NCERT-12.6.6.12

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Question: A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Theoretical Solution: Assume that θ be the angle between the hypotenuse and the height of the triangle, and x and y represent the portions of the hypotenuse corresponding to the lengths a and b projected along the hypotenuse.
The total length of the hypotenuse c can then be written as:

$$c = x + y. \tag{1}$$

The relationships between a, b, x, y, and θ are:

$$x = a\sec\theta, \quad y = b\csc\theta. \tag{2}$$

Thus,by substituting 2 in 1, the hypotenuse becomes:

$$c = a\sec\theta + b\csc\theta. \tag{3}$$

To minimize c, we differentiate it with respect to θ. Differentiating c gives:

$$\frac{dc}{d\theta} = a\sec\theta\tan\theta - b\csc\theta\cot\theta. \tag{4}$$

Setting $\frac{dc}{d\theta} = 0$ for critical points, we have:

$$a\sec\theta\tan\theta = b\csc\theta\cot\theta. \tag{5}$$

Using trigonometric identities, this simplifies to:

$$\frac{a\sin\theta}{\cos^2\theta} = \frac{b\cos\theta}{\sin^2\theta}. \tag{6}$$

Cross-multiplying:

$$a\sin^3\theta = b\cos^3\theta. \tag{7}$$

Dividing through by $\cos^3\theta\sin^3\theta$ gives:

$$\frac{\tan^3\theta}{1} = \frac{b}{a}. \tag{8}$$

Thus:

$$\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}. \tag{9}$$

Using $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, we calculate $\sin\theta$ and $\cos\theta$:

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{\left(\frac{b}{a}\right)^{\frac{1}{3}}}{\sqrt{1+\left(\frac{b}{a}\right)^{\frac{2}{3}}}}, \tag{10}$$

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+\left(\frac{b}{a}\right)^{\frac{2}{3}}}}. \tag{11}$$

Substituting these into 3, we get:

$$c = a\sqrt{1+\left(\frac{b}{a}\right)^{\frac{2}{3}}} + b\sqrt{1+\left(\frac{a}{b}\right)^{\frac{2}{3}}}. \tag{12}$$

Factoring and simplifying further, the result becomes:

$$c = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}. \tag{13}$$

Thus, the minimum length of the hypotenuse is:

$$\boxed{c_{\min} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}.} \tag{14}$$

Gradient Descent Method for finding local minima:
To find the value of θ for which the value of c will be the lowest, Gradient Descent Method can be used iteratively. For that, we need to choose a random value for θ, say θ_0, and apply the method on repeat until we can be as close as we can to the local minima. The process is as follows:
We treat the length of the hypotenuse as a function of the angle θ. 3

$$c = a\sec\theta + b\csc\theta. \tag{15}$$

By differentiating 3 and substituting the value of θ_0, we get to know the slope of the tangent at that point. By moving along the downward side of tangent, and by using the following Iteration,

$$\theta_{n+1} = \theta_n - \eta \times \left.\frac{dc}{d\theta}\right|_{\theta=\theta_n} \tag{16}$$

By substitution, we get:

$$\theta_{n+1} = \theta_n - \eta\left(\frac{a\sin\theta_n}{\cos^2\theta_n} - \frac{b\cos\theta_n}{\sin^2\theta_n}\right) \tag{17}$$

Performing iterations,

$$\theta_1 = \theta_0 - \eta \times \left(\frac{a\sin\theta_0}{\cos^2\theta_0} - \frac{b\cos\theta_0}{\sin^2\theta_0}\right) \tag{18}$$

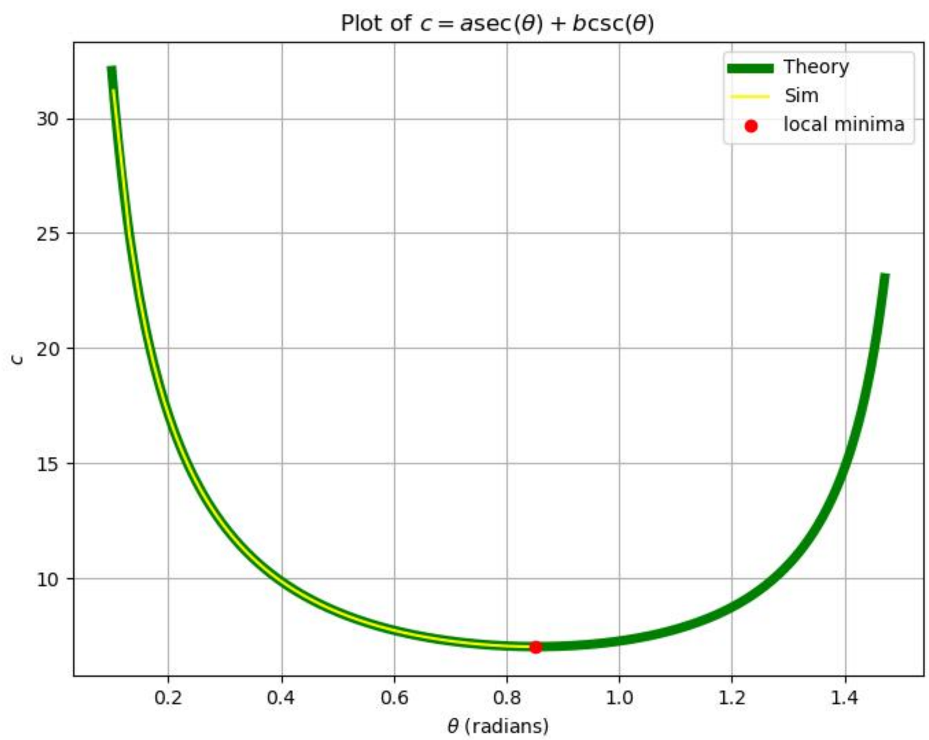
$$\theta_2 = \theta_1 - \eta \times \left(\frac{a\sin\theta_1}{\cos^2\theta_1} - \frac{b\cos\theta_1}{\sin^2\theta_1}\right) \tag{19}$$

$$\theta_3 = \theta_2 - \eta \times \left(\frac{a\sin\theta_2}{\cos^2\theta_2} - \frac{b\cos\theta_2}{\sin^2\theta_2}\right) \tag{20}$$

And so on ....
By opting certain value of ϵ such that

$$\left|\left(\frac{a\sin\theta_n}{\cos^2\theta_n} - \frac{b\cos\theta_n}{\sin^2\theta_n}\right)\right| < \epsilon \tag{21}$$

We can obtain the minimizing value of θ, i.e., θ^* such that $c(\theta^*)$ is minimum. For this iteration, the smaller the values of η and ϵ are, the greater the accuracy of the result will be.
Theory vs Simulation plot assuming certain values: In order to plot theory vs sim plot, I'm assuming the values of a and b to be 2 and 3 respectively. Theoretically, the value of θ^* is 0.844 radians. For simulation, I chose $\theta_0 = 0.1030$, $\epsilon = 0.000001$, and $\eta = 0.0001$. This resulted in θ^* to be approximately 0.851, which is almost the same value as the theoretical one. This shows the high efficiency of **Gradient Descent Method** in finding local minimas and maximas.



Plot of $c = a\sec(\theta) + b\csc(\theta)$
Theory
Sim
local minima
30
25
20
15
10
c
0.2
0.4
0.6
0.8
1.0
1.2
1.4
θ (radians)