8.1.11

EE24BTECH11063 - Y. Harsha Vardhan Reddy

Question:
Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$

Solution:

Variable	Description	values
V	Quadratic form of the matrix	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
u	Linear coefficient vector	$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$
f	constant term	0
m	The direction vector of line	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
h	Point on line	$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

TABLE 0: Variables used

Theoritical Solution:

The point of intersection of the line with the circle is $x_i = h + k_i m$, where, k_i is a constant and is calculated as follows:-

$$k_i = \frac{1}{m^\top Vm}\left(-m^\top (Vh+u) \pm \sqrt{[m^\top (Vh+u)]2 - g(h)(m^\top Vm)}\right)$$

Substituting the input parameters into k_i,

$$k_i = \frac{1}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}\left(-\begin{pmatrix} 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix}\right)\right) \pm$$
$$\sqrt{\left[\begin{pmatrix} 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix}\right)\right]^2 - g(h)\left(\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)} \tag{0.1}$$

We get,
$k_i = \sqrt{12}, -\sqrt{12}$

Substituting k_i into $x_i = h + k_i m$ we get

$$x_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \left(\sqrt{12}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.2}$$

$$\implies x_1 = \begin{pmatrix} 3 \\ \sqrt{12} \end{pmatrix} \tag{0.3}$$

$$x_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \left(-\sqrt{12}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.4}$$

$$\implies x_2 = \begin{pmatrix} 3 \\ -\sqrt{12} \end{pmatrix} \tag{0.5}$$

Area of the region bounded by $y^2 = 4x$ and $x = 3$,

$$2 \times \int_0^3 \left(\sqrt{4x}\right) \cdot dx \tag{0.6}$$

$$= 2 \times 2 \times \left[\frac{x^{3/2}}{3/2}\right]_0^3 \tag{0.7}$$

$$= \frac{8}{3} \times \sqrt{27} \tag{0.8}$$

$$= 13.856 \tag{0.9}$$

The area of the region bounded between the curve $y^2 = 4x$ and $x = 3$ is 13.856 sq.units

Computational Solution:

Taking trapezoid-shaped strips of small area and adding them all up. Say we have to find the area of y_x from $x = x_0$ to $x = x_n$, discretize the points on the x axis $x_0, x_1, x_2, \ldots, x_n$ such that they are equally spaced with the step size h.
Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2}h\left(y\left(x_1\right) + y\left(x_0\right)\right) + \frac{1}{2}h\left(y\left(x_2\right) + y\left(x_1\right)\right) + \cdots + \frac{1}{2}h\left(y\left(x_n\right) + y\left(x_{n-1}\right)\right) \tag{0.10}$$

$$= h\left[\frac{1}{2}\left(y\left(x_0\right) + y\left(x_n\right)\right) + y\left(x_1\right) + \cdots + y\left(x_{n-1}\right)\right] \tag{0.11}$$

Let $A\left(x_n\right)$ be the area enclosed by the curve $y\left(x\right)$ from $x = x_0$ to $x = x_n$, $\left(x_0, x_1, \ldots x_n\right)$ be equidistant points with step-size h.

$$A\left(x_n + h\right) = A\left(x_n\right) + \frac{1}{2}h\left(y\left(x_n + h\right) + y\left(x_n\right)\right) \tag{0.12}$$

We can repeat this till we get required area.
Discretizing the steps, making $A\left(x_n\right) = A_n, y\left(x_n\right) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h\left(y_{n+1} + y_n\right) \tag{0.13}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left((y_n + hy'_n) + y_n\right) \tag{0.14}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) \tag{0.15}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{0.16}$$

$$x_{n+1} = x_n + h \tag{0.17}$$

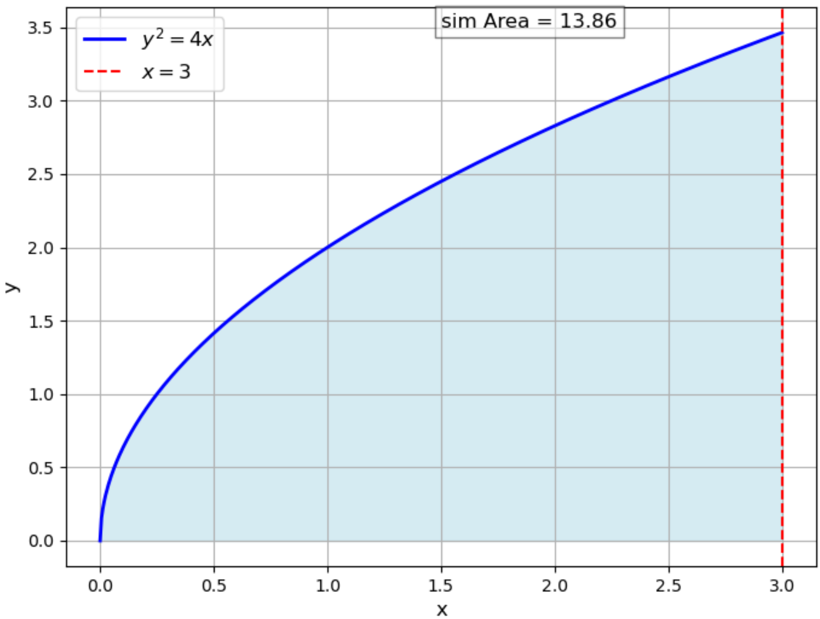
In the given question, $y_n = \sqrt{4x_n}$ and $y'_n = -\frac{4}{\sqrt{x}}$
The general difference equation will be given by

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{0.18}$$

$$A_{n+1} = A_n + h\left(\sqrt{4x_n}\right) + \frac{1}{2}h^2\left(-\frac{4}{\sqrt{x_n}}\right) \tag{0.19}$$

$$x_{n+1} = x_n + h \tag{0.20}$$

Iterating from $x_n = 0$ to $x_n = 3$ will return required area.(Upper half region)
The final result is multiplied by two include both half regions
Area obtained computationally: 13.856 sq. units
Area obtained theoretically: 13.856 sq. units


sim Area = 13.86
$y^2 = 4x$
$x = 3$
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0.0
0.5
1.0
1.5
2.0
2.5
3.0
x