10.4.2.3

EE24BTECH11021 - Eshan Ray

Question:
Find two numbers whose sum is 27 and product is 182
Solution: Let one of the numbers be x
So, the other number is $27 - x$
Given,

$$x(27 - x) = 182 \tag{1}$$
$$27x - x^2 = 182 \tag{2}$$
$$x^2 - 27x + 182 = 0 \tag{3}$$
$$(x - 13)(x - 14) = 0 \tag{4}$$
$$\implies x = 13, 14 \tag{5}$$

So, the numbers are 13 and 14

Computational Solution:
Using Newton- Raphson Method we get,

We start by taking an initial guess and then iteratively we us the following equation to find the roots of the quadratic equation :-

$$x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)} \tag{6}$$
$$f(x) = x^2 - 27x + 182 \tag{7}$$
$$f\prime(x) = 2x - 27 \tag{8}$$
$$x_{n+1} = x_n - \frac{x_n^2 - 27x_n + 182}{2x_n - 27} \tag{9}$$

After running the code, we obtained the following results:-

$$\text{Root-1: } 14.00000000 \tag{10}$$
$$\text{Root-2: } 13.00000000 \tag{11}$$

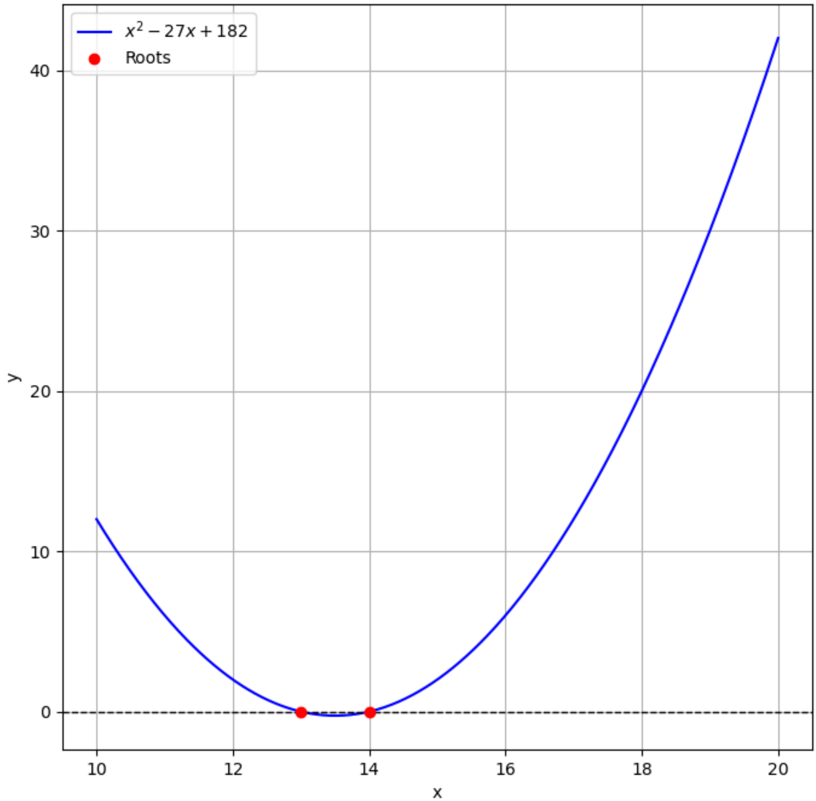

Fig. 0: Plot of the quadratic equation using newton-Raphson method

Alternate Method: Eigenvalues of Companion Matrix

In this method, we find the roots of any polynomial of the form $x^n + a_{n-1}x^{n-1} \dots ax + a_0 = 0$ by finding the eigenvalues of the Companion Matrix (C) given below:-

$$C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix} \tag{12}$$

For the Quadratic Equation $x^2 - 27x + 182 - 0$, we get the following companion Matrix

$$C = \begin{pmatrix} 0 & 1 \\ -182 & 27 \end{pmatrix} \tag{13}$$

The roots of the equation is the eigenvalues of the matrix C which has been calculated using the QR Decomposition with shifts process.
The details of the process is given below:-

QR Decomposition : Gram-schmidt Process

1) In the QR Decomposition, the matrix A is decomposed into matrices Q and R as:

$$A = QR \tag{14}$$

where ,Q is an orthogonal matrix and R is an upper triangular matrix.
2) We start by producing an orthogonal set of column vectors of Q $\{q_1, q_2, \ldots, q_n\}$ from a set of column vectors of A $\{a_1, a_2, \ldots, a_n\}$.
3) For orthogonalization we subtract each vector a_i with the projections of all previously obtained orthogonal vectors $q_1, q_2, \ldots, q_{i-1}$ to make q_i orthogonal to them.
The projection of a_i onto a vector q_j is calculated as:

$$proj_{q_j}(a_i) = \frac{\langle a_i, q_j \rangle}{\langle q_j, q_j \rangle} q_j \tag{15}$$

Then q_i is computed as:

$$q_i = a_i - \sum_{j=1}^{i-1} proj_{q_j}(a_i) \tag{16}$$

Then all the q_i's are normalized by :

$$q_i = \frac{q_i}{||q_i||} \tag{17}$$

The process is repeated for all the colums of A
3) As Q is an orthonormal matrix

$$Q^\top Q = I \tag{18}$$

So, R can be represented as follows

$$R = Q^\top A \tag{19}$$

$$r_{ij} = \langle a_j, q_i \rangle \text{ , for } i \leq j \tag{20}$$

QR algorithm:

In the QR algorithm, the matrix A_n is decomposed into matrices Q_n and R_n as:

$$A_n = Q_n R_n \tag{21}$$

Then, the new matrix A_{n+1} is computed as:

$$A_{n+1} = R_n Q_n \tag{22}$$

This process is repeated until the off-diagonal elements of the matrix become negligibly small, at which point the diagonal elements approximate the eigenvalues of the original matrix.

$$\text{Eigenvalues computed} : [14.0, 13.0] \tag{23}$$

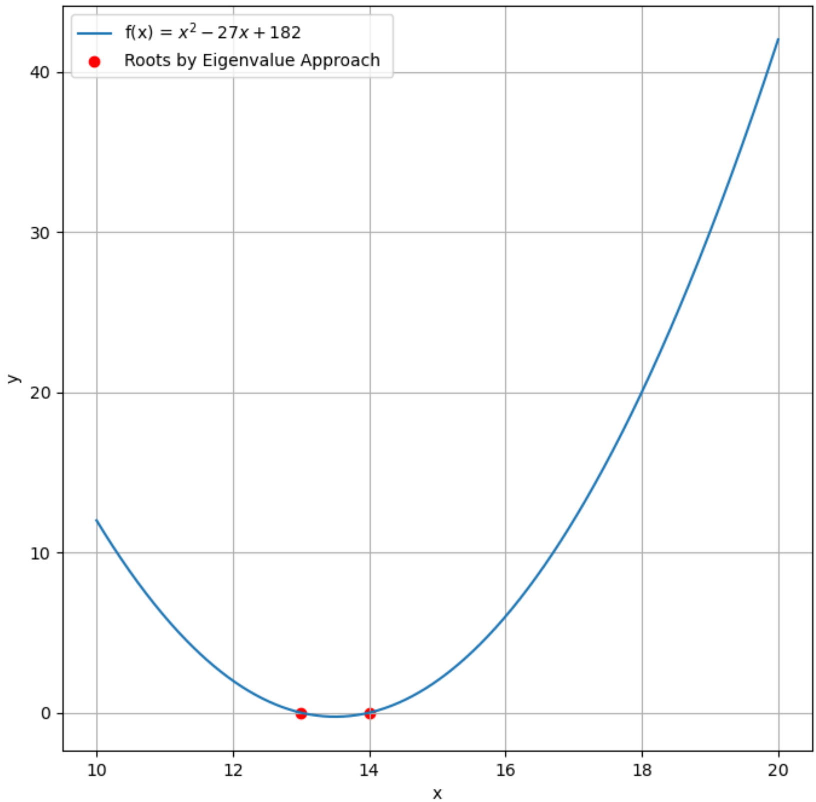

Fig. 3: Plot of the quadratic equation by eigenvalue approach