Damped LC Oscillations

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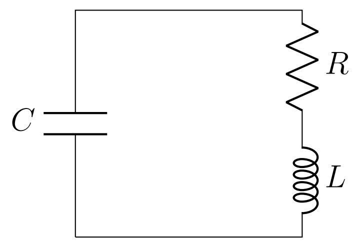
Abstract

This report explores the behavior of an RLC circuit exhibiting damped oscillations. The differential equation governing the circuit is derived, and solutions for different damping conditions are analyzed. Simulations and plots illustrate the circuit's response.

1 Introduction

An LC circuit consists of an inductor (L) and a capacitor (C), forming an oscillatory system. When a resistor (R) is added, the circuit exhibits damped oscillations, leading to energy dissipation over time.

2 Theory for Underdamped Case



Taking Kirchhoff's law in the circuit loop gives the equation

$$L\frac{di}{dt} + iR + \frac{1}{C}\int idt = 0$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

$$\frac{d^2q}{dt^2} + \frac{R}{L}\left(\frac{dq}{dt}\right) + \frac{q}{LC} = 0$$

This equation can be written as,

$$\frac{d^2q}{dt^2} + 2\xi\omega_n\left(\frac{dq}{dt}\right) + \omega_n\frac{q}{LC} = 0 \tag{1}$$

where ξ is the damping factor, $\xi = \frac{R}{2}\sqrt{\frac{C}{L}}$, and ω_n is the natural frequency, $\omega_n = \frac{1}{\sqrt{LC}}$.

According to the initial conditions, the initial current in the circuit is 0 and the initial voltage across capacitor is V_{C_0} which gives $\frac{dq}{dt}\Big|_{t=0} = 0$ and $q(0) = CV_{C_0}$.

The solution to the given differential equation can be given by,

$$q = c_1e^{s_1t} + c_2e^{s_2t}$$

where s_1, s_2 are the solutions for the quadratic equation $s^2 + 2\xi\omega_n s + (\omega_n)^2 = 0$ and c_1, c_2 are constants.

For an underdamped system ($\xi < 1$),

$$s_2 = {s_1}^* = s^* \tag{2}$$
$$c_2 = {c_1}^* = c^* \tag{3}$$
$$\implies q = ce^{st} + c^*e^{s^*t} \tag{4}$$

Let $c = a + ib$, then

$$\begin{aligned}
s &= \frac{-R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \\
q &= ce^{st} + \overline{c}e^{\overline{s}t} \\
q(0) = c + \overline{c} = 2a = V_{C_0} \implies a &= \frac{V_{C_0}}{2} \\
\frac{dq}{dt} &= cse^{st} + \overline{cs}e^{\overline{s}t} \\
\frac{dq}{dt}\Big|_{t=0} &= cs + \overline{cs} \\
2Re(cs) = 0 \implies Re(cs) &= 0 \\
Re((\frac{V_{C_0}}{2} + ib)(\frac{-R}{2L} + i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}})) &= 0 \\
\frac{-V_{C_0}R}{2L} - b\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} &= 0 \\
b &= \frac{-V_{C_0}R}{2L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}
\end{aligned}$$

Substituting the value of a and b gives the following

$$q = \left(\frac{V_{C_0}}{2} + ib\right) e^{\left(\frac{-R}{2L} + i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)t} + \left(\frac{V_{C_0}}{2} - ib\right) e^{\left(\frac{-R}{2L} - i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)t}$$

$$q = \frac{V_{C_0}}{2} e^{\frac{-R}{2L}t} \left(\left(1 + i\frac{2b}{V_{C_0}}\right) e^{i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}t} + \left(1 - i\frac{2b}{V_{C_0}}\right) e^{-i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}t}\right)$$

$$q = \frac{V_{C_0}}{2} e^{\frac{-R}{2L}t} \left(2\cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}t\right) - \frac{2b}{V_{C_0}} \sin\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}t\right)\right)$$

$$q = V_{C_0} e^{\frac{-R}{2L}t} \left(\cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}t\right) + \frac{R}{2L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \sin\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}t\right)\right)$$

3 Solution for Underdamped Case

For $\frac{R^2}{L^2} < \frac{4}{LC}$, the general solution is:

$$q = V_{C_0} e^{\frac{-R}{2L}t} \left(\cos\left(\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)t\right) + \frac{R}{2L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \sin\left(\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)t\right)\right)$$

4 Procedure

1. Connect the circuit as per the circuit diagram.
2. Measure the internal resitance of the inductor using a multimeter.
3. Oscilloscope's probes are to be connected across the inductor to measure the voltage across it. Set the display mode to $X - T$.
4. Disconnect the capacitor and inductor to charge the capacitor.
5. Use a Regulated DC Supply to provide $5V$ DC voltage across the capacitor.
6. Disconnect the voltage supply and connect the inductor back to the capacitor, one after the other.
7. Measure the Voltage drop across the capacitor using the Oscilloscope probe. Capture the wave which is being displayed on the screen. It is the graph of the response in steady state.
8. Enter the **Cursor** menu and set the cursor A to the current channel. Use the knob marked as ↻ to adjust the marker.
9. To check the voltage value, check the marker value at the maximum point of output wave.

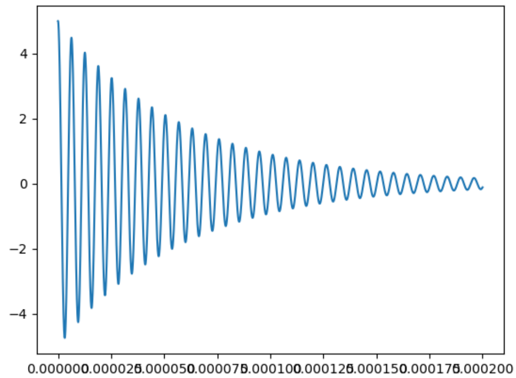
5 Simulation and Results

Numerical simulations were conducted using Python to visualize the circuit response.

$$R = 24.5\,\Omega \text{ (internal resistance of inductor)} \tag{5}$$
$$L = 2.2\,mH \tag{6}$$
$$C = 460\,pF \tag{7}$$

Oscilloscope

Simulation



6 Verification Codes

Python verification codes are provided below,
`https://github.com/AbhimanyuKoushik/EE1200/blob/main/Lab4/codes/rlc.py`

7 Conclusion

Damped LC circuits exhibit different responses based on resistance. Understanding these behaviors is crucial in designing stable electronic systems.