

4.4.2.7

EE24BTECH11024 - Abhimanyu Koushik

Question:

Find the equation of circle which passes through the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and the centre lies on the straight line $y - 4x + 3 = 0$

Variable	Description
\mathbf{x}_1	First point
\mathbf{x}_2	second point
\mathbf{n}	Normal to the diameter
\mathbf{x}	Vector which represents points on the diameter
\mathbf{c}	Centre of the circle
\mathbf{u}	$-\mathbf{c}$
r	radius of the circle
f	$\ \mathbf{u}\ ^2 - r^2$

TABLE I: Variables Used

Let the equation of circle be $\|\mathbf{x}\|^2 + 2\mathbf{u}^\top \mathbf{x} + f = 0$ where $\mathbf{u} = -\mathbf{c}$, $f = \|\mathbf{u}\|^2 - r^2$ and the equation of diameter be $\mathbf{n}^\top \mathbf{x} = c$. Then

$$\|\mathbf{x}_1\|^2 + 2\mathbf{u}^\top \mathbf{x}_1 + f = 0 \quad (1)$$

$$\|\mathbf{x}_2\|^2 + 2\mathbf{u}^\top \mathbf{x}_2 + f = 0 \quad (2)$$

$$-\mathbf{u}^\top \mathbf{n} = c \quad (3)$$

These equations can be written as

$$2\mathbf{x}_1^\top \mathbf{u} + f = -\|\mathbf{x}_1\|^2 \quad (4)$$

$$2\mathbf{x}_2^\top \mathbf{u} + f = -\|\mathbf{x}_2\|^2 \quad (5)$$

$$\mathbf{n}^\top \mathbf{u} = -c \quad (6)$$

Turning them into matrix form gives

$$\begin{pmatrix} 2\mathbf{x}_1^\top & 1 \\ 2\mathbf{x}_2^\top & 1 \\ \mathbf{n}^\top & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = - \begin{pmatrix} \|\mathbf{x}_1\|^2 \\ \|\mathbf{x}_2\|^2 \\ c \end{pmatrix} \quad (7)$$

$$(8)$$

Given the equation of diameter

$$y - 4x + 3 = 0 \quad (9)$$

$$\begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -3 \quad (10)$$

$$\begin{pmatrix} -4 \\ 1 \end{pmatrix}^T \mathbf{x} = -3 \quad (11)$$

$$\mathbf{n} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \quad (12)$$

$$c = -3 \quad (13)$$

Given $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. substituting into the matrix equation gives

$$\begin{pmatrix} 4 & 6 & 1 \\ 8 & 10 & 1 \\ -4 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ -41 \\ 3 \end{pmatrix} \quad (14)$$

The Augmented matrix is

$$\begin{pmatrix} 4 & 6 & 1 & -13 \\ 8 & 10 & 1 & -41 \\ -4 & 1 & 0 & 3 \end{pmatrix} \quad (15)$$

Solving the matrix equation

$$\begin{pmatrix} 4 & 6 & 1 & -13 \\ 8 & 10 & 1 & -41 \\ -4 & 1 & 0 & 3 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_1 + R_3} \begin{pmatrix} 4 & 6 & 1 & -13 \\ 8 & 10 & 1 & -41 \\ 0 & 7 & 1 & -10 \end{pmatrix} \quad (16)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 4 & 6 & 1 & -13 \\ 0 & -2 & -1 & -15 \\ 0 & 7 & 1 & -10 \end{pmatrix} \quad (17)$$

$$\xrightarrow{R_3 \leftarrow 2R_3 + 7R_2} \begin{pmatrix} 4 & 6 & 1 & -13 \\ 0 & -2 & -1 & -15 \\ 0 & 0 & -5 & -125 \end{pmatrix} \quad (18)$$

$$\xrightarrow{R_3 \leftarrow \frac{R_3}{-5}} \begin{pmatrix} 4 & 6 & 1 & -13 \\ 0 & -2 & -1 & -15 \\ 0 & 0 & 1 & 25 \end{pmatrix} \quad (19)$$

$$\xrightarrow{R_2 \leftarrow R_2 + R_3} \begin{pmatrix} 4 & 6 & 1 & -13 \\ 0 & -2 & 0 & 10 \\ 0 & 0 & 1 & 25 \end{pmatrix} \quad (20)$$

$$\xrightarrow{R_2 \leftarrow \frac{R_2}{-2}} \begin{pmatrix} 4 & 6 & 1 & -13 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 25 \end{pmatrix} \quad (21)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_3} \begin{pmatrix} 4 & 6 & 0 & -38 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 25 \end{pmatrix} \quad (22)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 6R_2} \begin{pmatrix} 4 & 0 & 0 & -8 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 25 \end{pmatrix} \quad (23)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 25 \end{pmatrix} \quad (24)$$

The value of u and f is

$$\mathbf{u} = \begin{pmatrix} -2 \\ -5 \end{pmatrix} \quad (25)$$

$$f = 25 \quad (26)$$

The centre and radius of circle is

$$\mathbf{c} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad (27)$$

$$r = \sqrt{2^2 + 5^2 - 25} \quad (28)$$

$$r = 2 \quad (29)$$

The equation of circle is $\|\mathbf{x}\|^2 - 2 \begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} + 25 = 0$

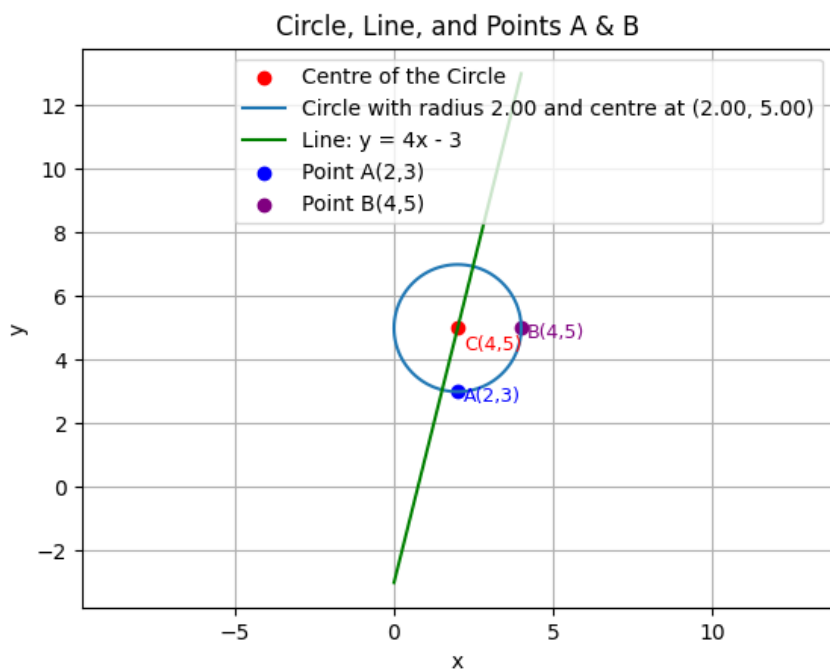


Fig. 1: The circle which satisfies the conditions