12.6.5.11

EE24BTECH11024 - G. Abhimanyu Koushik

Question:
It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of a.
Solution:
Theoritical Solution:
Given function

$$f(x) = x^4 - 62x^2 + ax + 9 \tag{0.1}$$

If a function has a local maxima at $x = x_0$ then $f'(x_0) = 0$ and $f''(x_0) \leq 0$

$$f'(x) = 4x^3 - 124x + a \tag{0.2}$$

$$f''(x) = 12x^2 - 124 \tag{0.3}$$

$$f'(1) = 4 - 124 + a \tag{0.4}$$

$$f''(1) = 12 - 124 \tag{0.5}$$

$$\tag{0.6}$$

$f''(1)$ is anyway negative so $f'(1)$ should be 0

$$a - 120 = 0 \tag{0.7}$$

$$a = 120 \tag{0.8}$$

When $a = 120$, it satisfies all the conditions.
Computational Solution:
Given function

$$f(x) = x^4 - 62x^2 + ax + 9 \tag{0.9}$$

At any critical point $x = x_0$, $(f'(x_0))^2$ is minimum. We need to minimize

$$L(a) = (f'(1))^2 \tag{0.10}$$

$$L(a) = (a - 120)^2 \tag{0.11}$$

We use the method of gradient descent to find the minimum of the above function, since the objective function is convex.

$$a_{n+1} = a_n - \mu L'(a_n) \tag{0.12}$$

$$L'(a_n) = 2(a_n - 120) \tag{0.13}$$

$$\rightarrow a_{n+1} = a_n - 2\mu(a_n - 120) \tag{0.14}$$

Applying unilateral Z-transform,

$$zX(z) - zx_0 = (1-2\mu)X(z) + \frac{240\mu}{1-z^{-1}} \tag{0.15}$$

The Unilateral Z-transform of a constant function $f(x) = c$ is $\frac{c}{1-z^{-1}}$ with Radius of convergence being $|z| > 1$

$$(z-(1-2\mu))X(z) = zx_0 + \frac{240\mu}{1-z^{-1}} \tag{0.16}$$

$$X(z) = \frac{zx_0}{z-(1-2\mu)} + \frac{240\mu}{(1-z^{-1})(z-(1-2\mu))} \tag{0.17}$$

$$X(z) = \frac{x_0}{1-(1-2\mu)z^{-1}} + \frac{240\mu}{z-2(1-\mu)+(1-2\mu)z^{-1}} \tag{0.18}$$

$$X(z) = \frac{x_0}{1-(1-2\mu)z^{-1}} + \frac{240\mu z^{-1}}{1-2(1-\mu)z^{-1}+(1-2\mu)z^{-2}} \tag{0.19}$$

$$= \sum_0^\infty (1-2\mu)^n z^{-n} + \sum_1^\infty \frac{1-(1-2\mu)^n}{2\mu} z^{-n} \tag{0.20}$$

$$= \sum_0^\infty (1-2\mu)^n z^{-n} + \sum_1^\infty \frac{1}{2\mu} z^{-n} - \sum_1^\infty \frac{(1-2\mu)^n}{2\mu} z^{-n} \tag{0.21}$$

$$\tag{0.22}$$

From the last equation, ROC is

$$|z| > |1-2\mu| \tag{0.23}$$

$$|z| > 1 \tag{0.24}$$

$$\rightarrow 0 < |1-2\mu| < 1 \tag{0.25}$$

$$\rightarrow \mu \in \left(0, \frac{1}{2}\right) \tag{0.26}$$

Now, if μ satisfies the previous condition,

$$\lim_{n\to\infty} (a_{n+1} - a_n) = 0 \tag{0.27}$$

$$\rightarrow \lim_{n\to\infty} (-2\mu(a_n - 120)) = 0 \tag{0.28}$$

$$\rightarrow -2\mu \lim_{n\to\infty} (a_n - 120) = 0 \tag{0.29}$$

$$\rightarrow \lim_{n\to\infty} (a_n) = 120 \tag{0.30}$$

$$\rightarrow \lim_{n\to\infty} a_n = 120 \tag{0.31}$$

Taking initial guess = 50
step size = 0.1
tolerance(minimum value of gradient) = 1e-5
We get
a_{min} = 119.99999528200934
The question minimum value of $L(a) = (a-120)^2$ can be viewed as a Quadratic

Programming problem as:

$$\min_{\mathbf{x}} \left| e_2^\top \mathbf{x} \right| \tag{0.32}$$

$$\text{s.t.} \tag{0.33}$$

$$\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.34}$$

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{0.35}$$

$$\mathbf{u} = \begin{pmatrix} -120 \\ -0.5 \end{pmatrix} \tag{0.36}$$

$$f = 14400 \tag{0.37}$$

The constraint here is non-convex since the constraint defines a set which is not convex, since points on the line joining any 2 points on the curve don't belong to the set. However, if we make the constraint

$$\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} \leq 0 \tag{0.38}$$

the constraint becomes convex. Using cvxpy to solve this convex optimization problem, we get

$$Optimalx : [[1.19999997e + 02] \tag{0.39}$$

$$[-3.74242669e - 05]] \tag{0.40}$$

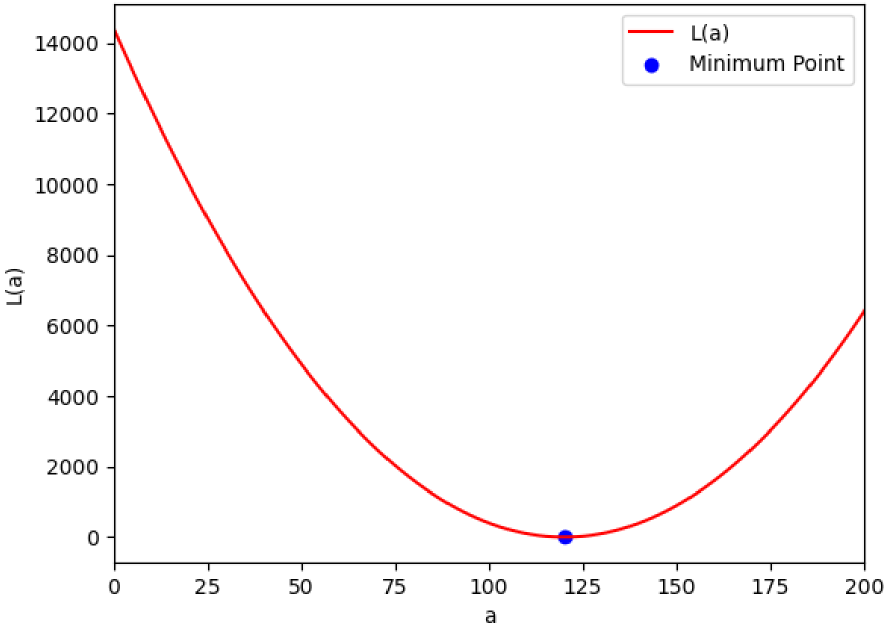

Fig. 0.1: Graph of $L(a) = (a - 120)^2$ and the value of a which satisfies the given conditions

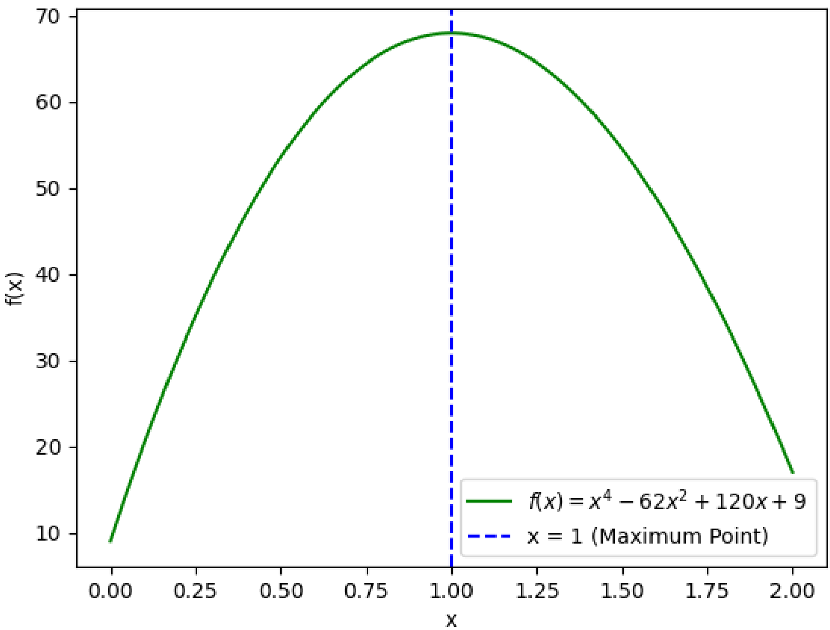

Fig. 0.2: Graph of $f(x)$ when $a = 120$