Ch8.Ex.6

EE24BTECH11015 - Dhawal

Question:
Find the area of the region founded by two parabolas $y = x^2$ and $y^2 = x$.

Solution:

Variable	Description	values
$\mathbf{V_1}$	Quadratic form of the matrix of $y = x^2$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\mathbf{u_1}$	Linear coefficient vector of $y = x^2$	$\begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix}$
f_1	constant term of $y = x^2$	0
$\mathbf{V_2}$	Quadratic form of the matrix of $x = y^2$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathbf{u_2}$	Linear coefficient vector of $x = y^2$	$\begin{pmatrix} \frac{-1}{2} \\ 0 \end{pmatrix}$
f_2	constant term of $x = y^2$	0

TABLE 0: Variables used

Theoritical Solution:

The intersection of two conics with parameters $V_i, u_i, f_i,\ i = 1, 2$ is defined as

$$x^T (V_1 + \mu V_2) x + 2 (u_1 + \mu u_2)^T x + (f_1 + \mu f_2) = 0 \tag{0.1}$$

we can get μ by solving the below equation

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{0.2}$$

$$\begin{vmatrix} 1 & 0 & \frac{-\mu}{2} \\ 0 & \mu & \frac{-1}{2} \\ \frac{-\mu}{2} & \frac{-1}{2} & 0 \end{vmatrix} = 0 \tag{0.3}$$

$$\frac{1}{4} = \frac{\mu^3}{4} \tag{0.4}$$

$$\mu = 1 \tag{0.5}$$

now solving the equation by placing the value of μ

$$x^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + 2 \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{2} \end{pmatrix}^T x = 0 \tag{0.6}$$

$$\left(x^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{2} \end{pmatrix} \right) x = 0 \tag{0.7}$$

The points of intersection are

$$x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.8}$$

The area of the region founded by two parabolas $y = x^2$ and $y^2 = x$ is

$$= \int_0^1 \sqrt{x} dx - \int_0^1 x^2 dx \tag{0.9}$$

$$= \left(\frac{2x\sqrt{x}}{3} - \frac{x^3}{3} \right)_0^1 \tag{0.10}$$

$$= \frac{2}{3} - \frac{1}{3} \tag{0.11}$$

$$= 0.333333 \tag{0.12}$$

Computational Solution:

Taking trapezoid-shaped strips of a small area and adding them all up. Say we have to find the area of y_x from $x = x_0$ to $x = x_n$, discretize the points on the x axis $x_0, x_1, x_2, \ldots, x_n$ such that they are equally spaced with the step size h.
Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2} h \left(y \left(x_1 \right) + y \left(x_0 \right) \right) + \frac{1}{2} h \left(y \left(x_2 \right) + y \left(x_1 \right) \right) + \cdots + \frac{1}{2} h \left(y \left(x_n \right) + y \left(x_{n-1} \right) \right) \tag{0.13}$$

$$= h \left[\frac{1}{2} \left(y \left(x_0 \right) + y \left(x_n \right) \right) + y \left(x_1 \right) + \cdots + y \left(x_{n-1} \right) \right] \tag{0.14}$$

Let $A \left(x_n \right)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $\left(x_0, x_1, \ldots x_n \right)$ be equidistant points with step-size h.

$$A \left(x_n + h \right) = A \left(x_n \right) + \frac{1}{2} h \left(y \left(x_n + h \right) + y \left(x_n \right) \right) \tag{0.15}$$

We can repeat this till we get required area.
Discretizing the steps, making $A \left(x_n \right) = A_n, y \left(x_n \right) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2} h \left(y_{n+1} + y_n \right) \tag{0.16}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left((y_n + hy'_n) + y_n\right) \tag{0.17}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) \tag{0.18}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.19}$$

$$x_{n+1} = x_n + h \tag{0.20}$$

In the given question, $y_n = \sqrt{x_n} - {x_n}^2$ and $y'_n = \frac{2x_n\sqrt{x_n}}{3} - \frac{{x_n}^3}{3}$
The general difference equation will be given by

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.21}$$

$$A_{n+1} = A_n + h\left(\sqrt{x_n} - {x_n}^2\right) + \frac{1}{2}h^2\left(\frac{2x_n\sqrt{x_n}}{3} - \frac{{x_n}^3}{3}\right) \tag{0.22}$$

$$x_{n+1} = x_n + h \tag{0.23}$$

Iterating till we reach $x_n = 1$ will return required area.
Area obtained computationally: 0.333039 sq. units
Area obtained theoretically: 0.333333 sq.unis

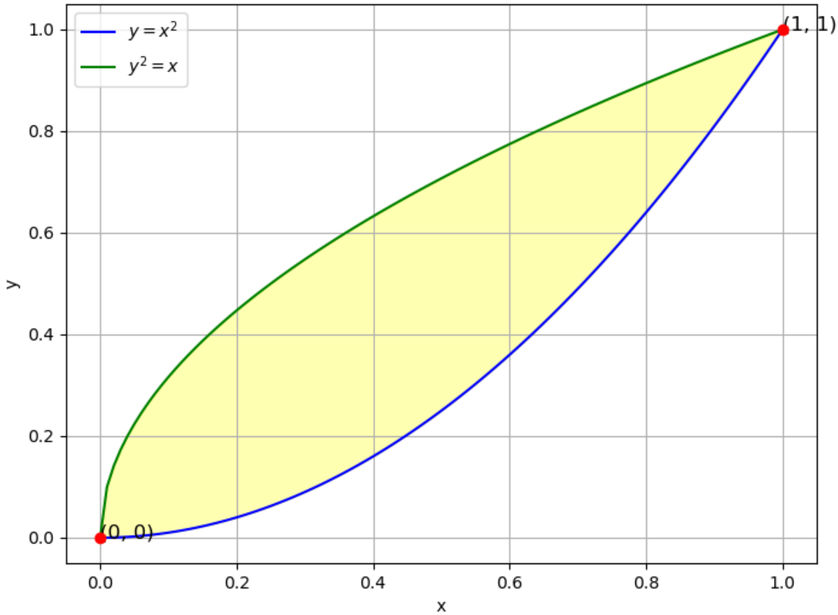

Fig. 0.1: Graph of the parabolas $y = x^2$ and $y^2 = x$ and the area enclosed between them