12.9.ex.1.1

EE24BTECH11003 - Akshara Sarma Chennubhatla

Question: Solve the differential equation $\frac{dy}{dx} - \cos x = 0$, with the initial condition $y(0) = 0$

Solution:
Theoretical Solution:

$$\frac{dy}{dx} = \cos x \tag{1}$$
$$\tag{2}$$

Integrating on both sides,

$$\int \frac{dy}{dx} dx = \int \cos x dx \tag{3}$$
$$y = \sin x + C \tag{4}$$
$$\tag{5}$$

Since $(0, 0)$ satisfies the function,

$$0 = \sin(0) + C \tag{6}$$
$$\implies 0 = 0 + C \tag{7}$$
$$\implies C = 0 \tag{8}$$
$$\tag{9}$$

So the function $y(x)$ is,

$$y = \sin x \tag{10}$$
$$\tag{11}$$

Simulated Solution:
The method being used here is the Bilinear Transform
First step is to apply Laplace Transform on both sides of the equation
Laplace transform by definition is,

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt \tag{12}$$

Properties of Laplace Transform used here are,

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1} \tag{13}$$
$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0) \tag{14}$$
$$\tag{15}$$

By applying Laplace Transform,

$$\mathcal{L}\left(\frac{dy}{dx}\right) = \mathcal{L}(\cos x) \tag{16}$$
$$sY(s) - y(0) = \frac{s}{s^2+1} \tag{17}$$
$$\tag{18}$$

By taking $y(0) = 0$,

$$Y(s) = \frac{1}{s^2+1} \tag{19}$$

Applying Bilinear transform, with $T = h$, we get,

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{20}$$
$$= \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} \tag{21}$$
$$Y(z) = \frac{1}{\left(\frac{2\left(1-z^{-1}\right)}{h\left(1+z^{-1}\right)}\right)^2 + 1} \tag{22}$$
$$Y(z) = \frac{h^2(z+1)^2}{4(z-1)^2 + h^2(z+1)^2} \tag{23}$$
$$Y(z)\left(\left(4+h^2\right)\left(z^2+1\right) + \left(h^2-4\right)2z\right) = h^2\left(z^2+1+2z\right) \tag{24}$$
$$z^2Y(z)\left(4+h^2\right) + Y(z)\left(4+h^2\right) + 2zY(z)\left(h^2-4\right) = h^2\left(z^2+1+2z\right) \tag{25}$$
$$\tag{26}$$

Properties of one sided z transform used here are,

$$\mathcal{Z}(y[n+2]) = z^2Y(z) - y[1]z - y[0] \tag{27}$$
$$\mathcal{Z}(y[n+1]) = zY(z) - zy[0] \tag{28}$$
$$\mathcal{Z}(y[n]) = Y(z) \implies \mathcal{Z}(y[n-n_0]) = z^{-n_0}Y(z) \tag{29}$$

By the time shift property (29),

$$\mathcal{Z}(\delta[n+2]) = z^2,\ z \neq 0 \tag{30}$$
$$\mathcal{Z}(\delta[n+1]) = z,\ z \neq 0 \tag{31}$$
$$\mathcal{Z}(\delta[n]) = 1 \tag{32}$$
$$\tag{33}$$

By rewriting the equation,

$$z^2 (Y(z) - y(1) z - y(0)) \left(4 + h^2\right) + Y(z) \left(4 + h^2\right) + \tag{34}$$

$$2 (zY(z) - zy(0)) \left(h^2 - 4\right) + \left(4 + h^2\right) (y(1) z + y(0)) + 2zy(0) \left(h^2 - 4\right) = h^2 \left(z^2 + 1 + 2z\right) \tag{35}$$

$$\tag{36}$$

For plotting the above difference equation, we need $y_0 = y(0)$ as well as y_1. To find $y_1 = y(0 + h) = y(h)$ we employ first principle of derivative,

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} \tag{37}$$

$$y(x+h) = y(x) + hy'(x), h \to 0 \tag{38}$$

$$y_1 = y(h) = y(0) + hy'(0) \tag{39}$$

$$y_1 = 0 + h \cos(0) \tag{40}$$

$$y_1 = h \tag{41}$$

Taking z inverse transform on both sides of the equation, we get the difference equation which is,

$$(y_{n+2} + y_n) \left(4 + h^2\right) + 2 \left(h^2 - 4\right) y_{n+1} + h \left(4 + h^2\right) \delta(n) = h^2 (\delta(n+2) + \delta(n) + 2\delta(n+1)) \tag{42}$$

Here, δ is given by,

$$\delta(n - n_0) = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases} \tag{43}$$

As $n > 0$,

$$\delta(n+2) = \delta(n+1) = 0 \tag{44}$$

Below is the simulated plot and the theoretical plot for given curve based on initial conditions, $x_0 = 0, y_0 = 0, y_1 = h$, obtained by iterating through the values of x with step size of h

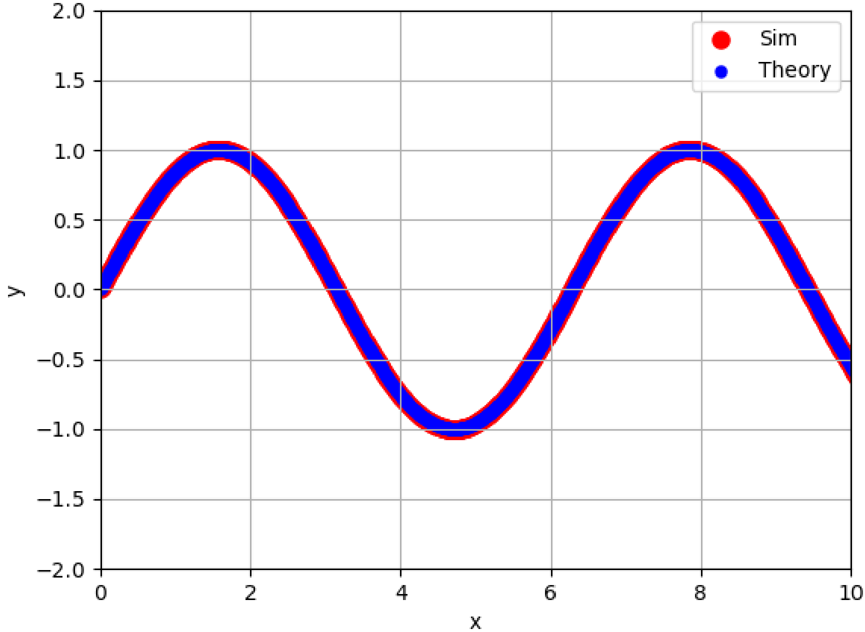

Fig. 1: Plot of the solution of $\frac{dy}{dx} - \cos x = 0$