Solving differential equation NCERT-12.8.3.2

EE24BTECH11056 - S.Kavya Anvitha

Question:
Find the area between the curves $y = x$ and $y = x^2$.
Exact Integral Solution:
The point of intersection of the line with the parabola is $x_i = h + k_i m$,
where,k_i is a constant and is calculated as follows:-

$$k_i = \frac{1}{m^\top V m}\left(-m^\top (Vh + u) \pm \sqrt{\left[m^\top (Vh + u)\right]^2 - g(h)\left(m^\top V m\right)}\right)$$

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

$$f = 0 \tag{2}$$

$$u = -\begin{pmatrix} 0 \\ 2a \end{pmatrix} \text{here } \ 4a = 1 \text{so} 2a = \frac{1}{2} \tag{3}$$

$$u = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} m = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4}$$

Substituting the input parameters in k_i,
we get: $x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ The area between the curves can be found using the definite integral:

$$A = \int_0^1 (x - x^2)\, dx \tag{5}$$

Calculating the integral term by term:

$$A = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \tag{6}$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \approx 0.1667 \tag{7}$$

By using trapezoidal rule:
The area between the curves $y = x$ and $y = x^2$ can be calculated using the trapezoidal

rule by integrating the difference between the curves. The area can be expressed as:

$$A = \int_a^b f(x)\, dx \tag{8}$$

where the curves intersect at $x = 0$ and $x = 1$

Trapezoidal rule formula:

The trapezoidal rule approximates the integral using the formula:

$$A \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(n)\right] \tag{9}$$

where:

1) $h = \frac{b-a}{n}$ is the width of each subinterval.
2) $f(x) = x - x^2$.
3) $a = 0, b = 1$.
4) n is the number of subintervals.

Taking trapezoid shaped strips of small area and adding them all up.. Say we have to find the area of $y(x)$ from $x = x_0$ to $x = x_n$,discretize points on the x axis $x_0, x_1, x_2, ..., x_n$ such that they are equally spaced with step-size h.Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2}h\left(y(x_1) + y(x_0)\right) + \frac{1}{2}h\left(y(x_2) + y(x_1)\right) + \frac{1}{2}h\left(y(x_3) + y(x_2)\right) + \cdots + \frac{1}{2}h\left(y(x_n) + y(x_{n-1})\right) \tag{10}$$

$$= h\left[\frac{1}{2}\left(y(x_0) + y(x_n)\right) + y(x_1) + y(x_2) + \cdots + y(x_{n-1})\right] \tag{11}$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$

$$A(x_n + h) = A(x_n) + \frac{1}{2}h\left[y(x_n + h) + y(x_n)\right] \tag{12}$$

we can repeat this till we get a required area

$$A_{n+1} = A_n + \frac{1}{2}h\left[y_{n+1} + y_n\right] \tag{13}$$

We can write y_{n+1} in terms of y_n as $y_{n+1} = y_n + h \cdot y'_n$ Substituting in the equation we get:

$$A_{n+1} = A_n + \frac{1}{2}h\left[(y_n + h \cdot y'_n) + y_n\right] \tag{14}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{15}$$

$$x_{n+1} = x_n + h \tag{16}$$

In the given question $y_n = x_n - x_n^2$ and $y'_n = 1 - 2x_n$

General difference equation will be given by:

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y_n' \tag{17}$$

$$A_{n+1} = A_n + h\left(x_n - x_n^2\right) + \frac{1}{2}h^2\left(1 - 2x_n\right) \tag{18}$$

$$A_{n+1} = A_n - hx_n^2 + \left(h - h^2\right)x_n + \frac{h^2}{2} \tag{19}$$

$$x_{n+1} = x_n + h \tag{20}$$

Using ($n = 4$) subintervals as an example:

$$h = \frac{1-0}{4} = 0.25 \tag{21}$$

The points are:

$$x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75, \quad x_4 = 1 \tag{22}$$

Function values:

$$f(0) = 0, \quad f(0.25) = 0.1875, \quad f(0.5) = 0.25, \quad f(0.75) = 0.1875, \quad f(1) = 0 \tag{23}$$

Applying the trapezoidal rule formula:

$$A \approx \frac{0.25}{2}\left[0 + 2(0.1875 + 0.25 + 0.1875) + 0\right] \tag{24}$$

$$A \approx \frac{0.25}{2} \times 1.25 = 0.15625 \tag{25}$$

Comparison with Exact Area:

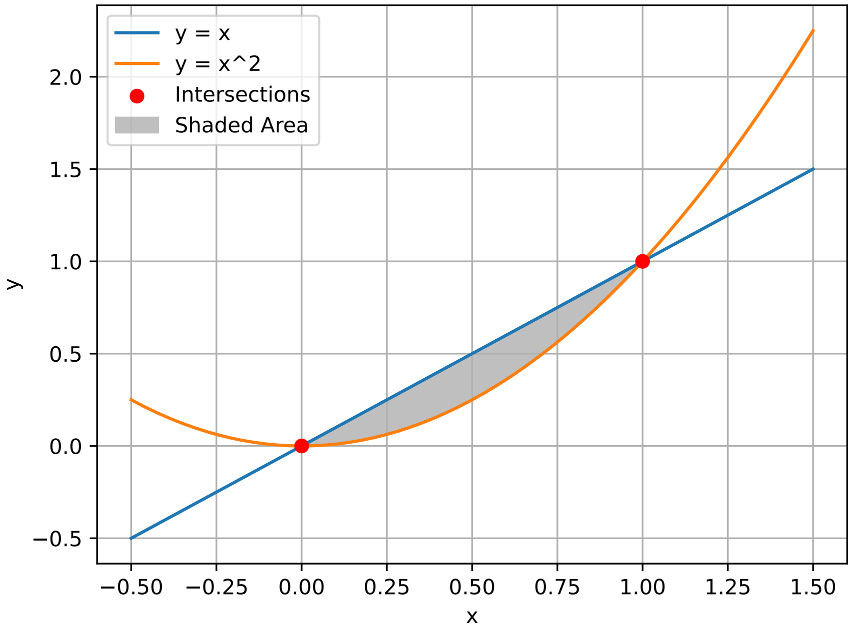
The exact area calculated earlier was:

$$A_{\text{exact}} = \frac{1}{6} \approx 0.1667 \tag{26}$$

The area calculated using the trapezoidal rule with ($n = 4$) is:

$$A_{\text{trapezoidal}} \approx 0.15625 \tag{27}$$

The approximation improves as the number of subintervals increases. Therefore, the trapezoidal rule provides a close estimate of the integral as the step size decreases.


y = x
y = x^2
Intersections
Shaded Area
2.0
1.5
1.0
0.5
0.0
−0.5
y
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
1.25
1.50
x