8.1.9

EE24BTECH11005 - Arjun Pavanje

Question: Find the area of the region bounded by the parabola $y = x^2$, and $y = |x|$.
Solution:
Expressing the equation of parabola in matrix form $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 0 = 0 \tag{1}$$

Given line equation can be expressed as,

$$\mathbf{x} = \begin{cases} \kappa \begin{pmatrix} 1 \\ 1 \end{pmatrix} & x \geq 0 \\ \kappa \begin{pmatrix} 1 \\ -1 \end{pmatrix} & x < 0 \end{cases} \tag{2}$$

Intersection of a line and a conic is given by,

$$\kappa_i = \frac{-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(h)(\mathbf{m}^\top \mathbf{V}\mathbf{m})}}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{3}$$

For the given conic, $\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f = 0$. For the given line, $\mathbf{h} = \mathbf{0}$, $\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ when $x \geq 0$, $\mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ when $x < 0$.
Equation (3) simplifies to be,

$$\kappa_i = -\mathbf{m}^\top \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \pm \mathbf{m}^\top \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \tag{4}$$

$\mathbf{x} = 0$ is a common solution to both parts of the line equation. When $x \geq 0$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution, when $x < 0$, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is also a solution.

The curve $y = x^2$ and the line $y = |x|$ meet at three points $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Equation for area enclosed is given by,

$$\int_{-1}^{1}\left(|x|-x^2\right)dx \tag{5}$$
$$=\int_{-1}^{0}\left(x+x^2\right)dx+\int_{0}^{1}\left(x-x^2\right)dx \tag{6}$$
$$=2\int_{0}^{1}\left(x-x^2\right)dx \tag{7}$$

There are two ways to solve the above integral, Theoretically and Computationally (trapezoid method).We shall compare the results obtained by both methods.

Theoretical Solution:

$$2\int_{0}^{1}\left(x-x^2\right)dx \tag{8}$$
$$=2\left(\left[\frac{x^2}{2}\right]_{x=0}^{x=1}-\left[\frac{x^3}{3}\right]_{x=0}^{x=1}\right) \tag{9}$$
$$=2\left(\frac{1}{2}-\frac{1}{3}\right) \tag{10}$$
$$=\frac{1}{3} \tag{11}$$

Computational Solution:

Taking trapezoid shaped strips of small area and adding them all up. Say we have to find the area of y_x from $x=x_0$ to $x=x_n$, discretize points on the x axis $x_0,x_1,x_2,\ldots,x_n$ such that they are equally spaced with step-size h.
Sum of all trapezoidal areas is given by,

$$A=\frac{1}{2}h\left(y\left(x_1\right)+y\left(x_0\right)\right)+\frac{1}{2}h\left(y\left(x_2\right)+y\left(x_1\right)\right)+\cdots+\frac{1}{2}h\left(y\left(x_n\right)+y\left(x_{n-1}\right)\right) \tag{12}$$
$$=h\left[\frac{1}{2}\left(y\left(x_0\right)+y\left(x_n\right)\right)+y\left(x_1\right)+\cdots+y\left(x_{n-1}\right)\right] \tag{13}$$

Let $A\left(x_n\right)$ be the area enclosed by the curve $y\left(x\right)$ from $x=x_0$ to $x=x_n$, $\left(x_0,x_1,\ldots x_n\right)$ be equidistant points with step-size h.

$$A\left(x_n+h\right)=A\left(x_n\right)+\frac{1}{2}h\left(y\left(x_n+h\right)+y\left(x_n\right)\right) \tag{14}$$

We can repeat this till we get required area.
Discretizing the steps, making $A\left(x_n\right)=A_n, y\left(x_n\right)=y_n$ we get,

$$A_{n+1}=A_n+\frac{1}{2}h\left(y_{n+1}+y_n\right) \tag{15}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left((y_n + hy'_n) + y_n\right) \tag{16}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) \tag{17}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{18}$$

$$x_{n+1} = x_n + h \tag{19}$$

In the given question, $y_n = x_n + x_n^2$ and $y'_n = 1 - 2x_n$
General Difference Equation will be given by,

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{20}$$

$$= A_n + h\left(x_n + x_n^2\right) + \frac{1}{2}h^2\left(1 - 2x_n\right) \tag{21}$$

$$= A_n + x_n\left(h - h^2\right) + x_n^2\left(h\right) + \frac{h^2}{2} \tag{22}$$

$$x_{n+1} = x_n + h \tag{23}$$

Iterating till we reach $x_n = 1$ will return required area. Note, Area obtained is to be multiplied by 2 as the calculated area only accounts for one half of the graph.

Area obtained computationally: 0.3333195149898529 sq. units
Area obtained theoretically: $\frac{1}{3} = 0.33333\ldots$ sq. units

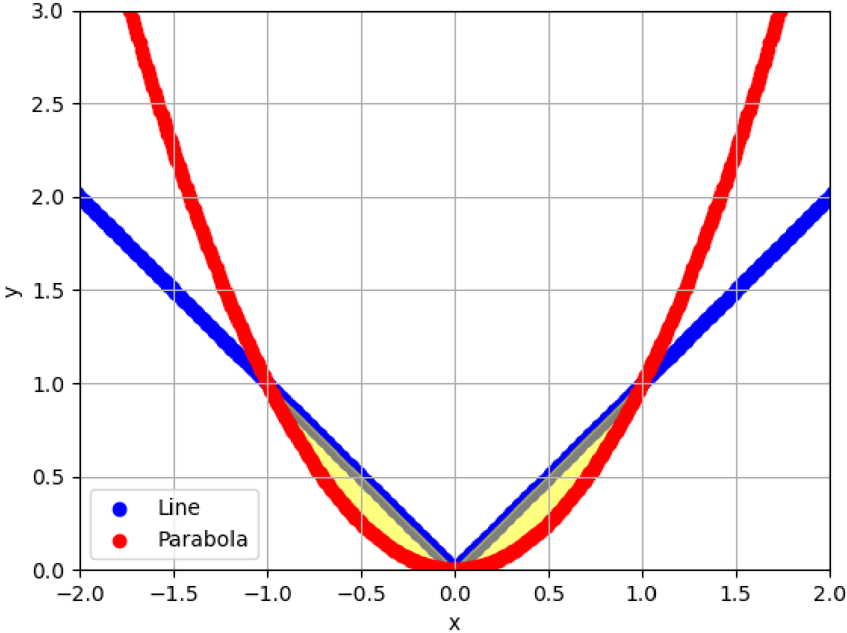

Fig. 1: Graph of the parabola $y = x^2$ and $y = |x|$ and the area enclosed between them