10.3.4.1.3

EE24BTECH11013 - MANIKANTA D

Problem:
Solve the system of linear equations:

$$3x - 5y - 4 = 0 \text{ and } 9x = 2y + 7. \tag{0.1}$$

Step 1: Represent the system in matrix form
The system of equations can be written as:

$$A\mathbf{x} = \mathbf{b}, \tag{0.2}$$

where

$$A = \begin{pmatrix} 3 & -5 \\ 9 & -2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}. \tag{0.3}$$

Step 2: Perform LU Decomposition
We decompose the matrix A into the product of a lower triangular matrix L and an upper triangular matrix U, i.e.,

$$A = LU, \tag{0.4}$$

where

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}, U = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}. \tag{0.5}$$

Now, let's compute the LU decomposition step by step.
First, we find the elements of U and L:

$$u_{11} = a_{11} = 3, u_{12} = a_{12} = -5. \tag{0.6}$$

Next, we compute l_{21} and u_{22}:

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{9}{3} = 3, \tag{0.7}$$

$$u_{22} = a_{22} - l_{21}u_{12} = -2 - 3 \times (-5) = 13. \tag{0.8}$$

So the LU decomposition is:

$$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, U = \begin{pmatrix} 3 & -5 \\ 0 & 13 \end{pmatrix}. \tag{0.9}$$

Step 3: Solve for x using LU decomposition
Now we solve the system in two steps using forward substitution and backward substitution.

First, solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$:

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}. \tag{0.10}$$

This gives:

$$y_1 = 4, 3y_1 + y_2 = -7 \Rightarrow 3(4) + y_2 = -7 \Rightarrow y_2 = -19. \tag{0.11}$$

Thus, $\mathbf{y} = \begin{pmatrix} 4 \\ -19 \end{pmatrix}$.

Next, solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$:

$$\begin{pmatrix} 3 & -5 \\ 0 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -19 \end{pmatrix}. \tag{0.12}$$

This gives:

$$13y = -19 \Rightarrow y = -\frac{19}{13}, \tag{0.13}$$

$$3x - 5y = 4 \Rightarrow 3x - 5\left(-\frac{19}{13}\right) = 4 \Rightarrow 3x = 4 - \frac{95}{13} \Rightarrow x = \frac{147}{39}. \tag{0.14}$$

Thus, the solution is $x = \frac{147}{39}$ and $y = -\frac{19}{13}$.

LU Decomposition using Doolittle's algorithm:

The LU decomposition can be efficiently computed using Doolittle's algorithm. This method generates the matrices L (lower triangular) and U (upper triangular) such that $A = LU$. The elements of these matrices are calculated as follows:

Elements of the U Matrix:

For each column j:

$$U_{ij} = A_{ij} \text{ if } i = 0, \tag{0.15}$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \text{ if } i > 0. \tag{0.16}$$

Elements of the L Matrix:

For each row i:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \text{ if } j = 0, \tag{0.17}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \text{ if } j > 0. \tag{0.18}$$

This systematic approach ensures that the matrix A is decomposed into L and U without requiring row swaps, provided A is nonsingular.

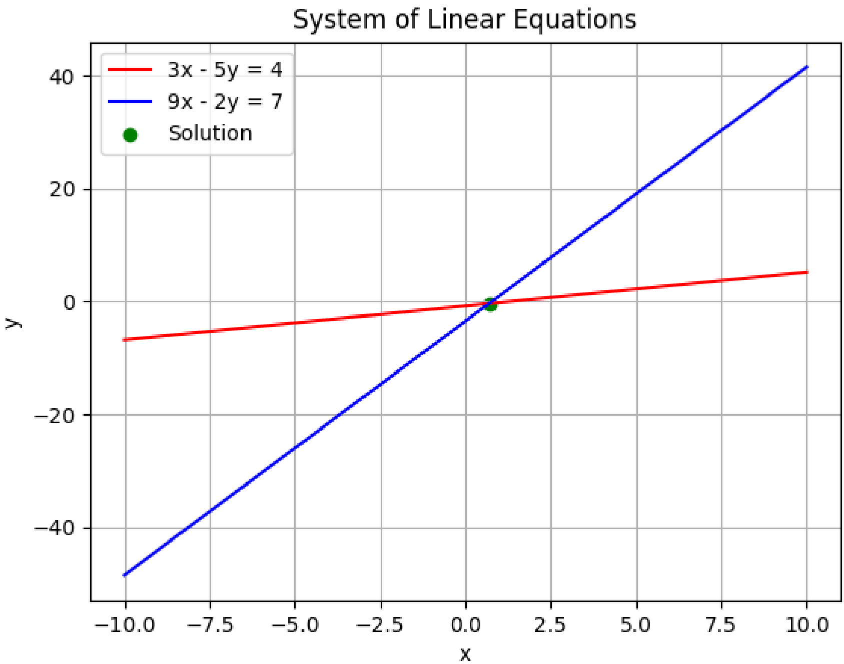

Fig. 0.1: Solving the system of equations