12.8.3.11

EE24BTECH11024 - G. Abhimanyu Koushik

Question: Using the method of integration find the area bounded by the curve $|x|+|y| = 1$
Solution:
Theoretical Solution:
The curve $|x| + |y| = 1$ consists of 4 lines

$$x + y = 1 \tag{1}$$
$$-x + y = 1 \tag{2}$$
$$x - y = 1 \tag{3}$$
$$-x - y = 1 \tag{4}$$

These lines intersect at points $(1, 0)$, $(-1, 0)$, $(0, 1)$ and $(0, -1)$ forming a square. To find the area we can integrate $x + y = 1$ from $x = 0$ to $x = 1$ and then multiply the area by 4 to get the total area

$$A_0 = \int_0^1 (1 - x)\, dx \tag{5}$$
$$A_0 = \left(x - \frac{x^2}{2}\right)\Big|_0^1 \tag{6}$$
$$A_0 = \frac{1}{2} \tag{7}$$
$$A = 4A_0 \tag{8}$$
$$A = 2 \tag{9}$$

Computational Solution:

Taking trapezoid shaped strips of small area and adding them all up. Say we have to find the area of $y(x)$ from $x = x_0$ to $x = x_n$, discretize points on the x axis $x_0, x_1, x_2, \dots, x_n$ such that they are equally spaced with step-size h.
Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2}h(y(x_1) + y(x_0)) + \frac{1}{2}h(y(x_2) + y(x_1)) + \cdots + \frac{1}{2}h(y(x_n) + y(x_{n-1})) \tag{10}$$
$$= h\left[\frac{1}{2}(y(x_0) + y(x_n)) + y(x_1) + \cdots + y(x_{n-1})\right] \tag{11}$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \dots x_n)$ be equidistant points with step-size h.

$$A(x_n + h) = A(x_n) + \frac{1}{2}h(y(x_n + h) + y(x_n)) \tag{12}$$

We can repeat this till we get required area.
Discretizing the steps, making $A(x_n) = A_n, y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h(y_{n+1} + y_n) \tag{13}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h((y_n + hy'_n) + y_n) \tag{14}$$

$$A_{n+1} = A_n + \frac{1}{2}h(2y_n + hy'_n) \tag{15}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{16}$$

$$x_{n+1} = x_n + h \tag{17}$$

In the given question, $y_n = 1 - x_n$ and $y'_n = -1$
General Difference Equation will be given by,

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{18}$$

$$A_{n+1} = A_n + h(1 - x_n) + \frac{1}{2}h^2(-1) \tag{19}$$

$$A_{n+1} = A_n - hx_n + \left(h - \frac{h^2}{2}\right) \tag{20}$$

$$x_{n+1} = x_n + h \tag{21}$$

Iterating till we reach $x_n = 1$ will return required area. Note, Area obtained is to be multiplied by 4 as the calculated area only accounts for one quater of the graph.

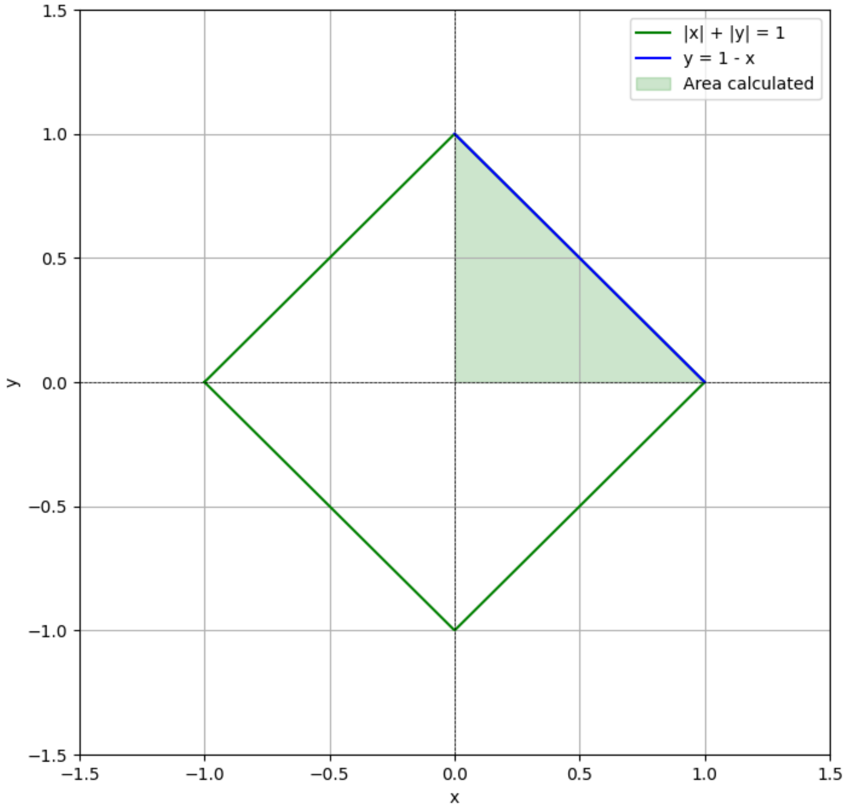

Fig. 1: Graph of the parabola $|x| + |y| = 1$ and $x + y = 1$ and the area of which the integral is calculated