9.3.11

EE24BTECH11024 - G. Abhimanyu Koushik

Question:
Solve the differential equation $\frac{d^2y}{dx^2} = y$ with initial conditions $y(0) = 1$ and $y'(0) = 0$
Solution:
Theoritical Solution:
Laplace Transform definition

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \tag{0.1}$$

Properties of Laplace tranform

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) \tag{0.2}$$

$$\mathcal{L}(1) = \frac{1}{s} \tag{0.3}$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{0.4}$$

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}\left(e^{at}f(t)\right) = F(s-a) \tag{0.5}$$

Applying the properties to the given equation

$$y'' - y = 0 \tag{0.6}$$

$$\mathcal{L}(y'') - \mathcal{L}(y) = 0 \tag{0.7}$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) - \mathcal{L}(y) = 0 \tag{0.8}$$

$$\tag{0.9}$$

Substituting the initial conditions gives

$$\left(s^2 - 1\right)\mathcal{L}(y) = s \tag{0.10}$$

$$\mathcal{L}(y) = \frac{s}{s^2 - 1} \tag{0.11}$$

$$\mathcal{L}(y) = \frac{1}{2(s+1)} + \frac{1}{2(s-1)} \tag{0.12}$$

$$y = \frac{1}{2}\left(\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)\right) \tag{0.13}$$

$$y = \frac{1}{2}\left(e^{-x} + e^{x}\right)u(x) \tag{0.14}$$

The theoritical solution is

$$f(x) = \frac{1}{2}\left(e^{-x} + e^{x}\right)u(x) \tag{0.15}$$

Computational Solution:
The given differential equation is

$$y'' - y = 0 \tag{0.16}$$

Let

$$y' = y_1 \tag{0.17}$$

$$y = y_2 \tag{0.18}$$

Then

$$\frac{dy_1}{dx} = y_2 \tag{0.19}$$

$$\frac{dy_2}{dc} = y_1 \tag{0.20}$$

$$\int_{y_{1,k}}^{y_{1,k+1}} dy_1 = \int_{x_k}^{x_{k+1}} y_2 dx \tag{0.21}$$

$$\int_{y_{2,k}}^{y_{2,k+1}} dy_2 = \int_{x_k}^{x_{k+1}} y_1 dx \tag{0.22}$$

Discretizing the steps using trapezoidal rule gives us

$$y_{1,k+1} - y_{1,k} = \frac{h}{2}\left(y_{2,k} + y_{2,k+1}\right) \tag{0.23}$$

$$y_{2,k+1} - y_{2,k} = \frac{h}{2}\left(y_{1,k} + y_{1,k+1}\right) \tag{0.24}$$

Then solving for $y_{1,k+1}$ and $y_{2,k+1}$ in terms of $y_{1,k}$, $y_{2,k}$ and h will help us to calculate the value of function at x_{k+1}

$$y_{1,k+1} = y_{1,k} + \frac{h}{2}\left(y_{2,k} + \left(y_{2,k} + \frac{h}{2}\left(y_{1,k} + y_{1,k+1}\right)\right)\right) \tag{0.25}$$

$$y_{1,k+1} = y_{1,k}\left(1 + \frac{h^2}{4}\right) + y_{2,k}h + y_{1,k+1}\left(\frac{h^2}{4}\right) \tag{0.26}$$

$$y_{1,k+1}\left(1 - \frac{h^2}{4}\right) = y_{1,k}\left(1 + \frac{h^2}{4}\right) + y_{2,k}h \tag{0.27}$$

$$y_{1,k+1} = \frac{\left(y_{1,k}\right)\left(4 + h^2\right) + 4h\left(y_{2,k}\right)}{4 - h^2} \tag{0.28}$$

Similarly

$$y_{2,k+1} = \frac{\left(y_{2,k}\right)\left(4 + h^2\right) + 4h\left(y_{1,k}\right)}{4 - h^2} \tag{0.29}$$

The difference equations are

$$y_{1,k+1} = \frac{(y_{1,k})\left(4+h^2\right)+4h\,(y_{2,k})}{4-h^2} \tag{0.30}$$

$$y_{2,k+1} = \frac{(y_{2,k})\left(4+h^2\right)+4h\,(y_{1,k})}{4-h^2} \tag{0.31}$$

Using the above formula, recording the value of y at each value of $x_k = x_0 + kh$ and taking $y(0) = 1$ and $y'(0) = 0$ and plotting gives

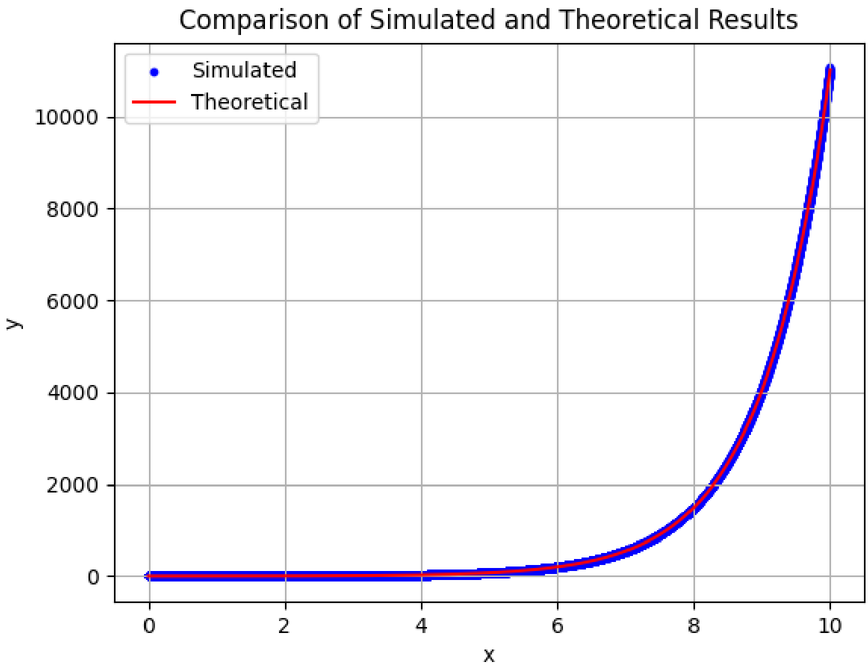

Fig. 0.1: Comparison between the Theoritical solution and Computational solution