10.3.6.1.8

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Question:
Solve the following pairs of equations by reducing them to a pair of linear equations:

$$\frac{1}{3x+y} + \frac{1}{3x-y} = \frac{3}{4}$$
$$\frac{1}{2(3x+y)} - \frac{1}{2(3x-y)} = \frac{-1}{8}$$

Solution:
Let's solve this using LU decomposition. First, let's substitute:

$$\frac{1}{3x+y} = u \tag{1}$$
$$\frac{1}{3y-y} = v \tag{2}$$

Then our equations become:

$$u + v = \frac{3}{4} \tag{3}$$
$$\frac{1}{2}u - \frac{1}{2}v = \frac{-1}{8} \tag{4}$$

$$4u + 4v = 3 \tag{5}$$
$$4u - 4v = -1 \tag{6}$$

This can be written in matrix form as:

$$\begin{pmatrix} 4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \tag{7}$$

Let A=$\begin{pmatrix} 4 & 4 \\ 4 & -4 \end{pmatrix}$
By multiplying A on both sides

$$\begin{pmatrix} 4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} 4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} \tag{8}$$

after solving

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} \tag{9}$$

Therefore:

$$\frac{1}{3x+y} = \frac{1}{4} \implies 3x+y=4 \tag{10}$$
$$\frac{1}{3x-y} = \frac{1}{2} \implies 3x-y=2 \tag{11}$$
$$\tag{12}$$

Any non-singular matrix can be represented as a product of a lower triangular matrix L and an upper triangular matrix U

$$\mathbf{Ax} = \mathbf{LUx} = \mathbf{b} \tag{13}$$

Factorization of LU:
Given a matrix $\mathbf{A}$ of size $n \times n$, LU decomposition is performed row by row and column by column. The update equations are as follows:

- Start by initializing $\mathbf{L}$ as the identity matrix $\mathbf{L} = \mathbf{I}$ and $\mathbf{U}$ as a copy of $\mathbf{A}$.
- For each column $j \geq k$, the entries of U in the k-th row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j} \quad \forall \quad j \geq k \tag{14}$$

- For each row $i > k$, the entries of L in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right) \quad \forall \quad i > k \tag{15}$$

By doing the following steps and solving we get :

$$\begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{16}$$

By doing the following factorization we get:

$$\mathbf{U} = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} \tag{17}$$
$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{18}$$

Using forward substitution:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{19}$$
$$\implies y_1 = 4 \tag{20}$$
$$\implies y_2 = -2 \tag{21}$$

Now using back substitution:

$$\begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{22}$$

This gives:

$$\implies y = 1 \tag{23}$$
$$3x + 1 = 4 \tag{24}$$
$$x = 1 \tag{25}$$

The solution is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{26}$$

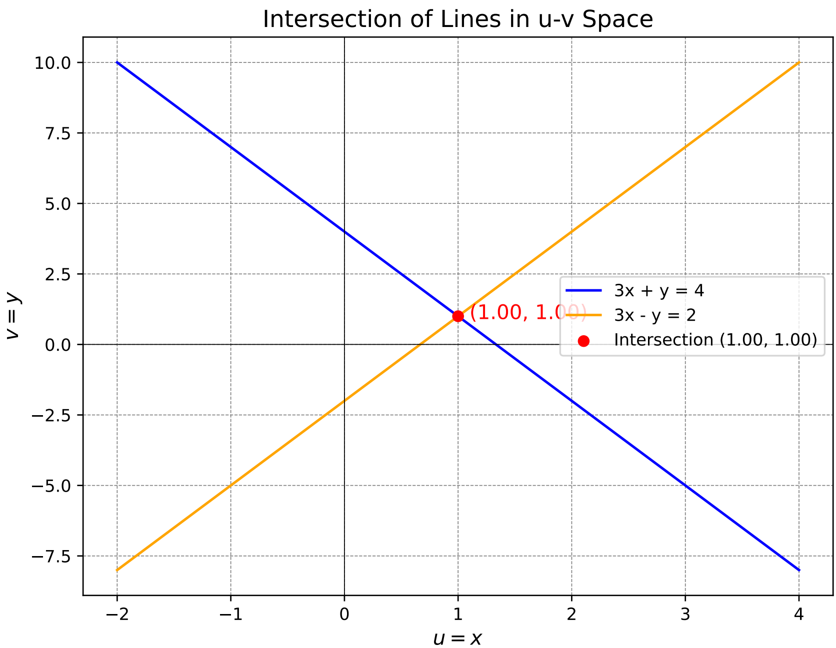

Fig. 1: Graph of the solution