12.8.1.8

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Question:
Prove that the area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = \frac{8}{3}$
Solution:
The equation of parabola is $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$. In matrix form, it is given by,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 0 = 0 \tag{1}$$

Line equation is,

$$\mathbf{x} = \kappa \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{2}$$

Intersection of a line and a conic is given by,

$$\kappa_i = \frac{-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(h)(\mathbf{m}^\top \mathbf{V}\mathbf{m})}}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{3}$$

For the given conic, $\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}, f = 0$. For the given line, $\mathbf{h} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$k_i = \frac{1}{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \left(-\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} \right) \right) \pm$$
$$\sqrt{\left[\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} \right) \right]^2 - g(h) \left(\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)} \tag{4}$$

by the solving the equation we get

$$\implies \kappa_i = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{5}$$

The 2 curves meet at the points $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. So, the area between the curves is given by,
By using the above formula for the line $x = \frac{8}{3}$ we get

$$\implies \kappa = \begin{pmatrix} \frac{8}{3} \\ \frac{2\sqrt{6}}{3} \end{pmatrix}, \begin{pmatrix} \frac{8}{3} \\ -\frac{2\sqrt{6}}{3} \end{pmatrix} \tag{6}$$

Theoretical Solution:
Area between $x = 4$ and $x = y^2$

$$\int \left(4 - y^2\right) dy \implies [4y]_{-2}^{2} - \left[\frac{y^3}{3}\right]_{-2}^{2} \tag{7}$$

$$\implies (16) - \left(\frac{16}{3}\right) \tag{8}$$

$$\implies \frac{32}{3} sq.units \tag{9}$$

Area between $x = \frac{8}{3}$ and $x = y^2$

$$\int \left(\frac{8}{3} - y^2\right) dy \tag{10}$$

$$\implies \left[\frac{8}{3}y\right]_{-\frac{2\sqrt{6}}{3}}^{\frac{2\sqrt{6}}{3}} - \left[\frac{y^3}{3}\right]_{-\frac{2\sqrt{6}}{3}}^{\frac{2\sqrt{6}}{3}} \tag{11}$$

$$\implies \frac{16}{3} sq.units \tag{12}$$

The first area is twice than the second area. Thus,the line $x = \frac{8}{3}$ divides the area into two equal parts.
Simulated Solution:
Using the Trapezoidal rule which approximates the ntegral of a function $f(x)$ over an interval $[a, b]$ by dividing the interval into n subintervals and approximating the area under the curve as a series of trapezoids

$$\int_a^b f(x)\, dx \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} \left(f(x_i) + f(x_n)\right)\right] \tag{13}$$

Where x_0 is semi-major axis of ellipse and x_n is semi-minor axis of the ellipse and h is the width of each subinterval.

$$x_n = x_0 + n \cdot h \tag{14}$$

$$\implies h = \frac{x_n - x_0}{n} \tag{15}$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \ldots x_n)$ be equidistant points with step-size h. Then,

$$A(x_n + h) = A(x_n) + \frac{1}{2}h\left(y(x_n + h) + y(x_n)\right) \tag{16}$$

where $\frac{1}{2}h\left(y(x_n + h) + y(x_n)\right)$ is area of difference trapezium We can repeat this till we get the required area.
Let $A(x_n) = A_n$ and $y(x_n) = y_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left(y_{n+1} + y_n\right) \tag{17}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left((y_n + hy'_n) + y_n\right) \tag{18}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) \tag{19}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{20}$$

$$\tag{21}$$

In the given question, $y^2 = x$ and $y' = \frac{1}{2\sqrt{x}}$
General Difference Equation is given by, from 20

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{22}$$

$$= A_n + h\left(\sqrt{x_n}\right) + \frac{1}{2}h^2\left(\frac{1}{2\sqrt{x_n}}\right) \tag{23}$$

$$x_{n+1} = x_n + h \tag{24}$$

By iterating through the required value of n, we get the area enclosed between the line and the parabola.
Theoretical area = $\frac{32}{3}$ sq.units
Calculated area through trapezoidal rule = 10.6667 sq.units

Below is the plot for line and the parabola