NCERT - 12.8.2.4

EE24BTECH11040 - Mandara Hosur

Question:
Using integration, find the area of region bounded by the triangle whose vertices are $(-1, 0)$, $(1, 3)$, and $(3, 2)$.

Solution (using the trapezoidal rule):
The trapezoidal rule approximates the area A under a curve $y(x)$ over an interval $[a, b]$ by dividing the area to be computed into multiple trapezoids.

First, we define h. This quantity represents the width of each trapezoid (the distance between the two parallel sides of the trapezoid).

$$h = \frac{b-a}{n} \tag{0.1}$$

Here, n represents the total number of trapezoids the area to be integrated is split into. The higher the value of n, the smaller the value of h. This increases the accuracy of the computed integral.

The area a_0 of any of the trapezoids can be calculated as illustrated:

$$a_0 = \frac{1}{2}(\text{sum of lengths of parallel sides})(\text{width}) \tag{0.2}$$

$$\implies a_0 = \frac{1}{2}\left[y(x) + y(x+h)\right](h) \tag{0.3}$$

Here, x is any value in the interval $[a, b]$.

Taking $a = x_0$, $b = x_n$, and defining A_k as the area under the curve $y(x)$ from $x = x_0$ to $x = x_k$, we can define the following relation (assume $x_{k+1} = x_k + h$ for $0 < k < n$):

$$A_{k+1} = A_k + \frac{h}{2}(y_k + y_{k+1}) \tag{0.4}$$

It is known that

$$y_{k+1} = y_k + hy'_k \tag{0.5}$$

Hence, equation (0.4) can be rewritten using equation (0.5) as

$$A_{k+1} = A_k + \frac{h}{2}\left(2y_k + hy'_k\right) \tag{0.6}$$

$$\implies A_{k+1} = A_k + hy_k + \frac{h}{2}y'_k \tag{0.7}$$

The final sum A_n gives us a good approximation of the area A that we were originally attempting to compute.

$$A = \int_a^b y(x)\ dx = \frac{h}{2}\left(y(a) + 2\sum_{i=1}^{n-1} y(x_i) + y(b)\right) \tag{0.8}$$

Now, the above concept is to be implemented to find the area of the triangle mentioned in the question.

$$A = \text{ area under } AB + \text{ area under } BC - \text{ area under } AC \tag{0.9}$$

Let the area under AB, BC, and AC be p, q, and r, respectively. Then,

$$A = p + q - r \tag{0.10}$$

Line equation of side AB can be expressed as:

$$y = \frac{3}{2}(x+1) \text{ and } y' = \frac{3}{2}. \tag{0.11}$$

Hence area equation can be written as:

$$A_{k+1} = A_k + h\left(\frac{3}{2}(x_k+1)\right) + \frac{h}{2}\left(\frac{3}{2}\right) \tag{0.12}$$

Taking $x_0 = -1$ and $x_n = 1$ gives

$$A_n = p = 3 \tag{0.13}$$

Line equation of side BC can be expressed as:

$$y = \frac{-1}{2}(x-7) \text{ and } y' = \frac{-1}{2}. \tag{0.14}$$

Hence area equation can be written as:

$$A_{k+1} = A_k + h\left(\frac{-1}{2}(x_k-7)\right) + \frac{h}{2}\left(\frac{-1}{2}\right) \tag{0.15}$$

Taking $x_0 = 1$ and $x_n = 3$ gives

$$A_n = q = 5 \tag{0.16}$$

Line equation of side CA can be expressed as:

$$y = \frac{1}{2}(x+1) \text{ and } y' = \frac{1}{2}. \tag{0.17}$$

Hence area equation can be written as:

$$A_{k+1} = A_k + h\left(\frac{1}{2}(x_k+1)\right) + \frac{h}{2}\left(\frac{1}{2}\right) \tag{0.18}$$

Taking $x_0 = -1$ and $x_n = 3$ gives

$$A_n = r = 4 \tag{0.19}$$

Therefore the required area can be found from (0.10) as

$$A = 3 + 5 - 4 \tag{0.20}$$
$$\implies A = 4 \tag{0.21}$$

Solution (using manual methods):
Equation (0.10) can be solved using the manual method of integration, as illustrated below:

$$A = \int_{-1}^{1} \frac{3}{2}(x+1)\,dx + \int_{1}^{3} \frac{-1}{2}(x-7)\,dx - \int_{-1}^{3} \frac{1}{2}(x+1) \tag{0.22}$$
$$\implies A = \left[\frac{3}{4}x^2 + \frac{3}{2}x\right]_{-1}^{1} + \left[\frac{-1}{4}x^2 + \frac{7}{2}x\right]_{1}^{3} - \left[\frac{1}{4}x^2 + \frac{1}{2}x\right]_{-1}^{3} \tag{0.23}$$
$$\implies A = 3 + 5 - 4 \tag{0.24}$$
$$\implies A = 4 \tag{0.25}$$

Clearly, from equations (0.21) and (0.25), we see that the area has been approximated by the trapezoidal rule well.

Plotting the triangle using difference equation:

$$\frac{dy}{dx} = \frac{y(x+h) - y(x)}{h} \tag{0.26}$$
$$\implies y(x+h) = y(x) + h \cdot \frac{dy}{dx} \tag{0.27}$$

Let x_0 and y_0 be the initial conditions. Let some $x_1 = x_0 + h$. Then

$$y_1 = y_0 + h \cdot \left(\frac{dy}{dx}\right)_{(x_0, y_0)} \tag{0.28}$$

Iterating through the above-mentioned process to generate y_2, y_3, y_4 and so on generalises equation (0.28) to

$$y_{n+1} = y_n + h \cdot \left(\frac{dy}{dx}\right)_{(x_n, y_n)} \tag{0.29}$$

The smaller the value of h, the more accurate the curve is.
For lines AB, BC, and CA

$$y_{n+1} = y_n + \frac{3}{2}h \tag{0.30}$$
$$y_{n+1} = y_n + \frac{-1}{2}h \tag{0.31}$$
$$y_{n+1} = y_n + \frac{1}{2}h \tag{0.32}$$

respectively.

The curve generated using method of finite differences is compared with the actual plot of the triangle in the figure below.

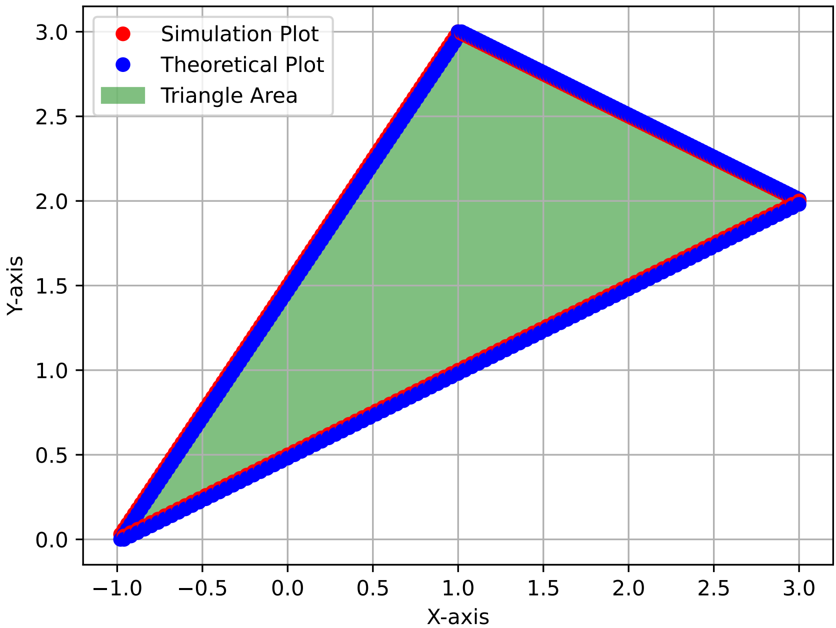

Fig. 0.1: Plot of Given Triangle