NCERT-10.3.6.2.3

EE24BTECH11039 - MANDALA RANJITH

Question: Roohi travels 300km to her home partly by train and partly by bus. She takes 4hours if she tarvels 60km by train and the remaining by bus. If she travels 100km by train and the remaining by bus, she takes 10minutes longer. Find the speed of the train and the bus separately.

Solution:

$$\frac{60}{x} + \frac{240}{y} = 4 \tag{1}$$

$$\frac{100}{x} + \frac{200}{y} = \frac{25}{6} \tag{2}$$

1 Theoretical Solution

We solve the above equations using elimination:
We get $x = 60 \quad y = 80$

2 Numerical Method:

3 LU Decomposition to Solve the System

We now solve the system of equations using LU decomposition.

3.1 Matrix Form

The system of equations can be expressed in matrix form as:

$$\begin{pmatrix} 60 & 240 \\ 100 & 200 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 25/6 \end{pmatrix}. \tag{3}$$

Here, the coefficient matrix is:

$$A = \begin{pmatrix} 60 & 240 \\ 100 & 200 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 25/6 \end{pmatrix}. \tag{4}$$

3.2 Step 1: Decomposing A into L and U

The matrix A can be decomposed into:

$$A = L \cdot U, \tag{5}$$

where:

$$L = \begin{pmatrix} 1 & 0 \\ \frac{3}{5} & 1 \end{pmatrix}, \tag{6}$$

$$U = \begin{pmatrix} 100 & 200 \\ 0 & 120 \end{pmatrix}. \tag{7}$$

Step 2: LU Factorization Using Update Equations

Given a matrix **A** of size $n \times n$, LU decomposition is performed row by row and column by column. The update equations are as follows:

Step-by-Step Procedure: 1. **Initialization:** - Start by initializing **L** as the identity matrix **L** = **I** and **U** as a copy of **A**.

2. **Iterative Update:** - For each pivot $k = 1, 2, \ldots, n$: - Compute the entries of U using the first update equation. - Compute the entries of L using the second update equation.

3. **Result:** - After completing the iterations, the matrix **A** is decomposed into **L**·**U**, where **L** is a lower triangular matrix with ones on the diagonal, and **U** is an upper triangular matrix.

1. Update for $U_{k,j}$ (Entries of U)

For each column $j \geq k$, the entries of U in the k-th row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j}, \quad \text{for } j \geq k.$$

This equation computes the elements of the upper triangular matrix **U** by eliminating the lower triangular portion of the matrix.

2. Update for $L_{i,k}$ (Entries of L)

For each row $i > k$, the entries of L in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right), \quad \text{for } i > k.$$

This equation computes the elements of the lower triangular matrix **L**, where each entry in the column is determined by the values in the rows above it.

LU Factorization of Matrix A

We decompose A as:

$$A = LU,$$

where L is a lower triangular matrix and U is an upper triangular matrix. For the given example, we calculate L and U as follows:

$$L = \begin{bmatrix} 1 & 0 \\ \frac{3}{5} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 100 & 200 \\ 0 & 120 \end{bmatrix}.$$

Theoretical Solution using LU Decomposition

Step 1: Convert to Matrix Form

Given the equations:

$$\frac{60}{x} + \frac{240}{y} = 4$$
$$\frac{100}{x} + \frac{200}{y} = \frac{25}{6}$$

where we define $a = \frac{1}{x}$ and $b = \frac{1}{y}$, transforming the system into:

$$60a + 240b = 4$$
$$100a + 200b = \frac{25}{6}$$

Rewriting in matrix form:

$$A \cdot X = B$$

where:

$$A = \begin{bmatrix} 60 & 240 \\ 100 & 200 \end{bmatrix}, \quad X = \begin{bmatrix} a \\ b \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ \frac{25}{6} \end{bmatrix}$$

Step 2: LU Decomposition Theory

LU decomposition expresses the matrix A as the product of a lower triangular matrix L and an upper triangular matrix U:

$$A = PLU$$

where:

- P is a permutation matrix accounting for row swaps,
- L is a lower triangular matrix,
- U is an upper triangular matrix.

For this system, performing LU decomposition yields:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 0.6 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 100 & 200 \\ 0 & 120 \end{bmatrix}$$

Step 3: Solve Using Forward and Back Substitution

We first solve for Y in:

$$LY = PB$$

where:

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad PB = \begin{bmatrix} \frac{25}{6} \\ 4 \end{bmatrix}$$

Using forward substitution:

$$y_1 = \frac{25}{6}$$
$$0.6y_1 + y_2 = 4$$

Solving for y_2:

$$y_2 = 4 - 0.6 \times \frac{25}{6} = 1.5$$

Thus, Y is:

$$Y = \begin{bmatrix} \frac{25}{6} \\ 1.5 \end{bmatrix}$$

Next, solve for X in:

$$UX = Y$$

Using back substitution:

$$120b = 1.5 \Rightarrow b = \frac{1}{80}$$

$$100a + 200b = \frac{25}{6}$$

Solving for a:

$$100a = \frac{25}{6} - 2.5 = \frac{10}{6} = \frac{5}{3}$$

$$a = \frac{5}{300} = \frac{1}{60}$$

Step 4: Compute Final Values

Since:

$$x = \frac{1}{a} = 60, \quad y = \frac{1}{b} = 80$$

Final Answer:

- Speed of the train = 60 km/h
- Speed of the bus = 80 km/h

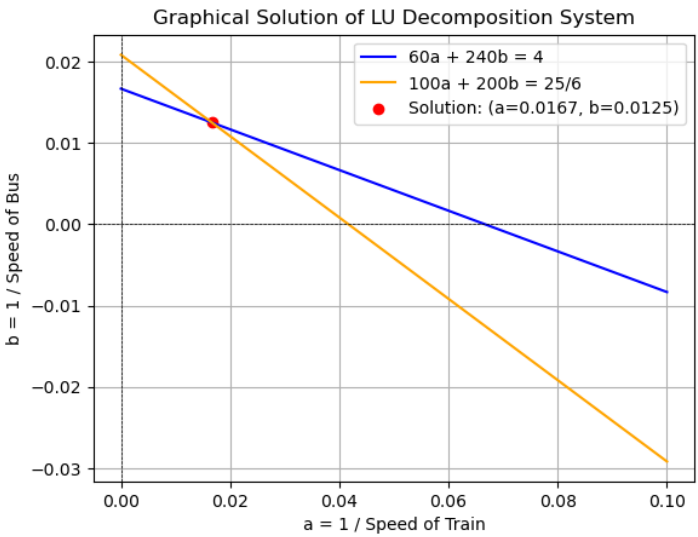

Fig. 0.1: Plot showing the relationship between a and b