NCERT-10.3.4.1.1

EE24BTECH11056 - S.Kavya Anvitha

Problem:

Solve the system of linear equations:

$$x + y = 5 \quad (1)$$

$$2x - 3y = 4 \quad (2)$$

Step 1: Represent the system in matrix form

The system of equations can be written as:

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

Step 2: Perform LU Decomposition

We decompose the matrix A into the product of a lower triangular matrix L and an upper triangular matrix U, i.e.,

$$A = LU$$

where

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}.$$

Now, let's compute the LU decomposition step by step.
First, we find the elements of U and L:

$$u_{11} = a_{11} = 1, \quad u_{12} = a_{12} = 1.$$

Next, we compute l_{21} and u_{22}:

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{2}{1} = 2,$$

$$u_{22} = a_{22} - l_{21}u_{12} = -3 - 2 \times 1 = -5.$$

So the LU decomposition is:

$$L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 \\ 0 & -5 \end{pmatrix}.$$

Step 3: Solve for $\mathbf{x}$ using LU decomposition

Now we solve the system in two steps using forward substitution and backward substitution.

First, solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$:

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

This gives:

$$y_1 = 5, \quad 2y_1 + y_2 = 4 \quad \Rightarrow \quad 2(5) + y_2 = 4 \quad \Rightarrow \quad y_2 = -6.$$

Thus, $\mathbf{y} = \begin{pmatrix} 5 \\ -6 \end{pmatrix}$.

Next, solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$:

$$\begin{pmatrix} 1 & 1 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \end{pmatrix}.$$

This gives:

$$-5y = -6 \quad \Rightarrow \quad y = \frac{6}{5}.$$

$$x + y = 5 \quad \Rightarrow \quad x + \left(\frac{6}{5}\right) = 5 \quad \Rightarrow \quad x = \frac{19}{5},$$

Thus, the solution is $x = \frac{19}{5}$ and $y = \frac{6}{5}$.

LU Decomposition using Doolittle's algorithm

The LU decomposition can be efficiently compted using Doolittle's algorithm. This method generates the matrices L (lower triangular) and U (upper triangular) such that $A = LU$. The elements of these matrices are calculated as follows:
Elements of the U Matrix:
For each column j:

$$U_{ij} = A_{ij} \quad \text{if } i = 0, \tag{1}$$

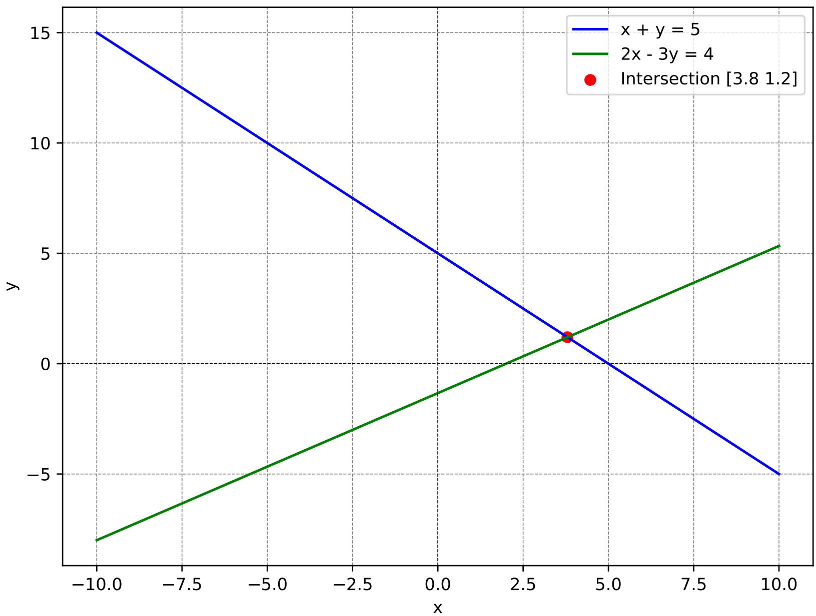
$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \quad \text{if } i > 0. \tag{2}$$

Elements of the L Matrix:
For each row i:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \quad \text{if } j = 0, \tag{3}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \quad \text{if } j > 0. \tag{4}$$

This systematic approach ensures that the matrix A is decomposed into L and U without requiring row swaps, provided A is nonsingular.


x + y = 5
2x - 3y = 4
Intersection [3.8 1.2]
15
10
5
0
−5
y
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
x