10.4.3.9

EE24BTECH11015 - Dhawal

Question:

Two water taps together can fill a tank in $\frac{75}{8}$ hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank separately. Find the time in which each tap can separately fill the tank.

Solution

Let time taken by each tap A, B to fill the tank be x, y respectively.
As tap A takes 10 hours less to fill the tank.

$$y = x + 10 \tag{1}$$

As total time taken by both to fill the tank is $\frac{75}{8}$

$$\left(\frac{1}{x} + \frac{1}{y}\right)\frac{75}{8} = 1 \tag{2}$$

$$\frac{1}{x} + \frac{1}{y} = \frac{8}{75} \tag{3}$$

Putting Eq. 1 in Eq. 3, we get

$$\frac{1}{x} + \frac{1}{x+10} = \frac{8}{75} \tag{4}$$

$$\frac{2x+10}{x(x+10)} = \frac{8}{75} \tag{5}$$

$$4x^2 - 35x - 375 = 0 \tag{6}$$

Theoretical Solution

Using quadratic formula, $a = 4, b = -35, c = -375$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7}$$

We get $x = 15$ and $x = -6.25$
We can't take negative values so $x = 15$ and $y = 25$ is the solution.

So, time taken by each tap A, B to fill the tank is $15, 25$ hours.

Computational Solution

Newton's Method

We will use Newton's Method for solving equations.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{8}$$

Where we define $f(x)$ as,

$$f(x) = 4x^2 - 35x - 375 \tag{9}$$
$$f'(x) = 8x - 35 \tag{10}$$

Thus, the new update equation is,

$$x_{n+1} = x_n - \frac{4x^2 - 35x - 375}{8x - 35} \tag{11}$$

This is a quadratic equation, it can have 2 solutions. As at $x = 0, f(x) \leq 0$. So we will iterate it from $(-100, 0)$ and $(0, 100)$. Take initial guess as $x_0 = 0$, we can see that x_n converges at $x = 15$ and $x = -6.25$.
Root 1 : -6.250000
Root 2 : 15.000000

Eigen Values

Companion matrix: A matrix is said to be the companion of a polynomial $f(x)$ if $det(A - \lambda I) = 0 \implies f(x) = 0$.
For,

$$f(x) = c_0 + c_1 x \cdots + x^n \tag{12}$$
$$f(x) = -93.75 - 8.75x + x^2 \tag{13}$$

The companion matrix is,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix} \tag{14}$$

For the equation at hand, the companion matrix is,

$$A = \begin{pmatrix} 0 & 93.75 \\ 1 & 8.75 \end{pmatrix} \tag{15}$$

QR-Algorithm

1) Initialization
   Let $A_0 = A$, where A is the given matrix.
2) QR Decomposition
   For each iteration $k = 0, 1, 2, \ldots$:
   a) Compute the QR decomposition of A_k, such that:

$$A_k = Q_k R_k \tag{16}$$

where:

i) Q_k is an orthogonal matrix ($Q_k^\top Q_k = I$).

ii) R_k is an upper triangular matrix.

The decomposition ensures $A_k = Q_k R_k$.

b) Form the next matrix A_{k+1} as:

$$A_{k+1} = R_k Q_k \tag{17}$$

3) Convergence

Repeat Step 2 until A_k converges to an upper triangular matrix T. The diagonal entries of T are the eigenvalues of A.

4) The eigenvalues of matrix will be the roots of the equation.

Using the QR algorithm we can now solve for the eigenvalues and thus the solutions for the given equation.

Eigenvalue 1 : $15.000000 + -0.000000j$

Eigenvalue 2 : $-6.250000 + 0.000000j$