12.8.3.8

EE24BTECH11006 - Arnav Mahishi

Question:
Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4} = 1$ and the line $\frac{x}{3}+\frac{y}{2} = 1$

input	Description	value
a	Length of semi major axis of ellipse	3
b	Length of semi minor axis of ellipse	2
v	Quadratic form of matrix	$\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}$
u	Linear coefficient vector	0
f	Constant Term	$-(a^2b^2)$
h	One of the points the line passes through	$\begin{pmatrix} a \\ 0 \end{pmatrix}$
m	Slope of line	$\begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix}$
n	number of subintervals we are taking	1000
x_0	x coordinate of first intersection point	3
x_n	y coordinate of second intersection point	2

TABLE 0: Variables Used

Theoretical Solution:
The point of intersection of the line with the ellipse is $x_i = h + k_i m$,
where,k_i is a constant and is calculated as follows:-

$$k_i = \frac{1}{m^\top Vm}\left(-m^\top (Vh+u) \pm \sqrt{[m^\top (Vh+u)]^2 - g(h)(m^\top Vm)}\right)$$

Substituting the input parameters in k_i,

$$k_i = \frac{1}{\begin{pmatrix} \frac{1}{b} & \frac{-1}{a} \end{pmatrix}\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix}}\left(-\begin{pmatrix} \frac{1}{b} & \frac{-1}{a} \end{pmatrix}\left(\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \pm \right.$$
$$\left.\sqrt{\left[\begin{pmatrix} \frac{1}{b} & \frac{-1}{a} \end{pmatrix}\left(\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)\right]^2 - g(h)\left(\begin{pmatrix} \frac{1}{b} & \frac{-1}{a} \end{pmatrix}\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix}\right)}\right) \quad (0.1)$$

We get,

$$k_i = 0, -ab$$

Substituting k_i in $x_i = h + k_i m$ we get,

$$x_1 = \begin{pmatrix} a \\ 0 \end{pmatrix} + (0) \begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix} \tag{0.2}$$

$$\implies x_1 = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{0.3}$$

$$x_2 = \begin{pmatrix} a \\ 0 \end{pmatrix} + (-ab) \begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix} \tag{0.4}$$

$$\implies x_2 = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} -a \\ b \end{pmatrix} \tag{0.5}$$

$$\implies x_2 = \begin{pmatrix} 0 \\ b \end{pmatrix} \tag{0.6}$$

The area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$ is

$$= \int_0^a \frac{b}{a} \sqrt{a^2 - x^2}\, dx - \int_0^a \frac{b}{a} (a - x)\, dx \tag{0.7}$$

$$= \frac{b}{a} \left(\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} - ax + \frac{x^2}{2} \right)_0^a \tag{0.8}$$

$$= \frac{b}{a} \left(\frac{\pi a^2}{4} - \frac{a^2}{2} \right) = \frac{ab}{2} \left(\frac{\pi}{2} - 1 \right) \tag{0.9}$$

The given area is $\frac{ab}{2} \left(\frac{\pi}{2} - 1 \right)$ sq. units
$\therefore$ Upon substituting $a = 3, b = 2$ the given area is $3 \left(\frac{\pi}{2} - 1 \right)$sq. units ≈ 1.712 sq. units

Computational Solution:
Using the Trapezoidal rule which approximates the integral of a function $f(x)$ over an interval $[a, b]$ by dividing the interval into n subintervals and approximating the area under the curve as a series of trapezoids

$$\int_a^b f(x)\, dx \approx \frac{h}{2} \left[f(x_0) + 2 \sum_{i=1}^{n-1} \left(f(x_i) + f(x_n) \right) \right] \tag{0.10}$$

Where x_0 is semi-major axis of ellipse and x_n is semi-minor axis of the ellipse and h is the width of each subinterval.

$$x_n = x_0 + n \cdot h \tag{0.11}$$

$$\implies h = \frac{x_n - x_0}{n} \tag{0.12}$$

In the case of our problem of the area between the line and ellipse the area is computed by:

$$A = \int_{x_0}^{x_n} \left(f_{ellipse}(x) - f_{line}(x)\right) dx \tag{0.13}$$

$$f_{ellipse}(x) = \sqrt{4\left(1 - \frac{x^2}{9}\right)} \tag{0.14}$$

$$f_{line}(x) = 2 - \frac{2x}{3} \tag{0.15}$$

Where $[x_0, x_n]$ are the intersection points.We need to find area of y_x from x_0 to x_n.Taking trapezoids of small width of h discretizing points on the x axis $x_0, x_1, x_2, \ldots, x_n$.The sum of the trapezoidal areas will be

$$A = \frac{1}{2}h(y(x_1) + y(x_0)) + \frac{1}{2}h(y(x_2) + y(x_1)) + \cdots + \frac{1}{2}h(y(x_n) + y(x_{n-1})) \tag{0.16}$$

$$= h\left[\frac{1}{2}(y(x_0) + y(x_n)) + y(x_1) + \cdots + y(x_{n-1})\right] \tag{0.17}$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \ldots x_n)$ be equidistant points with step-size h.

$$A(x_n + h) = A(x_n) + \frac{1}{2}h(y(x_n + h) + y(x_n)) \tag{0.18}$$

We can repeat this till we get required area.
Discretizing the steps, making $A(x_n) = A_n, y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h(y_{n+1} + y_n) \tag{0.19}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h((y_n + hy'_n) + y_n) \tag{0.20}$$

$$A_{n+1} = A_n + \frac{1}{2}h(2y_n + hy'_n) \tag{0.21}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.22}$$

$$x_{n+1} = x_n + h \tag{0.23}$$

In the given question, $y_n = \sqrt{4\left(1 - \frac{x_n^2}{9}\right)} + \frac{2x_n}{3} - 2$ and $y'_n = \frac{2}{3}\left(1 - \frac{x}{\sqrt{9-x^2}}\right)$
General Difference Equation will be given by,

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.24}$$

$$= A_n + h\left(\sqrt{4\left(1 - \frac{x_n^2}{9}\right)} + \frac{2x_n}{3} - 2\right) + \frac{1}{2}h^2\left(\frac{2}{3}\left(1 - \frac{x}{\sqrt{9 - x^2}}\right)\right) \tag{0.25}$$

$$x_{n+1} = x_n + h \tag{0.26}$$

Iterating till we reach $x_n = 3$ will return required area.

Area obtained computationally: 1.7123332003665432 sq. units
Area obtained theoretically: $3\left(\frac{\pi}{2} - 1\right) = 1.71238898038$ sq. units.
As n tends to infinity A_n will be the exact area of the ellipse.

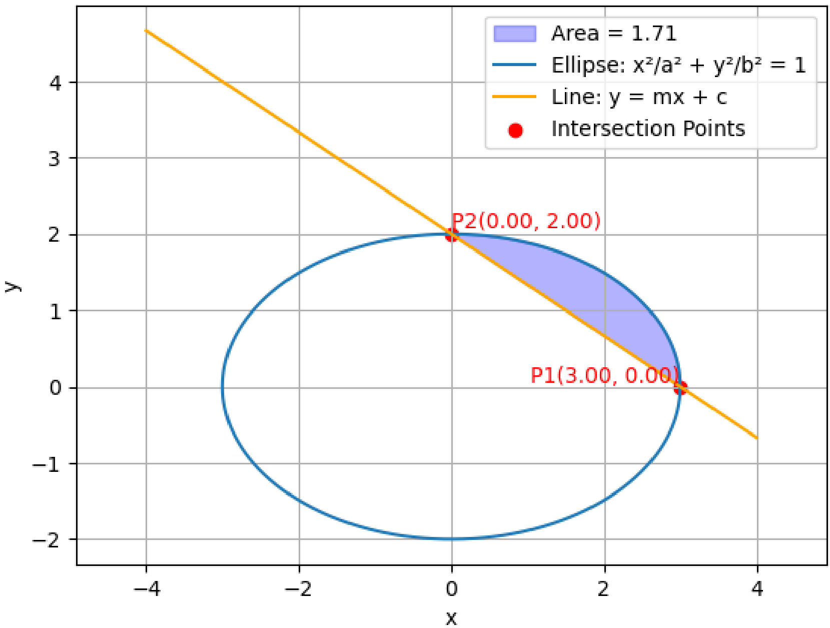

Fig. 0.1: Plot of the smaller area between ellipse and line