9.3.11.1

EE24BTECH11042 - M.SRUJANA

Question:
Solve the differential equation $\frac{d^2y}{dx^2} = -y$ with initial conditions $y(0) = 1$ and $y'(0) = 0$
Solution:
Theoretical Solution:
Laplace Transform definition

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt \tag{0.1}$$

Properties of Laplace Transform:

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) \tag{0.2}$$

$$\mathcal{L}(1) = \frac{1}{s} \tag{0.3}$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{0.4}$$

$$\mathcal{L}(e^{at} f(t)) = F(s-a) \tag{0.5}$$

Applying the Laplace Transform to the differential equation:

$$y'' + y = 0 \tag{0.6}$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = 0 \tag{0.7}$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = 0 \tag{0.8}$$

Substitute the initial conditions:

$$(s^2 + 1)\mathcal{L}(y) = s \tag{0.9}$$

$$\mathcal{L}(y) = \frac{s}{s^2+1} \tag{0.10}$$

$$\mathcal{L}(y) = \frac{1}{2}\left(\frac{1}{s+i} + \frac{1}{s-i}\right) \tag{0.11}$$

Taking the inverse Laplace transform:

$$y = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \tag{0.12}$$

$$y = \cos(x) \tag{0.13}$$

So, the theoretical solution is:

$$y(x) = \cos(x) \tag{0.14}$$

Now, we will find the difference equation using the Bilinear Z-transform.

Using the substitution for s:

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{0.15}$$

Applying this substitution to the Laplace transform expression:

$$Y(z) = \frac{1}{2}\left(\frac{T(1 + z^{-1})}{2(1 - z^{-1}) + T(1 + z^{-1})} + \frac{T(1 + z^{-1})}{2(1 - z^{-1}) - T(1 + z^{-1})}\right) \tag{0.16}$$

Simplify the expression:

$$Y(z) = \frac{T(1 + z^{-1})}{2(1 - z^{-1}) + T(1 + z^{-1})} - \frac{T(1 + z^{-1})}{2(1 - z^{-1}) - T(1 + z^{-1})} \tag{0.17}$$

Define the constants:

$$\alpha_1 = -\frac{T - 2}{T + 2} \tag{0.18}$$

$$\alpha_2 = -\frac{T + 2}{T - 2} \tag{0.19}$$

Thus, we have:

$$Y(z) = \frac{T}{2(T + 2)}\left(\frac{1}{1 - \alpha_1 z^{-1}} + \frac{z^{-1}}{1 - \alpha_1 z^{-1}}\right) - \frac{T}{2(T - 2)}\left(\frac{1}{1 - \alpha_2 z^{-1}} + \frac{z^{-1}}{1 - \alpha_2 z^{-1}}\right) \tag{0.20}$$

The radius of convergence is given by:

$$\text{Radius of convergence} = \max(|\alpha_1|, |\alpha_2|) \tag{0.21}$$

Finally, apply the inverse Z-transform:

$$(1 - \alpha_1 z^{-1})(1 - \alpha_2 z^{-1})Y(z) = \frac{T}{2}\left(\frac{1 - (\alpha_2 - 1)z^{-1} - \alpha_2 z^{-2}}{T + 2} - \frac{1 - (\alpha_1 - 1)z^{-1} - \alpha_1 z^{-2}}{T - 2}\right) \tag{0.22}$$

Rearrange to get the difference equation:

$$y_{n+2} - (\alpha_1 + \alpha_2)y_{n+1} + y_n = \frac{T}{2}\left(\frac{\alpha_1}{T - 2} - \frac{\alpha_2}{T + 2}\right)\delta(n) \tag{0.23}$$

Now, moving to the computational solution, discretize the equation using the trapezoidal rule:

$$y_{k+1} - y_k = \frac{h}{2}\left(y'_k + y'_{k+1}\right) \tag{0.24}$$

$$y'_{k+1} - y'_k = \frac{h}{2}\left(y_k + y_{k+1}\right) \tag{0.25}$$

Solve for y_{k+1} and y'_{k+1}:

$$y_{k+1} = \frac{(y_k)(4 + h^2) + 4hy'_k}{4 - h^2} \tag{0.26}$$

$$y'_{k+1} = \frac{(y'_k)(4 + h^2) + 4hy_k}{4 - h^2} \tag{0.27}$$

Use these difference equations to compute the values of y and y' at each step and plot the results. The figure below compares the theoretical and computational solutions. Comparison between the theoretical solution and computational solutions.