11.16.3.8.9

EE24BTECH11002 - Agamjot Singh

Question:
Three coins are tossed at once. Find the probability of getting atmost two tails.
Solution:
Sample space (Ω) is given by,

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \tag{1}$$

Event space ($\mathcal{F}$) is given by,

$$\mathcal{F} = 2^{\Omega} \tag{2}$$

Let X be the random variable,

$$X = \text{number of tails in the sequence} \tag{3}$$

We express this random variable as a sum of 3 bernoulli random variables.

$$X = X_1 + X_2 + X_3 \tag{4}$$

where,

$$X_i = \begin{cases} 0 & i^{\text{th}} \text{ toss is a Heads} \\ 1 & i^{\text{th}} \text{ toss is a Tails} \end{cases} \tag{5}$$

X models a binomial distribution.
For converting to z-domain, we use the property,

$$M_X(z) = M_{X_1}(z) \, M_{X_2}(z) \, M_{X_3}(z) \tag{6}$$

Extending this system to m tosses, we get,

$$M_X(z) = \prod_{k=1}^{m} M_{X_k}(z) \tag{7}$$

Let probablity mass function function for the bernoulli random variable X_i be given by,

$$p_{X_i}(n) = \begin{cases} p & n = 0 \\ 1 - p & n = 1 \\ 0 & n = \mathbb{Z} - \{0, 1\} \end{cases} \tag{8}$$

where p is the probablity of getting heads.

$$M_{X_1}(z) = \sum_{k=-\infty}^{\infty} p_{X_1}(n) z^{-k} = p + (1-p) z^{-1} \tag{9}$$

$$M_{X_2}(z) = \sum_{k=-\infty}^{\infty} p_{X_2}(n) z^{-k} = p + (1-p) z^{-1} \tag{10}$$

$$\vdots \tag{11}$$

$$M_{X_m}(z) = \sum_{k=-\infty}^{\infty} p_{X_m}(n) z^{-k} = p + (1-p) z^{-1} \tag{12}$$

$$\implies M_X(z) = \left(p + (1-p) z^{-1}\right)^m \tag{13}$$

$$M_X(z) = \sum_{k=-\infty}^{\infty} \left({}^{m}C_k p^{m-k} (1-p)^k\right) z^{-k} \tag{14}$$

$$\tag{15}$$

Taking z-inverse on both sides, we get,

$$p_X(n) = {}^{m}C_n p^{m-n} (1-p)^n \tag{16}$$

Taking $m = 3$ and $p = \frac{1}{2}$,

$$p_X(n) = {}^{3}C_n \left(\frac{1}{2}\right)^3 \tag{17}$$

Using this probability mass function, the cummulative distribution function C.D.F ($F_X(n)$) is given by,

$$F_X(n) = \sum_{k=-\infty}^{n} {}^{3}C_k \left(\frac{1}{2}\right)^3 \tag{18}$$

$$F_X(n) = \begin{cases} 0 & n < 0 \\ {}^{3}C_0 \left(\frac{1}{2}\right)^3 = \frac{1}{8} & 0 \le n < 1 \\ {}^{3}C_0 \left(\frac{1}{2}\right)^3 + {}^{3}C_1 \left(\frac{1}{2}\right)^3 = \frac{1}{2} & 1 \le n < 2 \\ {}^{3}C_0 \left(\frac{1}{2}\right)^3 + {}^{3}C_1 \left(\frac{1}{2}\right)^3 + {}^{3}C_2 \left(\frac{1}{2}\right)^3 = \frac{7}{8} & 2 \le n < 3 \\ {}^{3}C_0 \left(\frac{1}{2}\right)^3 + {}^{3}C_1 \left(\frac{1}{2}\right)^3 + {}^{3}C_2 \left(\frac{1}{2}\right)^3 + {}^{3}C_3 \left(\frac{1}{2}\right)^3 = 1 & 3 \le n \end{cases} \tag{19}$$

Let A be an event defined as,

$$A: \text{Getting atmost two tails} \tag{20}$$

$$\Pr(A) = \Pr(X <= 2) = F_X(2) \tag{21}$$

$$\Pr(A) = \frac{7}{8} = 0.875 \tag{22}$$

Simulation Running a simulation requires generating random numbers with uniform probability. This is done using OpenSSL's random byte generator.

1) 1 byte of randomly generated uniform data is generated using OpenSSL rand.h.
2) This random number is scaled down from $\{0, 1, 2, \ldots 255\}$ to $[0, 1]$ by dividing by 255.
3) For generating the bernoulli random variable, if this normalized number is less than p then 0 is returned, else 1 is returned.

As number of trials increase, the relative frequency converges to the actual probability of the event.

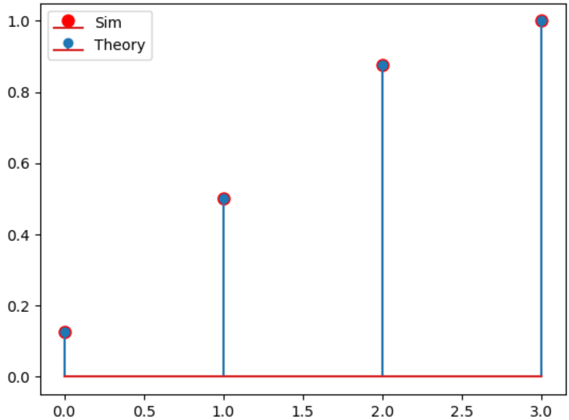

Fig. 3: CDF Plot

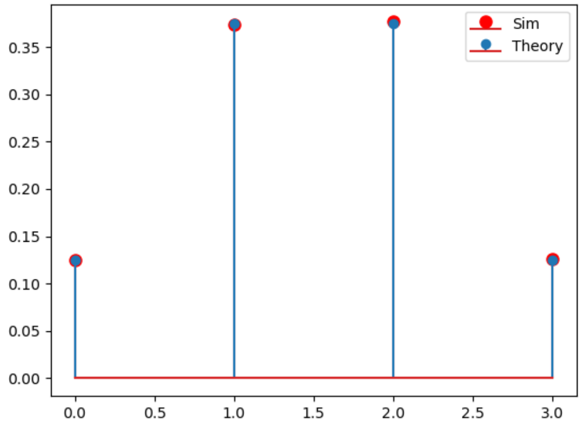

Fig. 3: PMF plot for $m = 3$

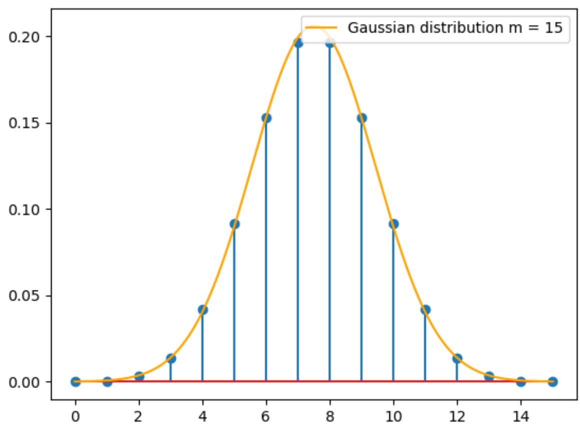

Fig. 3: PMF plot for $m = 15$ with the gaussian distrbution plot