10.3.3.1.3

EE24BTECH11002 - Agamjot Singh

Question:
Solve the following pair of linear equations,

$$3x - y = 3 \tag{1}$$
$$9x - 3y = 9 \tag{2}$$

Solution:
Let

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3}$$

Expressing the system in matrix form,

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} \tag{4}$$
$$\text{which is of the form } A\mathbf{x} = \mathbf{b} \tag{5}$$

Any non-singular matrix A can be expressed as a product of an upper triangular matrix U and a lower triangular matrix L, such that

$$A = LU \tag{6}$$
$$\implies LU\mathbf{x} = \mathbf{b} \tag{7}$$

U is determined by row reducing A using a pivot,

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \tag{8}$$

Let

$$L = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \tag{9}$$

l is the multiplier used to zero out a_{21} in A.

$$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \tag{10}$$

This LU decomposition could also be computationally found using Doolittle's algorithm.

The update equation is given by,

$$U_{ij} = \begin{cases} A_{ij} & i = 0 \\ A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} & i > 0 \end{cases} \tag{11}$$

$$L_{ij} = \begin{cases} \frac{A_{ij}}{U_{jj}} & j = 0, U_{jj} \neq 0 \\ \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} & j > 0 \end{cases} \tag{12}$$

$$\tag{13}$$

We see that there is a zero on the diagonal of the upper triangular matrix U which implies that A is singular and hence the system has either zero or infinitely many solutions. Let $\mathbf{y} = U\mathbf{x}$,

$$L\mathbf{y} = \mathbf{b} \tag{14}$$

After we find $\mathbf{y}$, we find $\mathbf{x}$ using the following equation,

$$U\mathbf{x} = \mathbf{y} \tag{15}$$

Applying forward substitution on equation (14), we get,

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} \tag{16}$$

$$\implies \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{17}$$

Substituting $\mathbf{y}$ in equation (15), we get,

$$\begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{18}$$

$$\implies 0\,(x) + 0\,(y) = 0 \tag{19}$$

This shows that the equation has infinitely many solutions.

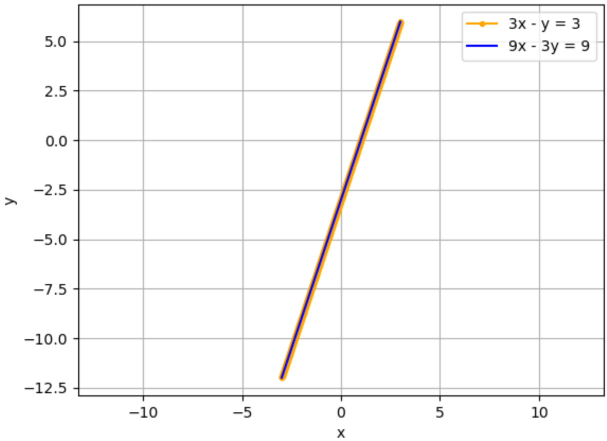

Fig. 0: Plotting the two lines, which come out as parallel