

12.9.3.11 Presentation

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- Laplace Transform properties
- Equation solving
- Computational Solution

3 Plot of the function

Problem Statement

Solve the differential equation $\frac{d^2y}{dx^2} = y$ with initial conditions $y(0) = 1$ and $y'(0) = 0$

Laplace Transform properties

Properties of Laplace transform

$$\mathcal{L}(y'') = s^2 \mathcal{L}(y) - sy(0) - y'(0) \quad (3.1)$$

$$\mathcal{L}(1) = \frac{1}{s} \quad (3.2)$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \quad (3.3)$$

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}(e^{at}f(t)) = F(s-a) \quad (3.4)$$

Equation solving

Applying the properties to the given equation

$$y'' - y = 0 \quad (3.5)$$

$$\mathcal{L}(y'') - \mathcal{L}(y) = 0 \quad (3.6)$$

$$s^2 \mathcal{L}(y) - sy(0) - y'(0) - \mathcal{L}(y) = 0 \quad (3.7)$$

$$(3.8)$$

Substituting the initial conditions gives

$$(s^2 - 1) \mathcal{L}(y) = s \quad (3.9)$$

$$\mathcal{L}(y) = \frac{s}{s^2 - 1} \quad (3.10)$$

$$\mathcal{L}(y) = \frac{1}{2(s+1)} + \frac{1}{2(s-1)} \quad (3.11)$$

$$y = \frac{1}{2} \left(\mathcal{L}^{-1} \left(\frac{1}{s+1} \right) + \mathcal{L}^{-1} \left(\frac{1}{s-1} \right) \right) \quad (3.12)$$

$$y = \frac{1}{2} (e^{-x} + e^x) u(x) \quad (3.13)$$

With Radius of convergence of being $\operatorname{Re}(s) > \operatorname{Re}(-1)$ and $\operatorname{Re}(s) > \operatorname{Re}(1)$ which is $\operatorname{Re}(s) > 1$.

The theoretical solution is

$$f(x) = \frac{1}{2} (e^{-x} + e^x) u(x) \quad (3.14)$$

Bilinear Transform

We can arrive at a difference equation by applying Bilinear Z-transform on the Laplace equations. Take

$$s = \frac{2}{T} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) \quad (3.15)$$

Here $T = h$

$$Y(z) = \frac{1}{2} \left(\frac{1}{\frac{2}{T} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) + 1} + \frac{1}{\frac{2}{T} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) - 1} \right) \quad (3.16)$$

$$Y(z) = \frac{1}{2} \left(\frac{T(1 + z^{-1})}{2(1 - z^{-1}) + T(1 + z^{-1})} + \frac{T(1 + z^{-1})}{2(1 - z^{-1}) - T(1 + z^{-1})} \right) \quad (3.17)$$

$$Y(z) = \frac{1}{2} \left(\frac{T(1 + z^{-1})}{(T - 2)z^{-1} + (T + 2)} - \frac{T(1 + z^{-1})}{(T + 2)z^{-1} + (T - 2)} \right) \quad (3.18)$$

$$Y(z) = \frac{T}{2(T+2)} \left(\frac{1}{1-\alpha_1 z^{-1}} + \frac{z^{-1}}{1-\alpha_1 z^{-1}} \right) - \frac{T}{2(T-2)} \left(\frac{1}{1-\alpha_2 z^{-1}} + \frac{z^{-1}}{1-\alpha_2 z^{-1}} \right) \quad (3.19)$$

Where $\alpha_1 = -\frac{T-2}{T+2}$, $\alpha_2 = -\frac{T+2}{T-2}$

The radius of convergence of $\frac{1}{1-\alpha_1 z^{-1}}$ and $\frac{z^{-1}}{1-\alpha_1 z^{-1}}$ is $|z| > |\alpha_1|$ and radius of convergence of $\frac{1}{1-\alpha_2 z^{-1}}$ and $\frac{z^{-1}}{1-\alpha_2 z^{-1}}$ is $|z| > |\alpha_2|$

The radius of convergence of $Y(z)$ is $\max(|\alpha_1|, |\alpha_2|)$

Applying the inverse Z-transform with some rearrangement gives

$$(1 - (\alpha_1 + \alpha_2)z^{-1} + z^{-2}) Y(z) = \frac{T}{2} \left(\frac{1 - (\alpha_2 - 1)z^{-1} - \alpha_2 z^{-2}}{T+2} \right) - \frac{T}{2} \left(\frac{1 - (\alpha_1 - 1)z^{-1} - \alpha_1 z^{-2}}{T-2} \right) \quad (3.20)$$

$$z^2 Y(z) - z(\alpha_1 + \alpha_2) Y(z) + Y(z) - z^2 y[0] - zy[1] - z(\alpha_1 + \alpha_2)y[0]$$

$$\begin{aligned}
+z^2 y[0] + zy[1] + z(\alpha_1 + \alpha_2)y[0] &= \frac{T}{2} \left(\frac{z^2 - (\alpha_2 - 1)z - \alpha_2}{T + 2} \right) \\
&\quad - \frac{T}{2} \left(\frac{z^2 - (\alpha_1 - 1)z - \alpha_1}{T - 2} \right)
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
y_{n+2} - (\alpha_1 + \alpha_2)y_{n+1} + y_n + \delta[n+2]y[0] + \delta[n+1](y[1] + (\alpha_1 + \alpha_2)y[0]) \\
= \frac{T}{2} \left(\delta[n+2] \left(\frac{1}{T+2} - \frac{1}{T-2} \right) - \delta[n+1] \left(\frac{\alpha_1 - 1}{T-2} + \frac{\alpha_2 - 1}{T+2} \right) \right) \\
\quad + \frac{T}{2} \left(\delta[n] \left(\frac{\alpha_1}{T-2} - \frac{\alpha_2}{T+2} \right) \right)
\end{aligned} \tag{3.22}$$

Since $n \geq 0$, $\delta[n+2] = 0$ and $\delta[n+1] = 0$

$$y_{n+2} - (\alpha_1 + \alpha_2)y_{n+1} + y_n = \frac{T}{2} \left(\frac{\alpha_1}{T-2} - \frac{\alpha_2}{T+2} \right) \delta[n] \tag{3.23}$$

As $y[0] = 1$ from initial condition and

$$y[1] = y(0) + hy'(0) \quad (3.24)$$

$$(3.25)$$

Hence $n \geq 2$ which gives the difference equation as

$$y_{n+2} = (\alpha_1 + \alpha_2) y_{n+1} - y_n \quad (3.26)$$

The given differential equation is

$$y'' - y = 0 \quad (3.27)$$

Let

$$y' = y_1 \quad (3.28)$$

$$y = y_2 \quad (3.29)$$

Then

$$\frac{dy_1}{dx} = y_2 \quad (3.30)$$

$$\frac{dy_2}{dx} = y_1 \quad (3.31)$$

$$\int_{y_{1,k}}^{y_{1,k+1}} dy_1 = \int_{x_k}^{x_{k+1}} y_2 dx \quad (3.32)$$

$$\int_{y_{2,k}}^{y_{2,k+1}} dy_2 = \int_{x_k}^{x_{k+1}} y_1 dx \quad (3.33)$$

Discretizing the steps using trapezoidal rule gives us

$$y_{1,k+1} - y_{1,k} = \frac{h}{2} (y_{2,k} + y_{2,k+1}) \quad (3.34)$$

$$y_{2,k+1} - y_{2,k} = \frac{h}{2} (y_{1,k} + y_{1,k+1}) \quad (3.35)$$

Then solving for $y_{1,k+1}$ and $y_{2,k+1}$ in terms of $y_{1,k}$, $y_{2,k}$ and h will help us to calculate the value of function at x_{k+1}

$$y_{1,k+1} = y_{1,k} + \frac{h}{2} \left(y_{2,k} + \left(y_{2,k} + \frac{h}{2} (y_{1,k} + y_{1,k+1}) \right) \right) \quad (3.36)$$

$$y_{1,k+1} = y_{1,k} \left(1 + \frac{h^2}{4} \right) + y_{2,k} h + y_{1,k+1} \left(\frac{h^2}{4} \right) \quad (3.37)$$

$$y_{1,k+1} \left(1 - \frac{h^2}{4} \right) = y_{1,k} \left(1 + \frac{h^2}{4} \right) + y_{2,k} h \quad (3.38)$$

$$y_{1,k+1} = \frac{(y_{1,k}) (4 + h^2) + 4h (y_{2,k})}{4 - h^2} \quad (3.39)$$

Similarly

$$y_{2,k+1} = \frac{(y_{2,k}) (4 + h^2) + 4h (y_{1,k})}{4 - h^2} \quad (3.40)$$

The difference equations are

$$y_{1,k+1} = \frac{(y_{1,k}) (4 + h^2) + 4h (y_{2,k})}{4 - h^2} \quad (3.41)$$

$$y_{2,k+1} = \frac{(y_{2,k}) (4 + h^2) + 4h (y_{1,k})}{4 - h^2} \quad (3.42)$$

Using the above formula, recording the value of y at each value of $x_k = x_0 + kh$ and taking $y(0) = 1$ and $y'(0) = 0$ and plotting gives

Plot of the function

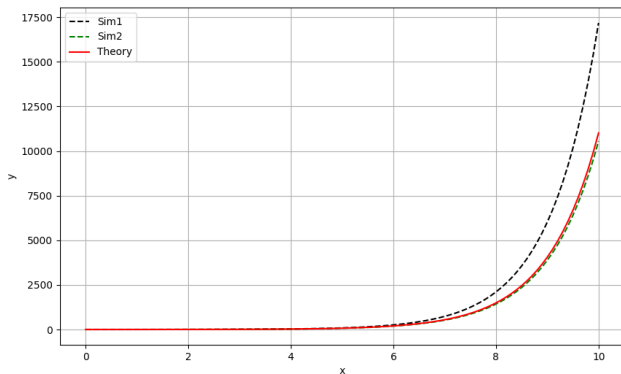


Figure: Comparison between the Theoretical solution and Computational solutions, Red is theory, Black line is derived from difference equation from Trapezoidal method, while Green line is from Z-transform while taking stepsize to be 0.1