10.4.1.1.8 Presentation

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Problem Statement

Find the roots of the equation $x^3 - 4x^2 - x + 1 = (x-2)^3$

Theoritical Solution

The given equation can be simplified to

$$x^3 - 4x^2 - x + 1 = (x-2)^3 \tag{3.1}$$

$$x^3 - 4x^2 - x + 1 = x^3 - 6x^2 + 12x - 8 \tag{3.2}$$

$$2x^2 - 13x + 9 = 0 \tag{3.3}$$

Applying quadratic formula gives solution as

$$x_1 = \frac{13 - \sqrt{97}}{4} \tag{3.4}$$

$$x_2 = \frac{13 + \sqrt{97}}{4} \tag{3.5}$$

Computational Solution

Two methods to find solution of a quadratic equation are:

Matrix-Based Method:

For a polynomial equation of form $x_n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0 = 0$ we construct a matrix called companion matrix of form

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ -b_0 & -b_1 & -b_2 & \dots & -b_{n-1} \end{pmatrix} \tag{3.6}$$

The eigenvalues of this matrix are the roots of the given polynomial equation.

The solution given by the code is

$$x_1 = 0.7878 \tag{3.7}$$

$$x_2 = 5.7122 \tag{3.8}$$

Newton-Raphson Method:
Start with an initial guess x_0, and then run the following logical loop,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3.9}$$

where,

$$f(x) = 2x^2 - 13x + 9 \tag{3.10}$$
$$f'(x) = 4x - 13 \tag{3.11}$$

The update equation will be

$$x_{n+1} = x_n - \frac{2{x_n}^2 - 13x_n + 9}{4x_n - 13} \tag{3.12}$$

$$\tag{3.13}$$

The problem with this method is if the roots are complex but the coeffcients are real, x_n either converges to an extrema or grows continuously without any bound. To get the complex solutions, however , we can just take the initial guess point to be a random complex number. The output of a program written to find roots is shown below:

$$r_1 = 0.7878 \tag{3.14}$$

$$r_2 = 5.7122 \tag{3.15}$$

Companion matrix

$$\Lambda_0 = \begin{pmatrix} 0 & 1 \\ -\frac{9}{2} & \frac{13}{2} \end{pmatrix} \tag{3.16}$$

The update equation will be

$$\Lambda_{n+1} = Q\left(\Lambda_n - \sigma I\right) Q^\top + \sigma I \tag{3.17}$$

$$\tag{3.18}$$

As n tends to infinite, Λ converges to Upper triangular matrix

$$\sigma = \frac{13}{2} \tag{3.19}$$

$$\Lambda_{\text{shifted}} = \begin{pmatrix} 0 & 1 \\ -\frac{9}{2} & \frac{13}{2} \end{pmatrix} - \sigma I \tag{3.20}$$

$$\Lambda_{\text{shifted}} = \begin{pmatrix} \frac{-13}{2} & 1 \\ \frac{-9}{2} & 0 \end{pmatrix} \tag{3.21}$$

Where σ is the last diagonal element.

$$Q = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \tag{3.22}$$

$$\tag{3.23}$$

Where

$$c = \cos\phi \tag{3.24}$$

$$s = \sin\phi \tag{3.25}$$

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \times \begin{pmatrix} \frac{-13}{2} & 1 \\ \frac{-9}{2} & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \tag{3.26}$$

$$c\left(\frac{9}{2}\right) + s\left(\frac{13}{2}\right) = 0 \tag{3.27}$$

$$c^2 + s^2 = 1 \tag{3.28}$$

Solving for c and s gives

$$c = \frac{\frac{-13}{2}}{\sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{9}{2}\right)^2}} \tag{3.29}$$

$$s = \frac{\frac{-9}{2}}{\sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{9}{2}\right)^2}} \tag{3.30}$$

Now, as we got the Q matrix we will do the following

$$\Lambda_{\text{new}} = Q\Lambda_{\text{shifted}}Q^\top + \sigma I \tag{3.31}$$

$$\Lambda_{\text{new}} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \frac{-13}{2} & 1 \\ \frac{-9}{2} & 0 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix} + \sigma I \tag{3.32}$$

$$\Lambda_{\text{new}} \approx \begin{pmatrix} 0.468 & 5.176 \\ -0.324 & 6.032 \end{pmatrix} \tag{3.33}$$

Run the same sequence of steps for 20 iterations after which you end up with the following matrix

$$\Lambda_{\text{new}} = \begin{pmatrix} 0.78778555 & 5.5 \\ 0 & 5.71221445 \end{pmatrix} \tag{3.34}$$

The eigenvalues are same as its diagonal elements
Hence the roots of given equation are 0.78778555 and 5.71221445

For any general matrix, we first do hessenberg reduction and then convert into QR form. We do the following
For a general matrix, we should bring to hessenberg form H, for that let

$$P = I - 2\mathbf{w}\mathbf{w}^\top \tag{3.35}$$

where w is a vector with $|w|^2 = 1$. The matrix P is orthogonal as

$$\begin{aligned} P^2 &= \left(I - 2\mathbf{w}\mathbf{w}^\top\right) \cdot \left(I - 2\mathbf{w}\mathbf{w}^\top\right) && (3.36) \\ &= I - 4\mathbf{w}\mathbf{w}^\top + 4\mathbf{w} \cdot \left(\mathbf{w}^\top \mathbf{w}^\top\right) \cdot \mathbf{w}^\top && (3.37) \\ &= I && (3.38) \end{aligned}$$

Therefore, $P = P^{-1}$ but $P = P^\top$, so $P = P^\top$
We can rewrite P as

$$P = I - \frac{\mathbf{u}\mathbf{u}^\top}{H} \tag{3.39}$$

where the scalar H is

$$H = \frac{1}{2} |\mathbf{u}|^2 \tag{3.40}$$

Where $\mathbf{u}$ can be any vector. Suppose $\mathbf{x}$ is the vector composed of the first column of A. Take

$$\mathbf{u} = \mathbf{x} \mp |\mathbf{x}|\, \mathbf{e}_1 \tag{3.41}$$

Where $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 & \ldots \end{pmatrix}^\top$, we will take the choice of sign later. Then

$$\begin{aligned} P \cdot \mathbf{x} &= \mathbf{x} - \frac{\mathbf{u}}{H} \cdot \left(\mathbf{u} \mp |\mathbf{x}|\, \mathbf{e}_1\right)^\top \cdot \mathbf{x} && (3.42) \\ &= \mathbf{x} - \frac{2\mathbf{u}\left(|x|^2 \mp |x|\, x_1\right)}{2\,|x|^2 \mp |x|\, x_1} && (3.43) \\ &= \mathbf{x} - \mathbf{u} && (3.44) \\ &= \mp |\mathbf{x}|\, \mathbf{e}_1 && (3.45) \end{aligned}$$

To reduce a matrix A into Hessenberg form, we choose vector $\mathbf{x}$ for the first householder matrix to be lower $n-1$ elements of the first column, then the lower $n-2$ elements will be zeroed.

$$P_1 \cdot A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & p_{11} & p_{12} & \cdots & p_{1n} \\ 0 & p_{21} & p_{22} & \cdots & p_{2n} \\ 0 & p_{31} & p_{32} & \cdots & p_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} a_{00} & \times & \times & \cdots & \times \\ a_{10} & \times & \times & \cdots & \times \\ a_{20} & \times & \times & \cdots & \times \\ a_{30} & \times & \times & \cdots & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0} & \times & \times & \cdots & \times \end{pmatrix} \tag{3.46}$$

$$= \begin{pmatrix} a'_{00} & \times & \times & \cdots & \times \\ 0 & \times & \times & \cdots & \times \\ 0 & \times & \times & \cdots & \times \\ 0 & \times & \times & \cdots & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \times & \times & \cdots & \times \end{pmatrix} \tag{3.47}$$

Now we choose the vector $\mathbf{x}$ for the householder matrix to be the bottom $n-2$ elements of the second column, and from it construct the P_2

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & p_{22} & \cdots & p_{2n} \\ 0 & 0 & p_{32} & \cdots & p_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & p_{n2} & \cdots & p_{nn} \end{pmatrix} \tag{3.48}$$

Continue the pattern

Let H be the Hessenberg matrix. To find Q, we do the following, let

$$G = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ 0 & \cdots & c & s & \cdots & 0 \\ 0 & \cdots & -s & c & \cdots & 0 \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{3.49}$$

Where the value of c and s are

$$c = \frac{\overline{x_{i,i}}}{\sqrt{|x_{i,i}|^2 + |x_{i,i+1}|^2}} \tag{3.50}$$

$$s = \frac{\overline{x_{i,i+1}}}{\sqrt{|x_{i,i}|^2 + |x_{i,i+1}|^2}} \tag{3.51}$$

Multiplying G and A nulls out the element in $(i+1)^{\text{th}}$ row and i^{th} column. Now

$$Q = G_{n-1} G_{n-2} \cdots G_2 G_1 \tag{3.52}$$

QR decomposition:

Given H_0

$$H_n = Q_n R_n \tag{3.53}$$

$$Q^{\top} H_n = R_n \tag{3.54}$$

$$H_{n+1} = R_n Q_n \tag{3.55}$$

The update equation will be

$$H_{n+1} = Q^{\top} H_n Q \tag{3.56}$$

As n tends to infinite, H will converge to upper triangular matrix, whose eigenvalues are the roots of the equation
QR decomposition with Shift:
Given H_0

$$H_n - \sigma I = Q_n R_n \tag{3.57}$$

$$Q^{\top} (H_n - \sigma I) = R_n \tag{3.58}$$

$$H_{n+1} = R_n Q_n + \sigma I \tag{3.59}$$

The update equation will be

$$H_{n+1} = Q^{\top} (H_n - \sigma I) Q_n + \sigma I \tag{3.60}$$

Where σ is the last diagonal element As n tends to infinite, H will converge to upper triangular matrix faster than QR decomposition

Plot

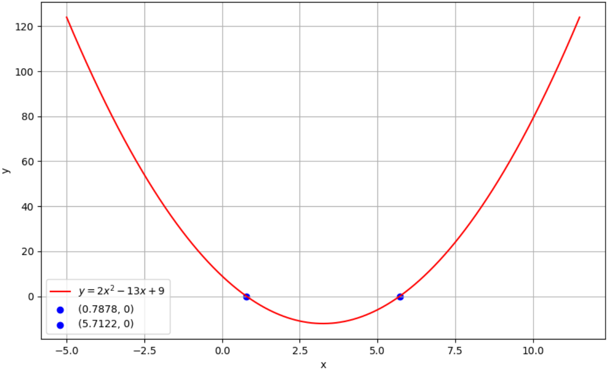

Figure: Graph of $y = 2x^2 - 13x + 2$