

9.7.4

EE24BTECH11012 - Bhavanisankar G S

QUESTION :

Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is the general solution of differential equation, $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$, where c is a parameter. Considering $y(1) = 1$, find the particular solution of the given differential equation.

SOLUTION :

$$(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy \quad (0.1)$$

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \quad (0.2)$$

Putting

$$y = vx \quad (0.3)$$

in (0.2), we have

$$v + x \frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} \quad (0.4)$$

$$x \frac{dv}{dx} = \frac{1 - v^4}{v^3 - 3v} \quad (0.5)$$

Separating the variables, we have

$$\frac{v^3 - 3v}{1 - v^4} dv = \frac{1}{x} dx \quad (0.6)$$

By the method of **partial fractions**, we have

$$\left(\frac{-2v}{1 + v^2} - \frac{v}{1 - v^2} \right) dv = \frac{1}{x} dx \quad (0.7)$$

Integrating on both the sides of (0.7), we have

$$\int \left(\frac{-2v}{1 + v^2} - \frac{v}{1 - v^2} \right) dv = \int \frac{1}{x} dx \quad (0.8)$$

$$\log \left(\frac{\sqrt{1 - v^2}}{1 + v^2} \right) = \log x + \log c, c - \text{constant} \quad (0.9)$$

$$\left(\frac{\sqrt{1 - v^2}}{1 + v^2} \right) = cx \quad (0.10)$$

Substituting (0.3) in (0.10) and rearranging, we have

$$\frac{\sqrt{x^2 - y^2}}{x^2 + y^2} = c \quad (0.11)$$

$$x^2 - y^2 = c(x^2 + y^2)^2 \quad (0.12)$$

Hence proved.

Substituting the initial conditions ,we have

$$1 - 1 = c(1 + 1) \quad (0.13)$$

$$c = 0 \quad (0.14)$$

Particular solution :

$$x^2 - y^2 = 0 \quad (0.15)$$

$$y = \pm x \quad (0.16)$$

By the **Forward-Euler method**, we have

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (0.17)$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \text{ for small } h \quad (0.18)$$

$$y_{n+1} \approx y_n + h(y') \quad (0.19)$$

Substituting (0.2) in (0.15), we have

$$y_{n+1} = y_n + h \left(\frac{x_n^3 - 3x_n y_n^2}{y_n^3 - 3x_n^2 y_n} \right) \quad (0.20)$$

which is the required difference equation.

Algorithm :

Take

$$x_0 = 1 \quad (0.21)$$

$$y_0 = 1 \quad (0.22)$$

$$h = 0.01 \quad (0.23)$$

The simulation plot can be plotted by iterating y_n ((0.16)) for different x_n as

$$x_{n+1} = x_n + h \quad (0.24)$$

The plot for computational and theoritical solution are given below.

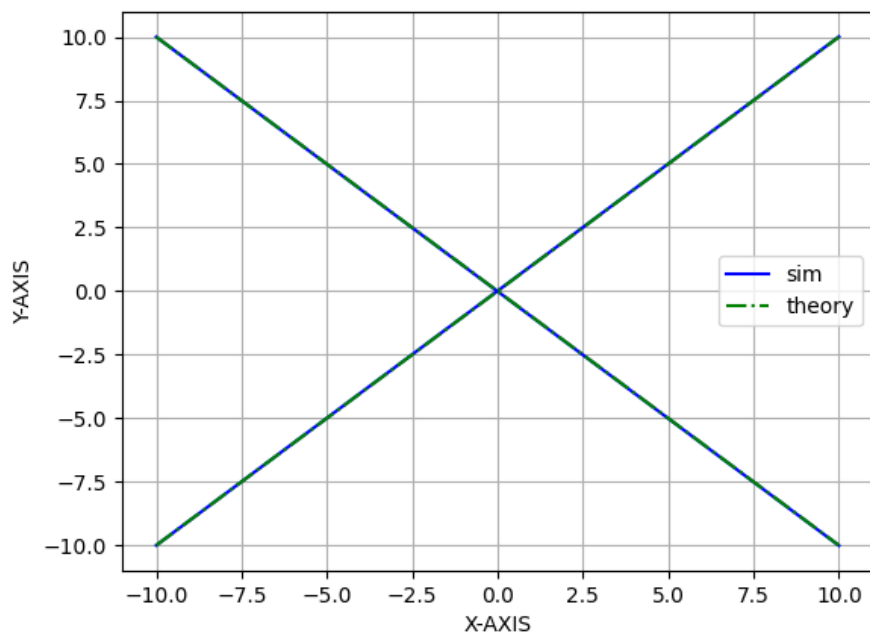


Fig. 0.1: Computational and theoretical solution.