10.4.4.4

EE24BTECH11029 - J SHRETHAN REDDY

Question: Is the following situation possible? If so, determine their present ages. The sum of the ages of two friends is 20 years. Four years ago, the product of their ages in years was 48.

Solution

Lets take present age of two friends as x, y

$$x + y = 20 \tag{0.1}$$

$$y = 20 - x \tag{0.2}$$

$$(x - 4)(y - 4) = 48 \tag{0.3}$$

equation (0.2) in (0.3)

$$(x - 4)(16 - x) = 48 \tag{0.4}$$

$$16x - x^2 - 64 + 4x = 48 \tag{0.5}$$

$$x^2 - 20x + 112 = 0 \tag{0.6}$$

we get roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{0.7}$$

$$= \frac{20 \pm \sqrt{400 - 448}}{2} \tag{0.8}$$

$$= 10 \pm 2\sqrt{3}j \tag{0.9}$$

Computational Solution

Eigenvalues of Companion Matrix:

The roots of a polynomial equation $x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0 = 0$ is given by finding eigenvalues of the companion matrix (C).

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{pmatrix} \tag{0.10}$$

Here $b_0 = 112$, $b_1 = -20$

$$C = \begin{pmatrix} 0 & 1 \\ -112 & 20 \end{pmatrix} \tag{0.11}$$

We find the eigenvalues using the QR algorithm. The basic principle behind this algorithm is a similarity transform,

$$A' = X^{-1}AX \tag{0.12}$$

which does not alter the eigenvalues of the matrix A.
We use this to get the Schur Decomposition,

$$A = Q^{-1}UQ = Q^{*}UQ \tag{0.13}$$

where Q is a unitary matrix $\left(Q^{-1} = Q^{*}\right)$ and U is an upper triangular matrix whose diagonal entries are the eigenvalues of A.
To efficiently get the Schur Decomposition, we first householder reflections to reduce it to an upper hessenberg form.
A householder reflector matrix is of the form,

$$P = I - 2\mathbf{u}\mathbf{u}^{*} \tag{0.14}$$

Householder reflectors transforms any vector $\mathbf{x}$ to a multiple of $\mathbf{e_1}$,

$$P\mathbf{x} = \mathbf{x} - 2\mathbf{u}\left(\mathbf{u}^{*}\mathbf{x}\right) = \alpha\mathbf{e_1} \tag{0.15}$$

P is unitary, which implies that,

$$\|P\mathbf{x}\| = \|\mathbf{x}\| \tag{0.16}$$

$$\implies \alpha = \rho\,\|\mathbf{x}\| \tag{0.17}$$

$$\tag{0.18}$$

As $\mathbf{u}$ is unit norm,

$$\mathbf{u} = \frac{\mathbf{x} - \rho\,\|\mathbf{x}\|\,\mathbf{e_1}}{\|\mathbf{x} - \rho\,\|\mathbf{x}\|\,\mathbf{e_1}\|} = \frac{1}{\|\mathbf{x} - \rho\,\|\mathbf{x}\|\,\mathbf{e_1}\|}\begin{pmatrix} x_1 - \rho\,\|\mathbf{x}\| \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{0.19}$$

Selection of ρ is flexible as long as $|\rho| = 1$. To ease out the process, we take $\rho = \frac{x_1}{|x_1|}$, $x_1 \neq 0$. If $x_1 = 0$, we take $\rho = 1$.
Householder reflector matrix (P_i) is given by,

$$P_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^{*} \\ \mathbf{0} & I_{n-i} - 2\mathbf{u_i}\mathbf{u_i}^{*} \end{bmatrix} \tag{0.20}$$

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{bmatrix} \xrightarrow{P_2} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \tag{0.21}$$

Next step is to do Given's rotation to get the QR Decomposition.
The Givens rotation matrix $G(i, j, c, s)$ is defined by

$$G(i, j, c, s) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\overline{s} & \cdots & \overline{c} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \tag{0.22}$$

where $|c|^2 + |s|^2 = 1$, and G is a unitary matrix.
Say we take a vector $\mathbf{x}$, and $\mathbf{y} = G(i, j, c, s)\,\mathbf{x}$, then

$$y_k = \begin{cases} cx_i + sx_j, & k = i \\ -\overline{s}x_i + \overline{c}x_j, & k = j \\ x_k, & k \neq i, j \end{cases} \tag{0.23}$$

For y_j to be zero, we set

$$c = \frac{\overline{x_i}}{\sqrt{|x_i|^2 + \left|x_j\right|^2}} = c_{ij} \tag{0.24}$$

$$s = \frac{\overline{x_j}}{\sqrt{|x_i|^2 + \left|x_j\right|^2}} = s_{ij} \tag{0.25}$$

Using this Givens rotation matrix, we zero out elements of subdiagonal in the hessenberg matrix H.

$$H = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(1,2,c_{12},s_{12})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$$

$$\xrightarrow{G(2,3,c_{23},s_{23})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(3,4,c_{34},s_{34})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$$

$$\xrightarrow{G(4,5,c_{45},s_{45})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix} = R \tag{0.26}$$

where R is upper triangular. For the given companion matrix,

Let $G_k = G(k, k+1, c_{k,k+1}, s_{k,k+1})$, then we deduce that

$$G_4 G_3 G_2 G_1 H = R \tag{0.27}$$

$$H = G_1^* G_2^* G_3^* G_4^* R \tag{0.28}$$

$$H = QR, \text{ where } Q = G_1^* G_2^* G_3^* G_4^* \tag{0.29}$$

Using this QR algorithm, we get the following update equation,

$$A_k = Q_k R_k \tag{0.30}$$

$$A_{k+1} = R_k Q_k \tag{0.31}$$

$$= (G_n \dots G_2 G_1) A_k (G_1^* G_2^* \dots G_n^*) \tag{0.32}$$

Running the eigenvalue code we get

$$x_1 = 0.0 + 3.464101615137755j \tag{0.33}$$

$$x_2 = 0.0 - 3.464101615137755j \tag{0.34}$$

Newton-Raphson iterative method:

$$f(x) = x^2 - 20x + 112 \tag{0.35}$$

$$f'(x) = 2x - 20 \tag{0.36}$$

Difference equation,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.37}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 20x_n + 112}{2x_n - 20} \tag{0.38}$$

$$x_{n+1} = \frac{x_n^2 - -112}{2x_n - 20} \tag{0.39}$$

Picking two initial guesses,

$$x_0 = 0 + i \text{ converges to } 0.0 + 3.464101615137755j \tag{0.40}$$

$$x_0 = 0 - i \text{ converges to } 0.0 - 3.464101615137755j \tag{0.41}$$

Acknowledgments:
I have took codes from [github:Dwarak A] available at [https://github.com/gadepall/sprog/tree/main/ncert/10/4/1/1/1/codes]