12.8.3.1.2

EE24BTECH11019 - Dwarak A

Question:

Find the area under the given curves and given lines:

$$y = x^4 \tag{0.1}$$
$$x = 1 \tag{0.2}$$
$$x = 5 \tag{0.3}$$
$$y = 0 \quad \text{(x-axis)} \tag{0.4}$$

Solution:

Theoretical Solution:

Area between lines and curve A,

$$A = \int_1^5 x^4 dx \tag{0.5}$$
$$= \frac{x^5}{5}\bigg|_{x=5} - \frac{x^5}{5}\bigg|_{x=1} \tag{0.6}$$
$$= \frac{3125}{5} - \frac{1}{5} \tag{0.7}$$
$$= 624.8 \tag{0.8}$$

Simulated Solution (Trapezoidal Rule):

Taking trapezoid shaped strips of small area and adding them all up. Say we have to find the area of $y(x)$ from $x = x_0$ to $x = x_n$, discretize points on the x axis $x_0, x_1, x_2, \ldots, x_n$ such that they are equally spaced with step-size h.

Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2}h(y(x_1) + y(x_0)) + \frac{1}{2}h(y(x_2) + y(x_1)) + \cdots + \frac{1}{2}h(y(x_n) + y(x_{n-1})) \tag{0.9}$$
$$= h\left[\frac{1}{2}(y(x_0) + y(x_n)) + y(x_1) + \cdots + y(x_{n-1})\right] \tag{0.10}$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \ldots x_n)$ be equidistant points with step-size h.

$$A(x_n + h) = A(x_n) + \frac{1}{2}h(y(x_n + h) + y(x_n)) \tag{0.11}$$

We can repeat this till we get required area.
Discretizing the steps, making $A(x_n) = A_n, y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h(y_{n+1} + y_n) \tag{0.12}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h((y_n + hy'_n) + y_n) \tag{0.13}$$

$$A_{n+1} = A_n + \frac{1}{2}h(2y_n + hy'_n) \tag{0.14}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.15}$$

$$x_{n+1} = x_n + h \tag{0.16}$$

In the given question, $y_n = x_n^4$ and $y'_n = 4x_n^3$
General Difference Equation will be given by,

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.17}$$

$$A_{n+1} = A_n + h\left(x_n^4\right) + \frac{1}{2}h^2\left(4x_n^3\right) \tag{0.18}$$

$$A_{n+1} = A_n + hx_n^4 + 2h^2 x_n^3 \tag{0.19}$$

$$x_{n+1} = x_n + h \tag{0.20}$$

Iterating from $x_0 = 1$ till we reach $x_n = 5$ will return required area.
For $h = 0.05$, Area under the curve is 627.9241

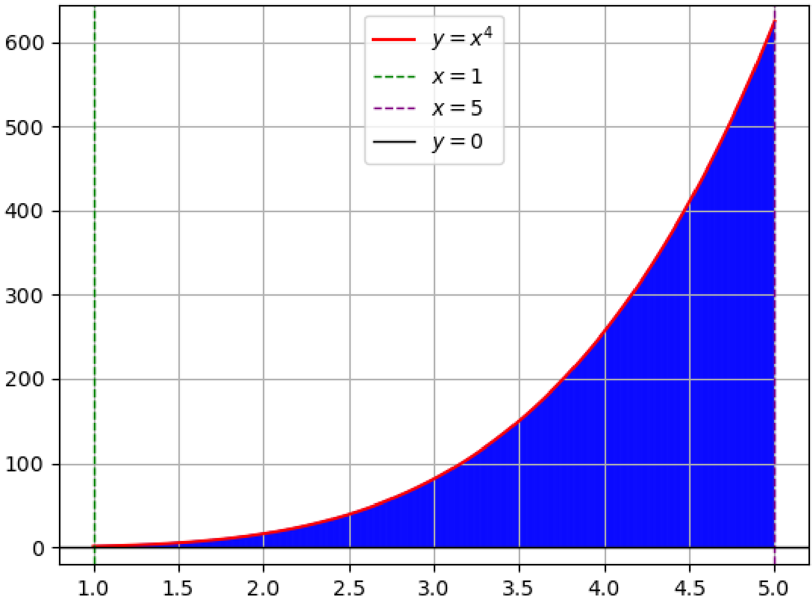

Fig. 0.1: Plot of the differential equation when $h = 0.01$