9.7.7

EE24BTECH11005 - Arjun Pavanje

Question: Solve the differential equation $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1} = 0$, with initial conditions $y(0) = 0$

Solution:

Theoretical Solution:

This is a linear differential equation of the first order.

$$\frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1} \tag{1}$$

$$\frac{dy}{y^2+y+1} = -\frac{dx}{x^2+x+1} \tag{2}$$

Integrating on both sides,

$$\int \frac{dy}{y^2+y+1} = -\int \frac{dx}{x^2+x+1} \tag{3}$$

Completing the square,

$$\int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = -\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \tag{4}$$

We know that $\int \frac{dx}{x^2+a^2} = \tan^{-1}\left(\frac{x}{a}\right)$

$$\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) = -\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c \tag{5}$$

$$\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) = c \tag{6}$$

We know $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\tan^{-1}\left(\frac{2x++2y+2}{2-4xy-2x-2y}\right) = c \tag{7}$$

On simplifying we get,

$$(x+y+1) = A\left(1-x-y-2xy\right) \tag{8}$$

Where A is a constant. On substituting initial conditions we get,

$$(x+y+1) = (1-x-y-2xy) \tag{9}$$

$$y = \frac{-x}{1+x} \tag{10}$$

Computational Solution:

By first principle of derivatives,

$$y'(t) = \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} \tag{11}$$

$$y(t+h) = y(t) + hy'(t) \tag{12}$$

If we repeat the above proccess iteratively, we obtain the points to plot. Taking smaller step-size h will give more accurate plots. On discretizing the proccess we get,

$$y(x_{n+1}) = y(x_n) + hy'(x_n) \tag{13}$$

$$x_{n+1} = x_n + h \tag{14}$$

If we denote $y(x_n)$ as y_n, the equation (14) becomes,

$$y_{n+1} = y_n + hy'_n \tag{15}$$

The above equation is the general difference equation.

In the given question,

$$y' = -\frac{y^2 + y + 1}{x^2 + x + 1} \tag{16}$$

Difference Equation can be written as,

$$y_{n+1} = y_n - h\left(\frac{y_n^2 + y_n + 1}{x_n^2 + x_n + 1}\right) \tag{17}$$

Below is a comparission between Simulated Plot and Theoretical Plot.

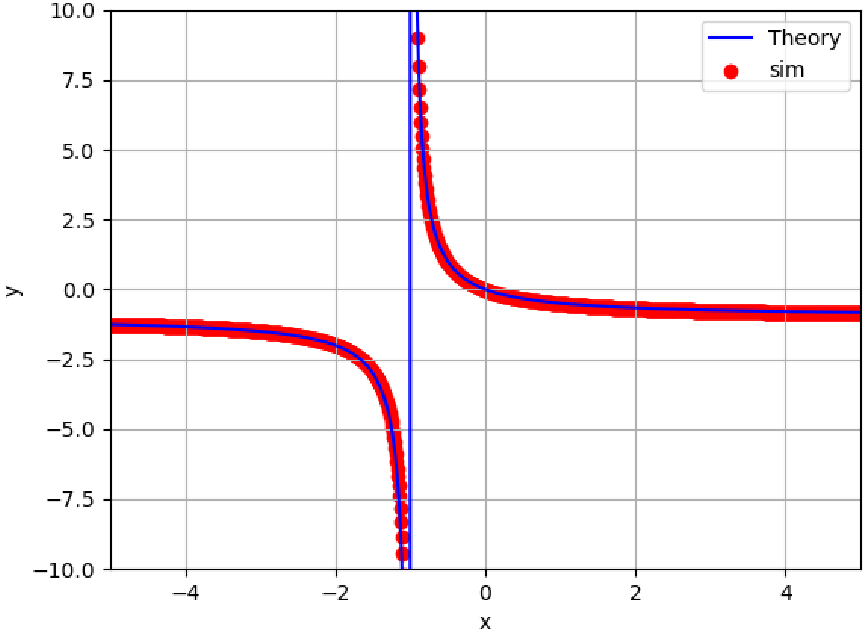

Fig. 1: Computational vs Theoretical solution of $\frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$