10.3.6.1.4

EE24BTECH11052 - Rongali Charan

Question:
Solve the following pairs of equations by reducing them to a pair of linear equations:

$$\frac{5}{x-1}+\frac{1}{y-2}=2$$
$$\frac{6}{x-1}-\frac{3}{y-2}=1$$

Solution:
Let's solve this using LU decomposition. First, let's substitute:

$$\frac{1}{x-1}=u \tag{0.1}$$
$$\frac{1}{y-2}=v \tag{0.2}$$

Then our equations become:

$$5u+v=2 \tag{0.3}$$
$$6u-3v=1 \tag{0.4}$$

This can be written in matrix form as:

$$\begin{pmatrix}5 & 1\\6 & -3\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix} \tag{0.5}$$

Any non-singular matrix can be represented as a product of a lower triangular matrix L and an upper triangular matrix U

$$\mathbf{Ax}=\mathbf{LUx}=\mathbf{b} \tag{0.6}$$

Factorization of LU:
Given a matrix $\mathbf{A}$ of size $n\times n$, LU decomposition is performed row by row and column by column. The update equations are as follows:

- Start by initializing $\mathbf{L}$ as the identity matrix $\mathbf{L}=\mathbf{I}$ and $\mathbf{U}$ as a copy of $\mathbf{A}$.
- For each column $j\geq k$, the entries of U in the k-th row are updated as:

$$U_{k,j}=A_{k,j}-\sum_{m=1}^{k-1}L_{k,m}\cdot U_{m,j}\quad\forall\quad j\geq k \tag{0.7}$$

- For each row $i > k$, the entries of L in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right) \quad \forall \quad i > k \tag{0.8}$$

By doing the following steps and solving we get :

$$\mathbf{U} = \begin{pmatrix} 5 & 1 \\ 0 & -\frac{21}{5} \end{pmatrix} \tag{0.9}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{6}{5} & 1 \end{pmatrix} \tag{0.10}$$

Now,

$$A = \begin{pmatrix} 5 & 1 \\ 6 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{6}{5} & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 0 & -\frac{21}{5} \end{pmatrix} \tag{0.11}$$

We can solve this using two steps:

$$L\mathbf{y} = \mathbf{b} \tag{0.12}$$

$$U\mathbf{x} = \mathbf{y} \tag{0.13}$$

Using forward substitution:

$$\begin{pmatrix} 1 & 0 \\ \frac{6}{5} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{0.14}$$

This gives:

$$y_1 = 2 \tag{0.15}$$

$$\frac{6}{5}(2) + y_2 = 1 \tag{0.16}$$

$$y_2 = -\frac{7}{5} \tag{0.17}$$

Now using back substitution:

$$\begin{pmatrix} 5 & 1 \\ 0 & -\frac{21}{5} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{7}{5} \end{pmatrix} \tag{0.18}$$

This gives:

$$v = \frac{1}{3} \tag{0.19}$$

$$5u + \frac{1}{3} = 2 \tag{0.20}$$

$$u = \frac{1}{3} \tag{0.21}$$

Therefore:

$$\frac{1}{x-1} = \frac{1}{3} \implies x = 4 \tag{0.22}$$

$$\frac{1}{y-2} = \frac{1}{3} \implies y = 5 \tag{0.23}$$

The solution is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{0.24}$$

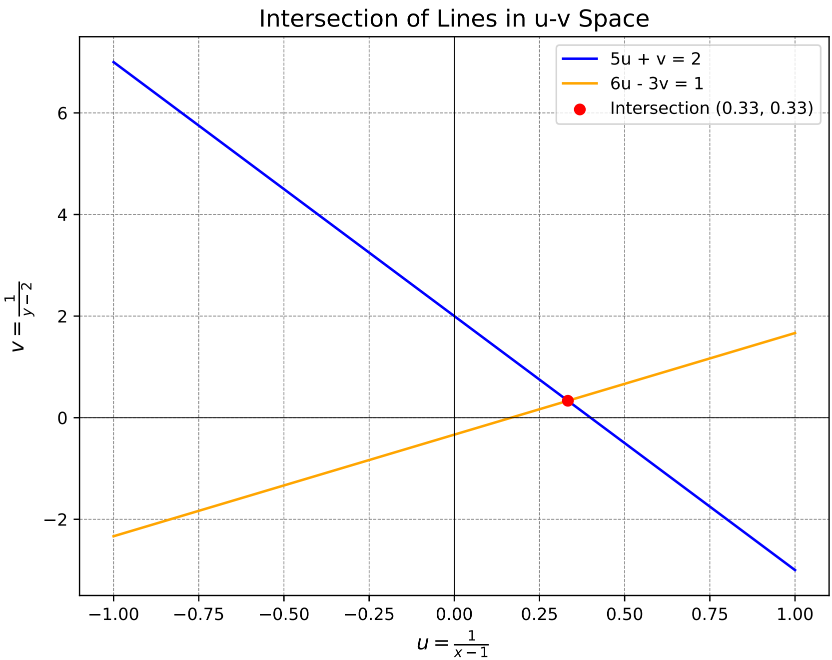

Fig. 0.1: Graph of the solution