9.9.2.28

EE24BTECH11024 - Abhimanyu Koushik

Question:
Find the area of the region $\left\{(x, y) : x^2 \leq y \leq x\right\}$, using integration.
Solution:

Variable	Description
$\mathbf{x_1}$	First intersection point
$\mathbf{x_2}$	Second intersection point
$\mathbf{h}$	Point on the given line
$\mathbf{m}$	Direction vector of given line
A_1	Area under the curve $y = x^2$
A_2	Area under the line $y = x$
A	Area of the region between $y = x^2$ and $y = x$

TABLE I: Variables Used

The equation of a parabola in Matrix form is

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

The equation of a line in vector form is

$$\mathbf{x} = \mathbf{h} + \kappa \mathbf{m} \tag{2}$$

For the given parabola $y = x^2$, The values of $\mathbf{V}$,$\mathbf{u}$,f are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix} \tag{4}$$

$$f = 0 \tag{5}$$

For the given line $x = y$, The values of $\mathbf{h}$, $\mathbf{m}$ are

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7}$$

Substituing the line equation in parabola equation gives the values of κ

$$(\mathbf{h}+\kappa\mathbf{m})^\top \mathbf{V}(\mathbf{h}+\kappa\mathbf{m})+2\mathbf{u}^\top(\mathbf{h}+\kappa\mathbf{m})+f=0 \tag{8}$$

$$\left(\begin{pmatrix}0\\0\end{pmatrix}+\kappa\begin{pmatrix}1\\1\end{pmatrix}\right)^\top\begin{pmatrix}1&0\\0&0\end{pmatrix}\left(\begin{pmatrix}0\\0\end{pmatrix}+\kappa\begin{pmatrix}1\\1\end{pmatrix}\right)+2\begin{pmatrix}0\\\frac{-1}{2}\end{pmatrix}^\top\left(\begin{pmatrix}0\\0\end{pmatrix}+\kappa\begin{pmatrix}1\\1\end{pmatrix}\right)+0=0 \tag{9}$$

$$\begin{pmatrix}\kappa&\kappa\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}\kappa\\\kappa\end{pmatrix}+2\begin{pmatrix}0&\frac{-1}{2}\end{pmatrix}\begin{pmatrix}\kappa\\\kappa\end{pmatrix}=0 \tag{10}$$

$$\begin{pmatrix}\kappa&\kappa\end{pmatrix}\begin{pmatrix}\kappa\\0\end{pmatrix}-(\kappa)=0 \tag{11}$$

$$\kappa^2-\kappa=0 \tag{12}$$

$$\kappa_1=0 \tag{13}$$

$$\kappa_2=1 \tag{14}$$

The intersection points are

$$\mathbf{x_1}=\mathbf{h}+\kappa_1\mathbf{m} \tag{15}$$

$$\mathbf{x_1}=\begin{pmatrix}0\\0\end{pmatrix} \tag{16}$$

$$\mathbf{x_2}=\mathbf{h}+\kappa_2\mathbf{m} \tag{17}$$

$$\mathbf{x_2}=\begin{pmatrix}1\\1\end{pmatrix} \tag{18}$$

The Area under the curve $y=x^2$ is given by

$$A_1=\int_0^1\left(x^2\right)dx \tag{19}$$

$$A_1=\left(\frac{1^3-\left(0^3\right)}{3}\right) \tag{20}$$

$$A_1=\frac{1}{3} \tag{21}$$

The Area under the line $y=x$ is given by

$$A_2=\int_0^1(x)\,dx \tag{22}$$

$$A_2=\left(\frac{1^2-\left(0^2\right)}{2}\right) \tag{23}$$

$$A_2=\frac{1}{2} \tag{24}$$

The area of region bounded by the line $x = y$ and the parabola $y = x^2$ is given by

$$A = A_2 - A_1 \tag{25}$$

$$A = \frac{1}{2} - \frac{1}{3} \tag{26}$$

$$A = \frac{1}{6} \tag{27}$$

The area of region bounded by the line $x = y$ and the parabola $y = x^2$ is $\frac{1}{6}$.

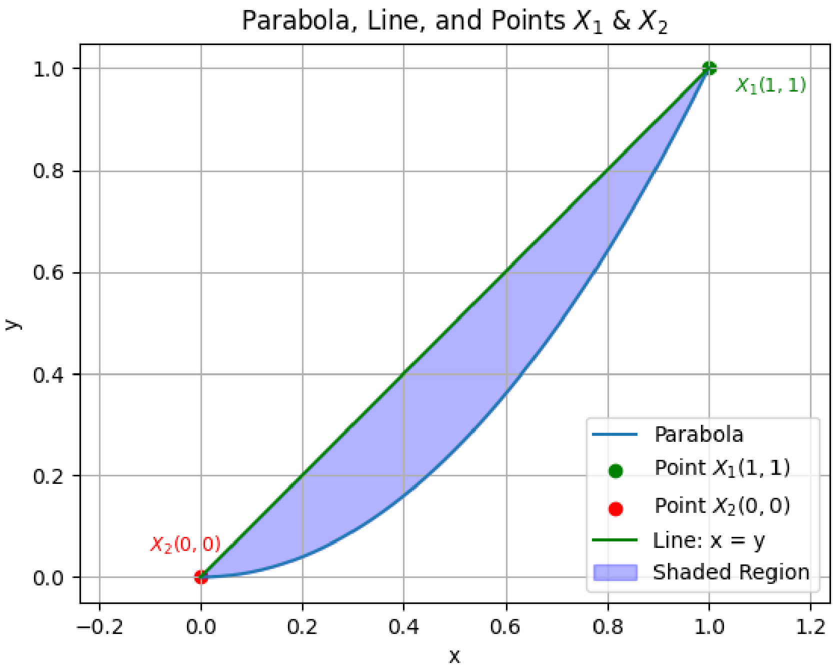

Fig. 1: The parabola along with the line