NCERT-10.3.5.1.3

S. Sai Akshita - EE24BTECH11054

Question: Check whether the following pair of equations has a unique solution, no solution, or infinitely many solutions. If there is a unique solution, find it by using Cross- Multiplication method.

$$3x - 5y = 20 \tag{1}$$
$$6x - 10y = 40 \tag{2}$$

Theoretical Solution: By comparing 1 and 2 with the following:

$$a_1x + b_1y = c_1 \tag{3}$$
$$a_2x + b_2y = c_2 \tag{4}$$

we get,

$$a_1 = 3, b_1 = -5, c_1 = 20 \tag{5}$$
$$a_2 = 6, b_2 = -10, c_2 = 40 \tag{6}$$

Calculating the ratios of x, ycoefficients and constants, we get:

$$\frac{a_1}{a_2} = \frac{3}{6} = \frac{1}{2} \tag{7}$$
$$\frac{b_1}{b_2} = \frac{-5}{-10} = \frac{1}{2} \tag{8}$$
$$\frac{c_1}{c_2} = \frac{20}{40} = \frac{1}{2} \tag{9}$$
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \tag{10}$$

This shows the condition met for the lines to have infinitely many solutions.
LU Decomposition:
LU Decomposition is the process of factoring a square matrix A into the product of two matrices:

$$A = L \cdot U, \tag{11}$$

where:

- L is a **lower triangular matrix** with ones on the diagonal.
- U is an **upper triangular matrix**.

This decomposition is used to solve systems of linear equations efficiently, and it forms the foundation for many numerical methods.

Step 1: Represent the Matrix A

Suppose A is a $n \times n$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}. \tag{12}$$

We want to decompose A into:

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & u_{nn} \end{bmatrix}. \tag{13}$$

Step 2: Decomposition Process

1. The first row of U is the same as the first row of A:

$$u_{1j} = a_{1j}, \quad \text{for } j = 1, 2, \ldots, n. \tag{14}$$

2. The first column of L is calculated as:

$$l_{i1} = \frac{a_{i1}}{u_{11}}, \quad \text{for } i = 2, 3, \ldots, n. \tag{15}$$

3. For the remaining rows of U, calculate:

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}, \quad \text{for } j = i, i+1, \ldots, n. \tag{16}$$

4. For the remaining columns of L, calculate:

$$l_{ij} = \frac{1}{u_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right), \quad \text{for } i = j+1, j+2, \ldots, n. \tag{17}$$

This process continues until A is fully decomposed into L and U.

Proving Dependency of Equations Using LU Decomposition:

We aim to demonstrate that the system of equations has infinitely many solutions using LU decomposition. The given system can be written in matrix form as:

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{18}$$

where:

$$\mathbf{A} = \begin{bmatrix} 3 & -5 \\ 6 & -10 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 20 \\ 40 \end{bmatrix}. \tag{19}$$

Step 1: Perform LU Decomposition

The goal is to decompose the coefficient matrix $\mathbf{A}$ into:

$$\mathbf{A} = \mathbf{L} \cdot \mathbf{U}, \tag{20}$$

where:

- $\mathbf{L}$ is a lower triangular matrix.
- $\mathbf{U}$ is an upper triangular matrix.

Let:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}. \tag{21}$$

Step 2: Start Decomposition

1) The first row of **U** is the same as the first row of **A**:

$$u_{11} = 3, \quad u_{12} = -5. \tag{22}$$

2) The first column of **L** is derived by dividing the corresponding element of the second row of **A** by u_{11}:

$$l_{21} = \frac{6}{3} = 2. \tag{23}$$

3) The second row of **U** is computed by eliminating the first column from the second row of **A**:

$$u_{22} = -10 - (l_{21} \cdot u_{12}) = -10 - (2 \cdot -5) = -10 + 10 = 0. \tag{24}$$

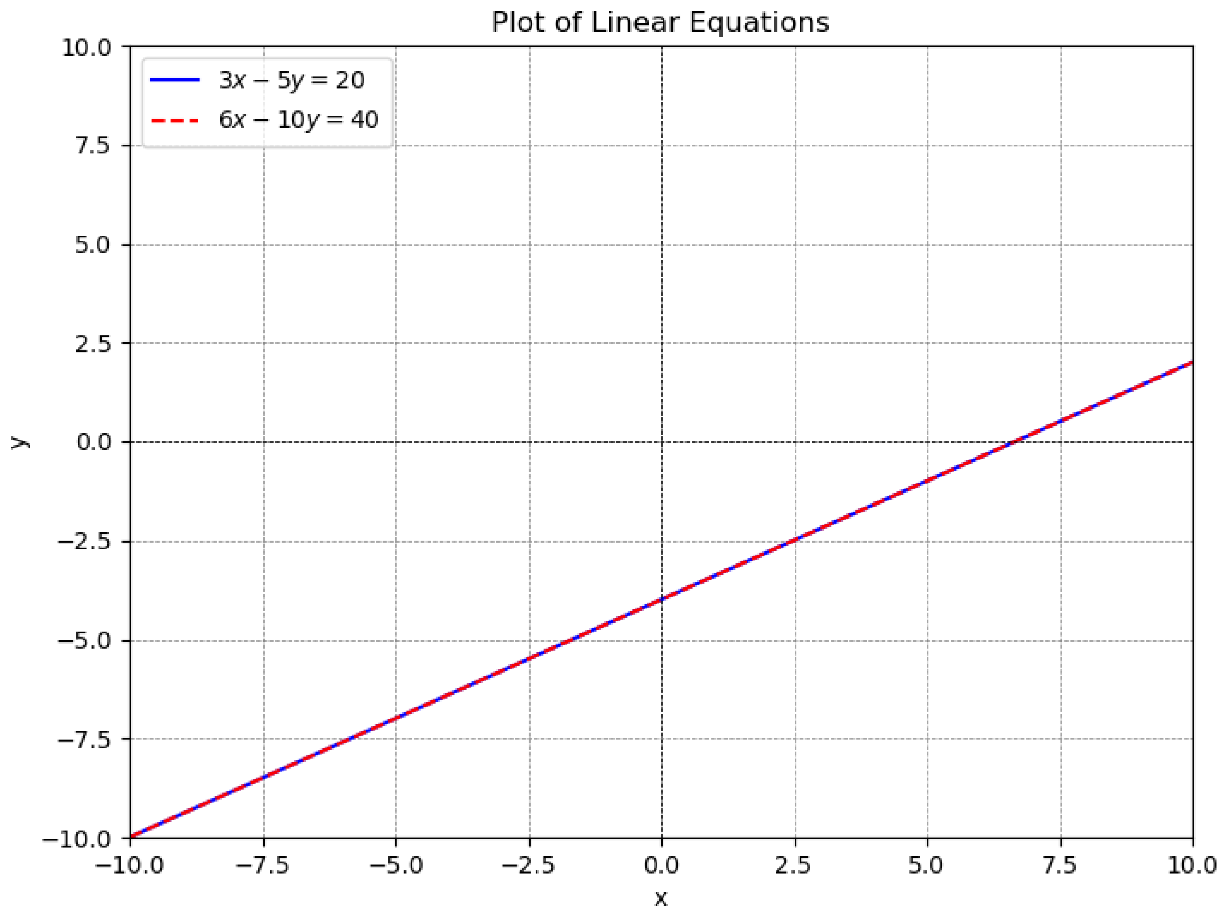
Thus, the matrices **L** and **U** are:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 3 & -5 \\ 0 & 0 \end{bmatrix}. \tag{25}$$

Step 3: Analyze the Result

The matrix **U** contains a zero in the bottom-right element ($u_{22} = 0$). This indicates that the rows of the original matrix **A** are **linearly dependent**. Therefore, The system of equations does not have a unique solution. Instead, it has infinitely many solutions because the two equations represent the same line. Through LU decomposition, we proved that the system of equations is **dependent** (since $u_{22} = 0$), and there are infinitely many solutions. The general solution can be expressed as:

$$x = \frac{5y + 20}{3}, \quad y \text{ is any real number.} \tag{26}$$



Plot of Linear Equations
$3x - 5y = 20$
$6x - 10y = 40$
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
x
y