10.3.2.4.4

EE24BTECH11024 - G. Abhimanyu Koushik

Question:
A three coins are tossed once, what is the probability of getting atmost 2 heads?
Solution:
Define a discrete random variable X = number of heads
We will assume our random variable as a sum of outcomes of three bernoulli random variables

$$X = X_1 + X_2 + X_3 \tag{0.1}$$

Where

$$X_i = \begin{cases} 1, & \text{Outcome in Heads} \\ 0, & \text{Outcome in Tails} \end{cases} \tag{0.2}$$

$$p_{X_i}(n) = \begin{cases} 1 - p, & n = 0 \\ p, & n = 1 \end{cases} \tag{0.3}$$

Where $p = \frac{1}{2}$
Using properties of Z-Transform of PMF

$$M_X(z) = M_{X_1}(z) M_{X_2}(z) M_{X_3}(z) \tag{0.4}$$

$$M_{X_1}(z) = \sum_{n=-\infty}^{\infty} p_{X_1}(n) z^{-n} = p + (1 - p) z^{-1} \tag{0.5}$$

$$M_{X_2}(z) = \sum_{n=-\infty}^{\infty} p_{X_2}(n) z^{-n} = p + (1 - p) z^{-1} \tag{0.6}$$

$$M_{X_3}(z) = \sum_{n=-\infty}^{\infty} p_{X_3}(n) z^{-n} = p + (1 - p) z^{-1} \tag{0.7}$$

$$M_X(z) = (p + (1 - p) z^{-1})^3 \tag{0.8}$$

$$= \sum_{k=-\infty}^{\infty} {}^3C_k p^{3-k} (1 - p)^k z^{-k} \tag{0.9}$$

$$p_X(k) = {}^3C_k p^{3-k} (1 - p)^k \tag{0.10}$$

$$p_X(k) = \frac{{}^3C_k}{8} \tag{0.11}$$

The Probability Mass Function (PMF) for the given random variable is

$$p_X(n) = \begin{cases} \frac{1}{8}, & n = 0 \\ \frac{3}{8}, & n = 1 \\ \frac{3}{8}, & n = 2 \\ \frac{1}{8}, & n = 3 \end{cases} \tag{0.12}$$

The Cumulative Distribution Function (CDF) for the given random variable is

$$F_X(n) = p(X \le n) = \begin{cases} 0, & n < 0 \\ \frac{1}{8}, & 0 \le n < 1 \\ \frac{4}{8}, & 1 \le n < 2 \\ \frac{7}{8}, & 2 \le n < 3 \\ 1, & 3 \le n \end{cases} \tag{0.13}$$

The probability of getting atmost 2 heads is

$$F_X(2) = \frac{7}{8} \tag{0.14}$$

Simulation:
To run a simulation we need to generate random numbers with uniform probability, which is done as shown below(Algorithm taken from OpenSSL's random_uniform.c):

1) Generate 32 bits of entropy using /dev/urandom.
2) Treat this as a fixed point number in the range [0, 1)
3) Scale this to desired range using fixed point multiplication and treat as 64bit number(upper 32 bits integer and rest as fractional part)
4) Return the integer part of the fixed point numbers

The following shows how the relative frequency reaches true probability with increasing number of trials of the event.

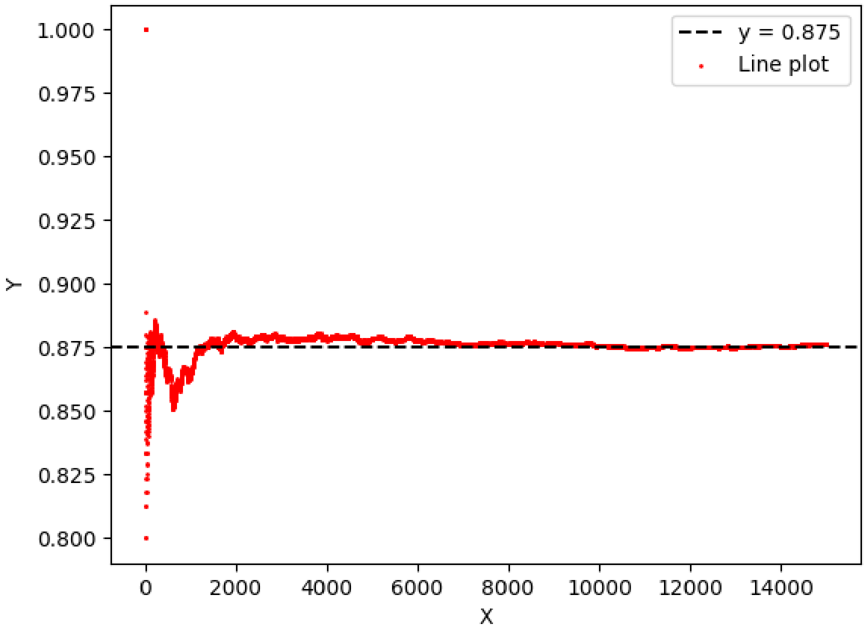

Fig. 4.1: Relative Frequency tends to True Probability

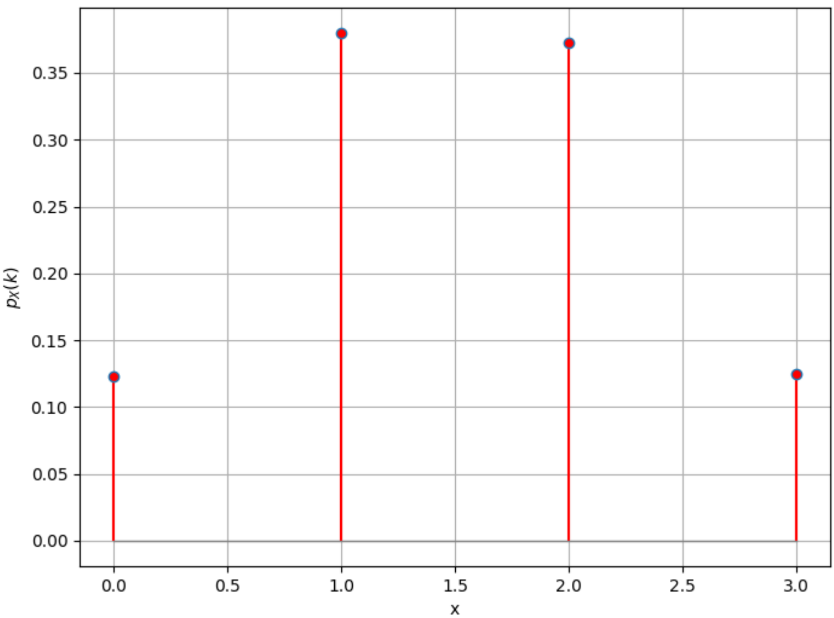

Fig. 4.2: Probability Mass Function of given Random variable

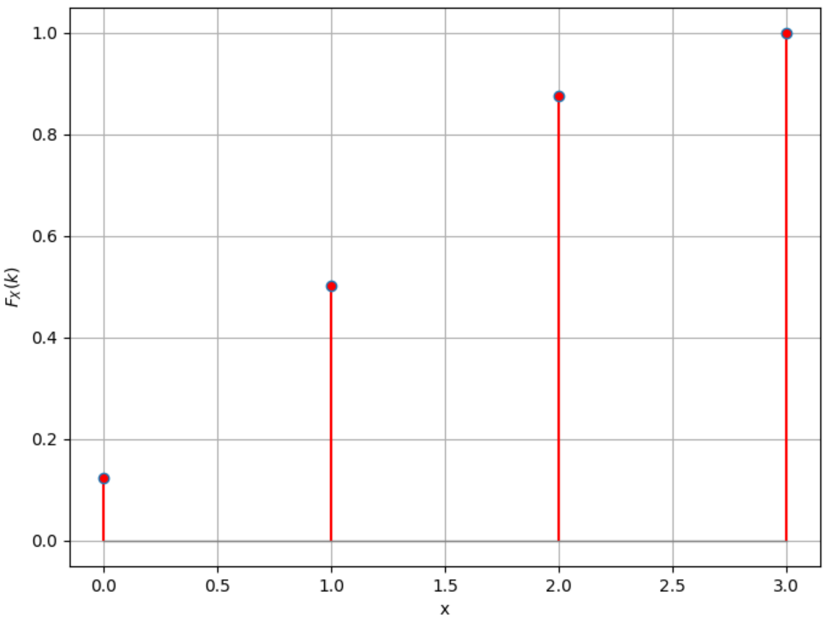

Fig. 4.3: Cumulative Distribution Function of given Random variable