10.3.6.1.6

EE24BTECH11020 - Ellanti Rohith

QUESTION :
Solve the system of equations$x \neq 0, y \neq 0$.

$$6x + 3y = 6xy \tag{1}$$
$$2x + 4y = 5xy \tag{2}$$

SOLUTION :
Dividing both equations by xy,

$$\frac{6x}{xy} + \frac{3y}{xy} = \frac{6xy}{xy} \tag{3}$$
$$\frac{2x}{xy} + \frac{4y}{xy} = \frac{5xy}{xy} \tag{4}$$

which simplifies to

$$\frac{6}{y} + \frac{3}{x} = 6 \tag{5}$$
$$\frac{2}{y} + \frac{4}{x} = 5 \tag{6}$$

Let $u = \frac{1}{x}$ $(x \neq 0)$ and $v = \frac{1}{y}(y \neq 0)$. Then the system transforms to

$$6v + 3u = 6 \tag{7}$$
$$2v + 4u = 5 \tag{8}$$

The matrix form is

$$\begin{pmatrix} 6 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} 6 & 3 & 6 \\ 2 & 4 & 5 \end{pmatrix} \xrightarrow{R_1 \div 6,\ R_2 - 2R_1} \begin{pmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 3 & 3 \end{pmatrix} \xrightarrow{R_2 \div 3,\ R_1 - \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix}$$

Thus, $x = \frac{1}{2}, y = 1$.
LU decomposition :
The matrix $\mathbf{A}$ can be decomposed as

$$\mathbf{A} = \mathbf{LU} \tag{10}$$

where,

$$\mathbf{L} = Lowertriangular \tag{11}$$
$$\mathbf{U} = Uppertriangular \tag{12}$$

Then the system of equations can be solved as

$$\mathbf{Ax} = \mathbf{B} \tag{13}$$
$$\mathbf{LUx} = \mathbf{B} \tag{14}$$
$$\implies \mathbf{Ly} = \mathbf{B} \tag{15}$$
$$\mathbf{Ux} = \mathbf{y} \tag{16}$$

Algorithm :

1) Let $\mathbf{A}$ be an $n \times n$ matrix. Initialize $\mathbf{L}$ to an $n \times n$ Identity matrix. Initialize $\mathbf{U}$ to a zero matrix.

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{17}$$

$$U = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{18}$$

2) For each row i from 0 to $n-1$:
   a) For each column j from i to $n-1$:

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \tag{19}$$

   b) For each row j from $i+1$ to $n-1$:

$$L_{ji} = \frac{1}{U_{ii}} \left(A_{ij} - \sum_{k=0}^{i-1} L_{jk} U_{ki} \right) \tag{20}$$

3) Repeat the above step for all $i = 0, 1, \ldots, n-1$
4) After all the iterations

$$\mathbf{A} = \mathbf{LU} \tag{21}$$

Decomposing $\mathbf{A}$ as $\mathbf{LU}$,

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} 6 & 3 \\ 0 & 3 \end{pmatrix} \tag{22}$$

Using forward substitution,

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{23}$$

Solving,

$$y_1 = 6, \quad y_2 = 5 - \frac{1}{3} \times 6 = 3 \tag{24}$$

Using backward substitution,

$$\begin{pmatrix} 6 & 3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \tag{25}$$

Solving,

$$u = 1, \quad v = \frac{6 - 3(1)}{6} = \frac{1}{2} \tag{26}$$

Thus,

$$\frac{1}{x} = 1 \implies x = 1, \quad \frac{1}{y} = \frac{1}{2} \implies y = 2 \tag{27}$$

QR decomposition :
Decomposing **A** as **QR**,

$$\mathbf{Q} = \begin{pmatrix} \frac{6}{\sqrt{45}} & \frac{3}{\sqrt{45}} \\ \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} \end{pmatrix}, \tag{28}$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{45} & \sqrt{5} \\ 0 & \frac{6}{\sqrt{5}} \end{pmatrix} \tag{29}$$

The system simplifies as:

$$\mathbf{R} \begin{pmatrix} v \\ u \end{pmatrix} = \mathbf{Q}^T \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{30}$$

Computing,

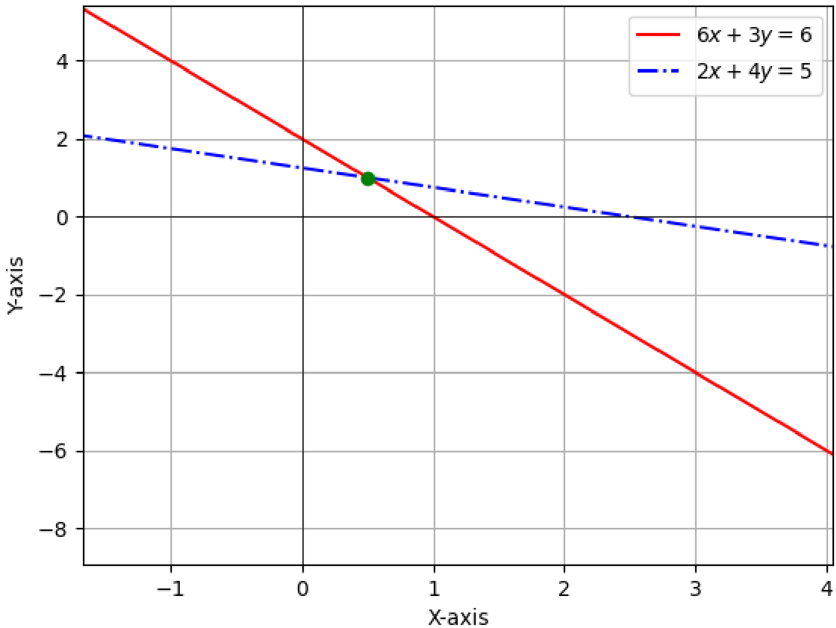
$$\begin{pmatrix} \sqrt{45} & \sqrt{5} \\ 0 & \frac{6}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} \frac{6}{\sqrt{45}} & \frac{3}{\sqrt{45}} \\ \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} \end{pmatrix}^T \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{31}$$

Solving for u, v:

$$u = 1, \tag{32}$$

$$v = \frac{1}{2} \tag{33}$$

Hence, the system is **consistent** with a unique solution $x = 1$, $y = 2$.



$6x + 3y = 6$
$2x + 4y = 5$
Y-axis
X-axis
4
2
0
−2
−4
−6
−8
−1
0
1
2
3
4