10.3.2.4.3

EE24BTECH11005 - Arjun Pavanje

Question: Solve the following system of equations,

$$2x + y - 6 = 0 \tag{1}$$

$$4x - 2y - 4 = 0 \tag{2}$$

Solution:

LU Decomposition

Representing using matrices,

$$\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \tag{3}$$

We shall solve this system of equations by LU Decomposition. Any non-sigular matrix can be represented as a product of a lower triangular matrix L and an upper triangular matrix U

$$A\mathbf{x} = LU\mathbf{x} = \mathbf{b} \tag{4}$$

Applying row reduction on A,

$$\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix} \tag{5}$$

Let

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \tag{6}$$

l_{21} is the multiplier used to zero a_{21}, so $l_{21} = 1$.

Now,

$$A = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix} \tag{7}$$

We have thus obtained LU Decomposition of the matrix A.
The LU Decomposition of matrix A can also be obtained by Doolittle's Algorithm. This gives us update equations to construct the L and U matrix.
Elements of U matrix:
For each column j,

$$U_{ij} = \begin{cases} A_{ij} & i = 0 \\ A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} & i > 0 \end{cases} \tag{8}$$

Elements of L matrix:
For each row i,

$$L_{ij} = \begin{cases} \frac{A_{ij}}{U_{ij}} & j = 0 \\ \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{ij}} & j > 0 \end{cases} \tag{9}$$

The above proccess decomposes any non-sigular matrix A into an upper-triangular matrix U and a lower-triangular matrix L. Now, let

$$U\mathbf{x} = \mathbf{y} \tag{10}$$

$$L\mathbf{y} = \mathbf{b} \tag{11}$$

Substituting L in equation (9),

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix} \tag{13}$$

Backsubstituting into equation (8),

$$\begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix} \tag{14}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{15}$$

Thus, the system of equations is solved at $\mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

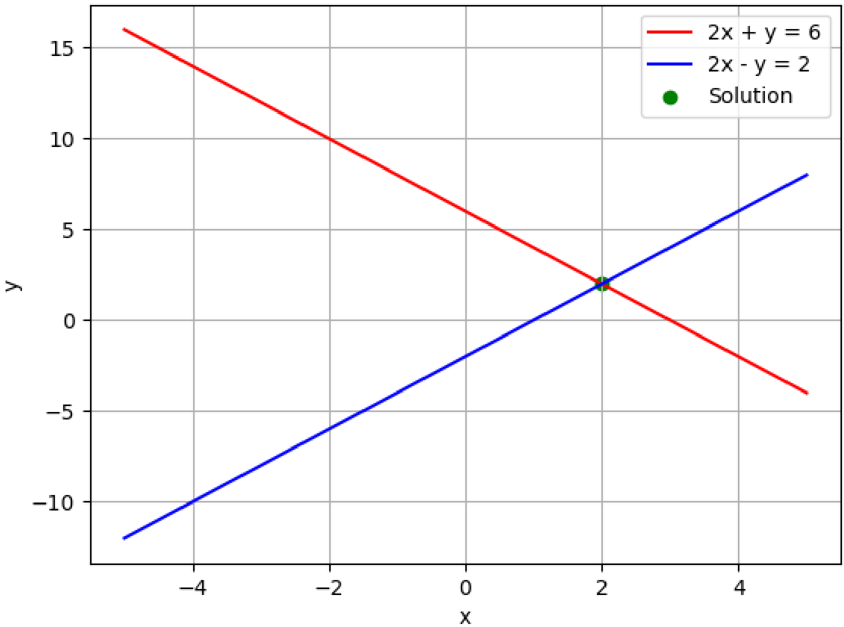

Fig. 1: Solving the system of equations, $2x + y = 6, 2x - y = 2$