10.3.2.4.1

EE24BTECH11012 - Bhavanisankar G S

QUESTION :
Determine if the system of equations

$$x + y = 5 \tag{0.1}$$
$$2x + 2y = 10 \tag{0.2}$$

is consistent or not. If consistent, obtain the solution graphically.
SOLUTION :
A linear equation is said to be **consistent** if it has atleast one solution.
A linear equation is said to be **inconsistent** if it has no solution.
Consider the given equations,

$$x + y = 5 \tag{0.3}$$
$$2x + 2y = 10 \tag{0.4}$$

The equations can be written in the matrix form as,

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} \tag{0.5}$$

This question can be attempted to solve using **LU decomposition** . The matrix **A** can be decomposed as

$$\mathbf{A} = \mathbf{LU} \tag{0.6}$$

where,

$$\mathbf{L} = Lowertriangular \tag{0.7}$$
$$\mathbf{U} = Uppertriangular \tag{0.8}$$

Then the system of equations can be solved as

$$\mathbf{Ax} = \mathbf{B} \tag{0.9}$$
$$\mathbf{LUx} = \mathbf{B} \tag{0.10}$$
$$\implies \mathbf{Ly} = \mathbf{B} \tag{0.11}$$
$$\mathbf{Ux} = \mathbf{y} \tag{0.12}$$

Algorithm :

1) Let **A** be an $n \times n$ matrix. Initialize **L** to an $n \times n$ Identity matrix. Initialize **U** to a

zero matrix.

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{1.1}$$

$$U = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{1.2}$$

2) For each row i from 0 to $n-1$:
   a) For each column j from i to $n-1$:

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \tag{2.1}$$

   b) For each row j from $i+1$ to $n-1$:

$$L_{ji} = \frac{1}{U_{ii}} \left(A_{ij} - \sum_{k=0}^{i-1} L_{jk} U_{ki} \right) \tag{2.2}$$

3) Repeat the above step for all $i = 0, 1, \ldots, n-1$
4) After all the iterations

$$\mathbf{A} = \mathbf{LU} \tag{4.1}$$

It can be seen that

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \tag{4.2}$$

$$\mathbf{U} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{4.3}$$

The matrix **U** can also be calculated by **Row-reduction**.

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{4.4}$$

$$\implies \mathbf{U} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{4.5}$$

The matrix **L** is such that the diagonal elements are 1, and is lower-triangular.

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \tag{4.6}$$

$$\implies \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} \tag{4.7}$$

It can be seen that

$$a = 2 \tag{4.8}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \tag{4.9}$$

Using (0.11), we have

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} \tag{4.10}$$

By the method of **Forward substitution**, we have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{4.11}$$

Similarly,

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{4.12}$$

It can be seen that

$$x_1 + x_2 = 5 \tag{4.13}$$

$$0 = 0 \tag{4.14}$$

indicating that the given system of equations has infinite number of solutions. It can also be seen that (4.13) and (0.1) are the same.
Hence, the given system of equations is **consistent**.
Thus, any point on the line (0.1) is a solution to the given system of equations.
Another method :
The ratio of the coefficients can be seen as

$$\frac{1}{2} = \frac{1}{2} = \frac{5}{10} \tag{4.15}$$

Hence, the given set of equations has infinitely many solutions.
QR decomposition :
This question can also be solved using the **QR decomposition**. Consider

$$\mathbf{A} = \mathbf{QR} \tag{4.16}$$

$$\mathbf{Q} - Orthonormal \tag{4.17}$$

$$\mathbf{R} - Uppertriangular \tag{4.18}$$

The equation (0.9) can be simplified to

$$\mathbf{QRx} = \mathbf{B} \tag{4.19}$$

$$\mathbf{Rx} = \mathbf{Q}^T\mathbf{B} \tag{4.20}$$

Using the **Gram-Schmidt orthogonalisation**, we have

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \tag{4.21}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ 0 & 0 \end{pmatrix} \tag{4.22}$$

$$\implies \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} \tag{4.23}$$

which again simplifies to (4.12) indicating that the given system is consistent with infinite number of solutions.

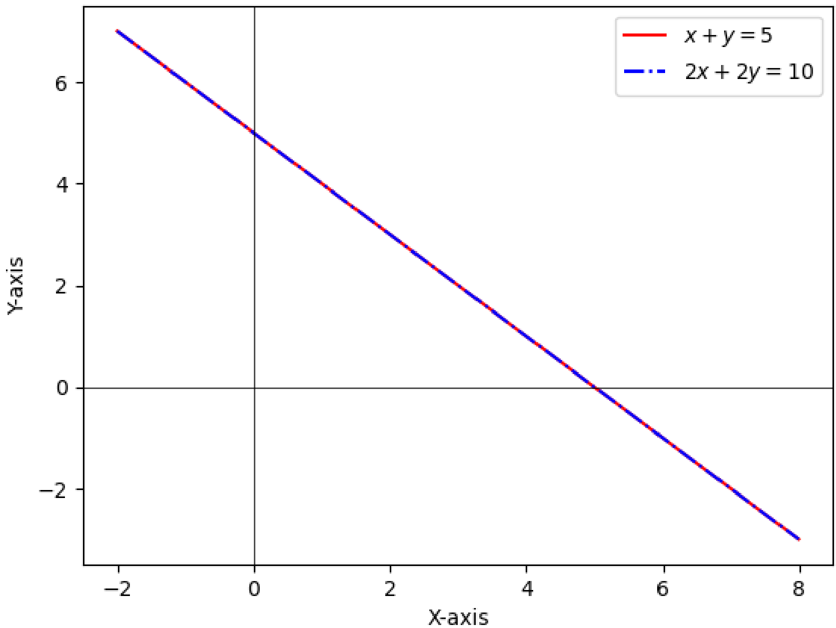

Fig. 4.1: Plot of the given question.