NCERT - 10.4.ex.14.1

EE1003

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Question: Find the roots of the equation : $x + \frac{1}{x} = 3,\ x \neq 0.$

Theoretical solution:

The equation we need to solve is :

$$x^2 - 3x + 1 = 0 \tag{1}$$

Using the quadratic formula, the roots will be $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ for the equation $ax^2+bx+c = 0$

Therefore, the roots are :

$$\frac{3 \pm \sqrt{(3)^2 - 4\,(1)}}{2} \tag{2}$$

$$\frac{3 + \sqrt{5}}{2} \approx 2.61803398875, \tag{3}$$

$$\frac{3 - \sqrt{5}}{2} \approx 0.38196601125 \tag{4}$$

Newton Raphson method

We can use the Newton Raphson method to find the roots of the equation $x^2 - 3x + 1 = 0$. The algorithm iterates as follows:

$$x_{n+1} = x_n - \frac{f\,(x_n)}{f'\,(x_n)} \tag{5}$$

We take an initial guess for the root as x_0 for first iteration and repeat iteration until $|f\,(x_n)|$ is very small.

$$x_{n+1} = x_n - \left(\frac{x_n^2 - 3x_n + 1}{2x_n - 3}\right) \tag{6}$$

I have taken $x_0 = 0$ and 3 for finding the two roots.

Using Companion matrix : For a polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, the companion matrix is of the form -

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \cdots & -\frac{a_{n-1}}{a_n} \end{pmatrix} \tag{7}$$

The eigen values of C are the required roots of the polynomial. We can apply the **QR-algorithm** to find the eigen values of the matrix.
We start with

$$C_0 = C \tag{8}$$

We can decompose it into a product

$$C_0 = Q_0 R_0 \tag{9}$$

Which are orthogonal and upper triangular matrices respectively.
Then

$$C_1 = R_0 Q_0 \tag{10}$$

We repeat this iteratively until C_k becomes nearly upper triangular, and the diagonal of it will be the required roots.
For this question,

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} \tag{11}$$

The eigen values obtained are plotted.
Plotting:
Taking

$$x_0 = 0, 3 \tag{12}$$

$$tolerance = 10^{-6} \tag{13}$$

$$n = 100 \tag{14}$$

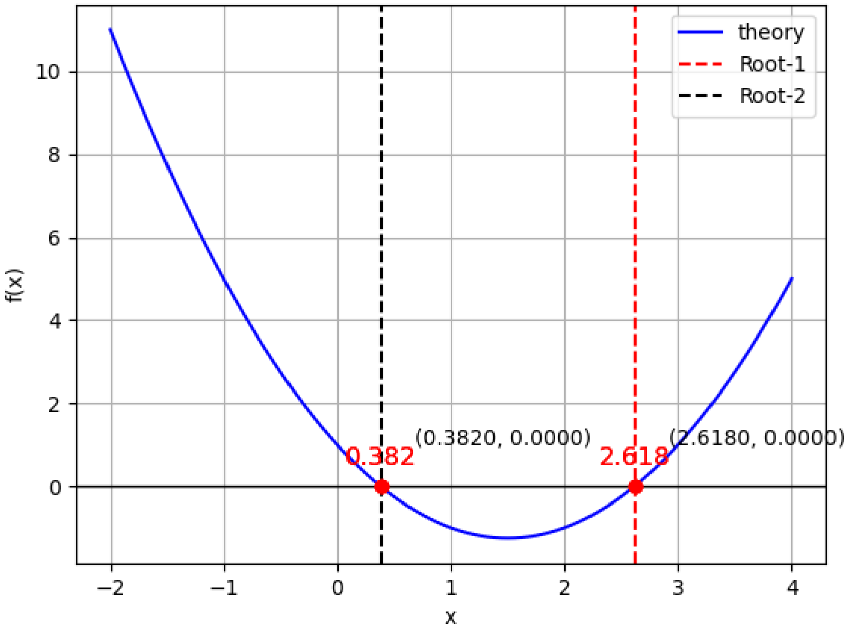

Fig. 0: Plot