9.1.7

EE24BTECH11006 - Arnav Mahishi

Question:
Solve the differential equation $y''' + 2y'' + y' = 0$ with initial conditions $y(0) = 1, y'(0) = -1$, and $y''(0) = 1$
Solution:

Variable	Description
n	Order of given differential equation
y_i	ith derivative of the function in the equation
c	constant in the equation
a_i	coefficient of ith derivative of the function in the equation
$\mathbf{y}(t)$	Vector containing all 1 and y_i from $i = 0$ to $i = n - 1$
$\mathbf{y}'(t)$	Vector containing 1 and y_i' from $i = 0$ to $i = n - 1$
A	the coefficient matrix that transforms each y_i to its derivative
h	the stepsize between each t we are taking
t_o	The start time from which we are plotting
t_f	The end time at which we stop plotting

TABLE 0: Variables Used

Theoretical Solution: We apply the Laplace transform to each term in the equation. The Laplace transforms for the derivatives of $y(t)$ are:

$$\mathcal{L}y'(t) = sY(s) - y(0) \tag{0.1}$$

$$\mathcal{L}y''(t) = s^2Y(s) - sy(0) - y'(0) \tag{0.2}$$

$$\mathcal{L}y'''(t) = s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \tag{0.3}$$

Now, applying the Laplace transform to the entire differential equation:

$$\mathcal{L}\{y''' + 2y'' + y'\} = 0 \tag{0.4}$$

$$\mathcal{L}\{y'''(t)\} + 2\mathcal{L}\{y''(t)\} + \mathcal{L}\{y'(t)\} = 0 \tag{0.5}$$

$$(s^3Y(s) - s^2y(0) - sy'(0) - y''(0)) + 2(s^2Y(s) - sy(0) - y'(0)) + (sY(s) - y(0)) = 0 \tag{0.6}$$

Substitute the initial conditions $y(0) = 1$, $y'(0) = -1$, and $y''(0) = 1$:

$$\left(s^3Y(s) - s^2 \cdot 1 - s \cdot (-1) - 1\right) + 2\left(s^2Y(s) - s \cdot 1 - (-1)\right) + (sY(s) - 1) = 0 \tag{0.7}$$

$$s^3Y(s) - s^2 + s - 1 + 2s^2Y(s) - 2s + 2 + sY(s) - 1 = 0 \tag{0.8}$$

Simplify the equation:

$$(s^3 + 2s^2 + s)Y(s) - (s^2 - s + 1) - (2s - 2) - 1 = 0 \tag{0.9}$$

$$(s^3 + 2s^2 + s)Y(s) - s^2 - s + 1 - 2s + 2 - 1 = 0 \tag{0.10}$$

$$(s^3 + 2s^2 + s)Y(s) - (s^2 + s) = 0 \tag{0.11}$$

Now, solve for $Y(s)$:

$$\left(s^3 + 2s^2 + s\right)Y(s) = s^2 + s \tag{0.12}$$

$$Y(s) = \frac{s^2 + s}{s(s + 1)^2} \tag{0.13}$$

$$\implies Y(s) = \frac{1}{s + 1} \tag{0.14}$$

Now, take the inverse Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{1}{s + 1}\right) = e^{-t}u(t) \tag{0.15}$$

Thus, the solution to the differential equation is:

$$y(t) = e^{-t}u(t) \tag{0.16}$$

Radius of Convergence:
The denominator indicates a pole at $s = -1$.To ensure convergence of the Laplace transform integral, the real part of s must satsify:

$$Re(s) > -1 \tag{0.17}$$

Since the ROC extends infinitely to the right in the s-plane, the radius of convergence is:

$$R = \infty \tag{0.18}$$

Computational Solution:
Consider the given linear differential equation

$$a_n y_n + a_{n-1} y_{n-1} + \cdots + a_1 y_1 + a_0 y_0 + c = 0 \tag{0.19}$$

Where y_i is the ith derivative of the function then

$$\begin{pmatrix} y_0' \\ y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \frac{-(\sum_{i=0}^{i=n-1} a_i y_i)-c}{a_n} \end{pmatrix} \tag{0.20}$$

$$\implies \begin{pmatrix} 1 \\ y_0' \\ y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \frac{-c}{a_n} & \frac{-a_0}{a_n} & \frac{-a_1}{a_n} & \frac{-a_2}{a_n} & \cdots & \cdots & \frac{-a_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} 1 \\ y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} \tag{0.21}$$

$$\implies \mathbf{y}_k' = A\mathbf{y}_k \tag{0.22}$$

Where $\mathbf{y}_k$ is the vector $\begin{pmatrix} 1 \\ y_{0,k} \\ y_{1,k} \\ y_{2,k} \\ \vdots \\ y_{n-1,k} \end{pmatrix}$ at a k.

Using the trapezoidal rule,

$$J = \int_a^b f(x)\, dx \tag{0.23}$$

$$\approx h\left(\frac{1}{2}f(x) + f(x_1) + f(x_2) \cdots + f(x_{n-1}) + \frac{1}{2}f(b)\right) \tag{0.24}$$

$$\frac{\mathbf{y}_{n+1} - \mathbf{y}_n}{1}\left(\frac{1}{2} + \frac{1}{2}\right) = A\frac{x_{n+1} - x_n}{1}\left(\frac{\mathbf{y}_n}{2} + \frac{\mathbf{y}_{n+1}}{2}\right) \tag{0.25}$$

$$\rightarrow \mathbf{y}_{k+1} = \left(I - \frac{h}{2}A\right)^{-1} \cdot \left(I + \frac{h}{2}A\right) \cdot \mathbf{y}_k \tag{0.26}$$

Bilinear Transform:
Another way we can arrive at the difference equation is by using the Bilinear transform,

applying Laplace transform to both sides of the differential equation,

$$s^3Y(s) - s^2 + s - 1 + 2s^2Y(s) - 2s + 2 + sY(s) - 1 = 0 \tag{0.27}$$

$$(s^3 + 2s^2 + s)Y(s) - (s^2 + s) = 0 \tag{0.28}$$

$$\left(s^3 + 2s^2 + s\right)Y(s) = s^2 + s \tag{0.29}$$

$$Y(s) = \frac{s^2 + s}{s(s+1)^2} \tag{0.30}$$

$$\implies Y(s) = \frac{1}{s+1} \tag{0.31}$$

Apply Bilinear Transform with $T = h$

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{0.32}$$

$$Y(z) = \frac{1}{\frac{2}{T} \cdot \frac{1-z^{-1}}{1+z^{-1}} + 1} \tag{0.33}$$

$$Y(z) = \frac{1 + z^{-1}}{\frac{2}{h}\left(1 - z^{-1}\right) + \left(1 + z^{-1}\right)} \tag{0.34}$$

$$\implies Y(z) = \frac{1 + z^{-1}}{\left(1 + \frac{2}{h}\right) - \left(1 - \frac{2}{h}\right)z^{-1}} \tag{0.35}$$

$$\implies \text{Taking } \alpha = \frac{1 - \frac{2}{h}}{1 + \frac{2}{h}} \tag{0.36}$$

$$\implies Y(z) = \frac{(1 - \alpha)\left(1 + z^{-1}\right)}{2\left(1 + \alpha z^{-1}\right)} \tag{0.37}$$

$$\left(1 + \alpha z^{-1}\right)Y(z) = \frac{1 - \alpha}{2}\left(1 + z^{-1}\right) \tag{0.38}$$

$$\tag{0.39}$$

Applying Inverse Z transform, we get the following difference equation

$$y_n = \frac{1 - \alpha}{2}\left(1 + y_{n-1}\right) \tag{0.40}$$

$$\implies y_n = \frac{2}{1 - \alpha}\delta(n) + \frac{2}{1 - 3\alpha}y_{n-1}, |\alpha| < 1 \tag{0.41}$$

Trapzoidal form:
Using trapezoidal form when h is the step-size For the system $\mathbf{y}' = A\mathbf{y}$, the equation gives:

$$\frac{\mathbf{y}_{k+1} - \mathbf{y}_k}{h} = A \cdot \frac{\mathbf{y}_{k+1} + \mathbf{y}_k}{2} \tag{0.42}$$

Rearranging:

$$\mathbf{y}_{k+1} = \left(I - \frac{h}{2}A\right)^{-1} \cdot \left(I + \frac{h}{2}A\right) \cdot \mathbf{y}_k \tag{0.43}$$

For any particular differential equation derive B_1 and B_2 to find $\mathbf{y}_{k+1}$ from $\mathbf{y}_k$

$$B_1 = \left(I - \frac{h}{2}A\right) \tag{0.44}$$

$$B_2 = \left(I + \frac{h}{2}A\right) \tag{0.45}$$

For $y''' + 2y'' + y' = 0$ we get

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix} \tag{0.46}$$

$$\implies B_1 = \begin{pmatrix} 1 & \frac{-h}{2} & 0 \\ 0 & 1 & \frac{-h}{2} \\ 0 & \frac{h}{2} & 1+h \end{pmatrix} \tag{0.47}$$

$$\implies B_2 = \begin{pmatrix} 1 & \frac{h}{2} & 0 \\ 0 & 1 & \frac{h}{2} \\ 0 & \frac{-h}{2} & 1-h \end{pmatrix} \tag{0.48}$$

$$\tag{0.49}$$

When k ranges from 0 to $\frac{t_o - t_f}{h}$ in increments of 1, discretizing the steps gives us all $\mathbf{y}_k$, Record the $y_{0,k}$ for each k we got and then plot the graph. The result will be as given below.As we can see from the two graphs below the graph for bilnear transform is more accurate than the one found by finite differences proving its superiority.

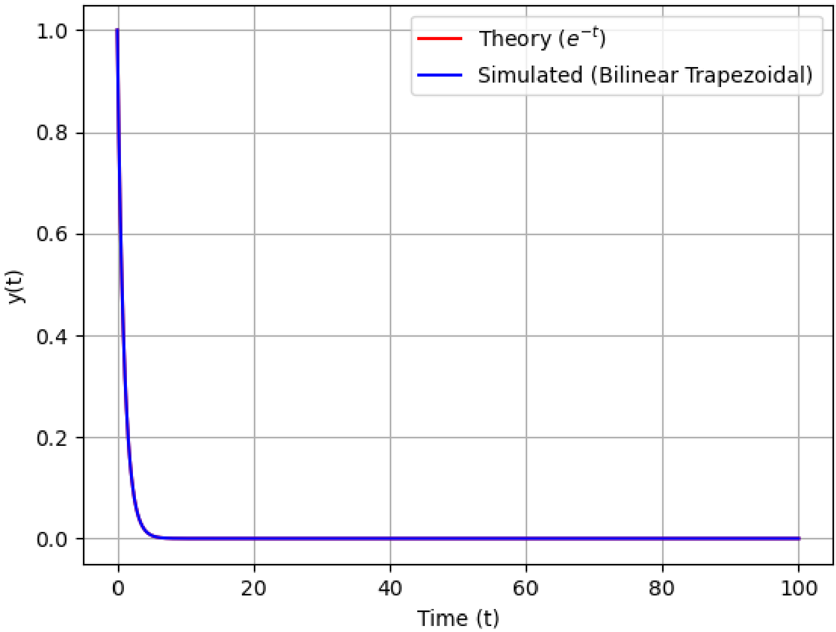

Fig. 0.1: Comparison between the Theoretical solution and Computational solution for Bilinear Transform

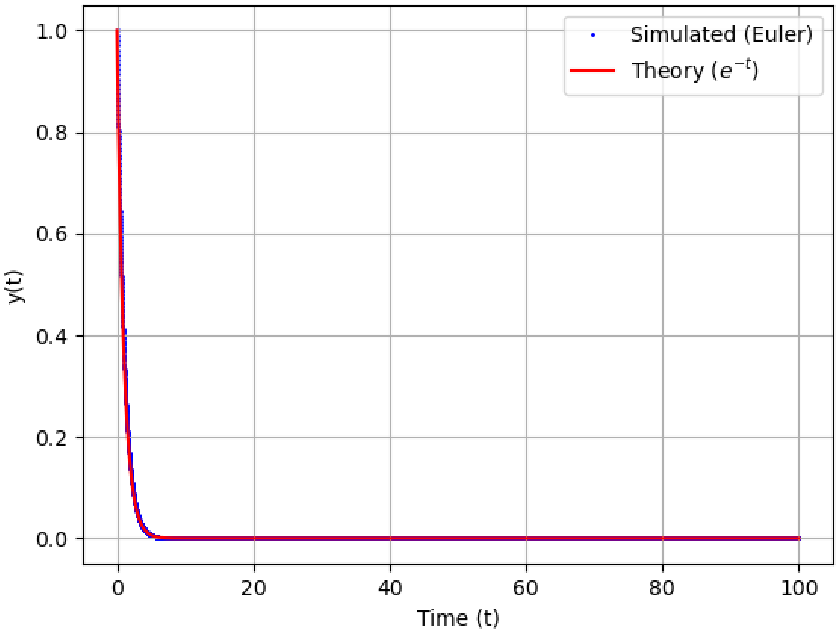

Fig. 0.2: Comparison between the Theoretical solution and Computational solution for Finite Differences