

12.6.5.11

EE24BTECH11024 - G. Abhimanyu Koushik

Question:

It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval $[0, 2]$. Find the value of a .

Solution:

Theoretical Solution:

Given function

$$f(x) = x^4 - 62x^2 + ax + 9 \quad (0.1)$$

If a function has a local maxima at $x = x_0$ then $f'(x_0) = 0$ and $f''(x_0) \leq 0$

$$f'(x) = 4x^3 - 124x + a \quad (0.2)$$

$$f''(x) = 12x^2 - 124 \quad (0.3)$$

$$f'(1) = 4 - 124 + a \quad (0.4)$$

$$f''(1) = 12 - 124 \quad (0.5)$$

$$(0.6)$$

$f''(1)$ is anyway negative so $f'(1)$ should be 0

$$a - 120 = 0 \quad (0.7)$$

$$a = 120 \quad (0.8)$$

When $a = 120$, it satisfies all the conditions.

Computational Solution:

Given function

$$f(x) = x^4 - 62x^2 + ax + 9 \quad (0.9)$$

At any critical point $x = x_0$, $(f'(x_0))^2$ is minimum. We need to minimize

$$L(a) = (f'(1))^2 \quad (0.10)$$

$$L(a) = (a - 120)^2 \quad (0.11)$$

We use the method of gradient descent to find the minimum of the above function, since the objective function is convex.

$$a_{n+1} = a_n - \mu L'(a_n) \quad (0.12)$$

$$L'(a_n) = 2(a_n - 120) \quad (0.13)$$

$$\rightarrow a_{n+1} = a_n - 2\mu(a_n - 120) \quad (0.14)$$

Applying unilateral Z-transform,

$$zX(z) - zx_0 = (1 - 2\mu)X(z) + \frac{240\mu}{1 - z^{-1}} \quad (0.15)$$

The Unilateral Z-transform of a constant function $f(x) = c$ is $\frac{c}{1 - z^{-1}}$ with Radius of convergence being $|z| > 1$

$$(z - (1 - 2\mu))X(z) = zx_0 + \frac{240\mu}{1 - z^{-1}} \quad (0.16)$$

$$X(z) = \frac{zx_0}{z - (1 - 2\mu)} + \frac{240\mu}{(1 - z^{-1})(z - (1 - 2\mu))} \quad (0.17)$$

$$X(z) = \frac{x_0}{1 - (1 - 2\mu)z^{-1}} + \frac{240\mu}{z - 2(1 - \mu) + (1 - 2\mu)z^{-1}} \quad (0.18)$$

$$X(z) = \frac{x_0}{1 - (1 - 2\mu)z^{-1}} + \frac{240\mu z^{-1}}{1 - 2(1 - \mu)z^{-1} + (1 - 2\mu)z^{-2}} \quad (0.19)$$

$$= \sum_{n=0}^{\infty} (1 - 2\mu)^n z^{-n} + \sum_{n=1}^{\infty} \frac{1 - (1 - 2\mu)^n}{2\mu} z^{-n} \quad (0.20)$$

$$= \sum_{n=0}^{\infty} (1 - 2\mu)^n z^{-n} + \sum_{n=1}^{\infty} \frac{1}{2\mu} z^{-n} - \sum_{n=1}^{\infty} \frac{(1 - 2\mu)^n}{2\mu} z^{-n} \quad (0.21)$$

$$(0.22)$$

From the last equation, ROC is

$$|z| > |1 - 2\mu| \quad (0.23)$$

$$|z| > 1 \quad (0.24)$$

$$\rightarrow 0 < |1 - 2\mu| < 1 \quad (0.25)$$

$$\rightarrow \mu \in \left(0, \frac{1}{2}\right) \quad (0.26)$$

Now, if μ satisfies the previous condition,

$$\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = 0 \quad (0.27)$$

$$\rightarrow \lim_{n \rightarrow \infty} (-2\mu(a_n - 120)) = 0 \quad (0.28)$$

$$\rightarrow -2\mu \lim_{n \rightarrow \infty} (a_n - 120) = 0 \quad (0.29)$$

$$\rightarrow \lim_{n \rightarrow \infty} (a_n) = 120 \quad (0.30)$$

$$\rightarrow \lim_{n \rightarrow \infty} a_n = 120 \quad (0.31)$$

Taking initial guess = 50

step size = 0.1

tolerance(minimum value of gradient) = 1e-5

We get

$a_{min} = 119.99999528200934$

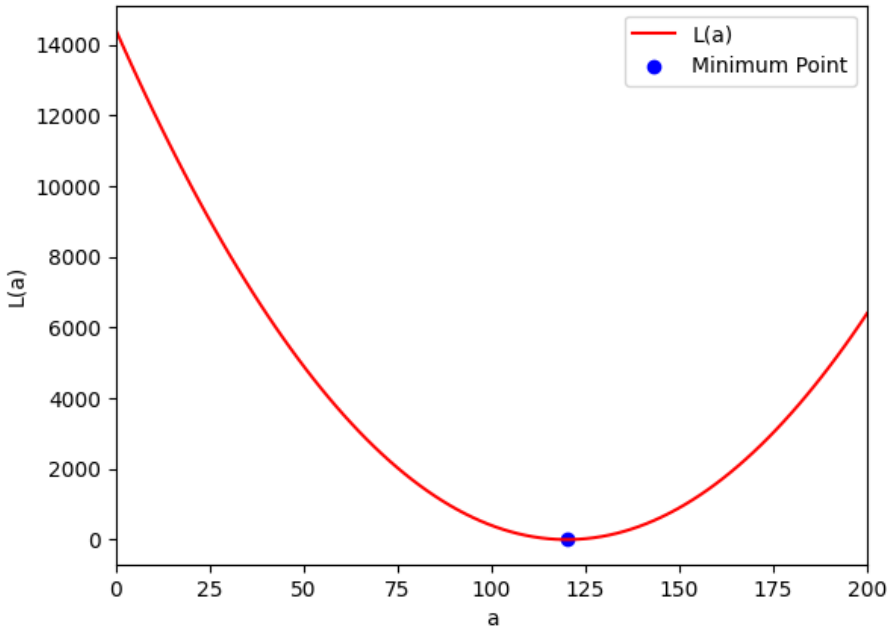


Fig. 0.1: Graph of $L(a) = (a - 120)^2$ and the value of a which satisfies the given conditions

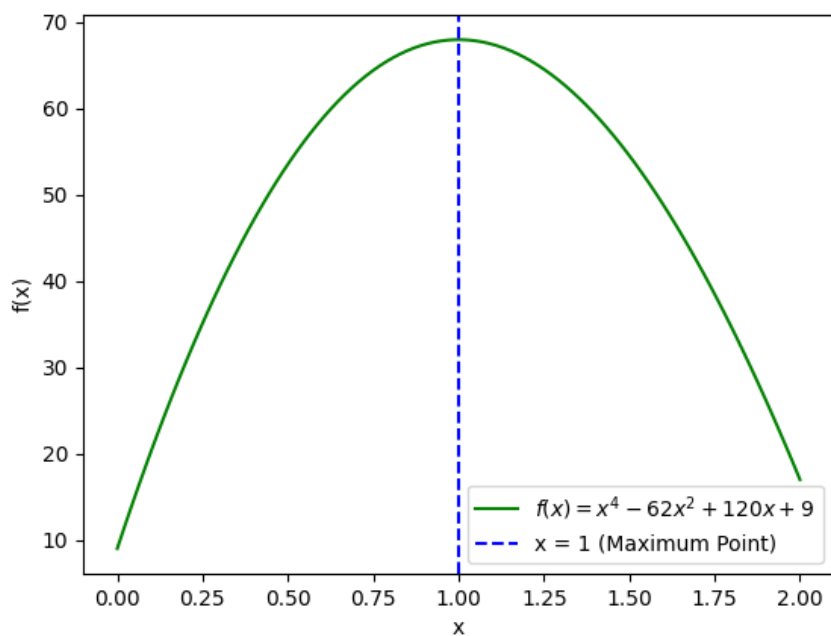


Fig. 0.2: Graph of $f(x)$ when $a = 120$