

# 9.3.11

EE24BTECH11024 - G. Abhimanyu Koushik

## Question:

Solve the differential equation  $\frac{d^2y}{dx^2} + 1 = 0$  with initial conditions  $y(0) = 0$  and  $y'(0) = 0$

## Solution:

Variable	Description
$c_1$	First Integration constant
$c_2$	Second Integration constant
$n$	Order of given differential equation
$a_i$	Coefficient of $i$ th derivative of the function in the equation
$c$	constant in the equation
$y^i$	$i$ th derivative of given function
$\mathbf{y}(t)$	$\begin{pmatrix} c \\ y(t) \\ y'(t) \\ \vdots \\ y^{n-1}(t) \end{pmatrix}$
$h$	stepsize, taken to be 0.001

TABLE 0: Variables Used

## Theoretical Solution:

$$\frac{d^2y}{dx^2} + 1 = 0 \quad (0.1)$$

$$\frac{d^2y}{dx^2} = -1 \quad (0.2)$$

$$\int \frac{d^2y}{dx^2} dx = \int -1 dx \quad (0.3)$$

$$\frac{dy}{dx} = -x + c_1 \quad (0.4)$$

$$\int \frac{dy}{dx} dx = \int (-x + c_1) dx \quad (0.5)$$

$$y = \frac{-x^2}{2} + c_1 x + c_2 \quad (0.6)$$

Substituting the initial conditions gives

$$y(0) = 0 \implies \frac{-0^2}{2} + c_1 0 + c_2 = 0 \quad (0.7)$$

$$c_2 = 0 \quad (0.8)$$

$$y'(0) = 0 \implies -0 + c_1 = 0 \quad (0.9)$$

$$c_1 = 0 \quad (0.10)$$

The theoretical solution is

$$f(x) = \frac{-x^2}{2} \quad (0.11)$$

Computational Solution:

Consider the given linear differential equation

$$a_n y^n + a_{n-1} y^{n-1} + \dots + a_1 y' + a_0 y + c = 0 \quad (0.12)$$

Then

$$y'(t) = \lim_{h \rightarrow 0} \frac{y(t+h) - y(t)}{h} \quad (0.13)$$

$$y(t+h) = y(t) + h y'(t) \quad (0.14)$$

Similarly

$$y^i(t+h) = y^i(t) + h y^{i+1}(t) \quad (0.15)$$

$$y^{n-1}(t+h) = y^{n-1}(t) + h y^n(t) \quad (0.16)$$

$$y^{n-1}(t+h) = y^{n-1}(t) + h \left( -\frac{a_{n-1}}{a_n} y^{n-1} - \frac{a_{n-2}}{a_n} y^{n-2} - \dots - \frac{a_0}{a_n} y - \frac{c}{a_n} \right) \quad (0.17)$$

Where i ranges from 0 to  $n-1$

$$\mathbf{y}(t+h) = \mathbf{y}(t) + \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -\frac{1}{a_n} & -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \dots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{pmatrix} (h \mathbf{y}'(t)) \quad (0.18)$$

$$\mathbf{y}(t+h) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ -\frac{h}{a_n} & -\frac{a_0 h}{a_n} & -\frac{a_1 h}{a_n} & -\frac{a_2 h}{a_n} & \dots & -\frac{a_{n-2} h}{a_n} & 1 - \frac{a_{n-1} h}{a_n} \end{pmatrix} (\mathbf{y}(t)) \quad (0.19)$$

Discretizing the steps gives us

$$\mathbf{y}_{k+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ -\frac{h}{a_n} & -\frac{a_0 h}{a_n} & -\frac{a_1 h}{a_n} & -\frac{a_2 h}{a_n} & \dots & -\frac{a_{n-2} h}{a_n} & 1 - \frac{a_{n-1} h}{a_n} \end{pmatrix} (\mathbf{y}_k) \quad (0.20)$$

where  $k$  ranges from 0 to number of data points with  $y_0^i$  being the given initial condition. For the given question

$$\mathbf{y}_{k+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & h \\ -h & 0 & 1 \end{pmatrix} \mathbf{y}_k \quad (0.21)$$

Record the  $y_k$  for

$$x_k = \text{lowerbound} + kh \quad (0.22)$$

and then plot the graph. The result will be as given below.

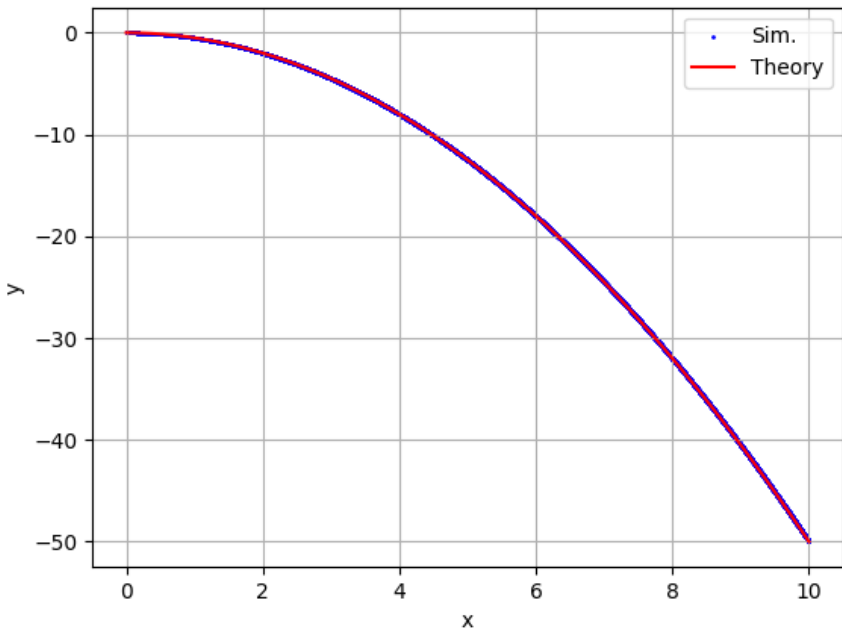


Fig. 0.1: Comparison between the Theoretical solution and Computational solution