11.16.1.4

Sai Akhila - EE24BTECH11055

Question: A fair coin is tossed and a fair six-sided die is rolled. Find the PMF of the random variable X, where X is the sum of the number on the die and 1 if the coin lands heads, and the number on the die if the coin lands tails. Verify through simulation.

Solution (Direct Calculation):
Let C be the outcome of the coin toss and D be the outcome of the die roll. We have:

$$C = \begin{cases} 1, & \text{if coin is heads} \\ 0, & \text{if coin is tails} \end{cases}$$

$D \in \{1, 2, 3, 4, 5, 6\}$

$X = C + D$

The possible values for X are $\{1, 2, 3, 4, 5, 6, 7\}$.

Since the coin and die are fair, we have: $P(C = 1) = 0.5$ $P(C = 0) = 0.5$ $P(D = i) = \frac{1}{6}$ for $i \in \{1, 2, 3, 4, 5, 6\}$

Now we can calculate the PMF of X:

- $P(X = 1) = P(C = 0, D = 1) = P(C = 0)P(D = 1) = 0.5 \times \frac{1}{6} = \frac{1}{12}$
- $P(X = 2) = P(C = 0, D = 2) + P(C = 1, D = 1) = (0.5 \times \frac{1}{6}) + (0.5 \times \frac{1}{6}) = \frac{2}{12} = \frac{1}{6}$
- $P(X = 3) = P(C = 0, D = 3) + P(C = 1, D = 2) = (0.5 \times \frac{1}{6}) + (0.5 \times \frac{1}{6}) = \frac{2}{12} = \frac{1}{6}$
- $P(X = 4) = P(C = 0, D = 4) + P(C = 1, D = 3) = (0.5 \times \frac{1}{6}) + (0.5 \times \frac{1}{6}) = \frac{2}{12} = \frac{1}{6}$
- $P(X = 5) = P(C = 0, D = 5) + P(C = 1, D = 4) = (0.5 \times \frac{1}{6}) + (0.5 \times \frac{1}{6}) = \frac{2}{12} = \frac{1}{6}$
- $P(X = 6) = P(C = 0, D = 6) + P(C = 1, D = 5) = (0.5 \times \frac{1}{6}) + (0.5 \times \frac{1}{6}) = \frac{2}{12} = \frac{1}{6}$
- $P(X = 7) = P(C = 1, D = 6) = P(C = 1)P(D = 6) = 0.5 \times \frac{1}{6} = \frac{1}{12}$

Solution (Using Z-transform):

Let C be the outcome of the coin toss (1 for Heads, 0 for Tails) and D be the outcome of the die roll (1 to 6). Then $X = C + D$.

The Z-transform of C is:

$$Z_C(z) = E[z^C] = 0.5z^0 + 0.5z^1 = 0.5 + 0.5z$$

The Z-transform of D is:

$$Z_D(z) = E[z^D] = \frac{1}{6}(z^1 + z^2 + z^3 + z^4 + z^5 + z^6)$$

Since C and D are independent, the Z-transform of $X = C + D$ is:

$$Z_X(z) = Z_C(z)Z_D(z) = (0.5 + 0.5z)\frac{1}{6}(z + z^2 + z^3 + z^4 + z^5 + z^6)$$

$$Z_X(z) = \frac{1}{12}(1 + z)(z + z^2 + z^3 + z^4 + z^5 + z^6)$$

$$Z_X(z) = \frac{1}{12}(z + 2z^2 + 2z^3 + 2z^4 + 2z^5 + 2z^6 + z^7)$$

The coefficient of z^k gives $P(X = k)$:

- $P(X = 1) = \frac{1}{12}$
- $P(X = 2) = \frac{2}{12} = \frac{1}{6}$
- $P(X = 3) = \frac{2}{12} = \frac{1}{6}$
- $P(X = 4) = \frac{2}{12} = \frac{1}{6}$
- $P(X = 5) = \frac{2}{12} = \frac{1}{6}$
- $P(X = 6) = \frac{2}{12} = \frac{1}{6}$
- $P(X = 7) = \frac{1}{12}$

Thus, the PMF is:

$$P_X(x) = \begin{cases} \frac{1}{12}, & x = 1 \\ \frac{1}{6}, & x = 2 \\ \frac{1}{6}, & x = 3 \\ \frac{1}{6}, & x = 4 \\ \frac{1}{6}, & x = 5 \\ \frac{1}{6}, & x = 6 \\ \frac{1}{12}, & x = 7 \\ 0, & \text{otherwise} \end{cases} \tag{0.1}$$

Simulation:

We simulate this process as follows:

1) Simulate the coin toss: Generate a uniform random number between $[0, 1)$. If the number is less than 0.5, the coin is heads (C=1), otherwise tails (C=0).
2) Simulate the die roll: Generate a discrete uniform random number between 1 and 6 (inclusive).
3) Calculate $X = C + D$.
4) Repeat steps 1-3 for a large number of trials (e.g., 10^5 simulations).
5) Count the occurrences of each possible value of X (1 through 7).
6) Divide the counts by the total number of trials to estimate the PMF.

The graph below shows the comparison between the theoretically calculated and simulated PMF of the given random variable.

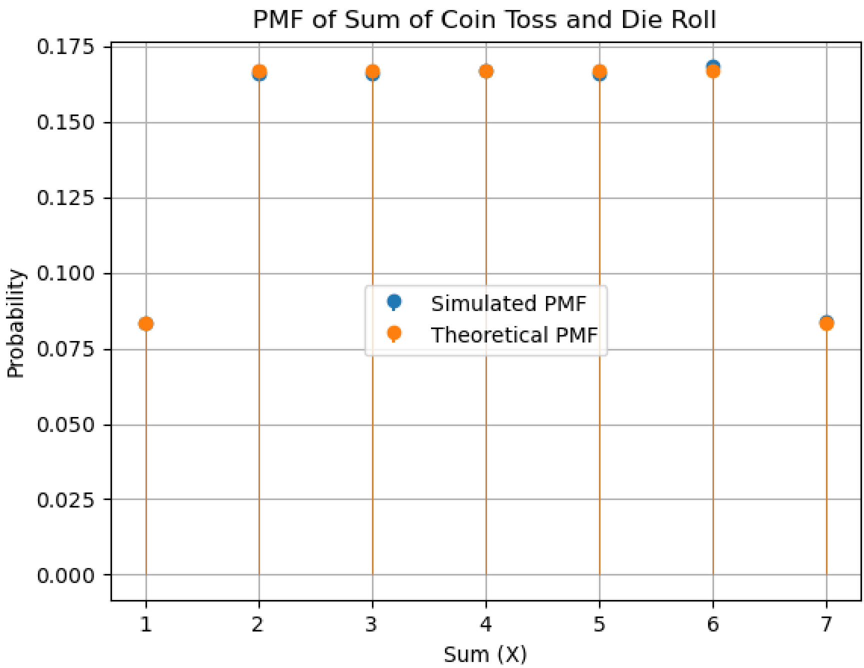

Fig. 6.1: Comparison of Theoretical and Simulated PMF