10.4.3.8

EE24BTECH11003 - Akshara Sarma Chennubhatla

Question: A train travels $360km$ at a uniform speed. If the speed had been $5km/h$ more, it would have taken 1 hour less for the same journey. Find the speed of the train.

Solution:

Theoretical Solution:

Let s be the speed of the train, then,

$$\frac{360}{s} - 1 = \frac{360}{s+5} \tag{1}$$

$$\implies s^2 + 5s = 1800 \tag{2}$$

This is a quadratic equation whose roots are the possible values of the speed.
Using the quadratic formula,

$$s = \frac{-5 \pm \sqrt{5^2 - 4(1)(-1800)}}{2(1)} \tag{3}$$

$$s = -45, 40 \tag{4}$$

Simulated Solution:

By Newton-Ralphson method,

Take initial guess s_0, then run the following loop,

$$s_{n+1} = s_n - \frac{f(s_n)}{f'(s_n)} \tag{5}$$

$$f(s) = s^2 + 5s - 1800 \tag{6}$$

$$f'(s) = 2s + 5 \tag{7}$$

$$s_{n+1} = s_n - \frac{s_n^2 + 5s_n - 1800}{2s_n + 5} \tag{8}$$

This method converges for real roots but when roots are complex, it can go to infinity as well. To avoid that, if our roots are complex, take initial guess as a complex number. The values of s got through this method are,

$$s = -45 \tag{9}$$

$$s = 40 \tag{10}$$

Alternatively, we can solve the question by using the eigen values of the companion matrix.

For a polynomial equation of form $x_n + c_{n-1}x^{n-1} + \cdots + c_2x^2 + c_1x + c_0 = 0$ the companion matrix if of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix} \tag{11}$$

The eigen values of this matrix are the roots of the polynomial equation. For this question,

$$n = 2 \tag{12}$$

$$c_0 = -1800 \tag{13}$$

$$c_1 = 5 \tag{14}$$

$$C = \begin{pmatrix} 0 & 1800 \\ 1 & -5 \end{pmatrix} \tag{15}$$

To find the eigen values of the matrix, we use the method of QR decomposition of the matrix.
The QR algorithm decomposes a matrix A into the product of an orthogonal matrix Q and an upper triangular matrix R, such that

$$A = QR \tag{16}$$

The matrix is then updated iteratively as:

$$A_{new} = RQ \tag{17}$$

This process is repeated until A converges to an upper triangular form.
Steps to perform QR decomposition with accelerated convergence,

1) Convert to Upper Hessenberg form via Householder Reflections
2) Performing QR decomposition via Givens Rotations with shifts
3) Read off diagonal elements

Householder Reflections:
A square matrix A of order $n \times n$ is said to be in upper Hessenberg form if all the entries below the first subdiagonal are zero. For example:

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\ 0 & h_{32} & h_{33} & \cdots & h_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_{n-1,n-1} & h_{n-1,n} \\ 0 & \cdots & \cdots & 0 & h_{nn} \end{bmatrix} \tag{18}$$

1) Select a Subvector **x**:

$$x = \begin{bmatrix} A_{2,1} \\ A_{3,1} \\ \vdots \\ A_{n,1} \end{bmatrix}. \tag{19}$$

2) Define the Target Vector: The goal is to transform **x** into a new vector **y** where only the first element is non-zero, and all the other elements are zero. First, compute $\|x\|$:

$$\mathbf{y} = \pm \|x\| \, e_1, \tag{20}$$

3) Construct the Householder Vector **v**: To generate a reflection that transforms x to y, the Householder vector v is defined as:

$$v = x - \operatorname{sign}(x_1) \, \|x\| \, e_1 \tag{21}$$

$$\operatorname{sign}(x_1) = \frac{x_1}{|x_1|}, \tag{22}$$

After defining v, it is normalized to a unit vector:

4) Construct the Householder Matrix H_k: The Householder matrix H_k is constructed as:

$$H_k = I - 2\frac{vv^*}{v^*v}, \tag{23}$$

5) Apply the Householder Transformation: The matrix H_k is applied to A as:

$$A' = H_k A H_k^*, \tag{24}$$

This will reduce the matrix to Hessenberg form by eliminating the sub-diagonal elements of the first column.

6) Repeat for Subsequent Columns: This Householder transformation approach ensures that the matrix is gradually transformed to a Hessenberg form, where all elements below the first sub-diagonal are zero.

Givens Rotations:

$$
\begin{bmatrix}
h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\
h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\
0 & h_{32} & h_{33} & \cdots & h_{3n} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & h_{n-1,n-1} & h_{n-1,n} \\
0 & \cdots & \cdots & 0 & h_{nn}
\end{bmatrix}
\longrightarrow
\begin{bmatrix}
r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\
0 & r_{22} & r_{23} & \cdots & r_{2n} \\
0 & 0 & r_{33} & \cdots & r_{3n} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & r_{n-1,n-1} & r_{n-1,n} \\
0 & \cdots & \cdots & 0 & r_{nn}
\end{bmatrix}
$$

Each Givens rotation zeros out a specific subdiagonal element, progressively transforming the Hessenberg matrix into an upper triangular matrix.

To choose the values of c and s for the Givens rotation in QR decomposition, let a_j be the element we wish to null out (i.e. make 0). Pick an arbitrary non-zero pivot element a_i (on a different row). Usually, if we wish to null a particular sub-diagonal element, we pick the principal diagonal element above it as a pivot.

$$c = \frac{\overline{a_i}}{\sqrt{a_i^2 + a_j^2}}, \quad s = \frac{-\overline{a_j}}{\sqrt{a_i^2 + a_j^2}} \tag{25}$$

Givens rotation essentially rotates the two rows that a_i and a_j are on such that $a_j = 0$ after rotation, other rows remain unaffected.

Visualizing the process,

$$
\begin{bmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
0 & \times & \times & \times \\
0 & 0 & \times & \times
\end{bmatrix}
\xrightarrow{G(3,2,\theta_1)}
\begin{bmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
0 & 0 & \times & \times \\
0 & 0 & \times & \times
\end{bmatrix}
\xrightarrow{G(4,3,\theta_2)}
\begin{bmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
0 & 0 & \times & \times \\
0 & 0 & 0 & \times
\end{bmatrix}. \tag{26}
$$

After all Givens rotations, the resulting matrix is upper triangular:

$$
R = \begin{bmatrix}
\times & \times & \times & \times \\
0 & \times & \times & \times \\
0 & 0 & \times & \times \\
0 & 0 & 0 & \times
\end{bmatrix}. \tag{27}
$$

The sequence of Givens rotations $G_1, G_2, \ldots, G_m$ satisfies:

$$G_m \cdots G_2 G_1 A = R, \tag{28}$$

where R is upper triangular. The QR decomposition is obtained by combining the transposes of the Givens rotations into Q:

$$A = QR, \quad Q = G_1^\top G_2^\top \cdots G_m^\top. \tag{29}$$

$$\begin{aligned} A_{k+1} &= R_k Q_k && (30)\\ &= (G_n \dots G_2 G_1) A_k (G_1^\top G_2^\top \dots G_n^\top) && (31)\\ &= (G_n \dots G_2 G_1) A_k (G_n \dots G_2 G_1)^\top && (32) \end{aligned}$$

Iteratively repeating this process causes the matrix to converge to upper triangular.

Handling Jordan Blocks:

Jordan blocks pose challenges in eigenvalue computation because the matrix cannot be diagonalized. A Jordan block for eigenvalue λ appears as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{33}$$

where a and b are the diagonal elements and c is a non zero sub-diagonal element.
To handle Jordan blocks, the QR algorithm implemented here solves for the eigenvalues directly using the characteristic polynomial of the block. For a 2×2 Jordan block, the eigenvalues are roots of:

$$\lambda^2 - (\text{trace})\lambda + \det = 0. \tag{34}$$

In this case, the eigen values of the matrix computed are,

$$\begin{aligned} \lambda_1 &= -45 && (35)\\ \lambda_2 &= 40 && (36)\\ & && (37) \end{aligned}$$

Below is the plot for given quadratic equation, obtained by iterating through the values of x with step size of h

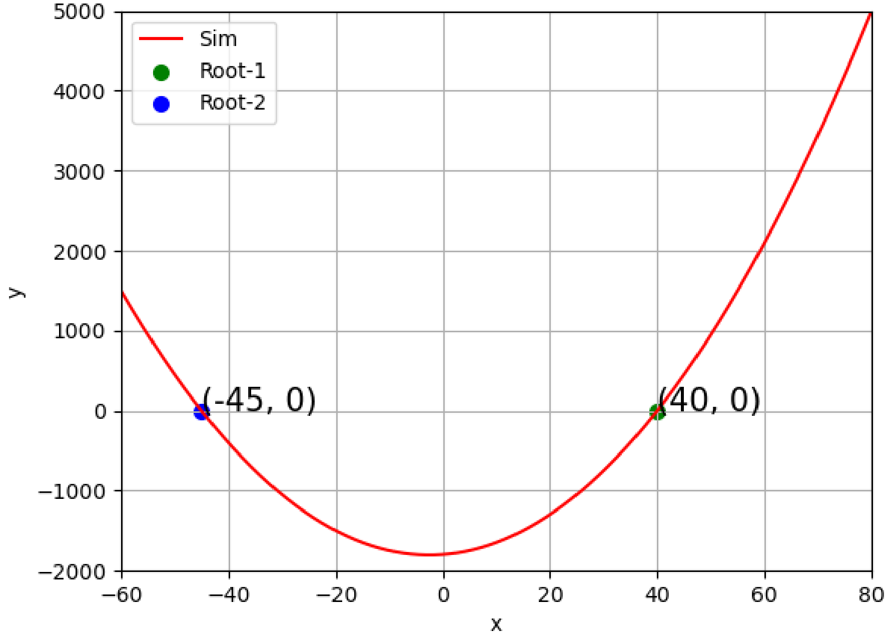

Fig. 1: Plot of $s^2 + 5s = 1800$