10.4.3.3.2

EE24BTECH11012 - Bhavanisankar G S

QUESTION :
Find the roots of the quadratic equation, $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, x \neq -4, 7.$
SOLUTION :

1) **QUADRATIC FORMULA** :
Consider an equation,

$$ax^2 + bx + c = 0 \tag{1.1}$$
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{1.2}$$
$$x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0 \tag{1.3}$$
$$\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right) = 0 \tag{1.4}$$
$$\left(x + \frac{b}{2a}\right) = \frac{\sqrt{b^2 - 4ac}}{2a} \tag{1.5}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.6}$$

which is the quadratic formula.
Given equation,

$$\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, x \neq -4, 7 \tag{1.7}$$
$$\frac{-11}{(x+4)\,x-7} = \frac{11}{30} \tag{1.8}$$
$$\implies x^2 - 3x + 2 = 0 \tag{1.9}$$
$$\implies x = 1 \tag{1.10}$$
$$x = 2 \tag{1.11}$$

2) **EIGENVALUE APPROACH** :
Consider the equation, (1.2). It can be rearranged as

$$\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0 \tag{2.1}$$
$$\lambda\left(\lambda + \frac{b}{a}\right) + \frac{c}{a} = 0 \tag{2.2}$$
$$-\lambda\left(-\lambda - \frac{b}{a}\right) - (-1)\frac{c}{a} = 0 \tag{2.3}$$

This can be considered equivalent to the determinant of the matrix,

$$\begin{pmatrix} -\lambda & 1 \\ -\frac{c}{a} & \frac{-b}{a} - \lambda \end{pmatrix} \tag{2.4}$$

Clearly, it can be seen that the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 \\ \frac{-c}{a} & \frac{-b}{a} \end{pmatrix} \tag{2.5}$$

are the roots of the required quadratic equation. This matrix, (2.5) is called the **Companion matrix** (**C**).
For the given question,

$$\mathbf{C} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \tag{2.6}$$

QR ALGORITHM : Eigenvalues of the companion matrix can be found using QR algorithm. Using the Gram-Schmidt orthogonalization, the matrix **C** can be factorized into

$$\mathbf{C} = \mathbf{QR} \tag{2.7}$$

where,

$$\mathbf{Q} = Orthonormalmatrix \tag{2.8}$$

$$\mathbf{R} = Uppertriangularmatrix \tag{2.9}$$

This process can be continues as

$$\mathbf{C_k} = \mathbf{Q_k R_k} \tag{2.10}$$

$$\mathbf{C_{k+1}} = \mathbf{R_k Q_k} \tag{2.11}$$

As $k \to \infty$, the diagonal elements of $\mathbf{Q_k}$ converge to the eigenvalues of the matrix. It can be seen that eigenvalues are 1 and 2.

3) **Newton-Raphson method** :
We have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3.1}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 3x_n + 2}{2x_n - 3} \tag{3.2}$$

Iterating and updating the value of x_n, we can obtain the roots of the quadratic equation.
The roots found using this method taking the initial guesses as 10 and 0 are 2.000000000905422 and 1.0000000022055868 respectively.

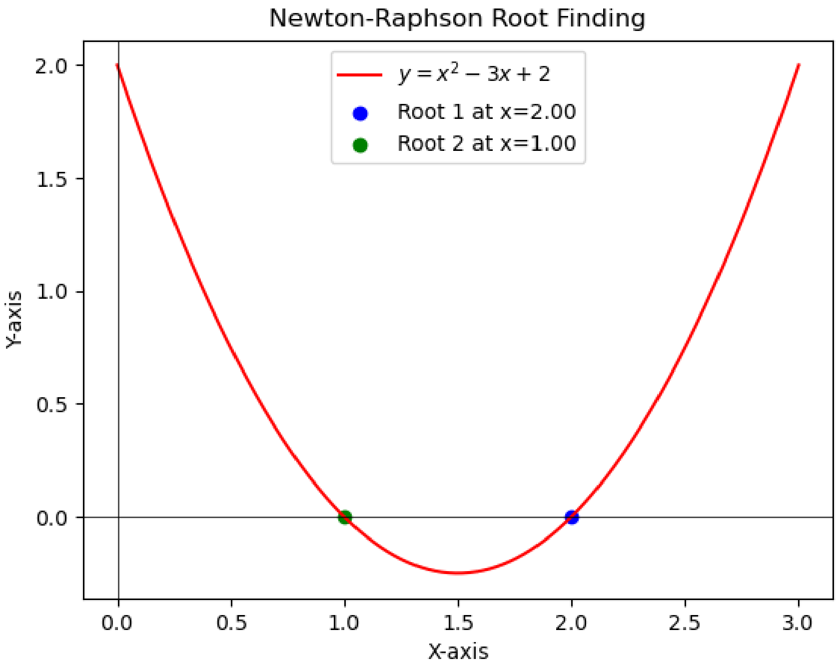

Fig. 3.1: Plot of the given question.