9.4.5

EE24BTECH11021 - Eshan Ray

Question:

For the Differential Equation $(e^{-x}+e^{x})\,dy-(e^{x}-e^{-x})\,dx=0$, find a general solution of the differential equation.

Solution: Solving the given D.E. , we get,

$$(e^{-x}+e^{x})\,dy-(e^{x}-e^{-x})\,dx=0 \tag{1}$$

$$\implies \frac{dy}{dx}=\frac{e^{x}-e^{-x}}{e^{-x}+e^{x}} \tag{2}$$

$$\implies \frac{dy}{dx}=\frac{e^{2x}-1}{e^{2x}+1} \tag{3}$$

Substituting, $t=e^{2x}$, we get,

$$\implies dt=2e^{2x}dx \tag{4}$$

$$\implies dx=\frac{dt}{2t} \tag{5}$$

$$\implies \frac{dy}{\left(\frac{dt}{2t}\right)}=\frac{t-1}{t+1} \tag{6}$$

$$\implies \frac{dy}{dt}=\frac{t-1}{2t\,(t+1)} \tag{7}$$

$$\implies \frac{dy}{dt}=\frac{1}{2}\left(\frac{1}{(t+1)}-\frac{1}{t\,(t+1)}\right) \tag{8}$$

$$\implies dy=\frac{1}{2}\left(\frac{2}{t+1}-\frac{1}{t}\right)dt \tag{9}$$

$$\tag{10}$$

Integrating both sides, we get,

$$\implies \int dy=\int\frac{dt}{t+1}-\int\frac{dt}{2t} \tag{11}$$

$$\implies y=\ln|t+1|-\frac{1}{2}\ln|t|+C \tag{12}$$

$$\tag{13}$$

substituting, $t=e^{2x}$

$$\implies y=\ln\left|e^{2x}+1\right|-\frac{1}{2}\ln\left|e^{2x}\right|+C \tag{14}$$

$$\implies y=\ln\left(e^{2x}+1\right)-x+C \tag{15}$$

Computational Solution:
Using Trapezoidal rule, we get,

$$f(x) = \frac{dy}{dx} \tag{16}$$

$$\int_{x_0}^{x_n} f(x) \approx \frac{h}{2}\left(\left(f(x_0) + f(x_1)\right) + \left(f(x_1) + f(x_2)\right) + \ldots \left(f(x_{n-1}) + f(n)\right)\right) \tag{17}$$

$$\int_{x_0}^{x_n} f(x) \approx \frac{h}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right) \tag{18}$$

$$\tag{19}$$

Using Trapezoid rule for discretizing the steps , we get,

$$y_{n+1} - y_n = \frac{h}{2}\left(f(x_n) + f(x_{n+1})\right) \tag{20}$$

$$\tag{21}$$

By the classical definition of derivative we know that $f(x_{n+1}) = f(x_n) + hf\prime(x_n)$

$$y_{n+1} - y_n = \frac{h}{2}\left(f(x_n) + f(x_n) + hf\prime(x_n)\right) \tag{22}$$

$$y_{n+1} = y_n + hf(x_n) + \frac{h^2}{2}f\prime(x_n) \tag{23}$$

The difference equation,

$$y_{n+1} = y_n + h\frac{e^{2x_n} - 1}{e^{2x_n} + 1} + \frac{h^2}{2}\frac{4e^{2x_n}}{\left(e^{2x_n} + 1\right)^2} \tag{24}$$

$$x_{n+1} = x_n + h \tag{25}$$

The initial conditions for the plotting of graph are as follows:-

$$x_0 = -5 \tag{26}$$

$$y_0 = 5 \tag{27}$$

$$h = 0.01 \tag{28}$$

$$C = 0 \tag{29}$$

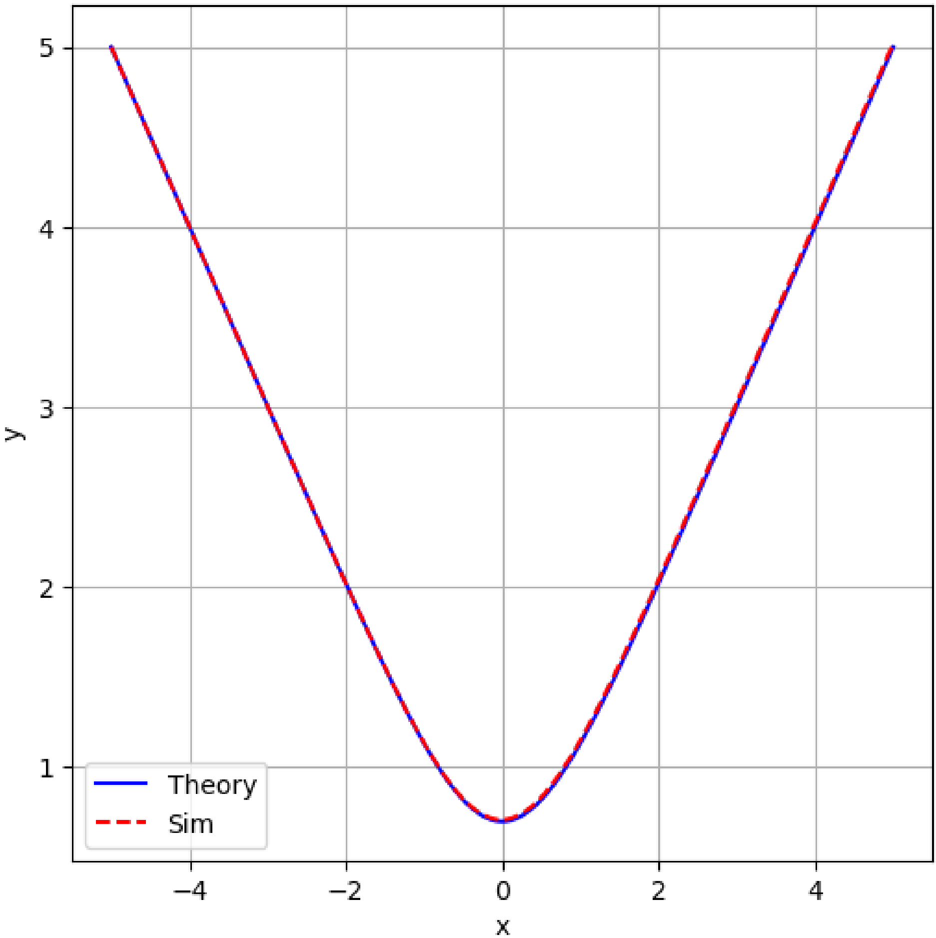

Fig. 0: Plot of the differential equation when $h = 0.01$