8.ex.8

EE24BTECH11049 - Patnam Shariq Faraz Muhammed

Question:
Find the area enclosed between the ellipse $\frac{x^2}{4} + \frac{y^2}{36} = 1$ and the line $\frac{x}{2} + \frac{y}{6} = 1$.
Solution:

Given	Formula
$\frac{x^2}{4} + \frac{y^2}{36} = 1$	$\mathbf{x^T V x} + 2\mathbf{u^T x} + f$
$\frac{x}{2} + \frac{y}{6} = 1$	$\mathbf{x} = \mathbf{h} + \kappa \mathbf{m}$

TABLE 0: Equations

Substituting the given values of we have
Conic:

$$\mathbf{V} = \begin{pmatrix} 36 & 0 \\ 0 & 4 \end{pmatrix} \tag{0.1}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.2}$$

$$f = -144 \tag{0.3}$$

Line:

$$\mathbf{h} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{0.4}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{0.5}$$

If a line intersects a conic, the κ value of the intersection points is given by

$$\kappa_i = \frac{-\mathbf{m}^\top (\mathbf{Vh} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{Vh} + \mathbf{u})]^2 - g(h)(\mathbf{m}^\top \mathbf{Vm})}}{\mathbf{m}^\top \mathbf{Vm}} \tag{0.6}$$

Substituting the given values, we get κ of the points of intersections as

$$\kappa_i = 0, 2 \tag{0.7}$$

Hence the points of intersection are $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Now $\frac{x^2}{4} + \frac{y^2}{36} = 1$ gives $y = \pm 3\sqrt{4 - x^2}$. But the common area lies in the first quadrant because the the points of intersection are on positive x and y axes.

The area bounded by the curve and the line is

Numerical solution:

$$= \int_0^2 3\sqrt{4-x^2} - (6-3x)\,dx \tag{0.8}$$

$$= 3\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2 - \left[6x - \frac{3x^2}{2}\right]_0^2 \tag{0.9}$$

$$= 3\left[0 + 2\sin^{-1}(1)\right] - [12-6] \tag{0.10}$$

$$= 3\pi - 6 \approx 3.42 \tag{0.11}$$

Computational method:

- Split the interval $[0,2]$ into N parts

$$h = \frac{2-0}{N} \tag{0.12}$$

- Consider the points

$$x_0 = 0 \tag{0.13}$$

$$x_N = 2 \tag{0.14}$$

$$x_{i+1} = x_i + h \tag{0.15}$$

- **Trapezoidal rule**
  Summing the areas of the trapezoids formed, we approximate the area between the line and curve Let

$$A = \int_0^2 \left(3\sqrt{4-x^2} - (6-3x)\right) dx \tag{0.16}$$

- It can be approximated as

$$f(x) = 3\sqrt{4-x^2} - 6 + 3x \tag{0.17}$$

$$A \approx \frac{h}{2}\sum_{i=1}^{N}\left(f(x_{i-1}) + f(x_i)\right) \tag{0.18}$$

$$j_{i+1} = j_i + \frac{h}{2}\left(f(x_i) + f(x_{i+1})\right) \tag{0.19}$$

$$j_{i+1} = j_i + \frac{h}{2}\left(3\sqrt{4-x_i^2} - 6 + 3x_i + 3\sqrt{4-x_{i+1}^2} - 6 + 3x_{i+1}\right) \tag{0.20}$$

Result:

Theoretical Area: 3.4247779607693793
Computed Area: 3.4247767135897336

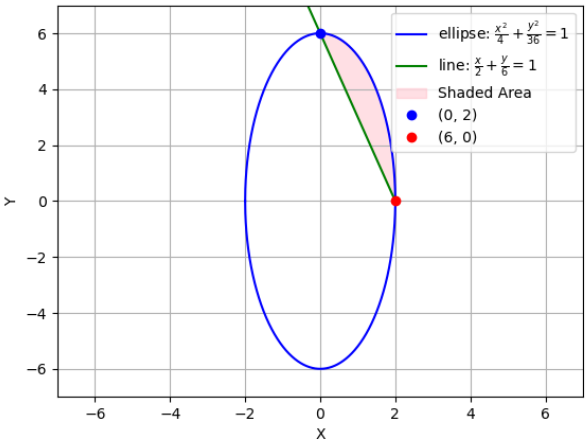

Fig. 0.1: Area between line and ellipse