Indian Institute of Technology, Gandhinagar

ME 605: Computational Fluid Dynamics

Project 2

Computational Solution of Viscous Flow over a Flat Plate

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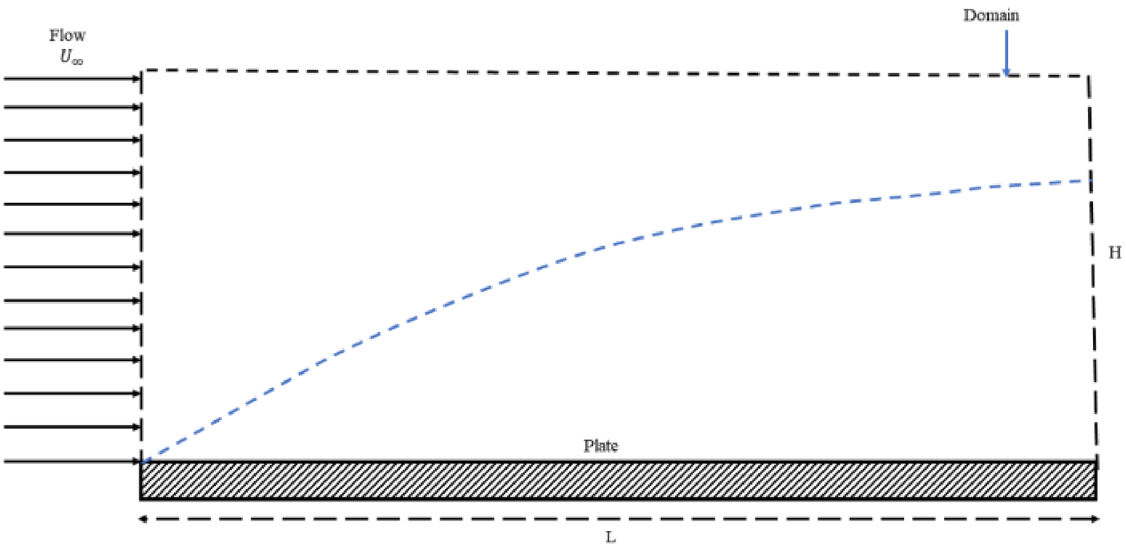
1. Problem Statement

Consider the flow of a viscous fluid over a flat plate of length L, as shown below. The fluid enters the simulation domain of height H at a uniform velocity, U_∞. Assume that the flow is steady, laminar, and incompressible. Further, assume that the Reynolds number is large enough such that the boundary layer approximation is valid.



The resulting governing equations for this specific problem are given below:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \vartheta\frac{\partial^2 u}{\partial y^2}$$

where ν is the kinematic viscosity of the fluid. You are required to write a computer program to solve the above boundary layer equations for this problem for a Reynolds number $Re_L = 10^4$, where:

$$Re_L = \frac{U_\infty L}{\vartheta}$$

The plate length (L), kinematic viscosity (ν), and free-stream velocity (U_∞) should be chosen such that ReL = 10^4. Use the finite difference method for discretization of the PDEs and solve the PDEs computationally using the following schemes:

(1) Euler Explicit scheme. You will have to choose Δx and Δy carefully, as the explicit scheme is not always stable.

(2) Euler Implicit scheme.

(3) Crank-Nicolson scheme.

For each of the above three schemes, you are required to:

1. Compute the x-velocity (u) and y-velocity (v) fields. fields. Show velocity fields as contour plots. Further, plot the normalized x-velocity (F′) as a function of similarity variable (η) as a line plot and compare with the Blasius solution. The variables are defined below:

$$F'(\eta) = \frac{u}{U_\infty} \quad \text{and} \quad \eta = y\sqrt{\frac{U_\infty}{\nu x}}$$

Note that you can vary the similarity variable (η) by varying y coordinate at a specific x-location or by varying x coordinate at a specific y location. You may pick one of the two

options to show the comparison between your simulation predictions with the Blasius solution.

2. Compute the boundary layer thickness. Plot the variation of boundary layer thickness with x coordinate. How does your simulation result compare with the Blasius flat plate boundary layer solution given below:

$$\frac{\delta}{x} = \frac{4.91}{\sqrt{Re_x}}$$

1. DISCRETIZING THE EQUATION

Consider the equation that is given below. The mesh that is given below is an approximate representation of the mesh that has been used in solving the 2D equations that governs the given fluid flow.

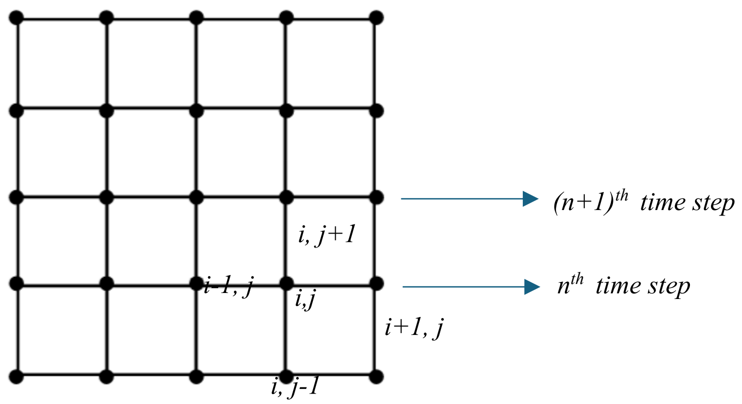


Figure 1: Sample Grid

The governing equations in this fluid flow problem are namely:

1. **Continuity Equation**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Understanding the physics of the given flow is an important aspect of discretizing the given governing equations. Hence, the continuity equation can be discretized in the following manner:

It is approximated as:

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} + \left(\frac{\partial v}{\partial y}\right)_{i,j} = 0$$

Applying forward differencing in the x- direction and central differencing in the y-direction, we get:

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} = 0$$

2. X-Momentum Equation

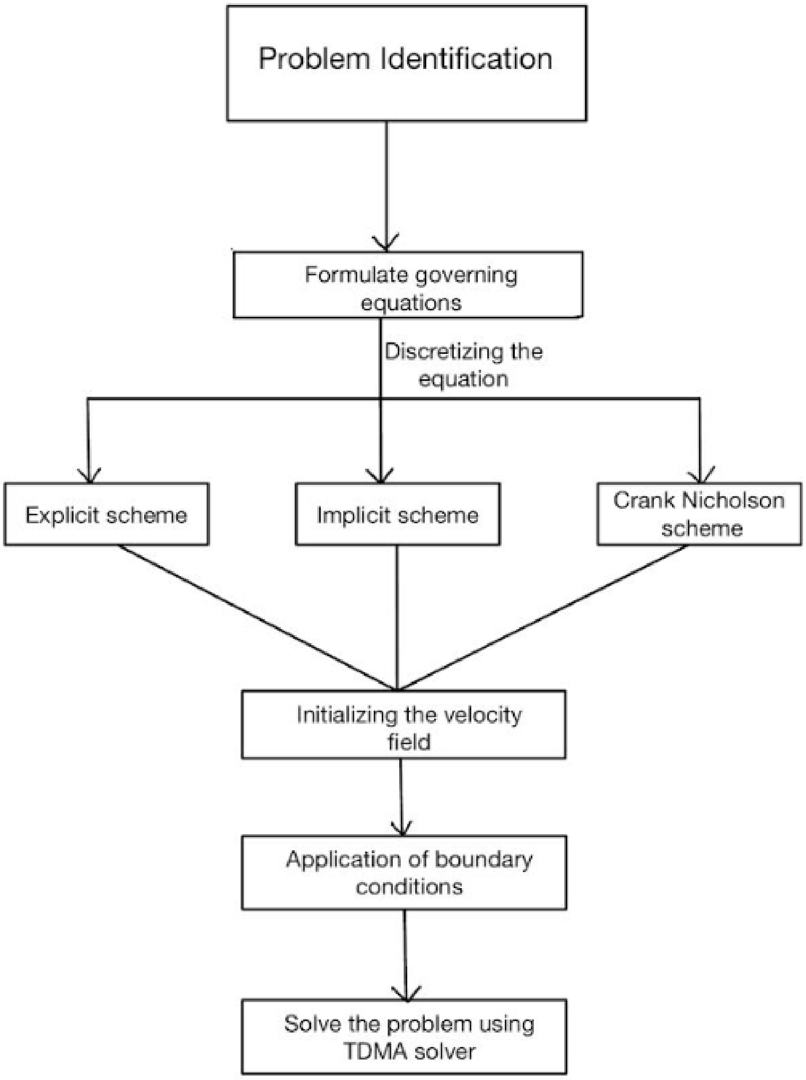
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \vartheta\frac{\partial^2 u}{\partial y^2}$$

The X-Momentum Equation is approximated as:

$$u_{i,j}\frac{u_{i+1,j} - u_{i,j}}{\Delta x} + v_{i,j}\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} = \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{\Delta y^2}$$

1.1. Algorithm

This is the algorithm that is used to implement the three schemes is as follows:



2. EXPLICIT SCHEME

The explicit scheme is the simplest method of solving discretized equations for a given flow, in which all the fluxes are evaluated using known values at the n^{th} time. Using these values obtained at n^{th} time, the solution at each point is calculated.

One of the key aspects of this scheme applied in this problem is that since the given problem is steady, there is no dependency on time. It marches in the x-direction, updating the velocity profile in the y-direction as it moves.

2.1. Methodology

The continuity and the *x*- momentum equation are as follows respectively:

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} = 0$$

$$u_{i,j}\frac{u_{i+1,j} - u_{i,j}}{\Delta x} + v_{i,j}\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} = \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{\Delta y^2}$$

Here, the convective terms are discretized using forward difference schemes and the diffusive terms are discretized using backward differencing schemes. The choice for the use of different types of differencing schemes is influenced by the physics of the given problem. In this, the convective and the diffusive terms are discretized using spatial finite differences, while the solution marches forward in time.

In the case of convection problems, the information flows in the downstream direction. In that case, it is ideal to use forward differencing schemes, as it respects the direction of the information flow. Central differencing schemes allows the information to flow through both directions. As a result, it can lead to numerical instabilities.

As for time-marching, the solution in explicit scheme marches forward in time. At each time step, the x-component of velocity, *u,* is computed at each nodal point using known values from the previous time step.

The Courant-Friedrichs-Lewy (CFL) condition ensures the stability of the numerical scheme. It is given by: -

$$\Delta t \le \frac{(\Delta y)^2}{2\vartheta}$$

The CFL condition ensures that the information flows in the right sense over the entire flow domain without any numerical instabilities.

In the code, the explicit scheme is written in two loops- the outer loop loops over the *x*-direction from 1 to *Nx,* which is the number of grids along the x-direction, while the inner loop loops over the *y*-direction from 1 to *Ny*-1, leaving the boundaries. The velocities at the boundaries are specified using the boundary conditions.

The *u_last* variable is used to store the last known values of the *x*-component of velocity, *u,* and to use it in successive iterations.

2.2. Results and Discussion

The contour plots obtained after solving the equations by the explicit scheme is given below: -

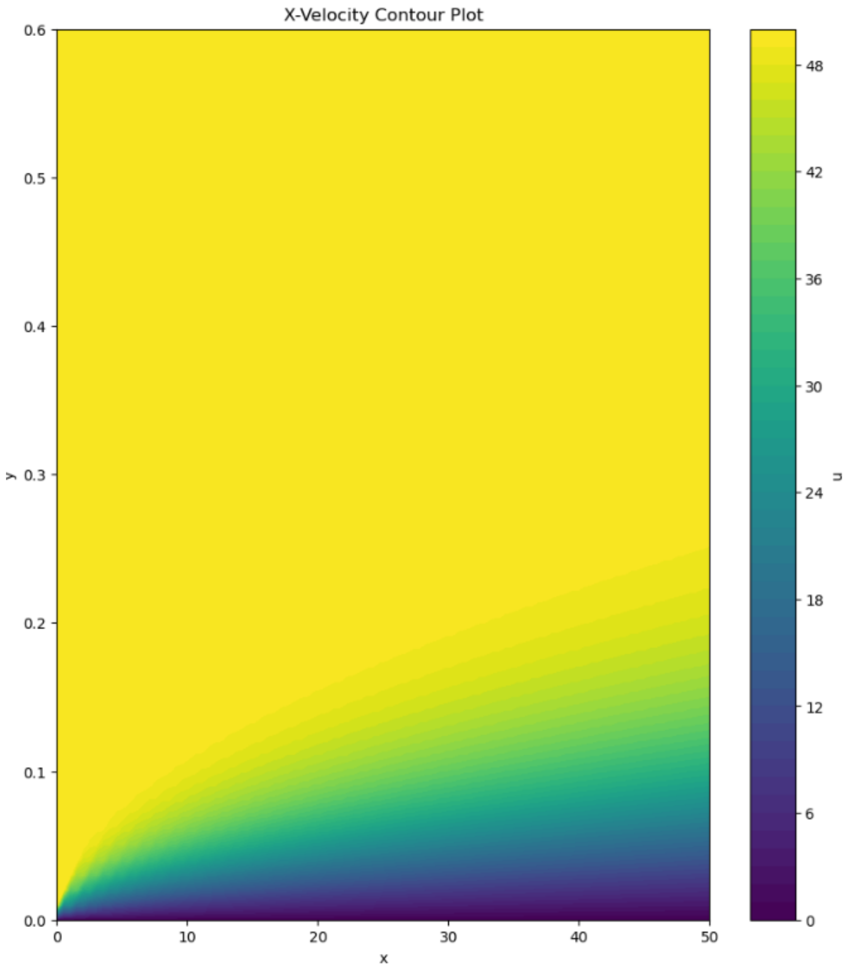


Figure 2: X-Velocity contour plot

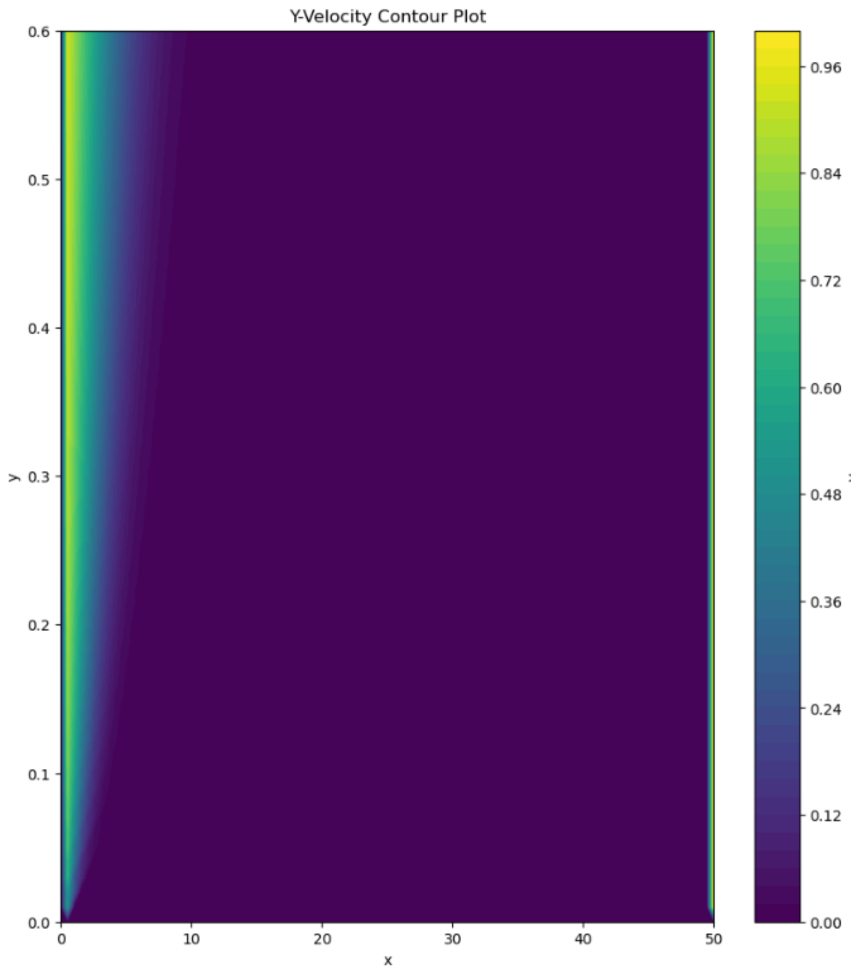


Figure 3: Contour Plot of y- velocity component

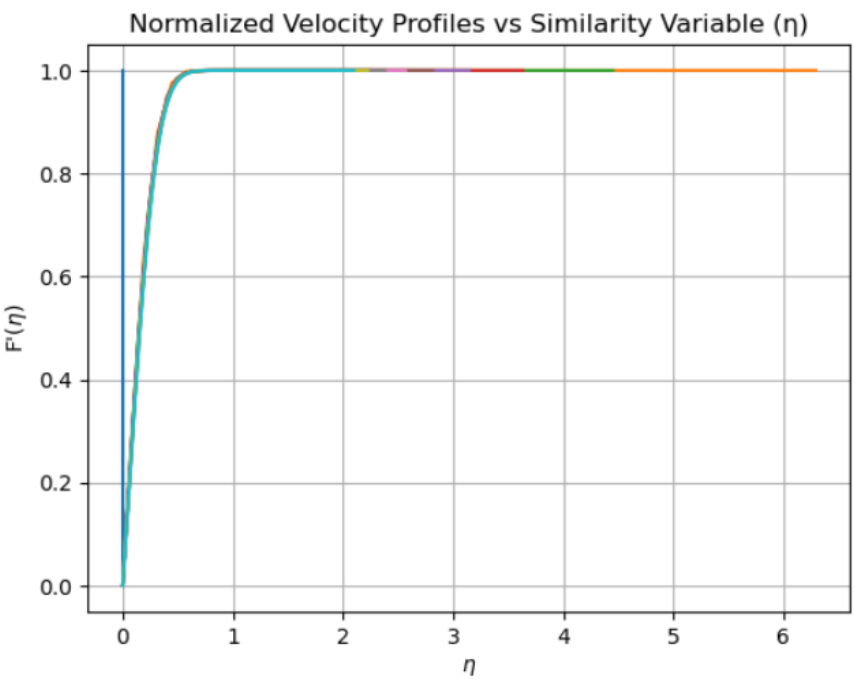


Figure 4: Normalized Velocity Profile with similarity variable

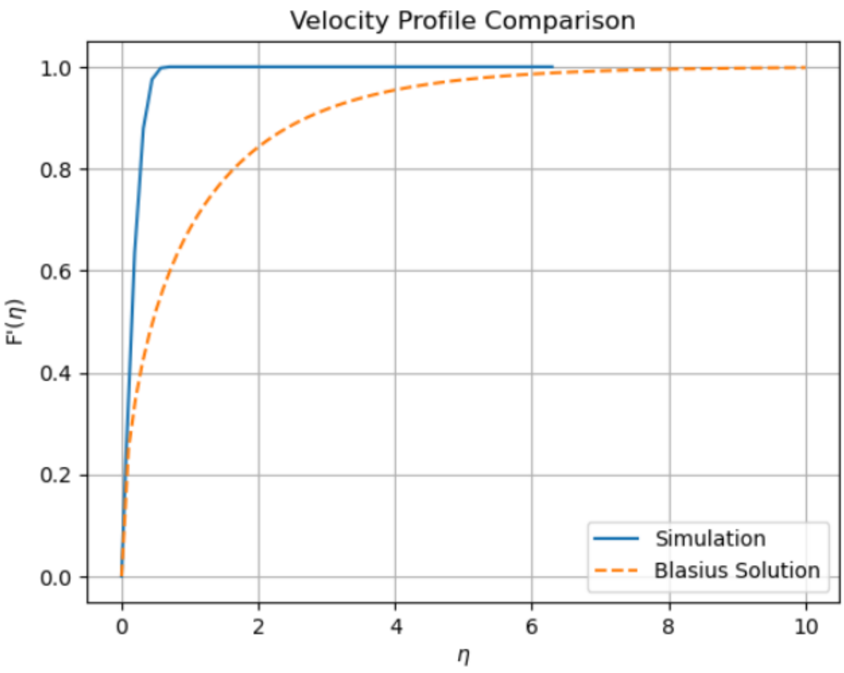


Figure 5: Velocity Profile Comparison

Figure 2 and 3 illustrates the x and y velocity components, u and v. It shows that near the bottom wall, i.e. at y=0, there is a distinct gradient in the velocity field, where the velocity is zero at the wall and increases as it moves away from the wall. This gradient in the x-direction diminishes as one moves along the x-direction, indicating the boundary layer development. The yellow region in the plot depicts the area where the velocity is the highest, where it is almost constant and uniform. This essentially shows the area that is unaffected by the wall.

In Figure 4, the plot represents the velocity at different points in the boundary layer plotted as a function of the similarity variable η. This depicts that there is a sudden increase in the velocity at $\eta = 0$. After $\eta \approx 2$, the curve stabilizes at $F'(\eta) = 1$, indicating the boundary layer has been developed fully.

In Figure 5, the plot shows a comparison between the simulation results and the analytical Blasius solution. Initially, at around $\eta = 0$, the simulation closely follows the the analytical Blasius solution, but as the simulations goes ahead, it is seen that there is a deviation from the Blasius solution. This may be attributed to the numerical instability associated with the Explicit Scheme or the number of grids along each direction.

Final Thoughts: *The Explicit Scheme is a numerical method that calculates values at a nodal point at a given time step, using the values obtained from the previous time steps. But the numerical instability associated with this scheme is often a cause to look for other methods which are numerically stable and can give even more accurate solutions, as can be inferred from the deviation of the simulation results from the Blasius solution.*

3. IMPLICIT EULER METHOD

Implicit Euler Method is commonly used in those problems, where accuracy is of prime importance for the analysis of the ordinary differential equations. In this method, usually all the flux and source terms in evaluated in terms of the unknown variables at the new time step.

3.1. Methodology

The continuity and the *x*-momentum equation is discretized as follows:

$$\frac{{u_{i,j}}^{n+1} - {u_{i-1,j}}^{n+1}}{\Delta x} + \frac{v_{i,j+1} - v_{i,j}}{\Delta y} = 0$$

$${u_{i,j}}^{n+1}\frac{{u_{i,j}}^{n+1} - {u_{i-1,j}}^{n+1}}{\Delta x} + {v_{i,j}}^{n+1}\frac{{u_{i,j}}^{n+1} - {u_{i,j-1}}^{n+1}}{2\Delta y} = \vartheta\,\frac{{u_{i,j+1}}^{n+1} + {u_{i,j-1}}^{n+1} - 2{u_{i,j}}^{n+1}}{\Delta y^2}$$

The method is used to discretize the continuity equation and the *x*-momentum equations, which are then represented as a system of equations in the form of a tridiagonal matrix. The *tdma* function takes in three arrays storing the values of the three diagonals of the tridiagonal matrix. The function has been written in such a way that it can handle error easily. It consists of forward elimination and back substitution.

The tridiagonal system formed for the velocity component u *(x,y)* is used to solve the resulting system at all the nodal points iteratively in the *x*-direction. The velocity component v(x,y) is then updated using the continuity equation. Convergence of the solution is checked at each point by providing a tolerance value of 10^{-6}.

The use of this method helps in taking large arbitrary time steps, which helps in studying transient or steady flows. But oscillatory solutions may arise when the central differencing scheme is applied on the solution domain where the grids are coarse. This can be attributed to the large Peclet number that is present in areas where there is large changes of the flow parameters such as velocity.

3.2. Results and Discussions

The obtained velocity contour plots along the *x* and *y* directions are as follows:

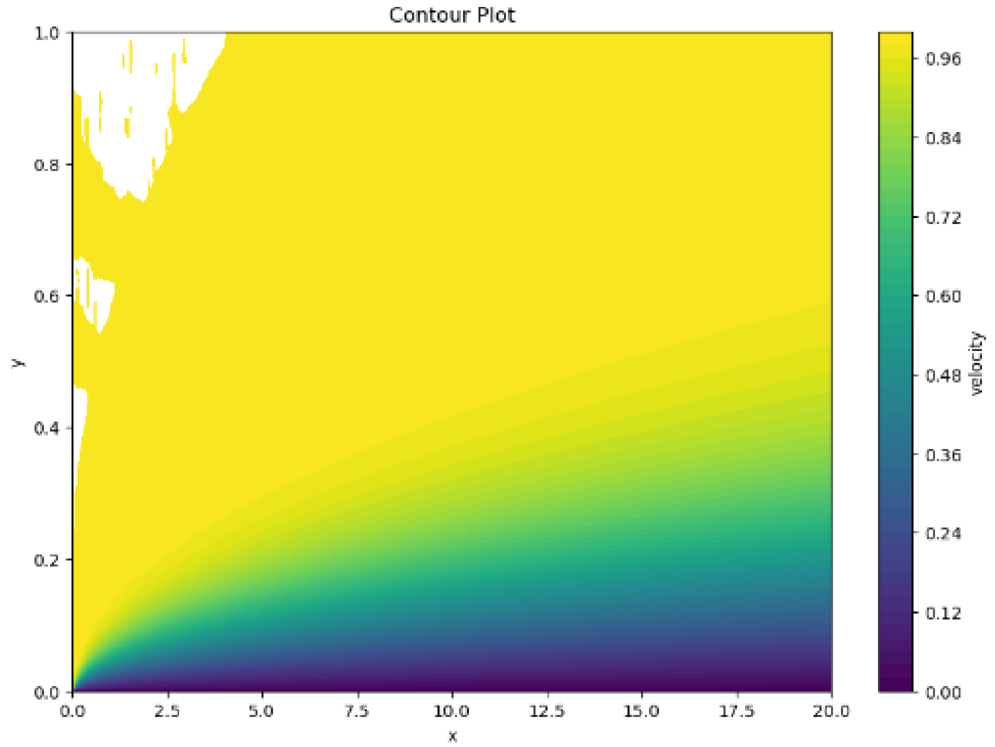


Figure 6: x-velocity contour plot

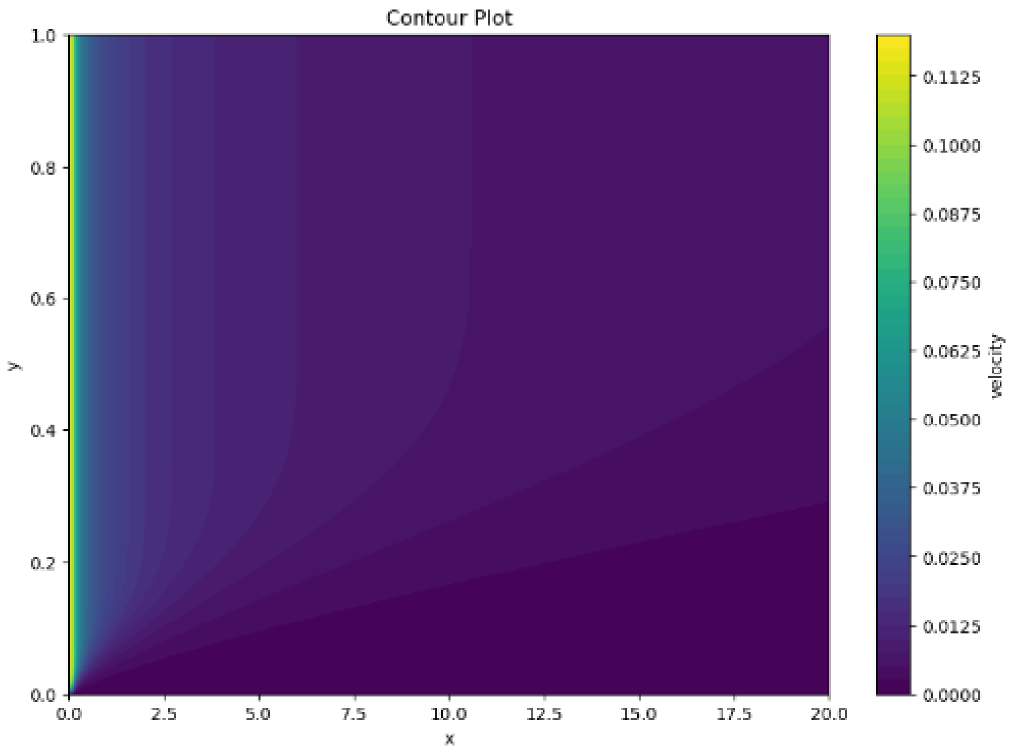


Figure 7: y-velocity contour plot

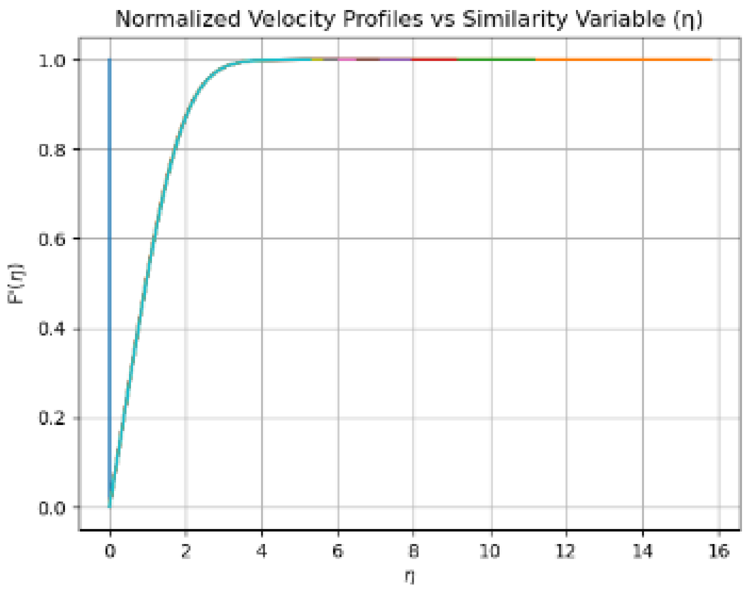


Figure 8: Normalized Velocity Profiles vs Similarity Variable

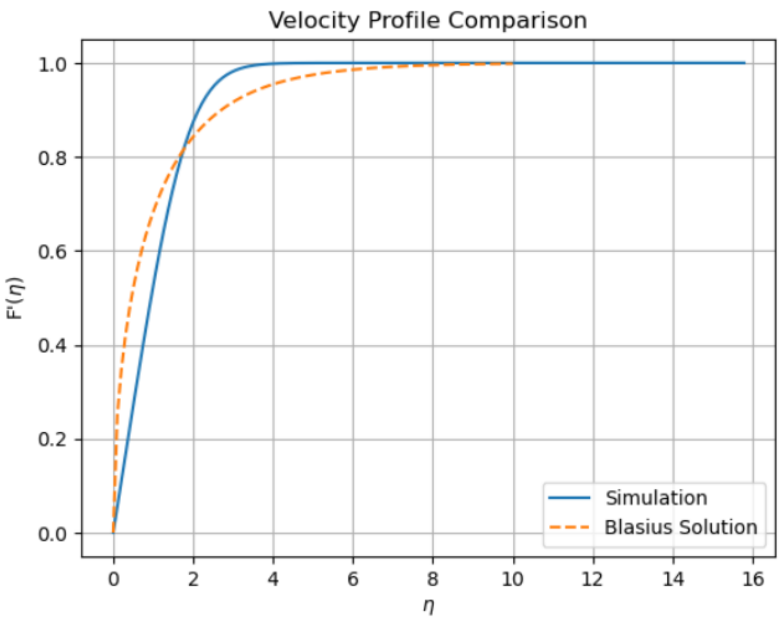


Figure 9: Velocity Profile Comparison between simulation results and Blasius Equation

In the x-velocity component, u, contour plot as shown in figure 6, one can observe a smooth development of the boundary layer along the downstream direction of the flat plate, starting from zero velocity at $y = 0$ to the smooth growth of the boundary layer along the x-direction. Farther away from the wall, the fluid layers experience less stress due to the wall and approaches the free-stream velocity as one moves in the y-direction. There is a presence of noise in the plot of u velocity contour in the upper portion, which might be an indication of slight numerical instability. This can be resolved using higher number of grids.

The y-velocity contour plot as shown in figure 7 shows lower magnitudes compared to the x-direction velocity contour plot, as the vertical component in boundary layer flows is expected to be small. The contour lines show a parabolic curve, with higher magnitudes near the wall and decreasing as the boundary layer grows. The smoothness of the curve depicts the fact that the numerical solution has been able to capture the intricate details of the fluid flow behaviour in the boundary layer.

In the normalized velocity profiles versus the similarity variable η as shown in figure 8, the plot demonstrates the velocity profile reaching 99% of the free stream velocity at approximately $\eta \approx 3$, which is supposed to consistent with the theoretical predictions of boundary layer thickness. On comparing the Blasius solution, which is the analytical solution, with the simulation results, one can see that there is a significant amount of agreement with the analytical and the numerical solution (as shown in figure 9). The slight differences may be attributed to the errors that may have crept in during truncation or iterations.

Final Thoughts: *The plots obtained from the Implicit Euler Scheme are in close agreement with that of the analytical solutions put forth by Blasius solutions, which depicts the accuracy, and the stability of the method used. The Implicit Method allows larger time steps without sacrificing stability, making them apt for implementation in flows involving high Reynolds number or in problems where the boundary conditions provided are stiff.*

4. Crank Nicolson Scheme

This numerical scheme is used to solve partial differential equations (particularly parabolic PDEs). It is classified as an implicit method because it requires solving a system of equations at each time step. This scheme is popular for its second-order accuracy and unconditional stability for linear problems. These characteristics makes it more reliable or long-time simulations compared to explicit methods.

4.1. Methodology

The continuity is as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

This equation can be approximated as:

$$\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} = 0$$

The value of the *y*-component of velocity, *v*, is given by:

$$v_{i,j} = v_{i,j-1} - \frac{\Delta y}{\Delta x}(u_{i+1,j} - u_{i-1,j})$$

The *x*-momentum equation is given by:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \vartheta\frac{\partial^2 u}{\partial y^2}$$

Upon discretizing the convection term, one gets:

$$\left(u\frac{\partial u}{\partial x}\right)_{i,j}^{n+1/2} = \frac{1}{2}\left(u_{i,j}^{n} + u_{i,j}^{n+1}\right)\left(\frac{\left(u_{i+1,j}^{n+1} - u_{i-1,j}^{n}\right)}{2\Delta x}\right)$$

$$\left(v\frac{\partial u}{\partial y}\right)_{i,j}^{n+1/2} = \frac{1}{2}\left(v_{i,j}^{n} + v_{i,j}^{n+1}\right)\left(\frac{\left(u_{i,j+1}^{n+1} - u_{i,j-1}^{n}\right)}{2\Delta y}\right)$$

Combining these, we get the final discretized equation for the *x*-momentum equation:

$$\frac{1}{2}\left(u_{i,j}^{n} + u_{i,j}^{n+1}\right)\left(\frac{\left(u_{i+1,j}^{n+1} - u_{i-1,j}^{n}\right)}{2\Delta x}\right) + \frac{1}{2}\left(v_{i,j}^{n} + v_{i,j}^{n+1}\right)\left(\frac{\left(u_{i,j+1}^{n+1} - u_{i,j-1}^{n}\right)}{2\Delta y}\right)$$
$$= \frac{\vartheta}{\Delta y^2}\left(u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1}\right)$$

4.2. Results and Discussion:

The following plots show the velocity contour plots (figure 10, 11), boundary layer thickness vs x-velocity (figure 12), similarity profiles (figure 13):

Contour of u-velocity:

The plot shown in the figure 10 is a contour plot of the u-velocity component in the flow domain. The x-axis represents the distance along the plate (in meters) and the y-axis shows the distance normal to the plate (in meters).
The colour scale on the right of the plot represents the magnitude of the u-velocity, ranging from zero to near the free-stream velocity.

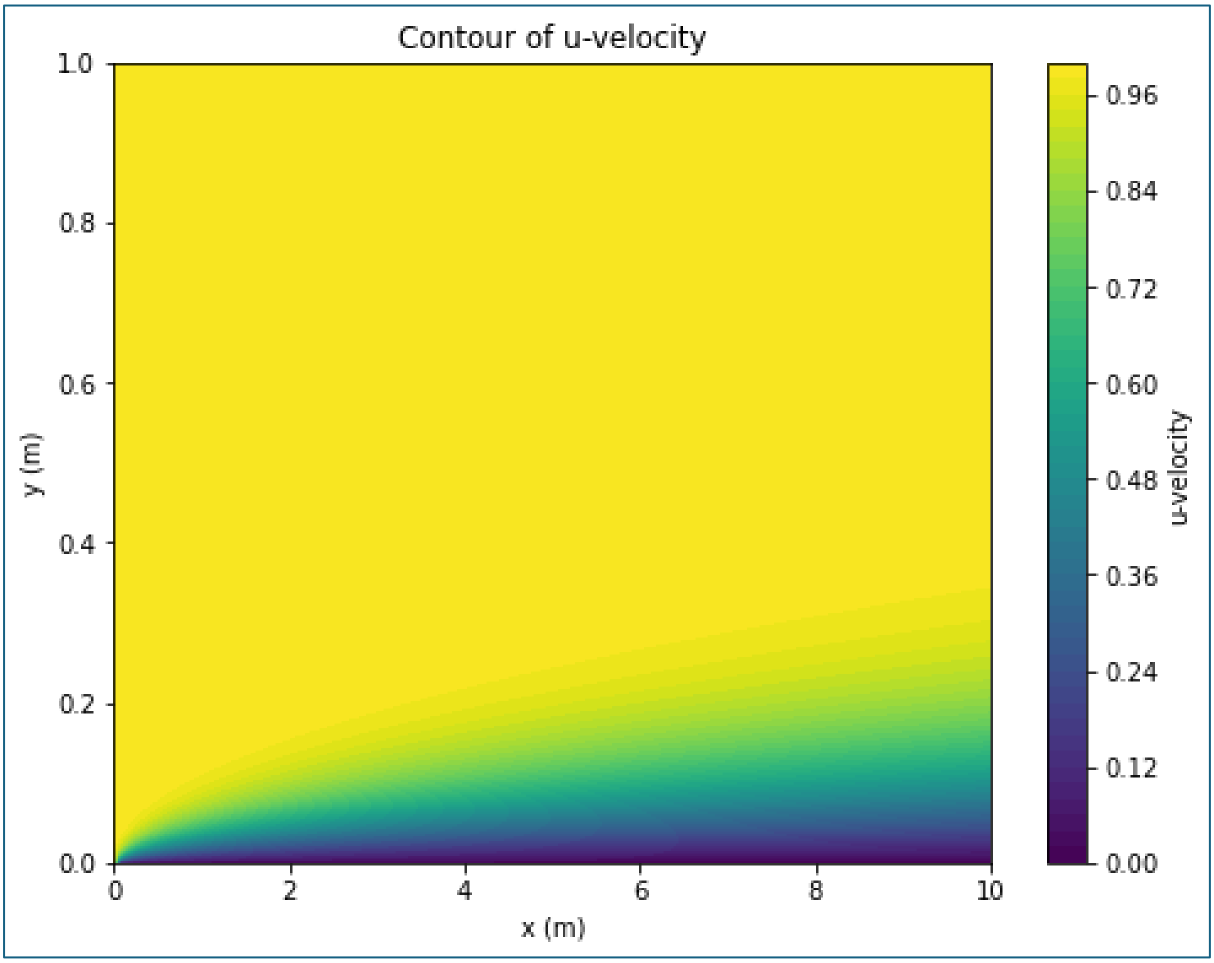


***Figure 10**: contour of u velocity*

It has been observed that near the leading edge i.e x = 0, the velocity close to the wall is almost zero due to the no slip boundary condition, while it increases quite rapidly at larger distances from the wall. The growth of the boundary layer with increasing x is seen from the region where velocity gradients near the wall become less steep. The yellow region in the plot depicts the free stream region where the velocity reaches the maximum value. This mostly remains unaffected.

Contour of v-velocity:

The plot shown in the figure 11 is a contour plot of the v-velocity component in the flow domain. It is observed with the colour scale that the v-velocity is significantly smaller compared to the u-velocity. This is expected in boundary layer flows.

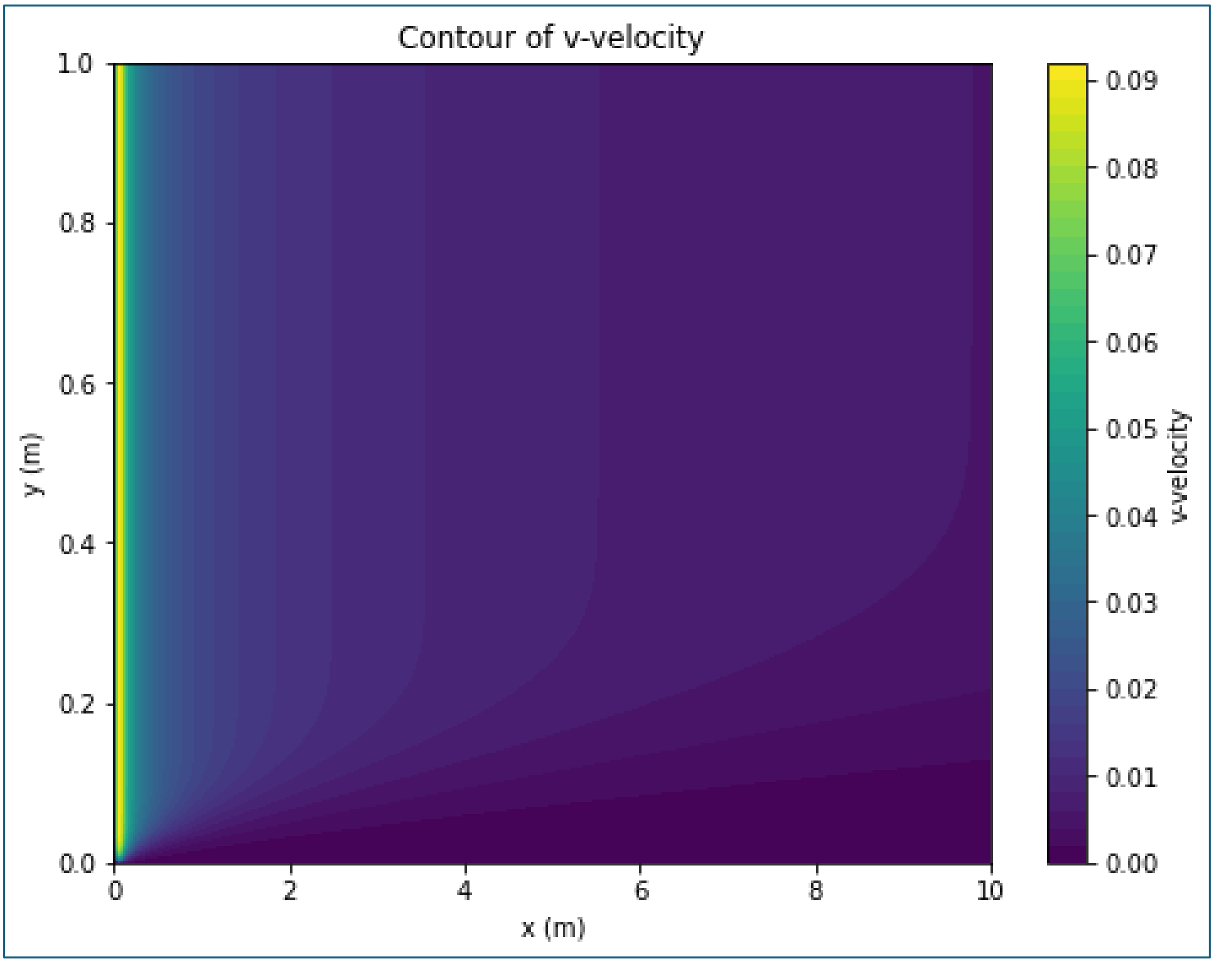


Figure 11: contour of v -velocity

At the leading edge the v-velocity is maximum and it rapidly decays as we move downstream. This small v-velocity component represents the movement of fluid perpendicular to the plate, which is due to the boundary layer growing in the vertical direction. The pattern (maximum near the wall, and decreasing to zero further away from plate) is typical for boundary layer flows, where the v-velocity increases slightly as the flow adjusts to the no slip condition at the wall.

Boundary layer thickness vs x:

The plot shown in figure 12 expresses the growth of boundary layer thickness along the length of the flat plate.

The numerical solution is represented by the blue curve and the Blasius solution(analytical) is represented by the red curve. The x-axis represents the position along the plate(x in meters), while the y-axis shows the boundary layer thickness.

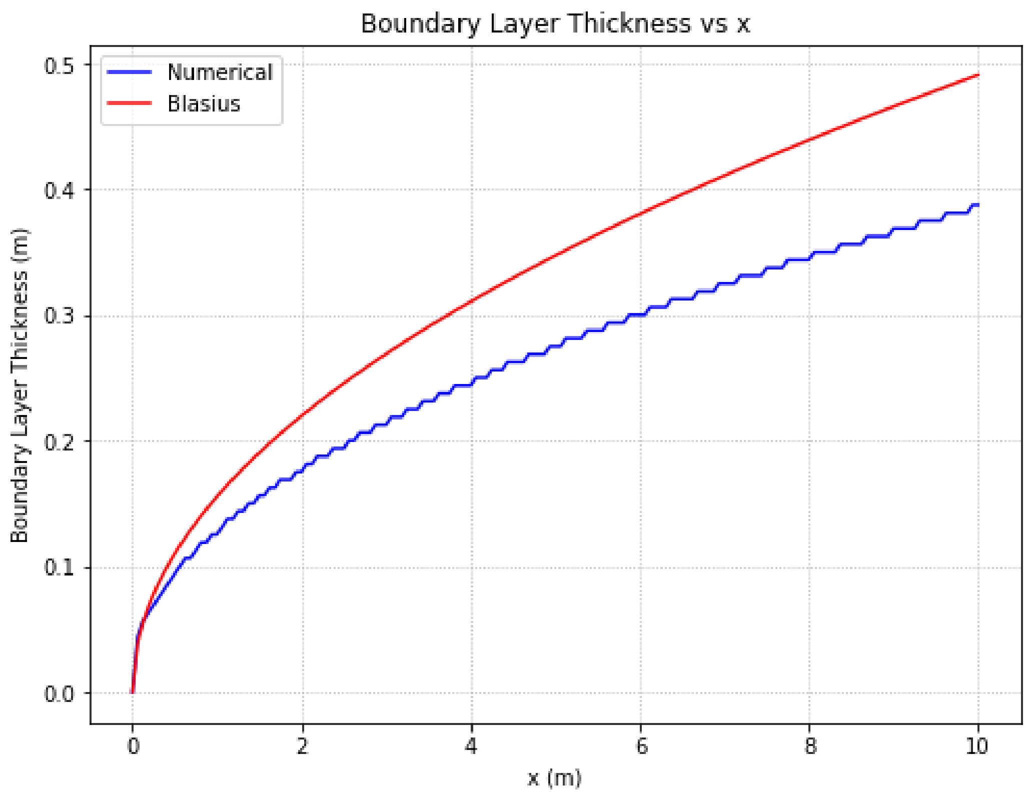


***Figure 12**: Boundary layer thickness vs x plot*

The boundary layer thickness starts at zero at the leading edge and grows along the plate.

It is observed that initially, the numerical solution closely follows the Blasius solution, but as we move forward in x, it deviates. This may be due to numerical dissipation, boundary conditions, or mesh resolution.

The curve of the Blasius solution is smooth and shows a predictable increase, while the numerical solution shows some steps and small oscillations. This can be resolved by increasing the no. of grids. The plot is observed with a grid size of 5000*5000 (longer computational time), which reduces the oscillation, thereby producing a smoother curve for the numerical solution.

Similarity Profiles:

This plot compares the similarity profiles at various positions along the plate.

The similarity variable is defined by the x-axis, while the dimensionless velocity profile(normalized u-velocity) is defined by the y axis.

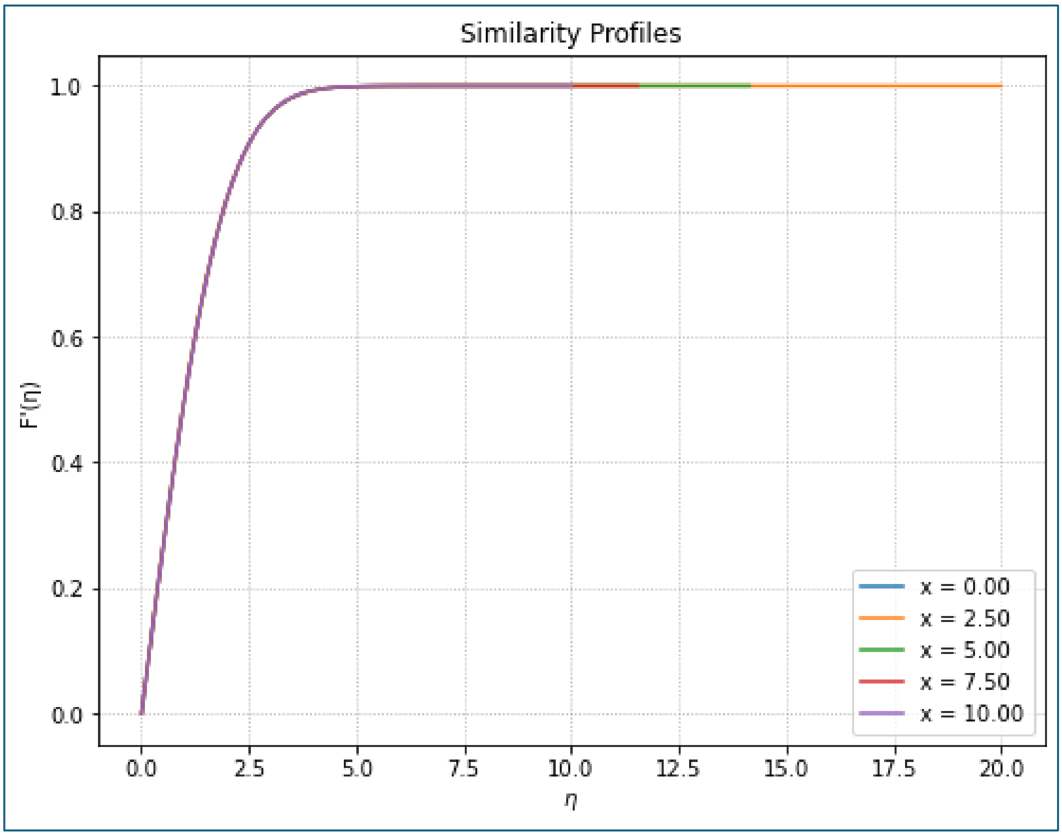


Figure 13: *Similarity profiles*

It is observed that the profiles collapse onto a single curve for all x, indicating that the flow is self-laminar in nature. This is again consistent with the boundary layer similarity solutions, where the velocity profiles at different positions can be described by the same function when scaled appropriately.

It can also be seen that for smaller values of the similarity variable the dimensionless velocity profile starts at zero and asymptotes to 1 at larger values. This shows that the velocity reaches the free stream velocity at a distance far from the plate.