NCERT - 10.3.6.1.1

EE1003

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(EE24BTECH11062)

Question: Solve the pair of linear equations

$$\frac{1}{2x} + \frac{1}{3y} = 2 \tag{1}$$

$$\frac{1}{3x} + \frac{1}{2y} = \frac{13}{6} \tag{2}$$

Solution:
Taking

$$\frac{1}{x} = u \tag{3}$$

$$\frac{1}{y} = v \tag{4}$$

We get the equations as

$$3u + 2v = 12 \tag{5}$$

$$2u + 3v = 13 \tag{6}$$

Writing them in matrix form

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \end{pmatrix} \tag{7}$$

LU decomposition:
For the system of linear equations $\mathbf{Ax} = \mathbf{b}$, if $\mathbf{A}$ is non-singular, we can decompose it as product LU where L is lower triangular matrix and U is an upper triangular matrix.
The equation becomes

$$\mathbf{LUx} = \mathbf{b} \tag{8}$$

Taking

$$\mathbf{y} = \mathbf{Ux} \tag{9}$$

Substituting in (8),

$$\mathbf{Ly} = \mathbf{b} \tag{10}$$

We solve for y in $\mathbf{Ly} = \mathbf{b}$ and then solve for $\mathbf{x}$ in $\mathbf{Ux} = \mathbf{y}$
Applying LU decomposition to matrix A,
For each column $j \geq k$, the entries of U in the kth row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m}.U_{m,j}, \forall j \geq k \tag{11}$$

For each row $i > k$, the entries of L in the kth column are updated as:

$$L_{j,k} = \frac{1}{U_{k,k}} \left(A_{j,k} - \sum_{m=1}^{k-1} .U_{m,k} \right), \forall i > k \tag{12}$$

We find $\mathbf{L}$ and $\mathbf{U}$ as follows:

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix} \tag{13}$$

$$\mathbf{U} = \begin{pmatrix} 3 & 2 \\ 0 & \frac{5}{3} \end{pmatrix} \tag{14}$$

Solving $\mathbf{Ly} = \mathbf{b}$ by forward substitution,

$$\begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \end{pmatrix} \tag{15}$$

$$y_1 = 12 \tag{16}$$

$$y_2 = 5 \tag{17}$$

$$y = \begin{pmatrix} 12 \\ 5 \end{pmatrix} \tag{18}$$

Solving $\mathbf{Ux} = \mathbf{y}$

$$\begin{pmatrix} 3 & 2 \\ 0 & \frac{5}{3} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix} \tag{19}$$

$$v = 3 \tag{20}$$

$$u = 2 \tag{21}$$

Hence, solution is

$$x = \frac{1}{u} = \frac{1}{2} \tag{22}$$

$$y = \frac{1}{v} = \frac{1}{3} \tag{23}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \tag{24}$$

Plotting:

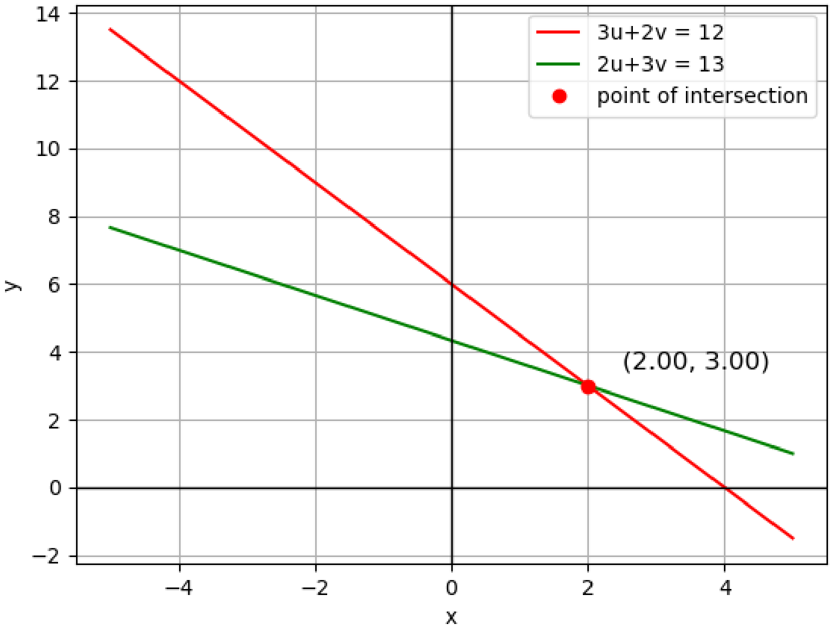
3u+2v = 12
2u+3v = 13
point of intersection
(2.00, 3.00)
14
12
10
8
6
4
2
0
−2
−4
−2
0
2
4
x
y

Fig. 0: Plot