10.3.3.1.5

EE24BTECH11003 - Akshara Sarma Chennubhatla

Question:
Solve the following pair of linear equations,

$$\sqrt{2}x + \sqrt{3}y = 0 \tag{1}$$
$$\sqrt{3}x - \sqrt{8}y = 0 \tag{2}$$

Solution:
Let

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3}$$

Expressing the system in matrix form,

$$\begin{pmatrix} \sqrt{2} & \sqrt{3} \\ \sqrt{3} & -\sqrt{8} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4}$$
$$\text{which is of the form } A\mathbf{x} = \mathbf{0} \tag{5}$$

Any non-singular matrix A can be expressed as a product of an upper triangular matrix U and a lower triangular matrix L, such that

$$A = LU \tag{6}$$
$$\implies LU\mathbf{x} = \mathbf{0} \tag{7}$$

U is determined by row reducing A using a pivot,

$$\begin{pmatrix} \sqrt{2} & \sqrt{3} \\ \sqrt{3} & -\sqrt{8} \end{pmatrix} \xrightarrow{R_2 \to R_2 - \sqrt{\frac{3}{2}} R_1} \begin{pmatrix} \sqrt{2} & \sqrt{3} \\ 0 & -\sqrt{8} - \frac{3}{2} \end{pmatrix} \tag{8}$$

Let

$$L = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \tag{9}$$

l is the multiplier used to zero out a_{21} in A.

$$L = \begin{pmatrix} 1 & 0 \\ \sqrt{\frac{3}{2}} & 1 \end{pmatrix} \tag{10}$$

This LU decomposition could also be computationally found using Doolittle's algorithm.

The update equation is given by,

$$U_{ij} = \begin{cases} A_{ij} & i = 0 \\ A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} & i > 0 \end{cases} \tag{11}$$

$$L_{ij} = \begin{cases} \frac{A_{ij}}{U_{jj}} & j = 0, U_{jj} \neq 0 \\ \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} & j > 0 \end{cases} \tag{12}$$

$$\tag{13}$$

Let $\mathbf{y} = U\mathbf{x}$,

$$L\mathbf{y} = \mathbf{0} \tag{14}$$

After we find $\mathbf{y}$, we find $\mathbf{x}$ using the following equation,

$$U\mathbf{x} = \mathbf{y} \tag{15}$$

Applying forward substitution on equation (14), we get,

$$\begin{pmatrix} 1 & 0 \\ \sqrt{\frac{3}{2}} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{16}$$

$$y_1 = 0 \tag{17}$$

$$\sqrt{\frac{3}{2}} y_1 + y_2 = 0 \tag{18}$$

$$\implies \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{19}$$

Substituting $\mathbf{y}$ in equation (15), we get,

$$\begin{pmatrix} \sqrt{2} & \sqrt{3} \\ 0 & -\sqrt{8} - \frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{20}$$

$$\sqrt{2}x + \sqrt{3}y = 0 \tag{21}$$

$$\left(-\sqrt{8} - \frac{3}{2}\right) y = 0 \tag{22}$$

$$\implies x = 0, y = 0 \tag{23}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{24}$$

This shows that the pair of linear equations have exactly one solution.

Below is the LU decomposition of this matrix got through the c code.

```
L:
1.000000 0.000000
1.224745 1.000000

U:
1.414214 1.732051
0.000000 −4.949748
```

Below is the plot of the pair of lines representing the linear equations and their point of intersection.

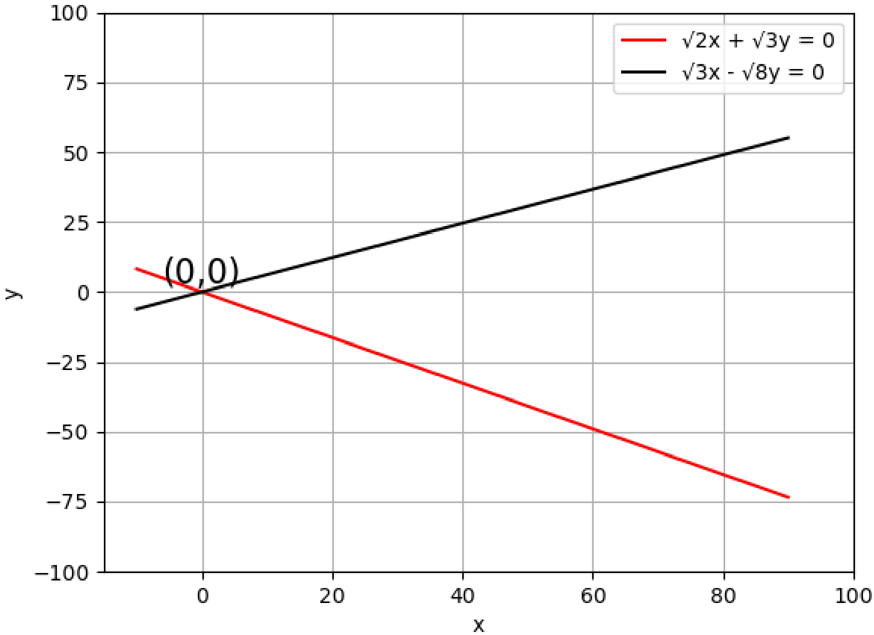

Fig. 0: Plot of the linear equations and their intersection point