9.ex.5

EE24BTECH11002 - Agamjot Singh

Question:
Solve the differential equation:

$$\frac{d^2y}{dx^2} + y = 0 \tag{1}$$

Solution:
Theoritical solution:
The given differential equation is a second-order linear ordinary differential equation. Let $y(0) = c_1$ and $y'(0) = c_2$. By definition of Laplace transform,

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \tag{2}$$

Some used properties of Laplace transform include,

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - sc_1 - c_2 \tag{3}$$

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1} \tag{4}$$

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1} \tag{5}$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{6}$$

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}\left(e^{at}f(t)\right) = F(s-a) \tag{7}$$

Applying Laplace transform on the given differential equation, we get,

$$y'' + y = 0 \tag{8}$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = 0 \tag{9}$$

$$s^2\mathcal{L}(y) - sc_1 - c_2 + \mathcal{L}(y) = 0 \tag{10}$$

$$\mathcal{L}(y) = \frac{sc_1 + c_2}{s^2+1} = c_1\frac{s}{s^2+1} + c_2\frac{1}{s^2+1} \tag{11}$$

$$\tag{12}$$

Taking laplace inverse on both sides, we get,

$$y = c_1\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + c_2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) \tag{13}$$

$$y = c_1\cos x + c_2\sin x \tag{14}$$

$$\implies y(x) = \sqrt{(c_1)^2 + (c_2)^2}\,\sin\left(x + \tan^{-1}\left(\frac{c_1}{c_2}\right)\right) \tag{15}$$

Computational Solution: Trapezoid Method
The given differential equation can be represented as

$$y'' + y = 0 \tag{16}$$

Let $y = y_1$ and $y' = y_2$, then,

$$\frac{dy_2}{dx} = -y_1 \text{ and } \frac{dy_1}{dx} = y_2 \tag{17}$$

$$\int_{y_{2,n}}^{y_{2,n+1}} dy_2 = \int_{x_n}^{x_{n+1}} -y_1 \, dx \tag{18}$$

$$\int_{y_{1,n}}^{y_{1,n+1}} dy_1 = \int_{x_n}^{x_{n+1}} y_2 \, dx \tag{19}$$

$$\tag{20}$$

Discretizing the steps (Trapezoid rule),

$$y_{2,n+1} - y_{2,n} = -\frac{h}{2}\left(y_{1,n} + y_{1,n+1}\right) \tag{21}$$

$$y_{1,n+1} - y_{1,n} = \frac{h}{2}\left(y_{2,n} + y_{2,n+1}\right) \tag{22}$$

Solving for $y_{1,n+1}$ and $y_{2,n+1}$, we get,

$$y_{1,n+1} = y_{1,n} + \frac{h}{2}\left(2y_{2,n} - \frac{h}{2}\left(y_{1,n} + y_{1,n+1}\right)\right) \tag{23}$$

$$\tag{24}$$

The difference equations can be written as,

$$y_{1,n+1} = \frac{\left(4 - h^2\right) y_{1,n} + 4h y_{2,n}}{\left(4 + h^2\right)} \tag{25}$$

$$y_{2,n+1} = \frac{\left(4 - h^2\right) y_{2,n} - 4h y_{1,n}}{\left(4 + h^2\right)} \tag{26}$$

$$\tag{27}$$

Iteratively plotting the above system taking intial conditions as

$$x_0 = 0 \ , \ y_{1,0} = 0 \ , \ y_{2,0} = 1 \tag{28}$$

we get the plot of the given differential equation.

Alternative Computational Solution: Bilinear transform
We have to apply laplace transformation on the given differential equation. From (11), we get,

$$Y(s) = \frac{sc_1 + c_2}{s^2 + 1} \tag{29}$$

$$Y(s) = \frac{sc_1 + c_2}{s^2 + 1} \tag{30}$$

Applying Bilinear transform, with $T = h$, we get,

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} \tag{31}$$

$$\implies Y(z) = \frac{2hc_1\left(z^2-1\right) + c_2h^2(z+1)^2}{(h^2+4)z^2 + 2(h^2-4)z + (h^2+4)} \tag{32}$$

$$\implies \left(z^2 + 2\frac{h^2-4}{h^2+4}z + 1\right)Y(z) = \frac{2hc_1\left(z^2-1\right) + c_2h^2\left(z^2+2z+1\right)}{h^2+4} \tag{33}$$

$$\implies z^2Y(z) + 2\frac{h^2-4}{h^2+4}zY(z) + Y(z) = \frac{\left(2hc_1 + c_2h^2\right)z^2 + \left(2h^2c_2\right)z + \left(h^2c_2 - 2hc_1\right)}{h^2+4} \tag{34}$$

Some properties of one sided z transform,

$$\mathcal{Z}(y[n+2]) = z^2Y(z) - y[1]z - y[0] \tag{35}$$

$$\mathcal{Z}(y[n+1]) = zY(z) - zy[0] \tag{36}$$

$$\mathcal{Z}(\delta[n]) = 1,\ z \neq 0 \tag{37}$$

$$\mathcal{Z}(y[n]) = Y(z) \implies \mathcal{Z}(y[n-n_0]) = z^{-n_0}Y(z) \tag{38}$$

By the time shift property (38),

$$\mathcal{Z}(\delta[n+2]) = z^2,\ z \neq 0 \tag{39}$$

$$\mathcal{Z}(\delta[n+1]) = z,\ z \neq 0 \tag{40}$$

Rewriting equation (34), we get,

$$z^2Y(z) + 2\frac{h^2-4}{h^2+4}zY(z) + Y(z) + (-y[1]z - y[0]) + 2\left(\frac{h^2-4}{h^2+4}\right)(-zy[0])$$

$$= \frac{\left(2hc_1 + c_2h^2\right)z^2 + \left(2h^2c_2 - y[1] - 2\left(\frac{h^2-4}{h^2+4}\right)y[0]\right)z + \left(h^2c_2 - 2hc_1 - y[0]\right)}{h^2+4} \tag{41}$$

$$\text{where } z \neq 0 \tag{42}$$

Region of convergence (**ROC**) is given by $z \neq 0$.
Taking z inverse transform on both sides of equation (41), we get the **difference equation** which is given by,

$$y[n+2] + 2\left(\frac{h^2-4}{h^2+4}\right)y[n+1] + y[n]$$

$$= \frac{\left(2hc_1 + c_2h^2\right)\delta[n+2] + \left(2h^2c_2 - y[1] - 2\left(\frac{h^2-4}{h^2+4}\right)y[0]\right)\delta[n+1] + \left(h^2c_2 - 2hc_1 - y[0]\right)\delta[n]}{h^2+4} \tag{43}$$

Here, δ is given by,

$$\delta[n-n_0] = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases} \tag{44}$$

As $n > 0$,

$$\delta[n+2] = \delta[n+1] = 0 \tag{45}$$

The equation (43) is now given by,

$$y[n+2] + 2\left(\frac{h^2-4}{h^2+4}\right)y[n+1] + y[n] = \frac{\left(h^2c_2 - 2hc_1 - y[0]\right)\delta[n]}{h^2+4} \tag{46}$$

At this point we drop the notation $y[n]$ and replace it with y_n, and we replace $c_1 = y(0)$ and $c_2 = y'(0)$,

$$y_{n+2} + 2\left(\frac{h^2-4}{h^2+4}\right)y_{n+1} + y_n = \frac{\left(h^2y'(0) - 2hy(0) - y_0\right)\delta[n]}{h^2+4} \tag{47}$$

Note that for computationally plotting the above difference equation, we need $y_0 = y(0)$ as well as y_1. To find $y_1 = y(0+h) = y(h)$ we employ first principle of derivative,

$$y'(x) = \lim_{h\to 0}\frac{y(x+h)-y(x)}{h} \tag{48}$$

$$y(x+h) = y(x) + hy'(x), h \to 0 \tag{49}$$

$$y_1 = y(h) = y(0) + hy'(0) \tag{50}$$

Iteratively plotting the above system taking intial conditions as

$$x_0 = 0 \ , \ y_0 = y(0) = 0 \ , \ y'(0) = 1 \tag{51}$$

we get the plot of the given differential equation.

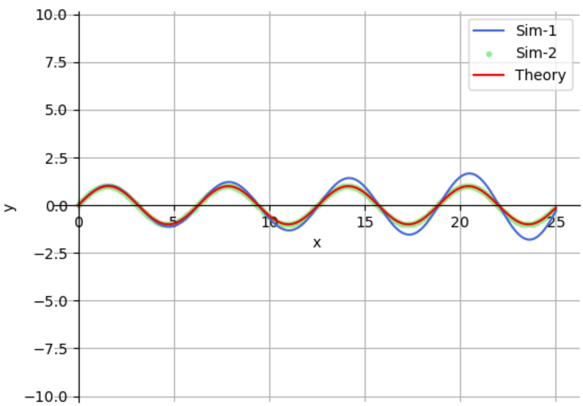

Fig. 0: Here Sim-1 plot represents the plot given by Trapezoid Method, and Sim-2 which is given by Bilinear transform using the same value of h. This plot clearly shows the accuracy of the Bilinear transform method.