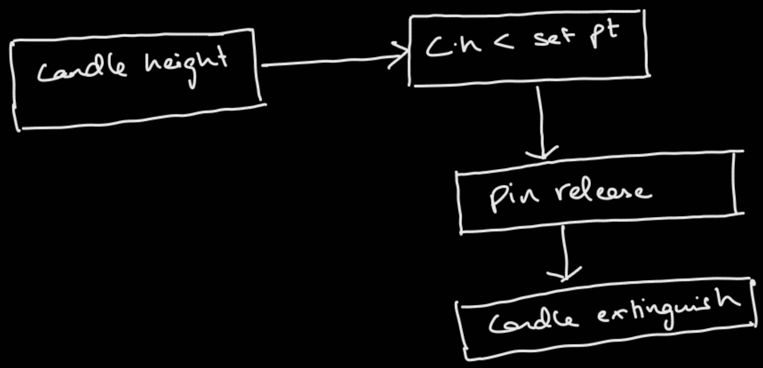
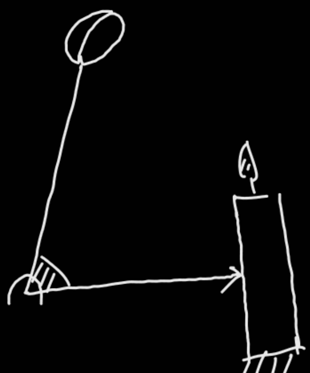
Q1,2 → 15+15

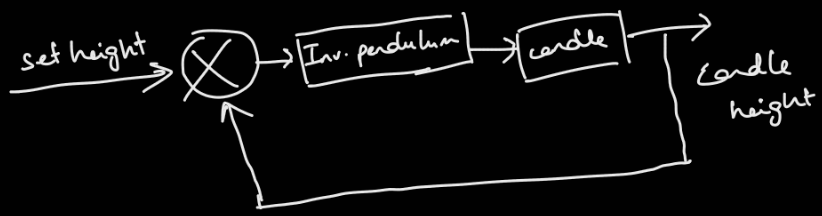
Tut, Ass → 20

End sem → 35

Project → 15







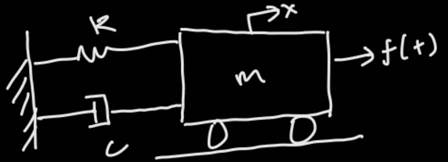
ON-OFF controller

dynamic action → pin slowly moves down near set point

In case where pin is embedded in candle: system is modelled based on length of pin that is embedded in the candle.

Laplace transforms:-

diff. eq'n → algebraic eq'n.



$ma = f(t) - kx - cv$

$m\frac{d^2x}{dt^2} = f(t) - kx - c\frac{dx}{dt}$

2^{nd} order non-homogenous ODE
Linear time invariant (LTI)

Laplace operator → $\mathcal{L}$

$\mathcal{L}[x(t)] = \int_0^\infty e^{-st} x(t)\, dt$

fourier $= \int e^{-i\omega t} x(t)\, dt$

$s = \alpha + i\omega$

$\mathcal{L}(\dot{x}) = ?$ $\quad \mathcal{L}(x(t)) = X(s)$

$\int_0^\infty e^{-st} \dot{x}\, dt = e^{-st}\int \dot{x}\, dt + s\int e^{-st}\left[\int \dot{x} dt\right] dt$

$= \left[e^{-st} x\right]_0^\infty + s\left[\int x e^{-st} dt\right] = sX - 1$

$f(t) = m\ddot{x} + c\dot{x} + kx$

$\mathcal{L}(m\ddot{x}+c\dot{x}+kx) = ms^2\mathcal{L}(x) + cs\mathcal{L}(x) + k\mathcal{L}(x) = \mathcal{L}(f(t))$

$$\mathcal{L}^{-1}[X(s)] = x(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)\, e^{st}\, ds$$

$$\int_0^\infty e^{-i\omega t}\sin t\, dt = e^{-i\omega t}[-\cos t] - \int i\omega\, e^{-i\omega t}\cos t\, dt$$

$$= \frac{1}{1+(i\omega)^2}$$

4/08/25

$$F(s) = \int_0^\infty e^{-t}\sin(t)\, e^{-st}\, dt = ?$$

$\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$ $\qquad$ $\mathcal{L}(e^{-t}\sin(t)) = \frac{1}{(s+1)^2+1}$

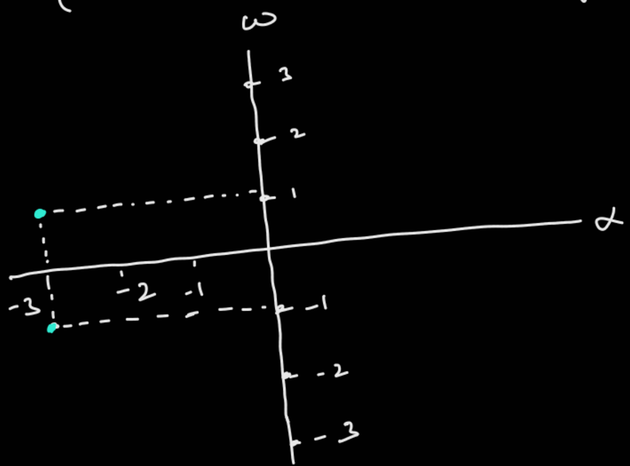
⇒ Laplace transform → goes to ∞ → poles of system

→ goes to 0 → zeroes of system.

$e^{-3t}\sin 2t$ $\qquad$ $\alpha = -3,\ \omega = \pm 1$ ⇒ pole

$\frac{1}{(s+3)^2+4} = \frac{1}{(\alpha+3+i\omega)^2+1}$

for oscillation: root = $\alpha \pm i\omega$ complex



if root = real ⇒ exponential decay.

if root: no real component, only complex.

⇒ no decay

⇒ fourier.

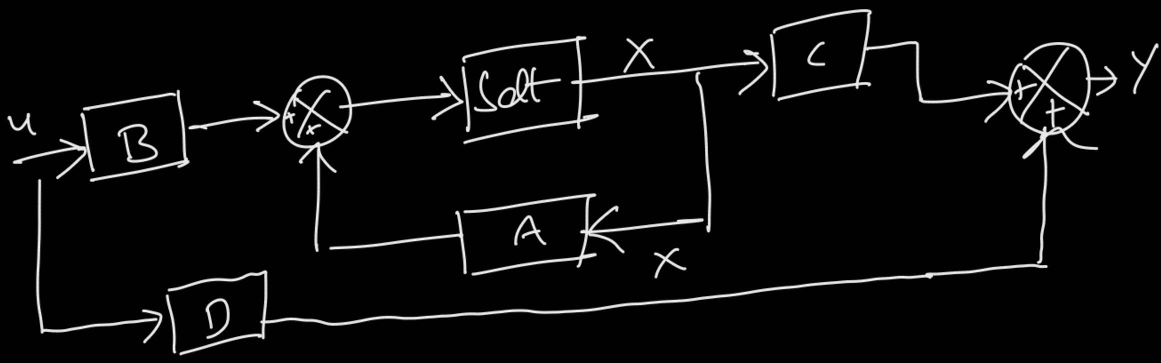
$$\frac{X(s)}{F(s)} = \text{transfer function of the system} = G(s)$$

($X(s)$ ↗ output; $F(s)$ ↓ input)

$$F(s) \longrightarrow \boxed{G(s)} \longrightarrow X(s)$$

State:- Min. set of variables, which if known at $t = t_0$; i/p for $t \geq t_0$. The system can be completely characterized.

State variable:- Variables making up the smallest set is state. ⇒ indirectly observable / measurable using estimators.



$$y = Cx + Du$$

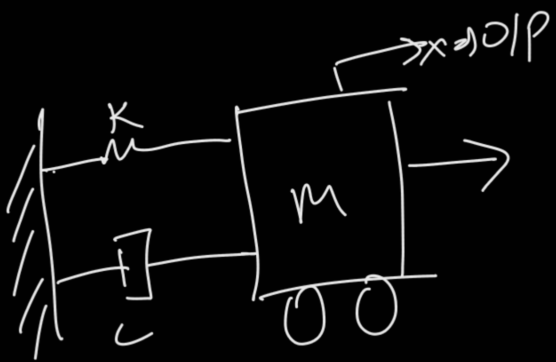
$$x = \int Bu + Ax \, dt \Rightarrow \dot{x} = Bu + Ax$$

$$sX(s) - \cancel{x(0)}^{\,0} = Ax(s) + BU(s) \Rightarrow x(s)\left[sI_n - A\right] = BU(s)$$

$$y(s) = Cx(s) + DU(s)$$

$$y(s) = C\left(BU(s)\right)\left(sI_n - A\right)^{-1} - DU(s)$$

$$= \left[CB(sI_n - A)^{-1} - D\right]U(s)$$



$$C = [1 \; 0]$$

$$u = \dot{x} \qquad u = \dot{v}$$
$$v = x$$

$$m\ddot{x} + Kx + c\dot{x} = 0$$

$$m\dot{u} + Kv + cu = 0$$

$$smU(s) + KV(s) + cU(s) = 0$$

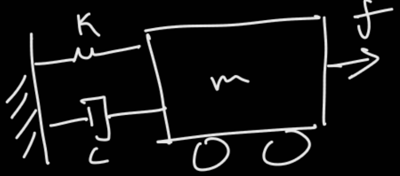
$$V(s) = -\frac{(ms + c)}{K}U(s)$$

Minimum no. of state variables:-

$\Rightarrow$ no. of independent variables

(or)

$\rightarrow$ no. of energy storing devices [from elec.]

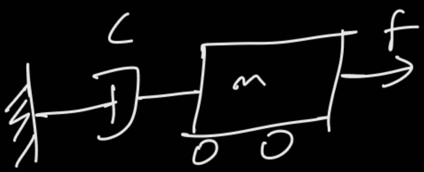


devices = 3

$$G(s) = \frac{1}{ms^2 + cs + K}$$

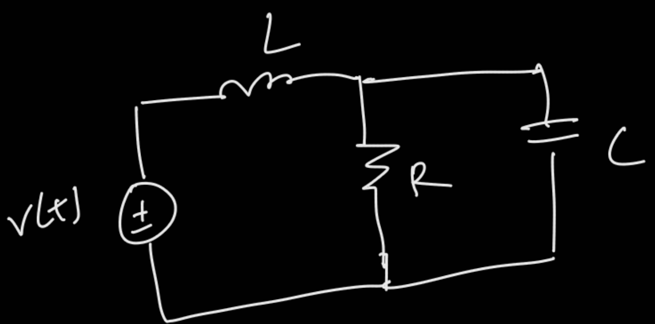
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$



$$m\ddot{x} + c\dot{x} = f \qquad x_1 = \dot{x}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{c}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f$$



$\boxed{\text{Output} = i_R}$

$$V(t) = L\frac{di}{dt} + i_R R$$

$$\Rightarrow V(t) = L\frac{di_R}{dt} + \frac{Li_c}{dt} + i_R R \rightarrow (1)$$

$$V(t) = \frac{Ldi_R}{dt} + \frac{Li_c}{dt} + \frac{\int i_L - i_R \, dt}{C} \rightarrow (2)$$

$$i_c = i_L - i_R$$

$$\frac{C dV_c}{dt} = i_L - \frac{V_c}{R}$$

$$\begin{bmatrix} \dot{i}_L \\ \dot{V}_c \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} i_c \\ V_c \end{bmatrix} + V_t \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$$

$$V_L + V_c = V_t$$

$$V_L = V_t - V_c$$

$$\Rightarrow L\frac{di_L}{dt} = V_t - V_c$$

Linear systems:-

$\dot{x} = Ax_{n\times 1}$

$Av = v\lambda$

$A = \lambda \underbrace{vv^{-1}}_{n\times n}$

$\dot{x} = Ax$

$A = V\lambda V^{-1}$

$\dot{x} = V\lambda V^{-1}x$

$(\dot{V^{-1}x}) = \lambda(V^{-1}x)$

$\Rightarrow \dot{z} = \lambda z$

$\Rightarrow$ x is transformed to z

$\boxed{x = Vz}$

$z = e^{\lambda t}$

$= 1 + \lambda t + \frac{(\lambda t)^2}{2!} + \dots$

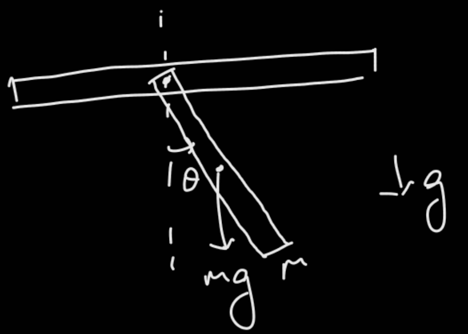
$$e^{\lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots \\ 0 & e^{\lambda_2 t} & & \vdots \\ 0 & & \ddots & \vdots \\ \vdots & \cdots & \cdots & e^{\lambda_n t} \end{bmatrix}$$

$x = V\left(1 + \lambda t + \frac{(\lambda t)^2}{2!} \dots\right)$

n unique $\Rightarrow$ min variables

All $\lambda > 0 \Rightarrow$ unstable system.

Pendulum:-



$\tau = I\alpha$

$\tau = mg\sin\theta \frac{l}{2} = -I\ddot{\theta}$

$\Rightarrow I\ddot{\theta} + mg\frac{l}{2}\sin\theta = 0$

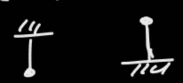
$\ddot{\theta} = \frac{-mgl\sin\theta}{2I}$ ideal pendulum

$\ddot{\theta} = \frac{-mgl\sin\theta}{2I} - \underbrace{\delta\dot{\theta}}_{\text{damping}}$

$$\dot{\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}} = \begin{bmatrix} \dot{\theta} \\ -\frac{mgl\sin\theta}{2I} - \delta\dot{\theta} \end{bmatrix}$$

$f(x)|_{x_0} = 0$ then x_0 is a fixed pt.

$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}$



Find jacobian of the sys about x_0, error dynamics about x_0 behave like linear sys.

$$J = \left[\frac{\partial f_i}{\partial x_i}\right]$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$ for 2nd order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

$$f_1 = \dot{\theta} \qquad f_2 = -\frac{mgl\sin\theta}{2I} + \delta\dot{\theta}$$

$$x_1 = \theta \qquad x_2 = \dot{\theta}$$

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{mgl\cos\theta}{2I} & -\delta \end{bmatrix}$$

fixed point

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad J = \begin{bmatrix} 0 & 1 \\ -\frac{mgl}{2I} & -\delta \end{bmatrix} \rightarrow -,- \text{ for } \delta > 2\sqrt{\alpha}$$

$$\begin{bmatrix} \pi \\ 0 \end{bmatrix} \quad J = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{2I} & -\delta \end{bmatrix} \rightarrow -,-$$

$$\alpha, \delta > 0$$

Stability:-

$$G(s) = \frac{1}{s(s+3)(s+8)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+8}$$

$\Rightarrow$ System is stable if roots of denominator are all -ve.

$$= A\sin t + e^{-3t}\sin t$$

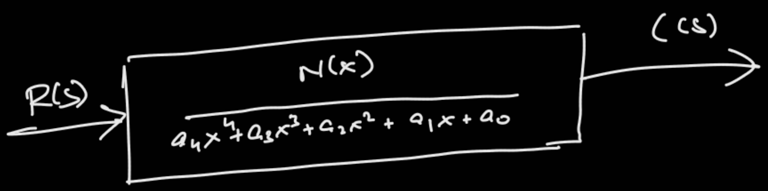
Natural response diverges to ∞ at $t \to \infty$ $\Rightarrow$ unstable

" " oscillates $\Rightarrow$ marginally stable.

" " decays to 0 $\Rightarrow$ stable. at $t \to \infty$

Routh Hurwitz stability criterion:-

1. Generate a routh table from the denominator of the pertinent.
2. Sign changes of the 1^{st} column in this table indicate no. of roots in the RHP.



$$
\begin{array}{llll}
s^4 & a_4 & a_2 & a_0 \\
s^3 & a_3 & a_1 & 0 \\
s^2 & \dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 & \dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_2 & \dfrac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_3 = 0 \\
s^1 & \dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 & \dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & b_3 \end{vmatrix}}{b_1} = c_2 = 0 & \dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = c_3 \\
s^0 & \dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}{b_1} & \dfrac{-\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}}{b_1} & \dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{b_1}
\end{array}
$$

$$G(s) = \frac{1000}{s^3+10s^2+31s+1030}$$

Routh table

s^3	1	31	0
s^2	10	1030	0
s^1	-72	0	0
s^0	1030	0	0

\# no. of poles in RHP = # of sign changes in 1^{st} column.

= 2

$$G(s) = \frac{10}{s^5+2s^4+3s^3+6s^2+5s+3}$$

					ε^+	ε^-
s^5	1	3	5	0	×	×
s^4	2	6	3	0	×	✓
s^3	$\cancel{0}\,\varepsilon$	$7/2$	0	0	✓	✓
s^2	$\frac{6\varepsilon-7}{\varepsilon}$	3	0	0	✓	×
s	$\frac{42\varepsilon-49-6\varepsilon^2}{12\varepsilon-14}$	0	0	0	×	×
s^0	3	0	0	0		

II method

$$s^5+2s^4+3s^3+6s^2+5s+3$$

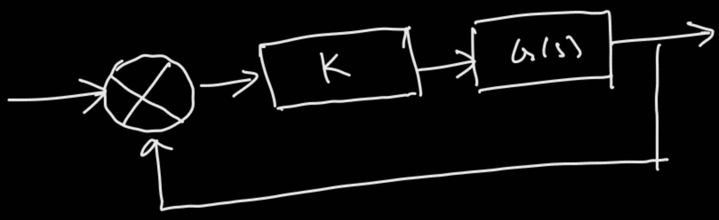
inverting roots

$$3s^5+5s^4+6s^3+3s^2+2s+1$$

s^5	3	6	2	0
s^6	5	3	1	0
s^4	$\frac{21}{5}$	$\frac{7}{5}$	0	0
s^3	$\frac{4}{3}$			
s^2				
s^1				
s^0				

Stability design via Routh Hurwitz criterion:-

$$G(s) = \frac{1}{(s)(s+3)(s+8)}$$



Find values of K for which system is stable, unstable, marginally stable.

$$s^3 + 11s^2 + 24s + K$$

s^3	1	24	0
s^2	11	K	0
s^1	$-\left(\frac{K-(24)(11)}{(11)}\right)$	0	0
s^0	K	0	0

$K < 264 \rightarrow$ stable

$K > 264 \rightarrow$ unstable

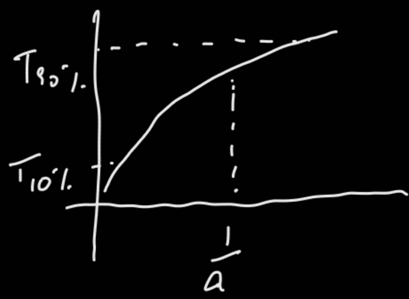
$K = 264 \rightarrow$ marginally stable

1st order system

t_R = rise time

$T_{90\%} - T_{10\%}$

$t_s \rightarrow 2\%$ (settling time)



$t_{90} \Rightarrow 0.9 = 1 - e^{-a t_{90}} \Rightarrow t_{90} = \frac{2.31}{a}$

$0.1 = 1 - e^{-a t_{10}} \Rightarrow t_{10} = \frac{0.11}{a}$

$$t_r = \frac{2.2}{a}$$

2nd order system.

$\Rightarrow$ 2 variables to define performance

$\frac{q}{s^2+ps+q}$ Steady state value $(s=0) = 1$

ex: $\frac{10}{s^2+7s+10} = \frac{10}{3}\left(\frac{1}{s+2} - \frac{1}{s+5}\right)$

I/p $= \frac{1}{s}$

$\frac{10}{\cancel{s}^2+10}$ $\quad D(s) = \frac{1}{s}\,\frac{10}{s^2+10}$

$\frac{10}{s^2+2\sqrt{10}s+10} = \frac{10}{(s)(s+\sqrt{10})^2} = \frac{10}{s+\sqrt{10}}\left(\left(\frac{1}{\sqrt{10}}\right)\left(\frac{1}{s} - \frac{1}{s+\sqrt{10}}\right)\right)$

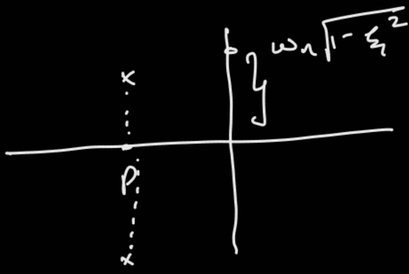
$$= \sqrt{10}\left[\frac{1}{\sqrt{10}}\left(\frac{1}{s} - \cancel{\frac{1}{s+\sqrt{10}}}\right) - \frac{1}{(s+\sqrt{10})^2}\right]$$

$$= \frac{1}{s} - \frac{1}{s+\sqrt{10}} - \frac{\sqrt{10}}{(s+\sqrt{10})^2} = 1 - e^{-\sqrt{10}t} - \sqrt{10}\,t e^{-\sqrt{10}t}$$

2nd order systems:-

$\frac{\omega_n^2}{s^2+2ps+\omega_n^2}$ natural freq: $\omega_n \to$ undamped freq. of oscillation

Damping ratio $(\xi) = \frac{p}{\omega_n} = \frac{\text{exp. freq}}{\text{nat. freq}}$

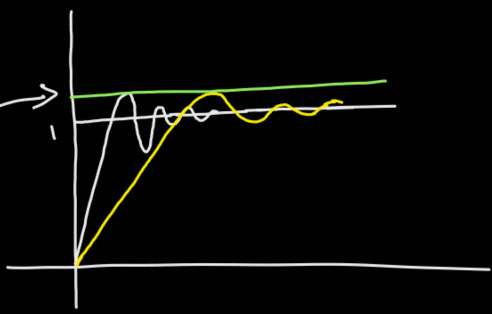


Rise %.

$$x(t_r) = 1 + \left(e^{\frac{n\xi 2\pi}{\sqrt{1-\xi^2}}} \sqrt{1-\xi^2}\right)$$

$$\hookrightarrow x(t_1) = \left(1 + e^{\frac{-\xi 2\pi}{\sqrt{1-\xi^2}}} \sqrt{1-\xi^2}\right)$$

↳ independant of freq.



Settling time:-

↳ within 2% of Oss

$O(t) = 0.98$ or 1.02

$$\frac{e^{-\xi\omega_n t_n}}{\sqrt{1-\xi^2}} = 0.02 \qquad \boxed{T_s = \frac{4}{\xi\omega_n}}$$

Design a system with < 10% overshoot, < 1s settling time, $T_p < 0.2$ sec

$$e^{\frac{-\xi 2\pi}{\sqrt{1-\xi^2}}} \sqrt{1-\xi^2} < 0.1 \qquad \frac{4}{\xi\omega_n} < 1 \Rightarrow \xi\omega_n > 4$$

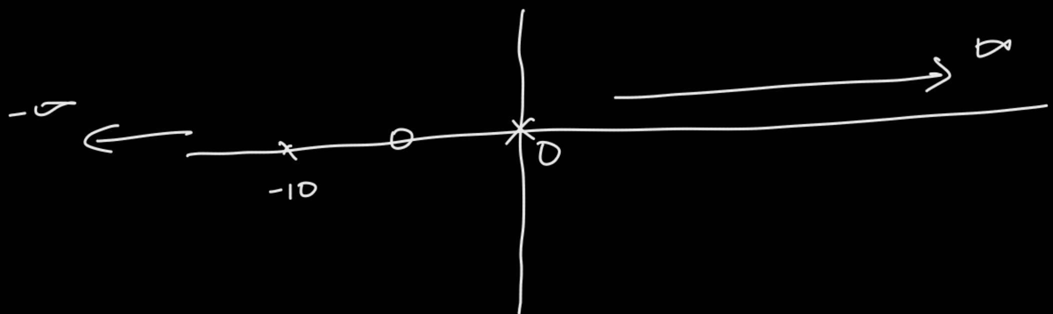
$$T_p = \frac{\pi}{\omega_d}$$

$$\Rightarrow \frac{\pi}{\omega_d} < 0.2 \Rightarrow \omega_d > 5\pi$$

$$\omega_n\sqrt{1-\xi^2} > 5\pi$$

Graphical, mathematical rules followed by root loci:-

1. # of roots of loci = # of poles
2. All branches start at poles of R·L system & end at zeros of R·L system   say $G(s) = \frac{s+5}{s+10}$



3. Root axis of R.L starts at poles on real axis, ends at zeros on the real axis