NCERT-10.4.ex.15

EE24BTECH11039 - MANDALA RANJITH

Question

A motor boat whose speed is 18 km/h in still water takes 1 hour more to go 24 km upstream than to return downstream to the same spot. Find the speed of the stream.

Theoritical Solution

Let the speed of the stream be x km/h.

Therefore, the speed of the boat upstream = $(18 - x)$ km/h and the speed of the boat downstream = $(18 + x)$ km/h.

The time taken to go upstream is:

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}} = \frac{24}{18 - x} \text{ hours.} \tag{1}$$

Similarly, the time taken to go downstream is:

$$\frac{24}{18 + x} \text{ hours.} \tag{2}$$

According to the question,

$$\frac{24}{18 - x} - \frac{24}{18 + x} = 1 \tag{3}$$

Multiplying throughout by $24(18 + x)(18 - x)$, we get:

$$24(18 + x) - 24(18 - x) = (18 - x)(18 + x) \tag{4}$$

$$x^2 + 48x - 324 = 0 \tag{5}$$

Using the quadratic formula:

$$x = \frac{-48 \pm \sqrt{48^2 + 1296}}{2} = \frac{-48 \pm \sqrt{3600}}{2} \tag{6}$$

$$= \frac{-48 \pm 60}{2} \tag{7}$$

$$= 6 \text{ or } -54 \tag{8}$$

Since x is the speed of the stream, it cannot be negative. So, we ignore the root $x = -54$.

Therefore, $x = 6$ gives the speed of the stream as **6 km/h**.

Theorem: (9)

Let $x = s$ be a solution of $x = g(x)$ and suppose that g has a continuous derivative in some interval J containing s. Then if $|g'| \leq K < 1$ in J, the iteration process defined above converges for any x_0 in J. The limit of the sequence $[x_n]$ is s

Since there is no solution (evident by quadratic formula) there exists no interval J for which the process converges to a point.

The same behaviour is shown by the Newton-Raphson Method,
Start with an initial guess x_0, and then run the following logical loop,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{10}$$

where ,

$$f(x) = x^2 + 48x - 324 \tag{11}$$

$$f'(x) = 2x + 48 \tag{12}$$

CODING LOGIC FOR FINDING EIGENVALUES :-

The quadratic equation

$$x^2 + 48x - 324 = 0 \tag{13}$$

is rewritten in matrix form:

$$\text{Matrix} = \begin{pmatrix} 0 & -\frac{c}{a} \\ 1 & -\frac{b}{a} \end{pmatrix} \tag{14}$$

$$a = 1, \quad b = 48, \quad c = -324. \tag{15}$$

Substituting the values of a, b and c, the matrix becomes:
Let

$$\text{A} = \begin{pmatrix} 0 & 324 \\ 1 & -\frac{48}{1} \end{pmatrix} \tag{16}$$

QR-DECOMPOSITION:-GRAM-SCHMIDT METHOD

1) QR decomposition

$$A = QR \tag{17}$$

   a) Q is an $m \times n$ orthogonal matrix
   b) R is an $n \times n$ upper triangular matrix.

Given a matrix $A = [a_1, a_2, \ldots, a_n]$, where each a_i is a column vector of size $m \times 1$.

2) Normalize the first column of A:

$$q_1 = \frac{a_1}{\|a_1\|} \tag{18}$$

3) For each subsequent column a_i, subtract the projections of the previously obtained orthonormal vectors from a_i :

$$a_i' = a_i - \sum_{k=1}^{i-1} \langle a_i, q_k \rangle q_k \tag{19}$$

Normalize the result to obtain the next column of Q:

$$q_i = \frac{a_i'}{\|a_i'\|} \tag{20}$$

Repeat this process for all columns of A.

4) Finding R:-
After constructing the ortho-normal columns $q_1, q_2, \ldots, q_n$ of Q, we can compute the elements of R by taking the dot product of the original columns of A with the columns of Q:

$$r_{ij} = \langle a_j, q_i \rangle \text{ , for } i \leq j \tag{21}$$

QR-Algorithm

1) Initialization
Let $A_0 = A$, where A is the given matrix.
2) QR Decomposition
For each iteration $k = 0, 1, 2, \ldots$:
   a) Compute the QR decomposition of A_k, such that:

$$A_k = Q_k R_k \tag{22}$$

   where:
      i) Q_k is an orthogonal matrix ($Q_k^\top Q_k = I$).
      ii) R_k is an upper triangular matrix.

   The decomposition ensures $A_k = Q_k R_k$.
   b) Form the next matrix A_{k+1} as:

$$A_{k+1} = R_k Q_k \tag{23}$$

3) Convergence
Repeat Step 2 until A_k converges to an upper triangular matrix T. The diagonal entries of T are the eigenvalues of A.

4) The eigenvalues of matrix will be the roots of the equation.

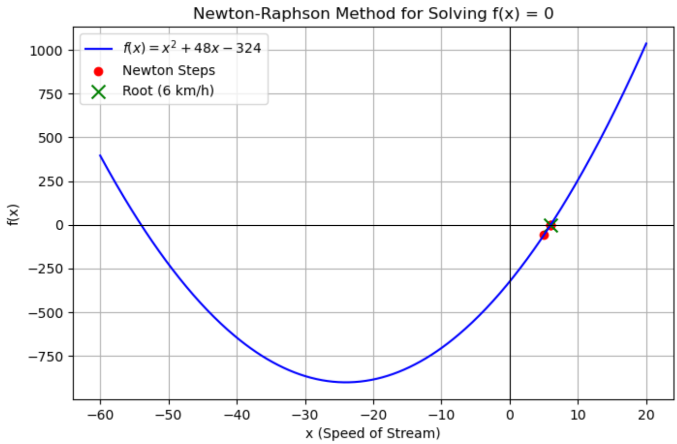

Fig. 4.1: Plot showing the relationship between $f(x)$ and speed of stream

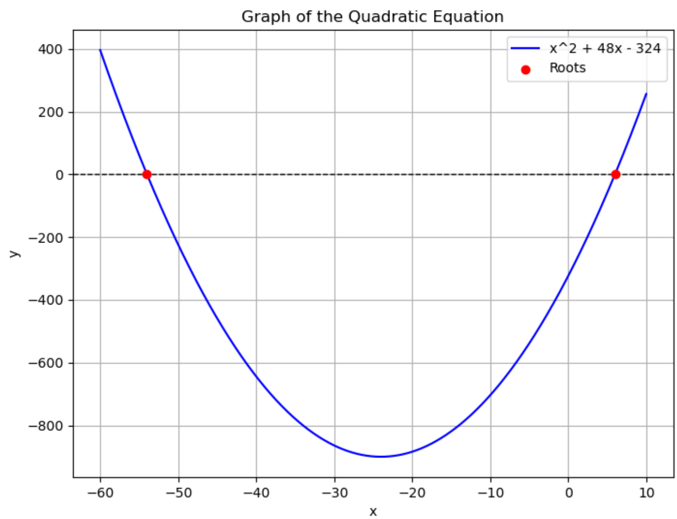

Fig. 4.2: Plot showing the relationship between $f(x)$ and speed of stream