10.4.ex-13.2

EE24BTECH11058 - P.Shiny Diavajna

Question: Solve the Quadratic equation

$$x^2 + 4x + 5 = 0 \tag{0.1}$$

Theoretical Solution:

The roots of a Standard Quadratic equation of the form $ax^2 + bx + c = 0$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{0.2}$$

Here, $a = 1$,$b = 4$ and $c = 5$.On substituting the values in (0.2), we have

$$x = \frac{-4 \pm \sqrt{16 - 20}}{2} \tag{0.3}$$

$$x = -2 \pm i \tag{0.4}$$

Computational Solution:

Eigenvalues of Companion matrix:

The eigenvalues of a companion matrix are the roots of the characteristic polynomial that defines the matrix. For a polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 \tag{0.5}$$

Companion matrix A is defined as:

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix} \tag{0.6}$$

Here, $a_0 = 5$, $a_1 = 4$

$$A = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix} \tag{0.7}$$

We find the eigenvalues using the QR algorithm. The basic principle behind this algorithm is a similarity transform,

$$A' = X^{-1}AX \tag{0.8}$$

which does not alter the eigenvalues of the matrix A.
We use this to get the Schur Decomposition,

$$A = Q^{-1}UQ = Q^{*}UQ \tag{0.9}$$

where Q is a unitary matrix $\left(Q^{-1} = Q^{*}\right)$ and U is an upper triangular matrix whose diagonal entries are the eigenvalues of A.
To efficiently get the Schur Decomposition, we first householder reflections to reduce it to an upper hessenberg form.
A householder reflector matrix is of the form,

$$P = I - 2\mathbf{u}\mathbf{u}^{*} \tag{0.10}$$

Householder reflectors transforms any vector $\mathbf{x}$ to a multiple of $\mathbf{e_1}$,

$$P\mathbf{x} = \mathbf{x} - 2\mathbf{u}\left(\mathbf{u}^{*}\mathbf{x}\right) = \alpha\mathbf{e_1} \tag{0.11}$$

P is unitary, which implies that,

$$||P\mathbf{x}|| = ||\mathbf{x}|| \tag{0.12}$$

$$\implies \alpha = \rho\,||\mathbf{x}|| \tag{0.13}$$

$$\tag{0.14}$$

As $\mathbf{u}$ is unit norm,

$$\mathbf{u} = \frac{\mathbf{x} - \rho\,||\mathbf{x}||\,\mathbf{e_1}}{||\mathbf{x} - \rho\,||\mathbf{x}||\,\mathbf{e_1}||} = \frac{1}{||\mathbf{x} - \rho\,||\mathbf{x}||\,\mathbf{e_1}||}\begin{pmatrix} x_1 - \rho\,||\mathbf{x}|| \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{0.15}$$

Selection of ρ is flexible as long as $|\rho| = 1$. To ease out the process, we take $\rho = \frac{x_1}{|x_1|}$, $x_1 \neq 0$. If $x_1 = 0$, we take $\rho = 1$.
Householder reflector matrix (P_i) is given by,

$$P_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^{*} \\ \mathbf{0} & I_{n-i} - 2\mathbf{u_i}\mathbf{u_i}^{*} \end{bmatrix} \tag{0.16}$$

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{bmatrix} \xrightarrow{P_2} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \tag{0.17}$$

Next step is to do Given's rotation to get the QR Decomposition.

The Givens rotation matrix $G(i, j, c, s)$ is defined by

$$G(i,j,c,s) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\overline{s} & \cdots & \overline{c} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \tag{0.18}$$

where $|c|^2 + |s|^2 = 1$, and G is a unitary matrix.
Say we take a vector $\mathbf{x}$, and $\mathbf{y} = G(i, j, c, s)\,\mathbf{x}$, then

$$y_k = \begin{cases} cx_i + sx_j, & k = i \\ -\overline{s}x_i + \overline{c}x_j, & k = j \\ x_k, & k \neq i, j \end{cases} \tag{0.19}$$

For y_j to be zero, we set

$$c = \frac{\overline{x_i}}{\sqrt{|x_i|^2 + \left|x_j\right|^2}} = c_{ij} \tag{0.20}$$

$$s = \frac{\overline{x_j}}{\sqrt{|x_i|^2 + \left|x_j\right|^2}} = s_{ij} \tag{0.21}$$

Using this Givens rotation matrix, we zero out elements of subdiagonal in the hessenberg matrix H.

$$H = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(1,2,c_{12},s_{12})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$$

$$\xrightarrow{G(2,3,c_{23},s_{23})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(3,4,c_{34},s_{34})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$$

$$\xrightarrow{G(4,5,c_{45},s_{45})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix} = R \tag{0.22}$$

where R is upper triangular. For the given companion matrix, Let $G_k = G(k, k+1, c_{k,k+1}, s_{k,k+1})$, then we deduce that

$$G_4G_3G_2G_1H = R \tag{0.23}$$

$$H = G_1^*G_2^*G_3^*G_4^*R \tag{0.24}$$

$$H = QR, \text{ where } Q = G_1^*G_2^*G_3^*G_4^* \tag{0.25}$$

Using this QR algorithm, we get the following update equation,

$$A_k = Q_kR_k \tag{0.26}$$

$$A_{k+1} = R_kQ_k \tag{0.27}$$

$$= (G_n \ldots G_2G_1) A_k (G_1^*G_2^* \ldots G_n^*) \tag{0.28}$$

Running the eigenvalue code we get

$$x_1 = -2.000000000000001 + -1.000000000000003j \tag{0.29}$$

$$x_2 = -2.000000000000001 + -1.000000000000003j \tag{0.30}$$

Newton-Raphson iterative method:

$$f(x) = x^2 + 4x + 5 \tag{0.31}$$

$$f'(x) = 2x + 4 \tag{0.32}$$

Difference equation,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.33}$$

$$x_{n+1} = x_n - \frac{x_n^2 + 4x + 5}{2x_n + 4} \tag{0.34}$$

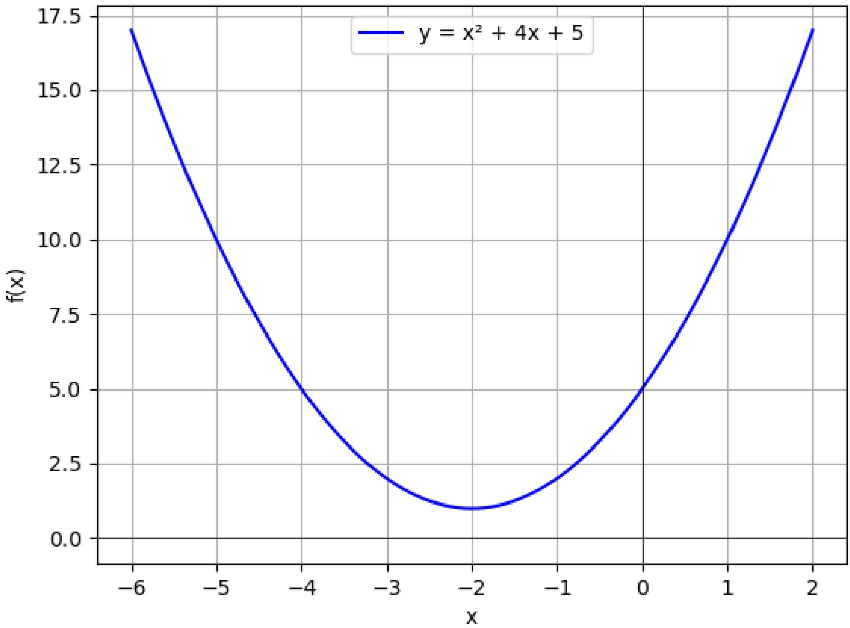
$$x_{n+1} = \frac{x_n}{2} - 1 - \frac{1}{2x_n + 4} \tag{0.35}$$

Picking two initial guesses,

$$x_0 = 0 + i \text{ converges to } -2.0 + 1.0000000000000002i \tag{0.36}$$

$$x_0 = 0 - i \text{ converges to } -2.0 + -1.0000000000000002i \tag{0.37}$$

Plot:



Acknowledgments:
Code from [https://github.com/Dwarak-A/sprog/tree/c6f873e29ab9910f22e49f7421dfc8e5fd5c60fc/ncert/10/4/1/1/2/codes].