NCERT - 12.9.6.6

EE24BTECH11040 - Mandara Hosur

Question:
For the differential equation given below, find the general solution:

$$x\frac{dy}{dx} + 2y = x^2 \ln x \tag{0.1}$$

Solution (using the method of finite differences):
A particular solution can be found using the method of finite differences, as illustrated.

$$\frac{dy}{dx} = \frac{y(x+h) - y(x)}{h} \tag{0.2}$$

$$\implies y(x+h) = y(x) + h \cdot \frac{dy}{dx} \tag{0.3}$$

As can be seen from the question above,

$$\frac{dy}{dx} = \frac{x^2 \ln x - 2y}{x} \tag{0.4}$$

$$\implies \frac{dy}{dx} = x \ln x - 2\frac{y}{x} \tag{0.5}$$

$$\tag{0.6}$$

Hence, we can rewrite equation (0.3) as

$$\implies y(x+h) = y(x) + h \cdot \left(x \ln x - 2\frac{y}{x}\right) \tag{0.7}$$

Let $x_0 = 0$ and $y_0 = 1$ (assume this as the initial condition as nothing is mentioned in the question).
Let some $x_1 = x_0 + h$. Then

$$y_1 = y_0 + h \cdot \left(x_0 \ln x_0 - 2\frac{y_0}{x_0}\right) \tag{0.8}$$

Iterating through the above-mentioned process to generate y_2, y_3, y_4 and so on generalises equation (0.8) to

$$y_{n+1} = y_n + h \cdot \left(x_n \ln x_n - 2\frac{y_n}{x_n}\right) \tag{0.9}$$

The smaller the value of h, the more accurate the curve is.

Solution (using manual methods):

Divide equation (0.1) by x in both LHS and RHS.

$$\frac{dy}{dx} + \frac{2}{x}y = x\ln x \tag{0.10}$$

This is a linear differential equation. To solve it, we must first calculate its integrating factor ($I.F.$).

$$I.F. = e^{\int \frac{2}{x}\,dx} \tag{0.11}$$
$$\implies I.F. = e^{2\ln x} \tag{0.12}$$
$$\implies I.F. = e^{\ln x^2} \tag{0.13}$$
$$\implies I.F. = x^2 \tag{0.14}$$

Multiply equation (0.14) on LHS and RHS of equation (0.10)

$$\frac{dy}{dx}x^2 + \frac{2}{x}yx^2 = x^3\ln x \tag{0.15}$$
$$\implies x^2dy + 2xydx = x^3\ln xdx \tag{0.16}$$
$$\implies D\left(x^2y\right) = x^3\ln xdx \tag{0.17}$$

Integrating on both sides,

$$x^2y = \int x^3\ln x\,dx \tag{0.18}$$

We can simplify RHS using integration by parts. Let

$$u = \ln x\ ,\ v = x^3 \tag{0.19}$$

$$\implies \frac{du}{dx} = \frac{1}{x}\ ,\ \int v\,dx = \frac{x^4}{4} \tag{0.20}$$

According to the rule of integration by parts,

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx + c \tag{0.21}$$

Here c is the constant of integration.

Using (0.19) and (0.20) in (0.21),

$$\int x^3\ln x\,dx = \frac{x^4}{4}\ln x - \int\left(\frac{1}{x}\cdot\frac{x^4}{4}\right)dx \tag{0.22}$$
$$\implies \int x^3\ln x\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + c \tag{0.23}$$

Substituting equation (0.23) in equation (0.18), we get

$$x^2 y = \frac{x^4}{4}\ln x - \frac{x^4}{16} + c \tag{0.24}$$

$$\implies y = \frac{x^2}{4}\ln x - \frac{x^2}{16} + cx^{-2} \tag{0.25}$$

Substituting the initial condition of $x = 1$, $y = 0$ in equation (0.25) gives

$$c = \frac{1}{16} \tag{0.26}$$

Substituting equation (0.26) in (0.25) gives

$$y = \frac{x^2}{4}\ln x - \frac{x^2}{16} + \frac{1}{16}x^{-2} \tag{0.27}$$

$$\implies y = \frac{x^2}{4}\ln x - \frac{1}{16}\left(x^2 - \frac{1}{x^2}\right) \tag{0.28}$$

Therefore, the equation of the curve found by manual methods is

$$y = \frac{x^2}{4}\ln x - \frac{1}{16}\left(x^2 - \frac{1}{x^2}\right) \tag{0.29}$$

The curve generated using both described methods for the given question, taking $h = 0.1$ and running iterations 100 times is given below.

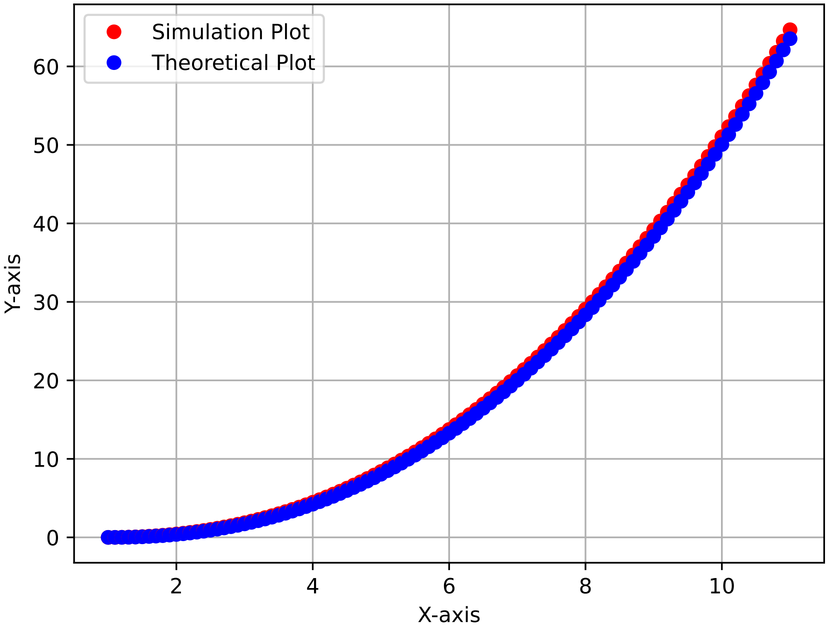

Fig. 0.1: Solution of given DE