7.7.2.24

EE24BTECH11024 - Abhimanyu Koushik

Question:

Find the equation of circle which passes through the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and the centre lies on the straight line $y - 4x + 3 = 0$

Solution:

Variable	Description
$\mathbf{x_1}$	First point
$\mathbf{x_2}$	second point
$\mathbf{n}$	Normal to the diameter
$\mathbf{c}$	Centre of the circle
$\mathbf{u}$	$-\mathbf{c}$
r	radius of the circle
f	$\|\mathbf{u}\|^2 - r^2$

TABLE I: Variables Used

Let the equation of circle be $||\mathbf{x}||^2 + 2\mathbf{u}^\top\mathbf{x} + f = 0$ where $\mathbf{u} = -\mathbf{c}$, $f = ||\mathbf{u}||^2 - r^2$ and the equation of diameter be $\mathbf{n}^\top\mathbf{x} = c$. Then

$$||\mathbf{x_1}||^2 + 2\mathbf{u}^\top\mathbf{x_1} + f = 0 \tag{1}$$

$$||\mathbf{x_2}||^2 + 2\mathbf{u}^\top\mathbf{x_2} + f = 0 \tag{2}$$

$$-\mathbf{u}^\top\mathbf{n} = c \tag{3}$$

These equations can be written as

$$2\mathbf{x_1}^\top\mathbf{u} + f = -||\mathbf{x_1}||^2 \tag{4}$$

$$2\mathbf{x_2}^\top\mathbf{u} + f = -||\mathbf{x_2}||^2 \tag{5}$$

$$\mathbf{n}^\top\mathbf{u} = -c \tag{6}$$

Turning them into matrix form gives

$$\begin{pmatrix} 2\mathbf{x_1}^\top & 1 \\ 2\mathbf{x_2}^\top & 1 \\ \mathbf{n}^\top & 0 \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = -\begin{pmatrix} ||\mathbf{x_1}||^2 \\ ||\mathbf{x_2}||^2 \\ c \end{pmatrix} \tag{7}$$

$$\tag{8}$$

Given the equation of diameter

$$y - 4x + 3 = 0 \tag{9}$$

$$\begin{pmatrix} -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -3 \tag{10}$$

$$\begin{pmatrix} -4 \\ 1 \end{pmatrix}^\top \mathbf{x} = -3 \tag{11}$$

$$\mathbf{n} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \tag{12}$$

$$c = -3 \tag{13}$$

Given $\mathbf{x_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{x_2} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. substituting into the matrix equation gives

$$\begin{pmatrix} 4 & 6 & 1 \\ 8 & 10 & 1 \\ -4 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ -41 \\ 3 \end{pmatrix} \tag{14}$$

The Augmented matrix is

$$\begin{pmatrix} 4 & 6 & 1 & -13 \\ 8 & 10 & 1 & -41 \\ -4 & 1 & 0 & 3 \end{pmatrix} \tag{15}$$

Solving the matrix equation

$$\begin{pmatrix} 4 & 6 & 1 & -13 \\ 8 & 10 & 1 & -41 \\ -4 & 1 & 0 & 3 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_1 + R_3} \begin{pmatrix} 4 & 6 & 1 & -13 \\ 8 & 10 & 1 & -41 \\ 0 & 7 & 1 & -10 \end{pmatrix} \tag{16}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 4 & 6 & 1 & -13 \\ 0 & -2 & -1 & -15 \\ 0 & 7 & 1 & -10 \end{pmatrix} \tag{17}$$

$$\xleftrightarrow{R_3 \leftarrow 2R_3 + 7R_2} \begin{pmatrix} 4 & 6 & 1 & -13 \\ 0 & -2 & -1 & -15 \\ 0 & 0 & -5 & -125 \end{pmatrix} \tag{18}$$

$$\xleftrightarrow{R_3 \leftarrow \frac{R_3}{-5}} \begin{pmatrix} 4 & 6 & 1 & -13 \\ 0 & -2 & -1 & -15 \\ 0 & 0 & 1 & 25 \end{pmatrix} \tag{19}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + R_3} \begin{pmatrix} 4 & 6 & 1 & -13 \\ 0 & -2 & 0 & 10 \\ 0 & 0 & 1 & 25 \end{pmatrix} \tag{20}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{-2}} \begin{pmatrix} 4 & 6 & 1 & -13 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 25 \end{pmatrix} \tag{21}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_3} \begin{pmatrix} 4 & 6 & 0 & -38 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 25 \end{pmatrix} \tag{22}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 6R_2} \begin{pmatrix} 4 & 0 & 0 & -8 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 25 \end{pmatrix} \tag{23}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 25 \end{pmatrix} \tag{24}$$

The value of $\mathbf{u}$ and f is

$$\mathbf{u} = \begin{pmatrix} -2 \\ -5 \end{pmatrix} \tag{25}$$

$$f = 25 \tag{26}$$

The centre and radius of circle is

$$\mathbf{c} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \tag{27}$$

$$r = \sqrt{2^2 + 5^2 - 25} \tag{28}$$

$$r = 2 \tag{29}$$

The equation of circle is $\|\mathbf{x}\|^2 - 2\begin{pmatrix} 2 & 5 \end{pmatrix}\mathbf{x} + 25 = 0$

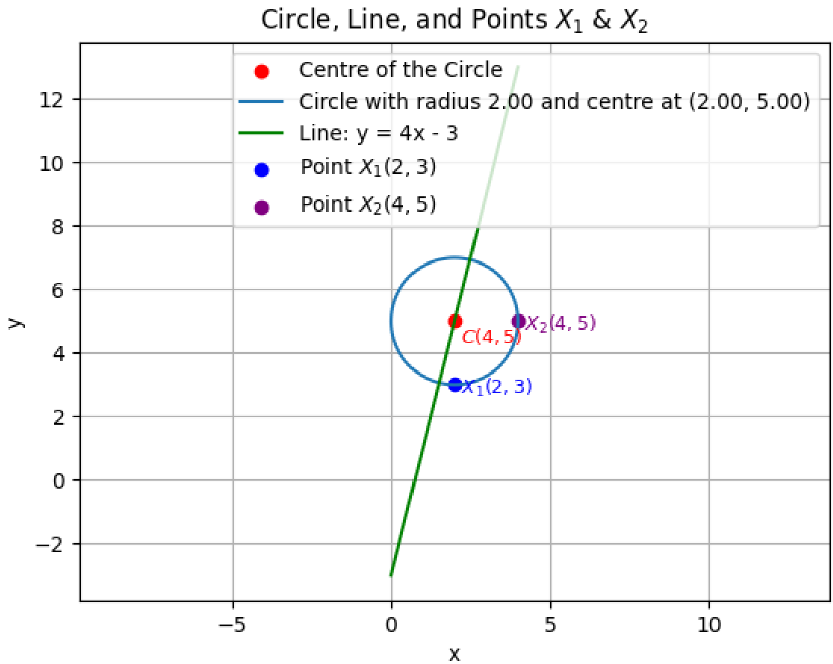

Fig. 1: The circle which satisfies the conditions