12.8.3.9

EE24BTECH11011- Pranay Kumar

Question:
Find the area of the smaller region bounded by the ellipse $\frac{x^2}{25}+\frac{y^2}{9} = 1$ and the line $\frac{x}{5}+\frac{y}{3} = 1$

Theoretical Solution:
The point of intersection of the line with the ellipse is $x_i = h + k_i m$,
where,k_i is a constant and is calculated as follows:-

$$k_i = \frac{1}{m^\top V m}\left(-m^\top (Vh+u) \pm \sqrt{\left[m^\top (Vh+u)\right]^2 - g(h)\left(m^\top V m\right)}\right)$$

Substituting the input parameters in k_i,

$$k_i = \frac{1}{\begin{pmatrix}\frac{1}{b} & \frac{-1}{a}\end{pmatrix}\begin{pmatrix}b^2 & 0\\ 0 & a^2\end{pmatrix}\begin{pmatrix}\frac{1}{b}\\ \frac{-1}{a}\end{pmatrix}}\left(-\begin{pmatrix}\frac{1}{b} & \frac{-1}{a}\end{pmatrix}\left(\begin{pmatrix}b^2 & 0\\ 0 & a^2\end{pmatrix}\begin{pmatrix}a\\ 0\end{pmatrix}+\begin{pmatrix}0\\ 0\end{pmatrix}\right)\pm\right.$$
$$\left.\sqrt{\left[\begin{pmatrix}\frac{1}{b} & \frac{-1}{a}\end{pmatrix}\left(\begin{pmatrix}b^2 & 0\\ 0 & a^2\end{pmatrix}\begin{pmatrix}a\\ 0\end{pmatrix}+\begin{pmatrix}0\\ 0\end{pmatrix}\right)\right]^2 - g(h)\left(\begin{pmatrix}\frac{1}{b} & \frac{-1}{a}\end{pmatrix}\begin{pmatrix}b^2 & 0\\ 0 & a^2\end{pmatrix}\begin{pmatrix}\frac{1}{b}\\ \frac{-1}{a}\end{pmatrix}\right)}\right) \tag{0.1}$$

We get,

$$k_i = 0, -ab$$

Substituting k_i in $x_i = h + k_i m$ we get,

$$x_1 = \begin{pmatrix}a\\ 0\end{pmatrix} + (0)\begin{pmatrix}\frac{1}{b}\\ \frac{-1}{a}\end{pmatrix} \tag{0.2}$$
$$\implies x_1 = \begin{pmatrix}a\\ 0\end{pmatrix} \tag{0.3}$$
$$x_2 = \begin{pmatrix}a\\ 0\end{pmatrix} + (-ab)\begin{pmatrix}\frac{1}{b}\\ \frac{-1}{a}\end{pmatrix} \tag{0.4}$$
$$\implies x_2 = \begin{pmatrix}a\\ 0\end{pmatrix} + \begin{pmatrix}-a\\ b\end{pmatrix} \tag{0.5}$$
$$\implies x_2 = \begin{pmatrix}0\\ b\end{pmatrix} \tag{0.6}$$

The area of the smaller region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and the line $\frac{x}{5} + \frac{y}{3} = 1$

is

$$= \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx - \int_0^a \frac{b}{a}(a - x)\,dx \tag{0.7}$$

$$= \frac{b}{a}\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - ax + \frac{x^2}{2}\right)_0^a \tag{0.8}$$

$$= \frac{b}{a}\left(\frac{\pi a^2}{4} - \frac{a^2}{2}\right) = \frac{ab}{2}\left(\frac{\pi}{2} - 1\right) \tag{0.9}$$

The given area is $\frac{ab}{2}\left(\frac{\pi}{2} - 1\right)$ sq. units
$\therefore$ Upon substituting $a = 5, b = 3$ the given area is $5\left(\frac{\pi}{2} - 1\right)$sq. units ≈ 2.712 sq. units

Computational Solution:
Using the Trapezoidal rule which approximates the integral of a function $f(x)$ over an interval $[a, b]$ by dividing the interval into n subintervals and approximating the area under the curve as a series of trapezoids

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1}\left(f(x_i) + f(x_n)\right)\right] \tag{0.10}$$

Where x_0 is semi-major axis of ellipse and x_n is semi-minor axis of the ellipse and h is the width of each subinterval.

$$x_n = x_0 + n \cdot h \tag{0.11}$$

$$\implies h = \frac{x_n - x_0}{n} \tag{0.12}$$

In the case of our problem of the area between the line and ellipse the area is computed by:

$$A = \int_{x_0}^{x_n}\left(f_{ellipse}(x) - f_{line}(x)\right)dx \tag{0.13}$$

$$f_{ellipse}(x) = \sqrt{9\left(1 - \frac{x^2}{25}\right)} \tag{0.14}$$

$$f_{line}(x) = 3 - \frac{3x}{5} \tag{0.15}$$

Where $[x_0, x_n]$ are the intersection points. We need to find the area of y_x from x_0 to x_n. Taking trapezoids of small width h and discretizing points on the x axis $x_0, x_1, x_2, \ldots, x_n$. The sum of the trapezoidal areas will be

$$A = \frac{1}{2}h\left(y(x_1) + y(x_0)\right) + \frac{1}{2}h\left(y(x_2) + y(x_1)\right) + \cdots + \frac{1}{2}h\left(y(x_n) + y(x_{n-1})\right) \tag{0.16}$$

$$= h\left[\frac{1}{2}\left(y(x_0) + y(x_n)\right) + y(x_1) + \cdots + y(x_{n-1})\right] \tag{0.17}$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \ldots x_n)$

be equidistant points with step-size h.

$$A(x_n + h) = A(x_n) + \frac{1}{2}h\left(y(x_n + h) + y(x_n)\right) \tag{0.18}$$

We can repeat this till we get the required area.
Discretizing the steps, making $A(x_n) = A_n, y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h\left(y_{n+1} + y_n\right) \tag{0.19}$$

We can write y_{n+1} in terms of y_n using the first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left(y_{n+1} + y_n\right) \tag{0.20}$$

We can write y_{n+1} in terms of y_n using the first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left(\left(y_n + hy'_n\right) + y_n\right) \tag{0.21}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) \tag{0.22}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.23}$$

$$x_{n+1} = x_n + h \tag{0.24}$$

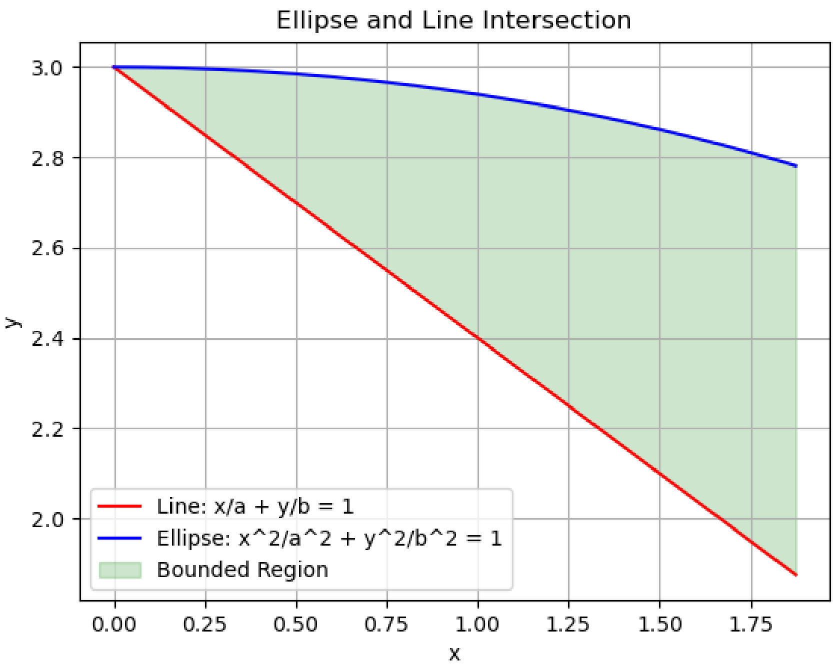
In the given question, $y_n = \sqrt{9\left(1 - \frac{x_n^2}{25}\right)} + \frac{3x_n}{5} - 3$ and $y'_n = \frac{3}{5}\left(1 - \frac{x_n}{\sqrt{25-x^2}}\right)$
General Difference Equation will be given by,

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y'_n \tag{0.25}$$

$$= A_n + h\left(\sqrt{9\left(1 - \frac{x_n^2}{25}\right)} + \frac{3x_n}{5} - 3\right) + \frac{1}{2}h^2\left(\frac{3}{5}\left(1 - \frac{x_n}{\sqrt{25 - x^2}}\right)\right) \tag{0.26}$$

$$x_{n+1} = x_n + h \tag{0.27}$$

Iterating till we reach $x_n = 5$ will return the required area.
Area obtained computationally: 2.7123332003665432 sq. units
Area obtained theoretically: $5\left(\frac{\pi}{2} - 1\right) = 2.71238898038$ sq. units.
As n tends to infinity A_n will be the exact area of the ellipse.


Ellipse and Line Intersection
3.0
2.8
2.6
2.4
2.2
2.0
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
x
Line: x/a + y/b = 1
Ellipse: x^2/a^2 + y^2/b^2 = 1
Bounded Region