

9.1.8

EE24BTECH11007 - Arnav Makarand Yadnopavit

Question: Solve the differential equation $y' + y = e^x$ with initial condition $y(0) = 1/2$.

Solution:

Theoretical Solution:

$$\frac{dy}{dx} + y = e^x \quad (1)$$

$$\frac{dy}{dx}e^x + ye^x = e^{2x} \quad (2)$$

$$d(ye^x) = e^{2x}dx \quad (3)$$

$$\int \frac{d(ye^x)}{dx} = \int e^{2x}dx \quad (4)$$

$$ye^x = \frac{e^{2x}}{2} + c \quad (5)$$

$$y = \frac{e^x}{2} + ce^{-x} \quad (6)$$

Substituting values from initial conditions

$$y(0) = 0 \quad (7)$$

$$\implies c = 0 \quad (8)$$

$$\therefore y = \frac{e^x}{2} \quad (9)$$

The theoretical solution is $f(x) = \frac{e^x}{2}$

Solution Using Laplace Transform:

Laplace Transform

$$\mathcal{L}(f(x)) = F(s) = \int_0^\infty f(x)e^{-sx}dx \quad (10)$$

Laplace Transform is a linear transformation, since integration is a linear operation.

Laplace transform of some functions:

$$f(x) = 0 \implies F(s) = 0 \quad (11)$$

$$f(x) = 1 \implies F(s) = \frac{1}{s} \text{ for } Re(s) > 0 \quad (12)$$

$$f(x) = x^n \implies F(s) = \frac{n!}{s^{n+1}} \text{ for } Re(s) > 0 \quad (13)$$

$$f(x) = e^{ax} \implies F(s) = \frac{1}{s-a} \text{ for } Re(s) > a \quad (14)$$

$$f(x) = \sin ax \implies F(s) = \frac{a}{s^2 + a^2} \text{ for } Re(s) > 0 \quad (15)$$

$$f(x) = \cos ax \implies F(s) = \frac{s}{s^2 + a^2} \text{ for } Re(s) > 0 \quad (16)$$

Some other useful results include :

$$\mathcal{L}(f'(x)) = sF(s) - f(0^-) \quad (17)$$

$$\mathcal{L}(f''(x)) = s^2F(s) - sf(0^-) - f'(0^-) \quad (18)$$

Applying Laplace transform to (1) on both sides:

$$\mathcal{L}\left(\frac{dy}{dx} + y\right) = \mathcal{L}(e^x) \quad (19)$$

$$sF(s) - f(0^-) + F(s) = \frac{1}{s-1} \quad (20)$$

$$F(s) = \frac{1}{s^2-1} + \frac{f(0^-)}{s+1} \quad (21)$$

Using initial condition, $y(0^-) = 1/2$, we have

$$F(s) = \frac{s+1}{2(s^2-1)} \quad (22)$$

$$F(s) = \frac{1}{2(s-1)} \quad (23)$$

$$f(x) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) \quad (24)$$

Using (14) we can deduce solution to be the following

$$f(x) = \frac{e^x}{2} \quad (25)$$

Computational Solution:

We can plot a curve with individual closely spaced points such that

$$x_0 = 0 \quad (26)$$

$$y_0 = \frac{1}{2} \quad (27)$$

$$h = 0.001 \quad (28)$$

$$x_{n+1} = x_n + h \quad (29)$$

$$y_{n+1} = y_n + hy' \quad (30)$$

$$\Rightarrow y_{n+1} = y_n + h(e^x - y_n) \quad (31)$$

