# 12.9.3.11 Presentation

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# Problem Statement

Solve the differential equation $\frac{d^2y}{dx^2} = y$ with initial conditions $y(0) = 1$ and $y'(0) = 0$

# Laplace Transform properties

Properties of Laplace tranform

$$\mathcal{L}\left(y''\right) = s^2\mathcal{L}(y) - sy(0) - y'(0) \tag{3.1}$$

$$\mathcal{L}(1) = \frac{1}{s} \tag{3.2}$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{3.3}$$

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}\left(e^{at}f(t)\right) = F(s-a) \tag{3.4}$$

# Equation solving

Applying the properties to the given equation

$$y'' - y = 0 \tag{3.5}$$

$$\mathcal{L}\left(y''\right) - \mathcal{L}\left(y\right) = 0 \tag{3.6}$$

$$s^2\mathcal{L}\left(y\right) - sy\left(0\right) - y'\left(0\right) - \mathcal{L}\left(y\right) = 0 \tag{3.7}$$

Substituting the initial conditions gives

$$\left(s^2 - 1\right) \mathcal{L}\left(y\right) = s \tag{3.8}$$

$$\mathcal{L}\left(y\right) = \frac{s}{s^2 - 1} \tag{3.9}$$

$$\mathcal{L}\left(y\right) = \frac{1}{2\left(s+1\right)} + \frac{1}{2\left(s-1\right)} \tag{3.10}$$

$$y = \frac{1}{2}\left(\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)\right) \tag{3.11}$$

$$y = \frac{1}{2}\left(e^{-x} + e^{x}\right) u\left(x\right) \tag{3.12}$$

With Radius of convergence of being $Re\left(s\right) > Re\left(-1\right)$ and $Re\left(s\right) > Re\left(1\right)$ which is $Re\left(s\right) > 1$.
The theoritical solution is

$$f\left(x\right) = \frac{1}{2}\left(e^{-x} + e^{x}\right) u\left(x\right) \tag{3.13}$$

# Bilinear Transform

We can arrive at a difference equation by applying Bilinear Z-transform on the Laplace equations. Take

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{3.14}$$

Here $T = h$

$$Y(z) = \frac{1}{2}\left(\frac{1}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1} + \frac{1}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)-1}\right) \tag{3.15}$$

$$Y(z) = \frac{1}{2}\left(\frac{T\left(1+z^{-1}\right)}{2\left(1-z^{-1}\right)+T\left(1+z^{-1}\right)} + \frac{T\left(1+z^{-1}\right)}{2\left(1-z^{-1}\right)-T\left(1+z^{-1}\right)}\right) \tag{3.16}$$

$$Y(z) = \frac{1}{2}\left(\frac{T\left(1+z^{-1}\right)}{(T-2)\,z^{-1}+(T+2)} - \frac{T\left(1+z^{-1}\right)}{(T+2)\,z^{-1}+(T-2)}\right) \tag{3.17}$$

$$Y(z) = \frac{T}{2(T+2)}\left(\frac{1}{1-\alpha_1 z^{-1}} + \frac{z^{-1}}{1-\alpha_1 z^{-1}}\right) - \frac{T}{2(T-2)}\left(\frac{1}{1-\alpha_2 z^{-1}} + \frac{z^{-1}}{1-\alpha_2 z^{-1}}\right) \tag{3.18}$$

Where $\alpha_1 = -\frac{T-2}{T+2}$, $\alpha_2 = -\frac{T+2}{T-2}$

The radius of convergence of $\frac{1}{1-\alpha_1 z^{-1}}$ and $\frac{z^{-1}}{1-\alpha_1 z^{-1}}$ is $|z| > |\alpha_1|$ and radius of convergence of $\frac{1}{1-\alpha_2 z^{-1}}$ and $\frac{z^{-1}}{1-\alpha_2 z^{-1}}$ is $|z| > |\alpha_2|$

The radius of convergence of $Y(z)$ is $max(|\alpha_1|, |\alpha_2|)$

Applying the inverse Z-transform with some rearrangement gives

$$\left(1-(\alpha_1+\alpha_2)z^{-1}+z^{-2}\right)Y(z) = \frac{T}{2}\left(\frac{1-(\alpha_2-1)z^{-1}-\alpha_2 z^{-2}}{T+2}\right) - \frac{T}{2}\left(\frac{1-(\alpha_1-1)z^{-1}-\alpha_1 z^{-2}}{T-2}\right) \tag{3.19}$$

$$z^2Y(z) - z(\alpha_1+\alpha_2)Y(z) + Y(z) - z^2y[0] - zy[1] - z(\alpha_1+\alpha_2)y[0]$$

$$+z^2 y[0] + zy[1] + z(\alpha_1+\alpha_2)\, y[0] = \frac{T}{2}\left(\frac{z^2-(\alpha_2-1)z-\alpha_2}{T+2}\right) - \frac{T}{2}\left(\frac{z^2-(\alpha_1-1)z-\alpha_1}{T-2}\right) \tag{3.20}$$

$$y_{n+2} - (\alpha_1+\alpha_2)\, y_{n+1} + y_n + \delta[n+2]\, y[0] + \delta[n+1]\left(y[1] + (\alpha_1+\alpha_2)\, y[0$$

$$= \frac{T}{2}\left(\delta[n+2]\left(\frac{1}{T+2}-\frac{1}{T-2}\right) - \delta[n+1]\left(\frac{\alpha_1-1}{T-2}+\frac{\alpha_2-1}{T+2}\right)\right) + \frac{T}{2}\left(\delta[n]\left(\frac{\alpha_1}{T-2}-\frac{\alpha_2}{T+2}\right)\right) \tag{3.21}$$

Since $n \geq 0$, $\delta[n+2] = 0$ and $\delta[n+1] = 0$

$$y_{n+2} - (\alpha_1+\alpha_2)\, y_{n+1} + y_n = \frac{T}{2}\left(\frac{\alpha_1}{T-2}-\frac{\alpha_2}{T+2}\right)\delta[n] \tag{3.22}$$

As $y[0] = 1$ from initial condition and

$$y[1] = y(0) + hy'(0) \tag{3.23}$$

$$\tag{3.24}$$

Hence $n \geq 2$ which gives the difference equation as

$$y_{n+2} = (\alpha_1 + \alpha_2)\, y_{n+1} - y_n \tag{3.25}$$

The given differential equation is

$$y'' - y = 0 \tag{3.26}$$

Let

$$y' = y_1 \tag{3.27}$$
$$y = y_2 \tag{3.28}$$

Then

$$\frac{dy_1}{dx} = y_2 \tag{3.29}$$
$$\frac{dy_2}{dx} = y_1 \tag{3.30}$$
$$\int_{y_{1,k}}^{y_{1,k+1}} dy_1 = \int_{x_k}^{x_{k+1}} y_2 dx \tag{3.31}$$
$$\int_{y_{2,k}}^{y_{2,k+1}} dy_2 = \int_{x_k}^{x_{k+1}} y_1 dx \tag{3.32}$$

Discretizing the steps using trapezoidal rule gives us

$$y_{1,k+1} - y_{1,k} = \frac{h}{2}\left(y_{2,k} + y_{2,k+1}\right) \tag{3.33}$$

$$y_{2,k+1} - y_{2,k} = \frac{h}{2}\left(y_{1,k} + y_{1,k+1}\right) \tag{3.34}$$

Then solving for $y_{1,k+1}$ and $y_{2,k+1}$ in terms of $y_{1,k}$, $y_{2,k}$ and $h$ will help us to calculate the value of function at $x_{k+1}$

$$y_{1,k+1} = y_{1,k} + \frac{h}{2}\left(y_{2,k} + \left(y_{2,k} + \frac{h}{2}\left(y_{1,k} + y_{1,k+1}\right)\right)\right) \tag{3.35}$$

$$y_{1,k+1} = y_{1,k}\left(1 + \frac{h^2}{4}\right) + y_{2,k}h + y_{1,k+1}\left(\frac{h^2}{4}\right) \tag{3.36}$$

$$y_{1,k+1}\left(1 - \frac{h^2}{4}\right) = y_{1,k}\left(1 + \frac{h^2}{4}\right) + y_{2,k}h \tag{3.37}$$

$$y_{1,k+1} = \frac{\left(y_{1,k}\right)\left(4 + h^2\right) + 4h\left(y_{2,k}\right)}{4 - h^2} \tag{3.38}$$

Similarly

$$y_{2,k+1} = \frac{(y_{2,k})\left(4+h^2\right)+4h\,(y_{1,k})}{4-h^2} \tag{3.39}$$

The difference equations are

$$y_{1,k+1} = \frac{(y_{1,k})\left(4+h^2\right)+4h\,(y_{2,k})}{4-h^2} \tag{3.40}$$

$$y_{2,k+1} = \frac{(y_{2,k})\left(4+h^2\right)+4h\,(y_{1,k})}{4-h^2} \tag{3.41}$$

Using the above formula, recording the value of $y$ at each value of $x_k = x_0 + kh$ and taking $y(0) = 1$ and $y'(0) = 0$ and plotting gives

# Plot of the function

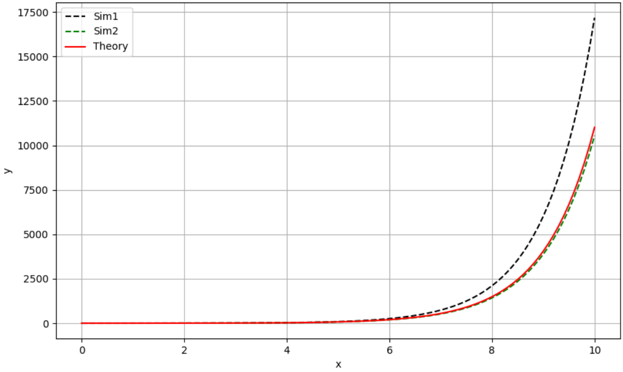


Figure: Comparison between the Theoritical solution and Computational solutions, Red is theory, Black line is derived from difference equation from Trapezoidal method, while Green line is from Z-transform while taking stepsize to be 0.1