Indian Institute of Technology Madras

# Control of Ball Balancing Robot

**Course: ME4010 Control Systems**

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# System Modeling

## 1 System Description

The system consists of a ball balancing robot modeled with three degrees of freedom. The mechanical structure is defined by the interaction between the wheel, the robot body, and a balancing mass.

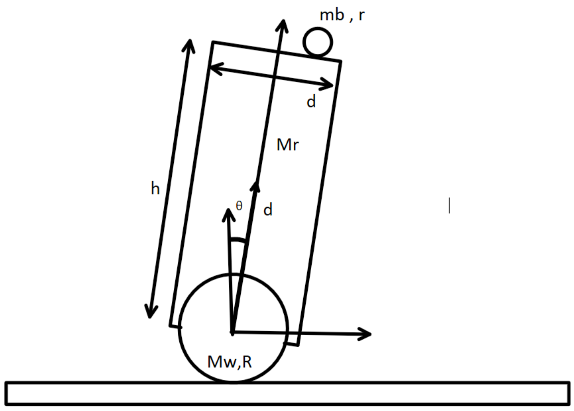



Figure 1: Diagram of the Ball Balancing Robot System

| Symbol | Description |
|---|---|
| $x_w(t)$ | Linear displacement of the wheel/cart |
| $\theta(t)$ | Tilt angle of the robot body |
| $x(t)$ | Position of the balancing mass |
| $M_w, I_w$ | Mass and Inertia of the wheel |
| $M_r, I_r$ | Mass and Inertia of the robot body |
| $m_b, I_b$ | Mass and Inertia of the balancing mass |
| $R$ | Radius of the wheel |
| $r$ | Radius of the balancing mass mechanism |
| $d$ | Distance to the robot body center of mass |
| $h$ | Height parameter for the balancing mass |
| $g$ | Gravitational acceleration |

Table 1: System Parameters

## 2 Equations of Motion Derivation

The equations of motion are derived using the Euler-Lagrange method. The Lagrangian $\mathcal{L}$ is defined as the difference between the total kinetic energy ($T$) and the total potential energy ($U$) of the system.

### 2.1 Kinetic Energy ($T$)

The total kinetic energy is the sum of the kinetic energies of the wheel ($T_w$), the robot body ($T_r$), and the balancing mass ($T_b$).

$$T_w = \frac{1}{2}M_w\dot{x}_w^2 + \frac{1}{2}I_w\left(\frac{\dot{x}_w}{R}\right)^2 \tag{1}$$

$$T_r = \frac{1}{2}M_r\dot{x}_w^2 + \frac{1}{2}I_r\dot{\theta}^2 \tag{2}$$

The kinetic energy of the balancing mass $T_b$ includes both translational and rotational components, accounting for the relative motion $x(t)$ and the body angle $\theta(t)$:

$$T_b = \frac{1}{2}m_b(v_{b,x}^2 + v_{b,y}^2) + \frac{1}{2}I_b\left(\frac{\dot{x}}{r}\right)^2 \tag{3}$$

### 2.2 Position and Velocity Analysis

The position of the balancing mass in the inertial frame is derived by first expressing its position in the body-fixed frame, then applying a rotation and translation.

#### 2.2.1 Body-Fixed Coordinates

In the body-fixed frame, the position of the balancing mass relative to the body center is:

$$\begin{bmatrix} x_{rel} \\ y_{rel} \end{bmatrix} = \begin{bmatrix} x \\ h+r \end{bmatrix} \tag{4}$$

#### 2.2.2 Rotation Matrix

The rotation from the body-fixed frame to the inertial frame is represented by the rotation matrix:

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{5}$$

where $\theta$ is the tilt angle of the robot body.

#### 2.2.3 Inertial Position Derivation

The position of the balancing mass in the inertial frame is obtained by rotating the body-fixed coordinates and adding the wheel position offset:

Performing the matrix multiplication:

$$\begin{bmatrix} x_b \\ y_b \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ h+r \end{bmatrix} + \begin{bmatrix} x_w \\ 0 \end{bmatrix} \tag{6}$$

Expanding the matrix product:

$$x_b = \cos(\theta)\cdot x + \sin(\theta)\cdot(h+r) + x_w \tag{7}$$
$$y_b = -\sin(\theta)\cdot x + \cos(\theta)\cdot(h+r) \tag{8}$$

Rearranging:

$$x_b = x\cos(\theta) + (h+r)\sin(\theta) + x_w \tag{9}$$
$$y_b = (h+r)\cos(\theta) - x\sin(\theta) \tag{10}$$

### 2.3 Potential Energy ($U$)

The total potential energy is the sum of the potential energies of the components:

$$U = U_b + U_r + U_w \tag{11}$$

where:

$$U_b = m_b g((h+r)\cos(\theta) - x\sin(\theta)) + R + \text{const} \tag{12}$$

$$U_r = M_r g d\cos(\theta) + R \tag{13}$$
$$U_w = R \quad \text{(constant)} \tag{14}$$

### 2.4 Lagrangian

The Lagrangian of the system is given by:

$$\mathcal{L} = T - U \tag{15}$$

### 2.5 Equations of Motion

Applying the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{16}$$

for the generalized coordinates $q = [x_w, \theta, x]$, we obtain the three equations of motion.

## 3 State-Space Formulation

### 3.1 Mass Matrix

The system dynamics can be expressed in the form:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{K}(\mathbf{q}, \dot{\mathbf{q}}) = 0 \tag{17}$$

The mass/inertia matrix $\mathbf{M}_1$ is:

$$\mathbf{M}_1 = \begin{bmatrix} \frac{I_b}{r^2} + m_b & m_b(h+r) \\ m_b(h+r) & I_r + m_b(h+r)^2 \end{bmatrix} \tag{18}$$

### 3.2 Stiffness Matrix

The stiffness/restoring force matrix $\mathbf{M}_2$ captures the gravitational restoring terms:

$$\mathbf{M}_2 = \begin{bmatrix} 0 & gm_b \\ gm_b & g(M_r d + m_b(h+r)) \end{bmatrix} \tag{19}$$

### 3.3 Input Coupling Matrix

The torque/input coupling matrix $\mathbf{M}_3$ represents the effect of the motor torque on the generalized coordinates:

$$\mathbf{M}_3 = \begin{bmatrix} -m_b(h+r) \\ -I_r - m_b(h+r)^2 \end{bmatrix} \tag{15}$$

### 3.4 Linearized State-Space Matrices

The system can be linearized around the equilibrium point $\theta = 0$ to obtain the linear state-space representation. Define the state vector as:

$$\mathbf{x} = \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} \tag{16}$$

The linearized state-space dynamics are:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{17}$$

where the state transition matrix $\mathbf{A}$ is:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ A_1(1,1) & 0 & A_1(1,2) & 0 \\ 0 & 0 & 0 & 1 \\ A_1(2,1) & 0 & A_1(2,2) & 0 \end{bmatrix} \tag{18}$$

and the input matrix $\mathbf{B}$ is:

$$\mathbf{B} = \begin{bmatrix} 0 \\ B_1(1) \\ 0 \\ B_1(2) \end{bmatrix} \tag{24}$$

where $\mathbf{A}_1 = -\mathbf{M}_1^{-1}\mathbf{M}_2$ and $\mathbf{B}_1 = \mathbf{M}_1^{-1}\mathbf{M}_3$.
The output equation is:

$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u \tag{25}$$

where $\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ and $\mathbf{D} = 0$.

# Control and State Estimation

| Dimension | Value | Units |
|---|---|---|
| $\mathbf{M_w}$ | 4.3 | kg |
| $\mathbf{M_r}$ | 10.12 | kg |
| $\mathbf{m_b}$ | 0.00271 | kg |
| $\mathbf{I_w}$ | 0.2725 | $\mathbf{kg \cdot m^2}$ |
| $\mathbf{I_r}$ | 0.4747 | $\mathbf{kg \cdot m^2}$ |
| $\mathbf{I_b}$ | $1.740 \times 10^{-6}$ | $\mathbf{kg \cdot m^2}$ |
| R | 0.356 | m |
| r | 0.04006 | m |
| d | 0.18865 | m |
| h | 0.31665 | m |
| g | 9.81 | $\mathbf{m/s^2}$ |

Table 2: Numerical Parameters for the Ball Balancing Robot System

## 4 Numerical State-Space Model

Using the parameters, the linearized state-space model $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u,\ y = \mathbf{C}\mathbf{x} + \mathbf{D}u$ is evaluated numerically as:

$$\mathbf{A} = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ -0.0143 & 0 & -3.0441 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0.0560 & 0 & 39.4486 & 0 \end{bmatrix} \tag{26}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{27}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{D} = 0 \tag{28}$$

## 5 LQR Controller Design

### 5.1 Optimal Control Problem

The Linear Quadratic Regulator (LQR) is designed to stabilize the unstable ball balancing robot while minimizing a cost function:

$$J = \int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) dt \tag{19}$$

### 5.2 Bryson's Rule for Weight Selection

$$\mathbf{Q} = \mathrm{diag}\left[\frac{1}{x_{\max}^2}, \frac{1}{\dot{x}_{\max}^2}, \frac{1}{\theta_{\max}^2}, \frac{1}{\dot{\theta}_{\max}^2}\right], \qquad \mathbf{R} = \frac{1}{u_{\max}^2} \tag{20}$$

$$\begin{aligned} x_{\max} &= d/8 = 0.0236 \text{ m} \\ \dot{x}_{\max} &= 0.5 \text{ m/s} \\ \theta_{\max} &= 15^\circ = 0.2618 \text{ rad} \\ \dot{\theta}_{\max} &= 2 \text{ rad/s} \\ u_{\max} &= 5 \text{ (actuator units)} \end{aligned}$$

### 5.3 Optimal Feedback Control Law

$$\mathbf{A}^T\mathbf{S} + \mathbf{S}\mathbf{A} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} + \mathbf{Q} = 0 \tag{21}$$

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{S}, \qquad u = -\mathbf{K}\mathbf{x} \tag{22}$$

### 5.4 Closed-Loop System

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} = \mathbf{A}_{cl}\mathbf{x} \tag{23}$$

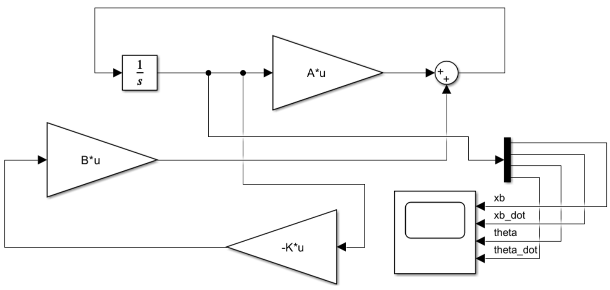



Figure 2: LQR Controller (simulink)

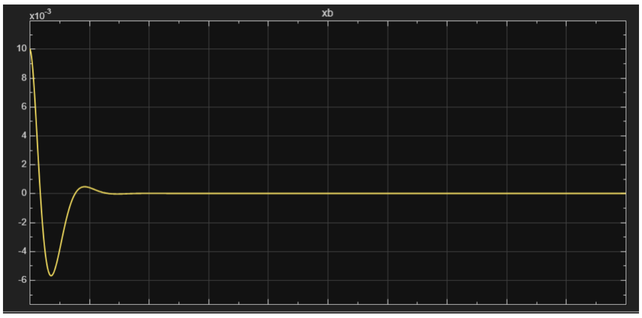



Figure 3: LQR Controller (xb plot)

## 6 State Estimation

### 6.1 Observability Analysis

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \mathbf{CA}^3 \end{bmatrix} \tag{24}$$

For fully observable, system rank(O) = 4.

## 7 Full-Order Kalman Observer

### 7.1 Observer Dynamics

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}) \tag{25}$$

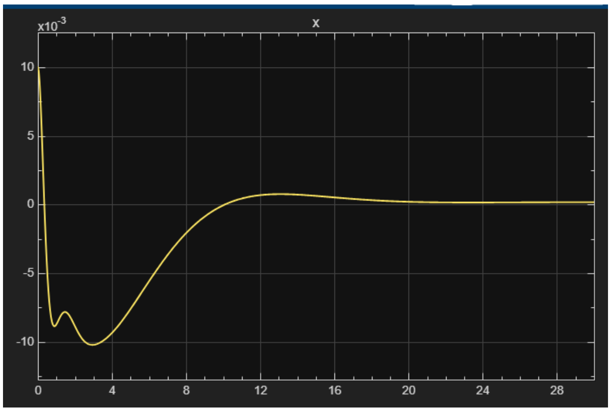
### 7.2 Kalman Filter Gain Design

$$\mathbf{Q}_{obs} = 0.1 \cdot \mathrm{diag}[10^7, 100, 10^6, 10], \qquad \mathbf{R}_{obs} = 0.00001 \tag{26}$$

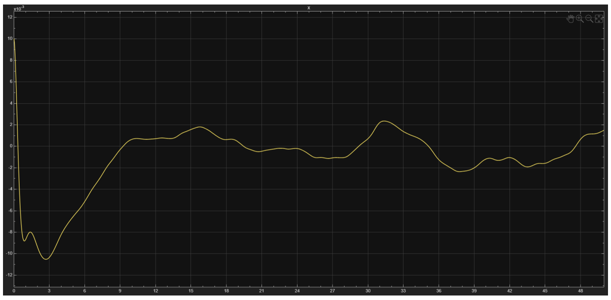
$$\mathbf{L} = \mathbf{P}\mathbf{C}^T\mathbf{R}_{obs}^{-1} \tag{27}$$

### 7.3 Estimation Error Dynamics

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e} \tag{28}$$




(a) No noise (xb plot)



(b) Measurement noise (xb plot)

(c) Noise filtered

Figure 4: Full-order observer responses

## 8 Minimum-Order(Reduced-Order) Observer

### 8.1 Motivation

Since $\theta$ is directly measured, a minimum-order observer estimates only unmeasured states $\dot{x}, x, \dot{\theta}$.

### 8.2 State Partitioning

$$\mathbf{x}_a = \begin{bmatrix}\theta\end{bmatrix}, \qquad \mathbf{x}_b = \begin{bmatrix} x \\ \dot{x} \\ \dot{\theta} \end{bmatrix} \tag{29}$$

$$\begin{bmatrix}\dot{\mathbf{x}}_a \\ \dot{\mathbf{x}}_b\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{aa} & \mathbf{A}_{ab} \\ \mathbf{A}_{ba} & \mathbf{A}_{bb}\end{bmatrix} \begin{bmatrix}\mathbf{x}_a \\ \mathbf{x}_b\end{bmatrix} + \begin{bmatrix}\mathbf{B}_a \\ \mathbf{B}_b\end{bmatrix} u \tag{30}$$

### 8.3 Reduced-Order Observer

$$\begin{aligned} \dot{\mathbf{z}} = \mathbf{A}_{bb}\mathbf{z} + \mathbf{A}_{ba}\mathbf{x}_a + \mathbf{B}_b u + \\ \mathbf{K}_{min}(\mathbf{x}_a - \mathbf{A}_{aa}\mathbf{x}_a - \mathbf{A}_{ab}\mathbf{z} - \mathbf{B}_a u) \end{aligned} \tag{31}$$

$$\hat{\mathbf{x}}_b = \mathbf{z} + \mathbf{K}_{min}\mathbf{x}_a \tag{32}$$

### 8.4 Noise Covariances

$$\mathbf{Q}_{min} = 10000 \cdot \mathrm{diag}[1, 100, 1], \quad \mathbf{R}_{min} = 0.005 \tag{33}$$

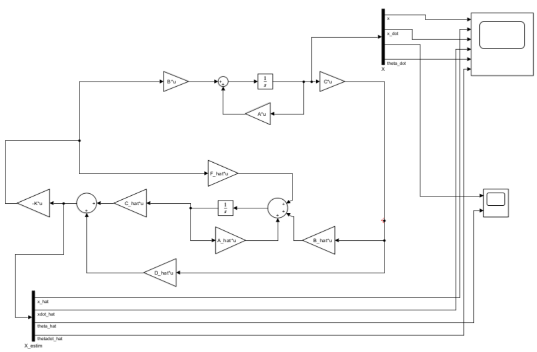


Figure 5: Minimum-Order Observer (simulink)

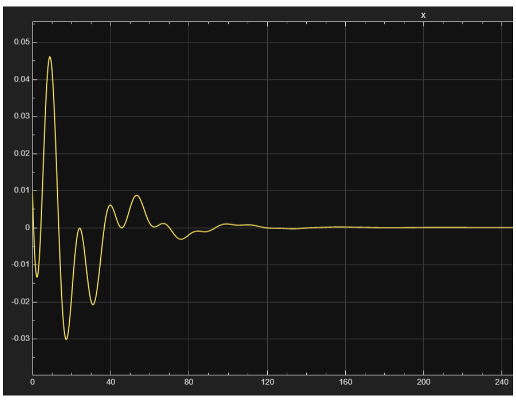



Figure 6: Minimum-Order Observer (xb hat)

## 9 Minimum-Order Observer with Noise Filtering

### 9.1 Motivation for Filtering

Measurement noise on $\theta$ can degrade observer performance. A low-pass filter is applied to the measured angle before feeding it to the minimum-order observer.

### 9.2 First-Order Low-Pass Filter (14 Hz)

A first-order low-pass filter is implemented with cutoff frequency $f_c = 14$ Hz:

$$\tau = \frac{1}{2\pi f_c} = \frac{1}{2\pi \cdot 14} \approx 0.01137 \text{ s} \tag{34}$$

The filter transfer function is:

$$H(s) = \frac{1}{\tau s + 1} = \frac{1}{0.01137s + 1} \tag{35}$$

The filtered measurement $y_{filt}$ is obtained from:

$$\dot{y}_{filt} = -\frac{1}{\tau}(y_{filt} - y_{meas}) \tag{36}$$

or in discrete form:

$$y_{filt}(k+1) = y_{filt}(k) + \frac{\Delta t}{\tau}(y_{meas}(k) - y_{filt}(k)) \tag{37}$$

### 9.3 Butterworth Filter (14 Hz)

For a second-order Butterworth filter:

$$H(s) = \frac{\omega_n^2}{s^2 + \sqrt{2}\,\omega_n s + \omega_n^2} \tag{38}$$

with

$$\omega_n = 2\pi f_c = 2\pi \cdot 14 \approx 87.96 \text{ rad/s} \tag{39}$$

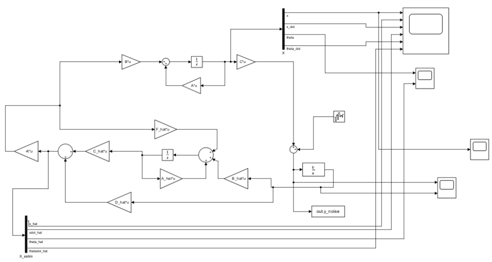


Figure 7: Minimum-Order Observer filtered (simulink)

### 9.4 Integrated Observation Architecture

$$\begin{aligned} \dot{\mathbf{z}} = & \mathbf{A}_{bb}\mathbf{z} + \mathbf{A}_{ba} y_{filt} + \mathbf{B}_b u \\ & + \mathbf{K}_{min}(y_{filt} - \mathbf{A}_{aa} y_{filt} - \mathbf{A}_{ab}\mathbf{z} - \mathbf{B}_a u) \end{aligned} \tag{40}$$

## 10 Integrated Control-Observer System

### 10.1 Separation Principle

$$u = -\mathbf{K}\hat{\mathbf{x}} \tag{41}$$

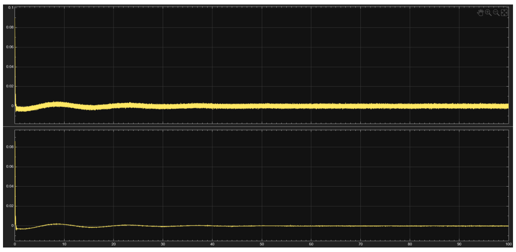


Figure 8: Minimum-Order Observer (measurement noise filtering)

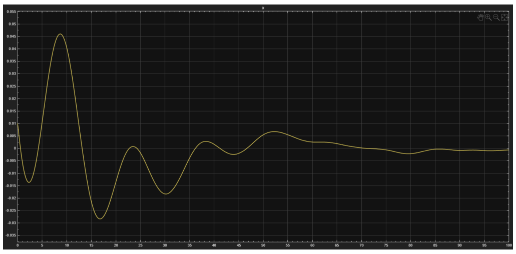


Figure 9: Minimum-Order Observer filtered (xb hat)

### 10.2 Combined Closed-Loop Dynamics

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ 0 & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \tag{42}$$

## 11 Summary of Observer Architectures

The key characteristics of the observer architectures are summarized below:

- **Full-Order Kalman Observer**
  **Order:** 4
  **States Estimated:** All ($\mathbf{x} = [x, \theta, \dot{x}, \dot{\theta}]^T$)
  **Advantages:** Provides complete state information; is the standard implementation.
- **Reduced-Order Observer**
  **Order:** 3
  **States Estimated:** $\dot{x}$, $x$, $\dot{\theta}$ (unmeasured states)
  **Advantages:** Offers better computational efficiency with less redundancy.
- **Reduced with Filter Observer**
  **Order:** 3
  **States Estimated:** Filtered $\theta$ plus unmeasured states
  **Advantages:** Noise rejection capability, leading to smooth estimates.

## 12 Root locus method: PID

### 12.1 Cascade Control Strategy

In addition to State-Space methods, a classical Cascade PID control architecture was designed using Root Locus techniques. This approach decomposes the unstable system into two loops:

1. **Inner Loop (Stabilization):** Controls the tilt angle $\theta$ to maintain upright stability. This loop must have faster dynamics.
2. **Outer Loop (Position):** Controls the ball position $x$. The output of this controller serves as the reference angle $\theta_{ref}$ for the inner loop.

### 12.2 Inner Loop Design ($\theta$)

The plant transfer function from input torque to angle $G_\theta(s)$ has a pair of Real axis poles (one stable, one unstable) and a pair of imaginary poles. A PID controller $C_\theta(s)$ is tuned to stabilize the system:

$$C_\theta(s) = K_{p,\theta} + \frac{K_{i,\theta}}{s} + K_{d,\theta} s \tag{43}$$

Based on the tuning parameters:

$$K_{p,\theta} = 3000, \quad K_{i,\theta} = 10, \quad K_{d,\theta} = 25 \tag{44}$$

The closed-loop inner system ensures all poles are in the Left Half Plane (LHP), guaranteeing stability for the pendulum dynamics.

### 12.3 Outer Loop Design ($x$)

The Outer loop regulates the position $x$ by manipulating the setpoint of the inner loop ($\theta_{ref}$). The controller $C_x(s)$ is defined as:

$$C_x(s) = K_{p,x} + \frac{K_{i,x}}{s} + K_{d,x} s \tag{45}$$

Using Root Locus analysis on the effective plant $G_x(s)$ (from $\theta_{ref} \to x$), the gains were selected to ensure tracking performance while maintaining stability boundaries:

$$K_{p,x} = 5.0, \quad K_{i,x} = 0.01, \quad K_{d,x} = 41 \tag{46}$$

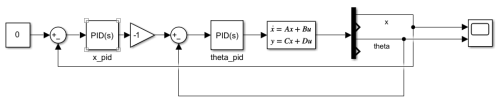



Figure 10: PID controller (simulink diagram)

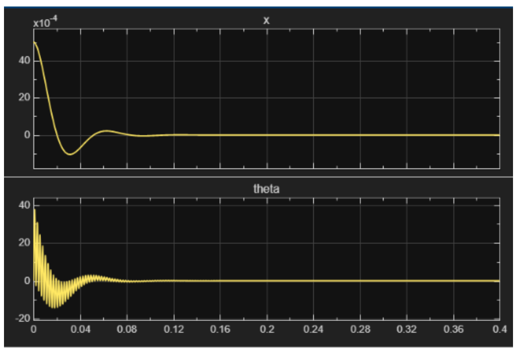



Figure 11: PID controller (simulink plot)

## 13 Performance Comparison (settling time)

| **Control Architecture** | **Settling Time ($T_s$)** |
|---|---|
| **PID (Cascade)** | 84.45 s |
| **LQR (State Feedback)** | 4.77 s |
| **LQR + Full Obs.** | 31.56 s |
| **LQR + Reduced Obs.** | 138.92 s |

Table 3: Comparison of Settling Times
Github link to project