9.7.12

EE24BTECH11052 - Rongali Charan

Question: Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1 \quad (x \neq 0)$

1) **Theoretical Solution:**
   Rearranging the equation to standard form:

$$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}} - y}{\sqrt{x}} \tag{1.1}$$

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \tag{1.2}$$

$$\text{This is in the form: } \frac{dy}{dx} + P(x)y = Q(x) \tag{1.3}$$

$$\text{where } P(x) = \frac{1}{\sqrt{x}} \text{ and } Q(x) = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \tag{1.4}$$

   The integrating factor is:

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{dx}{\sqrt{x}}} \tag{1.5}$$

$$= e^{2\sqrt{x}} \tag{1.6}$$

   Multiplying both sides by $\mu(x)$:

$$e^{2\sqrt{x}}\frac{dy}{dx} + \frac{e^{2\sqrt{x}}}{\sqrt{x}}y = \frac{e^{2\sqrt{x}}e^{-2\sqrt{x}}}{\sqrt{x}} \tag{1.7}$$

$$\frac{d}{dx}(e^{2\sqrt{x}}y) = \frac{1}{\sqrt{x}} \tag{1.8}$$

$$e^{2\sqrt{x}}y = 2\sqrt{x} + C \tag{1.9}$$

$$x_0 = 1, y_0 = 0 \implies C = -2 \tag{1.10}$$

$$\therefore y = e^{-2\sqrt{x}}(2\sqrt{x} - 2) \tag{1.11}$$

2) **Using method of finite differences:**
   The Method of Finite Differences approximates the solution using discrete steps.

We know that:

$$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}} - y}{\sqrt{x}} \tag{2.1}$$

$$\lim_{h\to 0} \frac{y_{n+1} - y_n}{h} = \frac{e^{-2\sqrt{x_n}} - y_n}{\sqrt{x_n}} \tag{2.2}$$

$$\approx \frac{y_{n+1} - y_n}{h} = \frac{e^{-2\sqrt{x_n}} - y_n}{\sqrt{x_n}} \tag{2.3}$$

$$\therefore y_{n+1} = y_n + h \cdot \frac{e^{-2\sqrt{x_n}} - y_n}{\sqrt{x_n}} \tag{2.4}$$

The following steps were used:

a) Initialize $x_0 = 1$ and $y_0 = 0$

b) Choose step size $h = 0.01$ and number of steps $n = 1000$ to ensure accuracy

c) Generate points iteratively using:

$$x_{n+1} = x_n + h \tag{2.5}$$

$$y_{n+1} = y_n + h \cdot \frac{e^{-2\sqrt{x_n}} - y_n}{\sqrt{x_n}} \tag{2.6}$$

The below graph shows the comparison between the curve that is obtained theoretically and the simulation curve(numerically generated points through iterations).