NCERT - 10.4.ex.18

EE24BTECH11040 - Mandara Hosur

Question:
Find the discriminant of the equation $3x^2 - 2x + \frac{1}{3} = 0$ and hence find the nature of its roots. Find them, if they are real.

Theoretical Solution:
Comparing with the standard form of a quadratic:

$$ax^2 + bx + c = 0 \tag{0.1}$$

We see that $a = 3$, $b = -2$, and $c = \frac{1}{3}$. Discriminant D is calculated as:

$$D = b^2 - 4ac = (-2)^2 - 4(3)\left(\frac{1}{3}\right) \tag{0.2}$$

$$\implies D = 0 \tag{0.3}$$

Since $D = 0$, the given quadratic equation has a single real solution.
From the quadratic formula, the solution x can be found as follows:

$$x = \frac{-b \pm \sqrt{D}}{2a} \tag{0.4}$$

$$\implies x = \frac{-b}{2a} \tag{0.5}$$

$$\implies x = \frac{-(-2)}{2(3)} \tag{0.6}$$

$$\implies x = \frac{1}{3} = 0.3333333333333333 \tag{0.7}$$

Newton-Raphson Method:
The formula for this method is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.8}$$

As per the given quadratic equation, define:

$$f(x) = 3x^2 - 2x + \frac{1}{3} \tag{0.9}$$

$$f'(x) = 6x - 2 \tag{0.10}$$

Therefore, the update equation for the Newton-Raphson method becomes:

$$x_{n+1} = x_n - \frac{3x_n^2 - 2x_n + \frac{1}{3}}{6x_n - 2} \tag{0.11}$$

Starting with an arbitrary initial guess $x_0 = 0$, x eventually converges to 0.3333326975492664 after 100 iterations. Therefore, by Newton-Raphson method:

$$x = 0.3333326975492664 \tag{0.12}$$

Companion Matrix:

Companion matrix can be written as

$$C = \begin{pmatrix} 0 & \frac{-c}{a} \\ 1 & \frac{-b}{-a} \end{pmatrix} \tag{0.13}$$

$$\implies C = \begin{pmatrix} 0 & \frac{1}{9} \\ 1 & \frac{2}{3} \end{pmatrix} \tag{0.14}$$

The eigenvalues of C are the roots of the given quadratic equation. The QR algorithm can be used to find the eigenvalues of C.
The QR algorithm repeatedly factorises the matrix C as:

$$C = Q_k R_k \tag{0.15}$$

Here, Q_k is an orthogonal matrix (from QR decomposition) and R_k is an upper triangular matrix.
The next iteration is

$$C_{k+1} = R_k Q_k \tag{0.16}$$

This process continues until the off-diagonal elements become negligibly small, revealing the eigenvalues of C along the diagonal.
Using this method and running 1000 iterations, the obtained eigenvalues are 0.3336667 and 0.33299997, which are both close. Taking their average,

$$x = \frac{0.3336667 + 0.33299997}{4} = 0.333333335 \tag{0.17}$$

Fixed-Point Iteration:

The fixed-point iteration method is based on rewriting the equation $f(x) = 0$ in the form $x = g(x)$, and iterating (until convergence) using the update equation

$$x_{n+1} = g(x_n) \tag{0.18}$$

As per the given quadratic equation, we have:

$$x = \frac{3x^2 + \frac{1}{3}}{2} \tag{0.19}$$

Therefore, the update equation is

$$x_{n+1} = \frac{3x_n^2 + \frac{1}{3}}{2} \tag{0.20}$$

Iterating 100 times, taking initial guess $x_0 = 0$, we get

$$x = 0.32706948008625647 \tag{0.21}$$

Plot:

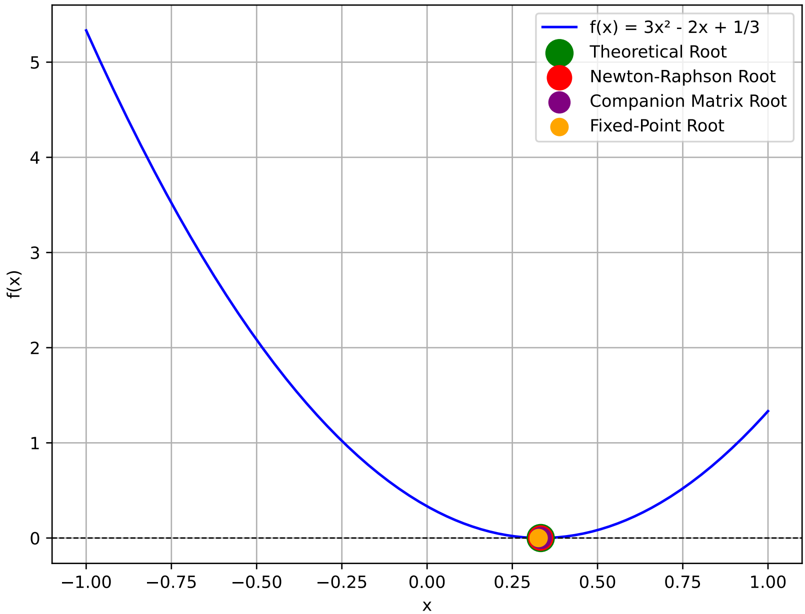

Fig. 0.1: Plot