10.3.2.3.5

EE24BTECH11015 - Dhawal

Question:
On comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the following pair of linear equations are consistent, or inconsistent .

$$\frac{4}{3}x + 2y = 8; 2x + 3y = 12 \tag{1}$$

Theoritical Solution: To determine whether the given pair of linear equations is consistent or inconsistent, we compare the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$, and $\frac{c_1}{c_2}$, where:

$$a_1x + b_1y = c_1 \quad \text{and} \quad a_2x + b_2y = c_2 \tag{2}$$

From the equations:

$$\frac{4}{3}x + 2y = 8 \quad \text{and} \quad 2x + 3y = 12, \tag{3}$$

we identify:

$$a_1 = \frac{4}{3},\ b_1 = 2,\ c_1 = 8,\ a_2 = 2,\ b_2 = 3,\ c_2 = 12. \tag{4}$$

Now calculate the ratios:

$$\frac{a_1}{a_2} = \frac{4}{6}, \quad \frac{b_1}{b_2} = \frac{2}{3}, \quad \frac{c_1}{c_2} = \frac{8}{12}. \tag{5}$$

Since:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}, \tag{6}$$

The given pair of equations is **consistent** and the lines represented by the equations are **coincident**. Therefore, the system of equations has infinitely many solutions.

Computational Solution:

Solution using LU Factorization

Given the system of linear equations:

$$\frac{4}{3}x + 2y = 8, \tag{7}$$

$$2x + 3y = 12. \tag{8}$$

We rewrite the equations as:

$$x_1 = x, \tag{9}$$

$$x_2 = y, \tag{10}$$

giving the system:

$$\frac{4}{3}x_1 + 2x_2 = 8, \tag{11}$$

$$2x_1 + 3x_2 = 12. \tag{12}$$

Step 1: Convert to Matrix Form

We write the system as:

$$A\mathbf{x} = \mathbf{b}, \tag{13}$$

where:

$$A = \begin{bmatrix} \frac{4}{3} & 2 \\ 2 & 3 \end{bmatrix}, \tag{14}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{15}$$

$$\mathbf{b} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}. \tag{16}$$

Step 2: LU factorization using update equaitons

Given a matrix $\mathbf{A}$ of size $n \times n$, LU decomposition is performed row by row and column by column. The update equations are as follows:

Step-by-Step Procedure:

1. Initialization: - Start by initializing $\mathbf{L}$ as the identity matrix $\mathbf{L} = \mathbf{I}$ and $\mathbf{U}$ as a copy of $\mathbf{A}$.

2. Iterative Update: - For each pivot $k = 1, 2, \ldots, n$: - Compute the entries of U using the first update equation. - Compute the entries of L using the second update equation.

3. Result: - After completing the iterations, the matrix $\mathbf{A}$ is decomposed into $\mathbf{L} \cdot \mathbf{U}$, where $\mathbf{L}$ is a lower triangular matrix with ones on the diagonal, and $\mathbf{U}$ is an upper triangular matrix.

1. Update for $U_{k,j}$ (Entries of U)

For each column $j \geq k$, the entries of U in the k-th row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j}, \quad \text{for } j \geq k.$$

This equation computes the elements of the upper triangular matrix $\mathbf{U}$ by eliminating the lower triangular portion of the matrix.

2. Update for $L_{i,k}$ (Entries of L)

For each row $i > k$, the entries of L in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right), \quad \text{for } i > k.$$

This equation computes the elements of the lower triangular matrix $\mathbf{L}$, where each entry in the column is determined by the values in the rows above it.

Step 2: LU Factorization of Matrix A

We decompose A as:

$$A = LU, \tag{17}$$

where L is a lower triangular matrix and U is an upper triangular matrix. by running the iteration code we get the L and U matrices :

$$L = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}, \tag{18}$$

$$U = \begin{bmatrix} \frac{4}{3} & 2 \\ 0 & 0 \end{bmatrix}. \tag{19}$$

Step 3: Solve $L\mathbf{y} = \mathbf{b}$ (Forward Substitution)

We solve:

$$L\mathbf{y} = \mathbf{b} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}. \tag{20}$$

From the first row:

$$y_1 = 8. \tag{21}$$

From the second row:

$$\frac{3}{2} y_1 + y_2 = 12 \quad \implies \quad \frac{3}{2} \cdot 8 + y_2 = 12 \quad \implies \quad y_2 = 0. \tag{22}$$

Thus:

$$\mathbf{y} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}. \tag{23}$$

Step 4: Solve $U\mathbf{x} = \mathbf{y}$ (Backward Substitution)

We solve:

$$U\mathbf{x} = \mathbf{y} \quad \text{or} \quad \begin{bmatrix} \frac{4}{3} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}. \tag{24}$$

From the second row:

$$0x_1 + 0x_2 = 0 \quad \implies \quad 0 = 0. \tag{25}$$

From the first row:

$$\frac{4}{3}x + 2y = 8 \tag{26}$$

Thus we get equation of the line. So we can say that both lines are **coincident**.

Final Solution

The solution is:

$$\frac{4}{3}x + 2y = 8 \tag{27}$$