NCERT-10.3.4.2.1

EE24BTECH11055 - Sai Akhila

Question:

If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes $\frac{1}{2}$ if we only add 1 to the denominator. What is the fraction?

Theoretical Solution:

Let the fraction be $\frac{x}{y}$.
Given, $\frac{x+1}{y-1} = 1$ and, $\frac{x}{y+1} = \frac{1}{2}$

Solving them:
$x + 1 = y - 1 \implies x - y + 2 = 0$

$$x - y + 2 = 0 \tag{0.1}$$

$$2x - y - 1 = 0 \tag{0.2}$$

$$y = x + 2$$

$$2x - (x + 2) - 1 = 0$$

$$x = 3$$

Substituting $x = 3$ in equation (0.1), we get $y = 5$.
Therefore the fraction is $\frac{3}{5}$

Using LU decomposition

LU Decomposition is used primarily to simplify the process of solving systems of linear equations and other matrix-related computations. The main reason for using LU decomposition is to break down a complex matrix operation into simpler steps.
The system of equations (0.1) and (0.2) can be written as:

$$A \cdot \mathbf{x} = \mathbf{b} \tag{0.3}$$

where,

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We decompose the matrix A into two simple triangular matrices L (lower triangular) and U (upper triangular).
Instead of solving the system directly, solve two systems:
$L \cdot y = b$ (forward substitution)
$U \cdot x = y$ (backward substitution)
Given below are the steps for implementation of this algorithm.

1. Initialize L as an identity matrix and U as A

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

2. Perform Gaussian elimination to make U upper triangular

For each column $j \geq k$, the entries of U in the kth row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m}.U_{m,j}, \forall j \geq k \tag{0.4}$$

For each row $i > k$, the entries of L in the kth column are updated as:

$$L_{j,k} = \frac{1}{U_{k,k}} \left(A_{j,k} - \sum_{m=1}^{k-1} .U_{m,k} \right), \forall i > k \tag{0.5}$$

Eliminate U_{21} (second row, first column) - The multiplier is:

$$l_{21} = \frac{2}{1} = 2$$

- Subtract 2 times the first row from the second row:

$$R_2 \rightarrow R_2 - 2R_1$$

$$U = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

- Store the multiplier in L :

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

3. Final Result

We have decomposed A into:

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Thus, we have $A = LU$.
Since $A = LU$, we rewrite it as:

$$LUx = b$$

Define y such that:

$$Ly = b$$

This gives two triangular systems:
1. Solve $Ly = b$ (Forward Substitution)
2. Solve $Ux = y$ (Backward Substitution)

Step 1: Solve $Ly = b$ (Forward Substitution)

Expanding:

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

This gives the equations:

$$y_1 = -2$$

$$2y_1 + y_2 = 1$$

Solving for y_2:

$$y_2 = 1 - 2(-2) = 1 + 4 = 5$$

Thus,

$$y = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Step 2: Solve $Ux = y$ (Backward Substitution)

Expanding:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

This gives the equations:

$$x_1 - x_2 = -2$$

$$x_2 = 5$$

Solving for x_1:

$$x_1 = -2 + x_2 = -2 + 5 = 3$$

Thus, the solution is:

$$x = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

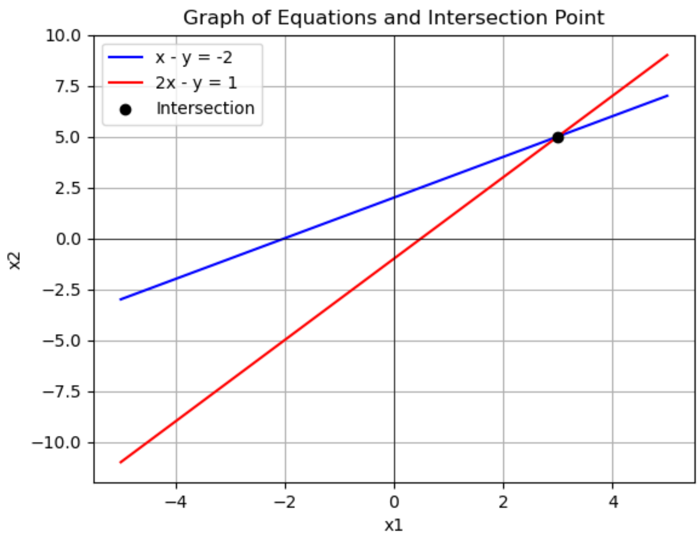

Fig. 0.1: Graph of the equations with the intersection point