9.1.11

EE24BTECH11010 - Balaji B

Question:
Plot a solution to the following differential equation: $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$

Solution: An exact theoretical solution using known methods of solving differential equations was not found; however, it can be approximated to a pretty good degree of precision. Euler's method will be used to obtain a plot of the solution
Computational Solution:
By first principle of derivatives,

$$y'(t) = \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} \tag{0.1}$$

$$y(t+h) = y(t) + hy'(t) \tag{0.2}$$

Let y^i be the i^{th} derivative of the function, m be the order of the differential equation. Set $y_1 = y, y_2 = y^1, y_3 = y^2 \ldots$, and so on.
We obtain the system,

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{m-1}' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \end{pmatrix} \tag{0.3}$$

$$y_m' = f(x, y_1, y_2, \ldots, y_m) \tag{0.4}$$

Generalizing the system according to Euler's form

$$\begin{pmatrix} y_1(x+h) \\ \vdots \\ y_{m-1}(x+h) \\ y_m(x+h) \end{pmatrix} = \begin{pmatrix} y_1(x) \\ \vdots \\ y_{m-1}(x) \\ y_m(x) \end{pmatrix} + h \begin{pmatrix} y_2(x) \\ \vdots \\ y_m(x) \\ f(x, y_1, y_2, \ldots, y_m) \end{pmatrix} \tag{0.5}$$

$$\mathbf{y}(x+h) = \mathbf{y}(x) + h \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & \frac{f(x,y_1,y_2,\ldots,y_m)}{y_m(x)} \end{pmatrix} \mathbf{y}(x) \tag{0.6}$$

$$\mathbf{y}(x+h) = \begin{pmatrix} 1 & h & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 + \frac{f(x,y_1,y_2,\dots,y_m)}{y_m(x)} \end{pmatrix} \mathbf{y}(x) \tag{0.7}$$

Discretizing the steps we get,

$$\mathbf{y}_{n+1} = \begin{pmatrix} 1 & h & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 + \frac{f(x,y_1,y_2,\dots,y_m)}{(y_m)_n} \end{pmatrix} \mathbf{y}_n \tag{0.8}$$

$$x_{n+1} = x_n + h \tag{0.9}$$

Where,

$$\mathbf{y}_n = \begin{pmatrix} y_1(x_n) \\ y_2(x_n) \\ \vdots \\ y_m(x_n) \end{pmatrix} \tag{0.10}$$

Smaller values of step size h will give more precise plots. We obtain points to plot by iterating repeatedly. The given differential equation can be written as

$$\left(\frac{d^2y}{dx^2}\right)^3 = -\left(\frac{dy}{dx}\right)^2 - \sin\left(\frac{dy}{dx}\right) - 1 \tag{0.11}$$

Here order m is 2.

$$\mathbf{y}_{n+1} = \begin{pmatrix} 1 & h & 0 \\ 0 & 1 & h \\ 0 & 0 & 1 + \frac{\left(-\left((y_1)_n^2 - \sin(y_1)_n - 1\right)\right)^{\frac{1}{3}}}{(y_2)_n} \end{pmatrix} \mathbf{y}_n \tag{0.12}$$

Note, here the vector $\mathbf{y}$ is not to be confused with y_i which represents a function, namely the $i+1^{th}$ derivative of $y(x)$ Below is the plot for given curve based on initial conditions, obtained by iterating through the above equation.

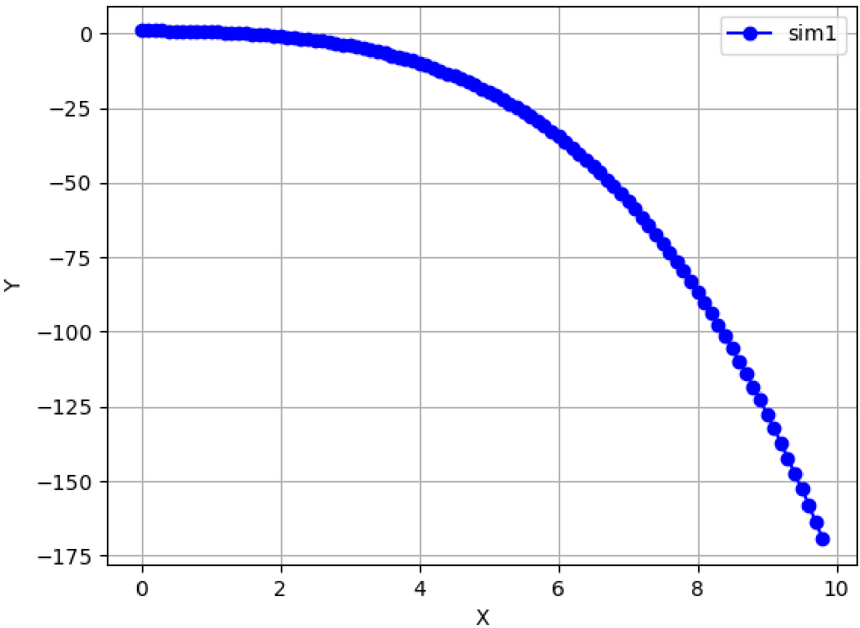

Fig. 0.1: Computational solution of $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$