# 8.3.7

EE24BTECH11003 - Akshara Sarma Chennubhatla

**Question:** Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.
**Solution:**

The equation of parabola is $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$. In matrix form, it is given by,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 0 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 0 = 0 \tag{1}$$

Line equation is,

$$\mathbf{x} = \kappa \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{2}$$

Intersection of a line and a conic is given by,

$$\kappa_i = \frac{-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(h)(\mathbf{m}^\top \mathbf{V}\mathbf{m})}}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{3}$$

For the given conic, $\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{2}{3} \end{pmatrix}, f = 0$. For the given line, $\mathbf{h} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$, $\mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\implies \kappa_i = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 12 \end{pmatrix} \tag{4}$$

The 2 curves meet at the points $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 12 \end{pmatrix}$. So, the area between the curves is given by,

$$\int \left(\frac{3}{2}x + 6\right) - \left(\frac{3}{4}x^2\right) \tag{5}$$

**Theoretical Solution:**

$$\int \left(\frac{3}{2}x + 6\right) - \left(\frac{3}{4}x^2\right) \tag{6}$$

$$= \left[\frac{3}{2}\left(\frac{x^2}{2}\right) + 6x\right]_{-2}^{4} - \left[\frac{3}{4}\left(\frac{x^3}{3}\right)\right]_{-2}^{4} \tag{7}$$

$$= (24 + 12 - 16) - (2 + 3 - 12) \tag{8}$$

$$= 27 sq.units \tag{9}$$

**Simulated Solution:**

First, we divide the interval $b-a$ into $n$ intervals of equal sizes, each of size $\frac{b-a}{n}$. We shall call each of the $x$ values at the boundaries as $x_1, x_2, x_3, ....x_{n+1}$, where $x_1 = a, x_{n+1} = b$ and the step size as $h$. Individual areas are in the shape of a trapezoid. So, we sum the values of the areas of the individual trapezoids to get the value of the definite integral between $a$ and $b$. For the $n_{th}$ trapezoid, the area is given by,

$$A_n = \frac{1}{2}(h)(y(x_n) + y(x_{n+1})) \tag{10}$$

Also,

$$A_{n+1} = A_n + \frac{1}{2}h(y_n + y_{n+1}) \tag{11}$$

Using this equation we can get the total area under the curve by taking the sum of $A_1$ to $A_n$.

$$A = \frac{1}{2}h(y(x_1) + y(x_0)) + \frac{1}{2}h(y(x_2) + y(x_1)) + \cdots + \frac{1}{2}h(y(x_n) + y(x_{n-1})) \tag{12}$$

$$= h\left[\frac{1}{2}(y(x_0) + y(x_n)) + y(x_1) + \cdots + y(x_{n-1})\right] \tag{13}$$

By the first principle of derivatives,

$$y(x+h) = y(x) + hy'(x) \tag{14}$$

In this case, to calculate the area enclosed between the line and the parabola, we subtract the $y$ coordinate of the parabola from the $y$ coordinate of the line and then apply the trapezoidal rule on that function.
For the parabola,

$$\frac{dy}{dx} = \frac{3x}{2} \tag{15}$$

For the line,

$$\frac{dy}{dx} = \frac{3}{2} \tag{16}$$

The general area element in this case is given by,

$$A_n = \frac{1}{2}h(y(x_n) + (y(x_n) + hy'(x_n)))\,(\text{for line}) - \frac{1}{2}h(y(x_n) + (y(x_n) + hy'(x_n)))\,(\text{for parabola}) \tag{17}$$

$$A_n = \frac{1}{2}h\left(3x_n + 12 + h\frac{3}{2} - \frac{3}{2}x_n^2 - h\frac{3x_n}{2}\right) \tag{18}$$

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The general difference equation is given by,

$$A_{n+1} = A_n + \frac{1}{2}h\left(3x_n + 12 + h\frac{3}{2} - \frac{3}{2}x_n^2 - h\frac{3x_n}{2}\right) \tag{20}$$

$$x_{n+1} = x_n + h \tag{21}$$

By iterating through the required value of $n$, we get the area enclosed between the line and the parabola.
Theoretical area = 27 sq.units
Calculated area through trapezoidal rule = 26.9815636 sq.units

Below is the plot for line and the parabola

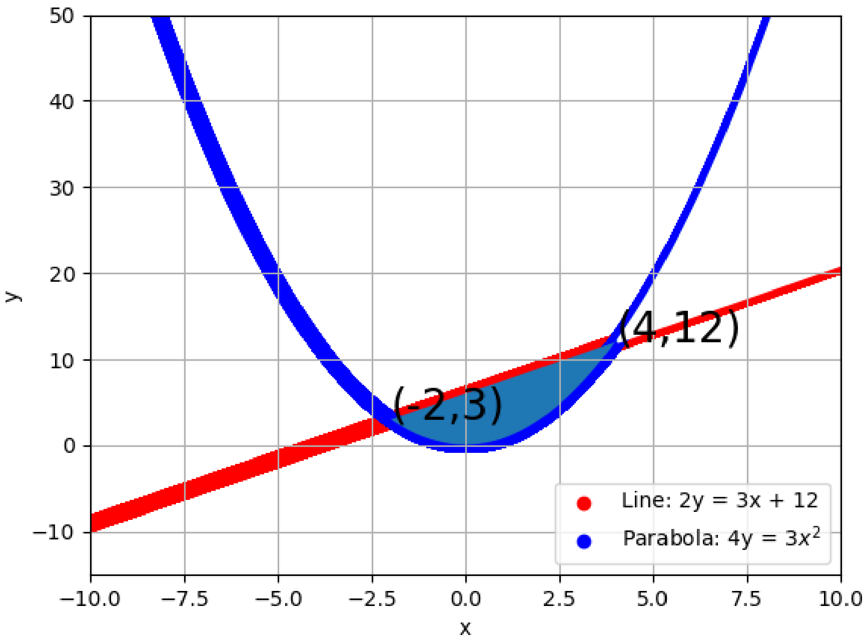


Fig. 1: Plot of the line $2y = 3x + 12$ and the parabola $= 4y = 3x^2$