

# 9.3.11

EE24BTECH11024 - G. Abhimanyu Koushik

## Question:

Solve the differential equation  $\frac{d^2 y}{dx^2} = y$  with initial conditions  $y(0) = 1$  and  $y'(0) = 0$

## Solution:

Theoretical Solution:

Laplace Transform definition

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t) dt \quad (0.1)$$

Properties of Laplace tranform

$$\mathcal{L}(y'') = s^2 \mathcal{L}(y) - sy(0) - y'(0) \quad (0.2)$$

$$\mathcal{L}(1) = \frac{1}{s} \quad (0.3)$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \quad (0.4)$$

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}(e^{at} f(t)) = F(s - a) \quad (0.5)$$

Applying the properties to the given equation

$$y'' - y = 0 \quad (0.6)$$

$$\mathcal{L}(y'') - \mathcal{L}(y) = 0 \quad (0.7)$$

$$s^2 \mathcal{L}(y) - sy(0) - y'(0) - \mathcal{L}(y) = 0 \quad (0.8)$$

$$(0.9)$$

Substituting the initial conditions gives

$$(s^2 - 1) \mathcal{L}(y) = s \quad (0.10)$$

$$\mathcal{L}(y) = \frac{s}{s^2 - 1} \quad (0.11)$$

$$\mathcal{L}(y) = \frac{1}{2(s+1)} + \frac{1}{2(s-1)} \quad (0.12)$$

$$y = \frac{1}{2} \left( \mathcal{L}^{-1} \left( \frac{1}{s+1} \right) + \mathcal{L}^{-1} \left( \frac{1}{s-1} \right) \right) \quad (0.13)$$

$$y = \frac{1}{2} (e^{-x} + e^x) u(x) \quad (0.14)$$

The theoretical solution is

$$f(x) = \frac{1}{2} (e^{-x} + e^x) u(x) \quad (0.15)$$

Computational Solution:

The given differential equation is

$$y'' - y = 0 \quad (0.16)$$

Let

$$y' = y_1 \quad (0.17)$$

$$y = y_2 \quad (0.18)$$

Then

$$\frac{dy_1}{dx} = y_2 \quad (0.19)$$

$$\frac{dy_2}{dx} = y_1 \quad (0.20)$$

$$\int_{y_{1,k}}^{y_{1,k+1}} dy_1 = \int_{x_k}^{x_{k+1}} y_2 dx \quad (0.21)$$

$$\int_{y_{2,k}}^{y_{2,k+1}} dy_2 = \int_{x_k}^{x_{k+1}} y_1 dx \quad (0.22)$$

Discretizing the steps using trapezoidal rule gives us

$$y_{1,k+1} - y_{1,k} = \frac{h}{2} (y_{2,k} + y_{2,k+1}) \quad (0.23)$$

$$y_{2,k+1} - y_{2,k} = \frac{h}{2} (y_{1,k} + y_{1,k+1}) \quad (0.24)$$

Then solving for  $y_{1,k+1}$  and  $y_{2,k+1}$  in terms of  $y_{1,k}$ ,  $y_{2,k}$  and  $h$  will help us to calculate the value of function at  $x_{k+1}$

$$y_{1,k+1} = y_{1,k} + \frac{h}{2} \left( y_{2,k} + \left( y_{2,k} + \frac{h}{2} (y_{1,k} + y_{1,k+1}) \right) \right) \quad (0.25)$$

$$y_{1,k+1} = y_{1,k} \left( 1 + \frac{h^2}{4} \right) + y_{2,k} h + y_{1,k+1} \left( \frac{h^2}{4} \right) \quad (0.26)$$

$$y_{1,k+1} \left( 1 - \frac{h^2}{4} \right) = y_{1,k} \left( 1 + \frac{h^2}{4} \right) + y_{2,k} h \quad (0.27)$$

$$y_{1,k+1} = \frac{(y_{1,k}) \left( 4 + h^2 \right) + 4h (y_{2,k})}{4 - h^2} \quad (0.28)$$

Similarly

$$y_{2,k+1} = \frac{(y_{2,k}) \left( 4 + h^2 \right) + 4h (y_{1,k})}{4 - h^2} \quad (0.29)$$

The difference equations are

$$y_{1,k+1} = \frac{(y_{1,k})(4 + h^2) + 4h(y_{2,k})}{4 - h^2} \quad (0.30)$$

$$y_{2,k+1} = \frac{(y_{2,k})(4 + h^2) + 4h(y_{1,k})}{4 - h^2} \quad (0.31)$$

Using the above formula, recording the value of  $y$  at each value of  $x_k = x_0 + kh$  and taking  $y(0) = 1$  and  $y'(0) = 0$  and plotting gives

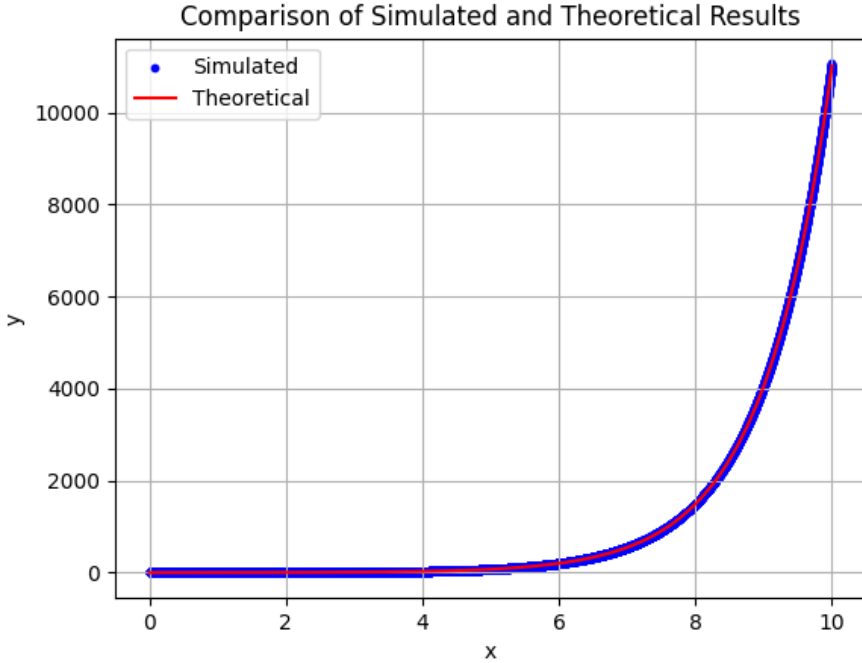


Fig. 0.1: Comparison between the Theoretical solution and Computational solution