8.1.6

EE24BTECH11002 - Agamjot Singh

Question:
Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$.

Theoritical solution:
Let the equation of the line be repsented as

$$\mathbf{x} = \mathbf{h} + \kappa\mathbf{m}, \text{ where } \mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{m} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \tag{1}$$

Let the equation of the circle be

$$\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0, \text{ where } V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } f = -4 \tag{2}$$

Point of intersection of the line with the circle is given by $x_i = h + \kappa_i m$, where κ_i is calculated as,

$$k_i = \frac{1}{\mathbf{m}^\top V\mathbf{m}} \left(-\mathbf{m}^\top (Vh + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^\top (Vh + \mathbf{u})\right]^2 - g(h)(\mathbf{m}^\top V\mathbf{m})} \right) \tag{3}$$

Substituting the values, we get,

$$k_i = \pm 2 \tag{4}$$

$$x_i = \pm \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{5}$$

As we are only considering the first quadrant, we take the point of intersection as $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$.

For finding the area enclosed A, we split the area into 2 parts A_1 and A_2 such that

$$A = A_1 + A_2 \tag{6}$$

$$A_1 = \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\,dx \tag{7}$$

$$= \left[\frac{x^2}{2\sqrt{3}}\right]_0^{\sqrt{3}} \tag{8}$$

$$= \frac{\sqrt{3}}{2} \approx 0.86602 \tag{9}$$

$$A_2 = \int_{\sqrt{3}}^{1} \sqrt{4-x^2}\,dx \tag{10}$$

$$= \left[\frac{x\sqrt{4-x^2}}{2} + 2\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2} \tag{11}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2} \approx 0.18117 \tag{12}$$

$$\implies A = A_1 + A_2 \approx 1.04719 \tag{13}$$

Computational Solution: Trapezoidal rule

For finding the approximate area enclosed using numerical methods, we use the Trapezoidal method. We split the area into multiple small trapeziums (like small strips), and we sum up all the trapezium areas to find the total area.

We discretize the range of x-coordinates with uniform step-size $h \to 0$, such that the discretized points are $x_0, x_1, \ldots, x_n$ and $x_{n+1} = x_n + h$.

Let the sum of trapizoidal areas till x_n be A_n and $y = y(x)$, then we write the **difference equation**,

$$A_n = \frac{h}{2}\left(y(x_0) + y(x_1)\right) + \frac{h}{2}\left(y(x_1) + y(x_2)\right) + \cdots + \frac{h}{2}\left(y(x_{n-1}) + y(x_n)\right) \tag{14}$$

$$A_n = h\left(\frac{y(x_0)}{2} + y(x_1) + y(x_2)\ldots\frac{y(x_n)}{2}\right) \tag{15}$$

$$A_{n+1} = A_n + \frac{h}{2}\left(y(x_{n+1}) + y(x_n)\right),\ x_{n+1} = x_n + h \tag{16}$$

$$A_{n+1} = A_n + \frac{h}{2}\left(y(x_n + h) + y(x_n)\right) \tag{17}$$

$$\tag{18}$$

By the first principle of derivative,

$$y'(x) = \lim_{h\to 0}\frac{y(x+h) - y(x)}{h} \tag{19}$$

$$y(x+h) = y(x) + h\left(y'(x)\right),\ h \to 0 \tag{20}$$

Rewriting the difference equation, we get,

$$A_{n+1} = A_n + \frac{h}{2}\left(y\left(x_n\right) + hy'\left(x_n\right) + y\left(x_n\right)\right) \tag{21}$$

$$A_{n+1} = A_n + h\left(y\left(x_n\right) + \frac{h}{2}y'\left(x_n\right)\right) \tag{22}$$

$$A_{n+1} = A_n + hy\left(x_n\right) + \frac{h^2}{2}y'\left(x_n\right) \tag{23}$$

For the given area enclosed, we take

$$y\left(x\right) = \begin{cases} \frac{x}{\sqrt{3}} & 0 < x < \sqrt{3} \\ \sqrt{4-x^2} & \sqrt{3} < x < 1 \end{cases} \tag{24}$$

Substituting $y\left(x\right)$, the equation becomes,

$$A_{n+1} = \begin{cases} A_n + h\frac{x_n}{\sqrt{3}} + \frac{h^2}{2\sqrt{3}} & 0 < x_n < \sqrt{3} \\ A_n + h\sqrt{4-x_n^2} + \frac{h^2}{2}\left(\frac{-x_n}{4-x_n^2}\right) & \sqrt{3} < x_n < 1 \end{cases} \tag{25}$$

$$x_{n+1} = x_n + 1 \tag{26}$$

Computational Area: 1.047367
Theoritical Area: 1.04719
Plotting the given equations, we get the following plot.

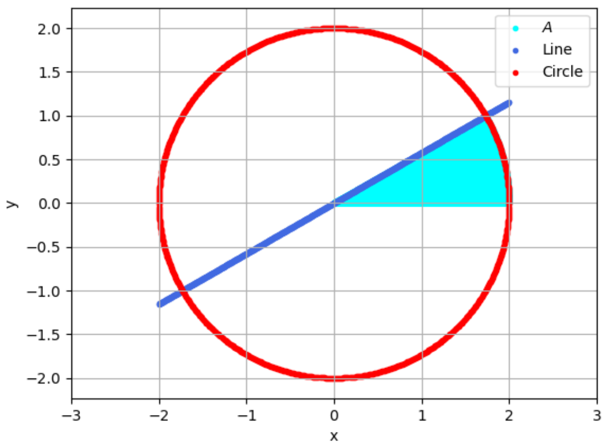

Fig. 0: Shaded area with the circle and line