NCERT - 10.3.6.1.5

EE24BTECH11040 - Mandara Hosur

Question:

Solve the given pair of equations by reducing them to a pair of linear equations.

$$\frac{7x-2y}{xy}=5 \tag{0.1}$$

$$\frac{8x+7y}{xy}=15 \tag{0.2}$$

Theoretical Solution:

The given two equations can be reduced as follows:

$$\frac{7}{y}-\frac{2}{x}=5 \tag{0.3}$$

$$\frac{8}{y}+\frac{7}{x}=15 \tag{0.4}$$

Make the following substitutions for convenience:

$$\frac{1}{x}=p \tag{0.5}$$

$$\frac{1}{y}=q \tag{0.6}$$

Substituting equations (0.5) and (0.6) in equations (0.3) and (0.4), we get:

$$7q-2p=5 \tag{0.7}$$

$$8q+7p=15 \tag{0.8}$$

Multiplying equation (0.7) with 7 and equation (0.8) with 2, we get:

$$49q-14p=35 \tag{0.9}$$

$$16q+14p=30 \tag{0.10}$$

Adding equations (0.9) and (0.10), we get:

$$65q=65 \tag{0.11}$$

$$\implies q=1 \tag{0.12}$$

Substituting equation (0.12) in equation (0.7), we get:

$$7 - 2p = 5 \tag{0.13}$$

$$\implies 2p = 2 \tag{0.14}$$

$$\implies p = 1 \tag{0.15}$$

Thus, from equations (0.5), (0.6), (0.15), (0.12) we get:

$$x = 1 \tag{0.16}$$

$$y = 1 \tag{0.17}$$

Solution via LU Decomposition:

LU decomposition is a method in linear algebra used to solve systems of linear equations. It factorises a given square, non-singular matrix A into the product of two matrices:

$$A = LU \tag{0.18}$$

Here, L is a lower triangular matrix (with ones on the diagonal) and U is an upper triangular matrix.

This factorisation allows solving $A\mathbf{x} = b$ by first solving two simpler systems $L\mathbf{y} = b$ (forward substitution) and $U\mathbf{x} = b$ (backward substitution).

$$A\mathbf{x} = b \implies LU\mathbf{x} = b \tag{0.19}$$

Take:

$$\mathbf{y} = U\mathbf{x} \tag{0.20}$$

$$\tag{0.21}$$

Then equation (0.19) becomes:

$$L\mathbf{y} = b \tag{0.22}$$

We first solve for $\mathbf{y}$ in $L\mathbf{y} = b$ and then solve for $\mathbf{x}$ in $U\mathbf{x} = \mathbf{y}$.

Applying LU decomposition to matrix A, for each column $j \geq k$, the entries of U in the kth row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m}.U_{m,j}, \forall j \geq k \tag{0.23}$$

For each row $i > k$, the entries of L in the kth column are updated as:

$$L_{j,k} = \frac{1}{U_{k,k}} \left(A_{j,k} - \sum_{m=1}^{k-1} .U_{m,k} \right), \forall i > k \tag{0.24}$$

The given equations (0.7) and (0.8) can be written in matrix form as:

$$\begin{pmatrix} -2 & 7 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} \tag{0.25}$$

Using the above-mentioned method, we find L and U as follows:

$$L = \begin{pmatrix} 1 & 0 \\ \frac{8}{7} & 1 \end{pmatrix} \tag{0.26}$$

$$U = \begin{pmatrix} 7 & -2 \\ 0 & \frac{65}{7} \end{pmatrix} \tag{0.27}$$

Solving $L\mathbf{y} = b$ by forward substitution:

$$\begin{pmatrix} 1 & 0 \\ \frac{8}{7} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} \tag{0.28}$$

$$\implies \mathbf{y} = \begin{pmatrix} 5 \\ \frac{65}{7} \end{pmatrix} \tag{0.29}$$

Solving $U\mathbf{x} = \mathbf{y}$:

$$\begin{pmatrix} 7 & -2 \\ 0 & \frac{65}{7} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 5 \\ \frac{65}{7} \end{pmatrix} \tag{0.30}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.31}$$

Hence, the required solution is:

$$x = \frac{1}{p} = 1 \tag{0.32}$$

$$y = \frac{1}{q} = 1 \tag{0.33}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 36 \end{pmatrix} \tag{0.34}$$

Plot:

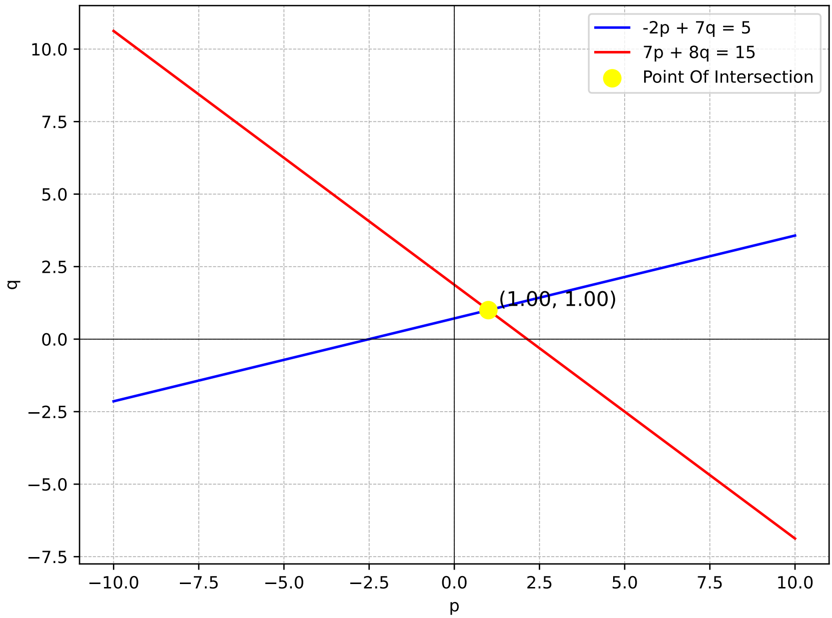

Fig. 0.1: Plot