8.3.4

EE24BTECH11013 - MANIKANTA D

Question:
Sketch the graph of $y = |x + 3|$ and evaluate $\int_{-6}^{0} |x + 3|\, dx$.
Solution:
Integral to calculate,

$$J = \int_{-6}^{0} |x + 3|\, dx \tag{0.1}$$

Using the trapezoidal rule,

$$J = \int_{a}^{b} f(x)dx \approx h\left(\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right) \tag{0.2}$$

Recursive formula for the numerical solution:

$$J = j_n, \text{ where } j_{i+1} = j_i + h\frac{f(x_{i+1}) + f(x_i)}{2}, \tag{0.3}$$

$$x_{i+1} = x_i + h. \tag{0.4}$$

$$j_{i+1} = j_i + h\frac{|x_{i+1} + 3| + |x_i + 3|}{2}, \tag{0.5}$$

$$x_{i+1} = x_i + h. \tag{0.6}$$

where $h = \frac{b-a}{n}$, and $x_i = a + ih$.
Let $a = -6$, $b = 0$, and $f(x) = |x + 3|$. Choose $n = 8$, so:

$$h = \frac{0 - (-6)}{8} = \frac{6}{8} = 0.75. \tag{0.7}$$

The points are:

$$x_0 = -6,\ x_1 = -5.25,\ x_2 = -4.5,\ x_3 = -3.75,\ x_4 = -3, \tag{0.8}$$

$$x_5 = -2.25,\ x_6 = -1.5,\ x_7 = -0.75,\ x_8 = 0. \tag{0.9}$$

The corresponding function values are:

$$f(x_0) = |-6 + 3| = 3,\ f(x_1) = |-5.25 + 3| = 2.25,\ f(x_2) = |-4.5 + 3| = 1.5, \tag{0.10}$$

$$f(x_3) = |-3.75 + 3| = 0.75,\ f(x_4) = |-3 + 3| = 0, \tag{0.11}$$

$$f(x_5) = |-2.25 + 3| = 0.75,\ f(x_6) = |-1.5 + 3| = 1.5, \tag{0.12}$$

$$f(x_7) = |-0.75 + 3| = 2.25,\ f(x_8) = |0 + 3| = 3. \tag{0.13}$$

Substitute into the trapezoidal rule formula:

$$J \approx 0.75\left(\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7) + \frac{1}{2}f(x_8)\right) \tag{0.14}$$

$$J \approx 0.75\left(\frac{1}{2}(3) + 2.25 + 1.5 + 0.75 + 0 + 0.75 + 1.5 + 2.25 + \frac{1}{2}(3)\right) \tag{0.15}$$

$$J \approx 0.75\,(1.5 + 2.25 + 1.5 + 0.75 + 0 + 0.75 + 1.5 + 2.25 + 1.5) \tag{0.16}$$

$$J \approx 0.75 \times 12 \times 2 = 18. \tag{0.17}$$

The approximate value of the integral using the trapezoidal rule with $n = 8$ is $J \approx 18$.

Theoretical Solution:

Using the properties of the absolute value function, split the integral at $x = -3$:

$$J = \int_{-6}^{-3} -(x+3)dx + \int_{-3}^{0} (x+3)dx. \tag{0.18}$$

Evaluate each part:

$$\int_{-6}^{-3} -(x+3)dx = \int_{-6}^{-3} (-x-3)dx = \left[-\frac{x^2}{2} - 3x\right]_{-6}^{-3} = \frac{27}{2}, \tag{0.19}$$

$$\int_{-3}^{0} (x+3)dx = \left[\frac{x^2}{2} + 3x\right]_{-3}^{0} = \frac{9}{2}. \tag{0.20}$$

Combine results:

$$J = \frac{27}{2} + \frac{9}{2} = \frac{36}{2} = 18. \tag{0.21}$$

The exact value of the integral is $J = 18$.

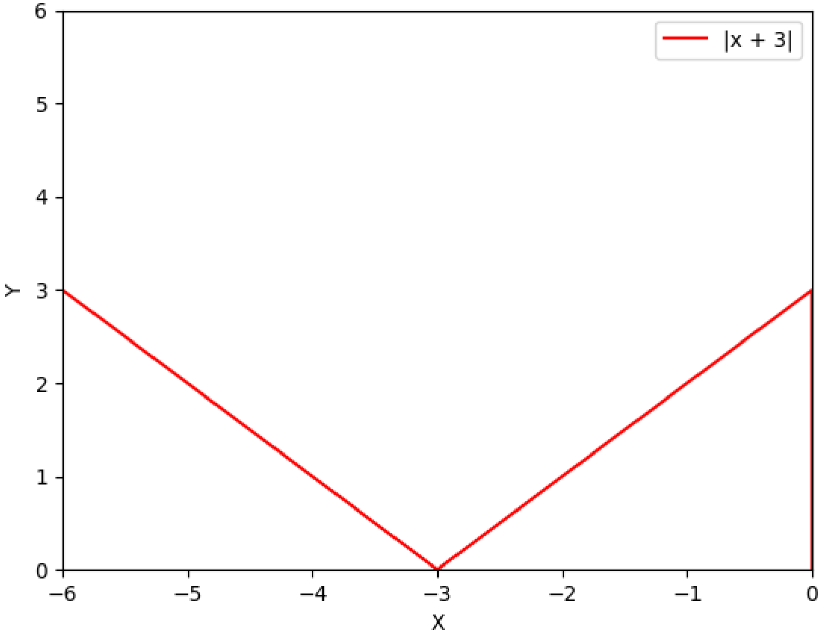

Fig. 0.1: Plot of the differential equation