12.8.3.14

EE24BTECH11020 - Ellanti Rohith

Question: Using the method of integration find the area of region bounded by lines: $2x + y = 4, 3x - 2y = 6$ and $x - 3y + 5 = 0$.

Theoretical Solution:
By Integral:
Let the three lines be defined as:
1) Line $L_1 : 2x + y = 4$,
2) Line $L_2 : 3x - 2y = 6$,
3) Line $L_3 : x - 3y + 5 = 0$.

$$L_1 : 2x + y = 4 \textit{ and } L_2 : 3x - 2y = 6$$

$$\begin{pmatrix} 2 & 1 & 4 \\ 3 & -2 & 6 \end{pmatrix} \xrightarrow{R_2 \to R_2 - \frac{3}{2}R_1} \begin{pmatrix} 2 & 1 & 4 \\ 0 & -\frac{7}{2} & 0 \end{pmatrix} \xrightarrow{R_1 \div 2, R_2 \div -\frac{7}{2}} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 - \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

Solution: $x = 2$, $y = 0$.

$$L_2 : 3x - 2y = 6 \textit{ and } L_3 : x - 3y = -5$$

$$\begin{pmatrix} 3 & -2 & 6 \\ 1 & -3 & -5 \end{pmatrix} \xrightarrow{R_2 \to R_2 - \frac{1}{3}R_1} \begin{pmatrix} 3 & -2 & 6 \\ 0 & -\frac{7}{3} & -7 \end{pmatrix} \xrightarrow{R_1 \div 3, R_2 \div -\frac{7}{3}} \begin{pmatrix} 1 & -\frac{2}{3} & 2 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{R_1 \to R_1 + \frac{2}{3}R_2} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}$$

Solution: $x = 4$, $y = 3$.

$$L_1 : 2x + y = 4 \textit{ and } L_3 : x - 3y = -5$$

$$\begin{pmatrix} 2 & 1 & 4 \\ 1 & -3 & -5 \end{pmatrix} \xrightarrow{R_2 \to R_2 - \frac{1}{2}R_1} \begin{pmatrix} 2 & 1 & 4 \\ 0 & -\frac{7}{2} & -7 \end{pmatrix} \xrightarrow{R_1 \div 2, R_2 \div -\frac{7}{2}} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_1 \to R_1 - \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Solution: $x = 1$, $y = 2$. The required area is the sum of:
1) The area bounded by L_1 and L_3 over the interval $x = 1$ to $x = 2$, and
2) The area bounded by L_2 and L_3 over the interval $x = 2$ to $x = 4$.

The required area can be calculated as the integral of the difference of the functions:

$$f(x) = \begin{cases} \frac{7}{3}x - \frac{7}{3} & \text{for } 1 \le x \le 2, \\ -\frac{3x}{2} + \frac{14}{3}, & \text{for } 2 < x \le 4. \end{cases} \tag{2.1}$$

The area is given by:

$$\text{Area} = \int_1^2 \left(\frac{7}{3}x - \frac{7}{3}\right) dx + \int_2^4 \left(-\frac{3x}{2} + \frac{14}{3}\right) dx \tag{2.2}$$

Thus, the total area is:

$$\text{Area} = \frac{7}{6} + \frac{1}{3} = \frac{9}{6} = \frac{3}{2}. \tag{2.3}$$

Area $= \frac{3}{2}$ square units.

Simulation:

Splitting the intervals with step size $h = 0.01$

Trapezoidal Rule:

Summing area of all Trapezoids to give area under a given curve $f(x)$

$$A \approx \frac{h}{2}\left((f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n))\right) \tag{2.4}$$

$$A \approx \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right) \tag{2.5}$$

$$A_{n+1} = A_n + \frac{h}{2}(y_n + y_{n+1}) \tag{2.6}$$

This is the required difference equation: (2.7)

$$x_{n+1} = x_n + h \tag{2.8}$$

$$y_{n+1} = y_n + hy'_n \tag{2.9}$$

$$A_{n+1} = A_n + \frac{h}{2}(2y_n + hy'_n) \tag{2.10}$$

$$y'_n = \begin{cases} \frac{7}{3} & \text{for } 1 \le x_n \le 2, \\ -\frac{3}{2}, & \text{for } 2 < x_n \le 4. \end{cases} \tag{2.11}$$

By simulation, the answer turns out to be $1.49933450000053 \approx \frac{3}{2}$.

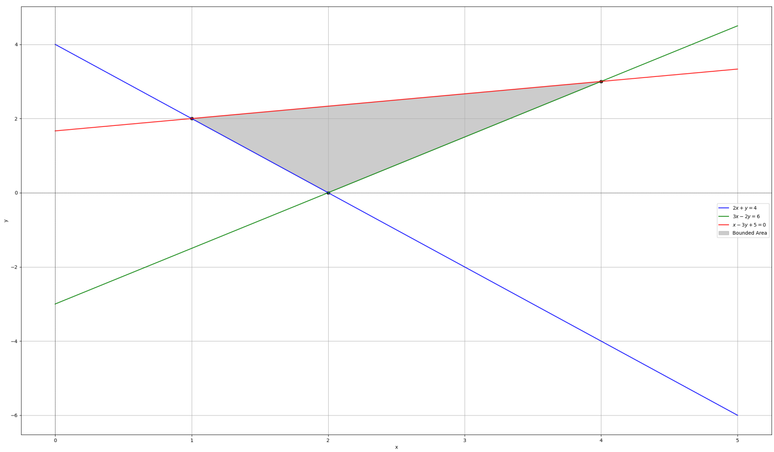

Fig. 2: Plot of three lines and the bounded area.