Abstract

This report examines the response of an RC circuit to a square wave input of time period T, with particular focus on three cases: $T \gg RC$, $T = RC$, and $T \ll RC$. The behavior of the output voltage across the capacitor is analyzed in each case, highlighting the effect of the relationship between the time constant RC and the period of the input signal.

1 Introduction

RC circuits, consisting of a resistor and a capacitor in series, exhibit dynamic behavior when subjected to time-varying inputs. The response of such circuits is governed by their time constant $\tau = RC$. When driven by a square wave input, the circuit's output depends on the interplay between the time constant and the time period T of the square wave. This document explores the circuit's response for three distinct scenarios.

2 Theory

The governing equation for the voltage $V_C(t)$ across the capacitor in an RC circuit with input voltage $V_{\text{in}}(t)$ is:

$$iR + \frac{q}{C} = V_{\text{in}}, \tag{1}$$

$$V_C = \frac{q}{C}, \tag{2}$$

$$RC\frac{dV_C}{dt} + V_C = V_{\text{in}}, \tag{3}$$

$$\frac{dV_C}{dt} + \frac{V_C}{RC} = \frac{V_{\text{in}}}{RC}. \tag{4}$$

For a square wave input, $V_{\text{in}}(t)$ alternates between two levels (e.g., V_0 and $-V_0$) with a period T. The response depends on the relationship between T and RC. Let:

$$V_{\text{in}}(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \right) \tag{5}$$

As V_{in} is a square, it will be in form $V_{in} = V_{dc} + V_{ac}$.
If the amplitude of V_{in} is $5V$, then the average value of V_{in} is $\frac{1}{T}\left(\int_0^T V_{in} dt\right) = \frac{5}{2}$. The value of $V_{dc} = \frac{5}{2}$.
As V_{ac} is an odd function, it will only contain sine terms

$$V_{\text{in}}(t) = a_0 + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right) \tag{6}$$

The values of b_n can be calculated as

$$b_n = \frac{2}{T}\left(\int_{\frac{T}{2}}^{-\frac{T}{2}} V_{in}(t) \sin\left(\frac{2\pi nt}{T}\right) dt\right) \tag{7}$$

$$b_n = \frac{2}{T}\left(\int_0^{\frac{T}{2}} V_0 \sin\left(\frac{2\pi nt}{T}\right) dt + \int_{\frac{T}{2}}^{0} (-V_0) \sin\left(\frac{2\pi nt}{T}\right) dt\right) \tag{8}$$

For even n, the terms cancel out, for odd n the terms add up giving

$$b_n = \frac{4V_0}{T}\left(\int_0^{\frac{T}{2}} \sin\left(\frac{2\pi nt}{T}\right) dt\right) \qquad \text{(for odd n)} \tag{9}$$

$$b_n = \frac{4V_0}{n\pi} \tag{10}$$

This gives

$$V_{in}(t) = a_0 + \frac{4V_0}{\pi}\left(\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\sin\left(\frac{2\pi nt}{T}\right)\right) \tag{11}$$

$$V_{in}(t) = \frac{5}{2} + \frac{10}{\pi}\left(\sin\left(\frac{2\pi t}{T}\right) + \frac{1}{3}\sin\left(\frac{6\pi t}{T}\right) + \frac{1}{5}\sin\left(\frac{10\pi t}{T}\right) + \ldots\right) \tag{12}$$

3 Analysis

3.1 Case 1: $T \gg RC$

When the period of the square wave is much greater than the time constant ($T \gg RC$), the capacitor has sufficient time to fully charge and discharge during each half-cycle. The output voltage $V_C(t)$ closely follows the input signal, with exponential transitions between V_0 and $-V_0$. The response can be expressed as:

$$V_C(t) = V_0\left(1 - e^{-t/(RC)}\right) \quad \text{(during charging),} \tag{13}$$

$$V_C(t) = -V_0\left(1 - e^{-t/(RC)}\right) \quad \text{(during discharging).} \tag{14}$$

3.2 Case 2: $T = RC$

When the period of the square wave is comparable to the time constant ($T \approx RC$), the capacitor does not fully charge or discharge during each half-cycle. The output voltage exhibits noticeable attenuation and rounded transitions, reflecting an intermediate behavior.

3.3 Case 3: $T \ll RC$

For a square wave with a period much smaller than the time constant ($T \ll RC$), the capacitor cannot respond significantly within a single cycle. The output voltage $V_C(t)$ remains nearly constant, approximating the average value of the input signal. This behavior effectively acts as a low-pass filter, smoothing the square wave into a nearly constant DC level.

4 Discussion

The response of the RC circuit demonstrates its filtering properties, with the time constant RC acting as a key determinant. For $T \gg RC$, the circuit behaves like a buffer, closely following the input. For $T \ll RC$, it acts as a low-pass filter, suppressing high-frequency components.

5 Conclusion

The analysis reveals the versatility of RC circuits in signal processing applications. By adjusting the time constant relative to the input signal's period, the circuit can function as a buffer, an attenuator, or a low-pass filter. This makes RC circuits fundamental in both analog and digital electronics.