NCERT-8.1.10

EE24BTECH11023 - RASAGNA

Question: Find the area bound by the curve $x^2 = 4y$ and the line $x = 4y - 2$.
Solution: The given parabola equation can be expressed as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0, \tag{0.1}$$

where,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0. \tag{0.2}$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 0 = 0 \tag{0.3}$$

Given line equation can be expressed as,

$$x = h + km \tag{0.4}$$

where,

$$\mathbf{h} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ \frac{1}{4} \end{pmatrix} \tag{0.5}$$

Intersection of a line and a conic is given by,

$$\kappa_i = \frac{-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(h)(\mathbf{m}^\top \mathbf{V}\mathbf{m})}}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{0.6}$$

simplifying and substituting the values,
we get,

$$k_1 = -1, k_2 = 2 \tag{0.7}$$

By substituting k_1 and k_2 values in equation (0.4),we get the points of intersection as, $\begin{pmatrix} -1 \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ Equation for area enclosed is given by,

$$A = \int_{-1}^{2} \left| \frac{x+2}{4} - \frac{x^2}{4} \right| dx \tag{0.8}$$

Theoritical Solution

$$A = \int_{-1}^{2} \left| \frac{x+2}{4} - \frac{x^2}{4} \right| dx \tag{0.9}$$

$$= \frac{1}{4} \int_{-1}^{2} \left(x + 2 - x^2 \right) dx. \tag{0.10}$$

$$A = \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}. \tag{0.11}$$

$$\begin{aligned} A &= \frac{1}{4}\left[\left(\frac{2^2}{2} + 2(2) - \frac{2^3}{3}\right) - \left(\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right)\right] && (0.12)\\ &= \frac{1}{4}\left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right] && (0.13)\\ &= \frac{9}{8} && (0.14)\end{aligned}$$

Computational Solution To calculate the area under the curve y_x from $x = x_0$ to $x = x_n$, we approximate it using trapezoidal strips of small areas. The x-axis is discretized into points $x_0, x_1, x_2, \ldots, x_n$, such that the spacing between consecutive points is equal, with a step size of h. The total area, as the sum of all trapezoidal areas, is given by:

$$\begin{aligned} A &= \frac{1}{2}h\left(y\left(x_1\right) + y\left(x_0\right)\right) + \frac{1}{2}h\left(y\left(x_2\right) + y\left(x_1\right)\right) + \cdots + \frac{1}{2}h\left(y\left(x_n\right) + y\left(x_{n-1}\right)\right) && (0.15)\\ &= h\left[\frac{1}{2}\left(y\left(x_0\right) + y\left(x_n\right)\right) + y\left(x_1\right) + \cdots + y\left(x_{n-1}\right)\right] && (0.16)\end{aligned}$$

Let $A\left(x_n\right)$ be the area enclosed by the curve $y\left(x\right)$ from $x = x_0$ to $x = x_n$, $\left(x_0, x_1, \ldots x_n\right)$ be equidistant points with step-size h.

$$A\left(x_n + h\right) = A\left(x_n\right) + \frac{1}{2}h\left(y\left(x_n + h\right) + y\left(x_n\right)\right) \tag{0.17}$$

We can iterate till we get required area. Discretizing the steps, making $A\left(x_n\right) = A_n, y\left(x_n\right) = y_n$,

$$A_{n+1} = A_n + \frac{1}{2}h\left(y_{n+1} + y_n\right) \tag{0.18}$$

Using first principle of derivative,$y_{n+1} = y_n + hy'_n$

$$\begin{aligned} A_{n+1} &= A_n + \frac{1}{2}h\left(\left(y_n + hy'_n\right) + y_n\right) && (0.19)\\ A_{n+1} &= A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) && (0.20)\\ A_{n+1} &= A_n + hy_n + \frac{1}{2}h^2y'_n && (0.21)\\ x_{n+1} &= x_n + h && (0.22)\end{aligned}$$

In the given question, $y_n = x_n + 2 - x_n^2$ and $y'_n = 1 - 2x_n$.

General Difference Equation is given by,

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.23}$$

$$= A_n + h\left(x_n + 2 - x_n^2\right) + \frac{1}{2}h^2\left(1 - 2x_n\right) \tag{0.24}$$

$$= A_n + x_n\left(h - h^2\right) - x_n^2\left(h\right) + \frac{h^2}{2} + 2h \tag{0.25}$$

$$x_{n+1} = x_n + h \tag{0.26}$$

We should iterate till $x_n = 2$ to get the required area.
The obtained theoritical solution is 1.125 sq.units.
The computational solution is 1.1249988750000004sq.units.