

9.4.12

EE24BTECH11012 - Bhavanisankar G S

QUESTION :

For the differential equation, $x(x^2-1)\frac{dy}{dx} = 1$, find the particular solution given that $y(2) = 0$.

SOLUTION :

Theoretical :

Consider the given equation,

$$x(x^2 - 1)\frac{dy}{dx} = 1 \quad (0.1)$$

$$\frac{dy}{dx} = \frac{1}{x(x^2 - 1)} \quad (0.2)$$

$$y(2) = 0 \quad (0.3)$$

By the method of **Partial fractions**, we have

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{x-1} \right) + \frac{1}{2} \left(\frac{1}{x+1} \right) - \frac{1}{x} \quad (0.4)$$

Integrating (0.4) on both the sides, we have

$$\int dy = \int \left(\frac{1}{2} \left(\frac{1}{x-1} \right) + \frac{1}{2} \left(\frac{1}{x+1} \right) - \frac{1}{x} \right) dx \quad (0.5)$$

$$y = \frac{1}{2} \log |x-1| + \frac{1}{2} \log |x+1| - \log |x| + \log c \quad (0.6)$$

$$y = \log \left(\frac{c \sqrt{x^2-1}}{x} \right) \quad (0.7)$$

Substituting the initial conditions in (0.3), we have

$$0 = \log \left(\frac{c \sqrt{2^2-1}}{2} \right) \quad (0.8)$$

$$c = \frac{2}{\sqrt{3}} \quad (0.9)$$

Hence, the required particular solution becomes

$$y = \log \left(\frac{2 \sqrt{x^2-1}}{\sqrt{3}x} \right) \quad (0.10)$$

Simulation :

1) For a general interval, say $[a, b]$, split up the intervals into n parts such that

$$h = \frac{b - a}{n} \quad (1.1)$$

2) Consider the points

$$x_0 = a \quad (2.1)$$

$$x_n = b \quad (2.2)$$

$$x_{i+1} = x_i + h \quad (2.3)$$

3) **Trapezoid rule :**

Summing the areas of the trapezoids formed, we have

$$f(x) = \frac{1}{x(x^2 - 1)} \quad (3.1)$$

$$A \approx \frac{h}{2} ((f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n))) \quad (3.2)$$

$$A \approx \frac{h}{2} \left(f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n) \right) \quad (3.3)$$

4) To set up the difference equation, we integrate the equation (0.2) from $n - 1$ to n .

$$a = n - 1 \quad (4.1)$$

$$b = n \quad (4.2)$$

On further simplifying the equation (3.3), we have

$$y_{n+1} - y_n = (x_{n+1} - x_n) \left(\frac{f(x_n)}{2} + \frac{f(x_{n+1})}{2} \right) \quad (4.3)$$

$$y_{n+1} = y_n + (x_{n+1} - x_n) \left(\frac{f(x_n)}{2} + \frac{f(x_{n+1})}{2} \right) \quad (4.4)$$

$$y_{n+1} = y_n + \frac{(x_{n+1} - x_n)}{2} \left(\frac{1}{x_n(x_n^2 - 1)} + \frac{1}{x_{n+1}(x_{n+1}^2 - 1)} \right) \quad (4.5)$$

which is the required difference equation.

5) Taking $x_0 = 2$ and $y_0 = 0$ and iterating (4.5), we can obtain the other points.

Another approach :

Consider (0.2). Let the Laplace transform of RHS be $X(s)$. Then,

$$g(t) = \frac{1}{t(t^2 - 1)} \quad (5.1)$$

$$\frac{dy}{dt} = g(t) \quad (5.2)$$

Applying Laplace transform on both the sides of (5.2) , we have

$$sY(s) = X(s) \quad (5.3)$$

The transfer function, $H(s)$ can then be defined as

$$H(s) = \frac{Y(s)}{X(s)} \quad (5.4)$$

$$H(s) = \frac{1}{s} \quad (5.5)$$

Applying **Bi-linear transform** on both sides of (5.5), i.e., converting s -domain into z -domain, we have

$$s = \frac{2}{h} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) \quad (5.6)$$

$$H(z) = \frac{h}{2} \left(\frac{1 + z^{-1}}{1 - z^{-1}} \right) \quad (5.7)$$

$$Y(z) = \frac{h}{2} \left(\frac{1 + z^{-1}}{1 - z^{-1}} \right) X(z) \quad (5.8)$$

$$(1 - z^{-1}) Y(z) = \frac{h}{2} (1 + z^{-1}) X(z) \quad (5.9)$$

Taking **Inverse z-transform** on both the sides of (5.9) , we have

$$y_n - y_{n-1} = \frac{h}{2} (g(x_n) + g(x_{n-1})) \quad (5.10)$$

$$y_n = y_{n-1} + \frac{h}{2} (g(x_n) + g(x_{n-1})) \quad (5.11)$$

which is the required difference equation.

It can be seen that (4.5) and (5.11) are essentially the same.

Using RK method :

By the method of **Finite differences**, we have

$$y(x + h) = y(x) + y'(x, y)h \quad (5.12)$$

According to the RK method, (5.12) can be written as

$$y'(x, y) = \frac{1}{x(x^2 - 1)} \quad (5.13)$$

$$y(x_0 + h) = y(x_0) + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \quad (5.14)$$

where,

$$k_1 = hy'(x_0, y_0) \quad (5.15)$$

$$k_2 = hy' \left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2} \right) \quad (5.16)$$

$$k_3 = hy' \left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2} \right) \quad (5.17)$$

$$k_4 = hy'(x_0 + h, y_0 + k_3) \quad (5.18)$$

Iterating (5.14), a graph can be plotted.

Theoretical and simulation graphs of the methods described above are plotted below.

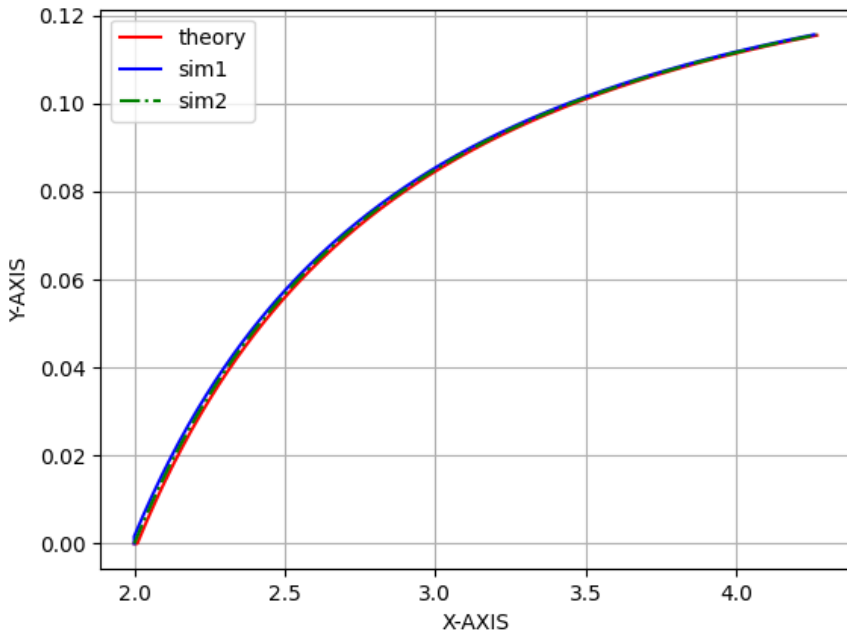


Fig. 5.1: Plot of the given question.