8.2.6

EE24BTECH11010 - Balaji B

Question:
Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

Solution:

Variable	Description	values
V	Quadratic form of the matrix	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
u	Linear coefficient vector	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
f	constant term	-4
m	The direction vector of line	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
h	Point on line	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

TABLE 0: Variables used

Theoritical Solution:

The point of intersection of the line with the circle is $x_i = h + k_i m$, where, k_i is a constant and is calculated as follows:-

$$k_i = \frac{1}{m^\top V m}\left(-m^\top (Vh + u) \pm \sqrt{[m^\top (Vh + u)]\, 2 - g(h)(m^\top V m)}\right)$$

Substituting the input parameters into k_i,

$$k_i = \frac{1}{\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}\left(-\begin{pmatrix} 1 & -1 \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)\right) \pm \\ \sqrt{\left[\begin{pmatrix} 1 & -1 \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)\right]^2 - g(h)\left(\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)} \tag{0.1}$$

We get,
$k_i = 0, -2$
Substituting k_i into $x_i = h + k_i m$ we get

$$x_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + (0) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{0.2}$$

$$\implies x_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{0.3}$$

$$x_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + (-2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{0.4}$$

$$\implies x_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{0.5}$$

$$\implies x_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{0.6}$$

The area of the smaller region bounded by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

$$= \int_0^2 \sqrt{4 - x^2} dx - \int_0^2 (2 - x) \, dx \tag{0.7}$$

$$= \left(\frac{x}{2} \sqrt{4 - x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} - 2x + \frac{x^2}{2} \right)_0^2 \tag{0.8}$$

$$= (\pi - 2) \tag{0.9}$$

Computational Solution:

Taking trapezoid-shaped strips of small area and adding them all up. Say we have to find the area of y_x from $x = x_0$ to $x = x_n$, discretize the points on the x axis $x_0, x_1, x_2, \ldots, x_n$ such that they are equally spaced with the step size h.
Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2} h \left(y \left(x_1 \right) + y \left(x_0 \right) \right) + \frac{1}{2} h \left(y \left(x_2 \right) + y \left(x_1 \right) \right) + \cdots + \frac{1}{2} h \left(y \left(x_n \right) + y \left(x_{n-1} \right) \right) \tag{0.10}$$

$$= h \left[\frac{1}{2} \left(y \left(x_0 \right) + y \left(x_n \right) \right) + y \left(x_1 \right) + \cdots + y \left(x_{n-1} \right) \right] \tag{0.11}$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \ldots x_n)$ be equidistant points with step-size h.

$$A(x_n + h) = A(x_n) + \frac{1}{2} h \left(y(x_n + h) + y(x_n) \right) \tag{0.12}$$

We can repeat this till we get required area.
Discretizing the steps, making $A(x_n) = A_n, y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2} h \left(y_{n+1} + y_n \right) \tag{0.13}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left((y_n + hy'_n) + y_n\right) \tag{0.14}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) \tag{0.15}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.16}$$

$$x_{n+1} = x_n + h \tag{0.17}$$

In the given question, $y_n = \sqrt{4 - x_n^2} + x_n - 2$ and $y'_n = \frac{-x_n}{\left(\sqrt{4-x_n^2}\right)} + 1$

The general difference equation will be given by

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.18}$$

$$A_{n+1} = A_n + h\left(\sqrt{4 - x_n^2} + x_n - 2\right) + \frac{1}{2}h^2\left(\frac{-x_n}{\left(\sqrt{4 - x_n^2}\right)} + 1\right) \tag{0.19}$$

$$x_{n+1} = x_n + h \tag{0.20}$$

Iterating till we reach $x_n = 2$ will return required area.
Area obtained computationally: 1.41555 sq. units
Area obtained theoretically: $(\pi - 2)$ sq. units = 1.14 sq.unis

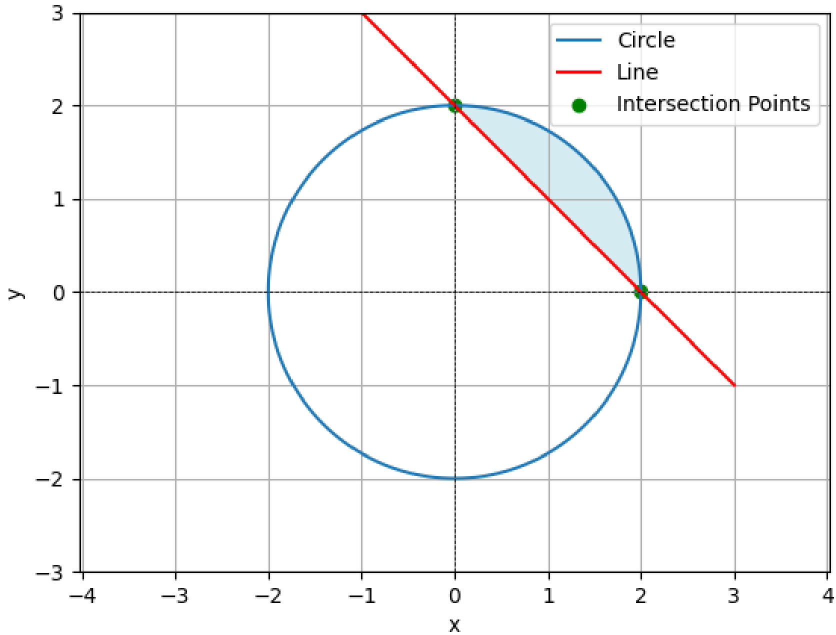

Fig. 0.1: Graph of the cirlce $x^2 + y^2 = 4$ and $x + y = 2$ and the area enclosed between them