9.3.12.A

EE24BTECH11026 - G.Srihaas

QUESTION

Which of the following differential equations has $y = x$ as one of its particular solution?

$$(A)\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x \tag{0.1}$$

Solution: NUMERICAL METHOD

Consider,

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x \tag{0.2}$$

Assuming the initial conditions $y(0) = 0$ and $y'(0) = 1$.
Solve it by splitting into two parts homogeneous and particulate parts.

$$y = y_p + y_h \tag{0.3}$$

HOMOGENEOUS PART:

The associated homogeneous equation is:

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0. \tag{0.4}$$

Assume a power series solution:

$$y_h = \sum_{n=0}^{\infty} a_n x^n. \tag{0.5}$$

The derivatives are:

$$\frac{dy_h}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \frac{d^2y_h}{dx^2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}. \tag{0.6}$$

Substitute into the homogeneous equation:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2\sum_{n=1}^{\infty} n a_n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n = 0. \tag{0.7}$$

Rewriting terms, we derive the recurrence relation:

$$a_{n+2} = \frac{n-2}{(n+2)(n+1)}a_{n-1}. \tag{0.8}$$

Apply the initial conditions

The initial conditions are: $y(0) = 0 \quad \Rightarrow \quad a_0 = 0$, $y'(0) = 1 \quad \Rightarrow \quad a_1 = 1$.

Using the recurrence relation:

$$a_{n+2} = \frac{n-2}{(n+2)(n+1)} a_{n-1}, \tag{0.9}$$

we compute the coefficients:

1) For $n = 0$: $a_0 = 0$
2) For $n = 1$: $a_1 = 1$
3) For $n = 2$: $a_2 = \frac{0-2}{(2+2)(2+1)} a_1 = \frac{-2}{12} = -\frac{1}{6}$
4) For $n = 3$: $a_3 = \frac{1-2}{(3+2)(3+1)} a_2 = \frac{-1}{20} \cdot \left(-\frac{1}{6}\right) = \frac{1}{120}$
5) For $n = 4$: $a_4 = \frac{2-2}{(4+2)(4+1)} a_3 = 0$

The pattern is:

$$a_{2k} = 0, \quad a_{2k+1} = \frac{(-1)^k}{(2k+1)!}. \tag{5.1}$$

Therefore, the homogeneous solution is:

$$y_h = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}. \tag{5.2}$$

PARTICULATE PART:

The nonhomogeneous term is x. Assume a particular solution of the form:

$$y_p = Ax + B. \tag{5.3}$$

Compute derivatives:

$$\frac{dy_p}{dx} = A, \quad \frac{d^2 y_p}{dx^2} = 0. \tag{5.4}$$

Substitute into the original equation:

$$0 - x^2 A + x(Ax + B) = x. \tag{5.5}$$

Simplify: $-x^2 A + Ax^2 + Bx = x$

Hence we get $B = 1$

Since A does not appear explicitly in the final equation, it is effectively irrelevant, and A can be chosen such that: $y_p = Ax + B = x$

This is done as to set a proper particulate equation as principle coefficient can not be zero

Therefore,

$$y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + x \tag{5.6}$$

$$y(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + ... \tag{5.7}$$

Clearly, $y = x$ is not the solution to the given equation.

COMPUTATIONAL METHOD

We use the finite difference method to approximate the solution of the given differential equation:

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x. \tag{5.8}$$

The finite difference approximations for derivatives are:

$$\frac{dy}{dx} = \frac{y(x+h) - y(x)}{h} \tag{5.9}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}(x+h) - \frac{dy}{dx}(x)}{h}. \tag{5.10}$$

Rewriting the derivatives:

$$\frac{dy}{dx}(x+h) = \frac{dy}{dx}(x) + h.\frac{d^2y}{dx^2}. \tag{5.11}$$

Substituting these approximations into the differential equation:

$$\frac{d^2y}{dx^2} = x^2\frac{dy}{dx} - xy + x. \tag{5.12}$$

$$\frac{\frac{y(x+2h)-y(x+h)}{h} - \frac{y(x+h)-y(x)}{h}}{h} - x^2\frac{y(x+h) - y(x)}{h} + xy = x. \tag{5.13}$$

Simplify the equation:

$$\frac{y(x+2h) - 2y(x+h) + y(x)}{h^2} - x^2\frac{y(x+h) - y(x)}{h} + xy = x. \tag{5.14}$$

After generalsing the above equations, we can:
1.Compute $\frac{d^2y}{dx^2}$ using:

$$\frac{d^2y}{dx^2}[n] = x_n^2\frac{dy}{dx}[n] - x_n y[n] + x_n. \tag{5.15}$$

2.Update $\frac{dy}{dx}$ using:

$$\frac{dy}{dx}[n+1] = \frac{dy}{dx}[n] + h.\frac{d^2y}{dx^2}[n]. \tag{5.16}$$

3.Update $y(x)$ using:

$$y_{n+1} = y_n + h.\frac{dy}{dx}(n). \tag{5.17}$$

Starting with initial conditions $x_0 = 0$, $y[0] = 0$, and $\frac{dy}{dx}[0] = 1$, and using $h = 0.1$, iteratively compute $y[n+1]$,$\frac{dy}{dx}[n+1]$, and $\frac{d^2y}{dx^2}[n]$ for successive n.

From the figure below,clearly they dont coincide hence $y = x$ is not a solution to the given differential equation.