9.3.11

EE24BTECH11024 - G. Abhimanyu Koushik

Question:
Solve the differential equation $\frac{d^2y}{dx^2} + 1 = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 0$
Solution:

Variable	**Description**
c_1	First Integration constant
c_2	Second Integration constant
n	Order of given differential equation
a_i	Coeefficient of ith derivative of the function in the equation
c	constant in the equation
y^i	ith derivative of given function
$\mathbf{y}(t)$	$\begin{pmatrix} c \\ y(t) \\ y'(t) \\ \vdots \\ y^{n-1}(t) \end{pmatrix}$
h	stepsize, taken to be 0.001

TABLE 0: Variables Used

Theoritical Solution:

$$\frac{d^2y}{dx^2} + 1 = 0 \tag{0.1}$$

$$\frac{d^2y}{dx^2} = -1 \tag{0.2}$$

$$\int \frac{d^2y}{dx^2} dx = \int -1 dx \tag{0.3}$$

$$\frac{dy}{dx} = -x + c_1 \tag{0.4}$$

$$\int \frac{dy}{dx} dx = \int (-x + c_1) dx \tag{0.5}$$

$$y = \frac{-x^2}{2} + c_1 x + c_2 \tag{0.6}$$

Substituting the initial conditions gives

$$y(0) = 0 \implies \frac{-0^2}{2} + c_1 0 + c_2 = 0 \tag{0.7}$$

$$c_2 = 0 \tag{0.8}$$

$$y'(0) = 0 \implies -0 + c_1 = 0 \tag{0.9}$$

$$c_1 = 0 \tag{0.10}$$

The theoritical solution is

$$f(x) = \frac{-x^2}{2} \tag{0.11}$$

Computational Solution:
Consider the given linear differential equation

$$a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y' + a_0 y + c = 0 \tag{0.12}$$

Where y^i is the ith derivative of the function then

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \tag{0.13}$$

$$y(t+h) = y(t) + hy'(t) \tag{0.14}$$

Similarly

$$y^i(t+h) = y^i(t) + hy^{i+1}(t) \tag{0.15}$$

$$y^{n-1}(t+h) = y^{n-1}(t) + hy^n(t) \tag{0.16}$$

$$y^{n-1}(t+h) = y^{n-1}(t) + h\left(-\frac{a_{n-1}}{a_n} y^{n-1} - \frac{a_{n-2}}{a_n} y^{n-2} - \cdots - \frac{a_0}{a_n} y - \frac{c}{a_n}\right) \tag{0.17}$$

Where i ranges from 0 to $n-1$

$$\mathbf{y}(t+h) = \mathbf{y}(t) + \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -\frac{1}{a_n} & -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \dots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{pmatrix} (h\mathbf{y}(t)) \tag{0.18}$$

$$\mathbf{y}(t+h) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ -\frac{h}{a_n} & -\frac{a_0 h}{a_n} & -\frac{a_1 h}{a_n} & -\frac{a_2 h}{a_n} & \dots & -\frac{a_{n-2} h}{a_n} & 1 - \frac{a_{n-1} h}{a_n} \end{pmatrix} (\mathbf{y}(t)) \tag{0.19}$$

Discretizing the steps gives us

$$\mathbf{y}_{k+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ -\frac{h}{a_n} & -\frac{a_0 h}{a_n} & -\frac{a_1 h}{a_n} & -\frac{a_2 h}{a_n} & \dots & -\frac{a_{n-2} h}{a_n} & 1 - \frac{a_{n-1} h}{a_n} \end{pmatrix} (\mathbf{y}_k) \tag{0.20}$$

where k ranges from 0 to number of data points with y_0^i being the given initial condition. For the given question

$$\mathbf{y}_{k+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & h \\ -h & 0 & 1 \end{pmatrix} \mathbf{y}_k \tag{0.21}$$

Record the y_k for

$$x_k = lowerbound + kh \tag{0.22}$$

and then plot the graph. The result will be as given below.

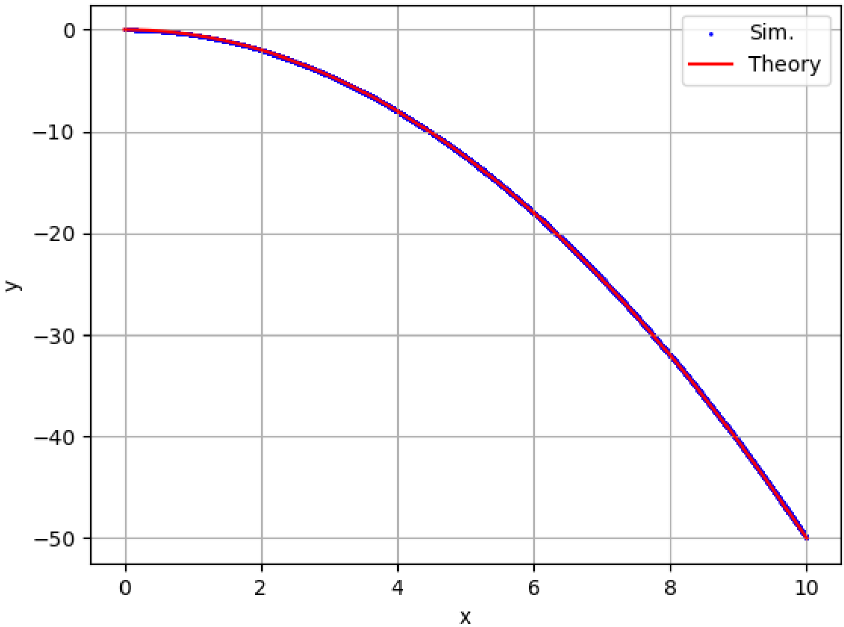

Fig. 0.1: Comparison between the Theoritical solution and Computational solution