9.7.2.4

EE24BTECH11021 - Eshan Ray

Question:
For the Differential Equation $\left(x^2+y^2\right)\frac{dy}{dx}-xy=0$, verify that $x^2=2y^2\log y$ is a solution of the differential equation.
Solution: Solving the given D.E. , we get,

$$\left(x^2+y^2\right)\frac{dy}{dx}-xy=0 \tag{1}$$
$$\implies \frac{dy}{dx}=\frac{xy}{x^2+y^2} \tag{2}$$
$$\tag{3}$$

Substituting, $y=vx$, we get,

$$\implies \frac{dy}{dx}=v+x\frac{dv}{dx} \tag{4}$$
$$\implies v+x\frac{dv}{dx}=\frac{x\left(vx\right)}{x^2+\left(vx\right)^2} \tag{5}$$
$$\implies v+x\frac{dv}{dy}=\frac{vx^2}{x^2\left(1+v^2\right)} \tag{6}$$
$$\implies x\frac{dv}{dx}=\frac{v}{1+v^2}-v \tag{7}$$
$$\implies x\frac{dv}{dx}=\frac{v-v-v^3}{1+v^2} \tag{8}$$
$$\implies x\frac{dv}{dx}=\frac{-v^3}{1+v^2} \tag{9}$$
$$\implies -\frac{\left(1+v^2\right)dv}{v^3}=\frac{dx}{x} \tag{10}$$
$$\implies -\frac{dv}{v^3}-\frac{v^2dv}{v^3}=\frac{dx}{x} \tag{11}$$
$$\tag{12}$$

Integrating both sides, we get,

$$\implies -\int\frac{dv}{v^3}-\int\frac{dv}{v}=\int\frac{dx}{x} \tag{13}$$
$$\implies \frac{1}{2v^2}-logv=logx+C \tag{14}$$
$$\tag{15}$$

putting $v = \frac{y}{x}$ we get,

$$\implies \frac{x^2}{2y^2} - \log\left(\frac{y}{x}\right) = \log x + C \tag{16}$$

$$\implies \frac{x^2}{2y^2} = \log y - \log x + \log x + C \tag{17}$$

$$\implies x^2 = 2y^2 \log y + 2Cy^2 \tag{18}$$

Putting $x = 0, y = 1$ we get,

$$\implies 0 = 2\,(1)^2\,(0) + 2C\,(1)^2 \tag{19}$$

$$\implies C = 0 \tag{20}$$

$$\implies x^2 = 2y^2 \log y \tag{21}$$

Computational Solution:

Using classical defination of derivative we get,

$$f\prime\,(x) = \frac{f\,(x+h) - f\,(x)}{h} \tag{22}$$

$$\implies f\,(x+h) = f\,(x) + f\prime\,(x)\,h \tag{23}$$

For $y = f\,(x)$, we can get the points of the required graph by iterating the equation obtained in (23) where values of x increases in each iteration by h and obtaining the y-coordinate of it.

For,

$$x_0 = -3 \tag{24}$$

$$y_0 = 2.31523 \tag{25}$$

$$h = 0.01 \tag{26}$$

$$\tag{27}$$

Using Euler Method, we get difference equation,

$$y_{n+1} = y_n + h\frac{dy}{dx}\Big|_{(x_n,y_n)} \tag{28}$$

$$y_{n+1} = y_n - \left(\frac{x_n y_n}{x_n^2 + y_n^2}\right)h \tag{29}$$

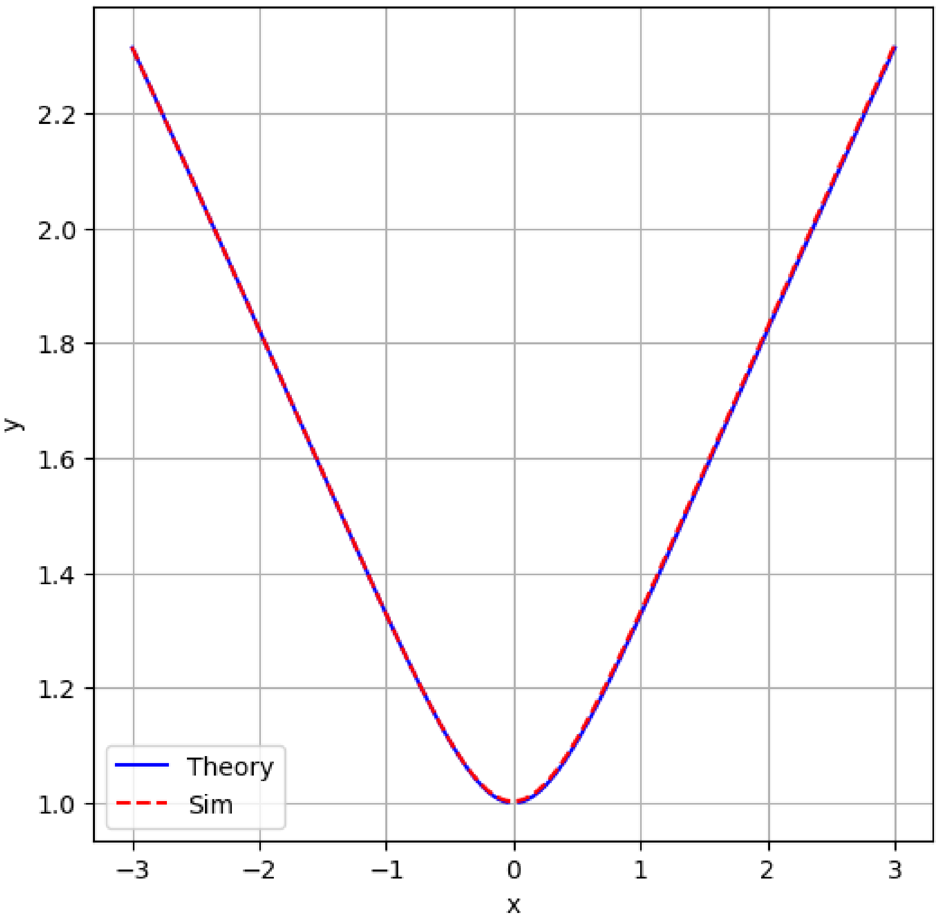

Fig. 0: Plot of the differential equation when $h = 0.01$