10.4.ex.8

EE24BTECH11013 - MANIKANTA D

Question:
Find the roots of the equation $5x^2 - 6x - 2 = 0$ by the method of completing the square.
Solution: Multiplying the equation throughout by 5, we get

$$25x^2 - 30x - 10 = 0. \tag{0.1}$$

This is the same as

$$(5x)^2 - 2 \cdot (5x) \cdot 3 + 3^2 - 3^2 - 10 = 0, \tag{0.2}$$
$$(5x - 3)^2 - 9 - 10 = 0, \tag{0.3}$$
$$(5x - 3)^2 - 19 = 0, \tag{0.4}$$
$$(5x - 3)^2 = 19. \tag{0.5}$$

Taking the square root on both sides, we get

$$5x - 3 = \pm \sqrt{19}, \tag{0.6}$$
$$5x = 3 \pm \sqrt{19}, \tag{0.7}$$
$$x = \frac{3 \pm \sqrt{19}}{5}. \tag{0.8}$$

Therefore, the roots are

$$x = \frac{3 + \sqrt{19}}{5} \quad \text{and} \quad x = \frac{3 - \sqrt{19}}{5}. \tag{0.9}$$

QR decomposition on Hessenberg matrix:
The QR decomposition method is a numerical algorithm to compute the eigenvalues of a matrix A. By iteratively factorizing the matrix A into the product of an orthogonal matrix Q and an upper triangular matrix R, and then recombining them in a specific order, the process converges to a diagonal matrix whose diagonal entries are the eigenvalues of A.

This document adapts the QR decomposition method specifically for finding the roots of the quadratic equation $5x^2 - 6x - 2 = 0$.

QR Decomposition for Quadratic Roots: Given the quadratic equation $5x^2-6x-2 = 0$:

1) Rewrite the equation in matrix form. For a quadratic equation $ax^2 + bx + c = 0$, the companion matrix is:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}.$$

For $5x^2 - 6x - 2 = 0$, this becomes:

$$A = \begin{bmatrix} 0 & 1 \\ -\left(\frac{-2}{5}\right) & -\left(\frac{-6}{5}\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2}{5} & \frac{6}{5} \end{bmatrix}.$$

2) Perform the QR decomposition of A:

$$A_n = Q_n R_n, \tag{2.1}$$

where Q_n is an orthogonal matrix and R_n is an upper triangular matrix.

3) Update the matrix:

$$A_{n+1} = R_n Q_n. \tag{3.1}$$

4) Repeat steps 2 and 3 until A_n converges to an upper triangular matrix. The diagonal elements of this matrix are the eigenvalues, which correspond to the roots of the quadratic equation.

Mathematical Description: At the n-th iteration, let A_n be the matrix:

$$A_n = Q_n R_n, \tag{4.1}$$

where Q_n and R_n are obtained via the QR decomposition of A_n. The matrix is updated as:

$$A_{n+1} = R_n Q_n. \tag{4.2}$$

Update Equation: The update equation for the $(n+1)$-th iteration in terms of the n-th iteration is:

$$A_{n+1} = Q_n^T A_n Q_n, \tag{4.3}$$

where Q_n is the orthogonal matrix from the QR decomposition of A_n, and R_n is an upper triangular matrix such that $A_n = Q_n R_n$.

Roots of the Quadratic Equation: The eigenvalues of the companion matrix A correspond to the roots of the quadratic equation $5x^2 - 6x - 2 = 0$. As the iterations progress, the diagonal elements of A_n will converge to the roots of the equation. The algorithm involves the following steps:

1) Initialize A_0 as the companion matrix:

$$A_0 = \begin{bmatrix} 0 & 1 \\ \frac{2}{5} & \frac{6}{5} \end{bmatrix}.$$

2) Perform the QR decomposition of A_n:

$$A_n = Q_n R_n, \tag{2.1}$$

where Q_n is orthogonal and R_n is upper triangular.

3) Compute A_{n+1} using the update equation:

$$A_{n+1} = R_n Q_n. \tag{3.1}$$

4) Repeat until A_n converges to an upper triangular matrix. The diagonal elements of this matrix are the eigenvalues, which correspond to the roots of the quadratic equation.

Conclusion:

The QR decomposition method applied to the companion matrix of $5x^2 - 6x - 2 = 0$ numerically finds the roots of the equation. The iterative process demonstrates how eigenvalue computation can be used effectively to determine the roots without relying on direct formulas.

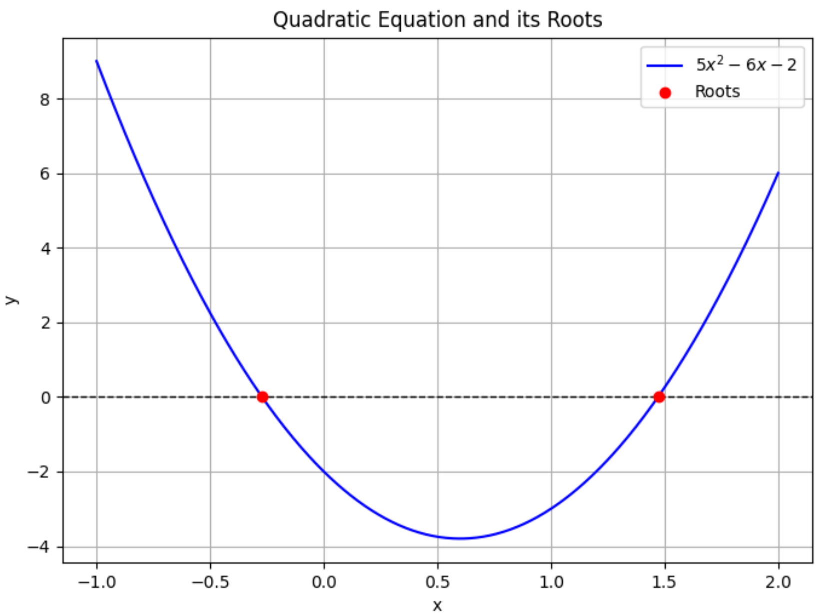

Fig. 4.1: Solution of the given function