12.9.7.15

EE24BTECH11024 - G. Abhimanyu Koushik

Question:
The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25,000 in the year 2004, what will be the population of the village in 2009?
Solution:

Variable	Description
n	Order of given differential equation
a_i	Coeefficient of ith derivative of the function in the equation
c	constant in the equation
y^i	ith derivative of given function
$\mathbf{y}(t)$	$\begin{pmatrix} c \\ y(t) \\ y'(t) \\ \vdots \\ y^{n-1}(t) \end{pmatrix}$
h	stepsize, taken to be 0.001
$u(x)$	Unit step function
k_0	proportionality constant

TABLE 0: Variables Used

Theoritical Solution:
Laplace Transform definition

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \tag{0.1}$$

Properties of Laplace tranform

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0) \tag{0.2}$$

$$\mathcal{L}(1) = \frac{1}{s} \tag{0.3}$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{0.4}$$

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}\left(e^{at} f(t)\right) = F(s-a) \tag{0.5}$$

Applying the properties to the given equation

$$y' = k_0 y \tag{0.6}$$
$$\mathcal{L}(y') - \mathcal{L}(k_0 y) = 0 \tag{0.7}$$
$$s\mathcal{L}(y) - y(0) - k_0\mathcal{L}(y) = 0 \tag{0.8}$$
$$\mathcal{L}(y) = \frac{y(0)}{s - k_0} \tag{0.9}$$
$$y = y(0)\, e^{k_0 x} u(x) \tag{0.10}$$

Taking 1999 to be the initial year, we get $y(0) = 20000$ and $y(5) = 25000$
Substituting the initial conditions gives

$$y(0) = 20000 \tag{0.11}$$
$$y(5) = 25000 \tag{0.12}$$
$$20000e^{5k_0} = 25000 \tag{0.13}$$
$$e^{5k_0} = \frac{5}{4} \tag{0.14}$$
$$5k_0 = \ln\frac{5}{4} \tag{0.15}$$
$$k_0 = \frac{1}{5}\ln\frac{5}{4} \tag{0.16}$$

The theoritical solution is

$$f(x) = 20000e^{\frac{1}{5}\left(\ln\frac{5}{4}\right)x} u(x) \tag{0.17}$$
$$f(x) = 20000\left(\frac{5}{4}\right)^{\frac{x}{5}} u(x) \tag{0.18}$$

Substituting $x = 10$ gives 31250
Computational Solution:
First we have to find the k value in the differential equation, for that

$$y(t+h) = y(t) + hy'(t) \tag{0.19}$$
$$y(t+h) = y(t) + hk_0 y(t) \tag{0.20}$$
$$y(t+2h) = y(t+h) + hk_0 y(t+h) \tag{0.21}$$
$$y(t+2h) = y(t) + hk_0 y(t) + hk_0\left(y(t) + hk_0 y(t)\right) \tag{0.22}$$
$$y(t+2h) = (1 + hk_0)^2\, y(t) \tag{0.23}$$

Similarly

$$y(t+nh) = (1 + hk_0)^n\, y(t) \tag{0.24}$$

Subtituting the initial condition and value of h gives

$$y(5) = (1 + 0.001k_0)^{5000}\, y(0) \tag{0.25}$$

$$25000 = 20000\,(1 + 0.001k_0)^{5000} \tag{0.26}$$

$$\left(\frac{5}{4}\right)^{\frac{1}{5000}} - 1 = 0.001k_0 \tag{0.27}$$

$$k_0 = \frac{\left(\frac{5}{4}\right)^{\frac{1}{5000}} - 1}{0.001} \tag{0.28}$$

$$k_0 \approx 0.04462 \tag{0.29}$$

Consider the given linear differential equation

$$a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y' + a_0 y + c = 0 \tag{0.30}$$

Then

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \tag{0.31}$$

$$y(t+h) = y(t) + hy'(t) \tag{0.32}$$

Similarly

$$y^i(t+h) = y^i(t) + hy^{i+1}(t) \tag{0.33}$$

$$y^{n-1}(t+h) = y^{n-1}(t) + hy^n(t) \tag{0.34}$$

$$y^{n-1}(t+h) = y^{n-1}(t) + h\left(-\frac{a_{n-1}}{a_n}y^{n-1} - \frac{a_{n-2}}{a_n}y^{n-2} - \cdots - \frac{a_0}{a_n}y - \frac{c}{a_n}\right) \tag{0.35}$$

Where i ranges from 0 to $n-1$

$$\mathbf{y}(t+h) = \mathbf{y}(t) + \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -\frac{1}{a_n} & -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \dots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{pmatrix} (h\mathbf{y}(t)) \tag{0.36}$$

$$\mathbf{y}(t+h) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ -\frac{h}{a_n} & -\frac{a_0 h}{a_n} & -\frac{a_1 h}{a_n} & -\frac{a_2 h}{a_n} & \dots & -\frac{a_{n-2} h}{a_n} & 1 - \frac{a_{n-1} h}{a_n} \end{pmatrix} (\mathbf{y}(t)) \tag{0.37}$$

Discretizing the steps gives us

$$\mathbf{y}_{k+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ -\frac{h}{a_n} & -\frac{a_0 h}{a_n} & -\frac{a_1 h}{a_n} & -\frac{a_2 h}{a_n} & \dots & -\frac{a_{n-2} h}{a_n} & 1 - \frac{a_{n-1} h}{a_n} \end{pmatrix} (\mathbf{y}_k) \tag{0.38}$$

where k ranges from 0 to number of data points with y_0^i being the given initial condition and vector $\mathbf{y}_0 = \begin{pmatrix} c \\ y(0) \\ y'(0) \\ \vdots \\ y^{n-1}(0) \end{pmatrix}$

for finding the value of a_0, we wil substitute the initial conditions as in theoritical solution

For the given question

$$\mathbf{y}_{k+1} = \begin{pmatrix} 1 & 0 \\ -h & 1 + hk_0 \end{pmatrix} \mathbf{y}_k \tag{0.39}$$

Record the y_k for

$$x_k = lowerbound + kh \tag{0.40}$$

and then plot the graph. The result will be as given below.

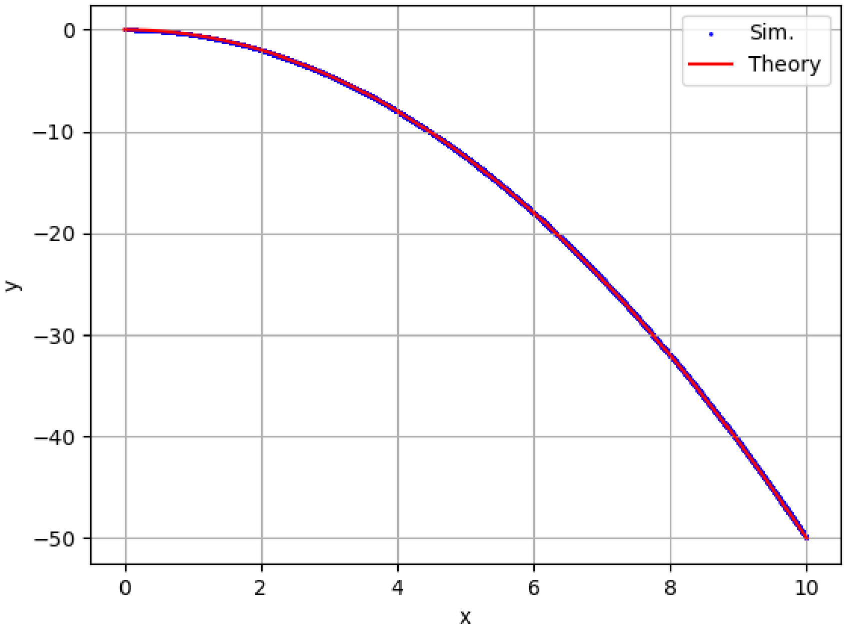

Fig. 0.1: Comparison between the Theoritical solution and Computational solution