12.8.3.10

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Question:
Find the area enclosed by the parabola $x^2 = y$ and the line $y = x + 2$ and the x-axis.
Solution:

The equation of parabola is $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$. In matrix form, it is given by,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 0 = 0 \tag{1}$$

Line equation is,

$$\mathbf{x} = \kappa \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{2}$$

Intersection of a line and a conic is given by,

$$\kappa_i = \frac{-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\right]^2 - g(h)(\mathbf{m}^\top \mathbf{V}\mathbf{m})}}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{3}$$

For the given conic, $\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f = 0$. For the given line, $\mathbf{h} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$k_i = \frac{1}{\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \left(-\begin{pmatrix} 1 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right) \right) \pm$$

$$\sqrt{\left[\begin{pmatrix} 1 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right) \right]^2 - g(h) \left(\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)} \tag{4}$$

by the solving the equation we get

$$\implies \kappa_i = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{5}$$

The 2 curves meet at the points $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. So, the area between the curves is given by,
Theoretical Solution:

$$\int (x + 2) - \left(x^2\right) \tag{6}$$

$$\int (x+2) - \left(x^2\right) \tag{7}$$

$$= \left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \left[\frac{x^3}{3}\right]_{-1}^{2} \tag{8}$$

$$= \left(\frac{15}{2}\right) - (3) \tag{9}$$

$$= \frac{9}{2} sq.units \tag{10}$$

Simulated Solution:

Using the Trapezoidal rule which approximates the integral of a function $f(x)$ over an interval $[a, b]$ by dividing the interval into n subintervals and approximating the area under the curve as a series of trapezoids

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} (f(x_i) + f(x_n))\right] \tag{11}$$

Where x_0 is semi-major axis of ellipse and x_n is semi-minor axis of the ellipse and h is the width of each subinterval.

$$x_n = x_0 + n \cdot h \tag{12}$$

$$\implies h = \frac{x_n - x_0}{n} \tag{13}$$

Using this equation we can get the total area under the cuve by taking the sum of A_1 to A_n.

$$A = \frac{1}{2}h(y(x_1) + y(x_0)) + \frac{1}{2}h(y(x_2) + y(x_1)) + \cdots + \frac{1}{2}h(y(x_n) + y(x_{n-1})) \tag{14}$$

$$= h\left[\frac{1}{2}(y(x_0) + y(x_n)) + y(x_1) + \cdots + y(x_{n-1})\right] \tag{15}$$

By the first principle of derivatives,

$$y(x+h) = y(x) + hy'(x) \tag{16}$$

In this case, to calculate the area enclosed between the line and the parabola, we subtract the y coordinate of the parabola from the y coordinate of the line and then apply the trapezoidal rule on that function.

For the parabola,

$$\frac{dy}{dx} = 2x \tag{17}$$

For the line,

$$\frac{dy}{dx} = 1 \tag{18}$$

The general area element in this case is given by,

$$A_n = \frac{1}{2}h\left(y\left(x_n\right) + \left(y\left(x_n\right) + hy'\left(x_n\right)\right)\right) - \frac{1}{2}h\left(y\left(x_n\right) + \left(y\left(x_n\right) + hy'\left(x_n\right)\right)\right) \tag{19}$$

$$A_n = \frac{1}{2}h\left(x_n + 2 + h - x_n^2 - h2x\right) \tag{20}$$

$$\tag{21}$$

The general difference equation is given by,

$$A_{n+1} = A_n + \frac{1}{2}h\left(x_n + 2 + h - x_n^2 - h2x\right) \tag{22}$$

$$x_{n+1} = x_n + h \tag{23}$$

By iterating through the required value of n, we get the area enclosed between the line and the parabola.
Theoretical area = $\frac{9}{2}$ sq.units
Calculated area through trapezoidal rule = 4.21875 sq.units

Below is the plot for line and the parabola