Solving differential equation NCERT-12.9.ex.12

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Question:
$xdy = \left(2x^2 + 1\right) dx$
Solution:
Using Bilinear Transform technique:
Original Differential Equation:

$$\frac{dy}{dx} = 2x + \frac{1}{x} \tag{1}$$

Let $\frac{dy}{dx} = x(t)$ where $x(t) = 2t + \frac{1}{t}$ Taking the Laplace transform:

$$sY(s) = X(s) \tag{2}$$

which gives the transfer function:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s}. \tag{3}$$

Using the Bilinear Transform Relation
The bilinear transform substitutes with a discrete-time equivalent:

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{4}$$

where T is the sampling period. Substituting this into the equation gives :

$$H(z) = \frac{1}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \tag{5}$$

Simplify the expression:

$$H(z) = \frac{T}{2} \times \frac{1 + z^{-1}}{1 - z^{-1}} \tag{6}$$

$H(z)$ describes the transfer function in the z-domain. Rewrite it in terms of input-output relation:

$$H(z) = \frac{Y(z)}{X(z)} \tag{7}$$

$$\frac{Y(z)}{X(z)} = \frac{T}{2} \times \frac{1+z^{-1}}{1-z^{-1}} \tag{8}$$

$$\text{Cross multiply:} \tag{9}$$

$$Y(z)\left(1 - z^{-1}\right) = \frac{T}{2} X(z)\left(1 + z^{-1}\right) \tag{10}$$

Expanding into time domain (replace with a delay operator):

$$y[n] - y[n-1] = \frac{T}{2}\left(x[n] + x[n-1]\right) \tag{11}$$

$x(n) = 2t(n) + \frac{1}{t(n)}$, where $t(n) = t(0) + n * h$ Substituting in the equation gives

$$y[n] - y[n-1] = \frac{h}{2}\left(2t(n) + \frac{1}{t(n)} + 2t(n-1) + \frac{1}{t(n-1)}\right) \tag{12}$$

$$y[n] - y[n-1] = \frac{h}{2}\left(2(t(0) + n \cdot h) + \frac{1}{t(0) + n \cdot h} + 2(t(0) + (n-1) \cdot h) + \frac{1}{t(0) + (n-1) \cdot h}\right) \tag{13}$$

$$\text{simplifying,} \tag{14}$$

$$y_n - y_{n-1} = \frac{h}{2}\left(4t_0 + 4n \cdot h - 2h + \frac{1}{t_0 + n \cdot h} + \frac{1}{t_0 + (n-1) \cdot h}\right) \tag{15}$$

here $t_0 = 1$ substituting we get

$$y_n - y_{n-1} = \frac{h}{2}\left(4 + 4n \cdot h - 2h + \frac{1}{1 + n \cdot h} + \frac{1}{1 + (n-1) \cdot h}\right) \tag{16}$$

trapezoidal solution: The aim is to find the difference equation using the trapezoidal law using the following initial conditions $x_0 = 1$ and $y_0 = 1$

$$\frac{dy}{dx} = 2x + \frac{1}{x} \tag{17}$$

1. Start with the Differential Equation
The given differential equation is:

$$\frac{dy}{dx} = f(x, y). \tag{18}$$

Integrate this over the interval :

$$\int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y)\, dx \tag{19}$$

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y)\, dx. \tag{20}$$

3. Generalize Over Multiple Intervals

Now, consider the integral over multiple intervals from to :

$$\int_{x_0}^{x_n} f(x)\, dx. \tag{21}$$

Using the trapezoidal rule, we approximate this integral as:

$$\int_{x_0}^{x_n} f(x)\, dx \approx \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]. \tag{22}$$

This can be expressed as:

$$\int_{x_0}^{x_n} f(x)\, dx \approx h\left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]. \tag{23}$$

The trapezoidal rule approximates an integral over an interval as:

$$\int_{x_n}^{x_{n+1}} f(x)\, dx \approx \frac{h}{2}\left[f(x_n) + f(x_{n+1})\right], \tag{24}$$

4. Substitute Back into the Differential Equation

Returning to the differential equation, , the difference equation becomes:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]. \tag{25}$$

The function is:

$$f(x, y) = 2x + \frac{1}{x}. \tag{26}$$

Substitute and into the general trapezoidal formula:

$$y_{n+1} = y_n + \frac{h}{2}\left[2x_n + \frac{1}{x_n} + 2x_{n+1} + \frac{1}{x_{n+1}}\right]. \tag{27}$$

Express x_{n+1} Explicitly
since

$$x_{n+1} = x_n + h, \tag{28}$$

substitute this into the second term:

$$2x_{n+1} + \frac{1}{x_{n+1}} = 2(x_n + h) + \frac{1}{x_n + h}. \tag{29}$$

Now rewrite the formula:

$$y_{n+1} = y_n + \frac{h}{2}\left[2x_n + \frac{1}{x_n} + 2(x_n + h) + \frac{1}{x_n + h}\right].y_{n+1} = y_n + \frac{h}{2}\left(4x_n + 2h + \frac{1}{x_n} + \frac{1}{x_n + h}\right) \tag{30}$$

theoretical solution:

$$\frac{dy}{dx} = 2x + \frac{1}{x} \tag{31}$$

$$dy = \left(2x + \frac{1}{x}\right)dx \tag{32}$$

$$\int dy = \int \left(2x + \frac{1}{x}\right)dx \tag{33}$$

$$y = x^2 + \log|x| + C \tag{34}$$

given it passes through $\left(1, 1\right)$ C = 0 Therefore:

$$y = x^2 + \log|x| \tag{35}$$