Generalized mode acceleration method

A Project report
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By

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Abstract

The time-domain analysis methods like the mode displacement method and mode acceleration method are discussed extensively in the literature, but their accuracy is limited by number of modes used to span solution space. The generalized mode acceleration method offers further improvement in these approaches while using the same number of modes for response prediction. The generalized mode acceleration method requires the problem to be transferred from the time domain to the frequency domain using Laplace transform and after solving the problem in the frequency domain, we again shift back to the time domain using inverse transform. The generalized mode acceleration method is found to be very effective in improving the solutions of mode displacement and mode acceleration methods. The improvement in solution is found to be significant in the lower order of approximations, but it slows down asymptotically as the order of approximation increases. The method offers an extra handle to the designer. the designer can then decide the order of approximation by taking into account the needed accuracy and available computational capacity. If the spatial variation of force gets changed, the correction term added in mode displacement method can be calculated just by changing the force vector. Because of this, the method gives very flexible approach to find the dynamic response.

Pledge

- I pledge on my honour that I have not received report and codes from any of my course mates for course project.
- I also pledge on my honor that I have not sent my report and codes to any of my course mates for the course project.
- Also, I have not indulged in any other unfair means while solving a course project.

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Chapter 1

Introduction

Predicting computationally efficient and accurate dynamic response of any structure has always posed challenges for the designer. Practically, all the structures are continuous systems, but it is common practice to convert them into a discretized system in order to compute their response. After discretization, the system reduces to a large number of lumped masses. The number of degrees of freedom of such systems is very large, which poses the problem of computing all mode shapes and eigenvalues. To predict the exact response of the structure, we compute all the modes shapes and say any response can be expressed as a linear combination of these mode shapes. For a small system having less number of DOFs, it is practical to perform modal analysis and compute all mode shapes and eigenvalues. In such cases, by using modal superposition method, one can predict a sufficiently accurate response. But a large system poses a problem while computing mode shapes and eigenvalues due to limitation on computational capacities and accuracy of higher modes. For such large systems, it is common practice to implement mode truncation methods[3].

The simplest form of mode truncation method is the mode displacement method. In the mode displacement method, We compute limited numbers of mode shapes and span the solution as a linear combination of these mode shapes, Actual number of mode shapes is very very greater than the number of mode shapes considered for approximation. So solution space spanned by all mode shapes cannot be accurately spanned by limited number of mode shapes. Now, to improve this solution, researchers have proposed many different techniques, mode acceleration method is one such technique, which takes into account the quasi-static response of system[3].

The improvement proposed by Daniel J. Rixen is a generalized mode acceleration method to get an improved solution and can help to get sufficiently accurate results[1]. The basic idea of this method is to transform the complete system into a frequency domain by using Laplace transform and perform analysis in the frequency domain. Later on, using the inverse Laplace transform, we can shift from frequency to time domain[2].

Scope of project

The project objective is to study and demonstrate of generalized mode acceleration method on project problem.

Outline of project

In chapter 2, The methodology of the generalized mode acceleration method(GMAM) is discussed and derivation of the mode displacement method and mode acceleration method from the GMAM is given. In chapter 3, the methodology discussed in Chapter 2 will be applied for a simple 3 DOF system to validate the octave code and derivation of GMAM and results of GMAM for project problem will be presented in section 3.2. Chapter 4 discusses the results obtained in previous chapters. Chapter 5 is dedicated to note the original work that went into the course project. The conclusion of the study of GMAM and the course project is given in Chapter 6.

Chapter 2

Methodology

In this section, we will see the methodology of the generalized mode acceleration method(GMAM) and try to derive the mode displacement method and mode acceleration method from the GMAM[1].
The linear equations of motion of a discrete or discretized system with N degrees of freedom is written as,

$$\mathbf{M}\ddot{q} + \mathbf{K}q = f(t) \tag{2.1}$$

where M and K are respectively the N x N symmetric linear mass and stiffness matrices, q is the vector of the degrees of freedom, and is the vector of external loads. The notation $\ddot{q}$ represents the second time-derivative of q. We assume that M and K are positive definite. The case where K is semi-definite can be handled using the concept of generalized inverses.

We will consider undamped system for the analysis. In order to calculate free vibration eigen modes and eigen values, we solve

$$\left(\mathbf{K} - \omega_s^2\mathbf{M}\right) x_s = 0$$

where ω_s and x_s are respectively the free vibration circular frequency and modes. Due to the symmetry and positiveness of the mass and stiffness matrices, there are N independent modes and we assume that the eigensolutions are numbered such that $(\omega_1 < \omega_2 \cdots < \omega_N)$. We assume that solution can be expressed as linear combination of all eigen modes.

$$q(t) = \sum_{s=1}^{N} \eta_s(t) x_s \tag{2.2}$$

Where, η_s are the modal co-ordinates, which represent temporal evolution of particular vibration mode. Now, we substitute equation 2.2 in equation 2.1 and pre-multiply by $x_r^T\mathbf{M}$. By using orthogonality conditions, we get N uncoupled equations of motion for vibration modes η_s.

$$\ddot{\eta_s} + \omega_s^2\eta_s = \phi_s \tag{2.3}$$

where, $s = 1, 2, 3 \cdots N$

Taking Laplace transform on both the sides,

$$s^2\overline{\eta}_s - s\eta_s(0) - \eta_s^{'}(0) + \omega_s^2(s\overline{\eta}_s - \eta_s(0)) = \overline{\phi}_s$$

Where, $\overline{x}$ is laplace transform of x.
Substituting initial conditions, $\eta_s(0) = \eta_s^{'}(0) = 0$,

$$\overline{\eta}_s(t) = \frac{\overline{\phi_s}}{\omega_s^2 + s^2}$$

substitute above equation in equation 2.2, we have

$$\overline{q}_{ex}(t) = \sum_{s=1}^{N} x_s \frac{\overline{\phi}_s}{\omega_s^2 + s^2} \tag{2.4}$$

Since $\overline{\phi}_s = x_s^T \overline{f}$

$$\overline{q}_{ex}(t) = \sum_{s=1}^{N} \frac{x_s x_s^T \overline{f}}{\omega_s^2 + s^2} \tag{2.5}$$

Equation 2.5 is a exact solution since all the mode shapes are considered. Above equation can be written as,

$$\overline{q}_{ex}(t) = \sum_{r=1}^{k} \frac{x_r x_r^T \overline{f}}{\omega_r^2 + s^2} + \sum_{r=k+1}^{N} \frac{x_r x_r^T \overline{f}}{\omega_r^2 + s^2} \tag{2.6}$$

Mode displacement method (MDM)

If we only retain only first r modes, we get a mode displacement method.

$$\overline{q}_{mdm}(t) = \sum_{r=1}^{k} \frac{x_r x_r^T \overline{f}}{\omega_r^2 + s^2}$$

By taking inverse transform on both side, we can get solution in time domain.

Generalized mode acceleration method

Following the steps given in paper [1], we can reduce equation 2.6 to n^{th} approximate solution as,

$$\overline{q}_n(t) = \sum_{r=1}^{k} \frac{x_r x_r^T \overline{f}}{\omega_r^2 + s^2} + \left(\mathbf{K}^{-1} - \sum_{r=1}^{k} \frac{x_r x_r^T}{\omega_r^2}\right) \sum_{j=1}^{n} \left(-s^2 \mathbf{M}\mathbf{K}^{-1}\right)^{j-1} \overline{f} \tag{2.7}$$

By putting appropriate value of n, we can get n^{th} order approximation from equation 2.7. Here, $\overline{f}$ is $N \times 1$ dimensional vector of laplace transform of forces at each DOF.

Mode acceleration method (MAM)

We can say mode displacement method is the 0^{th} order approximation of GMAM and mode acceleration method is 1^{th} order approximation. Put $n = 1$,

$$\overline{q}_n(t) = \sum_{r=1}^{k} \frac{x_r x_r^T \overline{f}}{\omega_r^2 + s^2} + \left(\mathbf{K}^{-1} - \sum_{j=1}^{n} \frac{x_r x_r^T}{\omega_r^2}\right) \overline{f}$$

Taking inverse transform of equation 2.7 to shift to time domain

$$q_n(t) = L^{-1}\left[\sum_{r=1}^{k} \frac{x_r x_r^T \overline{f}}{\omega_r^2 + s^2}\right] + L^{-1}\left[\left(\mathbf{K}^{-1} - \sum_{r=1}^{k} \frac{x_r x_r^T}{\omega_r^2}\right) \sum_{j=1}^{n} \left(-s^2 \mathbf{M}\mathbf{K}^{-1}\right)^{j-1} \overline{f}\right]$$

$$q_n(t) = \sum_{r=1}^{k} x_r x_r^T L^{-1}\left[\frac{\overline{f}}{\omega_r^2 + s^2}\right] + \left(\mathbf{K}^{-1} - \sum_{r=1}^{k} \frac{x_r x_r^T}{\omega_r^2}\right) L^{-1}\left[\sum_{j=1}^{n} \left(-s^2 \mathbf{M}\mathbf{K}^{-1}\right)^{j-1} \overline{f}\right]$$

$$q_n(t) = \sum_{r=1}^{k} x_r x_r^T L^{-1}\left[\frac{\overline{f}}{\omega_r^2 + s^2}\right] + \left(\mathbf{K}^{-1} - \sum_{r=1}^{k} \frac{x_r x_r^T}{\omega_r^2}\right) L^{-1}\left\{\left[1 - s^2 \mathbf{M}\mathbf{K}^{-1} + s^4 \left(\mathbf{M}\mathbf{K}^{-1}\right)^2 - s^6 \left(\mathbf{M}\mathbf{K}^{-1}\right)^3 \cdots\right] \overline{f}\right\}$$

For forcing function of the form $F sin(\omega t)$

$$\overline{f} = \frac{F\omega}{\omega^2 + s^2}$$

After substituting above equation and taking inverse transform, use method of partial fraction to solve inverse laplace problem.

$$q_n(t) = \sum_{r=1}^{k} \frac{x_r x_r^T F}{\omega_r^2 - \omega^2}\left[sin(\omega t) - \frac{\omega}{\omega_r} sin(\omega_r t)\right] + \left(\mathbf{K}^{-1} - \sum_{r=1}^{k} \frac{x_r x_r^T}{\omega_r^2}\right)\left[1 + \omega^2 \mathbf{MK}^{-1} + \omega^4 (\mathbf{MK}^{-1})^2 + \omega^6 (\mathbf{MK}^{-1})^3 \cdots\right] f \quad (2.8)$$

Above equation is valid for $t > 0$, since we have taken $\delta(t), \delta^{'}(t), \delta^{''}(t) \cdots$ equal to zero.

F is a vector of force amplitude of size $N \times 1$ and f is the force vector. The first term in the equation is the mode displacement response and the second term is the correction term. if we take 1^{st} order approximation, the formula reduces to mode acceleration method. If the spatial distribution of force changes, just by changing the force vector in correction term the response can be updated immediately. Therefore, the method is flexible for dynamic response calculation.

We will use the superposition principle to take into account the effect of individual forces. Therefore, f will be a vector having only one entry non-zero corresponding to the DOF where force is applied.

For a harmonic force, it can be shown that that error in exact response and response of the system with n^{th} order of approximation is given by,

$$z_{ex} - z_{napprox} = \sum_{r=k+1}^{N} \frac{x_r x_r^T}{\omega_r^2 - \omega^2}\left(\frac{\omega^2}{\omega_r^2}\right)^n S$$

From the above equation, we can say that, if we consider the eigenfrequencies lower or equal to the forcing frequency ω, we will get a convergent solution whose accuracy increases as the value of n increases.

Chapter 3

Results

In this chapter the methodology discussed in Chapter 2 will be applied to simple 3 DOF system (in section 3.1) to validate the approach and results of GMAM for project problem will be presented in section 3.2.

3.1 Validating results of GMAM with Newmark β method for simple 3 DOF system

We will now use GMAM methodology on a simple 3 DOF system to validate the results. The system shown below in figure 3.1, has $k_1 = 1, k_2 = 2, k_3 = 1, m_1 = 1, m_2 = 2, m_3 = 2$ The forcing function applied at DOF 1 in this example is $5sin(1.5t)$, so we will take eigen frequencies lower than this, i.e. $\omega_1 = 0.359rad/s$ and $\omega_2 = 1rad/s$. We will approximate the solution using two mode shapes. The response of DOF 2 is plotted and total squared error is evaluated. The system was solved using direct integration technique, Newmark β,

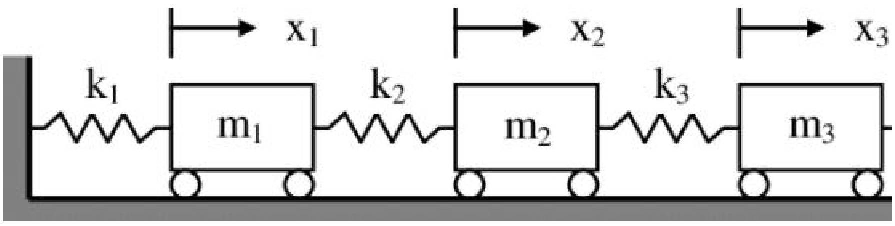

Figure 3.1: Three dof system [4]

with $\Delta t = 0.01$. For each solution squared error norm is calculated between exact solution and solution of n^{th} order.

- 0^{th} **order GMAM-MDM** - Results are shown in figure 3.2 with error norm of 12952.15.

- 1^{st} **order GMAM-MAM** - Results are shown in figure 3.3 with error norm of 7566.4.
- 2^{nd} **order GMAM** - Results are shown in figure 3.4 with error norm of 5849.6.
- 3^{rd} **order GMAM** - Results are shown in figure 3.5 with error norm of 5329.0.

We can see the error norm is reducing as the order of approximation is increasing. Results of Newmark β method were also cross checked with results of standard mode acceleration method independently.
We can conclude that the GMAM derivation and Octave code is correct and can be use for project problem.

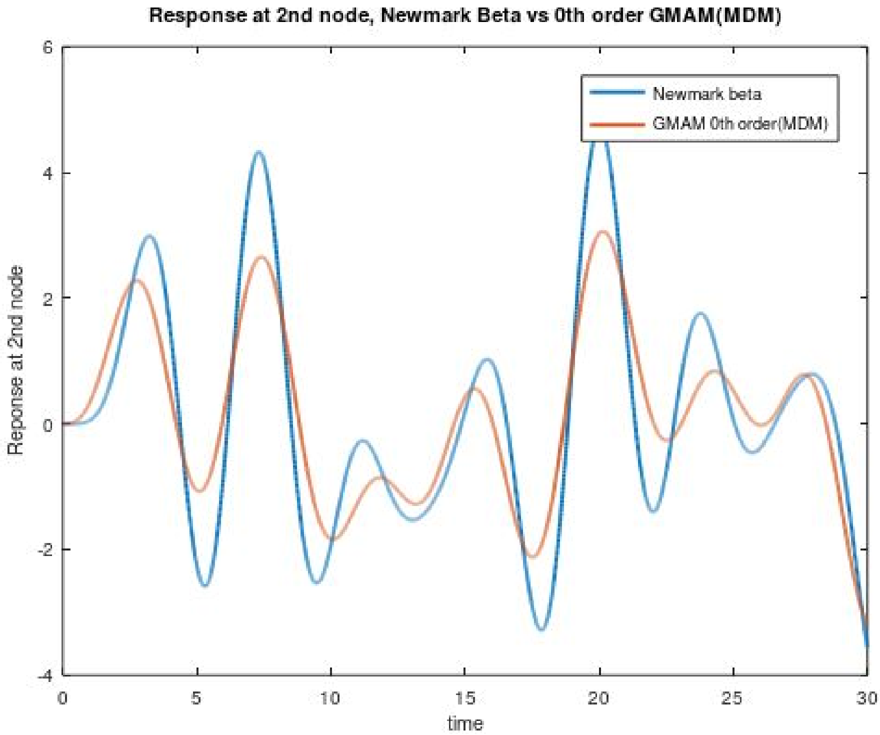

Figure 3.2: Response at 2nd node, Newmark Beta vs 0th order GMAM(MDM)

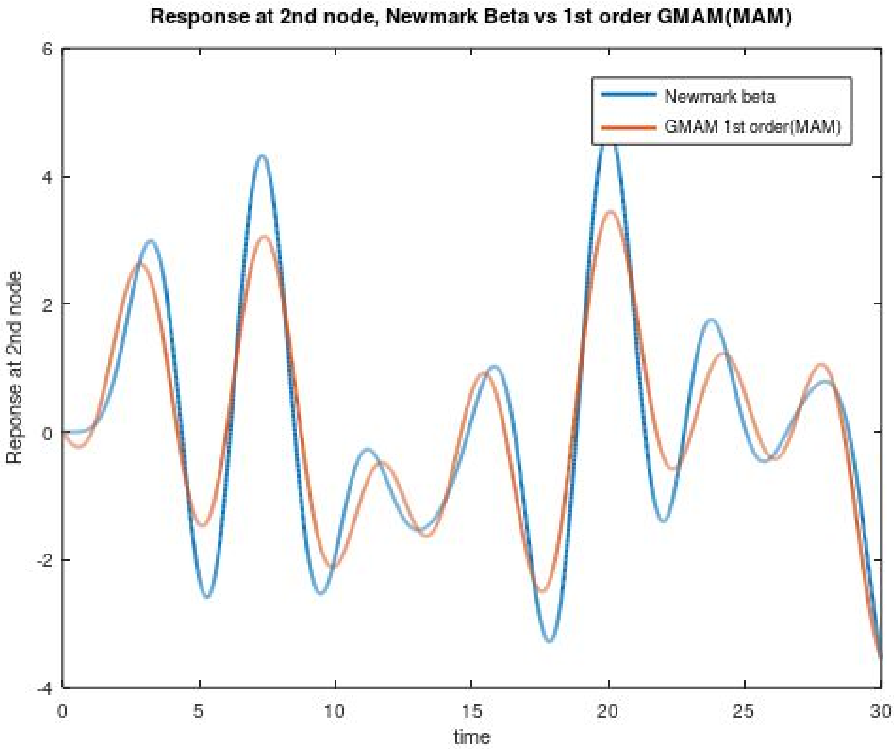

Figure 3.3: Response at 2nd node, Newmark Beta vs 1st order GMAM(MAM)

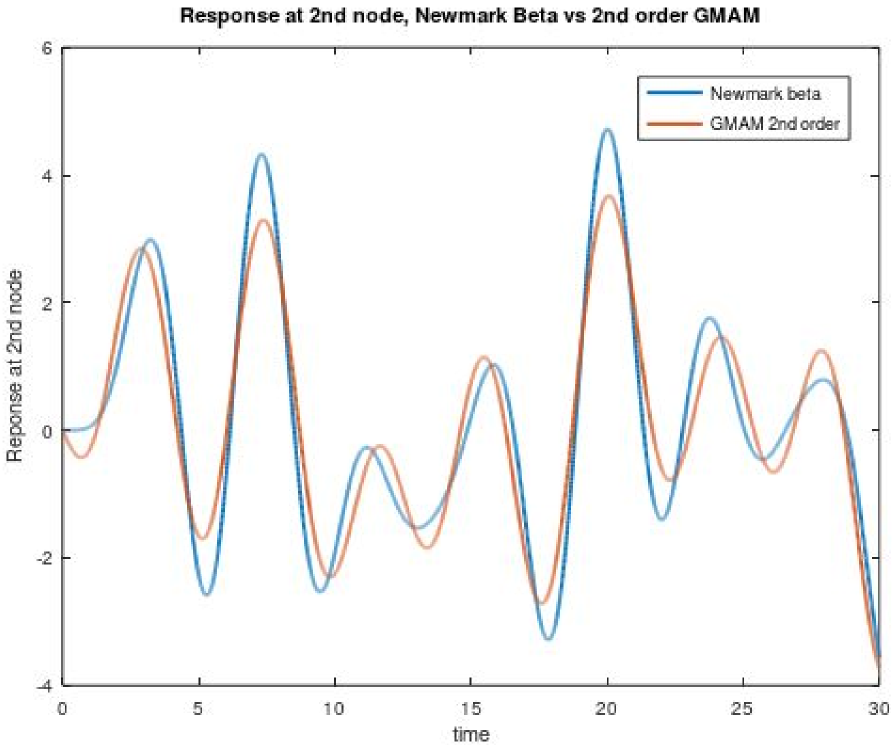

Figure 3.4: Response at 2nd node, Newmark Beta vs 2nd order GMAM

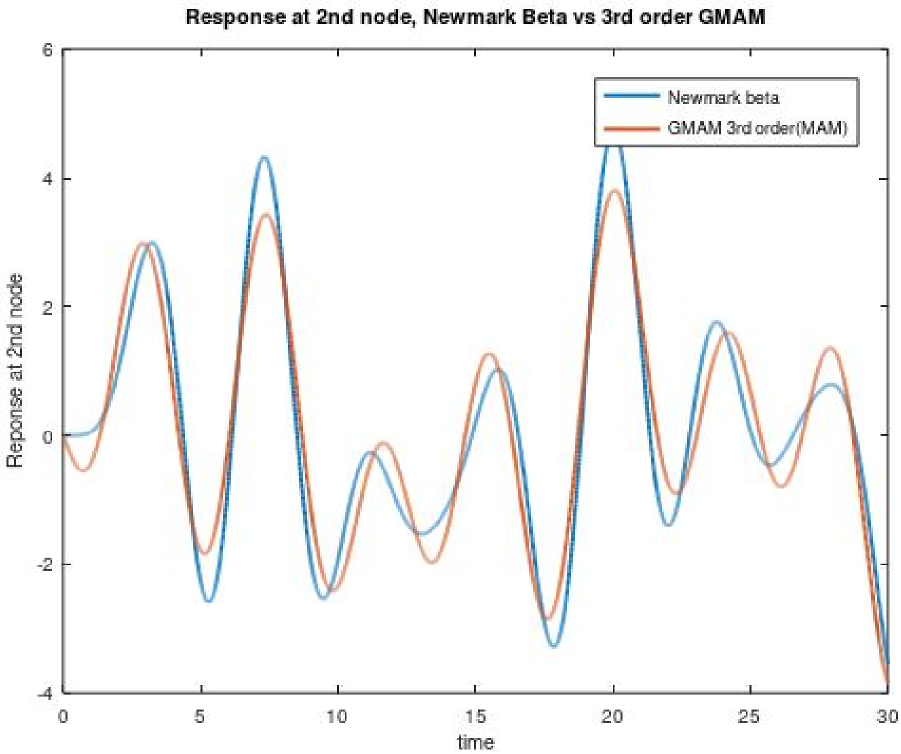

Figure 3.5: Response at 2nd node, Newmark Beta vs 3rd order GMAM

3.2 Implementing GMAM on project problem

The problem given in the project is two dimensional, we arrange the x and y coordinate one after the other, so that there are 3782 DOFs for 1891 nodes. In given problem forcing function at node 1891 is given as,

$$F_x = 10^5 sin(25000t), F_y = 10^5 sin(10000t)$$

We use superposition principle for forcing function, $F_x = 10^5 sin(25000t)$ $F_y = 10^5 sin(10000t)$

- **For $\mathbf{F_x}$**
  Use $F(3781) = 10^5$ and $\omega = 25000$ for calculating dynamics response. The 19^{th} eigen value of the system is 24799.32, so we will be considering only first 19 modes for response approximation.

- **For $\mathbf{F_y}$**
  Use $F(3782) = 10^5$ and $\omega = 10000$ for calculating dynamics response. The 4^{th} eigenvalue of the system is 8902.07, so we will be considering only the first 4 modes for response approximation.

We need to calculate the horizontal displacement at node 1891, that means, we have to calculate response at DOF 3781. The same problem is solved using the Newmark β method and results are compared. The error norms are $1.97 \times 10^{-8}, 0.966 \times 10^{-8}, 0.776 \times 10^{-8}$ and 0.736×10^{-8} for 0^{th},1^{st},2^{nd},3^{rd} order approximation respectively. We can see significant improvement for lower-order approximations and improvement decreases asymptotically for higher orders.

Horizontal displacement at node 1891 is plotted using 0^{th},1^{st},2^{nd},3^{rd} order approximation and compared with Newmark β method in figures 3.6,3.7,3.8,3.9 respectively.

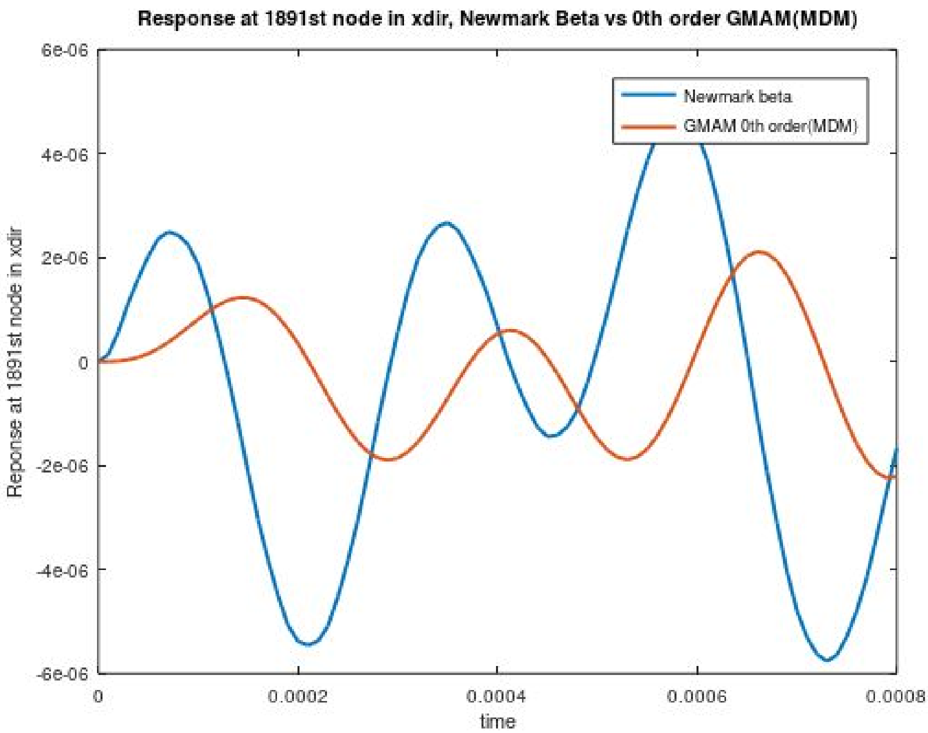

Figure 3.6: Horizontal response at 1891st node, Newmark Beta vs 0th order GMAM(MDM)

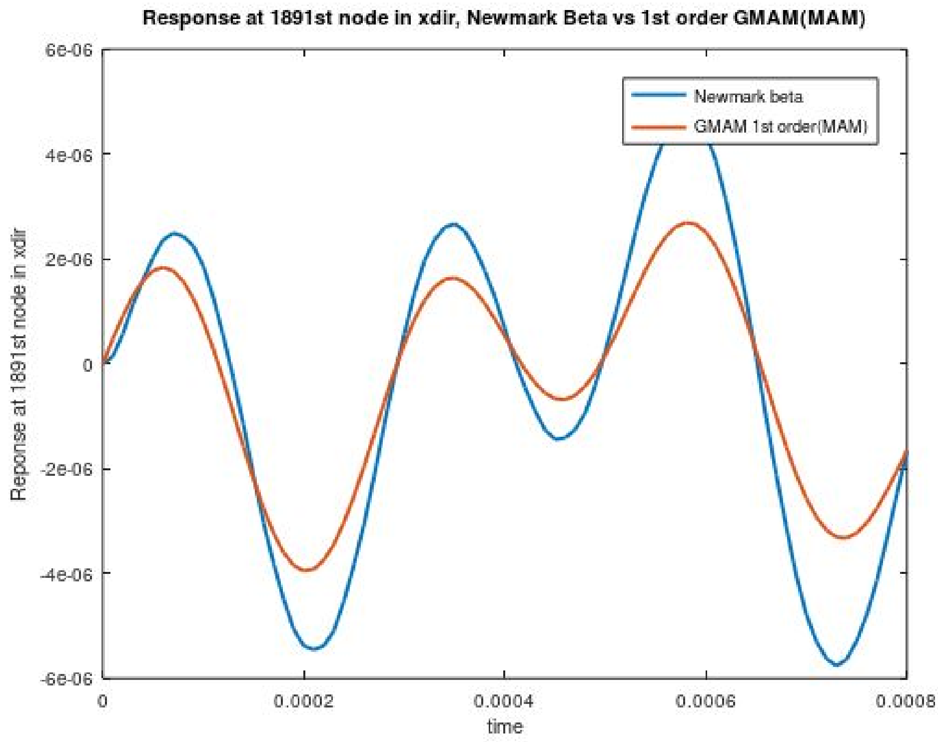

Figure 3.7: Horizontal response at 1891st node, Newmark Beta vs 1st order GMAM(MAM)

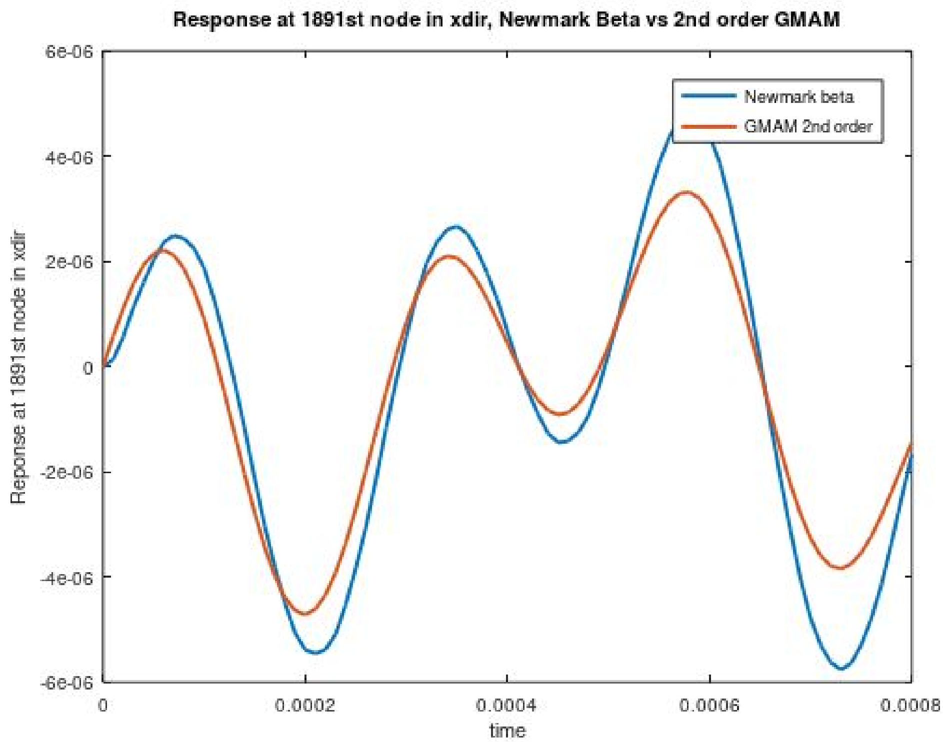

Figure 3.8: Horizontal response at 1891st node, Newmark Beta vs 2nd order GMAM

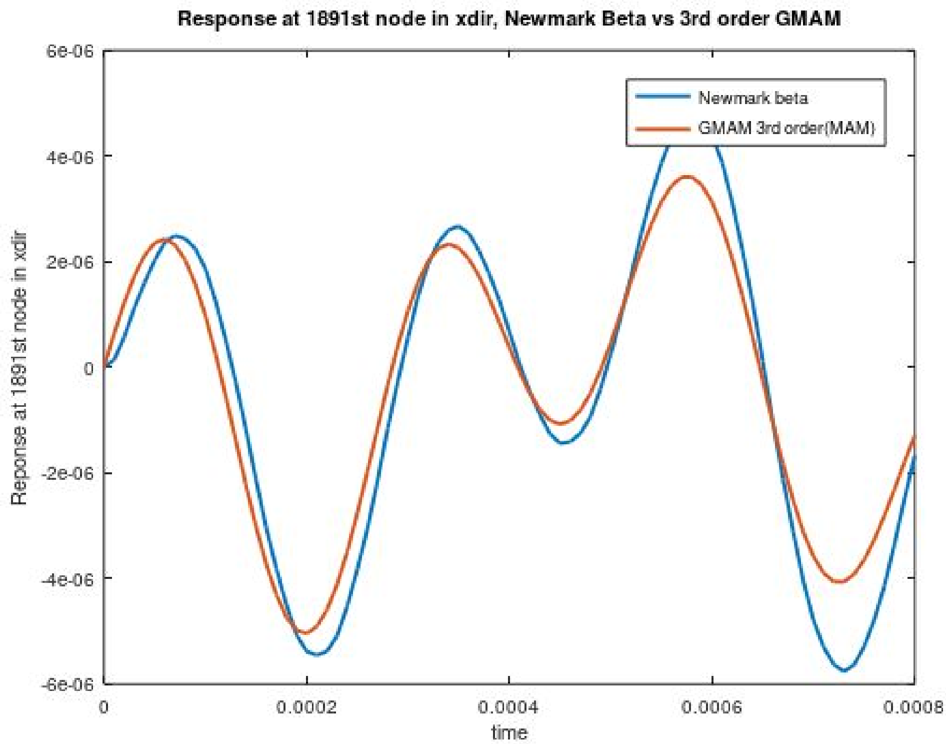

Figure 3.9: Horizontal response at 1891st node, Newmark Beta vs 3rd order GMAM

Chapter 4

Discussions

The generalized mode acceleration method requires the problem to be transferred from the time domain to the frequency domain using Laplace transform and after solving the problem in the frequency domain, we again shift back to the time domain using inverse transform. The improvement in solution is found to be significant in the lower order of approximations, but it slows down asymptotically as the order of approximation increases.

The generalized mode acceleration method proposed in the research paper is first implemented for a simple 3 DOF system in section 3.1, for which we can calculate the exact solution and can validate the results of the GMAM. The results are very convincing and as expected error norm is decreasing asymptotically. Visual inspection of the plot of the response of the 2nd DOF helps to confirm this claim. This example confirms the accuracy of the approach used to implement GMAM and octave code used for the same.

Now, the project problem is solved using the same method in section 3.2. The forcing function is expressed as the superposition of two forces and the resultant response is calculated as the superposition of responses to two forces. The same problem is solved using direct integration and results are found to be in close agreement. The error norm is decreasing asymptotically as expected.

Chapter 5

Original work

The derivation of generalized mode acceleration method i.e. equation 2.7 is referred from research paper [1]. The implementation of this approach is my original work as listed below.

- Derivation of response of the system in the time domain by taking Laplace inverse of equation 2.7 for the 3 DOF problem and project problem.
- Octave code used for Newmark β method.
- Octave code for MDM and MAM.
- Octave code for implementing GMAM.

Chapter 6

Conclusion

The generalized mode acceleration method proposed in research paper [1], is found to be very effective in improving the solutions of mode displacement method and mode acceleration method. The method offers an extra handle to the designer. the designer can then decide the order of approximation by taking into account the needed accuracy and available computational capacity.

The improvement in solution is found to be significant in the lower order of approximations, but it slows down asymptotically as the order of approximation increases. Therefore, the lower order of approximation can be used if there is a limit on computational capacity and desirable accuracy is obtained. The second term in equation 2.8 is the correction term added in the mode displacement method. If the spatial variation of force gets changes, this correction term can be calculated just by changing the force vector. because of this, the method offers very flexible approach to find the dynamic response.

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