10.4.ex.12

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Question:
A rectangular park is to be designed whose breadth is 3 m less than its length. Its area is to be 4 square metres more than the area of a park that has already been made in the shape of an isosceles triangle with its base as the breadth of the rectangular park and of altitude 12 m . Find its length and breadth.
Solution:
Let the length of the park is $x + 3$ and breadth is x.
Given that the altitude of triangle is $12m$
Area of the isosceles triangle $=\frac{1}{2}(x)12$
Given that area of the rectangle is = 4+area of triangle
Area of the rectangle $= x(x + 3)$
Theoretical solution: According to the question:

$$\implies x(x+3) = 4 + 6x \implies x^2 - 3x - 4 = 0 \tag{1}$$

Applying the quadratic formula,we get

$$x_1 = \frac{3 + \sqrt{9 - 4(-4)}}{2} \tag{2}$$

$$x_1 = \frac{3 + \sqrt{25}}{2} \tag{3}$$

$$x_1 = 4 \tag{4}$$

$$x_2 = \frac{3 - \sqrt{9 - (-4)}}{2} \tag{5}$$

$$x_2 = \frac{3 - \sqrt{25}}{2} \tag{6}$$

$$x_2 = -1 \tag{7}$$

Therefore, the breadth of the rectangle is 4m and length is 7m
Computational Solution:

Newton-Raphson Method

The Newton-Raphson method is defined as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{8}$$

Here:

$$f(x) = x^2 - 3x - 4, \quad f'(x) = 2x - 3 \tag{9}$$

Substitute into the formula:

$$x_{n+1} = x_n - \frac{x_n^2 - 3x_n - 4}{2x_n - 3} \tag{10}$$

The problem with this method is if the roots are complex but the coeffcients are real, x_n either converges to an extrema or grows continuously without any bound. To get the complex solutions, however, we can just take the initial guess point to be a random complex number.

Starting with an initial guess $x_0 = 3$:

$$x_1 = 3 - \frac{3^2 - 3(3) - 4}{2(3) - 3} = 3 - \frac{9 - 9 - 4}{6 - 3} = 3 + \frac{4}{3} \approx 4.33 \tag{11}$$

$$x_2 = 4.33 - \frac{4.33^2 - 3(4.33) - 4}{2(4.33) - 3} \approx 4.02 \tag{12}$$

The same for the other root the output of a program written to find roots is shown below:

$$x_1 = 4 \tag{13}$$

$$x_2 = -1 \tag{14}$$

$$\tag{15}$$

Companion Matrix

For a quadratic equation of the form:

$$ax^2 + bx + c = 0, \tag{16}$$

the corresponding companion matrix is given by:

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}. \tag{17}$$

Substitute the coefficients $a = 1$, $b = -3$, ad $c = -4$ into the companion matrix formula:

$$A = \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} \tag{18}$$

QR Algorithm

The QR algorithm iteratively decomposes the matrix A_n into an orthogonal matrix Q_n and an upper triangular matrix R_n, and updates the matrix as:

$$A_{n+1} = R_n Q_n. \tag{19}$$

This process continues until A_n converges to an upper triangular matrix, where the diagonal elements are the eigenvalues of A.

Steps of the Algorithm

1) Initialize the companion matrix:

$$A_0 = \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}. \tag{20}$$

2) Perform QR decomposition of A_n:

$$A_n = Q_n R_n, \tag{21}$$

where Q_n is orthogonal and R_n is upper triangular.

3) Update the matrix:

$$A_{n+1} = R_n Q_n. \tag{22}$$

4) Repeat the above steps until A_n converges to an upper triangular matrix. The diagonal elements of this matrix are the eigenvalues, which correspond to the roots of the quadratic equation.

Roots of the Quadratic Equation

The eigenvalues of the companion matrix A are the roots of the quadratic equation. Applying the QR algorithm numerically to A, we find:

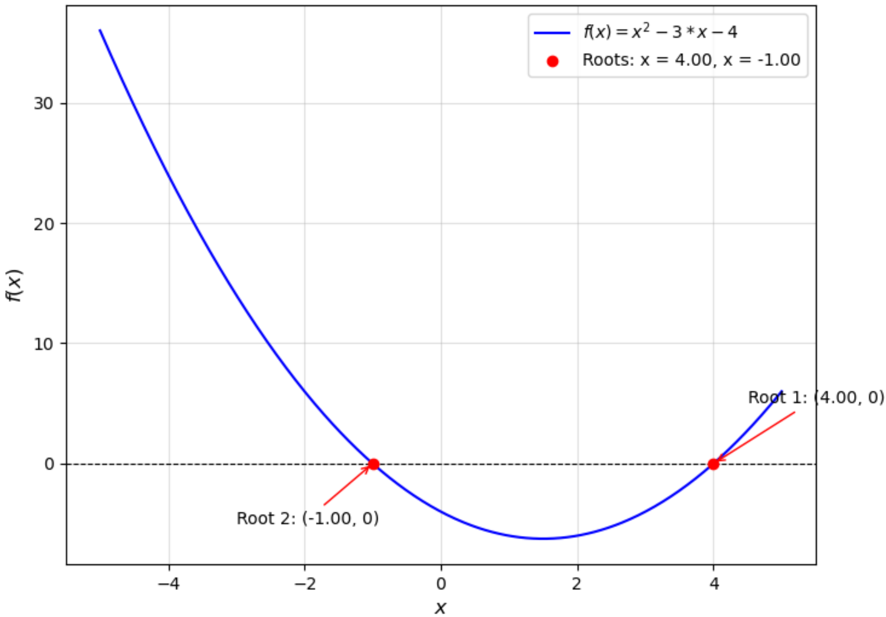
$$\lambda_1 = 4, \quad \lambda_2 = -1 \tag{23}$$

Conclusion

The QR decomposition method applied to the companion matrix of $x^2 - 3x - 4 = 0$ finds the roots of the equation. Both roots are real and distinct:

$$x_1 = 4, x_2 = -1 \tag{24}$$

This demonstrates the utility of the QR algorithm in computing eigenvalues, which are the roots of polynomial equations. Below is the plot for line and the parabola



$f(x) = x^2 - 3 * x - 4$
Roots: x = 4.00, x = -1.00
Root 1: (4.00, 0)
Root 2: (-1.00, 0)
30
20
10
0
$f(x)$
−4
−2
0
2
4
x