{15}$$

The matrix equations we get for lines $(1, 2)$ is

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \kappa_2 \\ \kappa_1 \end{pmatrix} \tag{16}$$

The augmented matrix for this will be

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \tag{17}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix} \tag{18}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{19}$$

$$\kappa_1 = 1 \tag{20}$$

Substituting κ_1 in line equation gives first intersection point to be

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{21}$$

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{22}$$

Similarly the other intersection points are $(1, 0)$, $(-1, 0)$, and $(0, -1)$ forming a square. To find the area we can integrate $x + y = 1$ from $x = 0$ to $x = 1$ and then multiply the area by 4 to get the total area

$$A_0 = \int_0^1 (1 - x)\, dx \tag{23}$$

$$A_0 = \left(x - \frac{x^2}{2} \right) \Big|_0^1 \tag{24}$$

$$A_0 = \frac{1}{2} \tag{25}$$

$$A = 4A_0 \tag{26}$$

$$A = 2 \tag{27}$$

Computational Solution:

Taking trapezoid shaped strips of small area and adding them all up. Say we have to find the area of $y(x)$ from $x = x_0$ to $x = x_n$, discretize points on the x axis $x_0, x_1, x_2, \ldots, x_n$ such that they are equally spaced with step-size h.
Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2} h \left(y(x_1) + y(x_0) \right) + \frac{1}{2} h \left(y(x_2) + y(x_1) \right) + \cdots + \frac{1}{2} h \left(y(x_n) + y(x_{n-1}) \right) \tag{28}$$

$$= h \left[\frac{1}{2} \left(y(x_0) + y(x_n) \right) + y(x_1) + \cdots + y(x_{n-1}) \right] \tag{29}$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \ldots x_n)$

be equidistant points with step-size h.

$$A(x_n + h) = A(x_n) + \frac{1}{2}h(y(x_n + h) + y(x_n)) \tag{30}$$

We can repeat this till we get required area.
Discretizing the steps, making $A(x_n) = A_n, y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h(y_{n+1} + y_n) \tag{31}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h((y_n + hy'_n) + y_n) \tag{32}$$

$$A_{n+1} = A_n + \frac{1}{2}h(2y_n + hy'_n) \tag{33}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{34}$$

$$x_{n+1} = x_n + h \tag{35}$$

In the given question, $y_n = 1 - x_n$ and $y'_n = -1$
General Difference Equation will be given by,

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{36}$$

$$A_{n+1} = A_n + h(1 - x_n) + \frac{1}{2}h^2(-1) \tag{37}$$

$$A_{n+1} = A_n - hx_n + \left(h - \frac{h^2}{2}\right) \tag{38}$$

$$x_{n+1} = x_n + h \tag{39}$$

Iterating till we reach $x_n = 1$ will return required area. Note, Area obtained is to be multiplied by 4 as the calculated area only accounts for one quater of the graph.
The calculated area is 4×0.5 which is 2.

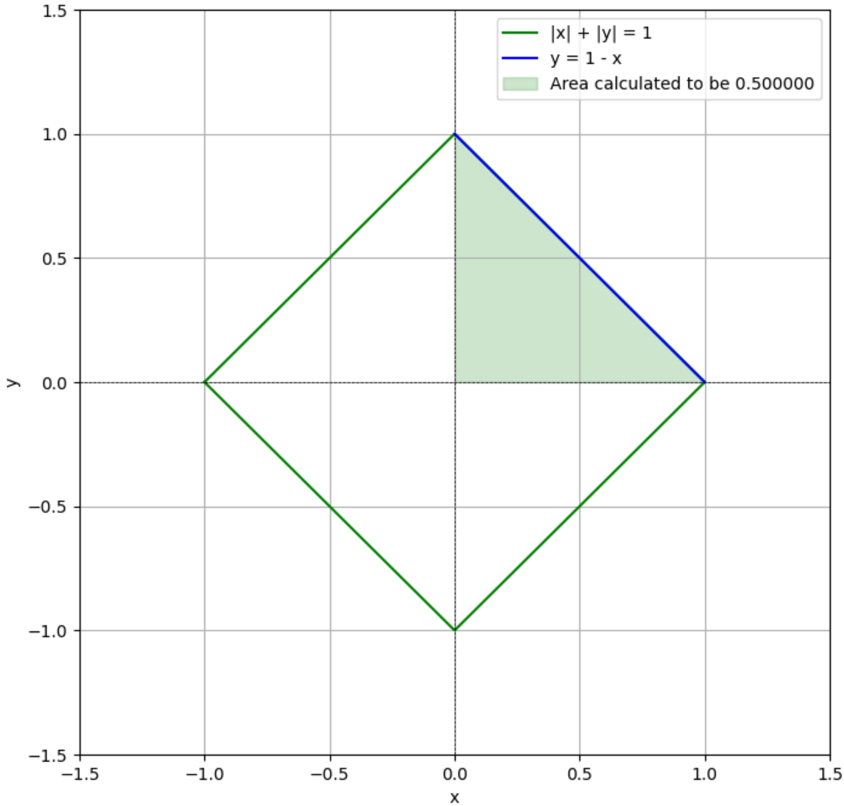

Fig. 1: Graph of the parabola $|x|+|y| = 1$ and $x+y = 1$ and the area of which the integral is calculated