NCERT-9.7.5

EE24BTECH11065 - Spoorthi yellamanchali

Question:
Form the differential equation of the family of circles in the first quadrant which touch the coordinate axes.
Solution:
The equation of a circle in the first quadrant, touching both the coordinate axes can be written as:

$$(x-r)^2+(y-r)^2=r^2; \tag{0.1}$$

where r is the radius of the circle,
Let

$$\frac{dy}{dx}=y' \tag{0.2}$$

On differentiating both LHS and RHS of equation (0.1), we get,

$$2(x-r)+2(y-r)\left(\frac{dy}{dx}\right)=0; \tag{0.3}$$

From this , we can write r as ,

$$r=\frac{x+yy'}{1+y'} \tag{0.4}$$

On substituting equation (0.5) in equation (0.1),i.e,on eliminating parameter r, we get,

$$\left[x-\left(\frac{x+yy'}{1+y'}\right)\right]^2+\left[y-\left(\frac{x+yy'}{1+y'}\right)\right]^2=\left(\frac{x+yy'}{1+y'}\right)^2 \tag{0.5}$$

$$\left[\frac{(x-y)\,y'}{(1+y)'}\right]^2+\left[\frac{y-x}{(1+y)'}\right]^2=\left[\frac{x+yy'}{(1+y')}\right]^2 \tag{0.6}$$

$$(x-y)^2y'^2+(x-y)^2=(x+yy')^2 \tag{0.7}$$

$$(x-y)^2\left[1+(y')^2\right]=(x+yy')^2 \tag{0.8}$$

On simplifying equation (0.8), we get,

$$(y')^2\left(x^2-2xy\right)-2xyy'+(x-y)^2-x^2=0 \tag{0.9}$$

$$(y')^2\left(x^2-2xy\right)-2xyy'+\left(y^2-2xy\right)=0 \tag{0.10}$$

$$\left(\frac{dy}{dx}\right)^2\left(x^2-2xy\right)-2xy\left(\frac{dy}{dx}\right)+\left(y^2-2xy\right)=0 \tag{0.11}$$

∴ The differential equation of the family of circles touching the coordinate axes in the first quadrant is given by:

$$\left(\frac{dy}{dx}\right)^2-\left(\frac{dy}{dx}\right)\left(\frac{2xy}{x^2-2xy}\right)+\left(\frac{y^2-2xy}{x^2-2xy}\right)=0 \tag{0.12}$$

It is a quadratic equation in terms of $\left(\frac{dy}{dx}\right)$ in the form $at^2+bt+c=0$,where,
$a=1$,$b=\left(\frac{2xy}{x^2-2xy}\right)$ and $c=\left(\frac{y^2-2xy}{x^2-2xy}\right)$
On solving this to find the roots of the equation (0.12), we get an expression for $\frac{dy}{dx}$,

$$\frac{dy}{dx}=\frac{\left(\frac{2xy}{x^2-2xy}\right)\pm\sqrt{\left(\frac{2xy}{x^2-2xy}\right)^2-4\times\left(\frac{y^2-2xy}{x^2-2xy}\right)}}{2} \tag{0.13}$$

$$\frac{dy}{dx}=\frac{y\pm\sqrt{\frac{2y}{x}\left(x-y\right)^2}}{x-2y} \tag{0.14}$$

$$\frac{dy}{dx}=\frac{y\pm\sqrt{\frac{2y}{x}}\left(x-y\right)}{x-2y} \tag{0.15}$$

we can see that,For given values of x,y,we get,two different values for $\frac{dy}{dx}$.
For few points (x, y),two different circles are possible.
for our plot,let,

$$\frac{dy}{dx}=\frac{y+\sqrt{\frac{2y}{x}}\left(x-y\right)}{x-2y} \tag{0.16}$$

From this differential equation,
On assuming initial conditions (x_0, y_0), we get the equation and plot of a unique circle,
Let us assume the initial conditions and on assuming a value for h close to zero,we get,

$$x_0 = 1 \tag{0.17}$$

$$y_0 = 2 \tag{0.18}$$

$$h = 0.01 \tag{0.19}$$

Then , by the finite difference method which is a numerical technique for solving differential equations by approximating derivatives with differences.
The first forward difference approximation of the derivative of $f(x)$ at x is given by:

$$\frac{dy}{dx}=\frac{f(x+h)-f(x)}{h} \tag{0.20}$$

Using this method we can write the expressions for (x_1, y_1) as :

$$x_1 = x_0 + h; \tag{0.21}$$

$$y_1 = y_0 + h\left(\frac{dy}{dx}|_{x=x_0}\right) \tag{0.22}$$

On substituting the expression of the derivative in equation (0.11), we get

$$y_1 = y_0 + h\left[\frac{y_0 + \sqrt{\frac{2y_0}{x_0}}(x_0 - y_0)}{x_0 - 2y_0}\right] \tag{0.23}$$

On substituting the values of x_0,y_0 and h in the above equations we get the point (x_1, y_1). what we have essentially done above is, obtaining a point which is very close to the initial point along the direction of derivative at that point.
similarly we get, The difference equations for the curve,which are,

$$x_n = x_{n-1} + h \tag{0.24}$$

$$y_n = y_{n-1} + h\left[\frac{y_{n-1} + \sqrt{\frac{2y_{n-1}}{x_{n-1}}}(x_{n-1} - y_{n-1})}{x_{n-1} - 2y_{n-1}}\right] \tag{0.25}$$

we can obtain points on the curve by using the above expressions for y_n and x_n
$\therefore$ we can plot the curve by the points obtained.